power iteration

$\rightarrow \vec{v}_1$

Deflation

$\lambda_1, \vec{v}_1$

$\lambda_2, \vec{v}_2$

$\vec{v}_1$

$(I - v_1 v_1^T)\, y$

$y - (v_1^T y)\, v_1$

x_0

for $k = 0, 1, 2, \ldots$

$y_{k+1} = A x_k$

$y_{k+1} \leftarrow y_{k+1} - (v_1^T y_{k+1}) v_1$

$x_{k+1} = y_{k+1} / \| y_{k+1} \|$

end

Deflation λ_1, v_1

$$H v_1 = \hat{e}_1 \Rightarrow v_1 = H^T \hat{e}_1 \qquad \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(H A H^T)\, \hat{e}_1 = H A H^T \hat{e}_1 = H A v_1$$

$$= \lambda_1 H v_1 = \lambda_1 \hat{e}_1$$

$$x \longmapsto Ax$$

$$e_1 \longmapsto Ae_1 = \vec{a}_1$$

$$e_k \longrightarrow Ae_k = \vec{a}_k$$

$$HAH^T \hat{e}_1 = \lambda_1 \hat{e}_1$$

$$HAH^T = \begin{bmatrix} \lambda_1 & u_1^T \\ 0 & \\ \vdots & A_1 \\ 0 & \end{bmatrix}$$

A_1 $(n-1) \times (n-1)$

$\rightarrow$ λ_2 $\vec{w}_2$ $\longrightarrow$ $\vec{v}_2$

v_1 $\lambda_1 = ?$

$\|v_1\| = 1$

A , v_1

$$A v_1 = \lambda_1 v_1$$

Rayleigh Quotient

$$v_1^T A v_1 = \lambda_1 v_1^T v_1$$

$$\lambda_1 = \frac{v_1^T A v_1}{v_1^T v_1}$$

$$A x \approx \lambda x$$

$$\lambda = \frac{x^T A x}{x^T x}$$

QR Iteration

A

$A_0 = A$

for $k = 0, 1, 2, \ldots$

$A_k = Q_k R_k$

$A_{k+1} = R_k Q_k$

end

$A_k \rightarrow$ upper triangular matrix

(block upper tri.

1x1, 2x2 blocks)

$\lambda, \bar{\lambda}$

Schur Form of A

$A = Q T Q^T$

similarity transformations

$A_k = Q_k R_k \Rightarrow R_k = Q_k^T A_k$

$A_{k+1} = R_k Q_k$

$$A_{k+1} = Q_k^T A_k Q_k$$

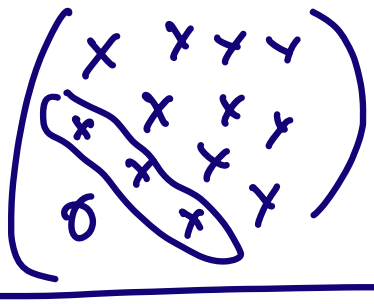
A

$$B = CAC^{-1}$$

$$A_k = Q^T A_0 Q$$

$$A_0 = A$$

Upper Hessenberg form

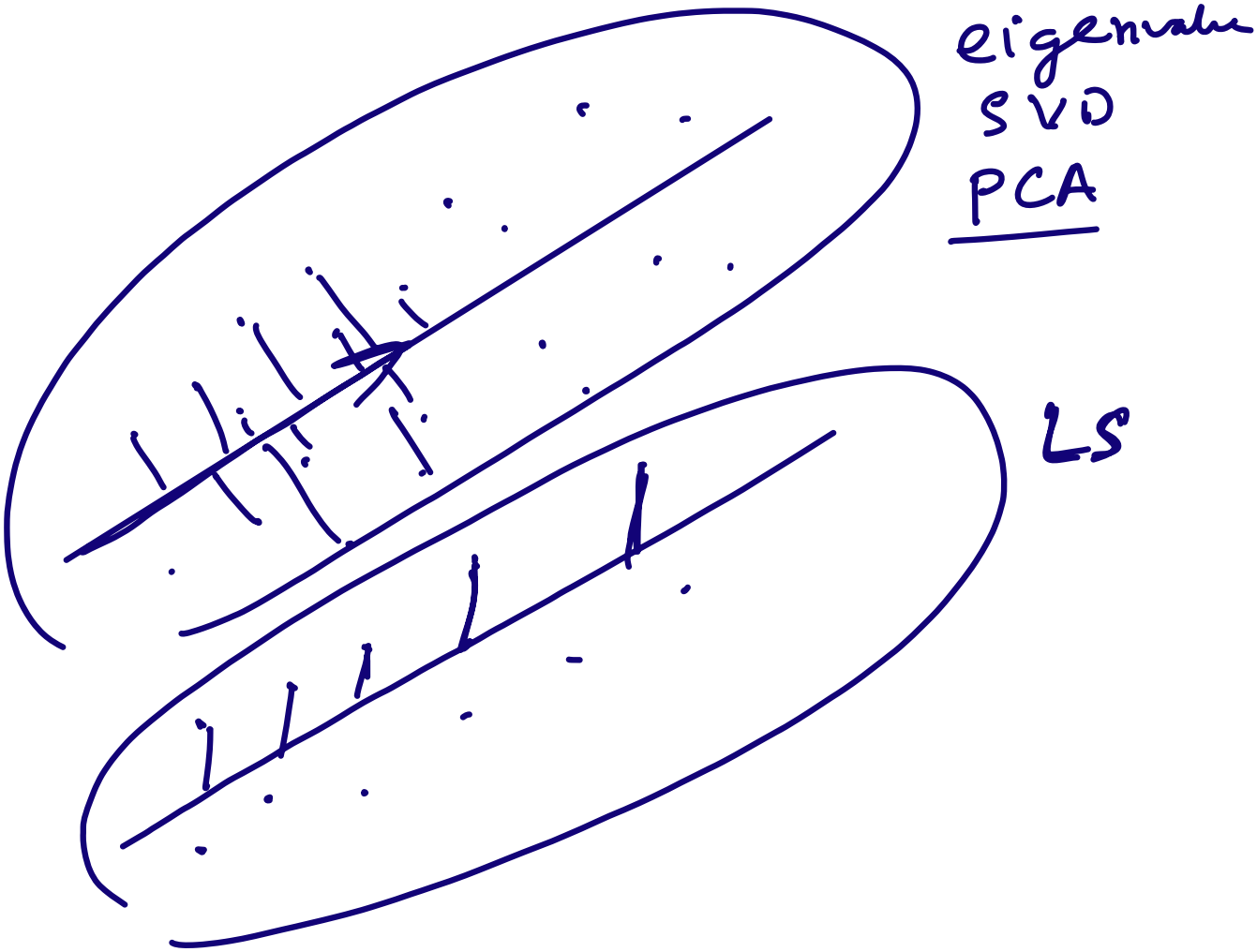


$$H_{n-2} \cdots H_1 \, A \, H_1^T \cdots H_{n-2}^T$$

Very large A

Krylov methods used

b, Ab, A^2b



Nonlinear Equations

scalar 1D

$ax = b$

$x = \frac{b}{a}$

$g(x) = b$

$f(x) = g(x) - b = 0$

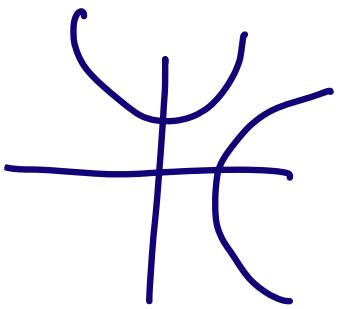
$f(x) = 0$

"root-finding problem"

x root of f.

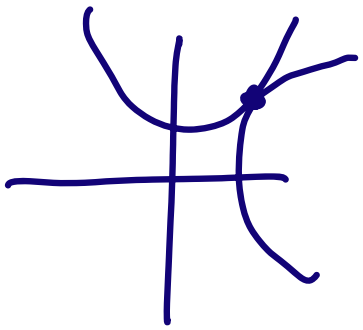
$x^2 - y + \alpha = 0$

$-x + y^2 + \alpha = 0$



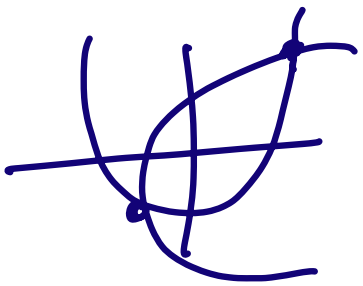
$\alpha = \frac{1}{2}$

no real solutions

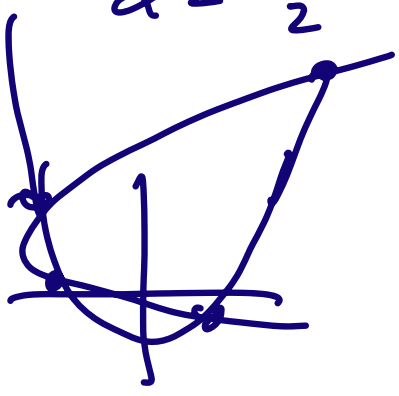


$\alpha = \frac{1}{4}$

1 solution



$\alpha = -\frac{1}{2}$

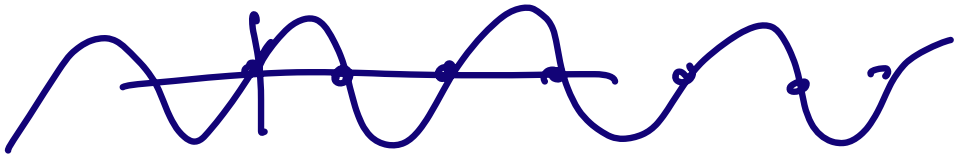


$e^x + 1 = 0$ no solutions

$e^x = -1$

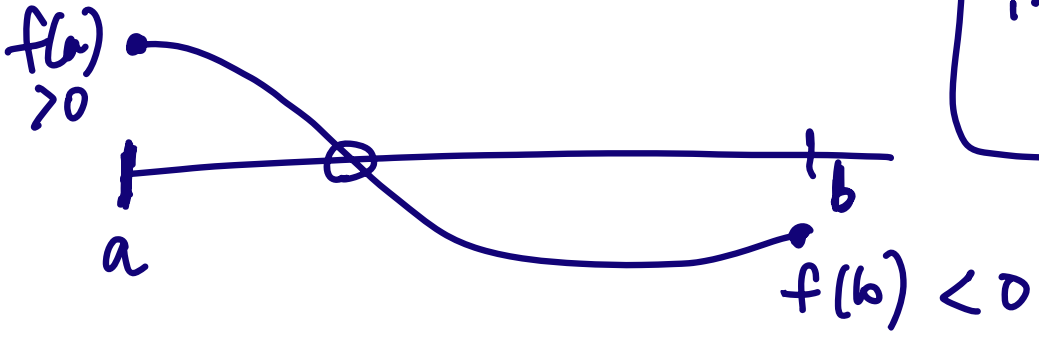
$e^{-x} - x = 0$ 1 solution

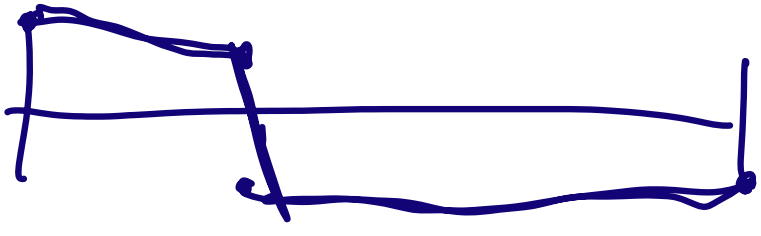
$\sin(x) - 0$



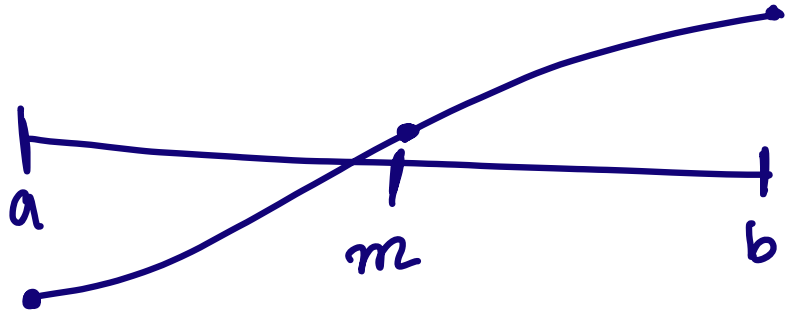
Intermediate Value Theorem f cont.

if $f(a) \cdot f(b) < 0$
$\exists$ root in $[a, b]$





Bisection Method



$[a, b]$

$f(a)\,f(b) < 0$

while $((b-a) > tol)$ do

$m = a + \frac{b-a}{2}$ $\qquad \frac{a+b}{2} = \left(a + \frac{b-a}{2}\right)$

if (sign $f(a)$ = sign $f(m)$)

$a \leftarrow m$

else

$b \leftarrow m$

end

end

- guaranteed to converge
- convergence rate:

$$e_{k+1} = \left[\frac{1}{2}\right] e_k$$

linear convergence
w/ constant $\frac{1}{2}$