$$A = LU \qquad\qquad Ax = b$$
$$\underbrace{LU}x = b$$

$$\begin{pmatrix} \boxed{0} & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

row permutations

For nonsingular A, $\exists$

$$PA = LU$$

$$A = \begin{pmatrix} \boxed{0} & 1 \\ 1 & 1 \end{pmatrix} \qquad a_{kk}$$

$$A = \begin{pmatrix} 10^{-20} & 1 \\ \boxed{1} & \boxed{1} \end{pmatrix}$$

$\cancel{a_{kk}}$

tiny pivots are bad

$$\underbrace{\begin{pmatrix} 1 \\ 10^{20} \end{pmatrix}}_{l_1} \underbrace{\begin{pmatrix} \boxed{10^{-20}} & 1 \end{pmatrix}}_{u_1^T} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & \boxed{10^{20}} \end{pmatrix}$$

$$A - l_1 u_1^T = \begin{pmatrix} 0 & 0 \\ 0 & 1-10^{20} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1-10^{20} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix} \begin{pmatrix} 10^{-20} & 1 \\ 0 & \boxed{1-10^{20}} \end{pmatrix} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}$$

$L \qquad U \qquad A$

$$\frac{|x - fl(x)|}{|x|} < \frac{\varepsilon}{2}$$

$$\begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix} \begin{pmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{pmatrix} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 0 \end{pmatrix}$$

$\tilde{L} \qquad \tilde{U} \qquad \tilde{A}$

backward error

$$A - \tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Rightarrow x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{\tilde{A}\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Rightarrow \tilde{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

LU factorization is "unstable"

add pivoting to stabilize (row permutation) row permutations: $PA = LU$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 7 \\ 2 & 4 & 8 \end{pmatrix} \qquad \boxed{3}$$

$$\underset{l_1}{\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}} \underset{u_1^T}{\begin{pmatrix} 2 & 4 & 8 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$$

$$A - l_1 u_1^T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boxed{1}$$

$$\underset{l_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \underset{u_2^T}{\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$A - l_1 u_1^T - l_2 u_2^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad 2$$

$$\underset{l_3}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \underset{u_3^T}{\begin{pmatrix} 0 & 0 & 2 \end{pmatrix}}$$

$$A = l_1 u_1^T + l_2 u_2^T + l_3 u_3^T = LU$$

3, 1, 2

$$\boxed{P} \underset{L}{\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 4 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}} = PA$$

$PA = LU$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{P} A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}}_{L} \begin{pmatrix} & u & \end{pmatrix}$$

$$L = P\hat{L}$$

$PA = LU$ partial pivoting = row pivoting

row permutation

---

complete pivoting

$$PAQ = LU$$

$$\boxed{PAx = Pb}$$

$$(PAQ)Q^{-1}x = Pb$$

$$LU\underbrace{(Q^{-1}x)} = Pb$$

$$LU\boxed{\tilde{x}} = Pb$$

$$\boxed{Q^{-1}x = \tilde{x}}$$

$$x = \underbrace{Q\tilde{x}}$$

$$\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

A

$$\begin{pmatrix} | & | & & | \\ a_2 & a_1 & \cdots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$AQQ^{-1}\vec{x} = \sum_{i=1}^{n} x_i \vec{a}_i$$

$$\underline{Q^{-1}} = Q^T$$

"orthogonal" matrix

$$\underline{P} = \underline{P_n} \cdots \underline{P_2}\,\underline{P_1}$$

Summary

Every nonsingular (i.e. invertible) $n\times n$ matrix A has

$$PA = LU$$

$$\underbrace{Lx = c}_{n^2}$$

To solve $Ax = b$

1. Find $PA = LU$
2. $PAx = L(Ux) = \underbrace{Pb}$
3. Solve $Ly = Pb$ $\quad (y = Ux)$
4. solve $U\vec{x} = y$

$$Ax = b$$

$$x = \underbrace{A^{-1}b}$$

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \square \\ \\ \\ \end{pmatrix}$$

$$n(\underbrace{n + n-1}) = \underline{2n^2} - n$$

Special linear systems

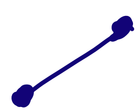
1. Symmetric

$$A = A^T$$

$$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 2 & -1 \\ 5 & -1 & \end{pmatrix}$$

Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$



2. Symmetric positive definite matrix

$$\underline{s.p.d.}$$

$$A = A^T$$

$\vec{x} \neq 0,\quad \underbrace{\vec{x}^T A \vec{x}} > 0$ A pos. def.

$x \neq 0 \quad x a x = \boxed{a x^2} > 0$

A spd

(... pivoting) $\Longrightarrow$ Cholesky factorization

LU (no pivoting) fac...

for $k = 1, \ldots, n$

  if $a_{kk} <= 0$ stop

  $\boxed{l_{kk} = (a_{kk})^{1/2}}$ ($l_{kk} = 1$)

  for $i = k+1, \ldots, n$

    $l_{ik} = a_{ik} / l_{kk}$

  end

  for $i = k+1, \ldots, n$

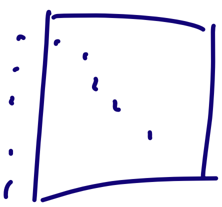
    for $j = k+1, \ldots, \cancel{n}\ i$

      $a_{ij} \leftarrow a_{ij} - l_{ik} l_{jk}$

    end

  end

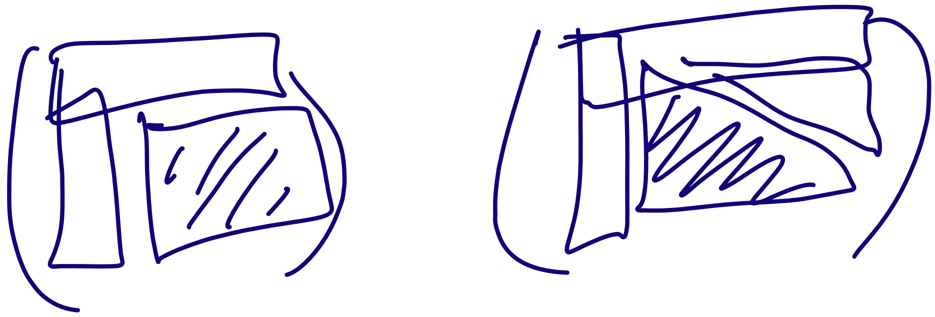
end



update only lower tri 1/2 ops

cuts work down by 1/2

$$A - \underline{l_1 u_1^T} - l_2 u_2^T \qquad \cancel{\frac{2}{3} n^3}$$



$$\frac{1}{3} n^3$$

LU

$$\begin{pmatrix} 1 & & \\ & 1 & \\ x & x & 1 \end{pmatrix} \begin{pmatrix} x & x & x \\ & x & x \\ & & x \end{pmatrix}$$

Cholesky

$$\begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{32} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ & l_{22} & l_{32} \\ & & l_{32} \end{pmatrix}$$

$$L \qquad U = L^T$$

$$A = \boxed{L L^T} \qquad \underbrace{R^T R}$$

$$= \boxed{\tilde{L} D \tilde{L}^T}$$

$\tilde{L}$ unit tri

$$\underbrace{\begin{pmatrix} \boxed{3} & & \\ 1 & \boxed{1} & \\ 1 & -1 & 2 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ & 1 & -1 \\ & & 2 \end{pmatrix}}_{L^T}$$

$$\begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 1/3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & & \\ 0 & 1 & \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & & \\ & 1 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ & 1 & -1 \\ & & 1 \end{pmatrix}$$

$$\underbrace{D^{1/2} \quad D^{1/2}}$$

$$\begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 1/3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 9 & & \\ & 1 & \\ & & 4 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ & 1 & -1 \\ & & 1 \end{pmatrix}$$

LL^T

$\tilde{L}D\tilde{L}^T$ — symm.

Cholesky factorization

A s.p.d.

1. $\exists\ A = LL^T$

2. stable w/o pivoting

3. $\frac{n^3}{3}$

4. can use Chol. fact.

to answer

A is spd ?

$\boxed{L \boxed{D} L^T}$