Nonlinear Eqs.

$$f(x) = 0$$

Bisection

Ex. (Newton's Method)

fixed point iteration

x is a "fixed point" of g if

$$x = g(x)$$

x_0

for $k = 0, 1, 2, \ldots$

$$x_{k+1} = g(x_k)$$

end

Example : $f(x) = x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x^* = 2, -1$

$$f(x) = 0 \Leftarrow \boxed{x = g(x)}$$

$$\underbrace{f(x) = x^2 - x - 2 = 0}$$

$x^2 = x + 2$

$x = \sqrt{x+2}$

(1) $g(x) = x^2 - 2$

$x = g(x) \implies x^2 - x - 2 = 0$ ✓

(2) $g(x) = \sqrt{x+2}$

$x = \sqrt{x+2} \implies x^2 = x + 2$ ✓

(3) $g(x) = 1 + \dfrac{2}{x}$

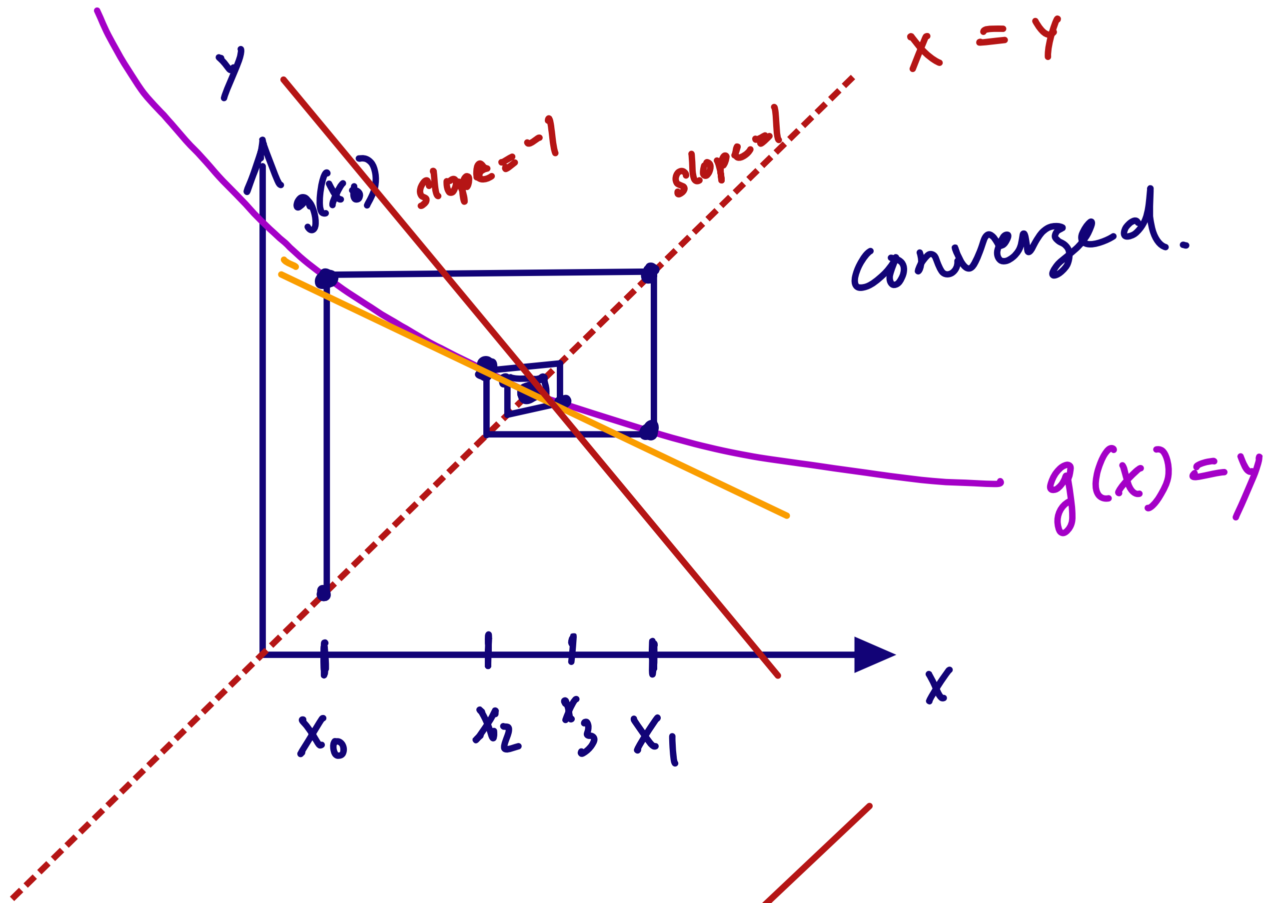
$x^2 = xx + \dfrac{2}{\cancel{x}}$ ✓

(4) $g(x) = \dfrac{(x^2+2)}{(2x-1)}$

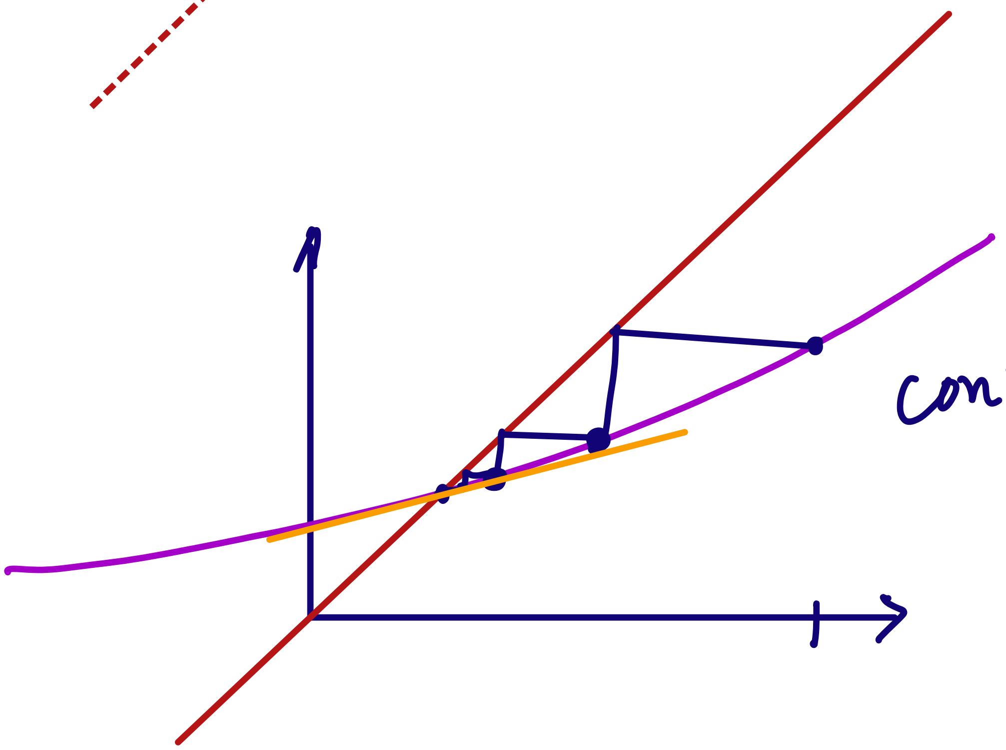
$(2x-1)x = x^2 + 2 \implies \cancel{2}x^2 - x = \cancel{x^2} + 2$

$x^2 - x - 2 = 0$ ✓

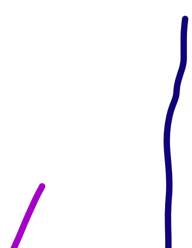
Fixed Pt. Iter

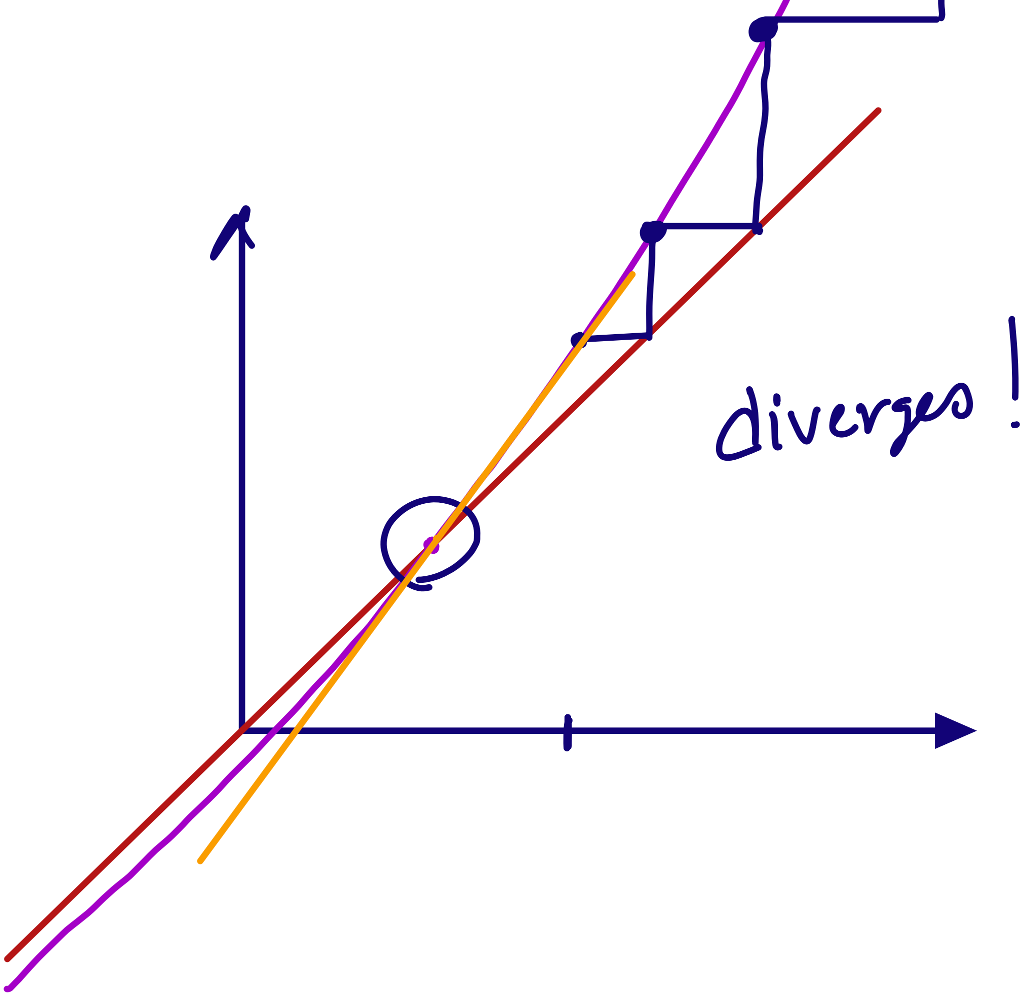


converged.



converge



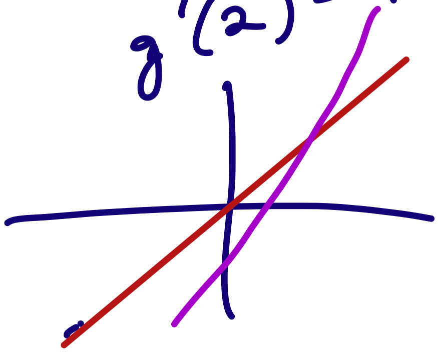

diverges!

$f(x) = x^2 - x - 2 = 0$

$g'(x) = 2x$ $\quad g'(2) = 4 \quad > 1$

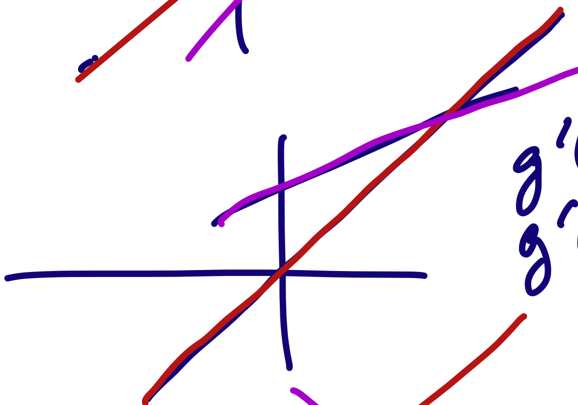
(1) $g(x) = x^2 - 2$

diverges



(2) $g(x) = \sqrt{x+2}$

converges



$g'(x) = \frac{1}{2}(x+2)^{-\frac{1}{2}}$

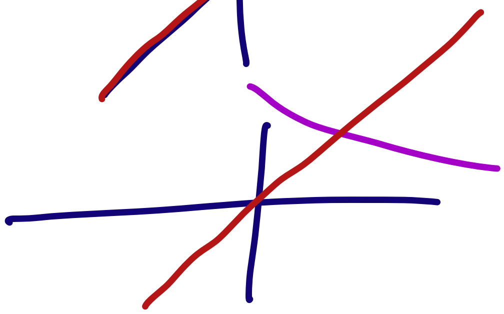
$g'(2) = \frac{1}{2}(4)^{-\frac{1}{2}}$

$= \frac{1}{4} < 1$

(3) $g(x) = 1 + \frac{2}{x}$

✓

converges

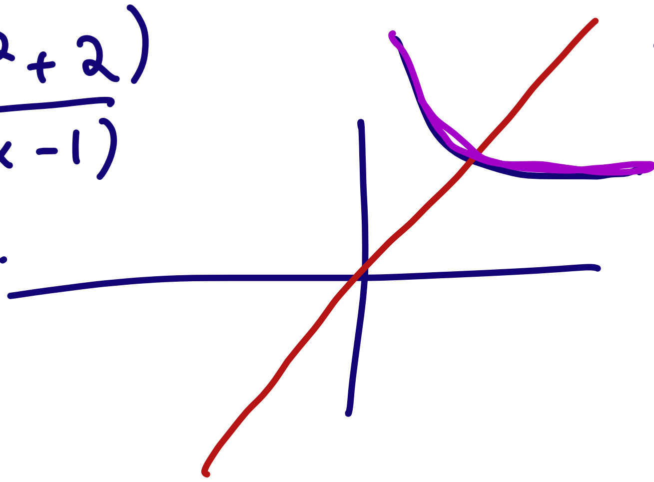


(4) $g(x) = \frac{(x^2+2)}{(2x-1)}$

✓

✓

converges



$g'(2) = 0$

Fixed Pt. Iteration Convergence

Taylor Series :

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \dots + \frac{f^{(n)}(x)}{n!}h^n + \dots$$

$$f(x+h) = f(x) + f'(x)h + \underbrace{O(h^2)}$$

$$= f(x) + f'(x)h + \frac{f''(\theta)}{2}h^2$$

$$x^* = g(x^*)$$

$$\frac{e_{k+1}}{e_k} = \frac{x_{k+1} - x^*}{x_k - x^*} = \frac{g(x_k) - x^*}{x_k - x^*}$$

$$= \frac{g(x_k) - g(x^*)}{x_k - x^*}$$

T.S.

$$g(\underbrace{x_k}_{x^* + \underbrace{(x_k - x^*)}_{h}}) = g(x^*) + g'(x^*)(x_k - x^*)$$

$$+ \frac{g''(x^*)}{2}\underbrace{(x_k - x^*)}_{h}{}^2$$

$$+ O(h^3)$$

$$\boxed{\frac{f(\overbrace{x+h}^{=y}) - f(x)}{h} = \frac{f(y) - f(x)}{y - x}}$$

$$\frac{e_{k+1}}{e_k} = \frac{\cancel{g(x^*)} + g'(x^*)\cancel{(x_k - x^*)} + \frac{g''(x^*)}{2}(x_k - x^*)^2 + O(h^3) \cancel{- g(x^*)}}{\cancel{(x_k - x^*)}}$$

$h = (x_k - x^*)$

$$\frac{e_{k+1}}{e_k} = \cancel{g'(x^*)} + \boxed{\frac{g''(x^*)}{2}\underbrace{(x_k - x^*)}_{e_k}} + O(h^2)$$

0

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} \to g'(x^*)$$

linear conv.

$$|g'(x^*)| < 1$$

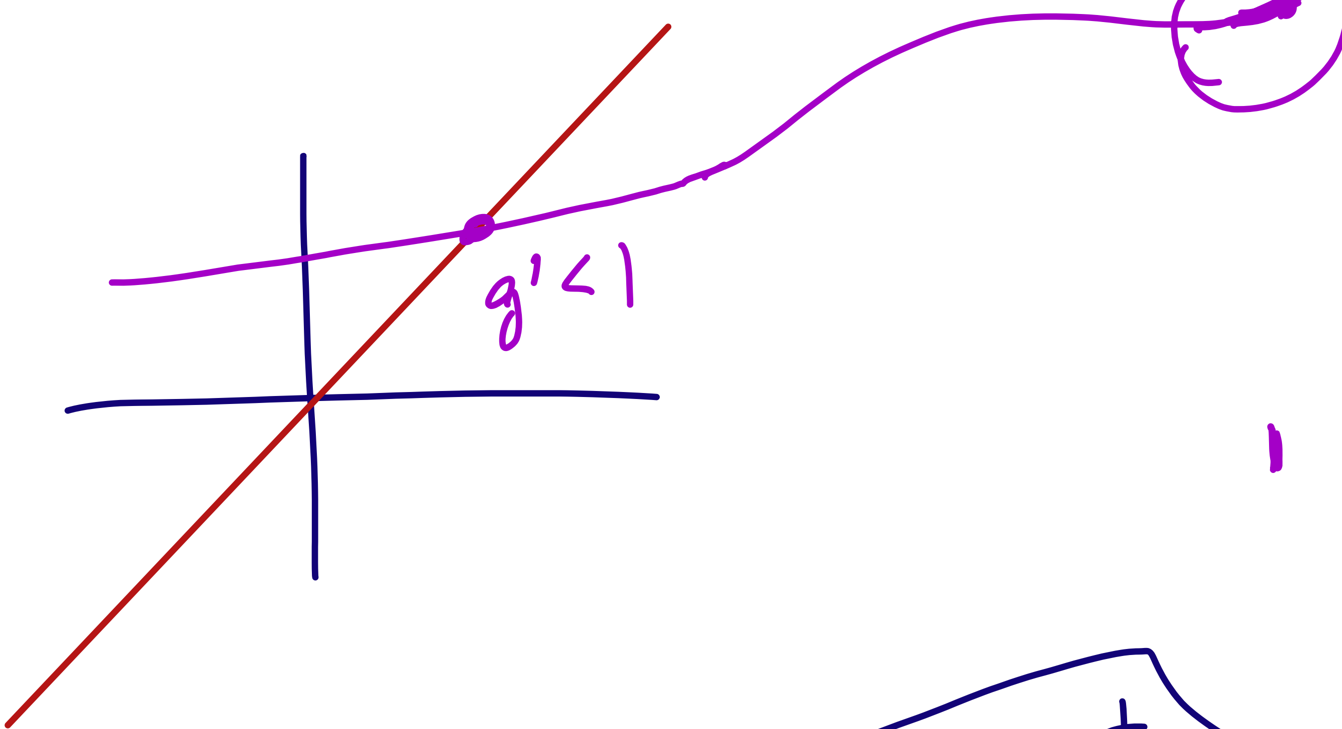
$g'(x^*) = 0$

$$\frac{e_{k+1}}{e_k^2} \to \boxed{\frac{g''(x^*)}{2}} \cancel{e_k}$$

quad. conv.

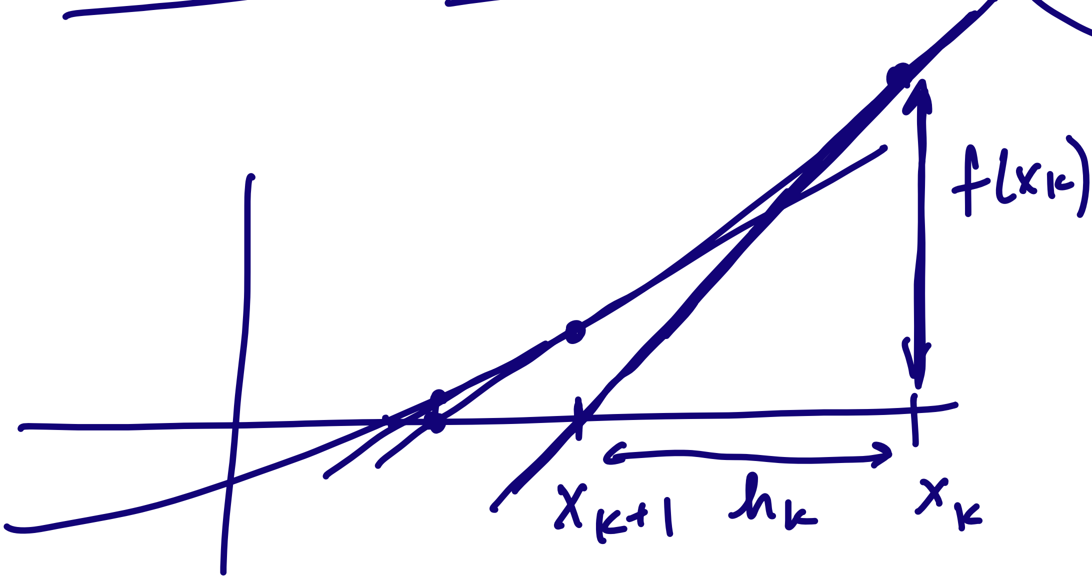
design g so that

$$g'(x^*) = 0$$



Newton's Method

fixed pt. $g'(x^*) = 0$



find x_{k+1} s.t.

$$\frac{f(x_k)}{h_k} = f'(x_k) \Rightarrow \boxed{h_k = \frac{f(x_k)}{f'(x_k)}}$$

$$x_{k+1} = x_k - h_k$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f(\boxed{x+h}) = f(x) + hf'(x) + O(h^2)$$

$$\underbrace{f(x+h)}_{=0} \approx f(x) + hf'(x)$$

$$h = -\frac{f(x)}{f'(x)}$$

$$x + h = x - \frac{f(x)}{f'(x)}$$

Newton's Method

x_0

decide stopping criteria

for $k = 0, 1, 2, \dots$

$$x_{k+1} = x_k - \underbrace{\frac{f(x_k)}{f'(x_k)}}$$

end

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$f(f')^{-1}$$

$$1$$

$$g'(x) = 1 - \left[-f(f')^{-2} f'' + f'\cancel{(f')^{-1}} \right]$$

$$= \cancel{1} + \frac{f f''}{(f')^2} + \cancel{1}$$

$$= \frac{f f''}{f'^2}$$

simple root

$$g'(x) = 0 \qquad (\text{if } f' \neq 0)$$

<u>root multiplicity</u>

$f(x) = 0$ x is a root of f

$f'(x) = 0$ multip. 2

$f''(x) = 0$ 3

mult. = 1 $\longrightarrow$ simple roots

Summary

simple root of f

we get $g'(x^*) = 0$

N.M. quadratic convergence

multiple root of f (mult. = m)

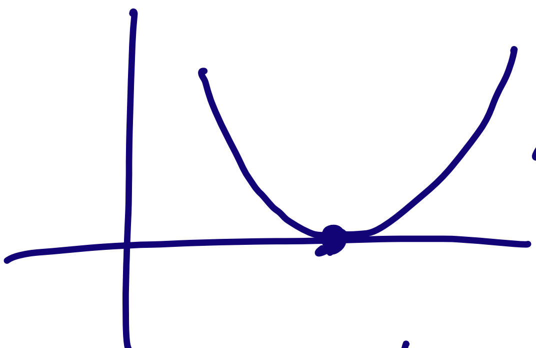
drop down to linear conv.

w/ constant

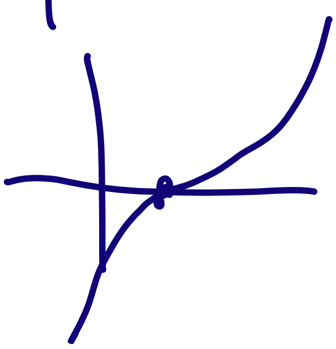
$$\underbrace{C = 1 - \frac{1}{m}}$$

$C = \frac{1}{2}$ $m=2$

$C = .8$ $m=5$



quad



cubic.