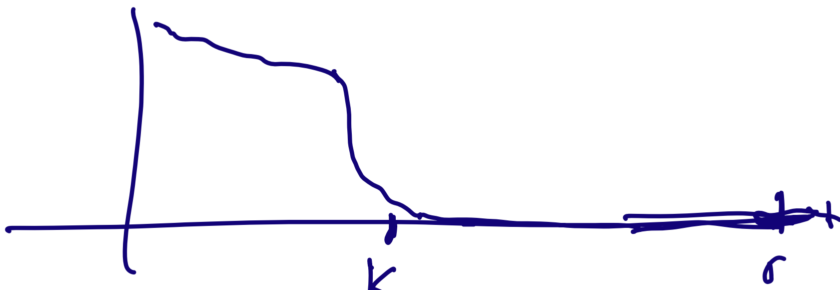
"best" rank k approximation A_k to a rank r matrix A, where $k<r$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$\sigma_1, \dots, \sigma_r,$

$u_1, \dots, u_r$

$v_1, \dots, v_r$



$B = A - A_k$

$$\min \|A - A_k\|_2 \le 0$$

$$\|A - A_k\|_F$$

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$\sigma_1 \geq \sigma_2 \geq \geq \sigma_k \geq \quad \geq \sigma_r > 0$$

$$\|B\| = ?$$

$$\|A\|_2 = \sigma_1$$

$$\|A\|_2 = \max_{\|\hat{x}\|_2 = 1} \|A\hat{x}\|_2$$

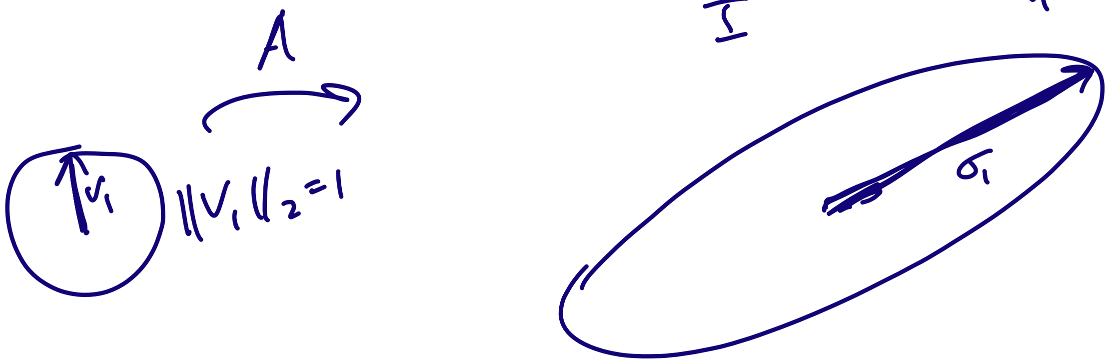
$$\|A\|_2 = \|U \Sigma V^T\|_2$$

$$\|A\|_2 = \|\Sigma\|_2$$

$$\|x\|_2^2 = \|Ux\|_2^2$$

$$x^T x = x^T \underbrace{\cancel{U^T U}}_{I} x$$

$$\|\sigma_1 u_1\|_2 = \sigma_1$$



Least Squares

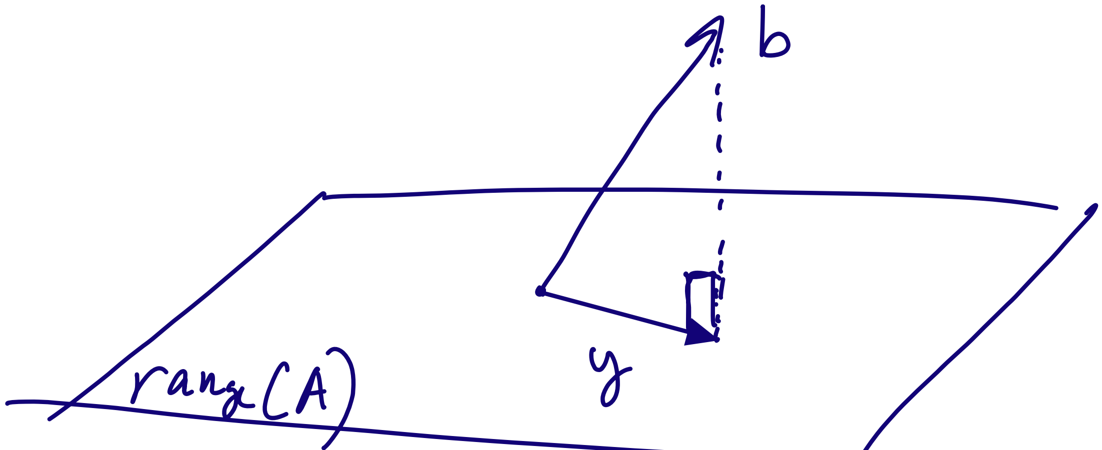
solution if $b \in \text{range}(A)$

no solution

$$Ax = b$$

"best" solution? $\hat{x}$

$$A\hat{x} \neq b$$



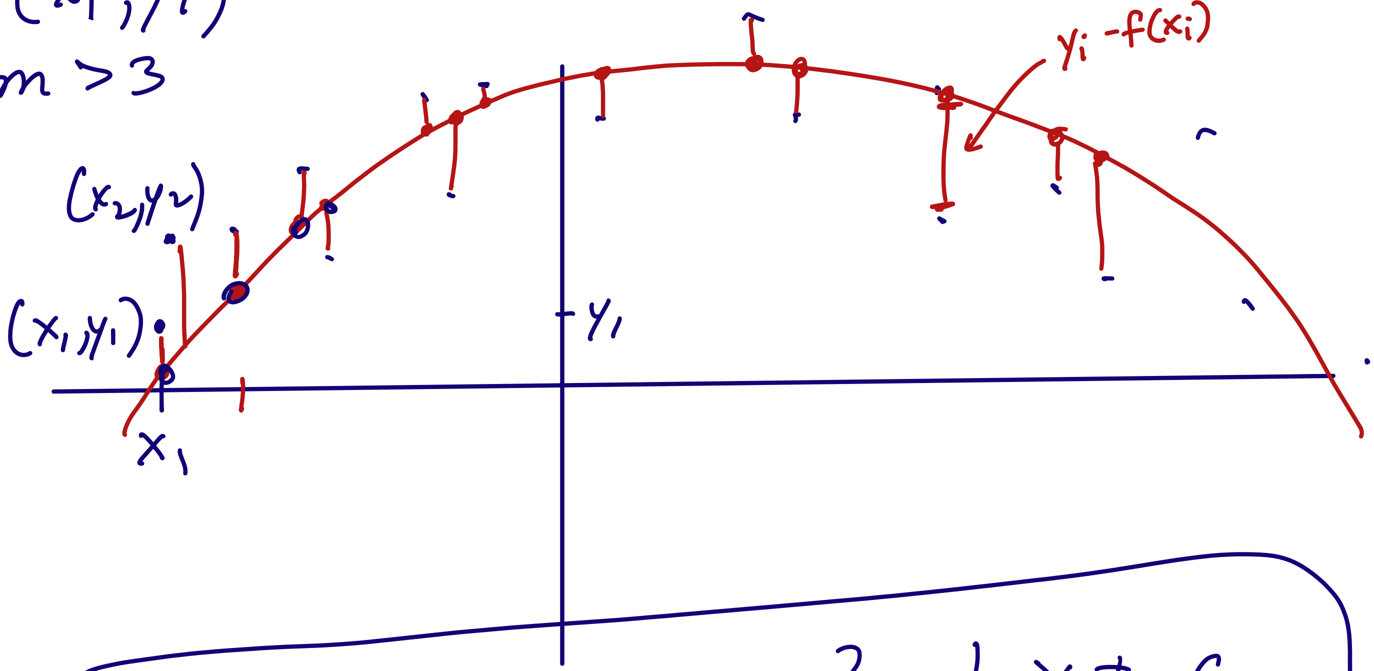
$$\text{range}(A) = \{ y \mid y = Ax \text{ for some } x \}$$

$$\min \|b - A\hat{x}\|_2^2$$

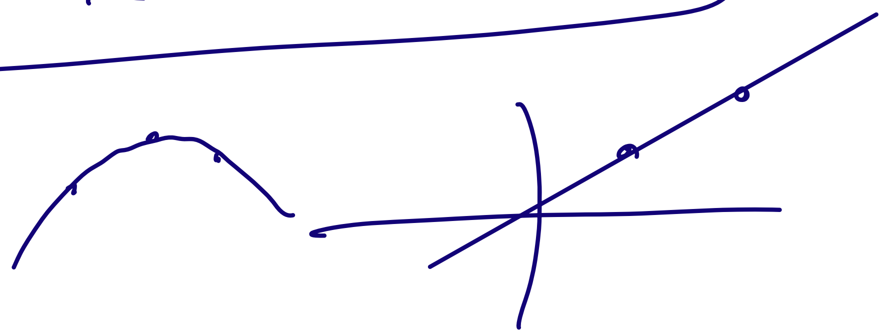
$$\|b - A\hat{x}\|_2^2 = \sum_{i=1}^{n} \underbrace{\left(b_i - \left(\sum_j A_{ij} x_j\right)\right)^2}$$

(x_i, y_i)

$m > 3$



$$\boxed{y = f(x) = \underline{a}x^2 + \underline{b}x + \underline{c}}$$



$$y_i \approx f(x_i) = a x_i^2 + b x_i + c$$

$$\boxed{\sum_{i=1}^{m} \underbrace{\left(y_i - f(x_i)\right)}^2 = \min}$$

$$a x_1^2 + b x_1 + c = y_1$$
$$a x_2^2 + b x_2 + c = y_2$$
$$\vdots$$
$$a x_m^2 + b x_m + c = y_m$$

$$\underbrace{\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & & \\ x_m^2 & x_m & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{b}$$

$$A \quad x \approx b$$

m eq's
3 unknowns

$$\min_x \underbrace{\| b - \underline{Ax} \|_2^2}_{\phi(x)}$$

$$\phi(x) = (b - Ax)^T (b - Ax)$$

$$= b^T b - \underbrace{b^T A x}_{1\times 1} - \underbrace{x^T A^T b}_{\overset{\shortparallel}{b^T A x}} + x^T A^T A x$$

$$\phi(x) = b^T b - 2 b^T A x + x^T A^T A x$$

$$\delta\phi = 0 - 2 b^T A \delta x + \delta x^T (A^T A) x + x^T A^T A \delta x$$

$$\boxed{\delta\phi = -2 b^T A \delta x + 2 x^T A^T A \delta x}$$

$(A^T A)^T = A^T A^{TT} = A^T A$

AA^T

$$= \boxed{(-2 b^T A + 2 x^T A^T A)} \delta x$$

$$\nabla\phi(x) = -2 b^T A + 2 x^T A^T A$$

at the min.

$$-\cancel{2} b^T A + \cancel{2} x^T A^T A = \cancel{0}\ \cancel{2} b^T A$$

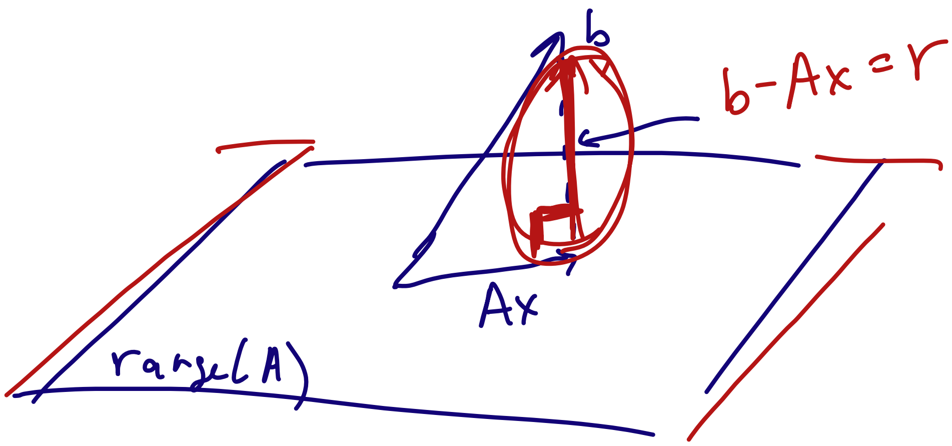
"normal equations" $\boxed{A^T A x = A^T b}$

$$Ax \approx b$$

$$A^T A x = A^T b$$

$$\boxed{A^T(\underbrace{b - Ax}_{r}) = 0}$$

r = residual = backward error



$$\boxed{A^T r = 0}$$

$$r \perp \text{range}(A)$$

$$\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \qquad \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{pmatrix} \begin{pmatrix} | \\ r \\ | \end{pmatrix} = 0$$

$$A^T A x = A^T b$$

is $(A^T A)$ invertible?

$$\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

$$\underset{m \times n}{A}$$

$$\underset{n \times m}{A^T}\ \underset{m \times n}{A} = \underset{n \times n}{\boxed{\quad\quad}}$$

if $\operatorname{rank}(A) = n$,

then $\operatorname{rank}(A^T A) = n$

i.e. $A^T A$ invertible

if A full rank (rank n),

solution is unique:

$$x = (A^T A)^{-1} A^T b$$

$$Ax = b$$

$$A^T A x = A^T b$$

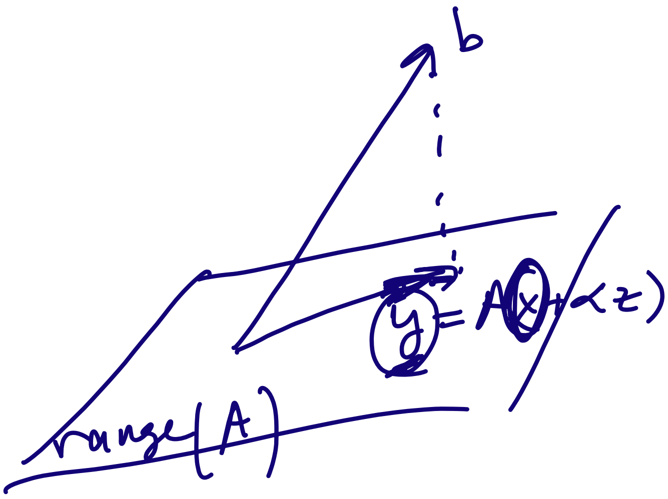
$$\underbrace{A^T b} \in \operatorname{range}(A^T A)$$

$\hat{x}$

$$\| b - A\hat{x} \| = \min$$

if A not full column rank
then $\exists\, z \neq 0$ s.t. $Az = 0$

$$\| b - A(\hat{x} + \alpha z) \| = \min$$



$$A^T A x = A^T b$$

B s.p.d

$$x^T B x > 0 \qquad \forall x \neq 0$$

$$x^T A^T A x = (Ax)^T (Ax) = w^T w = \|w\|_2^2 > 0$$

$$LL^T x = A^T b$$

$$\text{cond}(\underbrace{A^T A}) = (\text{cond}\, A)^2$$

$$\min \| b - \underline{A} x \|_2^2$$

$m > n$

$$\underset{m\times n}{A} = \underset{m\times m}{Q}\,\underset{m\times n}{R}$$

$$\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} = \left(\begin{array}{ccc|ccc} | & & | & | & & | \\ q_1 & \cdots & q_n & q_{n+1} & \cdots & q_m \\ | & & | & | & & | \end{array}\right) \begin{pmatrix} r_{11} & \cdots & r_{12} \\ & \ddots & \vdots \\ & & r_{nn} \\ \hline & 0 & \end{pmatrix}$$

$$\underset{m\times n}{A} = \underbrace{\left(\underset{\underline{m}\times n}{\boxed{\bar{Q}}} \;\middle|\; \underset{m\times(\underline{m-n})}{\boxed{\tilde{Q}}} \right)}_{Q} \underbrace{\begin{pmatrix} \hat{R} \\ \hline 0 \end{pmatrix}}_{m\times n} \begin{matrix} n\times n \\ \underline{(m-n)\times n} \end{matrix}$$

$m \times m$