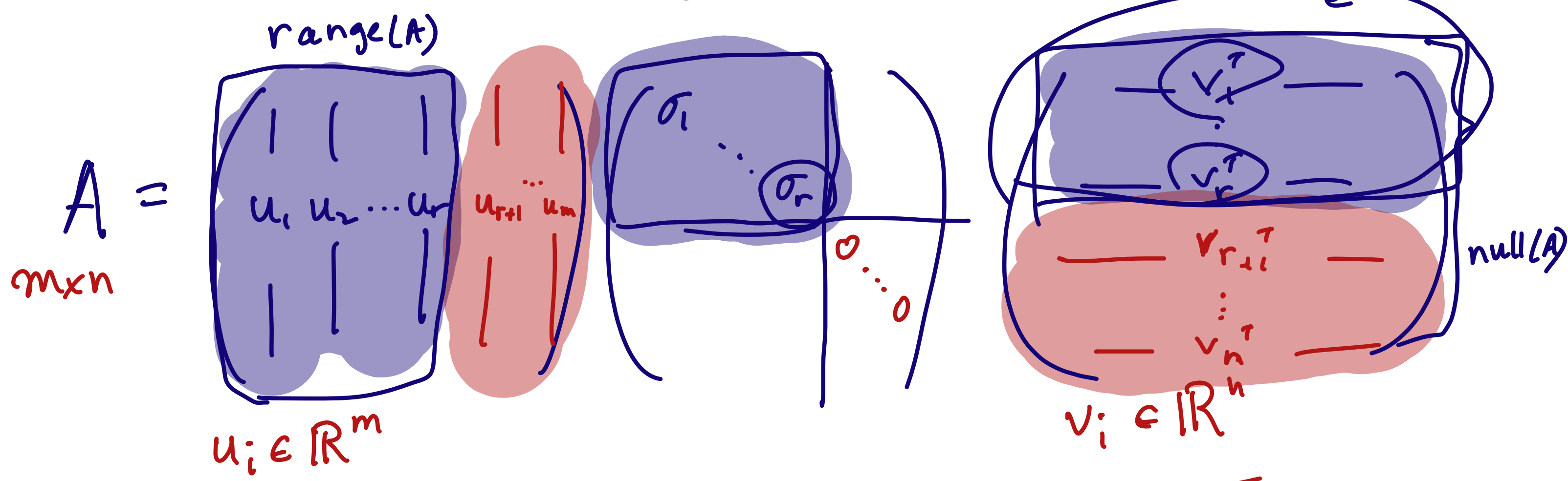
SVD

rank-revealing factorization



$$A = \underbrace{\sum_{i=1}^{r} \sigma_i u_i v_i^T}_{} + \underbrace{\cancel{0\, u_{r+1} v_{r+1}^T + \cdots + 0\, u_n v_n^T}}_{}$$

Fundamental subspaces of A

range(A) $\{ \underline{Ax} \mid x \in \mathbb{R}^n \}$

null(A) $= \{ \vec{z} \in \mathbb{R}^n \mid A\vec{z} = \vec{0} \}$

$$Ax = \left(\sum_{i=1}^{r} \sigma_i u_i v_i^T \right) x$$

Note: $\alpha_i = \cancel{\sigma_i} v_i^T x$, $\frac{}{\sigma_i}$, $\boxed{x = \frac{\alpha_i}{\sigma_i} v_i}$

$$Ax = \sum_{i=1}^{r} \underbrace{\left(\sigma_i (v_i^T x)\right)}_{\alpha_i} u_i = \sum_{i=1}^{r} \alpha_i u_i$$

$$\text{range}(A) = \text{span}(u_1, \ldots, u_r)$$

A $m \times n$

$z \in \mathbb{R}^n$ $\qquad v_1, \dots, v_n$

$$z = \beta_1 v_1 + \dots + \beta_n v_n$$

$$\boxed{\beta_i = v_i^T z}$$

$$Az = \left(\sum_{i=1}^{r} \sigma_i u_i v_i^T\right)\left(\sum_{j=1}^{n} \beta_j v_j\right)$$

$$= \sum_{i=1}^{r} \left[\sigma_i u_i \sum_{j=1}^{n} \beta_j v_i^T v_j\right]$$

$$v_i^T v_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$= \boxed{\left(\sum_{i=1}^{r} \sigma_i u_i v_i^T\right)\left(\sum_{j=1}^{r} \beta_j v_j\right)}$$

$$+ \boxed{\left(\sum_{i=1}^{r} \sigma_i u_i v_i^T\right)\left(\sum_{j=r+1}^{n} \beta_j v_j\right)} = 0$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T \beta_i v_i + 0$$

$$\text{null}(A) = \underbrace{\beta_{r+1} v_{r+1} + \dots + \beta_n v_n}_{\text{span}(v_{r+1}, \dots, v_n)}$$

$$A = U \Sigma V^T$$

$$\underset{m \times n}{A} = \underset{u \subset \mathbb{R}^m}{\begin{pmatrix} \overbrace{\begin{matrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{matrix}}^{\text{range}(A)} & \begin{matrix} | & & | \\ u_{r+1} & \cdots & u_m \\ | & & | \end{matrix} \end{pmatrix}} \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ \hline & & & 0 \\ & & & \ddots \\ & & & 0 \end{array}\right) \underset{v \subset \mathbb{R}^n}{\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_r^T & - \\ - & v_{r+1}^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}} \text{null}(A)$$

$$A^T = (U \Sigma V^T)^T = \underbrace{V \Sigma^T U^T}_{\text{SVD of } A^T!}$$

$$A^T = \begin{pmatrix} \overbrace{\begin{matrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{matrix}}^{\text{range}(A^T)} & \begin{matrix} | & & | \\ v_{r+1} & \cdots & v_n \\ | & & | \end{matrix} \end{pmatrix} \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ \hline & & & 0 \\ & & & \ddots \\ & & & 0 \end{array}\right) \begin{pmatrix} - & u_1^T & - \\ & \vdots & \\ - & u_r^T & - \\ - & u_{r+1}^T & - \\ & \vdots & \\ - & u_m^T & - \end{pmatrix}$$

$$\text{range}(A^T) = \text{span}(v_1, \dots, v_r)$$
$$\text{null}(A^T) = \text{span}(u_{r+1}, \dots, u_n)$$

$$\boxed{\begin{array}{c} \text{range}(A) \perp \text{null}(A^T) \\ \text{range}(A^T) \perp \text{null}(A) \end{array}}$$

$$Az = 0$$

$$\begin{pmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{pmatrix} \begin{pmatrix} | \\ z \\ | \end{pmatrix} = \begin{pmatrix} a_1^T z \\ a_2^T z \\ \vdots \\ a_m^T z \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\boxed{\text{null}(A) \perp \text{rowspace}(A)}$$

$$\text{rowspace}(A) = \text{colspace}(A^T) = \text{range}(A^T)$$

SVD(A) $\quad$ eigenvalue decomp. $\quad A^TA$

AA^T

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\qquad U \qquad\quad \Sigma \qquad\quad V^T$$

$$A = 3\, e_1 e_1^T + \underline{(-2)}\, e_2 e_2^T$$

$$A = 3\, e_1 e_1^T + 2 e_2 (-e_2^T)$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= 2\, e_1 e_1^T + 3\, e_2 e_2^T$$

$$= 3\, \underset{u_1}{e_2}\, \underset{v_1^T}{e_2^T} + 2\, \underset{u_2}{e_1}\, \underset{v_2^T}{e_1^T}$$

$$A = \begin{pmatrix} | & | \\ e_2 & e_1 \\ | & | \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} - & e_2^T & - \\ - & e_1^T & - \end{pmatrix}$$

$$\boxed{A = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V^T}}$$

U, V permutations

$$\underset{3\times 2}{A} = \underset{3\times 3}{U}\ \underset{3\times 2}{\Sigma}\ \underset{2\times 2}{V^T}$$

$u_i \in \mathbb{R}^3$

$v_j \in \mathbb{R}^2$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

1 2
2 4
3 6

$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (1\ 2)\frac{1}{\alpha}$

$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} (\frac{1}{2}\ 1)$

$$\underset{3\times 2}{A} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V^T}$$

$$= 2\, u_1 v_1^T$$

$$A = 2\, \underline{e}_1^3\, \underline{e}_2^{2\,T} +$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0\ 1)$$

$$\begin{pmatrix} | & | \\ e_1 & e_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} - e_2^T - \\ - e_1^T - \end{pmatrix}$$

$$A = \boxed{\underline{x}\, \underline{y}^T} = \sigma_1 \underline{u}_1 v_1^T$$

$$\begin{cases} \sigma_1 > 0 \\ \|u_1\|_2 = 1 \\ \|v_1\|_2 = 1 \end{cases}$$

$$A = \|x\| \left(\frac{x}{\|x\|}\right) y^T$$

$$= \|x\| \|y\| \left(\frac{x}{\|x\|}\right)\left(\frac{y}{\|y\|}\right)^T$$

$$A = \sigma_1 \quad \underline{u_1} \quad \underline{v_1^T}$$ reduced SVD

full SVD

$$A = \begin{pmatrix} | & \boxed{} \\ u_1 & \\ | & \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \begin{pmatrix} — v_1^T — \\ \boxed{} \end{pmatrix}$$

$$A = \sum_{i=1}^{r} \sigma_i (u_i)(v_i)^T$$

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & \\ & 2 \end{matrix}} & \\ & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$U \Sigma V^T$$

Uniqueness?

- σ_i are uniquely determined
- some freedom in singular vectors
- singular vectors associated w/ distinct singular values are unique (up to sign)

$$U \begin{pmatrix} \boxed{\begin{matrix} 4 & \\ & 3 \end{matrix}} & & \\ & 1 & \\ & & .1 \end{pmatrix} V^T$$

$\sigma_1 (u_1 v_1^T) + \ldots + \sigma_4 u_4 v_4^T$

$A = U \Sigma V^T$

if A invertible, square

$$\underset{n \times n}{A^{-1}} = (\underset{n \times n}{\underline{U}}\ \underset{n \times n}{\underline{\Sigma}}\ \underset{n \times n}{\underline{V^T}})^{-1}$$

$$= \underbrace{V \Sigma^{-1} U^T}$$ SVD of A^{-1}

if A is non-square or not invertible

"pseudo inverse" of A

$$\boxed{A^+ = V \Sigma^+ U^T}$$

$$(\Sigma^+)_{ii} = \begin{cases} \frac{1}{\sigma_{ii}} & \text{if } \sigma_{ii} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$