# Max Flow-Min Cut (Erickson 10)

Suppose we are given a directed graph $G=(V,E)$ that have "capacity" $c: E \to \mathbb{R}^{>0}$.

$\{x \in \mathbb{R} : x > 0\}$

A **flow** is a function $f: E \to \mathbb{R}^{\geq 0}$ ← $\{x \in \mathbb{R} : x \geq 0\}$

$f$ is **feasible** (for $G$) if $\forall e \in E$, $0 \leq f(e) \leq c(e)$

Can model

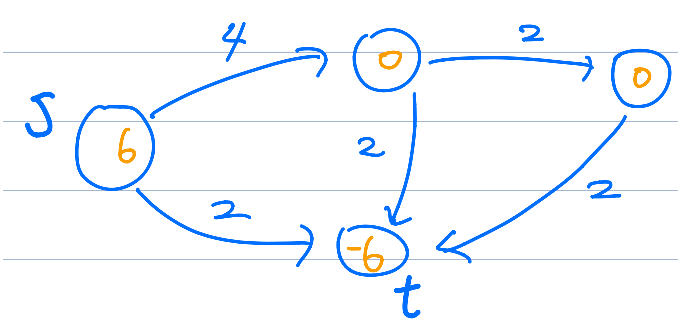
- railroads (kg/day)
- water/oil pipes (L/min)
- electrical network (A)
- communication network (MB/sec)

Given a flow $f: E \to \mathbb{R}^{\geq 0}$ and $v \in V$, define "net flow at $v$" to be

$$\Delta f(v) := \underbrace{\sum_{(v,w) \in E} f(v,w)}_{\text{outgoing flow}} - \underbrace{\sum_{(u,v) \in E} f(u,v)}_{\text{incoming flow}}$$

A vertex $v$ is called a "source" if $\Delta f(v) > 0$

"sink" if $\Delta f(v) < 0$.




$$\sum_{v \in V} \Delta f(v) = \sum_{e \in E} (f(u,w) - f(u,v)) = 0$$

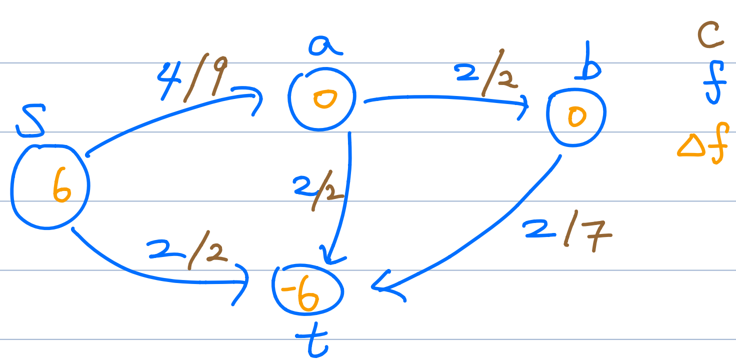
$$\sum_{v \text{ source}} \Delta f(v) = -\sum_{v \text{ sink}} \Delta f(v)$$

Call a flow $f$ "s-t flow" if the only source is $s$
and the only sink is $t$, and
let $\|f\| := \Delta f(s)$

## s-t Max Flow

**Input:** Directed graph $G=(V,E)$ with capacity $c: E \to \mathbb{R}^{\geq 0}$, $s, t \in V$.

**Output:** Find a s-t flow $f$ feasible for $G$ s.t. $\|f\|$ is maximized.



# Max-Flow Min-Cut Thm

**s-t Max Flow**

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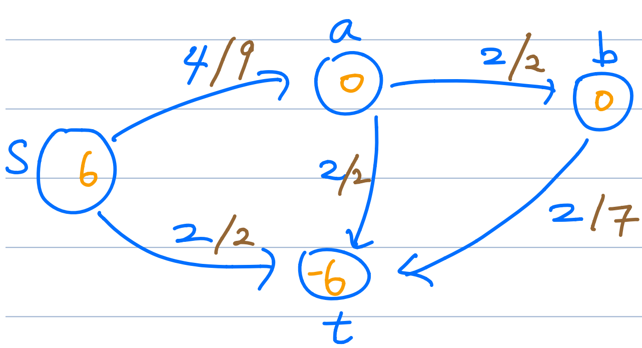



How can we "prove" that 6 is the best?

$(A,B) \neq (B,A)$

**Def.** A (directed) cut is a (ordered) pair $(A,B)$ s.t. $A, B \subseteq V$, $A \cap B = \emptyset$, $A \cup B = V$. It is called a s-t cut if $s \in A$, $t \in B$.

(E.g., $(\{s,a\},\{b,t\})$ and $(\{s,b\},\{a,t\})$ and $(\{s\},\{a,b,t\})$)

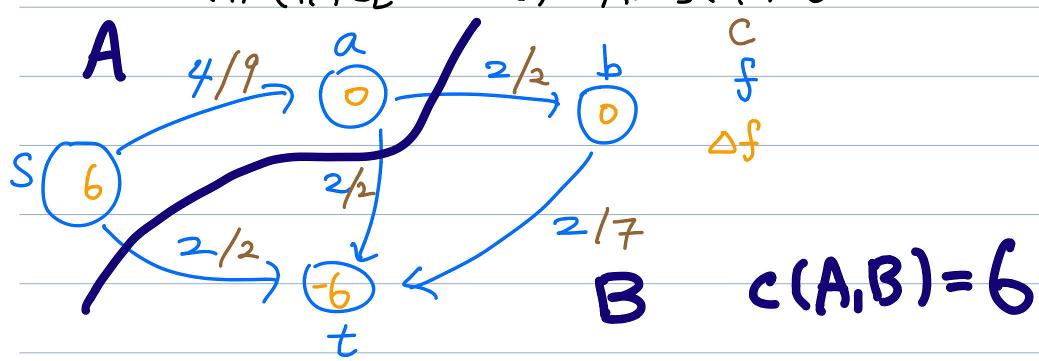
"Capacity" of $(A,B)$ $c(A,B) := \sum_{u \in A} \sum_{v \in B} c(u,v)$ (if $(u,v) \notin E$, assume $c(u,v) = 0$)

**Lemma 1.** Let $(A,B)$ be a $s$-$t$ cut and $f$ be feasible $s$-$t$ flow. Then $\|f\| \le c(A,B)$.

Furthermore, inequality becomes equality if

(i) $\forall (u,v) \in E$ with $u \in A, v \in B$ : $f(u,v) = c(u,v)$

(ii) $\forall (u,v) \in E$ with $u \in B, v \in A$: $f(u,v) = 0$




**Proof.** Let $f: E \to \mathbb{R}^{\ge 0}$ be any feasible $s$-$t$ flow.

$$
\begin{aligned}
\|f\| = \Delta f(s) &= \sum_{v \in A} \Delta f(v) \\
&= \sum_{v \in A} \Big( \sum_{u \in V} f(v,u) - \sum_{w \in V} f(w,v) \Big) && \left(\begin{array}{l}\text{if } a,b \in A, \\ \text{then } f(a,b) \text{ and} \\ -f(a,b) \text{ are added}\end{array}\right) \\
&= \sum_{v \in A} \Big( \sum_{u \in B} f(v,u) - \sum_{w \in B} f(w,v) \Big) \\
&\le \sum_{v \in A} \sum_{u \in B} f(v,u) && (f(e) \ge 0 \ \forall e) \\
&\le \sum_{v \in A} \sum_{u \in B} c(v,u) = c(A,B) && (f(e) \le c(e) \ \forall e)
\end{aligned}
$$

The "Furthermore" statement holds because with assumptions (i) and (ii), every inequalities must be equalities □

So, $$\max_{\substack{f:\text{ feasible} \\ s\text{-}t \text{ flow}}} \|f\| \le \min_{(A,B):\, s\text{-}t \text{ cut}} c(A,B).$$

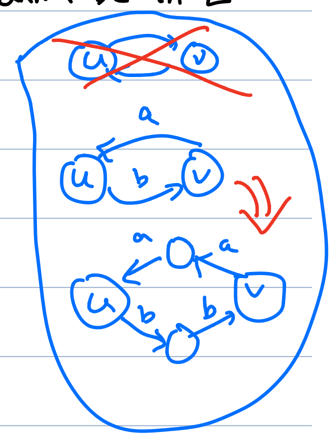
Other direction?

# Max-Flow Min-Cut Thm

**Theorem.** For any input $G=(V,E)$, $c: E \to \mathbb{R}^{\geq 0}$, $s,t \in V$ of s-t max flow.

$$\max_{f:\text{ feasible s-t flow}} \|f\| = \min_{(A,B):\text{ s-t cut}} c(A,B).$$

**Proof.** (Without loss of generality) will prove the thm when between any pair $u,v \in V$, $(u,v)$ and $(v,u)$ cannot be in $E$ simultaneously.




Let $f$ be a s-t flow. Will present an "algo." that (1) "improves $f$" or (2) "certify optimality".

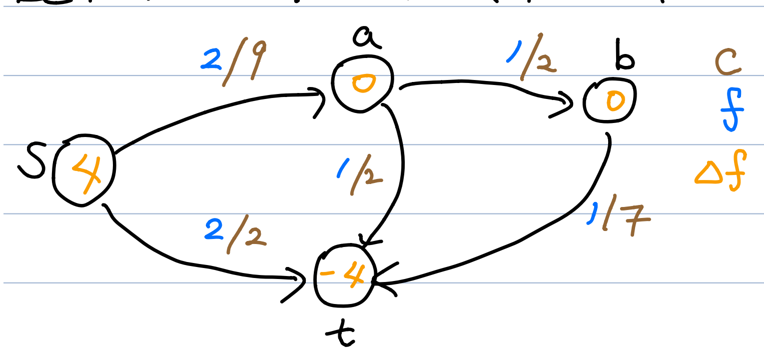
For each $u,v \in V$,

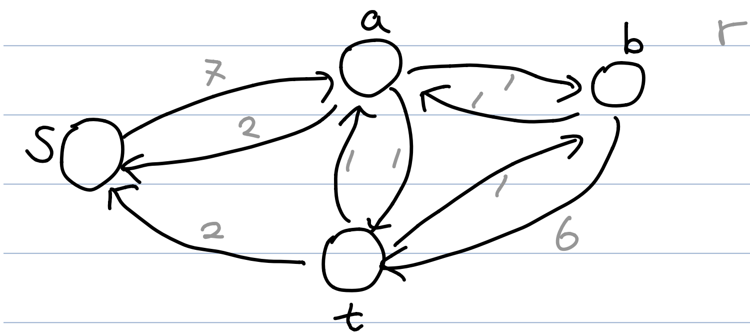
$$r(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{o.w.} \end{cases}$$

Then

$$r(u,v) + r(v,u) = \begin{cases} c(u,v) & \text{if } (u,v) \in E \\ c(v,u) & \text{if } (v,u) \in E \\ 0 & \text{o.w.} \end{cases}$$

Let $E' := \{(u,v) : r(u,v) > 0\}$





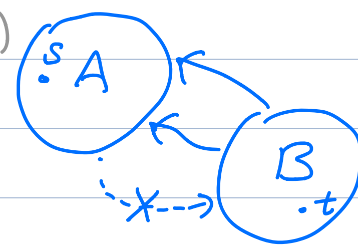

**Claim.** If $\nexists$ s-t path in $G' = (V, E')$, $\exists$ s-t cut $(A, B)$ s.t. $\|f\| = c(A,B)$.

**Proof.**

Let $A$ be the set of vertices reachable from $s$. Let $B = V \setminus A$.

For every $(u,v) \in A \times B$, $(= \{(a,b) : a \in A, b \in B\})$

$(u,v) \notin E'$.




if $(u,v) \in E$, then $r(u,v) = c(u,v) - f(u,v) = 0$

if $(v,u) \in E$, then $r(u,v) = f(v,u) = 0$

By "Furthermore" part of Lemma 1,

$$\|f\| = c(A,B).$$

□

↙ augmenting path.

So, let $p = (s, v_1, \ldots, v_{k-1}, t)$ be a s-t path in $G' = (V, E')$.

Also, let $F = \min_{(u,v) \in p} r(u,v) > 0$.

Consider a new flow $f' : E \to \mathbb{R}^{\geq 0}$ s.t. $\forall (u,v) \in E$

$$f'(u,v) = \begin{cases} f(u,v) + F & \text{if } (u,v) \in p \\ f(u,v) - F & \text{if } (v,u) \in p \\ f(u,v) & \text{o.w.} \end{cases}$$

**Claim.** $f'$ is feasible.

**Proof.** Fix $(u,v) \in E$.

If $(u,v) \in p$, $f'(u,v) = f(u,v) + F \leq f(u,v) + r(u,v)$

$$= f(u,v) + (c(u,v) - f(u,v)) \leq c(u,v)$$

If $(v,u) \in p$, $f'(u,v) = f(u,v) - F \geq f(u,v) - r(v,u)$

$$= f(u,v) - f(u,v) = 0. \qquad \square$$

**Claim.** $f'$ is s-t flow and $\|f'\| > \|f\|$.

**Proof.** $\Delta f'(s) = \Delta f(s) + F$,

$\Delta f'(v) = \Delta f(v) \quad \forall v \in V \setminus \{s,t\}$

$\Delta f'(t) = \Delta f(t) - F \qquad \square$

Therefore, $\exists$ feasible s-t flow $f'$ with $\|f'\| > \|f\|$.

(end of theorem) $\square$

# Ford-Fulkerson

Max-Flow

Given: $G=(V,E)$, $c: E \to \mathbb{R}^{>0}$, $s,t \in V$

Output: $f: E \to \mathbb{R}^{\geq 0}$

Let $f=0$ $(f(e)=0 \ \forall e \in E)$

While

Construct $G'=(V,E')$ and $r: E' \to \mathbb{R}^{>0}$ as previous theorem.

If $\nexists$ s-t path in $G'$

return

Else

$p = $ (s-t path in $G'$)

$F = \min_{e \in E'} r(e)$

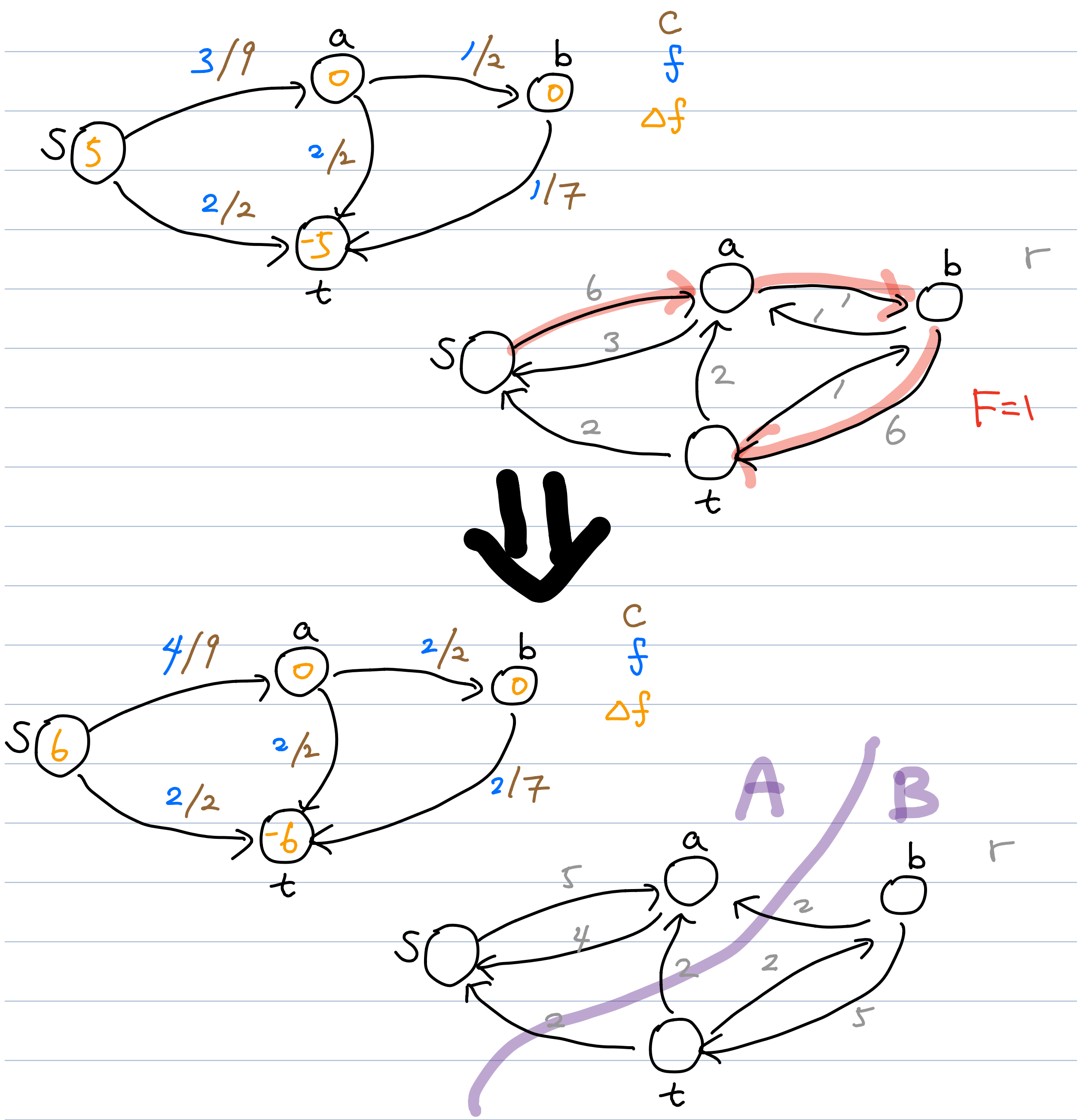
$$f'(u,v) = \begin{cases} f(u,v)+F & \text{if } (u,v) \in p \\ f(u,v)-F & \text{if } (v,u) \in p \\ f(u,v) & \text{o.w.} \end{cases}$$

$f = f'$

If current $f$ is not optimal, by Max-Flow Min-Cut thm, one can strictly increase $\|f\|$.

If all capacities are integers, then $F \geq 1$ in each iteration, so (# while loops) $\leq$ OPT ← (optimal $\|f\|$)

So running time $= O(m \cdot \text{OPT})$. ← pseudo-poly-time



c
f
Δf
S
a
b
t
3/9
1/2
2/2
2/2
1/7
5
0
0
-5
r
6
3
1
1
2
2
1
6
F=1
4/9
2/2
2/2
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6
0
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-6
A
B
5
4
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