① conditioning of the mathematical problem

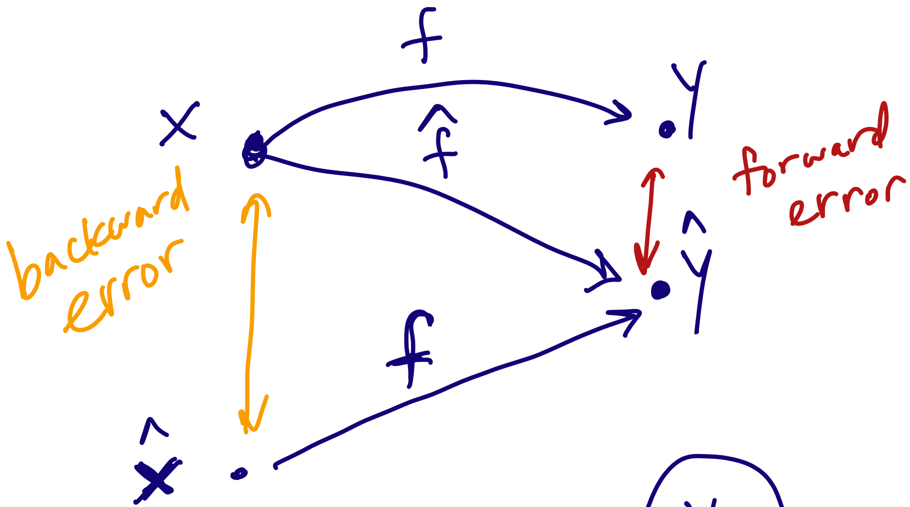
② stability of the numerical method

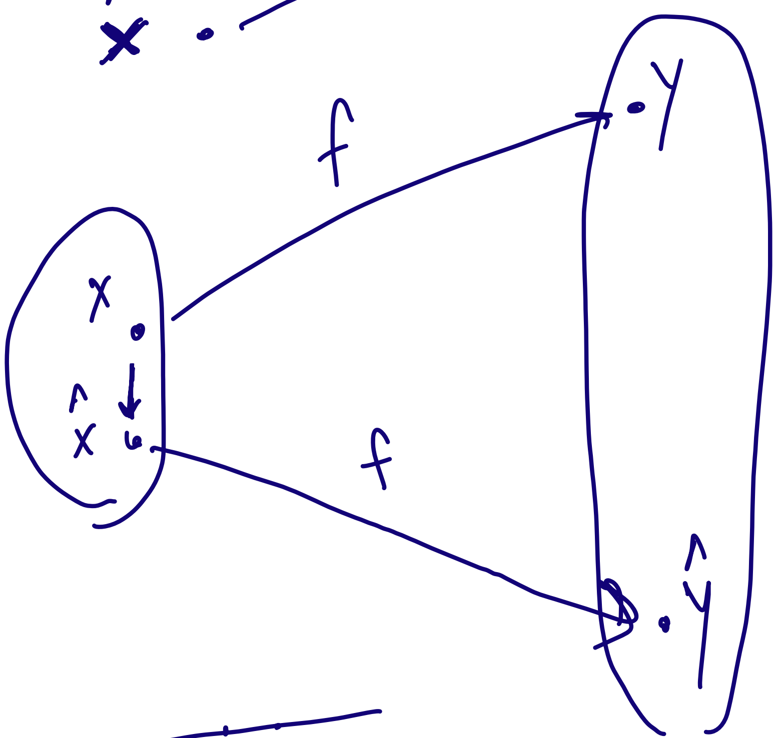
sensitivity to perturbations in inputs

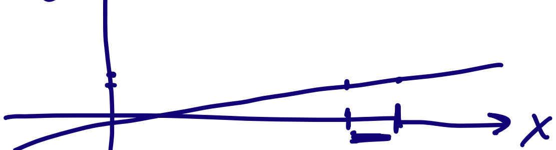
Conditioning

$$y = f(x)$$

$$x \longrightarrow \boxed{f} \longrightarrow y$$







$$\boxed{|f'(x)| \, \frac{|x|}{|f(x)|}}$$

10^4

$10^6 \times .01$

$$\frac{\|y-\hat{y}\|}{\|y\|} \leq \kappa \frac{\|x-\hat{x}\|}{\|x\|}$$

relative forward error

amplification factor

relative backward error

"condition number"

$$\frac{\|y-\hat{y}\|/\|y\|}{\|x-\hat{x}\|/\|x\|} \leq \kappa$$

$$x \mapsto \boxed{A} \to y$$

$$y = Ax \Rightarrow x = A^{-1}y$$

assuming: A invertible

$$\frac{\text{rel. for. err}}{\text{rel. bac err}}$$

$$\frac{\|Ax - A\hat{x}\|/\|y\|}{\|x-\hat{x}\|/\|A^{-1}y\|}$$

$$= \frac{\|A(x-\hat{x})\|}{\|x-\hat{x}\|} \cdot \frac{\|A^{-1}y\|}{\|y\|}$$

$$\|A\| = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|\hat{x}\|=1} \|A\hat{x}\|$$

$\langle w = x - \hat{x} \rangle$

$$= \frac{\|Aw\|}{\|w\|} \cdot \frac{\|A^{-1}y\|}{\|y\|}$$

$$\le \|A\| \, \|A^{-1}\|$$

$$x \to \boxed{Ax} \to y.$$

$$y = Ax$$

$$\boxed{K(A) = \|A\| \, \|A^{-1}\|}$$

$$y \to \boxed{A^{-1}y} \to x.$$

$$K(A^{-1}) = \|A^{-1}\| \, \|\underbrace{(A^{-1})^{-1}}\|$$

$$\boxed{K(A^{-1}) = \|A^{-1}\| \, \|A\|}$$

$A^{-1}y \Rightarrow$

$$\boxed{\text{find } x \text{ s.t. } Ax = y}$$

given x, find y: $Ax = y$

given y, find x: $x = \boxed{A^{-1}}y$

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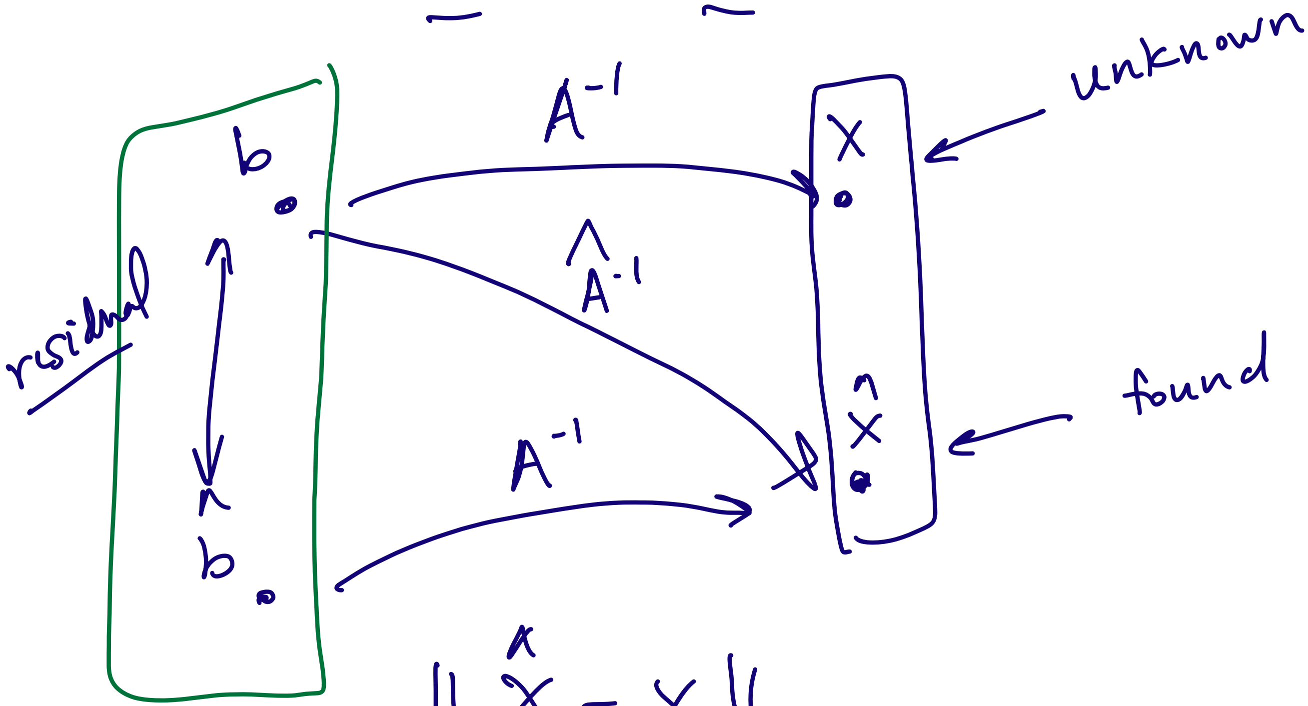
rel for err. $\le K$ rel back err

$$\underset{\sim}{A}x = \underset{\sim}{b}$$



$$\|\hat{x} - x\|$$

$$\underset{\sim}{A}\hat{x} = \hat{b}$$

$$r = b - \hat{b} = b - A\hat{x} = \boxed{\text{residual}}$$

$$\|b - \hat{b}\|$$

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \underbrace{\kappa(A)} \frac{\|r\|}{\|b\|}$$

well-conditioned ✓

ill-conditioned ✗ → add "regularization"

preconditioner

$$\underline{A}x = b$$

$M \approx A^{-1}$ $\qquad$ M invertible

$$\boxed{(MA)x = Mb}$$

$$x = (MA)^{-1}Mb$$

$\boxed{\kappa(MA) < \kappa(A)}$

$$= (A^{-1}M^{-1})Mb$$

$$= A^{-1}\cancel{M^{-1}M}b$$

$$\boxed{x = A^{-1}b}$$

$$\boxed{\kappa_2(I) = 1}$$

properties

$$\underbrace{\|A\|\,\|A^{-1}\|}$$

$$\|\cancel{\alpha} A\|\,\|\cancel{\tfrac{1}{\alpha}} A^{-1}\|$$

1. $\mathrm{cond}(A) \geq 1$

2. $\mathrm{cond}(I) = 1$

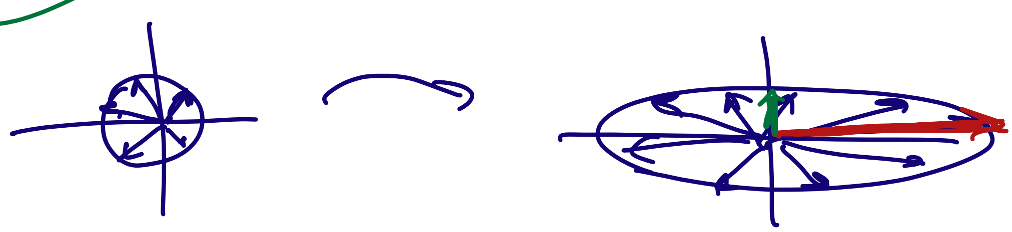
3. $\mathrm{cond}(\alpha A) = \mathrm{cond}(A)$

4. D diagonal $\quad D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$

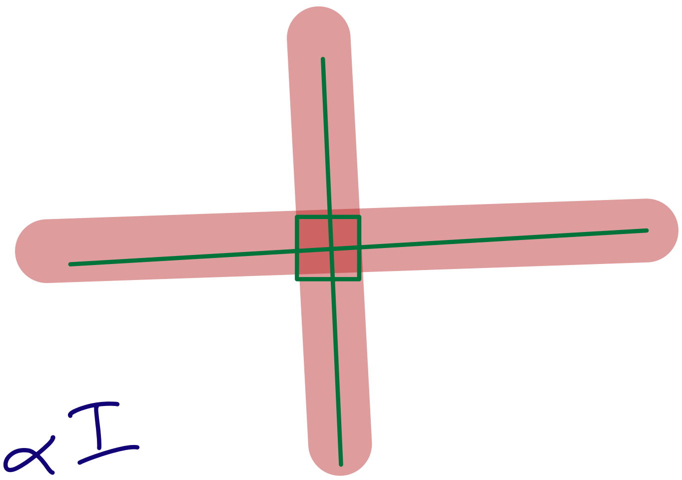
$$\mathrm{cond}_2(D) = \frac{\max_i |d_i|}{\min_i |d_i|}$$

$$= \underbrace{\max_i |d_i|}_{\|D\|} \cdot \underbrace{\frac{1}{\min_i |d_i|}}_{\|D^{-1}\|}$$

$$\begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} = \alpha I$$



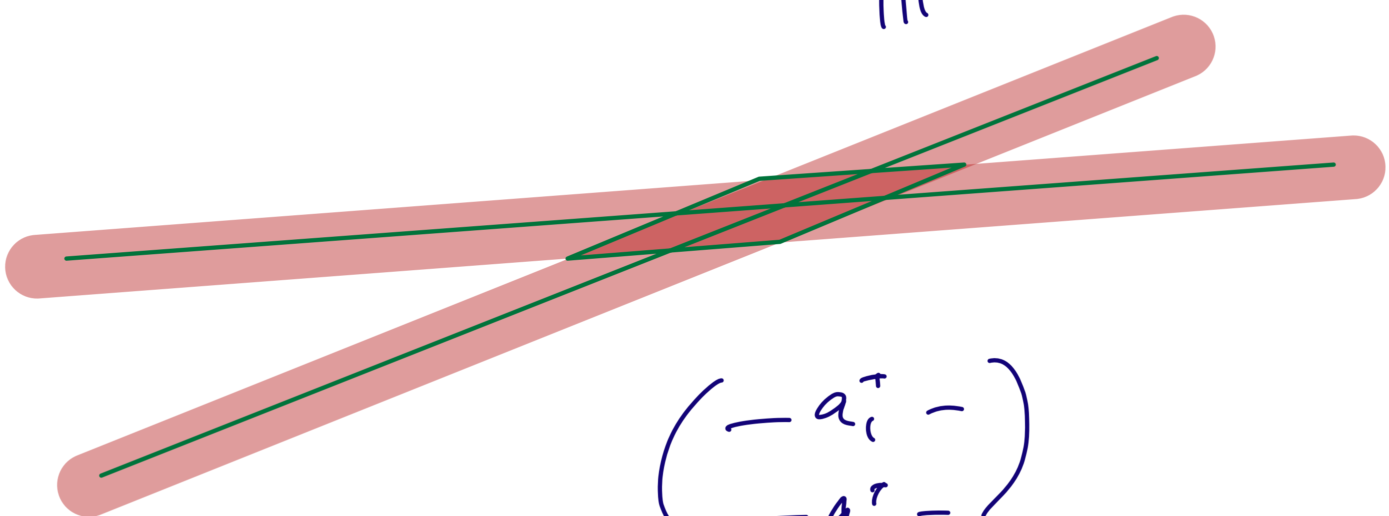
② $Ax = b$



well conditioned

αI

ill-conditioned



$$\begin{pmatrix} - & a_1^T & - \\ - & a_2^T & - \end{pmatrix}$$

A singular

$\text{cond}(A) = \infty$

① conditioning of a problem $\dfrac{\text{rel. for err}}{\text{rel backw err}}$

② cond # of a matrix

③ residual

Stability Ex.: LU without pivoting

unstable algorithm that is too sensitive to pertubations gives garbage

to compute soln:

✓ well conditioned

✓ stable algorithm

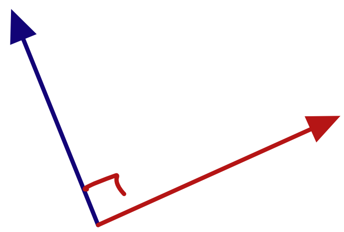
Orthogonality

1. orthogonal vectors $\vec{x}, \vec{y}$

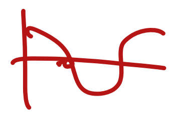
$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = 0$$

(complex $\bar{x}^T y = 0$)

$$\vec{x} \perp \vec{y}$$

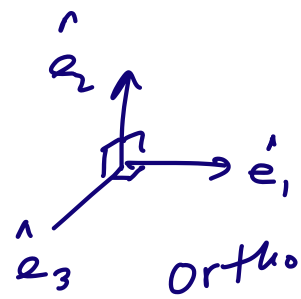


$$x \cdot y = \|x\| \|y\| \cos\theta$$



2. orthogonal basis

$v_1, \dots, v_n$



orthonormal

$$v_i^T v_j = 0 \qquad i \neq j$$

orthonormal basis:

$$v_i^T v_j = 0 \qquad i \neq j$$

$$v_i^T v_i = 1$$