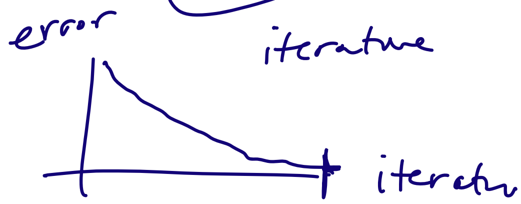
direct methods

iterative methods

$\boxed{\text{eigenvalues eigenvectors}}$ +

$$Ax = b$$

$$x_0 \downarrow x_1 \downarrow \vdots \downarrow x_n$$



$n \times n$ matrix

$$\det(A - \lambda I) = 0$$

$$A\vec{x} = \lambda \vec{x} \implies (A - \lambda I)\vec{x} = 0$$

$p(\lambda)$ of degree n

n eigenvalues

$\underbrace{A x} = \lambda x$ eigenvalues? eigenvectors?

shifts

$$\underbrace{(A + sI)}\vec{x} = A\vec{x} + s\vec{x}$$
$$= \lambda\vec{x} + s\vec{x}$$
$$= \underbrace{(\lambda + s)}\vec{x}$$

- eigenval's shifted
- eigenvec's same

inverse

A^{-1} exists

$\Rightarrow \lambda \neq 0$
$A\vec{x} = 0 \cdot \vec{x} = \vec{0}$

$$Ax = \lambda x$$
$$\underbrace{A^{-1}A}_{I} x = \lambda A^{-1} x \qquad \frac{1}{\lambda} x = \cancel{\lambda} A^{-1} x$$
$$A^{-1} x = \frac{1}{\lambda} x$$

A^{-1}

- eigenvalues are the reciprocals
- eigenvectors same

powers of A

$$\boxed{A\vec{x} = \lambda\vec{x}}$$
$$\underbrace{AA}\vec{x} = \lambda \underbrace{A\vec{x}}$$
$$A^2\vec{x} = \lambda\lambda\vec{x} = \lambda^2\vec{x}$$
$$A\, A^2\vec{x} = \underline{A}\lambda^2 \underline{\vec{x}}$$
$$\boxed{A^k\vec{x} = \lambda^k\vec{x}}$$

polynomials of A $\qquad A \quad n\times n \qquad Ax=\lambda x$

$$p(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_k t^k$$

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \dots + c_k A^k$$

$n\times n$

$$p(A)\vec{x} = c_0\vec{x} + c_1 Ax + c_2 A^2 x + \dots + c_k A^k x$$

$$= c_0\vec{x} + c_1\lambda\vec{x} + c_2\lambda^2\vec{x} + \dots + c_k\lambda^k\vec{x}$$

$$= \underbrace{(c_0 + c_1\lambda + \dots + c_k\lambda^k)}_{p(\lambda)}\vec{x}$$

- eigenvalues $p(\lambda)$
- $\vec{x}$ unchanged

Similarity $\quad A, B \quad n\times n$ are "similar"

$\exists$ invertible T $n\times n$

$$A = TBT^{-1}$$

$$Ax = \lambda x$$

$$Ax = \underbrace{TBT^{-1}x = \lambda x}$$

$$BT^{-1}x = T^{-1}(\lambda x)$$

$$B(T^{-1}x) = \lambda(T^{-1}x)$$

"

$$\boxed{B y = \lambda y}$$

- same eigenvalues λ
- eigenvectors transformed by T^{-1}

If A has a full set of eigenvectors

$$\boxed{A \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}_{\substack{\parallel \\ V}} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}$$

$$A v_i = \lambda_i v_i$$

$\{v_i \mid i = 1, \dots, n\}$ are lin. indep. set, then V is invertible

$$A V = V \Lambda$$

$$\boxed{A = V \Lambda V^{-1}}$$

eigen decomposition of A

$$\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \qquad \lambda\text{'s}: \; d_1, d_2, \dots, d_n$$

$$\begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & u_{22} & \vdots \\ & & \ddots \\ & & & u_{nn} \end{pmatrix} \qquad \lambda\text{'s}: \; u_{11}, u_{22}, \dots, u_{nn}$$

Special A for which

A real symmetric $A = A^T$

A complex Hermitian $A = \bar{A}^T = A^H = A^*$

$$A = Q \Lambda Q^T$$

$$A = Q \Lambda Q^H$$

λ_i's are real

$$U \Sigma V^T \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

$$A = V \Lambda V^{-1}$$

Jordan normal form

$$\boxed{A = V J V^{-1}}$$

defective

$$\begin{pmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & & & & \\ & & \lambda_2 & 1 & & \\ & & & \lambda_2 & 1 & \\ & & & & \lambda_2 & \\ & & & & & \lambda_3 \end{pmatrix}$$

$$p(\lambda) = (\lambda - \lambda_1)^{2} (\lambda - \lambda_2)^{3} (\lambda - \lambda_3)$$

	algebraic multiplicity
λ_1	2
λ_2	" 3
λ_3	1

[geometric multiplicity λ
$\dim(S_\lambda)$]

$\vec{v}_1$, $\vec{v}_2$ $\qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad S_\lambda = \{ \vec{v} \mid S\vec{v} = \lambda \vec{v} \}$

defective

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}}$$

$$\boxed{2x + y = 2x}$$

$2y = 2y$ ✓ $\qquad \dim(S_2) = 1 < 2$ (alg. mult.)

(geom. mult.)

$$2x + 1 = 2x$$
$$1 = 0$$

defective matrices = nondiagonalizable matrices

Normal Matrix

$$AA^H = A^H A$$

$$A = \underline{Q} \Lambda \underline{Q^H}$$

Examples:

A	real symm	Λ real
A	Hermitian	Λ real
A	skew-Hermitian $A^H = -A$	Λ complex
A	unitary	Λ complex

Spectrum of A

$$= \{ \lambda_i \mid Ax_i = \lambda_i x_i \}$$

spectral radius

$$\rho(A) = \max_i |\lambda_i|$$

power iteration

$$x_1 = A x_0$$

$$x_2 = A x_1$$

$$\vdots$$

converges to an eigenvector v_1

s.t. $$A v_1 = \lambda_1 v_1$$

λ_1 largest magnitude eigenvalue

Why? Assume A has a full set of eigenvectors: $v_1, \dots, v_n$ form a basis for $\mathbb{R}^n$

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$\underline{A^k x_0} = \alpha_1 \underline{A^k v_1} + \alpha_2 A^k v_2 + \dots + \alpha_n A^k v_n$$

$$= \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \dots + \alpha_n \lambda_n^k v_n$$

$$= \lambda_1^k \left(\alpha_1 v_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n \right)$$

assume

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

Algorithm normalized power iteration

$$\begin{cases} x_0 \\ \text{for } k = 1, 2, \dots \\ \qquad y_k = A x_{k-1} \\ \qquad x_k = y_k / \|y_k\|_\infty \\ \text{end} \end{cases}$$

$$A v = \lambda v$$

$$A(\alpha v) = \lambda(\alpha v)$$