Homework 4

Symmetric positive definiteness, Cholesky factorization

1. (Heath 2.37) Suppose that the symmetric $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} \alpha & \mathbf{a}^T \\ \mathbf{a} & A \end{pmatrix}$$

is positive definite.

(a) Show that the scalar α must be positive and the $n \times n$ matrix A must be positive definite.

(b) What is the Cholesky factorization of B in terms of α, $\mathbf{a}$, and the Cholesky factorization of $A - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$?

Sparse matrices

2. *LU factorization and fill-in.* Consider a sparse matrix generated in Matlab (or Octave) as follows

```
n = 100;
A = diag(rand(n,1));
A(1,:) = rand(1,n);
A(:,1) = rand(1,n);
```

(a) Use the command `spy(A)` to visualize the sparsity pattern of A.

(b) Compute the LU factorization of the matrix using the following function `my_lu`. Run `spy(L)` and `spy(U)` . Are L and U also sparse? Now try the built-in Matlab command *lu* on A. Are the L and U generated by the Matlab command sparse?

```
function [L,U] = my_lu(A)
n = size(A,1);
L = zeros(size(A));
A2 = A;
for k = 1:n
    if A2(k,k) == 0
        'Encountered 0 pivot. Stopping'
        return
    end
    L(k,k) = 1;
    for i = k+1:n
        L(i,k) = A2(i,k)/A2(k,k);
    end
    for i = k+1:n
        for j = k+1:n
            A2(i,j) = A2(i,j) - L(i,k)*A2(k,j);
        end
    end
end
U = triu(A2);
end
```

(c) A *tridiagonal* matrix is a matrix that has non-zeros only on its main diagonal and its first sub- and super-diagonals. It is an example of a *banded* matrix. Consider the tridiagonal matrix B given by the following Matlab commands:

```
n = 100;
B = diag(10*ones(n,1)) + diag(3*ones(n-1,1),1) + diag(2*ones(n-1,1),-1);
```

Run `my_lu` and Matlab's `lu` on this matrix. What are the sparsity patterns of L and U using the two different commands?

Matrix and vector norms

3. Let $\mathbf{x} \in \mathbb{R}^n$. Two vector norms, $||\mathbf{x}||_a$ and $||\mathbf{x}||_b$, are *equivalent* if $\exists\, c_1, d_1 \in \mathbb{R}$ such that

$$c_1||\mathbf{x}||_b \leq ||\mathbf{x}||_a \leq d_1||\mathbf{x}||_b.$$

Matrix norm equivalence is defined analogously to vector norm equivalence, i.e., $||\cdot||_a$ and $||\cdot||_b$ are equivalent if $\exists\, c_2, d_2$ s.t. $c_2||A||_b \leq ||A||_a \leq d_2||A||_b$.

(a) Let $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. For each of the following, derive the inequality and give an example of a non-zero vector or matrix for which the bound is achieved (showing that the bound is tight):

i. $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$
ii. $||\mathbf{x}||_2 \leq \sqrt{n}||\mathbf{x}||_\infty$
iii. $||A||_\infty \leq \sqrt{n}||A||_2$
iv. $||A||_2 \leq \sqrt{n}||A||_\infty$

This shows that $||\cdot||_\infty$ and $||\cdot||_2$ are equivalent, and that their induced matrix norms are equivalent.

(b) Prove that the equivalence of two vector norms implies the equivalence of their induced matrix norms.

Conditioning and stability

4. For each of the following statements, indicate whether the statement is true or false.

T/F A problem is ill-conditioned if its solution is highly sensitive to changes in its data.

T/F We can improve conditioning of a problem by switching from single to double precision arithmetic.

T/F In order to numerically solve a problem accurately, it is necessary to have both a well-conditioned problem and a stable algorithm.

T/F A condition number of 1 means the problem is well-conditioned.

5. (Heath 2.58) Suppose that the $n \times n$ matrix A is perfectly well-conditioned, i.e., cond(A) $= 1$. Which of the following matrices would then necessarily share this same property?

(a) cA, where c is any nonzero scalar
(b) DA, where D is a nonsingular diagonal matrix
(c) PA, where P is any permutation matrix
(d) BA, where B is any nonsingular matrix
(e) A^{-1}, the inverse of A

Orthogonality

6. (Strang I.5 1) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors, under what condition is $a\mathbf{u} + b\mathbf{v}$ orthogonal to $c\mathbf{u} + d\mathbf{v}$ (where a, b, c, d are scalars)? What are the lengths of those vectors (expressed using a, b, c, d)?

7. (Strang I.5 4) Prove this key property of every orthogonal matrix Q: $\|Q\mathbf{x}\|^2 = \|\mathbf{x}\|^2$ for every vector $\mathbf{x}$. More than this, show that $(Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ for every vector $\mathbf{x}$ and $\mathbf{y}$. So *lengths and angles are not changed by Q. Computations with Q never overflow!*.

8. (Strang I.5 6) A permutation matrix has the same columns as the identity matrix (in some order). Explain why this permutation matrix and every permutation is orthogonal:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

9. Let $A \in \mathbb{R}^{n \times m}$. Show that range$(A)$ is orthogonal to nullspace(A^T) . I.e., show that for any $\mathbf{y} \in$ range$(A), \mathbf{z} \in$ nullspace(A), $\mathbf{y}^T\mathbf{z} = 0$.

Projections

10. Given a vector $\mathbf{v} \in \mathbb{R}^n$ and an orthonomal set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in $\mathbb{R}^n$, find $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ that minimize $\|\mathbf{v} - \sum_{i=1}^k \alpha_i \mathbf{u}_i\|_2^2$.

11. Let P be a projection matrix (i.e., $P^2 = P$).

    (a) Prove that $I - P$ is also a projection matrix. This is called the *complementary projector* to P.

    (b) Prove that if P is a symmetric matrix, then $P\mathbf{x}$ is orthogonal to $(I - P)\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Such P is called an *orthogonal projection*. (Note also that the converse holds, i.e., $P\mathbf{x} \perp (I - P)\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \Rightarrow P = P^T$.)