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Halley's method

In numerical analysis, **Halley's method** is a root-finding algorithm used for functions of one real variable with a continuous second derivative. It is named after its inventor Edmond Halley.

The algorithm is second in the class of Householder's methods, after Newton's method. Like the latter, it iteratively produces a sequence of approximations to the root; their rate of convergence to the root is cubic. Multidimensional versions of this method exist.

Halley's method exactly finds the roots of a linear-over-linear Padé approximation to the function, in contrast to Newton's method or the Secant method which approximate the function linearly, or Muller's method which approximates the function quadratically.[1]

Method

Edmond Halley was an English mathematician who introduced the method now called by his name. Halley's method is a numerical algorithm for solving the nonlinear equation $f(x) = 0$. In this case, the function f has to be a function of one real variable. The method consists of a sequence of iterations:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

beginning with an initial guess x_0.[2]

If f is a three times continuously differentiable function and a is a zero of f but not of its derivative, then, in a neighborhood of a, the iterates x_n satisfy:

$$|x_{n+1} - a| \le K \cdot |x_n - a|^3, \text{ for some } K > 0.$$

This means that the iterates converge to the zero if the initial guess is sufficiently close, and that the convergence is cubic.[3]

The following alternative formulation shows the similarity between Halley's method and Newton's method. The expression $f(x_n)/f'(x_n)$ is computed only once, and it is particularly useful when $f''(x_n)/f'(x_n)$ can be simplified:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{2}} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 - \frac{f(x_n)}{f'(x_n)} \cdot \frac{f''(x_n)}{2f'(x_n)}\right]^{-1}.$$

When the second derivative is very close to zero, the Halley's method iteration is almost the same as the Newton's method iteration.

Derivation

Consider the function

$$g(x) = \frac{f(x)}{\sqrt{|f'(x)|}}.$$

Any root r of f that is *not* a root of its derivative is a root of g (i.e., $g(r) = 0$ when $f(r) = 0 \neq \sqrt{|f'(r)|}$), and any root r of g must be a root of f provided the derivative of f at r is not infinite. Applying Newton's method to g gives

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

with

$$g'(x) = \frac{2[f'(x)]^2 - f(x)f''(x)}{2f'(x)\sqrt{|f'(x)|}},$$

and the result follows. Notice that if $f'(c) = 0$, then one cannot apply this at c because $g(c)$ would be undefined.

Cubic convergence

Suppose a is a root of f but not of its derivative. And suppose that the third derivative of f exists and is continuous in a neighborhood of a and x_n is in that neighborhood. Then Taylor's theorem implies:

$$0 = f(a) = f(x_n) + f'(x_n)(a - x_n) + \frac{f''(x_n)}{2}(a - x_n)^2 + \frac{f'''(\xi)}{6}(a - x_n)^3$$

and also

$$0 = f(a) = f(x_n) + f'(x_n)(a - x_n) + \frac{f''(\eta)}{2}(a - x_n)^2,$$

where ξ and η are numbers lying between a and x_n. Multiply the first equation by $2f'(x_n)$ and subtract from it the second equation times $f''(x_n)(a - x_n)$ to give:

$$\begin{aligned} 0 = {} & 2f(x_n)f'(x_n) + 2[f'(x_n)]^2(a - x_n) + f'(x_n)f''(x_n)(a - x_n)^2 + \frac{f'(x_n)f'''(\xi)}{3}(a - x_n)^3 \\ & - f(x_n)f''(x_n)(a - x_n) - f'(x_n)f''(x_n)(a - x_n)^2 - \frac{f''(x_n)f''(\eta)}{2}(a - x_n)^3. \end{aligned}$$

Canceling $f'(x_n)f''(x_n)(a - x_n)^2$ and re-organizing terms yields:

$$0 = 2f(x_n)f'(x_n) + \left(2[f'(x_n)]^2 - f(x_n)f''(x_n)\right)(a - x_n) + \left(\frac{f'(x_n)f'''(\xi)}{3} - \frac{f''(x_n)f''(\eta)}{2}\right)(a - x_n)^3.$$

Put the second term on the left side and divide through by

$$2[f'(x_n)]^2 - f(x_n)f''(x_n)$$

to get:

$$a - x_n = \frac{-2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} - \frac{2f'(x_n)f'''(\xi) - 3f''(x_n)f''(\eta)}{6(2[f'(x_n)]^2 - f(x_n)f''(x_n))}(a - x_n)^3.$$

Thus:

$$a - x_{n+1} = -\frac{2f'(x_n)f'''(\xi) - 3f''(x_n)f''(\eta)}{12[f'(x_n)]^2 - 6f(x_n)f''(x_n)}(a - x_n)^3.$$

The limit of the coefficient on the right side as $x_n \to a$ is:

$$-\frac{2f'(a)f'''(a)-3f''(a)f''(a)}{12[f'(a)]^2-6f(a)f''(a)}.$$

If we take K to be a little larger than the absolute value of this, we can take absolute values of both sides of the formula and replace the absolute value of coefficient by its upper bound near a to get:

$$|a-x_{n+1}|\leq K|a-x_n|^3$$

which is what was to be proved.

To summarize,

$$\Delta x_{i+1}=\frac{3(f'')^2-2f'f'''}{12(f')^2}(\Delta x_i)^3+O[\Delta x_i]^4,\qquad \Delta x_i\triangleq x_i-a.$$ [4]

References

1. Boyd, J. P. (2013). "Finding the Zeros of a Univariate Equation: Proxy Rootfinders, Chebyshev Interpolation, and the Companion Matrix" (http://epubs.siam.org/doi/pdf/10.1137/110838297). *SIAM Review*. **55** (2): 375–396. doi:10.1137/110838297 (https://doi.org/10.1137%2F110838297).
2. Scavo, T. R.; Thoo, J. B. (1995). "On the geometry of Halley's method". *American Mathematical Monthly*. **102** (5): 417–426. doi:10.2307/2975033 (https://doi.org/10.2307%2F2975033). JSTOR 2975033 (https://www.jstor.org/stable/2975033).
3. Alefeld, G. (1981). "On the convergence of Halley's method". *American Mathematical Monthly*. **88** (7): 530–536. doi:10.2307/2321760 (https://doi.org/10.2307%2F2321760). JSTOR 2321760 (https://www.jstor.org/stable/2321760).
4. Proinov, Petko D.; Ivanov, Stoil I. (2015). "On the convergence of Halley's method for simultaneous computation of polynomial zeros". *J. Numer. Math*. **23** (4): 379–394. doi:10.1515/jnma-2015-0026 (https://doi.org/10.1515%2Fjnma-2015-0026). S2CID 10356202 (https://api.semanticscholar.org/CorpusID:10356202).

External links

- Weisstein, Eric W. "Halley's method" (https://mathworld.wolfram.com/HalleysMethod.html). *MathWorld*.
- *Newton's method and high order iterations (http://numbers.computation.free.fr/Constants/Algorithms/newton.html)*, Pascal Sebah and Xavier Gourdon, 2001 (the site has a link to a Postscript version for better formula display)