CS 210 Practice Final

Name	
Student ID	
Signature	

Rules:

- Work individually. No notes, calculators, etc., permitted. Scratch paper will be provided by the proctor.
- The proctor is not allowed to answer individual questions during the exam, but may choose to make an announcement if something requires correction/clarification.
- If you see a possible error or ambiguity in an exam question, you may bring it to the attention of the proctor.
- You should answer every question to the best of your ability even if you think there is a issue/mistake in the question.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
11	3	
12	3	
13	3	
14	3	
15	9	
16	12	
17	12	
18	12	
Total	77	

True/False

For •	each	question.	indicate	whether	the statement	t is tru	e or	false l	bv cii	cling	T	or F	respectively.

- 1. (T/F) For any matrix $A \in \mathbb{R}^{n \times n}$, $\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A.
- 2. (TF) Every matrix $A \in \mathbb{R}^{m \times n}$ has a singular value decomposition factorization of the form $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal and $\Sigma \in \mathbb{R}^{n \times n}$ is diagonal.
- 3. (T) F) If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then $AU = U\Lambda$ from some orthogonal matrix $U \in \mathbb{R}^{n \times n}$ A = UNUT is the spectral decomposition of A \mathbf{a} d diagonal matrix Λ .
- 4. (T/F) Given a real, symmetric, invertible matrix A, power iteration, inverse power iteration, and Rayleigh quotient iteration are all algorithms that can be used to find an eigenvalue of A.
- 5. (T) If $A \in \mathbb{R}^{n \times n}$ is symmetric, then its singular value decomposition is the same as its eigenvalue decomposition.
- 6. $(TF) \|A^{-1}\|_2 = \frac{1}{\sigma_1}$, where σ_1 is the largest singular value of an invertible, real matrix A.

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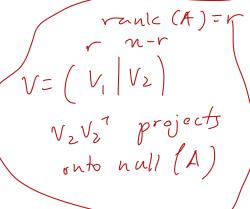
 7. $(TF) \|A^{-1}\|_2 = \frac{1}{\sigma_1}$, where σ_1 is the largest singular value of an invertible, real matrix A. composition of A^TA .
- 8. (TF) A nonlinear equation of a single variable f(x) = 0 is guaranteed to have one unique solution. 9. (TF) If f(x) is any function defined on a nonempty interval $[a,b] \subset \mathbb{R}^n$, with f(a) < 0 and f(b) > 0,
- then f has a root in [a, b]. Not necessarily. There are counterexamples if f is not continuous.
- 10. (T/F) In an iterative solver for $A\mathbf{x} = \mathbf{b}$, a small residual indicates that the error must be small as well.

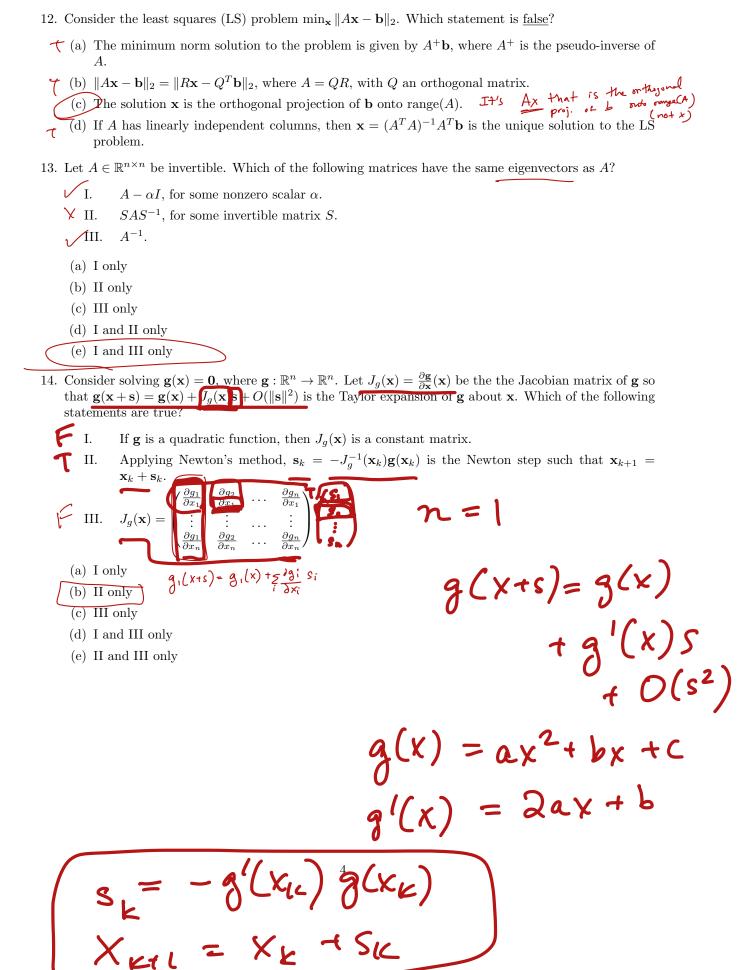
Multiple Choice

This depends on A. The error could be larse for poorly conditioned A.

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

- 11. Let $A \in$ and let $A = U\Sigma V^T$ be a singular value decomposition of A.
 - The singular values of A are uniquely determined.
 - II. U is always invertible.
 - VV^T is a projection matrix onto the nullspace of A.
 - (a) I and II only
 - (b) II and III only
 - (c) I and III only
 - (d) I, II, and III only





Written Response

15. Find the singular value decompositions of the matrices below. Show your work.

(a)
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

(b)
$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

(c)
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

(a)
$$A = e_1 e_1^{-1} + 2 e_2^{-1} e_2^{-1}$$

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -e_2 & e_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -e_1^{-1} & -e_2^{-1} & -e_2^{-1} \end{pmatrix}$$

$$(b) A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$= \sqrt{16} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1/55 & 2/55 \\ 2/55 & 1/55 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/52 & 1/52 \\ -1/52 & 1/52 \end{pmatrix}$$

$$A = \begin{pmatrix} 1/55 & 2/55 \\ 2/55 & 1/55 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & 0 & -1/52 \\ -1/52 & 1/52 \end{pmatrix}$$

$$(1-\lambda)^{2}-4=0 \Rightarrow \lambda^{2}-\lambda\lambda+1-4=0 \Rightarrow \lambda=-1,3$$

$$(\lambda-3)(\lambda+1)=0$$

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$$\lambda$$

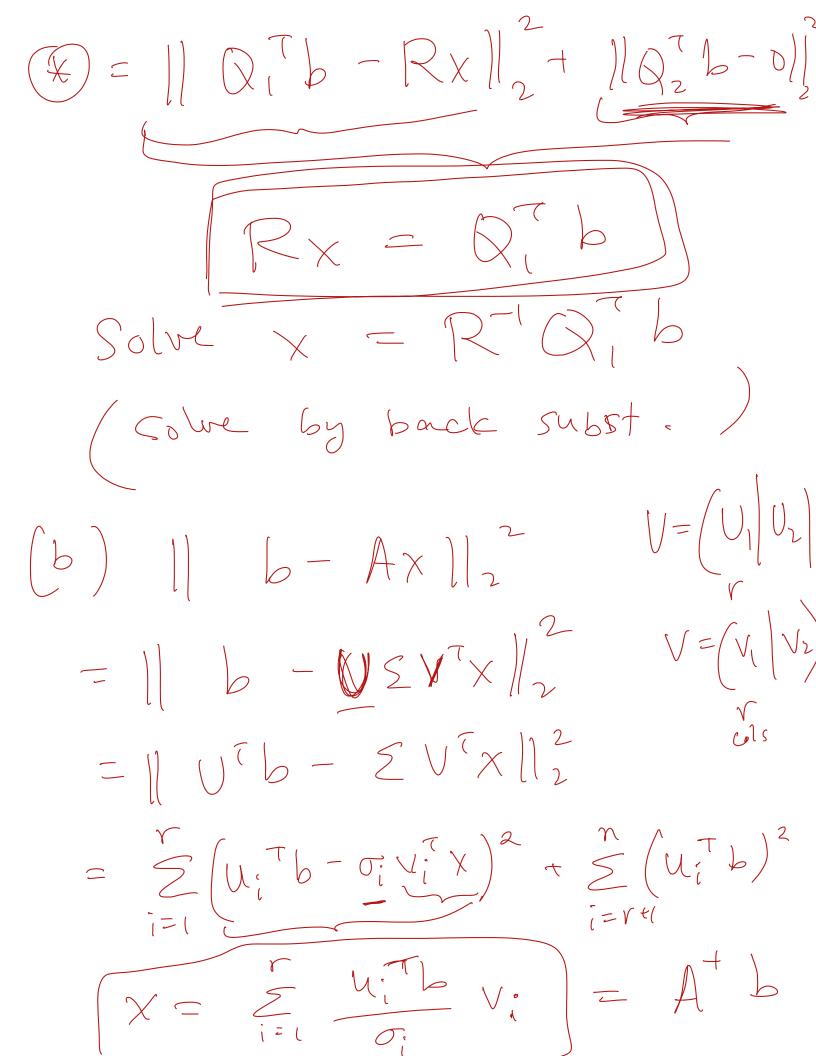
2 note - moved here so that singular values > 0 16. Least squares. Let $A \in \mathbb{R}^{m \times n}$, where m > n. Consider the least squares (LS) problem

$$\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2.$$

- Assume A has full rank. Show how you would use the QR decomposition $A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$ to solve the LS problem.
- Now assume A is rank-deficient with rank r < n. Show how you would use the Singular Value Decomposition $A = U\Sigma V^T$, with $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, to solve the LS problem.
 - (c) In parts (a) and (b) is the solution unique? Why or why not?
- (d) What does it say about **b** if $\min_{\mathbf{x}} ||\mathbf{b} A\mathbf{x}||_2 = 0$?

(a)
$$\|b - Ax\|_{2}^{2} = \|b - Ax\|_{2}^{2}$$

$$= \left[\left(\begin{array}{c} 27 \\ 0 \end{array} \right) \times \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \times \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$$



Ax=b

17. Iterative methods: matrix splitting. An iterative solver based on matrix splitting is given by

$$\mathbf{x}_{k+1} = T\mathbf{x}_k + \mathbf{b},$$

Sx = Tx+6

where A = S - T is the splitting. Consider the matrix

$$\begin{pmatrix} 24 & 4 & 9 \\ 4 & 16 & 9 \\ 9 & 9 & 21 \end{pmatrix}$$

Sx-Tx=b S-T)x=bAx=b

- (a) What are S and T for the Jacobi method?
- (b) Given that the eigenvalues of the iteration matrix $S^{-1}T$ are approximately $\{0.74273, -0.19897, -0.54376\}$ for the Jacobi method, does Jacobi converge for this matrix. At what rate?
- (c) What are S and T for the Gauss-Seidel method?
- (d) Given that the eigenvalues of the iteration matrix $S^{-1}T$ are approximately $\{0.00000, -0.17975, -0.22352\}$ for the Gauss-Seidel method, does Gauss-Seidel converge for this matrix? At what rate?

[S = diag(A) = D]

Six the diagonal part of D

$$T = -(L+U)$$

$$- x = S^{-1}T \times + S^{-1}b$$

ek11 = 5-17 ek

Yes, Jacobi 7 converges. because

converses linearly (rate = 1)

(with constant ~
$$|\lambda_1|$$
)

(C) $S \times_{k+1} = T \times_k + b$

$$S = D + L \quad (defined above)$$

(d) Yes, because
$$\rho(S^{-1}T) < 1$$
Quear, with a smaller constant ~ |\lambda_i|

lim 11 exall = C11 ex 11

$$f'(x) = 3x^2 - x + 2$$

- 18. Fixed point iteration. Let $f(x) = x^3 .5x^2 + 2x 1$. We want to find a real x such f(x) = 0. For convenience, you are given that the one real root of f occurs at x = .5.
 - (a) Find a fixed point iteration rule for computing x that uses only \times , +, and -.
 - (b) Show that convergence of your fixed point iteration is guaranteed sufficiently close to the solution.

(a)
$$x = g(x)$$

 $x = g(x)$
 $f(x) = 0 \iff 0$
 $x = .5 \times 3 + .5 \times 2 + .5$
 $x = g(x) = -.5 \times 3 + .5 \times 2 + .5$
(b) $|g'(x^{*})| \leq |g'(x^{*})| \leq$