CS 210
Practice Midterm

Name:	
Student ID:	

I agree to abide by the UCR Academic Integrity Policy.

Signature:	

Rules:

- Work individually. No notes, calculators, etc., permitted. Scratch paper will be provided by the proctor.
- The proctor is not allowed to answer individual questions during the exam, but may choose to make an announcement if something requires correction/clarification.
- If you see a possible error or ambiguity in an exam question, you may bring it to the attention of the proctor.
- You should answer every question to the best of your ability even if you think there is a issue/mistake in the question.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
11	4	
12	4	
13	4	
14	4	
15	4	
16	10	
17	8	
18	12	
Total	70	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.
Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $A, B \in \mathbb{R}^{n \times n}$.

1. (T/$\boxed{\text{F}}$) $\mathbf{u}\mathbf{v}^T$ has rank n.
2. ($\boxed{\text{T}}$/F) $(\mathbf{u}\mathbf{v}^T)\mathbf{w} = \alpha \mathbf{u}$, where $\alpha = \mathbf{v} \cdot \mathbf{w}$.
3. ($\boxed{\text{T}}$/F) $\mathbf{e}_i^T A \mathbf{e}_j = a_{ij}$, where $\mathbf{e}_i, \mathbf{e}_j$ are the i^{th} and j^{th} canonical vectors, respectively, and a_{ij} is the entry of A in row i and column j.
4. (T/$\boxed{\text{F}}$) $AB = BA$ for all $A, B \in \mathbb{R}^{n \times n}$.
5. ($\boxed{\text{T}}$/F) $(AB)^T = B^T A^T$ for all $A, B \in \mathbb{R}^{n \times n}$.
6. ($\boxed{\text{T}}$/F) The relative error in representing a number as a denormalized floating point number could be higher than the relative error in representing a number as a normalized floating point number.
7. (T/$\boxed{\text{F}}$) In floating point arithmetic, it is guaranteed that $(a + b) + c = a + (b + c)$.
8. (T/$\boxed{\text{F}}$) In the IEEE-754 floating point standard, `5/Inf` is undefined.
9. ($\boxed{\text{T}}$/F) Cancellation error can occur when two very close numbers are subtracted.
10. ($\boxed{\text{T}}$/F) Cholesky factorization without pivoting is a stable algorithm.

Multiple Choice

Instructions: For the multiple choice problems, circle <u>exactly one</u> of (a) - (e).

11. Consider non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^n$, with $1 \leq k < n$. Which scenario is possible?

    (a) $\boxed{\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n).}$
    (b) $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbb{R}^n$.
    (c) $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n) = \mathbb{R}^n$ and $\mathbf{v}_i^T \mathbf{u}_j = 0 \ \forall i \in \{1, \ldots, k\}, j \in \{1, \ldots, n\}$.
    (d) $\dim \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$.
    (e) None of the above are possible.

12. Which conditions necessarily imply that a matrix $A \in \mathbb{R}^{n \times n}$ is singular?

    I. $\det(A) < 0$.
    II. There exists a $\mathbf{b} \in \mathbb{R}^n$, such that $A\mathbf{x} = \mathbf{b}$ has more than one solution $\mathbf{x} \in \mathbb{R}^n$.
    III. $\|A\| = 0$.

    (a) I only
    (b) II only
    (c) I and III only
    (d) $\boxed{\text{II and III only}}$
    (e) I, II and III

13. Which of the following statements are true?

    I. The diagonal matrix $10^6 I$, where I is the $n \times n$ identity matrix, is well-conditioned.

    II. An unstable algorithm may compute a solution that has a large backward error.

    III. LU factorization without pivoting is an example of an unstable algorithm.

    (a) I only
    (b) II only
    (c) I and III only
    (d) II and III only
    (e) $\boxed{\text{I, II and III}}$

14. Let $\mathbf{q}, \mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{q}$ be a unit vector (i.e., $\|\mathbf{q}\|_2 = 1$). Let $\mathbf{v}_{||}$ be the component of $\mathbf{v}$ parallel to $\mathbf{q}$ and $\mathbf{v}_\perp$ be the component of $\mathbf{v}$ orthogonal to $\mathbf{q}$, so that $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_{||}$. Which statement is <u>false</u>?

    (a) $\mathbf{v}_{||} = (\mathbf{q}^T\mathbf{v})\mathbf{q}$
    (b) $(I - \mathbf{q}\mathbf{q}^T)(I - \mathbf{q}\mathbf{q}^T) = (I - \mathbf{q}\mathbf{q}^T)$
    (c) $\boxed{\mathbf{v}_\perp = (I - 2\mathbf{q}\mathbf{q}^T)\mathbf{v}}$
    (d) $(I - \mathbf{q}\mathbf{q}^T)(\mathbf{q}\mathbf{q}^T)\mathbf{v} = \mathbf{0}$

15. Let $A, P, M \in \mathbb{R}^{n\times n}$, and let $\mathbf{x}, \mathbf{b}, \mathbf{y} \in \mathbb{R}^n$. Let $A\mathbf{x} = \mathbf{b}$, and let P be a permutation matrix. Which statement is <u>false</u>?

    (a) $\boxed{PAP^T\mathbf{x} = P\mathbf{b}.}$
    (b) If $AP\mathbf{y} = \mathbf{b}$, then $\mathbf{y} = P^T\mathbf{x}$.
    (c) $MA\mathbf{x} = M\mathbf{b}$, where M is a singular matrix.
    (d) PA is a row permutation of A.
    (e) $PP^T = I$, where $I \in \mathbb{R}^{n\times n}$ is the $n \times n$ identity.

Written Response

16. Let $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & 1 \end{pmatrix}$. Compute the following:

    (a) $\|\mathbf{x}\|_\infty$

    (b) $\|\mathbf{x}\|_1$

    (c) $\|A\|_\infty$

    (d) $\|A\|_1$

    (e) $\|A\|_F$ (Frobenius norm of A)

    Solution:

    (a) $\|\mathbf{x}\|_\infty = \max(|1|, |2|, |-3|) = 3$

    (b) $\|\mathbf{x}\|_1 = |1| + |2| + |3| = 6$

    (c) $\|A\|_\infty = \max_i \sum_j |a_{ij}| = \max(|1| + |2| + |3|, |-1| + |0| + |4|, |2| + |-2| + |1|) = \max(6, 5, 5) = 6$

    (d) $\|A\|_1 = \max_j \sum_i |a_{ij}| = \max(|1| + |-1| + |2|, |2| + |0| + |-2|, |3| + |4| + |1|) = \max(4, 4, 8) = 8$

    (e) $\|A\|_F = (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}} = (1 + 4 + 9 + 1 + 16 + 4 + 4 + 1)^{\frac{1}{2}} = \sqrt{40}$

17. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$

(a) What is the rank of A?

(b) Give a basis for the column space A.

(c) Give a basis for the null space of A.

Solution:

(a) A has two columns which can be seen to be linearly dependent by inspection. Therefore, the column space of A is spanned by the first column of A and has dimension 1. The rank of A is the dimension of its column space, so rank(A) = 1.

(b) Either column of A, or any nonzero multiple thereof, comprises a basis for the column space of A. E.g., $\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}$

(c) The domain of A is $\mathbb{R}^2$. A has rank 1, so the null space has dimension 2-1 = 1. Note that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore a basis for the null space of A is

$$\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\}$$

18. Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & -4 & 12 \\ -4 & 13 & 6 \\ 12 & 6 & 73 \end{pmatrix}.$$

(a) Find the Cholesky factorization of A. Specifically, express A as $A = LL^T$ where L is a lower triangular matrix.

(b) It is possible to avoid square roots by instead computing the LDL^T decomposition $A = LDL^T$, where L is <u>unit</u> lower triangular, and D is a diagonal matrix. Find the LDL^T decomposition of A from your result in part (a).

(c) Give pseudocode for computing the LDL^T decomposition of a matrix.

Solution:

(a)

$$l_1 l_1^T = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} \begin{pmatrix} 2 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -4 & 12 \\ -4 & 4 & -12 \\ 12 & -12 & 36 \end{pmatrix}$$

$$A - l_1 l_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 18 \\ 0 & 18 & 37 \end{pmatrix}$$

$$l_2 l_2^T = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \begin{pmatrix} 0 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 18 \\ 0 & 18 & 36 \end{pmatrix}$$

$$A - l_1 l_1^T - l_2 l_2^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$l_3 l_3^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Putting this together,

$$A = l_1 l_1^T + l_2 l_2^T + l_3 l_3^T = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 6 & 6 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & -2 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

(b) We rescale the colums of L and the rows of L^T to get

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 6 & 6 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

(c) Note that if $A = LDL^T$, then

$$A = \sum_{i=1}^{n} d_{ii} \mathbf{l}_i \mathbf{l}_i^T,$$

where $\mathbf{l}_i$ are the columns of L and d_{ii} are the diagonal entries of D. Below is pseudocode for computing the LDL^T decomposition of a matrix.

```
function [L,D] = LDL(A)

[m,n] = size(A);

for k=1:n
    D(k,k) = A(k,k);

    % compute the vector l_k
    for i = k:n
        L(i,k) = A(i,k) / D(k,k);
    end

    % update A <- A - d_kk l_k l_k^T
    for i = k:n
        for j = k:n
            A(i,j) = A(i,j) - L(i,k) * D(k,k)* L(j,k);
        end
    end
end
```