Solving LP using MWU

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Vars: $x \in \mathbb{R}^n$.

$$\boxed{\begin{array}{ll} \max & \langle c, x \rangle \\ \text{s.t.} & Ax \le b \\ & x \ge 0. \end{array}}$$

Say $A = \begin{bmatrix} — a_1 — \\ \vdots \\ — a_m — \end{bmatrix}$

Assume we know the optimal value OPT (by binary search)

Let $K = \{x : \langle c, x \rangle = \text{OPT},\ x \ge 0\}$.

Then, the problem is equivalent to find $x \in K$ s.t.

$$\langle a_i, x \rangle \le b_i \quad \forall i \in [m].$$

Our goal: Find $x \in K$ s.t.

$$\langle a_i, x \rangle \le b_i + \varepsilon \quad \forall i \in [m]$$

satisfying constraints approximately

(if $b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $c \ge 0$, $y = x / (1+\varepsilon)$ satisfies all constraints exactly and $\langle c, y \rangle \ge \text{OPT} / (1+\varepsilon)$)

approximating obj. function.

Need one "Oracle".

Oracle. Given $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$, find $x \in K$ s.t. $\langle \alpha, x \rangle \leq \beta$, or report there is no such x.

Lemma. The Oracle can be implemented in $O(n)$ time.

Pf (When $\alpha, \beta \geq 0$, $c > 0$). Let $i \in [n]$ be the coordinate minimizing α_i / c_i. Consider $x = e_i \cdot OPT/c_i$ so that $x \in K$.

$e_i = (0, \dots, \underset{i\text{-th}}{1}, \dots, 0)$

If $\langle \alpha, x \rangle \leq \beta$, done. Otherwise, no y satisfies $y \in K$ and $\langle \alpha, y \rangle \leq \beta$ since $\forall y \in K$, $\langle \alpha, x \rangle / \langle c, x \rangle \leq \langle \alpha, y \rangle / \langle c, y \rangle$, $\langle c, x \rangle = \langle c, y \rangle$, so $\langle \alpha, y \rangle \geq \langle \alpha, x \rangle > \beta$ □

by def of i and x.

Def. width $\rho := \max_{x \in K, i \in [m]} \{ |\langle a_i, x \rangle - b_i| \}$.

(assume $\rho \geq 1$)

indeed, K can be even smaller set as long as
① K contains an optimal solution of the LP
② The Oracle can be efficiently implementable.

LP-Solver(ε).

$w^{(1)} \leftarrow (1, \ldots, 1) \in \mathbb{R}^m$, $T \leftarrow \rho^2 \ln m / \varepsilon^2$.

For $t = 1, \ldots, T$.

$p^{(t)} \leftarrow w^{(t)} / \left(\sum_{i \in [m]} w_i^{(t)}\right)$

$\left.\begin{array}{l} \alpha^{(t)} := \sum_{i \in [m]} p_i^{(t)} a_i \in \mathbb{R}^n \\ \beta^{(t)} := \sum_{i \in [m]} p_i^{(t)} b_i. \end{array}\right\} \rightarrow$ corresponds to $\langle p^{(t)}, Ax \rangle \leq \langle p^{(t)}, b \rangle$.

$x^{(t)} \leftarrow \text{Oracle}(\alpha, \beta)$.

if Oracle says infeasible.

Output infeasible

else

$\ell_i^{(t)} \leftarrow \overbrace{(b_i - \langle a_i, x^{(t)} \rangle)/\rho}^{\in [-1, +1].} \quad \forall i \in [m].$

$w_i^{(t+1)} \leftarrow w_i^{(t)} \cdot \exp\left(-(\varepsilon/\rho)\, \ell_i^{(t)}\right)$

Output $x = (x^{(1)} + \cdots + x^{(T)})/T$.

Intuition: "expert" i = i^{th} constraint for LP.

loss $\ell_i^{(t)}$ = "slack" of i^{th} constraint by $x^{(t)}$.

(LP solver wants $\sum_t \ell_i^{(t)} \gtrsim 0 \rightarrow$ every expert loses a lot!)

So, LP solver (as "adversary" in MWU) creates x s.t.

"MWU player's loss" $= \langle p^{(t)}, b - Ax^{(t)} \rangle \geq 0$.

$\therefore$ MWU Thm guarantees "main player's loss $\lesssim$ experts' loss",

which implies $\sum_t \ell_i^{(t)} \gtrsim 0$!

LP Solver = MWU adversary $\xrightarrow{\ell^{(t)}}$ MWU Player

Loss of i's expert $\ell_i^{(t)} = (b_i - \langle a_i, x^{(t)}\rangle)/\rho$

Choose $x^{(t)} \in K$ to ensure

MWU player's loss $= \sum_i p_i^{(t)} b_i - \langle \sum_i p_i^{(t)} a_i, x^{(t)} \rangle \ge 0$

Update weights $w^{(t)}$ and $p^{(t)}$

for experts (constraints)

$p^{(t)}$

[or constraint

$$\sum_i p_i^{(t)} b_i - \left\langle \sum_i p_i^{(t)} a_i, x \right\rangle \ge 0. \;]$$

Of course, $\langle c, x\rangle = \frac{1}{T}\sum_{t\in[T]} \langle c, x^{(t)}\rangle = OPT$. And $x \ge 0$ since $x^{(1)} \dots x^{(T)} \ge 0$.

Lemma. $\forall i \in [m]$, $\langle a_i, x\rangle \le b_i + 2\varepsilon$.

Pf. MWU Guarantee (with parameter (ε/ρ)) shows that

$$\forall i\in[m],\ \underbrace{\frac{1}{T}\left(\sum_{t\in[T]}\langle p^{(t)}, \ell^{(t)}\rangle\right)}_{\ge 0 \text{ since } \forall t\in[T],} \le \underbrace{\frac{1}{T}\sum_{t\in[T]} \ell_i^{(t)}}_{=\frac{1}{\rho}(b_i - \langle a_i, x\rangle)} + \underbrace{\ln m/((\varepsilon/\rho)T)}_{= \le (\varepsilon/\rho)} + \varepsilon/\rho$$

$$\langle p^{(t)}, \ell^{(t)}\rangle = \frac{1}{\rho}\sum_{i\in[m]} p_i^{(t)}(b_i - \langle a_i, x^{(t)}\rangle) = \frac{1}{\rho}\left(\beta^{(t)} - \langle \alpha^{(t)}, x^{(t)}\rangle\right) \ge 0.$$

Dividing by ρ. $0 \le b_i - \langle a_i, x\rangle + 2\varepsilon$. □