HW 2

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1

$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Column Space

$x_1\begin{pmatrix}1\\3\end{pmatrix} + x_2\begin{pmatrix}2\\4\end{pmatrix} \quad x \in \mathbb{R}^2$

$\Rightarrow x_1\begin{pmatrix}1\\3\end{pmatrix} + 2x_2\begin{pmatrix}1\\2\end{pmatrix}$

Null Space

$Ax = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = -2x_2 \rightarrow 1$

$x_1 = -\frac{4}{3}x_2 \rightarrow 2$

Solution not possible

We can get the same result using Rank Nullity Theorem

$A_2 = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

Column Space

$x_1\begin{pmatrix}2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\6\end{pmatrix} \quad x \in \mathbb{R}^2$

$(x_1 + 2x_2)\begin{pmatrix}2\\3\end{pmatrix}$

Null Space

$2(x_1 + 2x_2) = 0 \rightarrow 1$

$3(x_1 + 2x_2) = 0$

~~A~~ Hence Null Space

$\begin{bmatrix} -2t \\ t \end{bmatrix}$

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix} + x_3\begin{pmatrix}2\\1\end{pmatrix} \quad x \in \mathbb{R}^3 \qquad [\text{Column Space}]$$

$$\Rightarrow x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix} + (2x_1 + x_2)$$

Null Space

$Ax = 0$

$x_1 + 2x_3 = 0 \rightarrow ①$
$x_2 + x_3 = 0 \rightarrow ②$

Using ① and ② we get

$$\begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} \Rightarrow t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

② $$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Column Space

$$x\begin{pmatrix}-1\\0\\1\\0\\0\\1\end{pmatrix} + x_2\begin{pmatrix}1\\1\\0\\-1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\-1\\-1\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\-1\\-1\end{pmatrix}$$

$$x_1\begin{pmatrix}-1\\0\\1\\0\\0\\1\end{pmatrix}+x_2\begin{pmatrix}1\\1\\0\\-1\\0\\0\end{pmatrix}+x_3\begin{pmatrix}0\\-1\\-1\\0\\1\\0\end{pmatrix}+(-1)\times(x_2+x_3+x_4)$$

$\therefore$ Column Rank = 3

__Nullity__

$\dim(N(A))=1$ [Rank Nullity Theorem]

$Ax=0$

$-x_1+x_2=0 \quad x_1-x_3=0 \quad x_3-x_4=0 \quad x_4-x_2=0$

$x_2+(-x_3)=0 \quad \cancel{x_2} \quad x_4-x_4=0$

$$\therefore\ N(A)=\begin{bmatrix}t\\+t\\t\\t\end{bmatrix}\Rightarrow t\begin{bmatrix}1\\+1\\1\\1\end{bmatrix}$$

Null space of A^T

$Rank(A) = Rank(A^T) = 3$

Using Rank Nullity Theorem

$Rank(A^T) + dim(N(A^T)) = 6$

$dim(N(A^T)) = 3$

Now

$$A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0$$

$$\begin{aligned} -x_1 + x_3 + x_6 &= 0 \\ x_1 + x_2 - x_4 &= 0 \\ -x_2 - x_3 + x_5 &= 0 \\ x_4 - x_5 - x_6 &= 0 \end{aligned} \implies \begin{bmatrix} a+b \\ c-a \\ a \\ c+b \\ c \\ b \end{bmatrix} = N(A^T)$$

(b) Non Zero Vector in N(A)

$$N(A) = t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

let $t = 2$

$$\Rightarrow \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

(c) $k = \dim(N(A^T)) = 3$

We have

$$N(A^T) = \begin{bmatrix} a+b \\ c-a \\ a \\ c+b \\ c \\ b \end{bmatrix} \Rightarrow a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$ are 3 independent vectors

(3) Subspaces S and T of $\mathbb{R}^{20}$

$\dim(S) = 5$

$\dim(T) = 10$

~~(i) S ∩ T~~

Since S and T are subspaces of $\mathbb{R}^{20}$

$\dim(S+T) \leq \dim \mathbb{R}^{20}$

$\therefore \dim(S+T) \leq 20$

To prove $S+T$ is a subspace of $\mathbb{R}^{20}$

Subspace $S+T$. $\{\vec{u}+\vec{w} \mid \vec{u} \in S \text{ and } \vec{w} \in T\} \rightarrow$ (1)

~~Since $\vec{0} \in U$ and $\vec{0} \in$~~

Since $\vec{0} \in S$ and $\vec{0} \in T$

$\therefore \vec{0}+\vec{0} = \vec{0} \in S+T$

Since $\vec{x} \in S \Rightarrow \vec{x} \in \mathbb{R}^{20}$

Similarly

$\vec{y} \in T \Rightarrow \vec{y} \in \mathbb{R}^{20}$

$\vec{x}+\vec{y} \in \mathbb{R}^{20}$ (Subspace property)

But we also know $\vec{x}+\vec{y} \in S+T$ [From (1)]

$$\dim(S+T) = \dim(S) + d(T) - \dim(S \cap T) \Rightarrow (111)$$

From (111) we can get that

$$\dim(S \cap T) \leq \min(\dim(S), \dim(T))$$

$$\therefore \dim(S \cap T) \leq 5$$

Hence min value of (111) is

$$5 + 10 - 5 = 10$$

$$\therefore \quad 10 \leq \dim(S+T)$$

Min value of $\dim(S \cap T)$ can be 0 because then
{It is possible that no vector is common between S and T hence $S \cap T = \phi$}
$\dim(S+T) = 15$ which is still ≤ 20

Hence ~~dim~~

$0 \leq \dim(S \cap T) \leq 5 \rightarrow$ Answer for (i)

$10 \leq \dim(S+T) \leq 15 \rightarrow$ Answer for (ii)

For $S^{\perp}$ $\{v \in S^{\perp} \mid \langle v.w \rangle = 0 \ \forall \vec{w} \in S\}$

(IV) $\leftarrow$ We know $\dim(S) + \dim(S^{\perp}) = \dim(\mathbb{R}^{20})$ [The original vector space]

$\therefore \ 5 + \dim(S^{\perp}) = 20$

$\Rightarrow \dim(S^{\perp}) = 15$

For proof of (IV) we can show that ~~Bases~~ $\text{Bases}(S) \uplus \text{Bases}(S^{\perp})$
$= \text{Bases}(\mathbb{R}^{20})$ [Here it is $\mathbb{R}^{20}$ but it can be for any vector space]

④

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ -1 & ? & ? \\ 2 & ? & ? \end{bmatrix}$$

$\text{Rank}(A_1) = 1$

Assume Bases for $A_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}$ (Using Column ~~Row~~ Rank)

$$\begin{aligned} cx_1 &= 2 \\ cx_2 &= -1 \\ cx_3 &= 2 \end{aligned} \Rightarrow x_1 = x_3$$

Hence matrix will then become

$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & ? & ? \\ 2 & 2 & 3 \end{bmatrix}$$

Since Row Rank (A_1) = Column Rank $(A_1) = 1$

Assume Bases for $A_1 = \{(y_1, y_2, y_3)\}$ (Using Row Rank)

$$\begin{aligned} \alpha y_1 &= 2 \\ \alpha y_2 &= 2 \\ \alpha y_3 &= 3 \end{aligned} \Rightarrow y_1 = y_2 \rightarrow (1) \qquad \frac{y_1}{y_3} = 2/3 \rightarrow (11)$$

Hence matrix will then become

$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & -1 & ? \\ 2 & 2 & 3 \end{bmatrix}$$

Assuming its the scalar β such that

$\beta(y_1, y_2, y_3) = (1, 1, ?)$

$\beta y_1 = 1$

$\beta y_2 = 1$ $\Big\} \to$ From (1)

$\beta y_3 = ?$

Using (11)

$$\frac{\beta y_1}{\beta y_3} = \frac{1}{?} = 2/3$$

$\therefore\ ? = 3/2$

Hence matrix is

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ -1 & -1 & 3/2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} ? & 3 & ? \\ 2 & 1 & 4 \\ ? & -1 & ? \end{bmatrix}$$

Assume Bases for $A_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}$ [Column Rank = 1]

Let c be a constant such that $c\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$cx_1 = 3$

$cx_2 = 1 \quad \Rightarrow \quad x_2 = -x_3 \rightarrow$ (I)

$cx_3 = -1 \qquad x_1 = -3x_3 \rightarrow$ (II)

Hence matrix will then become [Using (I) and (II)]

$$A_2 = \begin{bmatrix} \cancel{6} & 3 & ? \\ 2 & 1 & 4 \\ -2 & -1 & -4 \end{bmatrix}$$

Assume Bases of $A_2 = \{(y_1, y_2, y_3)\}$ [Column Rank = Row Rank $= 1$]

Let β be a constant such that $\beta(y_1, y_2, y_3) = (2 \; 1 \; 4)$

$\beta y_1 = 2$

$\beta y_2 = 1 \quad \Rightarrow \quad \dfrac{y_1}{y_3} = \dfrac{1}{2} \rightarrow$ (III)

$\beta y_3 = 4$

Let γ be a constant such that

$$\gamma y_1 = 6$$

$$\gamma y_2 = 3$$

$$\gamma y_3 = ?$$

Using (11)

$$\frac{y_1}{y_3} = \frac{6}{?} = \frac{1}{2}$$

$$? = 12$$

Hence the matrix is

$$A_2 = \begin{bmatrix} 6 & 3 & 12 \\ 2 & 1 & 4 \\ -2 & -1 & -4 \end{bmatrix}$$

(5) $A = I + uv^T$

Given

A is non singular

Vectors u, v has dimensionality n

To prove

$$A^{-1} = I + \alpha uv^T \quad \alpha \in \mathbb{R}$$

$AA^{-1} = I$ The form of $A \Rightarrow \left\{ I + \overset{u}{[\;\;]} \overset{v^T}{[\;\;\;\;]} \right\} \rightarrow (1)$

$(I + uv^T)(I + \alpha uv^T) = I$

First of all finding conditions for nullity

$Ax = 0$

$(I + uv^T)x = 0 \rightarrow (1)$

$x + uv^Tx = 0 \rightarrow (11)$

From (1)

$$x + u \underbrace{\underset{1\times m}{v^T} \cdot \underset{m\times 1}{x}}_{\text{Scalar}} = 0$$

$x + u\alpha = 0$

$\therefore \; x = -\alpha u$

$\Rightarrow \; Null(A) = \{x \mid x \in \alpha u\}$

$\therefore \dim(Null(A)) = 1$

From (1)

$\cancel{\det(A) = 0 \Rightarrow \det(I + uv^T) = 0}$

$\det(Ax) = 0$

$\cancel{\Rightarrow \det(A)\det(x) = 0}$

Using (11)

$x + uv^T x = 0$

Since $x = \alpha u$

$\alpha u + uv^T \alpha u = 0$

$\alpha u(1 + \underbrace{v^T u}_{\text{Scalar}}) = 0 \Rightarrow (1 + v^T u)\alpha u = 0 \Rightarrow (1 + v^T u)x = 0$

Hence $Ax = 0$ is now $(1 + v^T u)x = 0$

$\therefore$ For $\det(A) = 0$, $v^T u = -1$

If we assume $v^T u = -1$ and put it in (11) along with $x = \alpha u$

We get $Ax = 0 \ \forall x \in \alpha u$

Hence we get $\det(A) = 0$ implying A is singular

$\therefore$ $\det(A) = 0 \Leftrightarrow A$ is singular $\Leftrightarrow v^T u = -1$

(6)

Base β Lower Limit = L

Precision p Upper Limit = U

(a) $\beta = 10$

$2365.27 \Rightarrow 2.36527 \times 10^{3}$

$0.0000512 \Rightarrow 5.12 \times 10^{-5}$

$\therefore L = -5$

$U = 3$

$p = 6$

(b) $\beta = 10$

$2365.27 \Rightarrow 2.36527 \times 10^{3}$

$0.0000512 \Rightarrow 0.00512 \times 10^{-2}$

$\therefore L = -2$

$U = 3$

$p = 6$

(7) Single precision $t_1 = 24$

Double precision $t_2 = 53$

~~Mantissa m Base β~~ Number $x = \pm (m/\beta^{t}) \beta^{e}$ $\left[\begin{array}{l} m \in \mathbb{Z}, e \in \mathbb{Z} \\ \beta^{t} \leq m \leq \beta^{t} - 1 \end{array} \right]$

Adjacent Pair of single precision

$\left(\dfrac{m}{\beta^{t_1}} \times \beta^{e} \right)$ and $\left(\dfrac{m+1}{\beta^{t_1}} \right) \beta^{e}$

Distance between these two $= \dfrac{(m+1)\beta^{e}}{\beta^{t_1}} - \dfrac{m\beta^{e}}{\beta^{t_1}} \Rightarrow \beta^{e-t_1}$

No. of double precision (including endpoints)

$$\Rightarrow \frac{\beta^{e-t_1}}{\beta^{e-t_2}} \mp 1$$

$$\Rightarrow \beta^{t_2 - t_1} - 1 \Rightarrow \beta^{29} - 1$$

(8)

$$\frac{x}{\sqrt{1+x^2}} = S \qquad S = \left(\sqrt{1 + \frac{1}{x^2}}\right)^{-1} \rightarrow (b)$$

$\downarrow$

Very large value of x will max x^2+1 overflow hence accurate division will not be possible, the machine will think

$$\frac{\text{finite number}}{\text{infinite number}} = 0$$

Which clearly here is wrong

For (b)

It will turn out to be

$$\left(\sqrt{1 + \frac{1}{\text{infinite}}}\right)^{-1} = 1.0$$

Below is a python implementation in which I have taken a very large x such that x^2 overflows in python

```
└─$ python
Python 3.11.6 (main, Dec 29 2023, 12:58:00) [Clang 15.0.0 (clang-1500.1.0.2.5)] on darwin
Type "help", "copyright", "credits" or "license" for more information.
>>> float i
  File "<stdin>", line 1
    float i
          ^
SyntaxError: invalid syntax
>>> import sys
>>> f = sys.float_info.max
>>> print(f)
1.7976931348623157e+308
>>> float i = 1.7976931348623157e+305
  File "<stdin>", line 1
    float i = 1.7976931348623157e+305
          ^
SyntaxError: invalid syntax
>>> i = 1.7976931348623157e+305
>>> print(i)
1.7976931348623156e+305
>>> print(i/(1 + i*i))
0.0
>>> print(1/(sqrt(1 + (1/(i*i)))
...
KeyboardInterrupt
>>> print(1/(sqrt(1 + (1/(i*i)))))
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
NameError: name 'sqrt' is not defined
>>> from math import sqrt
>>> print(1/(sqrt(1 + (1/(i*i)))))
1.0
>>> print(1/(sqrt(1 + (1/(i*i)))))
1.0
>>>
```

Q9

- The difference between 2 formulas is that when a will overflow the second one will still work correctly but not the first one.
- Both the formulas have the problem of catastrophic cancellation.
- I wrote a cpp program to solve this, fixing catastrophic cancellation and handling the edge cases of when a or b or c becomes 0.
- If roots are not real my function directly gives “roots are not real” output, otherwise it gives the answers
- The cases of 10^154 or 10^-155 are not working in my code as this is where the variables which store these numbers overflow or underflow. So does the subsequent computation.
- However some python libraries like sympy can solve them, they treat the representation of mathematical equations differently.

```cpp
#include<iostream>
#include<vector>
#include<algorithm>
#include<cmath>
#include<cfloat>
using namespace std;

int sign(float a)
```

```
{
   if(a < 0)
   {
       return -1;
   }
   if(a > 0)
   {
       return 1;
   }
   else
   {
       return 0;
   }
}
vector<float> naive(float a, float b, float c)
{
   float D = b * b - 4 * a * c;
   vector<float> answer;
   if(D>=0)
   {
       float root_1 = (-b + sqrt(D)) / (2 * a);
       float root_2 = (-b - sqrt(D)) / (2 * a);
       answer.push_back(root_1);
       answer.push_back(root_2);
   }
   return answer;
}
vector<float> advanced(float a, float b, float c)
{
   vector<float> answer;
   // Solving catastrophic cancellation of -b +- sqrt(D)
   // Solving catastrophic cancellation in Discriminant D
   float v1 = 4*a*c;
   float v2 = fma(-(4*a), c, v1);
   float v3 = fma(b, b, -v1);
   float corrected_D = sqrt(v3 + v2);
   if(corrected_D >= 0)
   {
       float r1 = (-b - sign(b) * corrected_D) / (2*a);
       float r2 = c / (r1 * a);
       answer.push_back(r1);
       answer.push_back(r2);
```

```
    }
    return answer;
}
int main()
{
    vector<float> answer;
    cout << "Input format:" << endl << "Write values for a, b, c individually, press
enter after writing value for each variable" << endl;
    float a, b, c;
    cin >> a;
    cin >> b;
    cin >> c;
    cout << "Written values (in float) are" << endl << "a is " << a << endl << "b is "
<< b << endl << "c is " << c << endl;
    // Taking case of edge cases of one of the a,b or c == 0
    // NOTE: Unreliable function isnan
    if(isnan(a) || isnan(b) || isnan(c))
    {
        cout << "one of the a or b or c is NaN" << endl;
        exit(1);
    }
    else if(a == 0)
    {
        cout << "EDGE CASES ARE TREATED DIFFERENTLY" << endl;
        if((c == 0) && (b == 0))
        {
            cout << "all coefficients 0" << endl;
            exit(1);
        }
        float r1 = -c / b;
        answer.push_back(r1);
        cout << answer[0] << endl;
        return 0;
    }
    else if(b == 0)
    {
        cout << "EDGE CASES ARE TREATED DIFFERENTLY" << endl;
        int sign_output = sign(a) * sign(c);
        if(sign_output <= 0)
        {
            float r1 = sqrt(-c/a);
            float r2 = -r1;
```

```
            answer.push_back(r1);
            answer.push_back(r2);
            cout << answer[0] << " " << answer[1] << endl;
        }
        else
        {
            cout << "roots are complex" << endl;
        }
        return 0;
    }
    else if(c == 0)
    {
        cout << "EDGE CASES ARE TREATED DIFFERENTLY" << endl;
        float r1 = 0;
        float r2 = -b/a;
        answer.push_back(r1);
        answer.push_back(r2);
        cout << answer[0] << " " << answer[1] << endl;
        return 0;
    }
    cout << "First, the naive function" << endl;
    answer = naive(a, b, c);
    if(answer.size() == 0)
    {
        cout << "Roots are not real" << endl;
    }
    else
    {
        cout << answer[0] << " " << answer[1] << endl;
    }
    cout << "Now trying the advance function" << endl;
    answer = advanced(a, b, c);
    if(answer.size() == 0)
    {
        cout << "Roots are not real" << endl;
    }
    else
    {
        cout << answer[0] << " " << answer[1] << endl;
    }
    return 0;
}
```