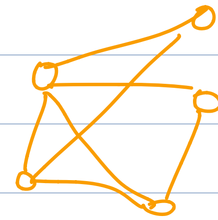
SDP and Max Cut

Max-cut

Input: Graph $G=(V,E)$

Output: Find $S\subseteq V$ to maximize $|\partial S|$.



Thm. $\exists$ 0.878-approximation algorithm.

Consider symmetric matrix $A\in\mathbb{R}^{n\times n}$.

$(A\succeq 0)$

Def. A is called positive semidefinite if $x^TAx\geq 0$ $\forall x\in\mathbb{R}^n$.

Recall that $A=\sum_{i=1}^{n}\lambda_i\cdot(v_iv_i^T)$ where $v_i\in S^{n-1}$, $v_i\perp v_j$ for $i\neq j$.

say $\lambda_1\geq\cdots\geq\lambda_n$.

$(v_n)^T\left(\sum_i\lambda_i v_i v_i^T\right)v_n$

$=\sum_i\lambda_i\langle v_n,v_i\rangle^2$

Lem. A psd $\Leftrightarrow$ $\lambda_n\geq 0$

Pf. $\Rightarrow$) If $\lambda_n<0$, $(v_n)^TA(v_n)=\lambda_n\cdot\langle v_n,v_n\rangle^2<0$.

$\Leftarrow$) For any x, $x=\sum_{i=1}^{n}\alpha_i v_i$.

$$x^TAx=\sum_i\alpha_i^2\lambda_i\geq 0$$

□

Example: I, VV^T for any $n\times k$ matrix V

$(x^TAx=x^TVV^Tx=\|V^Tx\|_2^2\geq 0)$

A,B psd. $\alpha,\beta\geq 0\Rightarrow\alpha A+\beta B$ psd

SDP

LP with one more structure.

Let $\mathcal{S}^n$ = set of $(n\times n)$-symmetric matrices

n^2 - dim vector.
with "deg. of freedom" $n(n+1)/2$.

LP	SDP
vars : $x \in \mathbb{R}^n$	vars : $X \in \mathcal{S}^n$
$\max \langle c, x\rangle$	$\max : \langle C, X\rangle$
s.t. $\langle a_1, x\rangle \le b_1$	s.t. : $\langle A_1, X\rangle \le b_1$
$\vdots$	$\vdots$
$\langle a_m, x\rangle \le b_m$	$\langle A_m, X\rangle \le b_m$
$x \geq 0$.	$X \succeq 0$
$c, a_1, \dots, a_m \in \mathbb{R}^n$	$C, A_1, \dots, A_m \in \mathcal{S}^n$
$b_1, \dots, b_m \in \mathbb{R}$.	$b_1, \dots, b_m \in \mathbb{R}$.

$\langle C, X\rangle = \sum_{ij} C_{ij} X_{ij}$

$$a_i = \begin{pmatrix} | \end{pmatrix} \Rightarrow A_i = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

$X_{ij} = 0 \quad i \neq j$

$$X = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \succeq 0$$

$\iff (x_1, \dots, x_n) \geq 0$

LP $\leq$ SDP

<u>Thm</u>, "Simple" SDPs can be "approximately" solved in poly-time. $\square$

Why is SDP useful (for us)?

<u>Lem</u>, $X \succeq 0 \iff X = VV^T$ for some V.

<u>Pf</u>, $\Leftarrow$ easy.

$\Rightarrow$) $X = \sum \lambda_i v_i v_i^T$ let $V = \begin{bmatrix} | & & | \\ \sqrt{\lambda_1} v_1 & \cdots & \sqrt{\lambda_n} v_n \\ | & & | \end{bmatrix}$.

$\square$

SDP and Vectors

vars : $X \in \mathbb{S}^n$

max : $\langle C, X\rangle$

s.t. : $\langle A_1, X\rangle \le b_1$

$\vdots$

$\langle A_m, X\rangle \le b_m$

$X \succeq 0$

$C, A_1, \dots, A_m \in \mathbb{S}^n$

$b_1, \dots, b_m \in \mathbb{R}$.

$X \succeq 0 \iff X = UU^T$

Now, let $U = \begin{bmatrix} — u_1 — \\ \vdots \\ — u_n — \end{bmatrix}$

So that $X_{ij} = \langle u_i, u_j\rangle$.

Then $\langle C, X\rangle$ and $\langle A_i, X\rangle$

become "deg-2 poly in $u_1, \dots, u_n$!".

Relaxation for Max Cut (Say $G = ([n], E)$)

Var : $\{x_1, \dots, x_n\}$ having ± 1 value

$$\max \sum_{(i,j)\in E} (1 - x_i x_j)/2$$

$$\text{s.t. } x_i^2 = 1 \qquad \forall i.$$

Exact if $x_i \in \mathbb{R}$. Relax x_i to vector u_i?

(dim unrestricted)

Var : $\{u_1, \dots, u_n\}$ having $\mathbb{S}^{n-1}$ value $\{u \in \mathbb{R}^n : \|u\|_2 = 1\}$

$$\max \sum_{(i,j)\in E} (1 - \langle u_i, u_j\rangle)/2$$

$$\text{s.t. } \langle u_i, u_i\rangle = 1. \qquad \forall i.$$

$$X = \begin{bmatrix} — u_1 — \\ \vdots \\ — u_n — \end{bmatrix} \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$$

$$\begin{aligned} \max \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = 1 \quad \forall i \\ & X \succeq 0 \end{aligned}$$

(i,i)

$$A_i = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

$$C = \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} i & j & & \end{array} \\ \begin{bmatrix} 1/4 & -1/4 & & \\ -1/4 & 1/4 & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

$(i,j) \in E$

Therefore, SDP is a relaxation!

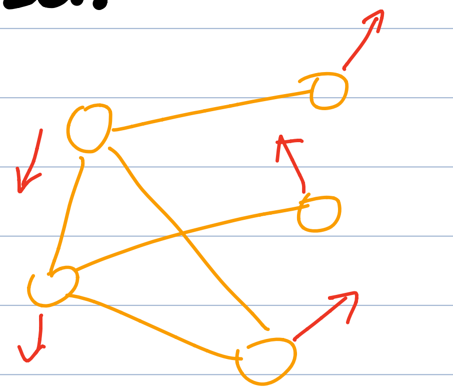
SDP $\geq$ OPT

Goemans-Williamson

Var : $\{u_1, \ldots, u_n\}$ having S^{n-1} value

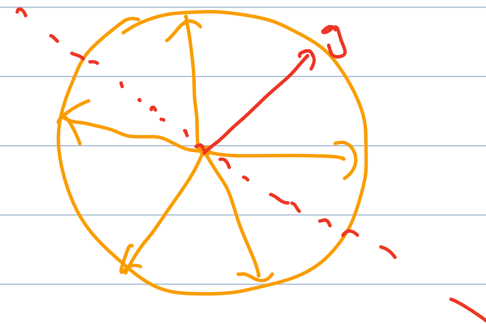
$$\max \sum_{(i,j)\in E} (1-\langle u_i,u_j\rangle)/2$$

s.t. $\langle u_i,u_i\rangle = 1$. $\forall i$.



Rounding

- Choose a random vector $g \in S^{n-1}$
- $A \leftarrow \{ i : \langle u_i, g\rangle \geq 0 \}$
- Output A.



Analysis. Fix $(i,j) \in E$.

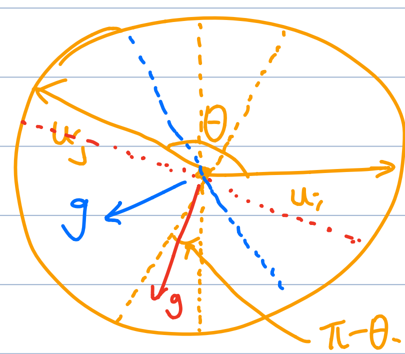
It contributes $(1-\langle u_i,u_j\rangle)/2$ to SDP

" " $\underbrace{\Pr[i,j \text{ seperated}]}_{p_{ij}}$ to ALG.

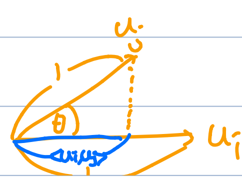
Can we say $p_{ij} / ((1-\langle u_i,u_j\rangle)/2) > ?$

WLOG, $u_i = (1,0,\ldots,0)$

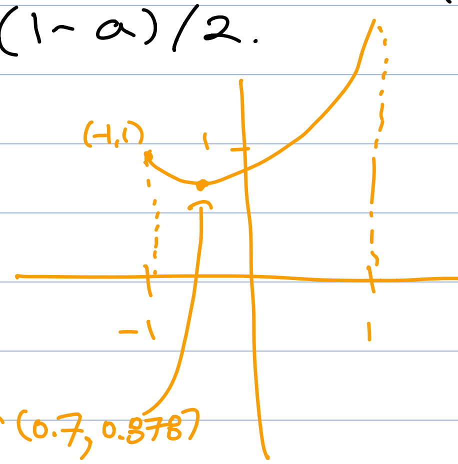
$u_j = (\alpha,\beta,0,\ldots,0)$



$$p_{ij} = \frac{2\theta}{2\pi} = \frac{\theta}{\pi} = \frac{\arccos(\langle u_i,u_j\rangle)}{\pi}$$



So, approx ratio $\geq \min_{a\in[-1,+1]} \dfrac{\arccos(a)/\pi}{(1-a)/2} =: \alpha_{GW}$.



$\therefore \exists$ 0.878-approx. algo. for Max-cut.

<u>Thm</u>, Assuming Unique Games Conjecture, $\forall \varepsilon > 0$,
$\nexists (\alpha_{GW}+\varepsilon)$-approx. algo.