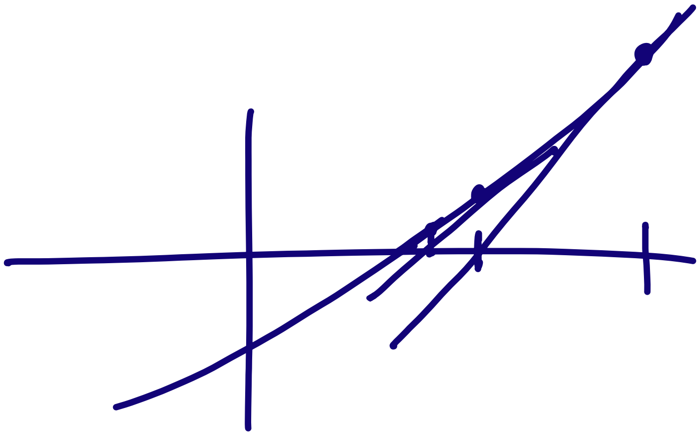
Newton's Method



① linearizing the problem at each iteration

$$f(x+h) \approx \boxed{f(x) + f'(x)\,\underline{h} = 0}$$

$$f'(x)\,h = -f(x)$$

$$h = -\frac{f(x)}{f'(x)}$$

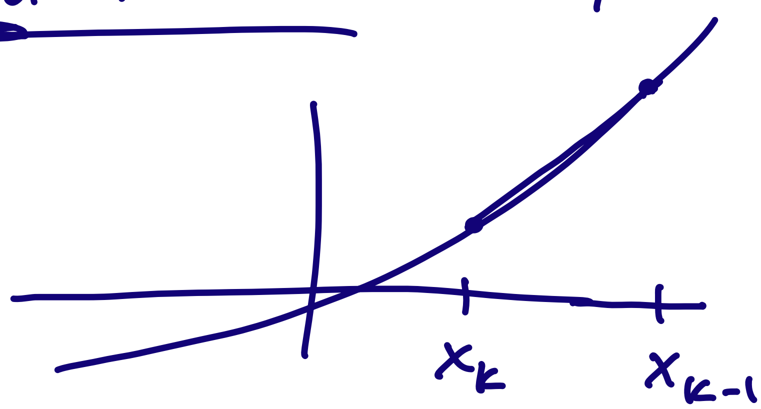
$$x \leftarrow x + h = x - \frac{f(x)}{f'(x)}$$

Drawback:

need $f(x)$

also $f'(x)$ $\qquad$ $J(x)$ $\quad n \times n$

secant method — replace $f'(x)$ w/ approx.



$$\boxed{f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}$$

Secant Method (Quasi-Newton Method)

x_0

for $k = 0, 1, 2, \ldots$

$$x_{k+1} = x_k - \frac{f(x_k)}{\left[\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right]}$$

end

$f'(x_k)$

save this cost

x_0, x_1 need two pts. to start

$r \approx 1.618$ convergence rate?

more iter. than N.M.

but... each iter is <u>cheaper</u>

<u>Safeguarded Methods</u>

Bisection method for conv. guarantees
\+
Secant method for speed

$[a, b]$ x_0, x_1

1. use secant method to generate x_{k+1}

2. $x_{k+1} \in [a, b]$?

   if yes
   $$m = x_{k+1}$$

   otherwise
   $$m = a + \left(\frac{b-a}{2}\right)$$

(3) update the bracket with m

Systems of Nonlinear Equations

$$\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$$

$$n = m$$

$$\vec{f}(\vec{x}) = A\vec{x} - \vec{b} = \vec{0}$$

$$A\vec{x} = \vec{b}$$

$$\vec{f}(\vec{x}) = \vec{f}(x_1, \ldots, x_n)$$

$$= \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$$

$$\vec{f}(\vec{x}) = \vec{0}$$

Jacobian

$$J(\vec{x}) = \frac{\partial \vec{f}}{\partial \vec{x}}$$

$$J_{ij}(\vec{x}) = \frac{\partial f_i(\vec{x})}{\partial x_j}$$

$$= \begin{pmatrix} \boxed{\frac{\partial f_1}{\partial x_1}} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$$f_1(x_1, x_2) = x_1^2 + \sin x_2 + 5$$

$$f_2(x_1, x_2) = x_1 + x_2^3$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & \cos x_2 \\ 1 & 3x_2^2 \end{pmatrix}$$

fixed point iteration

Newton's Method

Fixed Pt. Iter.

$$\vec{f}(\vec{x}) = \vec{0}$$

$$\Downarrow$$

$$\vec{g}(\vec{x}) = \vec{x}$$

$$\vec{x}_{k+1} = \vec{g}(\vec{x}_k)$$

$$|\underline{\vec{g}'(\vec{x}^*)}| < 1$$

$$\rho(J_g(x^*)) < 1$$

$$\rho(A) = \max_i \{ |\lambda_i| \mid \lambda_i \text{ eigenvalue of } A \}$$

$$J_g(x^*) = 0$$

N.M. for systems

Taylor Series

$$f(x+s) = f(x) + f'(x)s + \frac{f''(x)}{2}s^2 + \dots$$

$$\boxed{\vec{f}(\underbrace{\vec{x}+\vec{s}}) = \vec{f}(\vec{x}) + \boxed{J_f(\vec{x})\vec{s}}} + O(\|s\|^2)$$

$$\left\{\frac{1}{2}s^T H_f s\right\} + \dots$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

$$\frac{\partial f_i}{\partial x_j} : \delta x_j$$

find a step s such that x+s is a root of the linearized problem

$$\vec{f}(\vec{x}) + J_f(\vec{x})\vec{s} = 0$$

$$\underbrace{J_f(\vec{x})}_{n\times n}\vec{s} = \underbrace{-\vec{f}(\vec{x})}_{n}$$

solve an $n\times n$ linear system

$$\boxed{\begin{aligned} f'(x_k)\,h_k &= -f(x_k) \\ x_{k+1} &= x_k + h_k \end{aligned}}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

<u>N. M.</u>

$\vec{x}_0$

for $k = 0, 1, 2, \ldots$

Solve:

$$\underline{J_f(x_k)}\, s_k = -f(x_k)$$

$$x_{k+1} = x_k + s_k$$

end

$$\|A\| < \rho(A)$$

$$\underline{\|A\|} \leq \frac{\|Ax\|}{\|x\|}$$

$$\|Av\| = \|\lambda v\| = |\lambda| \, \|v\|$$

$$\underline{|\lambda|} = \frac{\|Av\|}{\|v\|}$$

Quasi-Newton Methods

- don't reevaluate J each iteration
- don't solve $Js = -f$ exactly

Damped Newton Method

$$x_{k+1} = \cancel{x_k + s_k}$$

$$x_k + \alpha_k s_k$$

$$0 < \alpha_k \leq 1$$

Broyden's Method

$$\underline{J} s = -f$$

$$s = -\boxed{J}^{-1} f$$

$$(A + xy^T)^{-1} \qquad A^{-1}$$

Directional Derivative

$$J_f = \begin{bmatrix} \boxed{\frac{\partial f_1}{\partial x_1} \quad \cdots \quad \frac{\partial f_1}{\partial x_n}} \\ \\ \frac{\partial f_n}{\partial x_1} \quad \cdots \quad \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

∇f_1 $\qquad\qquad\qquad$ $\vec{u}$

$\nabla f_1 \cdot \vec{u} = \square$ rate of change of f along direction $\vec{u}$.

$J_f(x)\vec{s}$ directional derivatives along $\vec{s}$.

$$S_k$$

$$\vec{x}_{k+1} = \vec{x}_k + S_k$$

$\vec{f}(\vec{x}_{k+1})$ $\qquad$ $\vec{f}(\vec{x}_k)$

$$s_k = x_{k+1} - x_k$$
$$y_k = f(x_{k+1}) - f(x_k)$$

$$\boxed{\boxed{J_f(x_{k+1})}\,\vec{s}_k \approx \vec{y}_k}$$

$$f'(x_{k+1})\cancel{s_k} \approx \frac{y_k}{s_k} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

Approximate J with a matrix B

B_k $\qquad x_{k+1} = x_k + s_k$

$$\boxed{B_{k+1} s_k = y_k}$$

"secant equation"

B_0

$$B_{k+1} = B_k \underbrace{\left(I - \frac{s_k s_k^T}{s_k^T s_k}\right)} + \frac{y_k s_k^T}{s_k^T s_k}$$

$$\boxed{B_{k+1} s_k} = 0 + \frac{y_k s_k^T \cancel{s_k}}{\cancel{s_k^T s_k}} = \boxed{y_k}$$

Let $z \perp s_k$

$$B_{k+1} z = B_k \left(I - \boxed{\frac{s_k s_k^T}{s_k^T s_k}} \right) z + \frac{y_k \cancel{s_k^T z}^{\,0}}{s_k^T s_k}$$

$$\boxed{B_{k+1} z = B_k z}$$

x_0 = initial guess

B_0 = initial estimate of J

for $k = 0, 1, 2, \ldots$

solve $B_k \vec{s}_k = -f(\vec{x}_k)$

$x_{k+1} = x_k + s_k$

$$B_{k+1} = B_k + \frac{y_k - B_k s_k}{s_k^T s_k} s_k^T$$

end

B^{-1}