$\text{rank}(A)$ $\qquad A \in \mathbb{R}^{m \times n}$

$\text{rank}(A) = \dim(\text{col}(A)) = r$

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{col}(A) = \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \middle| \ \alpha \in \mathbb{R} \right\}$$

$\dim(\text{col}(A)) = 1 = \text{rank}(A)$

$$A = \begin{pmatrix} 1 & \boxed{\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}} \\ 0 & \\ 0 & \end{pmatrix} \qquad \alpha \vec{a}_1 + \beta \vec{a}_2$$

rank $= 1$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{rank } 2$$

$A \in \mathbb{R}^{m \times n}$

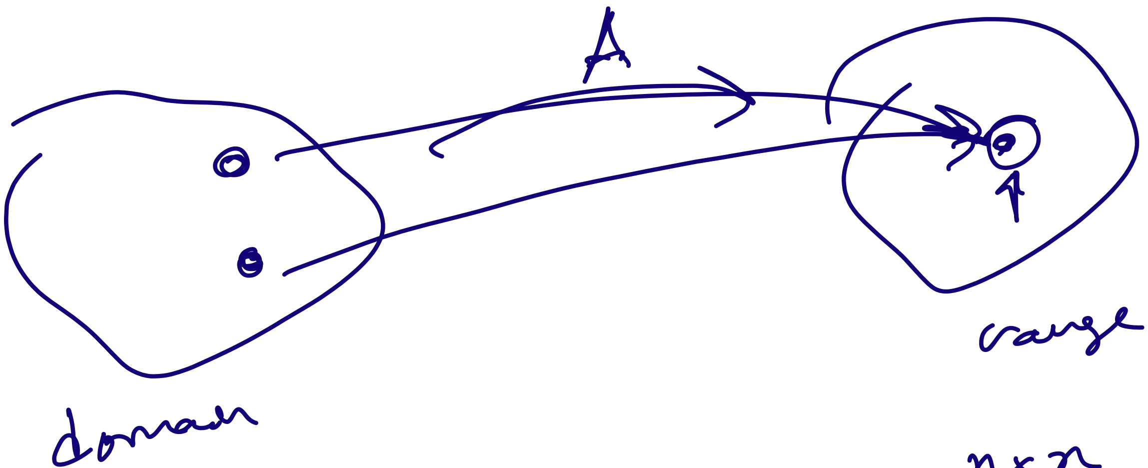
$$\underbrace{\text{rank}(A) \leq \min(m, n)}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

"full-rank"

$$\text{rank}(A) = \min(m, n)$$

invertible matrix :



- square matrix $\mathbb{R}^{n \times n}$
- full rank

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

$$\vec{y} = \underbrace{A\vec{x}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \boxed{\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}}\right) = A$$

$$\vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq A\vec{y}$$

$\vec{y} \in \mathbb{R}^3$

$\boxed{A \text{ square \& full rank} \Rightarrow}$ $\boxed{A^{-1} \text{ exists}}$

$\vec{y} = A\vec{x}$

$$\vec{y} = A\vec{x}_1$$
$$\vec{y} = A\vec{x}_2$$

$$\boxed{\vec{x} = A^{-1}\vec{y}}$$

$$A^{-1}Ax = x$$

$$AA^{-1}x = x$$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} \quad Ax = y$$

$$y = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

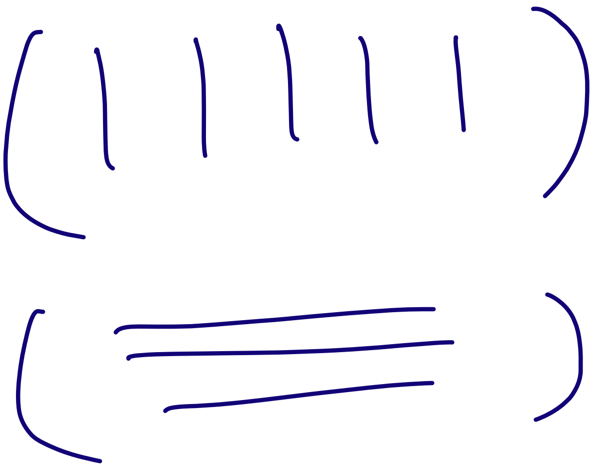
$$Ax = y \checkmark \quad x = \begin{pmatrix} 2-\alpha \\ 1-\alpha \\ \alpha \end{pmatrix}$$

$\underline{\text{col}(A)}$

$\text{row}(B)$

$\mathrm{col}(B^T)$

$\mathbb{R}^{m \times n}$ $\quad m \neq n$



$\mathbb{R}^m$

$\mathbb{R}^n$

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

indep rows = 2

indep cols = 2

$$(AB)^T = B^T A^T$$

---

if A, B are each invertible

A^{-1}, B^{-1} exist

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(AB)(B^{-1}A^{-1}) \overset{?}{=} I$$

$A(BB^{-1})A^{-1}$

$\downarrow$

$\vdots$

$(AB)C = A(BC)$ yes associative

$AB \neq BA$ not commutative

$\underset{3\times 2}{A} \quad \underset{2\times 5}{B}$

$\underset{2\times \underline{5}}{B} \quad \underset{\underline{3}\times 2}{A}$

outer product matrix

rank 1 matrix

$$\vec{a}^T a = a \cdot a$$
$(1 \times n \;\; n \times 1)$

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\underset{m\times 1}{\vec{a}}\;\underset{1\times n}{\vec{b}^T} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$$

$m \times n$

$$= \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & \ddots & \vdots \\ a_m b_1 & \cdots & a_m b_n \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix}$$

$$\underbrace{(\vec{a}\vec{b}^T)}_{\text{rank}=1}\vec{x} = a\underbrace{(\vec{b}^T\vec{x})}_{\beta} = \underbrace{\beta\,\vec{a}}$$

dim = 1

Matrix - matrix multiplication

$$\underset{m\times n}{C} = \underset{m\times p}{A}\ \underset{p\times n}{B} =$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[rr, bend right, "C"'] & \mathbb{R}^p \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

$$C_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$$

$$C = \begin{pmatrix} — & \vec{a}_1^T & — \\ — & \vec{a}_2^T & — \\ & \vdots & \\ — & \vec{a}_m^T & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_n \\ \vdots & & \ddots & \vdots \\ a_m^T b_1 & & \cdots & a_m^T b_n \end{pmatrix}$$

dot products of rows of A w/ cols of B

$$C = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix}$$

$$\vec{c}_j = A \vec{b}_j$$

$$= \begin{pmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{pmatrix} \begin{pmatrix} & B & \end{pmatrix} = \begin{pmatrix} — & a_1^T B & — \\ & \vdots & \\ — & a_m^T B & — \end{pmatrix}$$

$$\boxed{a_i^T B}$$

$1 \times p \quad p \times n$

$$C = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_p \\ | & & | \end{pmatrix} \begin{pmatrix} — & b_1^T & — \\ & \vdots & \\ — & b_p^T & — \end{pmatrix}$$

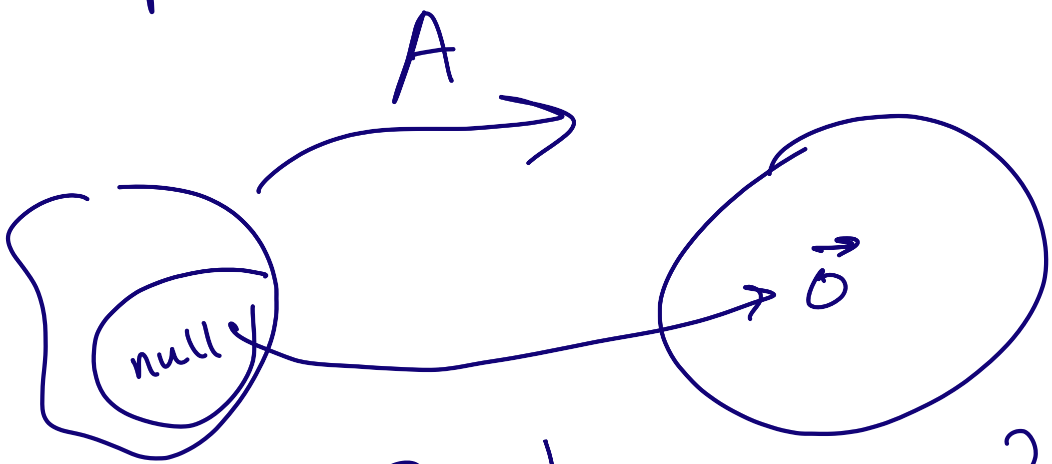
$m \times p \qquad\qquad p \times n$

$$\boxed{C = \sum_{i=1}^{p} \vec{a}_i \, \vec{b}_i^T}$$

sum of outer products

$\mathrm{col}(A) = \mathrm{range}(A)$

nullspace A = $\mathrm{null}(A)$



$$\mathrm{null}(A) = \{ \vec{z} \mid A\vec{z} = \vec{0} \}$$

$L(0) = 0$ $\quad$ L linear

$$A\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} = \vec{0}$$

$$\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}-2\alpha\\ \alpha\end{pmatrix} = (-2\alpha)\begin{pmatrix}1\\0\end{pmatrix} + \alpha\begin{pmatrix}2\\0\end{pmatrix}$$

$$= \begin{pmatrix}-2\alpha+2\alpha\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\mathrm{null}(A)$ subspace

$$\underbrace{\begin{pmatrix}| & | & & |\\ a_1 & a_2 & \cdots & a_n\\ | & | & & |\end{pmatrix}}\begin{pmatrix}z_1\\ \vdots\end{pmatrix} = \begin{pmatrix}|\\ \vec{0}\\ |\end{pmatrix}$$

z_n

$$\boxed{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \qquad \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = z$$

$A \in \mathbb{R}^{n \times n}$

$$\operatorname{rank}(A) = r$$

$$\dim(\operatorname{null}(A)) = n - r$$