$A = LU$

2 triang. lin. sys.

$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} = \underbrace{\begin{pmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix}}_{U} = \begin{pmatrix} | & | & | \\ l_1 & l_2 & l_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} - & u_1^T & - \\ - & u_2^T & - \\ - & u_3^T & - \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x \\ x \\ x \end{pmatrix} (x \ x \ x)} = \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} = \underline{l_1 u_1^T} \qquad xy^T$$

$$( \ \vdots \ )$$

$$\boxed{\begin{pmatrix} 0 \\ x \\ x \end{pmatrix} (0 \ x \ x)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{pmatrix} = l_2 u_2^T$$

$$\boxed{\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} (0 \ 0 \ x)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix} = l_3 u_3^T$$

$$A = \boxed{LU} = \cancel{\alpha}\,\frac{l_1}{\cancel{\alpha}}(u_1^T \alpha) + \underline{l_2 u_2^T} + \underline{l_3 u_3^T}$$

require : main diagonal of $L$ to be 1's

$$\underline{\begin{pmatrix} \boxed{1 \quad 2 \quad 3} \\ 2 & \boxed{\begin{matrix} 5 & 7 \\ 7 & 8 \end{matrix}} \\ 2 \end{pmatrix}}_{A} \checkmark$$

$$l_1 u_1^T$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 6 \end{pmatrix}$$

$$A = \underline{l_1 u_1^T} + \underline{l_2 u_2^T} + l_3 u_3^T$$

$$\underbrace{A - l_1 u_1^T}_{\shortparallel} = \underline{l_2 u_2^T} + \underline{l_3 u_3^T}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & \boxed{2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}}_{l_2 u_2^T} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & \boxed{3} \end{pmatrix}}$$

$$A - l_1 u_1^T - l_2 u_2^T$$

$$\parallel$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = l_3 u_3^T$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 7 & 8 \end{pmatrix}$$

$L$ $\qquad$ $U$ $\qquad$ $A$ ✓

unit lower triangular

upper tri

$$A = LU = l_1 u_1^T + l_2 u_2^T + \dots + l_n u_n^T$$

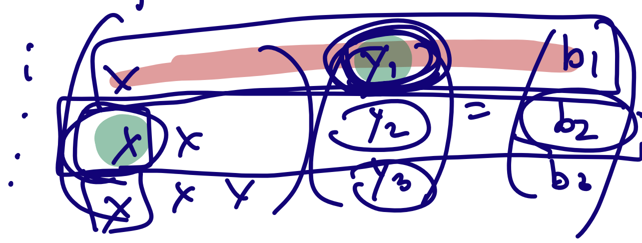
$$A - l_1 u_1^T = l_2 u_2^T + \dots + l_n u_n^T$$

$l_{jj} \neq 0$

## forward substitution

$Ly = b$ ①

$Ux = y$ ②

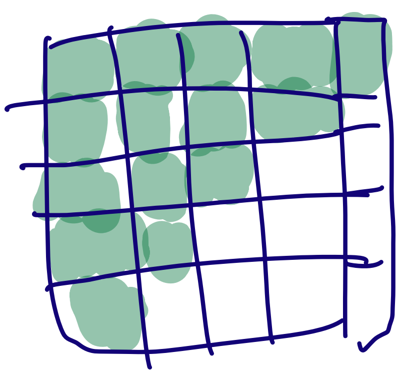



```
for j = 1,..,n
    y_j = b_j / l_jj
    for i = j+1, ..., n
        b_i <- b_i - y_j l_ij
    end
end
```

$O(n^2)$

triangular $L$

$$\det(L) = \prod_{i=1}^{n} l_{ii}$$



$$\sum_{j=1}^{n}\left[1+\sum_{i=j+1}^{n} 2\right]$$

$$= n + \sum_{j=1}^{n} 2\left(n-(j\cancel{+1})\cancel{+1}\right)$$

$$= n + \sum_{j=1}^{n}\left(2n - 2j\right)$$

$$= n + 2n\cdot n - 2\sum_{j=1}^{n} j$$

$$= n + 2n^2 - \cancel{2}\,\frac{n(n+1)}{\cancel{2}}$$

$$= \cancel{n} + 2n^2 - n^2 \cancel{- n} = n^2$$

$$LU \sim \frac{2}{3}n^3 + \dots$$

$$\sim \frac{1}{3}n^3$$

# LU factorization

$$\boxed{A \leftarrow A - \boxed{\ell_k u_k^T}}$$

$\ell_k u_k^T$

for $k = 1, \ldots, n$

$\quad$ if $\boxed{a_{kk}} == 0$ stop $\qquad$ ?? $\qquad$ "pivot"

$\quad \vec{\ell}_k$ $\quad \ell_{kk} = 1$

$\quad\quad$ for $i = k+1, \ldots, n$

$\quad\quad\quad \ell_{ik} = a_{ik} / \boxed{a_{kk}}$

$\quad\quad$ end

$\quad \vec{u}_k$

$\quad\quad$ for $j = k, \ldots, n$

$\quad\quad\quad u_{kj} = a_{kj}$

$\quad\quad$ end

$\quad$ update $A$

$\quad\quad$ for $i = k+1, \ldots, n$

$\quad\quad\quad$ for $j = k+1, \ldots, n$

$$A_{ij} \leftarrow A_{ij} - \ell_{ik} a_{kj}$$

$\quad\quad\quad$ end

$\quad\quad$ end

end

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{a_{ik}}{a_{kk}} \end{bmatrix} \ell_{kk}$$

$L, U$

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ l_{21} & & & \\ \vdots & & & \\ l_{n1} & & l_{n,n-1} & u_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ l_{21} & \ddots & & \\ \vdots & & & \\ l_{n1} & & l_{n,n-1} & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & a_{22} & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ \frac{a_{32}}{a_{22}} \end{pmatrix}$$

# Note: LU <u>in place</u>

## LU factorization

$\ell_k u_k^T$

$A \leftarrow A - \ell_k u_k^T$

for $k = 1, \dots, n$

→ if $\boxed{a_{kk}} == 0$ stop

$\vec{\ell}_k$: for $i = k+1, \dots, n$

$\quad a_{ik} \leftarrow a_{ik} / \boxed{a_{kk}}$

end

$\vec{u}_k$: ~~for $j = k, \dots, n$~~

~~$\quad u_{kj} = a_{kj}$~~

~~end~~

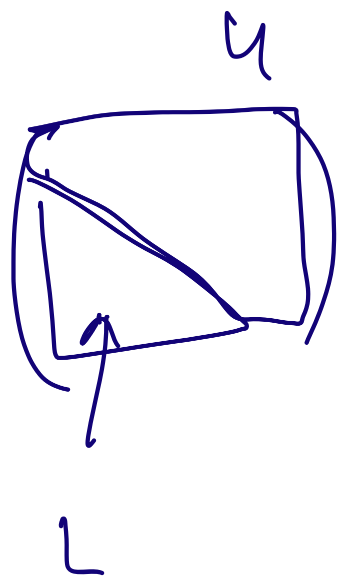
update $A$: for $i = k+1, \dots, n$

$\quad$ for $j = \boxed{k+1, \dots, n}$

$\quad\quad \boxed{A_{ij} \leftarrow A_{ij} - l_{ik} a_{kj}}$

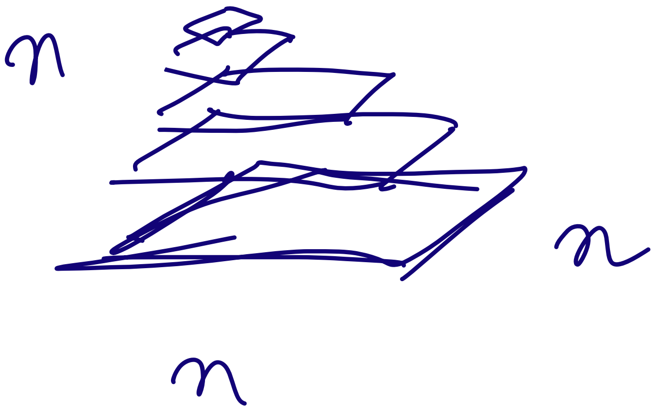
$\quad$ end

end




end

end




$$V = \frac{1}{3} b h$$

$$\sim \frac{1}{3} n^3$$

$$\boxed{\frac{2}{3} n^3 +}$$

$$\sum_{k=1}^{n} \Big($$

$$\begin{pmatrix} \boxed{0} & 1 \\ 2 & 3 \end{pmatrix}$$

no LU decomp.

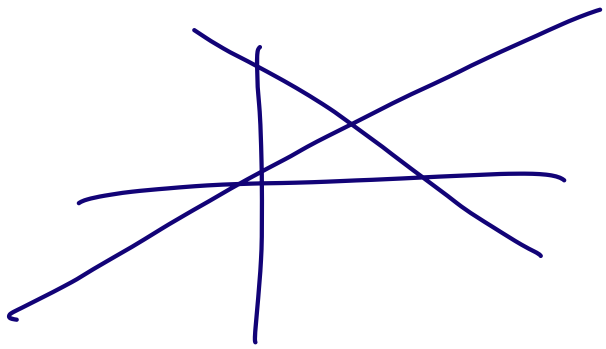
det $= -2$

nonsingular

$$\boxed{A}x = \boxed{b}$$

$$\downarrow$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$



row permutation

$P$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$\begin{pmatrix} & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & \end{pmatrix} \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_k^T & - \\ & \vdots & \\ - & a_n^T & - \end{pmatrix}$$

$$e_i^T A = a_i^T$$

$$e_1^T, \ldots, e_n^T$$

$$P = \begin{pmatrix} - & e_2^T & - \\ - & e_1^T & - \\ - & e_4^T & - \\ - & e_3^T & - \end{pmatrix}$$

$$I = \begin{pmatrix} - & e_1^T & - \\ & \vdots & \\ - & e_n^T & - \end{pmatrix}$$

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$$PA = LU \quad \checkmark$$

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