Homework 2

Column space, null space, and rank-one matrices

1. (adapted from Strang I.3 4) For each matrix A_i given below, give the column space and null space of A_i:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

2. (adapted from Strang I.3 9) Consider the directed graph given in Figure 1, The 6×4 incidence matrix A of the graph is given by

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

   (a) Compute the rank of this matrix, dimensions of $\mathbf{N}(A)$ and $\mathbf{N}(A^T)$ ($\mathbf{N}(A)$ is the nullspace of A) .
   (b) Find a nonzero vector $\mathbf{x}$ in $\mathbf{N}(A)$.
   (c) Let k be the dimension of $\mathbf{N}(A^T)$, find k independent vectors in $\mathbf{N}(A^T)$. Hint: consider the structure of the graph.

3. (adapted from Strang I.3 11) For subspaces $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{R}^{20}$ with dimensions 5 and 10 respectively, what are all the possible dimensions of

   (i) $\mathbf{S} \cap \mathbf{T} = \{$all vectors that are in both subspaces$\}$
   (ii) $\mathbf{S} + \mathbf{T} = \{$all sums $s + t$ with s in $\mathbf{S}$ and t in $\mathbf{T}\}$
   (iii) $\mathbf{S}^{\perp} = \{$all vectors in $\mathbf{R}^{20}$ that are perpendicular to every vector in $\mathbf{S}\}$.

4. (adapted from Strang I.4 2) Given a row and column of a rank-1 matrix A_i, find the missing entries of A_i:

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ -1 & ? & ? \\ 2 & ? & ? \end{bmatrix} \qquad A_2 = \begin{bmatrix} ? & 3 & ? \\ 2 & 1 & 4 \\ ? & -1 & ? \end{bmatrix}$$

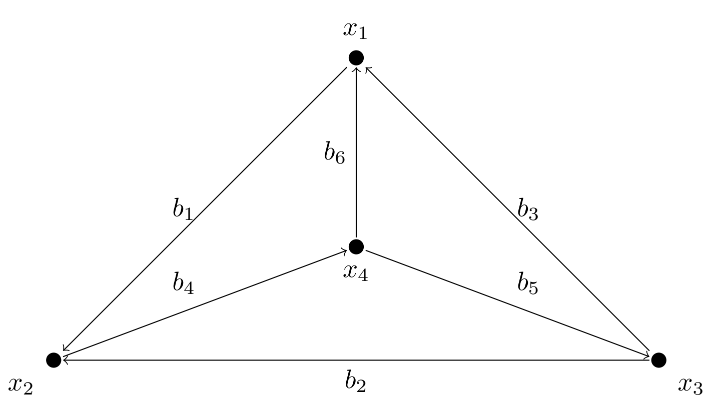

Figure 1: Directed graph for Problem 2

5. (Trefethen&Bau 2.6) If $\mathbf{u}$ and $\mathbf{v}$ are $m-$vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^T$ is known as a rank-one pertubation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is null(A)?

Floating point

6. (Heath 1.7) A floating point number system is characterized by four integers: the base β, the precision p, and the lower and upper limits L and U of the exponent range.

   (a) If $\beta = 10$, what are the smallest values of p and U, and the largest value of L, such that both 2365.27 and 0.0000512 can be represented *exactly* in a *normalized* floating-point system?

   (b) How would your answer change if the system is not normalized, i.e., if gradual underflow is allowed?

7. (Trefethen & Bau 13.1) Between an adjacent pair of nonzero IEEE single precision real numbers, how many IEEE doule precision numbers are there?

8. Let $x > 0$. Consider computing s, where

$$s = \frac{x}{\sqrt{1+x^2}} \tag{1}$$

   Note that s can also be expressed as

$$s = \left(\sqrt{1+\frac{1}{x^2}}\right)^{-1} \tag{2}$$

   Experiment with computing s for very large values of x using both expressions. How do they behave differently? Why?

9. *Computer problem* (Heath 1.10)

   (a) Write a program to solve the quadratic equation $ax^2 + bx + c = 0$ using the standard quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   or the alternative formula

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}.$$

   Your program should accept values for the coefficients a, b, and c as input and produce the two roots of the equation as output. Your program should detect when the roots are not real, but need not use complex arithmetic explicitly (for example, you could return the real and imaginary parts of the complex conjugate roots in this case). You should guard against unnecessary overflow, underflow, and cancellation. Try to make your program robust when given unusual input values, such as $a = 0$ or $c = 0$, which otherwise would make one of the formulas fail. Any root that is within the range of the floating-point system should be computed accurately, even if the other is out of range. Submit a copy of your code. You may use the language of your choice.

   (b) When should you use each of the two formulas?

   (c) Test your program using the following values for the coefficients and give your results:

a	b	c
6	5	-4
6×10^{154}	5×10^{154}	-4×10^{154}
0	1	1
1	-10^5	1
1	-4	3.999999
10^{-155}	-10^{155}	10^{155}