CS 210
Practice Midterm

Name:	
Student ID:	

I agree to abide by the UCR Academic Integrity Policy.

Signature:	

Rules:

- Work individually. No notes, calculators, etc., permitted. Scratch paper will be provided by the proctor.
- The proctor is not allowed to answer individual questions during the exam, but may choose to make an announcement if something requires correction/clarification.
- If you see a possible error or ambiguity in an exam question, you may bring it to the attention of the proctor.
- You should answer every question to the best of your ability even if you think there is a issue/mistake in the question.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
11	4	
12	4	
13	4	
14	4	
15	4	
16	9	
17	9	
18	12	
Total	70	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) Addition of two positive floating point numbers may cause underflow.
2. (T/F) If two numbers are exactly representable in floating point, then the result of an arithmetic operation on them is also an exactly representable floating point number.
3. (T/F) Floating point cancellation errors can be a source of instability in an algorithm.
4. (T/F) A problem that has a condition number of 1 can be said to be well-conditioned.
5. (T/F) If $A\mathbf{x} = \mathbf{b}$ then $\mathbf{x}$ is necessarily in the range of A.
6. (T/F) Solving $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n\times n}$ a diagonal matrix, requires $\sim n^2$ operations.
7. (T/F) Any symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$ has a Cholesky factorization of the form $A = LL^T$, where $L \in \mathbb{R}^{n\times n}$ is lower triangular.
8. (T/F) If $\|\cdot\|_q$ and $\|\cdot\|_p$ are both vector p-norms, then they are equivalent, i.e., there exist constants C_1 and C_2 such that $C_1\|\mathbf{x}\|_q \le \|\mathbf{x}\|_p \le C_2\|\mathbf{x}\|_q$ for all vectors $\mathbf{x}$.
9. (T/F) For any vector $\mathbf{x}$, $\|\mathbf{x}\|_1 \ge \|\mathbf{x}\|_2$.
10. (T/F) If $P \in \mathbb{R}^{n\times n}$ is a projection matrix, then so is P^T.

1, 3

$\frac{1}{3}$

$\begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}$

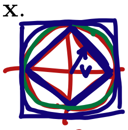
$b \in \text{range}(A)$

$A \in \mathbb{R}^{m\times n}$

$\boxed{\text{range}(A) \subseteq \mathbb{R}^m}$

$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$\boxed{x_i = \frac{b_i}{d_i}}$



$\|v\|_1 = 1$

$\|v\|_2 < 1$

$\boxed{P^2 = P}$

$(PP)^T = P^T \Rightarrow P^T P^T = P^T \checkmark$

$(P^T)^2 = P^T$

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

11. Which of the following statements are necessarily true?

F I. A small backward error implies a small forward error.

T II. It is possible for rounding errors to catastrophically destroy the accuracy of an algorithm.

F III. A large absolute error implies a large relative error.

(a) I only
(b) II only
(c) III only
(d) I and II only
(e) II and III only

LU $\begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}$

percentage error

poorly conditioned

10^6 $\quad 10^{-9}$

$z = \frac{|x|}{|y|}$ ~~$|x-y|$~~ ~~$|x-$~~

abs $\hat{x} - \hat{x}$

$\frac{|x - \hat{x}|}{|x|}$

$\begin{pmatrix} 0 & 0 & 0 \\ a & c & 0 \\ b & d & e \end{pmatrix}$ $\det(L) = \prod l_{ii}$

$\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\boxed{\det A = a_{11} a_{22} a_{33}}$

12. Which of the following statements are true?

T I. A triangular matrix with a zero on its diagonal is singular.

F II. A matrix with a zero on its diagonal is singular.

T III. If $A\mathbf{z} = \mathbf{0}$ for a non-zero vector $\mathbf{z}$, then $\det(A) = 0$.

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ det = 1

(a) I only
(b) II only
(c) I and III only
(d) II and III only
(e) I, II and III

- det(A)
- nontrivial null space
- singular A

13. Which of the following statements is false?

(a) The number of solutions of $A\mathbf{x} = \mathbf{b}$ may depend on $\mathbf{b}$.
(b) If A is singular, then $A\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.
(c) If $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$, then $\mathbf{b}$ must be in the columnspace of A.
(d) Solving a triangular system by forward or backward substitution requires $O(n^2)$ flops.
(e) The LU and Cholesky factorizations of a symmetric, positive definite matrix would be exactly the same. L in LU is unit lower, L in LL^T is not.

14. Which one of the following statements is false?

(a) A symmetric matrix, A, satisfies $||A||_1 = ||A||_\infty$.
(b) A permutation matrix, P, satisfies $||P||_2 = 1$.
(c) An orthogonal matrix, Q, satisfies $||Q||_2 = 1$.
F (d) If A is singular matrix, then $||A||_2 = 0$.
(e) For any vector $\mathbf{x}$, $||\mathbf{x}||_1 >= ||\mathbf{x}||_\infty$.

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$||A||_2 = \max_{||x||_2 = 1} ||Ax||_2$

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

15. Consider an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$. Which statement is false?

(a) Q must be nonsingular.
(b) $||Q\mathbf{x}||_2 = ||\mathbf{x}||_2$ for any vector $\mathbf{x} \in \mathbb{R}^n$.
(c) If $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n\}$ are the columns of Q, then for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = \mathbf{q}_1\mathbf{q}_1^T\mathbf{x} + ... + \mathbf{q}_n\mathbf{q}_n^T\mathbf{x}$.
(d) $Q = Q^T$.
(e) The condition number of Q with respect to the 2-norm is 1.

Counterexample:

$Q = P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

P is orthogonal:

$PP^T = I$

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

But $P \neq P^T$

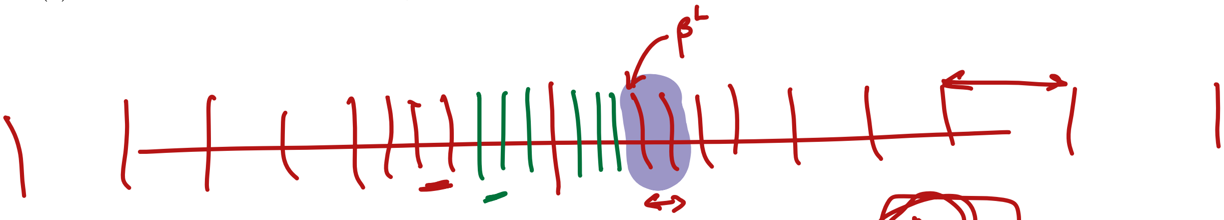
Written Response

16. Consider a normalized floating point number system with p digits of precision, base β and integer exponent E, $L \leq E \leq U$. Let x be a given nonzero floating-point number in this system and let y be an adjacent floating-point number, also nonzero.

    (a) What is the minimum possible distance ($|x - y|$) between x and y?

    (b) What is the maximum possible distance between x and y?

    (c) If we allow denormalization, what will be the distance between the denormalized numbers?



(a) $E = L$

$\underset{\beta^0}{\_}\ .\ \underset{\beta^{-1}}{\_}\ \ldots\ \underset{\beta^{-(p-1)}}{\boxed{1}}$

$\underbrace{1 \times \beta^{-(p-1)} \times \beta^{L}}$

$\beta = 10$

(b) $E = U$

$\underbrace{1 \times \beta^{-(p-1)} \times \beta^{U}}$

(c) Same as (a)

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normalized

$\boxed{\_}\ .\ \_\ \ldots\ \_$

nonzero

$1\ .\ \_\ \ldots\ \_$

$1 \times \beta^{L}$ smallest pos. normalized

denormalized

$0.0 \ldots 01$

smallest pos. denormal.

$\beta^{-(p-1)} \times \beta^{L}$

17. Let $\mathbf{x}^T = (1, 2, 3)$, $\mathbf{y}^T = (0, 1, -2)$.

(a) Write down the values of $\mathbf{x}^T\mathbf{y}$ and $\mathbf{x}\mathbf{y}^T$.

(b) What is $\text{rank}(\mathbf{x}\mathbf{y}^T)$?

(c) Give a basis for $\text{range}(\mathbf{x}\mathbf{y}^T)$?

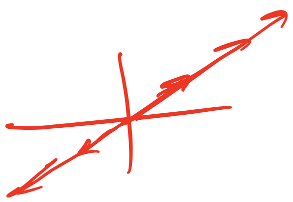
(a) $x^Ty = (1\ 2\ 3)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = 1\cdot 0 + 2\cdot 1 + 3(-2) = -4$

$$xy^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(0\ 1\ -2) = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{pmatrix}$$

(b) $\text{rank}(xy^T) = \dim(\text{range}(xy^T))$
$= \dim(\text{col}(xy^T))$

$\text{range}(xy^T) = \{xy^Tv \mid \forall v \in \mathbb{R}^3\}$

$(xy^T)v = x(y^Tv) = (y^Tv)\vec{x}$
$= \alpha \vec{x}$



$\dim(\text{range}(xy^T)) = \text{rank}(xy^T) = 1$

(c) every element in $\text{range}(xy^T)$ can be expressed $\alpha\vec{x}$ for some $\alpha \in \mathbb{R}$

So $\{\vec{x}\}$ is a basis for $\text{range}(xy^T)$

$$A = \underline{l_1 u_1^T} + l_2 u_2^T + l_3 u_3^T$$

18. Consider the 3×3 matrix

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 6 & 14 & 10 \\ 4 & 10 & 10 \end{pmatrix}.$$

(a) Express A as $A = LU$ where L is a unit lower triangular matrix, and U is ~~the~~ an upper triangular matrix ~~you found above~~.

(b) Explain how you would use the factors L and U to solve the linear equations $A\mathbf{x} = \mathbf{b}$.

(c) For a general invertible $n \times n$ matrix A with LU factorization $A = LU$, how many operations does it takes to solve the linear system $A\mathbf{x} = \mathbf{b}$ using the LU factoriation of A?

(a)

$$\underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{l_1} \underbrace{\begin{pmatrix} 2 & 4 & 3 \end{pmatrix}}_{u_1^T} = \begin{pmatrix} 2 & 4 & 3 \\ 6 & 12 & 9 \\ 4 & 8 & 6 \end{pmatrix}$$

$$\boxed{l_1 u_1^T}$$

$$\alpha l_1 \; \frac{1}{\alpha} u_1$$

$$A - l_1 u_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{l_2} \underbrace{\begin{pmatrix} 0 & 2 & 1 \end{pmatrix}}_{u_2^T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$l_2 u_2^T$$

$$A - l_1 u_1^T - l_2 u_2^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{l_3} \underbrace{\begin{pmatrix} 0 & 0 & 3 \end{pmatrix}}_{u_3^T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{U}$$

check:

$$\begin{pmatrix} 2 & 4 & 3 \\ 6 & 14 & 10 \\ 4 & 10 & 10 \end{pmatrix} \checkmark$$

(b) $Ax = b$

$(LU)x = b$

$L(\underbrace{Ux}_{y}) = b$

① Solve $Ly=b$ for y by forward substitution

② Solve $Ux=y$ for x by back substitution

(c) Do 2 triangular solves

Each triangular solve takes n^2 operations.

So need $2n^2$ operations.