CS 210
Practice Final

Name	
Student ID	
Signature	

Rules:

- Work individually. No notes, calculators, etc., permitted. Scratch paper will be provided by the proctor.
- The proctor is not allowed to answer individual questions during the exam, but may choose to make an announcement if something requires correction/clarification.
- If you see a possible error or ambiguity in an exam question, you may bring it to the attention of the proctor.
- You should answer every question to the best of your ability even if you think there is a issue/mistake in the question.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
11	3	
12	3	
13	3	
14	3	
15	9	
16	12	
17	12	
18	12	
Total	77	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) For any matrix $A \in \mathbb{R}^{n\times n}$, $\det(A) = \lambda_1\lambda_2 \ldots \lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

2. (T/F) Every matrix $A \in \mathbb{R}^{m\times n}$ has a singular value decomposition factorization of the form $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal and $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal.
   $\Sigma \in \mathbb{R}^{m\times n}$

3. (T/F) If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, then $AU = U\Lambda$ from some orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and diagonal matrix Λ.
   Yes $A = U\Lambda U^T$ is the spectral decomposition of A

4. (T/F) Given a real, symmetric, invertible matrix A, power iteration, inverse power iteration, and Rayleigh quotient iteration are all algorithms that can be used to find an eigenvalue of A.

5. (T/F) If $A \in \mathbb{R}^{n\times n}$ is symmetric, then its singular value decomposition is the same as its eigenvalue decomposition.
   σ_i are all positive, λ_i need not be.

6. (T/F) $\|A^{-1}\|_2 = \frac{1}{\sigma_1}$, where σ_1 is the largest singular value of an invertible, real matrix A.
   Actually $\|A^{-1}\|_2 = 1/\sigma_n$

7. (T/F) If $A = U\Sigma V^T$ is the singular value decomposition of A, then $A^TA = V(\Sigma^T\Sigma)V^T$ is an eigendecomposition of A^TA.

8. (T/F) A nonlinear equation of a single variable $f(x) = 0$ is guaranteed to have one unique solution.
   nonlinear equation could have 0, 1, or more solutions.

9. (T/F) If $f(x)$ is any function defined on a nonempty interval $[a, b] \subset \mathbb{R}^n$, with $f(a) < 0$ and $f(b) > 0$, then f has a root in $[a, b]$.
   Not necessarily. There are counterexamples if f is not continuous.
   True if f is continuous.

10. (T/F) In an iterative solver for $A\mathbf{x} = \mathbf{b}$, a small residual indicates that the error must be small as well.
    This depends on A. The error could be large for poorly conditioned A.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

11. Let $A \in$ and let $A = U\Sigma V^T$ be a singular value decomposition of A.

    T I. The singular values of A are uniquely determined.

    T II. U is always invertible.

    F III. VV^T is a projection matrix onto the nullspace of A.

    (a) I and II only

    (b) II and III only

    (c) I and III only

    (d) I, II, and III only

    rank$(A) = r$

    $r \quad n-r$

    $V = (V_1 \mid V_2)$

    $V_2V_2^T$ projects onto null(A)

12. Consider the least squares (LS) problem $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2$. Which statement is false?

T (a) The minimum norm solution to the problem is given by $A^+\mathbf{b}$, where A^+ is the pseudo-inverse of A.

T (b) $\|A\mathbf{x} - \mathbf{b}\|_2 = \|R\mathbf{x} - Q^T\mathbf{b}\|_2$, where $A = QR$, with Q an orthogonal matrix.

(c) The solution $\mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto range(A). It's Ax that is the orthogonal proj. of b onto range(A) (not x)

T (d) If A has linearly independent columns, then $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$ is the unique solution to the LS problem.

13. Let $A \in \mathbb{R}^{n\times n}$ be invertible. Which of the following matrices have the same eigenvectors as A?

✓ I. $A - \alpha I$, for some nonzero scalar α.

X II. SAS^{-1}, for some invertible matrix S.

✓ III. A^{-1}.

(a) I only

(b) II only

(c) III only

(d) I and II only

(e) I and III only

14. Consider solving $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$. Let $J_g(\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x})$ be the the Jacobian matrix of $\mathbf{g}$ so that $\mathbf{g}(\mathbf{x}+\mathbf{s}) = \mathbf{g}(\mathbf{x}) + J_g(\mathbf{x})\mathbf{s} + O(\|\mathbf{s}\|^2)$ is the Taylor expansion of $\mathbf{g}$ about $\mathbf{x}$. Which of the following statements are true?

F I. If $\mathbf{g}$ is a quadratic function, then $J_g(\mathbf{x})$ is a constant matrix.

T II. Applying Newton's method, $\mathbf{s}_k = -J_g^{-1}(\mathbf{x}_k)\mathbf{g}(\mathbf{x}_k)$ is the Newton step such that $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$.

F III. $J_g(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix}$ $\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$

$n = 1$

$g_1(x+s) = g_1(x) + \sum_i \frac{\partial g_i}{\partial x_i} s_i$

(a) I only

(b) II only

(c) III only

(d) I and III only

(e) II and III only

$g(x+s) = g(x) + g'(x)s + O(s^2)$

$g(x) = ax^2 + bx + c$

$g'(x) = 2ax + b$

$$\boxed{\begin{aligned} s_k &= -g'(x_k)\, g(x_k) \\ x_{k+1} &= x_k + s_k \end{aligned}}$$

Written Response

15. Find the singular value decompositions of the matrices below. Show your work.

   (a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(a) $A = e_1 e_1^T + 2(-e_2)e_2^T$

$$A = \begin{pmatrix} | & | \\ -e_2 & e_1 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} - & e_1^T & - \\ - & e_2^T & - \end{pmatrix}$$

(b) $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \sqrt{10} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(c) Since A is real, symmetric, we can find $A = U \Lambda U^T$ first.

$$(1-\lambda)^2 - 4 = 0 \Rightarrow \lambda^2 - 2\lambda + 1 - 4 = 0 \Rightarrow \boxed{\lambda = -1, 3}$$
$$(\lambda - 3)(\lambda + 1) = 0$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \Rightarrow \begin{matrix} x + 2y = -x \\ 2x + y = -y \end{matrix} \Rightarrow 2x + 2y = 0 \Rightarrow x = -y$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \Rightarrow -2x + 2y = 0 \Rightarrow x = y$$

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$ is the eigendecomposition.

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$ is the SVD

note - moved here so that singular values ≥ 0

16. *Least squares.* Let $A \in \mathbb{R}^{m\times n}$, where $m > n$. Consider the least squares (LS) problem

$$\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2.$$

(a) Assume A has full rank. Show how you would use the QR decomposition $A = Q\boxed{\begin{pmatrix} R \\ 0 \end{pmatrix}} = \bar{R}$ to solve the LS problem.

(b) Now assume A is rank-deficient with rank $r < n$. Show how you would use the Singular Value Decomposition $A = U\Sigma V^T$, with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, to solve the LS problem.

(c) In parts (a) and (b) is the solution unique? Why or why not?

(d) What does it say about $\mathbf{b}$ if $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2 = 0$?

(a) $$||b - Ax||_2^2 =$$

$$= \left|\left| b - Q\begin{pmatrix} R \\ 0 \end{pmatrix} x \right|\right|_2^2$$

$$= \left|\left| Q^T b - \begin{array}{c} \scriptstyle n \\ \scriptstyle m-n \end{array}\begin{pmatrix} R \\ \hline 0 \end{pmatrix} x \right|\right|_2^2$$

$$A^T A x = A^T b$$

$$\bar{R}^T \cancel{Q^T Q} \bar{R} x = \bar{R}^T Q^T b$$

$$\boxed{\bar{R}^T \bar{R}} x = \bar{R}^T Q^T b$$

solves: forward + backward!

$$(*) = \left|\left| \begin{pmatrix} Q_1^T \\ \hline Q_2^T \end{pmatrix}_{m-n}^{n} b - \begin{pmatrix} R \\ 0 \end{pmatrix} x \right|\right|_2^2$$

$$\left|\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right|\right|_2^2 = \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} \right|\right|_2^2 + ||(z)||_2^2$$

$$x^2 + y^2 + z^2 = x^2 + y^2 + z^2 \quad \checkmark$$

$$(*) = \underbrace{\| Q_1^T b - Rx \|_2^2}_{\boxed{Rx = Q_1^T b}} + \underline{\| Q_2^T b - 0 \|_2^2}$$

Solve $x = R^{-1} Q_1^T b$

(solve by back subst.)

(b) $\| b - Ax \|_2^2$

$U = (\underset{r}{U_1} | U_2 |$

$= \| b - U \Sigma V^T x \|_2^2$

$V = (\underset{r \text{ cols}}{V_1} | V_2)$

$= \| U^T b - \Sigma V^T x \|_2^2$

$$= \sum_{i=1}^{r} (u_i^T b - \sigma_i v_i^T x)^2 + \sum_{i=r+1}^{n} (u_i^T b)^2$$

$$\boxed{x = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i} = A^+ b$$

$Ax = b$

17. *Iterative methods: matrix splitting.* An iterative solver based on matrix splitting is given by

$$S\mathbf{x}_{k+1} = T\mathbf{x}_k + \mathbf{b},$$

$Sx = Tx + b$

$Sx - Tx = b$

$(S-T)x = b$

$Ax = b$

where $A = S - T$ is the splitting. Consider the matrix

$$\begin{pmatrix} 24 & 4 & 9 \\ 4 & 16 & 9 \\ 9 & 9 & 21 \end{pmatrix}$$

(a) What are S and T for the Jacobi method?

(b) Given that the eigenvalues of the iteration matrix $S^{-1}T$ are approximately $\{0.74273, -0.19897, -0.54376\}$ for the Jacobi method, does Jacobi converge for this matrix? At what rate?

$|\lambda_i| < 1 \ \forall i$

(c) What are S and T for the Gauss-Seidel method?

(d) Given that the eigenvalues of the iteration matrix $S^{-1}T$ are approximately $\{0.00000, -0.17975, -0.22352\}$ for the Gauss-Seidel method, does Gauss-Seidel converge for this matrix? At what rate?

(a) Jacobi

$$\boxed{S = \text{diag}(A) = D}$$

S is the diagonal part of D

$$A = \underset{\text{lower}}{L} + \underset{\text{upper}}{U} + \underset{\text{diagonal}}{D}$$

$$\boxed{T = -(L+U)}$$

(b)

$$S x_{k+1} = T x_k + b$$

$$x_{k+1} = S^{-1}T x_k + S^{-1} b$$

$$- \quad x = S^{-1}T x + S^{-1} b$$

$$e_{k+1} = \boxed{S^{-1}T} e_k$$

$\|e_{k+1}\| \le \|S^{-1}T\| \, \|e_k\|$

Yes, Jacobi converges because

$\rho(S^{-1}T) < 1$

converges linearly (rate = 1)
(with constant $\sim |\lambda_1|$)

(c) $S x_{k+1} = T x_k + b$

$$\begin{cases} S = D + L \\ T = -U \end{cases}$$ (defined above)

(d) Yes, because

$$\rho(S^{-1}T) < 1$$

linear, with a smaller constant $\sim |\lambda_1|$

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rate:

$$\lim_{k \to \infty} \| e_{k+1} \| = C \| e_k \|^r$$

$$f'(x) = 3x^2 - x + 2$$

18. *Fixed point iteration.* Let $f(x) = x^3 - .5x^2 + 2x - 1$. We want to find a real x such $f(x) = 0$. For convenience, you are given that the one real root of f occurs at $x = .5$.

    (a) Find a fixed point iteration rule for computing x that uses only $\times$, $+$, and $-$.

    (b) Show that convergence of your fixed point iteration is guaranteed sufficiently close to the solution.

(a) $x = g(x)$

$x_{k+1} = g(x_k)$

$f(x) = 0 \iff$

$x = -.5x^3 + .25x^2 + .5$

$x = g(x) = \underline{-.5x^3 + .25x^2 + .5}$

(b) $|g'(x^*)| < 1$ condition for convergence

Given $x^* = .5$

$g'(x) = -1.5x^2 + .5x$

$g'(.5) = -\frac{3}{2}\frac{1}{4} + \frac{1}{4} = -\frac{3}{8} + \frac{2}{8} = -\frac{1}{8}$

$|g'(.5)| < 1 \implies$ convergence

Newton iteration

$$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$$

$$g(x) = x - \frac{\left(x^3 - \frac{1}{2}x^2 + 2x - 1\right)}{\left(3x^2 - x + 2\right)}$$

$$g'(.5) \overset{?}{<} 1$$

$$g'(x) = 1 - \left[\frac{(3x^2 - x + 2)^2 - (x^3 - \frac{1}{2}x^2 + 2x - 1)(6x - 1)}{(3x^2 - x + 2)^2}\right]$$

$$= \frac{\left(x^3 - \frac{1}{2}x^2 + 2x - 1\right)(6x - 1)}{(3x^2 - x + 2)^2}$$

$$\boxed{g'\left(\tfrac{1}{2}\right) = 0}$$