Homework 7

Iterative Methods

1. (Heath 4.42)

   (a) If a matrix A has a simple dominant eigenvalue λ_1, what quantity determines the convergence rate of the power method for computing λ_1?

   (b) How can the convergence rate of power iteration be improved?

2. (Heath 4.24) Let A be an $n \times n$ real matrix of rank one.

   (a) Show that $A = \mathbf{u}\mathbf{v}^T$ for some nonzero real vectors $\mathbf{u}$ and $\mathbf{v}$.

   (b) Show that $\mathbf{u}^T\mathbf{v}$ is an eigenvalue of A.

   (c) If power iteration is applied to A, how many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue?

3. The asymptotic convergence rate of an iterative method is r if

   $$\lim_{k\to\infty} \|e_{k+1}\| = C\|e_k\|^r,$$

   where $e_k = x_k - x^*$ is the error at step k. Taking the log of both sides of the above equality

   $$\log \|e_{k+1}\| = \log C + r \log \|e_k\|.$$

   Thus the convergence rate can be seen to be the slope, r, of the log-log plot of error at $k+1$ vs error at k. Study the convergence rate of a few simple iterative methods in Matlab by implementing them and creating such plots. Include your code for each case.

   (a) Jacobi iteration to solve the system $A\mathbf{x} = \mathbf{b}$, with

   ```
   A = [ 3.5, 2, -1;  1 , 2.5, 0; 1, 2, -3.5]
   b = [ 1; 2; 3]
   ```

   and initial guess $\mathbf{x} = \mathbf{0}$.

   (b) The power method to find the dominant eigenvector of the symmetric matrix

   ```
   rng(12)
   A = rand(4,4);
   A = A + A';
   ```

   with initial guess

   ```
   x0 = rand(4,1);
   ```

   (c) Rayleigh quotient iteration to find an eigenvalue of the same matrix A from (b) starting with the same `x0`.

Nonlinear Equations

4. (Heath 5.1) Consider the nonlinear equation

   $$f(x) = x^2 - 2 = 0.$$

   (a) With $x_0 = 1$, as a starting point, what is the value of x_1 if you use Newton's method for solving this problem?

(b) With $x_0 = 1$ and $x_1 = 2$ as a starting points, what is the value of x_2 if you use the secant method for the same problem?

5. (Heath 5.12) Newton's method for solving a scalar nonlinear equation $f(x) = 0$ requires computation of the derivative of f at each iteration. Suppose that we instead replace the true derivative with a constant value d, that is, we use the iteration scheme

$$x_{k+1} = x_k - \frac{f(x)}{d}.$$

   (a) Under what condition on the value of d will this scheme be locally convergent?

   (b) What will be the convergence rate, in general?

   (c) Is there any value of d that would still yield quadratic convergence?

6. *Computer problem* (Heath 5.3) Implement the bisection, Newton, and secant methods for soving nonlinear equations in one dimension, and test your implementation by finding at least one root for each of the following equations. What termination criterion should you use? What convergence rate is achieved in each case?

   (a) $x^3 - 2x - 5 = 0$.

   (b) $e^{-x} = x$.

   (c) $x\sin(x) = 1$.

   (d) $x^3 - 3x^2 + 3x - 1 = 0$.

7. (Heath 5.10) Carry out one iteration of Newton's method applied ot the system of nonlinear equations

$$\begin{aligned} x_1^2 - x_2^2 &= 0 \\ 2x_1x_2 &= 1 \end{aligned}$$

   with starting value $\mathbf{x}_0 = (0, 1)^T$.

Optimization (Extra Credit)

8. (Heath 6.8) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2.$$

   (a) At what point does f attain a minimum?

   (b) Perform one iteration of Newton's method for minimizing f using as starting point $\mathbf{x}_0 = (2, 2)^T$.

   (c) In what sense is this a good step?

   (d) In what sense is this a bad step?

9. Given a quadratic function $\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, we will make one steepest gradient descent step from $\mathbf{x}_0 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$ to minimize ϕ.

   (a) What is the gradient direction $\mathbf{g}$ at $\mathbf{x}_0$?

   (b) Find the step size α along $\mathbf{d} = -\mathbf{g}$ that minimizes $\phi(\mathbf{x}_0 + \alpha\mathbf{d})$.

10. (Heath 6.9) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b} + c$$

   where A is an $n \times n$ symmetric positive definite matrix, $\mathbf{b}$ is an n-vector, and c is a scalar.

   (a) Show that Newton's method for minimizing this function converges in one iteration from any starting point $\mathbf{x}_0$.

   (b) If the steepest descent method is used on this problem, what happens if the starting value $\mathbf{x}_0$ is such that $\mathbf{x}_0 - \mathbf{x}^*$ is an eigenvector of A, where $\mathbf{x}^*$ is the solution?

11. *Programming assigment.* Write a program to find a minimum of Resenbrock's function,

$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2$$

using each of the following method:

(a) Steepest descent

(b) Newton

(c) Damped Newton (Newton's method with a line search).

You should try each of the methods from each of the three starting points:

(a) (-1,1)

(b) (0,1)

(c) (2,1)

Plot the path taken in the plane by the approximate solutions for each method from each starting point. In particular, your submission should consist of three plots, each one for each method, and the three paths (one for each starting point) in each plot.