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Homework 1 862468325

Q1

(a) $\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} \Rightarrow \underset{1\times 1}{\begin{bmatrix} \ \end{bmatrix}}$

(b) $\underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}}$

(d) $\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}}$

(c) $\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} \ \\ \ \end{bmatrix}}$

(e) $\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} \ \end{bmatrix}}$

(f) $\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} \ \end{bmatrix}}$

(g) $\alpha V_{n\times 1} \Rightarrow \begin{bmatrix} \\ \\ \end{bmatrix}_{n\times 1}$

(h) $\alpha B_{n\times n} = \begin{bmatrix} & \cdots & \\ \vdots & & \vdots \\ & \cdots & \end{bmatrix}_{n\times n}$

(i) $\underset{1\times n}{V^T} W_{n\times 1} = [\ ]_{1\times 1}$

(j) $A_{n\times n} V_{n\times 1} = \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix}_{n\times 1}$

(k) $\cancel{CV}_{m\times n} = C_{m\times n} V_{n\times 1} = \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix}_{m\times 1}$

(l) $V^T_{1\times n} A_{n\times n} W_{n\times 1} = [\ ]_{1\times 1}$

(m) $\underset{1\times n}{V^T} \underset{m\times n}{C} \underset{n\times 1}{V} = \underset{1\times n}{V^T} D_{m\times 1}$ ⇒ Only possible if $n=m$ then answer will be 1×1

(n) $\underset{1\times n}{V^T} C^T_{n\times m} C_{m\times n} V_{n\times 1} \Rightarrow [\ ]_{1\times 1}$

Q2

(a) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 1\times1 + 2\times0 + 3\times0 = 1$

(b) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 1\times(-1) + 2\times0 + 3\times0 = -1$

(c) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow 1\times1 + 2\times1 + 3\times1 = 7$

Q3

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\times1 & 1\times2 & 1\times3 \\ 0\times1 & 0\times2 & 0\times3 \\ 0\times1 & 0\times2 & 0\times3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$

(b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1\times1 & -1\times2 & -1\times3 \\ 0\times1 & 0\times2 & 0\times3 \\ 0\times1 & 0\times2 & 0\times3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\times1 & 1\times2 & 1\times3 \\ 1\times1 & 1\times2 & 1\times3 \\ 1\times1 & 1\times2 & 1\times3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}_{3\times3}$

(d) $\begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4\times1 & 4\times2 & 4\times3 \\ -1\times1 & -1\times2 & -1\times3 \\ 10\times1 & 10\times2 & 10\times3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ -1 & -2 & -3 \\ 10 & 20 & 30 \end{bmatrix}_{3\times3}$

Q4

$\cancel{(a) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times1 & 2\times0 & 3\times0 \\ 4\times1 & 5\times0 & 6\times0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}_{2\times3}}$

$\cancel{(b) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =}$

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times1 + 2\times0 + 3\times0 \\ 4\times1 + 5\times0 + 6\times0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2\times1}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times0 + 2\times0 + 3\times1 \\ 4\times0 + 5\times0 + 6\times1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}_{2\times1}$

(c) $Ae_1 = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}_{n\times n} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times1} = a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}_{n\times1}$

(d) $Ae_2 = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}_{n\times n} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times1} = a_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}_{n\times1}$

(e) $\underset{n\times n}{A}\ \underset{n}{e} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} \begin{bmatrix} 0 \\ 0 \\ | \\ 1 \end{bmatrix}_{n\times n} = a_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ | \\ a_{nn} \end{bmatrix}_{n\times 1}$

Q5

(a) $\underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1\times 1+4\times 0 & 1\times 2+0\times 5 & 1\times 3+0\times 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$

(b) $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0\times 1+1\times 4 & 0\times 2+1\times 5 & 0\times 3+1\times 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}_{1\times 3}$

(c) $e_1^T B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{1\times n} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} = b_1^T = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \end{bmatrix}_{1\times n}$

(d) $e_j^T B = \begin{bmatrix} 0 & 0 & \dots & \underset{\substack{\uparrow \\ j\text{th index}}}{1} & \dots & 0 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} = b_j^T = \begin{bmatrix} b_{j1}, & b_{j2}, & \dots & b_{jn} \end{bmatrix}_{1\times n}$

Q6

LHS

$\cancel{vu^T Aw}\ V_{n\times 1}\ u^T_{1\times n}\ A_{n\times n}\ w_{n\times 1} \Rightarrow [\ ]_{n\times 1}$

RHS

$(u^T_{1\times n}\ A_{n\times n}\ w_{n\times 1})\ V_{n\times 1} \Rightarrow [\ ]_{n\times 1}$

Dimensionality is the same

Also $u^T_{1\times n} A_{n\times n} W_{n\times 1} = \text{Scalar} = [\ ]_{1\times 1}$

Assume it is α

RHS then is

$\alpha V_{n\times 1}$

In LHS we can use the property $ABC = A(BC)$

$\therefore\ V_{n\times 1}(u^T_{1\times n} A_{n\times n} w_{n\times 1})$

$\Rightarrow V_{n\times 1}\alpha$

Since α is scalar

$V_{n\times 1}\alpha = \alpha V_{n\times 1}$

Hence both sides are always equal

Q7

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow AB \neq BA$

$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence we have a counterexample

Q8

$e_i^T A e_j$

$$\underbrace{\begin{bmatrix} 0 & \_ & 1 & \_ & 0 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n \times n}} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow j^{th} \text{ index}$$

i^{th} index

$$a_i^T = [a_{i1} \quad a_{i2} \quad \_ \quad a_{in}]_{1 \times n}$$

j^{th} index

$\therefore$ We now have $a_i^T e_j \Rightarrow [a_{i1} \quad a_{i2} \quad \_ \quad a_{in}]_{1 \times n} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

$\Rightarrow a_{ij}$

Q9

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$

$$\Rightarrow \begin{bmatrix} 1\times1+2\times0 & 1\times(-1)+2\times1 & 1\times1+2\times2 \\ 3\times1+4\times0 & 3\times(-1)+4\times1 & 3\times1+4\times2 \\ -1\times1+0\times0 & -1\times(-1)+0\times1 & -1\times1+0\times2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 3 & 1 & 11 \\ -1 & 1 & -1 \end{bmatrix}_{3\times 3}$$

(b) $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 3 \\ -1 & 1 & -1 \end{bmatrix}_{3\times 3} + \begin{bmatrix} 0 & 2 & 4 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 3 & 1 & 11 \\ -1 & 1 & -1 \end{bmatrix}_{3\times 3}$$

Same answer for both (a) and (b)

Q 10

LHS | RHS

$(AB)^T$ | $B^T A^T$

Assume
$C = AB$

$c_{ij} = \sum_{1\le k\le n} a_{ik} \times b_{kj}$

$C^T \Rightarrow c'_{ij} \Rightarrow c_{ji} = \sum_{1\le k\le n} a_{jk} \times b_{ki} \rightarrow 3$

$B^T \Rightarrow b'_{ij} \Rightarrow b_{ji} \rightarrow 1$

$A^T \Rightarrow a'_{ij} = a_{ji} \rightarrow 2$

Assume $D = B^T A^T$

$d_{ij} = \sum_{1\le k\le n} b'_{ik} \, a'_{kj}$

Using 1 and 2

$d_{ij} = \sum_{1\le k\le n} b_{ki} \, a_{jk}$

$\therefore d_{ij} = \sum_{1\le k\le n} a_{jk} \, b_{ki} \rightarrow 4$

Since 3 = 4

Hence LHS = RHS

Q11 LHS RHS

$= (AB)^{-1}$ $B^{-1}A^{-1}$

If LHS = RHS then

$(AB)^{-1}(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB)^{-1} = I$

$\therefore (B^{-1}A^{-1})(AB)$

$\Rightarrow B^{-1}(A^{-1}A)B$

$\Rightarrow B^{-1}IB$

$\Rightarrow B^{-1}B$

$\Rightarrow I$

$\therefore B^{-1}A^{-1}$ is the inverse of (AB)

Hence $(AB)^{-1} = B^{-1}A^{-1}$

Q13 Given

$② \leftarrow A = A^T$

$① \leftarrow \therefore (A^T)^{-1} = (A^{-1})^T$ $\left[\begin{array}{l} AA^{-1} = I \\ (AA^{-1})^T = (I)^T \\ (A^{-1})^T A^T = I \end{array}\right]$

(a) A^{-1}

$(A^{-1})^T$

Using ① $\Rightarrow (A^{-1})^T = (A^T)^{-1} \Rightarrow (A)^{-1}$ [Using ②]

Symmetric

(b) A^T

$(A^T)^T = A = A^T$ [Using (2)]

Symmetric

(c) A^2

$\Rightarrow (AA)^T$

$\Rightarrow A^T A^T \Rightarrow AA$ [Using (2)]

$\Rightarrow A^2$

Symmetric

Q12

Given

$P = I - C^T(CC^T)^{-1}C$

$C \in \mathbb{R}^{m \times n}$

C has full rank

CC^T is invertible

(a) $CP \rightarrow$ To find

$C(I - C^T(CC^T)^{-1}C)$

$CI - CC^T(CC^T)^{-1}C)$

$C - IC$

$\Rightarrow 0$

$\therefore CP = 0$

(b) <u>To Prove</u>

$$PP = P$$

<u>L.H.S</u>

$$(I - C^T(CC^T)^{-1}C)(I - C^T(CC^T)^{-1}C) \rightarrow (1)$$

$$\cancel{I.I - (C^T(CC^T)^{-1}C).I - I.(C^T(CC^T)^{-1}C) + (C^T(CC^T)^{-1}C)(C(CC^T)^{-1}C)}$$

$$\cancel{I - 2(C^T(CC^{-1})^{-1}C) +}$$

Assume $A = I$

$B = C^T(CC^T)^{-1}C$

$D = I - C^T(CC^T)^{-1}C = P \rightarrow (2)$

$\therefore$ (1) becomes

$$(A - B)D$$

$$\Rightarrow AD - BD$$

$$\Rightarrow ID - BD$$

$$\Rightarrow D - C^T(CC^T)^{-1}CD$$

$\Rightarrow D - C^T(CC^T)^{-1}CP$ [Using (2)]

$\Rightarrow D$ [Because $CP = 0$]

$\Rightarrow P$ [Using (2)] Hence proved