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① $Ax \approx b$
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{pmatrix}$$
$\Big] \to$ Given $\quad r = b - Ax$

Residual Vector is orthogonal to Range(A)

$$① \leftarrow \text{Range}(A) \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

(a) $r \overset{?}{=} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$r \cdot ①$ should be 0

$\Rightarrow c_1 \times 0 + (-2)c_2 + 3c_3 \neq 0 \;\; \forall c_1, c_2, c_3 \in \mathbb{R}$

Hence NO

(b) $r \overset{?}{=} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$r \cdot ①$ should be 0

$\Rightarrow c_1 \times 0 + c_2 \times 0 + c_3 \times 0 = 0 \;\; \forall c_1, c_2, c_3 \in \mathbb{R}$

Hence Yes

(c) $r \overset{?}{=} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$r \cdot ①$ should be 0

$c_1 \times (-2) + c_2 \times (-4) + c_3 \times 0 \neq 0 \quad \forall c_1, c_2, c_3 \in \mathbb{R}$

Hence NO

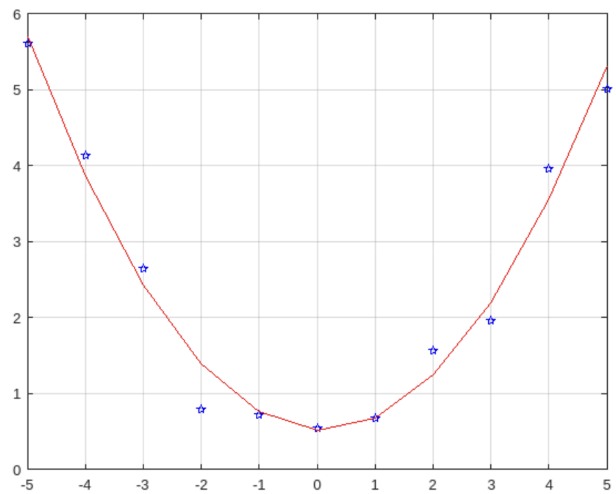
Q2

Code is

```
>> x = -5:5

x =

    -5    -4    -3    -2    -1     0     1     2     3     4     5
>> rng(12)
>> y = .2 * x . ^2 + rand(size(x))
y = .2 * x . ^2 + rand(size(x))
               ↑
Invalid use of operator.

>> y = .2 * x^2 + rand(size(x))
Error using ^ (line 52)
Incorrect dimensions for raising a matrix to a power. Check that the matrix is square and the power is a
scalar. To operate on each element of the matrix individually, use POWER (.^) for elementwise power.

>> y = .2 * x .^2  + rand(size(x));
>> y = .2 * x .^2  + rand(size(x))

y =

    5.6061    4.1442    2.6527    0.8023    0.7212    0.5520    0.6854    1.5681    1.9607    3.9646    5.0208
>> p = polyfit(x, y, 2);
>> v = polyval(p, x);
>> TSE = sum((v - y).^2);
>> figure(1)
>> plot(x, y, 'bp')
>> hold on
>> plot(x, v, '-r')
>> hold off
```

```
>> grid
>> p

p =

    0.1993   -0.0384    0.5227
```

Hence a = 0.1993, b = -0.0384 and c = 0.5227



Q4

```
>> rng(12)
>> A = rand(10,3)

A =

    0.1542    0.2838    0.7646
    0.7400    0.6061    0.0208
    0.2633    0.9442    0.1352
    0.5337    0.8527    0.1163
    0.0146    0.0023    0.3099
    0.9187    0.5212    0.6715
    0.9007    0.5520    0.4712
    0.0334    0.4854    0.8162
    0.9569    0.7681    0.2896
    0.1372    0.1607    0.7331
>> b = rand(10,1)

b =
```

```
    0.7026
    0.3276
    0.3346
    0.9781
    0.6246
    0.9503
    0.7675
    0.8250
    0.4066
    0.4513
>> rank(A)

ans =

     3
>> x = A \ b

x =

    0.1436
    0.4203
    0.7552
>> [Q, R] = qr(A)

Q =

   -0.0822    0.1526    0.4818   -0.1554   -0.1509   -0.4285   -0.3327
-0.4309   -0.2643   -0.3755
   -0.3946   -0.0135   -0.2502   -0.0558    0.1045   -0.3564   -0.4010
0.5017   -0.4216    0.2239
   -0.1404    0.7139   -0.2053   -0.5160    0.1118    0.2313    0.1214
-0.1939   -0.1270    0.1665
   -0.2846    0.4002   -0.2251    0.8000   -0.0181    0.0050    0.0012
-0.2423   -0.0574   -0.0761
   -0.0078   -0.0098    0.2341    0.0743    0.9648   -0.0290   -0.0026
-0.0405    0.0410   -0.0674
   -0.4899   -0.2449    0.2500   -0.0198   -0.0454    0.7313   -0.2310
-0.0310   -0.1843   -0.1097
   -0.4803   -0.1996    0.0903   -0.0742   -0.0245   -0.2385    0.7822
-0.0233   -0.2020   -0.0715
   -0.0178    0.4512    0.4764    0.0900   -0.1135    0.0475    0.1151
0.6620    0.1892   -0.2420
```

```
   -0.5103   -0.0329   -0.1184   -0.1706   -0.0093   -0.1908   -0.1876
-0.0815    0.7837   -0.0532
   -0.0732    0.0452    0.4960    0.1336   -0.0838   -0.0681   -0.0084
-0.1350    0.0812    0.8320


R =

   -1.8754   -1.5707   -0.8968
         0    1.0137    0.3898
         0         0    1.3104
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
>> x = R / Q .'*b
Error using /
Matrix dimensions must agree.

>> x = R\Q.'*b;
>> x = R\Q.'*b

x =

    0.1436
    0.4203
    0.7552
>> R(3,3) = 0

R =

   -1.8754   -1.5707   -0.8968
         0    1.0137    0.3898
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
```

```
         0         0         0
>> As = Q*R

As =

    0.1542    0.2838    0.1332
    0.7400    0.6061    0.3486
    0.2633    0.9442    0.4042
    0.5337    0.8527    0.4112
    0.0146    0.0023    0.0031
    0.9187    0.5212    0.3439
    0.9007    0.5520    0.3529
    0.0334    0.4854    0.1919
    0.9569    0.7681    0.4448
    0.1372    0.1607    0.0832
>> rank(As)

ans =

     2
>> x = A \ b

x =

    0.1436
    0.4203
    0.7552
>> x = As \ b
Warning: Rank deficient, rank = 2, tol =  4.164117e-15.


x =

    0.2615
    0.7107
         0
>> [Qs, Rs] = qr(As)

Qs =

   -0.0822    0.1526   -0.1069   -0.3758   -0.0057   -0.5250   -0.4494
-0.1088   -0.5618   -0.1104
   -0.3946   -0.0135   -0.8536   -0.1384    0.0008    0.2545   -0.0199
```

```
-0.1291    0.0985    0.0697
   -0.1404    0.7139    0.2558   -0.3495    0.0089    0.2766    0.1915
-0.4052    0.0709   -0.0230
   -0.2846    0.4002   -0.0554    0.8294    0.0003   -0.1363   -0.0844
-0.1061   -0.1683   -0.0510
   -0.0078   -0.0098    0.0024    0.0000    0.9999   -0.0050   -0.0049
0.0024   -0.0041   -0.0002
   -0.4899   -0.2449    0.3466   -0.0186   -0.0076    0.5738   -0.3341
0.1507   -0.3368   -0.0492
   -0.4803   -0.1996    0.0921   -0.0849   -0.0050   -0.2944    0.7429
0.0518   -0.2664   -0.0386
   -0.0178    0.4512   -0.1090   -0.1132    0.0028    0.0304    0.0546
0.8756   -0.0151   -0.0256
   -0.5103   -0.0329    0.2104   -0.0957   -0.0055   -0.3754   -0.2907
0.0614    0.6727   -0.0582
   -0.0732    0.0452    0.0820   -0.0104   -0.0011   -0.0816   -0.0522
0.0179   -0.0648    0.9858


Rs =

   -1.8754   -1.5707   -0.8968
         0    1.0137    0.3898
         0         0    0.0000
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
         0         0         0
>> x = Rs\Qs.'*b;
Warning: Rank deficient, rank = 2, tol =  4.164117e-15.
>> x = Rs\Qs.'*b
Warning: Rank deficient, rank = 2, tol =  4.164117e-15.

x =

    0.2615
    0.7107
         0
```

(3)

Given

$A \to m \times n \quad (m \geq n)$

$\text{rank}(A) \Rightarrow r < n$

$$A \Rightarrow \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{m \times n}$$

To Find

dim of set (x) that minimize $\|Ax - b\|_2$

We are trying to find solutions of $A\vec{x} = \vec{b}$

$\Rightarrow \quad A^T A \vec{x} = A^T b \qquad\qquad A^T A \to n \times n$

Now, we know that

$R(A^T) = R(A) = R(A^T A) \Rightarrow r$

$N(A) = N(A^T A) \Rightarrow n - r$

Let $\vec{w} \in R(A^T A)$ and $v \in N(A^T A)$ (II) such that

$A^T A(\vec{w} + \vec{v}) = A^T b$

$A^T A \vec{w} + 0 = A^T b$

$\Rightarrow A^T A \vec{w} = A^T b \to$ (1)

So we choose a specific $\vec{w}$ which satisfies (1)
but v can be any vector which satisfies (II)

$\therefore \vec{x} = \vec{w} + \vec{v}$

$\dim(v) = n - r$, $\vec{w}$ is constant

Hence $\vec{x}$ will be a set having dimension $n - r$

(5) $A = U\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^T$ (m,n) matrix, $\text{Rank}(A) = r$

$U, V \rightarrow$ orthogonal

$R \rightarrow$ invertible upper triangular

$\rightarrow$ Given

$\rightarrow$ Perform QR decomposition we get

$$AP = U\begin{bmatrix} R_{11} & S \\ 0 & 0 \end{bmatrix}$$

P → Permutation Matrix

$R \rightarrow r \times r$ upper triangular

$S \rightarrow r \times (n-r)$ matrix

Then another QR decomposition on $\begin{bmatrix} R_{11}^T \\ S^T \end{bmatrix}$

(1) $\Leftarrow \Rightarrow \begin{bmatrix} R_{11}^T \\ S^T \end{bmatrix} = V' \begin{bmatrix} R_2 \\ 0 \end{bmatrix}$ where $V' \rightarrow m \times m$ unitary

$R_2 \rightarrow$ non-singular upper triangular

$\Rightarrow A = U\begin{bmatrix} R & S \\ 0 & 0 \end{bmatrix} P^T \quad \{PP^T = I\}$

From (1)

$$A = U\begin{bmatrix} R_2^T & 0 \\ 0 & 0 \end{bmatrix} V'P^T \quad \{\text{From (1)}\}$$

$$\Rightarrow A = U\begin{bmatrix} R_2^T & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Hence Algorithm Succesful

☆ P and $P_1, \ldots P_r$ are different

<u>Alternate Formulation</u> {Assume $Rank(A) = r$}

→ Use QR decomposition with column pivoting

$$A = Q \begin{bmatrix} R \\ 0 \end{bmatrix} P^T \Rightarrow [Y \; Z] \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} P^T \quad \text{where:}$$

$R = [R_1, R_2] \in \mathbb{R}^{r \times t}$ {Upper Trapezoidal}

$R_1 \in \mathbb{R}^{r \times r}$ {Upper Triangular} is invertible

$R_2 \in \mathbb{R}^{r \times s}$

$Q = [Y, Z] \Rightarrow$ Orthogonal

$Y \in \mathbb{R}^{t \times r}$ $\qquad$ $P \rightarrow$ permutation matrix of A

$Z \in \mathbb{R}^{t \times s}$

<u>Assume</u>

$$L_{12} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = R^T \in \mathbb{R}^{t \times r} \quad \begin{cases} rank(L_{12}) = rank(L_1) \Rightarrow r \\ L_1 \in \mathbb{R}^{r \times r} \text{ Lower Triangular}, \; L_2 \in \mathbb{R}^{s \times r} \\ L_{12} \in \mathbb{R}^{t \times r} \text{ is pre-multiplied by Householder transform } P_1, \ldots P_r \end{cases}$$

$$\Rightarrow P_r \ldots P_1 \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} L \\ 0 \end{bmatrix}$$

Hence we get

$$\left. \begin{array}{l} A = Q \begin{bmatrix} L^T & 0 \\ 0 & 0 \end{bmatrix} P^T H^T \Rightarrow Y [L' \; 0] P^T H^T \\ \text{with } H = P P_r \ldots P_1 \in \mathbb{R}^{t \times t}, \; L^T \rightarrow \text{Upper Triangular} \end{array} \right\} \rightarrow \text{Answer}$$

(b) $\|Ax-b\|^2$ {The Normal Equation to

$$\underset{x}{\operatorname{argmin}} \left\| U \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} V^T x - b \right\|^2 \Rightarrow U^T(\quad)$$ {Multiply by U^T}

$$\Rightarrow \underset{x}{\operatorname{argmin}} \left\| \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} V^T x - U^T b \right\|^2$$ [Orthogonal matrix do not change ~~magn~~ ℓ^2 norm]

Let $U^T b = c$

Let $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} V^T x = y$

$$\Rightarrow \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} y \quad [V_1^T x = y]$$

$$\Rightarrow \begin{bmatrix} R \\ 0 \end{bmatrix} y$$

$$\therefore \begin{bmatrix} R \\ 0 \end{bmatrix} y = [U_1, U_2] b \Rightarrow [c_1, c_2]$$

Assume residual $r = b - Ax$

$\Rightarrow \|r\| = \|b - Ax\|$

$\Rightarrow \|U^T(b - Ax)\|$

$\Rightarrow \left\| \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} Ry \\ 0 \end{pmatrix} \right\|$

$\left\| \begin{matrix} c_1 - Ry \\ c_2 \end{matrix} \right\|$

$\Rightarrow \|r\|^2 = \|c_1 - Ry\|^2 + \|c_2\|^2$

Can be easily solved using substitution

Residual is minimized when $\underbrace{Ry}$ $= c_1 = U_1^T b$, also the value will be

$\|r\| = \|c_2\|$

Then from y we can obtain x because $V^T x = y$

(7) (a) and (b) Given

$$A = \begin{pmatrix} 6 & 3 & 3 & 1 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

If 5 is an eigenvalue then

$A\vec{\lambda} = 5\vec{\lambda}$ [$\vec{\lambda}$ is eigenvector] and $\vec{\lambda} \neq 0$

$$\begin{pmatrix} 6 & 3 & 3 & 1 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix} \lambda = 5\lambda$$

$\Rightarrow (A - 5I)\lambda = 0$

$$\begin{bmatrix} 6-5 & 3 & 3 & 1 \\ 0 & 7-5 & 4 & 0 \\ 0 & 0 & 5-5 & 4 \\ 0 & 0 & 0 & 8-5 \end{bmatrix} \lambda = 0$$

$$\begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0 \quad \forall c_i \in \mathbb{R}$$

$c_1 + 3c_2 + 3c_3 + c_4 = 0$
$2c_2 + 4c_3 = 0$
$4c_4 = 0$
$3c_4 = 0$

$\therefore c_4 = 0,\ c_3,\ c_2 = -2c_3,\ c_1 = 3c_3$

$\therefore$ eigenvector of the form

$$\begin{bmatrix} 3t \\ -2t \\ t \\ 0 \end{bmatrix} \neq 0$$

$t \in \mathbb{R}$

(8) Given

$A\vec{u} = \lambda\vec{u} \rightarrow (1)$

To Find

Eigenvector and Eigenvalues

(a) A^{-1}

$\Rightarrow A^{-1}A\vec{u} = A^{-1}\lambda\vec{u}$ [From (1)]

$\Rightarrow \vec{u} = \lambda A^{-1}\vec{u}$ [λ is scalar]

$\frac{1}{\lambda}\vec{u} = A^{-1}\vec{u}$

$\therefore$ eigenvalue is $1/\lambda$, eigenvector is $\vec{u}$

b) cA

$\Rightarrow cA\vec{u} = c\lambda\vec{u}$ [From (1)]

$\therefore$ eigenvalue is $c\lambda$; eigenvector is $\vec{u}$

(c) $A + cI$

$\Rightarrow (A + cI)\vec{x} = \alpha\vec{x}$

$\Rightarrow A\vec{x} + c\vec{x} = \alpha\vec{x}$

$A\vec{x} = (\alpha - c)\vec{x}$

$\therefore \vec{x} = \vec{u}$ and $\alpha - c = \lambda$ [From (1)]

Hence eigenvalue is $\lambda + c$, eigenvector is $\vec{u}$

(d) A^2

$A^2\vec{x} = \lambda\vec{x}$

$Au = \lambda u$ [From ①]

$A^2 u = \lambda A u$

From ①

$A^2 u = \lambda^2 u$

$\therefore$ Eigenvalue is λ, eigenvalue is $\vec{u}$

(e) $(A + 2I)(A - I)^2$

$\Rightarrow (A + 2I)(A^2 + I - 2A)$

$\Rightarrow A^3 + A - \cancel{2A^2} + \cancel{2A^2} + 2I - 4A$

eigenvector is $\lambda^3 - \lambda - 2$ and eigenvalue is $\vec{u}$

$\Rightarrow A^3 - 3A + 2I$

$\Rightarrow A^3 - 2A - (A - cI)$

$\Rightarrow (A^3 - 2A - (A - cI))\vec{u}$ $[c = 2]$

$$\Rightarrow \underbrace{A^3u}_{\lambda^3} \underbrace{- 2Au}_{-2\lambda} - \underbrace{(A - 2I)}_{\lambda - 2}\vec{u}$$

$\Rightarrow (A^3 - 2A - (A - 2I))\vec{u} = (\lambda^3 - 2\lambda + \lambda - 2)\vec{u}$

$\Rightarrow (\lambda^3 - \lambda - 2)\vec{u}$

9 (a) Given   To Find

$B = A^T A$ eigenvalue and eigenvector for C
$C = AA^T$
$Bu = \lambda u \rightarrow (1)$

$A \rightarrow m, n \quad u \rightarrow n \times 1$

$\lambda \neq 0$

$A^T A u = \lambda u$ [From (1)]

$A\underbrace{A^T A\vec{u}}_{m\times 1} = \lambda \underbrace{A\vec{u}}_{m \times 1}$

$\left.\begin{array}{l} AA^T\vec{y} = \lambda \vec{y} \quad \text{where } \vec{y}_{m\times 1} = A\vec{u} \\ \therefore \vec{y} \text{ is eigenvector and } \lambda \text{ is eigenvalue} \end{array}\right\} \rightarrow (IV)$

(b) $\lambda = 0$

Then $B\vec{u} = 0 \rightarrow (1)$
$\therefore u \in N(A^T A)$
$\dim(N(A^T A)) = \dim(N(A)) = n - r$ [$r = Rank(A)$]

From (1)
$\det(B) = 0$
$\Rightarrow \det(A^T A) = 0$

If A is square matrix then $\det(A^T A) = \det(AA^T) \Rightarrow 0$

PREVIOUS SOLUTION NOT COMPLETE

For matrix A, take $n \neq m$ [Part (a)]

In part (a) we know $\det(B) = \det(A^TA) \neq 0$ [From ① and $\lambda \neq 0$]

If $m \geq n$

A^TA has n columns
AA^T has m columns

We know $Rank(AA^T) = Rank(A) = Rank(A^T) = r$

clearly $r \leq n$

(III) $\Leftarrow \therefore \det(AA^T) = 0 \Rightarrow C$ has no non-zero eigenvalues [Because AA^T has m columns and max rank is n]

Conv. If $m < n$ then $\det(A^TA) = 0$

~~Then $\det(A^TA) = 0$~~ but since $\lambda \neq 0 \therefore Null(A^TA) = 0$. hence $\det(A^TA) \neq 0$

$\therefore m \leq n$ is not possible for part (a)

For matrix A, take $n = m$ [Part (a)]

then $\det(A^TA) = \det(AA^T)$ [$\det(AB) = \det(BA)$ if B and A are square matrices]

Since $\lambda \neq 0$
$\det(A^TA) = \det(AA^T) \neq 0$

Hence (IV) applies

For matrix A, assume $n \neq m$ [For Part (b)]

Assume $n < m$

$\lambda = 0$
$\Rightarrow \det(A^T A) = 0$

$\det(A^T A) = 0 \Rightarrow A^T A$ has n columns [From above]
AA^T has m columns $\rightarrow \det(AA^T) = 0$ [From (11)]

Hence both B and C have 0 non-zero eigenvalues

Assume $m < n$

$\lambda = 0$
$\Rightarrow \det(A^T A) = 0$

$A^T A \rightarrow n$ columns
$AA^T \rightarrow m$ columns

$\text{Rank}(A) = r < n$

C can have non-zero eigenvalue if $r = m$

Assume $m = n$

$\det(A^T A) = \det(AA^T) \Rightarrow 0$

Hence Both B and C have 0 non-zero eigenvalues

(10)

Given

Q is orthogonal

$Qu = \lambda u$

$\lambda \in C$

To Prove

$|\lambda| = 1$

We have

$Qu = \lambda u \quad u \neq 0 \rightarrow (1)$

$\Rightarrow u^{+}Qu = \lambda u^{+}u$

$\Rightarrow \lambda . \lambda u^{+}u \Rightarrow \lambda u^{+}Qu$

$\lambda^{2} u^{+}u \Rightarrow (\lambda u)^{+}Qu$ [λ is scalar]

From (1)

$(Qu)^{+}Qu$

$\Rightarrow u^{+}u$ [$Q^{T}Q = I$]

$\therefore \lambda^{2}u^{+}u = u^{+}u$

$\Rightarrow \lambda^{2} = 1$

$\Rightarrow |\lambda| = 1$

Hence $\lambda = e^{i\phi}$ where $\phi \in \mathbb{R}$

(6) Given

To Find

Corresponding EigenValue

$A \rightarrow$ matrix (m, n)

$x \rightarrow$ Approx. Eigenvector

We want to minimize norm of residual vector

$$\Rightarrow \underset{x}{\operatorname{argmin}} \|Ax - \lambda x\|$$

Equivalent to solving the least square problem of the form

$$\lambda x = Ax$$

$$\Rightarrow x\lambda = Ax \quad [\text{Since } \lambda \text{ is scalar}]$$

$$\Rightarrow \underbrace{x^T x}_{\text{Scalar}} \lambda = x^T A x$$

$$\Rightarrow \lambda = \frac{x^T A x}{x^T x}$$

$\therefore$ Corresponding eigenvalue is $\frac{x^T A x}{x^T x}$