Homework 1

Review of matrix and vector algebra

Some definitions for the following problems. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be scalars, $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be vectors, and $A, B \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times n}$ be matrices. Let the j^{th} *column* of A be denoted by $\mathbf{a}_j$, so that $A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix}$. Similarly, let the i^{th} *row* of B be denoted by $\mathbf{b}_i^T$, so that $B = \begin{bmatrix}\mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T\end{bmatrix}$. Let a_{ij} denote the entry of A in the i^{th} row and j^{th} column.

Let $\mathbf{e}_i$ represent the i^{th} canonical (standard basis) vector, every entry of which is 0, except for the i^{th} entry which is 1. E.g., $\mathbf{e}_1 = (1, 0, \dots, 0)^T$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)^T$.

1. It is important to stay aware of the dimensions of things, both as a sanity check and for understanding. These problems give you practice with this.

   For each problem, state the size of the resultant quantity (e.g., 1×1, 1×3, ...) . You do <u>not</u> need to compute the result.

   (a) $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4 \\ 5 \\ 6\end{bmatrix}$

   (b) $\begin{bmatrix}4 \\ 5 \\ 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$

   (c) $\begin{bmatrix}1 & 2 & 3 \\ 1 & 2 & 2\end{bmatrix}\begin{bmatrix}4 \\ 5 \\ 6\end{bmatrix}$

   (d) $\begin{bmatrix}1 & 2 & 3 \\ 1 & 2 & 2\end{bmatrix}\begin{bmatrix}1 & 4 \\ 2 & 5 \\ 3 & 6\end{bmatrix}$

   (e) $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 4 & -1 \\ 2 & 5 & 0 \\ 3 & 6 & 1\end{bmatrix}\begin{bmatrix}4 \\ 5 \\ 6\end{bmatrix}$

   (f) $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 4 \\ 2 & 5 \\ 3 & 6\end{bmatrix}\begin{bmatrix}4 \\ 5\end{bmatrix}$

   (g) $\alpha\mathbf{v}$

   (h) αB

   (i) $\mathbf{v}^T\mathbf{w}$

   (j) $A\mathbf{v}$

   (k) $C\mathbf{v}$

   (l) $\mathbf{v}^T A\mathbf{w}$

   (m) $\mathbf{v}^T C\mathbf{v}$

   (n) $\mathbf{v}^T C^T C\mathbf{v}$

2. Compute these <u>inner</u> products:

(a) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} =$

(c) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

3. Compute these <u>outer</u> products:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$

(b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$

(d) $\begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$

4. It is useful to know the result of a matrix times a canonical vector.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$

(c) $A\mathbf{e}_1 =$

(d) $A\mathbf{e}_2 =$

(e) $A\mathbf{e}_n =$

5. Similar results hold for a canonical row vector times a matrix. Find the following.

(a) $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} =$

(b) $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} =$

(c) $\mathbf{e}_1^T B =$

(d) $\mathbf{e}_j^T B =$

6. Is it true in general that $\mathbf{v}\mathbf{u}^T A\mathbf{w} = (\mathbf{u}^T A\mathbf{w})\mathbf{v}$? Explain.

7. Is it true in general that $AB = BA$? Explain if true, or give a counterexample if false.

8. What is the value of $\mathbf{e}_i^T A\mathbf{e}_j$ in terms of the entries of A?

9. It is very helpful for matrix algorithms to be able to think of matrix-matrix multiplication $C = AB$ in two different ways. The first is the way you were probably taught: The entries of the result C are $c_{ij} = \mathbf{a}_i^T\mathbf{b}_j$, where $\mathbf{a}_i^T$ are rows of A and $\mathbf{b}_j$ are columns of B. The second is as a sum of outer products $C = \sum_{j=i=1}^{n} \mathbf{a}_j\mathbf{b}_i^T$, where $\mathbf{a}_j$ are columns of A and $\mathbf{b}_i^T$ are rows of B. To practice these, do the following.

(a) *Inner products.* Compute using inner products,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

(b) *Outer products.* Verify that you get the same result using outer products (show the two intermediate matrices),

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} =$$

10. Show that $(AB)^T = B^T A^T$.

11. Assume A and B are invertible. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

12. Let $C \in \mathbb{R}^{m\times n}$, where $m < n$, and C is full rank so that CC^T is invertible. Let $P = I - C^T(CC^T)^{-1}C$.

    (a) Simplify CP.

    (b) Show that $PP = P$.

13. Let A be symmetric, i.e., $A = A^T$. Which of the following are necessarily also symmetric? Why?

    (a) A^{-1}

    (b) A^T

    (c) A^2