Homework 6

Note: For all problems below, you may use another language (e.g., Python) and supporting libraries where Matlab/Octave is specified. You will need to translate the given code snippets into that language.

Least Squares

1. (Heath 3.5-adapted) Let $\mathbf{x}$ be the solution to the linear least squares problem $A\mathbf{x} \approx \mathbf{b}$, where
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{pmatrix}.$$
Let $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ be the corresponding residual vector. Which of the following three vectors is a possible value for $\mathbf{r}$? Why?
(a) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

2. Let the vectors $\mathbf{x}$ and $\mathbf{y}$ be given by the following Matlab code:
```
x = -5:5;
rng(12);
y = .2 * x .^2  + rand(size(x));
```
Find the best fitting parabola $ax^2 + bx + c$ to the data by solving the least squares problem in Matlab/Octave. What are the coefficients a, b, c that you found? Include your code.

3. If A is a $m \times n$ matrix ($m \geq n$) and it is not full rank (rank $(A) = r < n$), what is the dimension of the set of vectors $\mathbf{x}$ that minimize $\|A\mathbf{x} - \mathbf{b}\|_2$? Why?

4. We will explore solving a least squares problem with the QR factorization. Generate a pseudorandom 10×3 matrix and 10×1 vector in Matlab/Octave as follows:

```
rng(12);
A = rand(10,3);
b = rand(10,1);
rank(A);  % note A has full rank (rank = 3)
```

   (a) Let Matlab find the LS solution with backslash:

   ```
   x = A \ b;
   ```

   Now find the solution yourself using the QR decomposition of A. Get the QR decomposition as follows:

   ```
   [Q,R] = qr(A);
   ```

   Write Matlab/Octave code that uses Q and R to find the same solution x that Matlab determined. Include your code and results from the Matlab LS solve and your LS solve.

   (b) Now construct a matrix As which is rank-deficient as follows.

   ```
   R(3,3) = 0;
   As = Q*R; % make As a matrix with linearly dependent columns
   rank(As); % note As is rank-deficient (rank = 2)
   ```

   What solution does Matlab give in this case? Find this solution yourself with the QR decomposition of As.

5. The QR factorization is useful for solving least squares problems when the matrix has full rank. In class, we showed that the SVD can be used to find the minimum norm least squares solution when the matrix does not have full rank. While this works well, it is rather expensive due to the cost of computing the SVD. In this problem we will work out a more efficient solution. The complete orthogonal factorization of A has the form

$$A = U \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^T,$$

   where U and V are orthogonal matrices and R is invertible and upper triangular.

   (a) Devise an algorithm for computing the factorization efficiently. (Hint: this is straightforward to do using QR.)

   (b) Show that the factorization above can be used to solve the minimum norm least squares problem $\|A\mathbf{x} - \mathbf{b}\|_2$.

Eigenvalues and Eigenvectors

6. (Heath 4.3) Given an approximate eigenvector $\mathbf{x}$ of A, what is the best estimate (in the least squares sense) for the corresponding eigenvalue?

7. (Heath 4.1)

   (a) Prove that 5 is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & 3 & 3 & 1 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

   (b) Exhibit an eigenvector of A corresponding to the eigenvalue 5.

8. Let $A\mathbf{u} = \lambda\mathbf{u}$ for some eigenvalue λ and eigenvector $\mathbf{u}$. Find an eigenvalue/eigenvector pair for each of the following:

   (a) A^{-1}

(b) cA

(c) $A + cI$

(d) A^2

(e) $(A + 2I)(A - I)^2$

9. Let $B = A^T A$, $C = AA^T$, and $B\mathbf{u} = \lambda\mathbf{u}$.

   (a) Find an eigenvalue/eigenvector pair for C. You may assume $\lambda \neq 0$.

   (b) What happens if instead $\lambda = 0$?

10. Let Q be an orthogonal matrix and $Q\mathbf{u} = \lambda\mathbf{u}$. Show that $|\lambda| = 1$. Note that λ will in general be complex. (Hint: it is easier to assume that Q unitary instead. It is actually easier to treat Q as possibly complex-valued.)