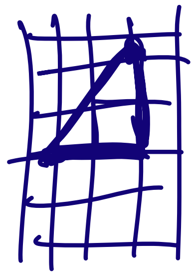
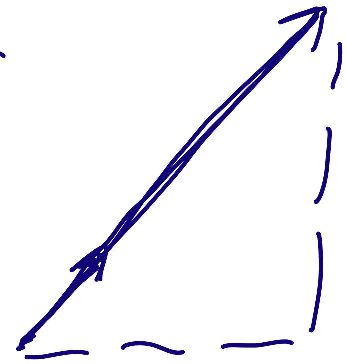
Norms - Vector norms
Matrix norms

$$\|\vec{v}\|_2 = \left(v_1^2 + v_2^2 + \dots + v_n^2\right)^{1/2}$$

Vector norms

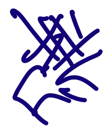




① $\|\vec{v}\| > 0$ if $\vec{v} \neq \vec{0}$

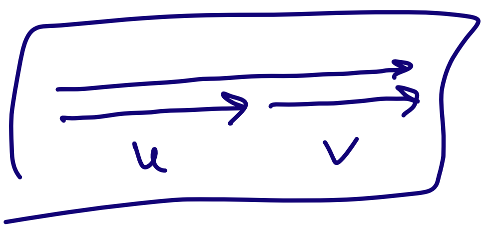
$\|\vec{v}\| = 0$ $\vec{v} = \vec{0}$

② $\|\alpha\vec{v}\| = |\alpha| \, \|\vec{v}\|$ scaling

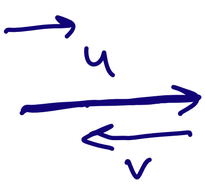


③ triangle inequality

$$\|\vec{v} + \vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\|$$



$\|u+v\| = \|u\| + \|v\|$



$\|u+v\| < \|u\| + \|v\|$

$$\|\vec{v}\| = |v_1| + |v_2| + \dots + |v_n| \qquad \text{norm?}$$

① $\|\vec{v}\| \geq 0$ ✓

$\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$

$\vec{v} = \vec{0} \Rightarrow \|\vec{v}\| = 0$

$\|\vec{v}\| = 0 \Rightarrow v_i = 0 \ \forall i \quad \vec{v} = \vec{0}$ ✓

② scaling

$$\|\alpha\vec{v}\| = \|(\alpha v_1, \alpha v_2, \dots, \alpha v_n)\|$$

$$\cancel{= \|\alpha(v_1, \dots, v_n)\|}$$

$$= |\alpha v_1| + |\alpha v_2| + \dots + |\alpha v_n|$$

$$=$$

$$\left(|xy| = |x||y| \right)$$

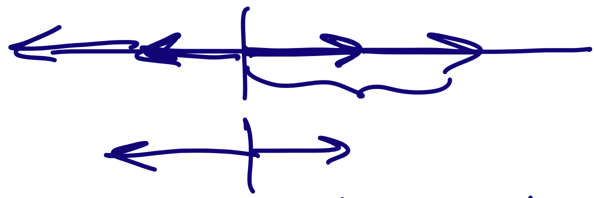
$$= |\alpha||v_1| + |\alpha||v_2| + \dots + |\alpha||v_n|$$

$$= |\alpha|\left(|v_1| + \dots + |v_n| \right)$$

$$= |\alpha| \, \|\vec{v}\| \quad ✓$$

③ $\|\vec{u}+\vec{v}\| = |u_1+v_1| + \cdots + |u_n+v_n|$

$|\alpha+\beta| \leq |\alpha| + |\beta|$



$\leq |u_1|+|v_1| + \cdots + |u_n|+|v_n|$

$\leq |u_1|+\cdots+|u_n| + |v_1|+\cdots+|v_n|$

$= \|u\| + \|v\|$

$\|u+v\| \leq \|u\| + \|v\|$ ✓

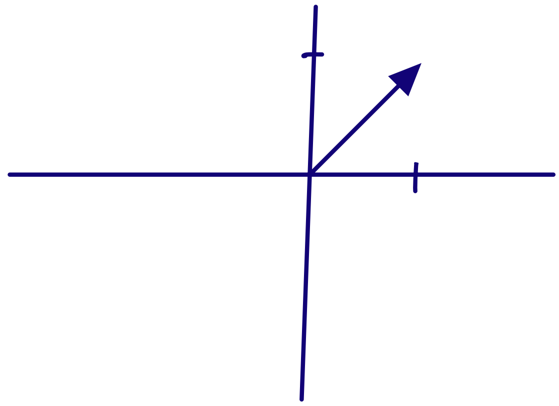
$\|\vec{u}\|_1 = |u_1|+|u_2|+\cdots+|u_n| = \left(\sum_{i=1}^{n} |u_i|\right)^{\frac{1}{1}}$

$\|\vec{u}\|_2 = \left(\sum_{i=1}^{n} |u_i|^2\right)^{1/2}$

$\|\vec{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}}$ $\quad p \geq 1$

$\lim_{p\to\infty} \quad \|\vec{u}\|_\infty = \max_i |u_i|$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$



$$\|v\|_1 = |v_1| + |v_2| + \dots + |v_n| = n$$

$$\|v\|_2 = \left(1^2 + 1^2 + \dots + 1^2\right)^{1/2} = n^{1/2}$$

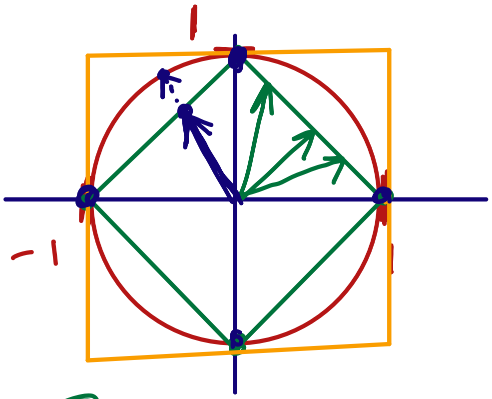
$$\|v\|_\infty = \max_i v_i = 1$$

"unit set"

Set of vector with norm = 1

$\|v\|_2 = 1$ $\qquad$ $\|v\|_1 = 1$

$|v_1| + |v_2| + \dots + |v_n| = 1$



$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\|v\|_\infty = 1$

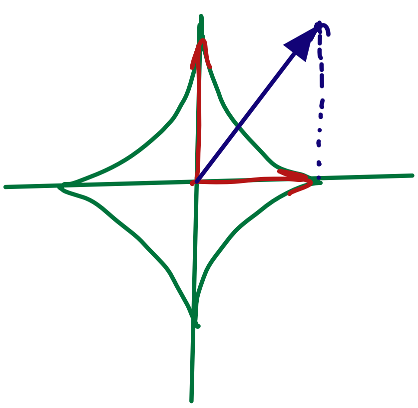
$$\|v\|_\infty^2 \le \|v\|_2^2 \le \|v\|_1^2$$

All p-norms are "equivalent"

$$\underline{C}\|v\|_q \le \|v\|_p \le \underline{D}\|v\|_q$$

C, D are independent of $\vec{v}$

$p = \frac{1}{2}$ $\qquad$ $\underline{\|v\|_{\frac{1}{2}}} = \left(\sum_{i=1}^{n} \sqrt{|v_i|} \right)^{\frac{1}{(\frac{1}{2})}}$



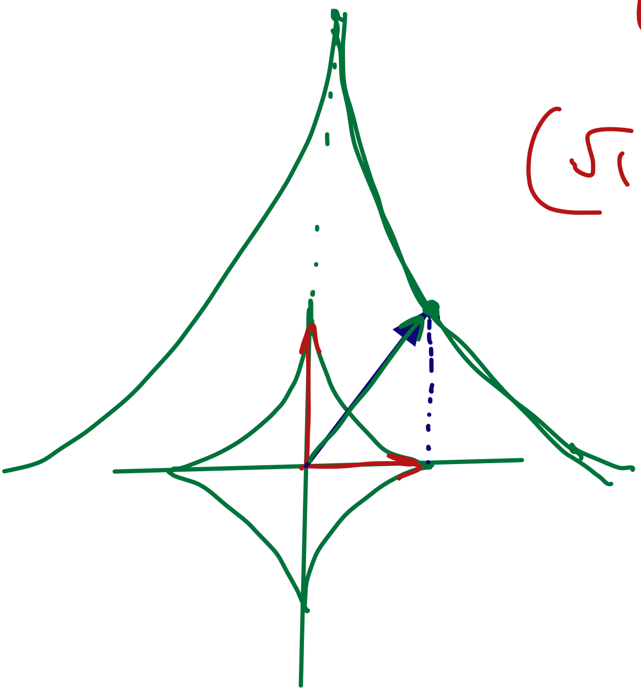
norm? no.

$$\underbrace{\left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|}_{4} \le \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|$$

$$1 + 1$$

2

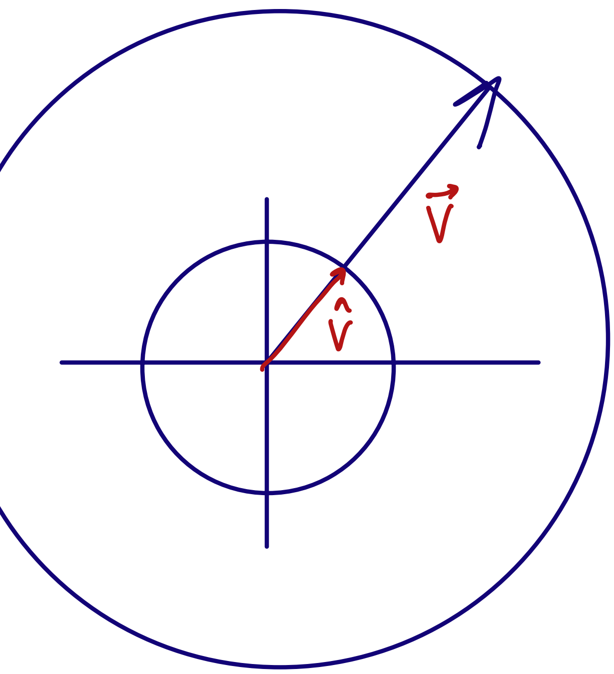
$$(\sqrt{1}+\sqrt{1})^2 = 2^2 = 4$$



$p = 0$ count of the non-zero entries

$$\|\alpha \vec{v}\| = |\alpha| \vec{v}$$

$$\left\| 10\begin{pmatrix}1\\1\\0\\2\\1\end{pmatrix} \right\|_0 = 4$$



$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{v} = \|\vec{v}\| \, \hat{v}$$