LP Duality

Linear Programming (vars : $x \in \mathbb{R}^n$)

$$\boxed{\begin{aligned} \max\ & \langle c, x\rangle \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned}}$$

($c \in \mathbb{R}^n$)

($A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$)

How do you "prove" that optimal value is at most blah?

$$\begin{aligned} \max\ & 4x_1 + x_2 + 3x_3 \\ \text{s.t.}\ & 2x_1 + 2x_2 + x_3 \le 5 \quad \text{(i)} \\ & x_1 + 4x_2 + 3x_3 \le 7 \quad \text{(ii)} \\ & x \ge 0 \end{aligned}$$

Suppose we multiply (i) by $y_1 \ge 0$ and

(ii) by $y_2 \ge 0$.

Can we get some upper bound on OPT?

<u>YES.</u> say $y_1 = 1$, $y_2 = 2$

$$\begin{aligned} & 2x_1 + 2x_2 + x_3 \le 5 \quad \cdots \times 1 \\ + \quad & x_1 + 4x_2 + 3x_3 \le 7 \quad \cdots \times 2 \\ \hline & 4x_1 + 10x_2 + 7x_3 \le 19 \end{aligned}$$

used $y_1, y_2 \ge 0$ to preserve "direction" of inequalities.

but objective function was

$$4x_1 + x_2 + 3x_3$$

used $x \ge 0$ and $\forall i$ (coefficient of x_i in LHS $\ge$ coeff of x_i in RHS)

Since $x \ge 0$, $19 \ge 4x_1 + 10x_2 + 7x_3 \ge 4x_1 + x_2 + 3x_3$!

Therefore, OPT ≤ 19.

But, can we find a better upper bound?
Above, we used that
(i) $y_1, y_2 \ge 0$
(ii) $\forall i$, coeff. of x_i in derived inequality $\ge$
" " " " objective function.
(iii) $x \ge 0$ (guaranteed by LP)

If y_1, y_2 satisfy (i) and (ii), upper bound we get is $5y_1 + 7y_2$.

$$\begin{aligned}
\max \quad & 4x_1 + x_2 + 3x_3 \\
\text{s.t.} \quad & 2x_1 + 2x_2 + x_3 \le 5 \quad \ldots y_1 \\
& x_1 + 4x_2 + 3x_3 \le 7 \quad \ldots y_2 \\
& x \ge 0
\end{aligned}$$

Write (ii) explicitly.

for x_1 : $2y_1 + y_2 \ge 4$
x_2 : $2y_1 + 4y_2 \ge 1$
x_3 : $y_1 + 3y_2 \ge 3$.

Therefore, if y_1, y_2 satisfy $y_1, y_2 \ge 0$ and above three inequalities, $5y_1 + 7y_2$ is an upper bound!
And all ineqs are "linear" again!

So, dual: Primal

$$\min \; 5y_1 + 7y_2 \qquad\qquad \max \; 4x_1 + x_2 + 3x_3$$

$$\text{s.t. } 2y_1 + y_2 \ge 4 \;\cdots x_1 \qquad\qquad \text{s.t. } 2x_1 + 2x_2 + x_3 \le 5 \;\cdots y_1$$

$$2y_1 + 4y_2 \ge 1 \;\cdots x_2 \qquad\qquad x_1 + 4x_2 + 3x_3 \le 7 \;\cdots y_2$$

$$y_1 + 3y_3 \ge 3 \;\cdots x_3 \qquad\qquad x \ge 0$$

$$y \ge 0$$

If Primal is $\max \langle c, x\rangle$ s.t. $Ax \le b$, $x \ge 0$

$(x, c \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m)$

Dual is $\min \langle b, y\rangle$ s.t. $A^T y \ge c$, $y \ge 0$

$(y \in \mathbb{R}^m$, A, b, c same$)$

If y is feasible for Dual and

x " " " Primal, then

$$\langle b, y\rangle \ge \langle Ax, y\rangle = \langle x, A^T y\rangle \ge \langle x, c\rangle$$

$b \ge Ax$, $y \ge 0$ $\qquad = \sum_{\substack{i\in[n]\\ j\in[m]}} x_i y_j A_{ji}$ $\qquad A^T y \ge c$, $x \ge 0$

Weak Duality, $OPT_{primal} \le OPT_{dual}$.

Strong Duality, $OPT_{primal} = OPT_{dual}$.

Max-Flow Min-Cut Duality

Given directed $G=(V,E)$, capacities $c: E \to \mathbb{R}^{\geq 0}$, $s,t \in V$.
(WLOG, no edge goes into s and no edge coming out of t.)

Max-Flow (vars: $f \in \mathbb{R}^E$)

$$\max \sum_{(s,v)\in E} f(s,v)$$

$$\text{s.t.} \quad \sum_{(u,v)\in E} f(u,v) - \sum_{(v,u)\in E} f(v,u) = 0 \quad \cdots y_u \quad \forall u \in V \setminus \{s,t\}$$

$$f(e) \leq c(e) \quad \cdots z_e \quad \forall e \in E$$

$$f \geq 0$$

Dual (vars: $y \in \mathbb{R}^{V \setminus \{s,t\}}$, $z \in \mathbb{R}^E$)

$$\min \sum_{e \in E} c(e) z_e$$

$$\text{s.t.} \quad z_e - y_u \geq 1 \quad \forall e = (s,u) \in E$$

$$z_e + y_u - y_v \geq 0 \quad \forall e = (u,v) \in E \ (u,v \notin \{s,t\})$$

$$z_e + y_u \geq 0 \quad \forall e = (u,t) \in E$$

$$z \geq 0$$

(y is unconstrained since primal constraints corresponding to y are equalities!)

since we're minimizing, $z_{(s,u)} = \max(1 + y_u, 0)$

$z_{(u,v)} = \max(y_v - y_u, 0)$

$z_{(u,t)} = \max(-y_u, 0)$

if $y_v \in \{-1, 0\}$ $\forall v \neq s,t$, and $S = \{v : y_v = -1\} \cup \{s\}$, then
$\sum_e c(e) z_e = c(S, V \setminus S)$. so (min s-t cut capacity) $\geq \text{OPT}_{\text{dual LP}}$!

So, Max-Flow Min-Cut Thm says

$$OPT_{\text{max flow}} = OPT_{\text{dual-LP}} = OPT_{\text{min cut}}.$$

Matching / Vertex Cover

(IP-M) maximize $\sum_{e \in E} x_e$

s.t. $\sum_{e: v \in e} x_e \le 1 \quad \forall v \in A \cup B$

$x \in \{0,1\}^E$

(LP-M) maximize $\sum_{e \in E} x_e$

s.t. $\sum_{e: v \in e} x_e \le 1 \quad \forall v \in A \cup B \quad \ldots y_v$

$x \ge 0$

Dual of LP-M? Vars: $y \in \mathbb{R}^{A \cup B}$.

Constraints: $\forall e \in E$, if we sum up $y_v \left(\sum_{e: v \in e} x_e \right) \le y_v$ over all $v \in V$, the coeff. of each x_e must be ≥ 1.

$\Rightarrow \forall e = (u,v) \in E,\ y_u + y_v \ge 1.$

(LP-VC) minimize $\sum_{v \in A \cup B} y_v$

s.t. $y_u + y_v \ge 1 \quad \forall e = (u,v) \in E$

$y \ge 0.$

What happens if we restrict $y \in \{0,1\}^{A \cup B}$, any combinatorial meaning?

(IP-VC) minimize $\sum_{v \in A \cup B} y_v$

s.t. $y_u + y_v \ge 1 \quad \forall e = (u,v) \in E$

$y \in \{0,1\}^{A \cup B}$

Vertex Cover.

Input: undirected $G=(V,E)$

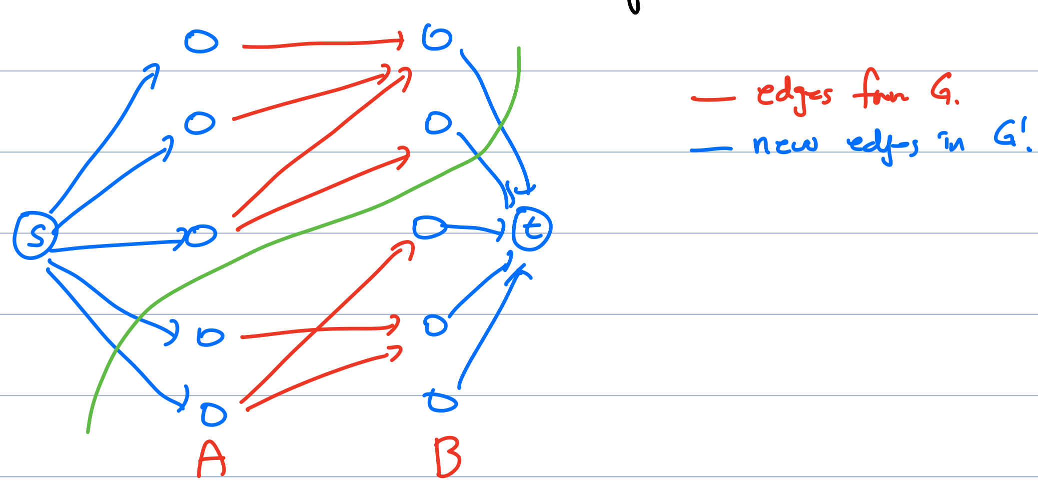
Output: $S \subseteq V$ that touches every edge.

Goal: Min $|S|$

So, $OPT_{IP\text{-}M} \le OPT_{LP\text{-}M} \le OPT_{LP\text{-}VC} \le OPT_{IP\text{-}V}$.

Lemma $OPT_{IP\text{-}M} = OPT_{IP\text{-}V}$.

Pf. Reconsider reduction from matching (in G) to max flow (in G').



Max-Flow Min-cut thm says (max matching size in G)

$=$ (max flow value in G') $=$ (min s-t cut size in G').

Let (S,T) be a s-t mincut in G'. ($s \in S$, $t \in T$).

If $\exists (u,v) \in E$ s.t. $u \in S$, $v \in T$, move v to S — $c(S,T)$ doesn't increase!

At the end, $\nexists (u,v) \in E$ s.t. $u \in S$, $v \in T$.

$c(S,T) = (|S|-1) + (|T|-1)$. So, $(S \setminus \{s\}) \cup (T \setminus \{t\})$ is a vertex cover of size $c(S,T) = OPT_{IP\text{-}M}$ $\square$

Hall's condition.

Corollary. Bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = n$ has a perfect matching iff $\forall A' \subseteq A$, $|A'| \leq |N(A')|$, where

$$N(A') := \{ b \in B : \exists (a,b) \in E \text{ for some } a \in A' \}$$

Pf. $\Rightarrow$) If $\exists A'$ s.t. $|A'| > |N(A')|$, then $N(A') \cup (A \setminus A')$ is a vertex cover with strictly less than n vertices.

$\Leftarrow$) If $(A' \cup B')$ is a vertex cover with $|A'| + |B'| < n$, $A' \subseteq A$, $B' \subseteq B$, then $N(A \setminus A') \subseteq B'$, which implies $n - |A'| \leq |B'|$. contradiction. $\square$

Primal-Dual Algo for Min-cost Perfect matching.

Min-cost Perfect Matching.

Input: Bipartite graph $G=(A\cup B, E)$, $w: E \to \mathbb{R}^{\geq 0}$.

Output: Perfect matching $M \subseteq E$ minimizing $w(M)$.

matching every vertex.

max-weight (non-perfect) matching can be reduced to this

Primal

$$\min \sum_{e\in E} w_e x_e$$

$$\text{s.t.} \quad \sum_{e \ni v} x_e = 1 \quad \forall v \in V = A\cup B. \quad \cdots y_v$$

$$x \geq 0.$$

Dual

$$\max \sum_v y_v$$

$$\text{s.t.} \quad y_u + y_v \leq w_e \quad \forall e=(u,v)\in E$$

If x is primal-feasible and y is dual-feasible,

$$\sum_e w_e x_e \geq \sum_{(u,v)\in E} (y_u+y_v) x_e = \sum_{u\in V} y_u \sum_{e\ni u} x_e = \sum_u y_u.$$

So, if x is (indicator vector of) a perfect matching in

$$E_y = \{(u,v)\in E : y_u + y_v = w_e\}.$$

this inequality becomes $=$ and x is optimal!

Primal-Dual algorithm: maintain both primal (integral) and dual (fractional) solutions guiding each other!

Primal-Dual.

$y \leftarrow 0$

While $\nexists$ perfect matching in E_y (otherwise, we're done.)

neighbors of A' in E_y.

Find $A' \subseteq A$ s.t. $|N_y(A')| < |A'|$. Hall's condition.

Let $\varepsilon = \min_{(u,v) \in E \cap (A' \times (B \setminus N_y(A')))} 1 - (y_u + y_v)$.

>0 since no edge of E_y exists between A' and $B \setminus N_y(A')$

For all v, $y_v \leftarrow \begin{cases} y_v + \varepsilon & \text{if } v \in A \\ y_v - \varepsilon & \text{if } v \in N_y(A') \\ y & \text{o.w.} \end{cases}$

Correctness: * y is feasible for dual always. (why?)

* if $\exists$ perfect matching in E_y, it is opt as argued above.

Running time: If w is integer and $W = \max$ weight,

ε is always ≥ 1 and # iterations $\leq nW$.

(again Pseudo-poly)