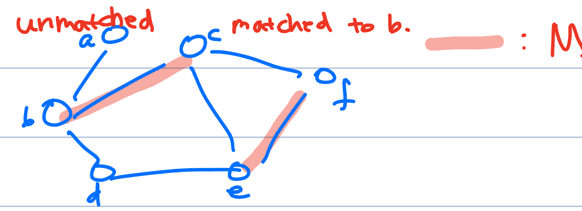
Bipartite Matching

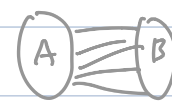
Let $G=(V,E)$ be undirected graph. $M \subseteq E$ is called a **matching** if every $v \in V$ is incident on at most one edge in M.

If v is incident on one $e=(u,v) \in M$, then v is "matched to u" (vice versa). Otherwise, v is "unmatched".



Maximum (Unweighted) Bipartite Matching

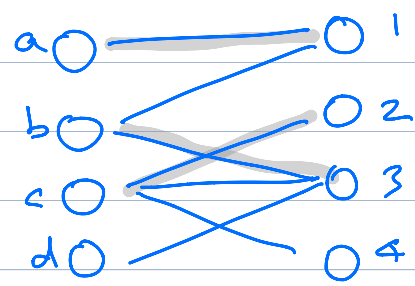
Input: Undirected bipartite graph $G=(A \cup B, E)$.



Output: Matching $M \subseteq E$ with largest $|M|$.

Applications
- Jobs/Applicants
- Donors/Recipients

⋮



Generalizations

- Weighted: $w: E \to \mathbb{R}$ and maximize $w(M) = \sum_{e \in M} w(e)$.
- General graph.
- Even weighted general graph matching can be solved in poly-time!

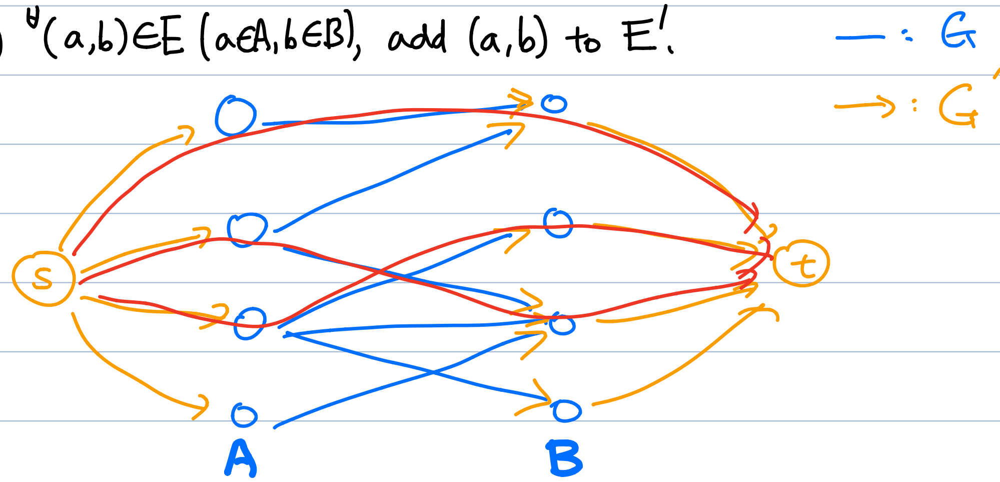
ALGO: Use "Reduction" from Matching to Max-Flow.

Given $\underbrace{G=(A\cup B, E)}_{\text{input for matching}}$, produce $\underbrace{G'=(A\cup B\cup\{s,t\}, E')}_{\text{input for Max Flow}}$ directed

s.t. (1) $\forall a\in A$, add (s,a) to E'

(2) $\forall b\in B$, add (b,t) to E'

(3) $\forall (a,b)\in E$ $(a\in A, b\in B)$, add (a,b) to E'.



and let $c: E'\to \mathbb{R}^{\geq 0}$ s.t. $c(e)=1$ $\forall e\in E'$.

Then G', c, s, t form a Max Flow instance!

Claim: (Size of max matching in G)

= (value of max flow in G')

Proof: 1-to-1 correspondence between matching M and "integral" feasible s-t flow $f: E'\to\{0,1\}$. But by previous lecture, since all capacities are 1, max flow value can be achieved by an integral flow. $\square$

Running Time: $O(|E'|\cdot OPT_{flow}) \leq O((|V|+|E|)\cdot|V|)$.

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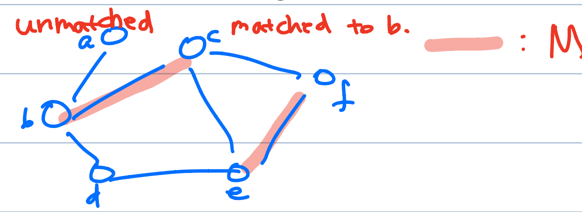
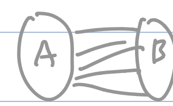


If v is incident on one $e=(u,v) \in M$, then v is "matched to u" (vice versa). Otherwise, v is "unmatched".

Maximum (Unweighted) Bipartite Matching

Input: Undirected bipartite graph $G=(A \cup B, E)$.



Output: Matching $M \subseteq E$ with largest $|M|$.

Equivalently, can write as an "Integer program" (IP).

$$
\text{(IP-M)} \quad \text{maximize} \quad \sum_{e \in E} x_e
$$
$$
\text{s.t.} \quad \sum_{e: v \in e} x_e \le 1 \qquad \forall v \in A \cup B
$$
$$
x \in \{0,1\}^E
$$

Then x is feasible $\Leftrightarrow$ it is (indicator vector) of matching.

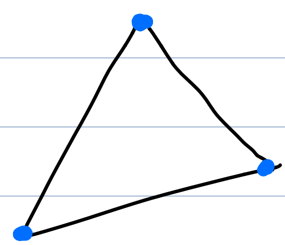
LP Relaxation: "Relax" $x \in \{0,1\}^E$!

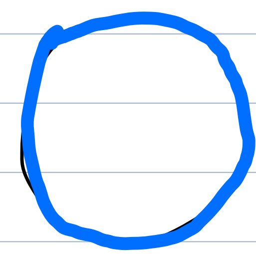
$$
\text{(LP-M)} \quad \text{maximize} \quad \sum_{e \in E} x_e
$$
$$
\text{s.t.} \quad \sum_{e: v \in e} x_e \le 1 \qquad \forall v \in A \cup B
$$
$$
x \ge 0
$$

Then, $OPT_{LP\text{-}M} \ge OPT_{IP\text{-}M}$. Tight?

Let $S = \{x \in \mathbb{R}^E : \sum_{e: v \in e} x_e \le 1 \ \forall v \in A \cup B,\ x \ge 0\}$ be the
feasible region

Def. Given a convex set $S \subseteq \mathbb{R}^n$, $x \in S$ is an "extreme point" if $\nexists\, y, z \in S$, $\lambda \in [0,1]$ s.t.
$x \ne y, z$ and $x = \lambda y + (1-\lambda) z$.





For polyhedra, "vertices"

Lemma. For any LP, there is an optimal $x \in \mathbb{R}^n$, which is an extreme point of the feasible set.

(also it can be found in $O((n+m)^{2.38})$ time.

Theorem. Let $x \in S$ be an extreme optimal solution to LP-M.
Then, $x_e = 0$ or 1 for all $e \in E$.
So, x is a matching and $OPT_{LP\text{-}M} = OPT_{IP\text{-}M}$!

(relaxation is "tight")

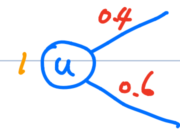
Proof. Call $e \in E$ "fractional" if $0 < x_e < 1$.
$\forall v \in V$, let $z_v = \sum_{e \ni v} x_e \le 1$, and call v "tight" if $z_v = 1$.

Assume $\exists$ fractional e. Will show x is not optimal or not extreme.

(i) $\forall$ fractional $e=(u,v)$, either u or v is tight, because otherwise x_e can be strictly increased and x is not optimal.

$\ncong$ ✗

(ii) $\forall$ tight v, if it has an fractional edge, it has ≥ 2.



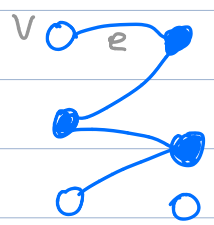
$\ncong$ ✗

Then, $\exists$ one of

① cycle $(v_1, v_2, \dots, v_k, v_1)$ of fractional edges (G bipartite $\Rightarrow$ k even)

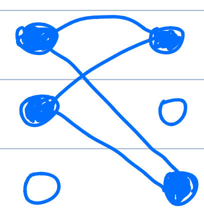
② path $(v_1, \dots, v_k)$ of fractional edges where v_1, v_k are NOT tight.

Case 1



$\exists$ fractional $e=(u,v)$ with v not tight.

Case 2



$\forall$ fractional edges both endpoints tight.

— : fractional edges

● : tight vertices

① cycle $(v_1, v_2, \dots, v_k, v_1)$ of fractional edges (k even)

Say $(v_1,v_2), (v_3,v_4) \dots$ odd edges

$(v_2, v_3), (v_4, v_5) \dots (v_k, v_1)$ even edges

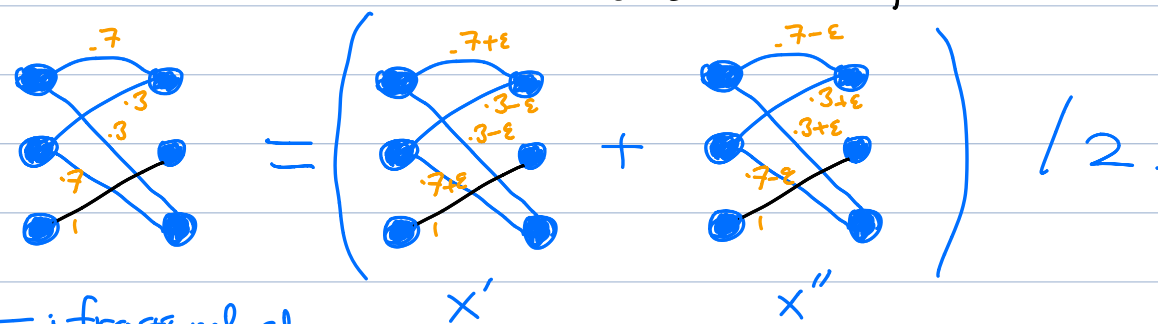
Consider x' and x'' s.t.

$$x'_e = \begin{cases} x_e + \varepsilon & e \text{ odd} \\ x_e - \varepsilon & e \text{ even} \\ x_e & \text{o.w.} \end{cases} \qquad x''_e = \begin{cases} x_e - \varepsilon & e \text{ odd} \\ x_e + \varepsilon & e \text{ even} \\ x_e & \text{o.w.} \end{cases}$$

Then, for small enough $\varepsilon > 0$, both x' and x'' are feasible

and $x = (x' + x'')/2$.

$\Rightarrow x$ is not an extreme point!



— : fractional edges

● : tight vertices

x_e

② path $(v_1, \dots, v_k)$ of fractional edges where v_1, v_k are NOT tight.

If k = odd (even # edges), similar trick will show that x is not an extreme point.

If k = even (odd # edges) will show that x is not optimal.

$\therefore$ If x has a fractional edge, x is not optimal or not extreme $\square$