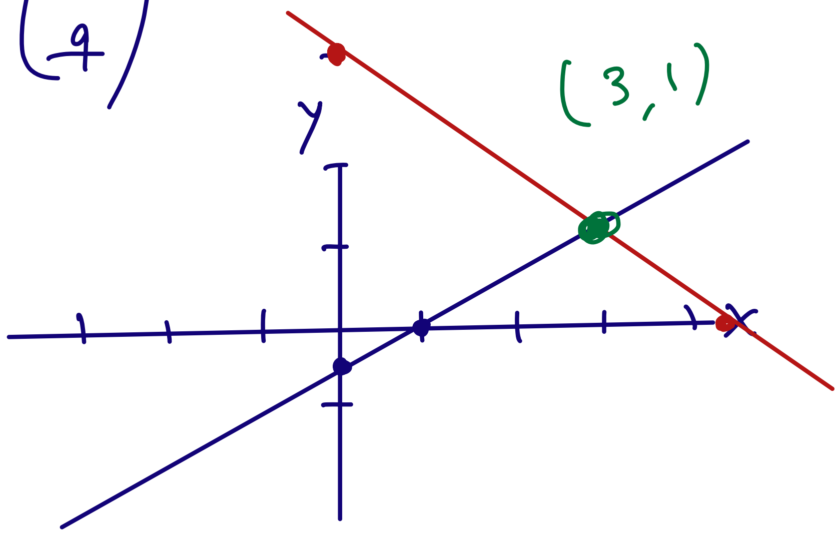
Linear Systems

det=7

$$\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 9\end{pmatrix}$$

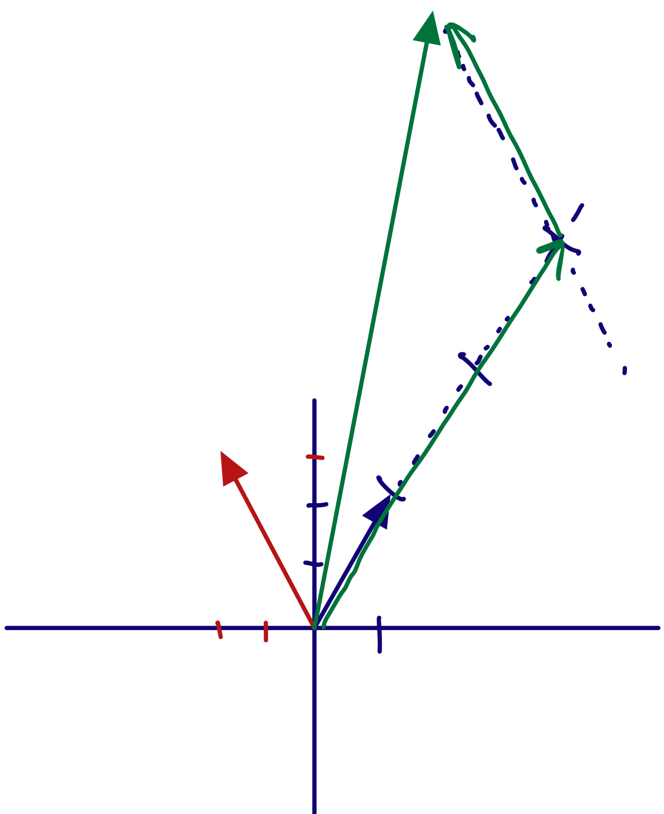
"row view"



$$\begin{cases} x-2y = 1\\ 2x+3y = 9\end{cases}$$

$$\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 9\end{pmatrix}$$

$$x\underbrace{\begin{pmatrix}1\\ 2\end{pmatrix}}+y\begin{pmatrix}-2\\ 3\end{pmatrix}=\underbrace{\begin{pmatrix}1\\ 9\end{pmatrix}}$$



$x=3$

$y=1$

"col view"

A singular

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

$\det(A) = 2\cdot 6 - 3\cdot 4 = 0$

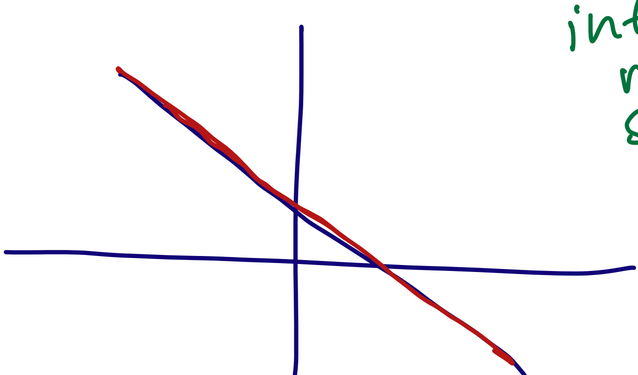
$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$\cancel{2\alpha} + \cancel{3}y = \frac{4-2\alpha}{3}$

$2x + 3y = 4$

$\frac{4x + 6y}{2} = \frac{8}{2}$

$2x + 3y = 4$



infinitely many soln's

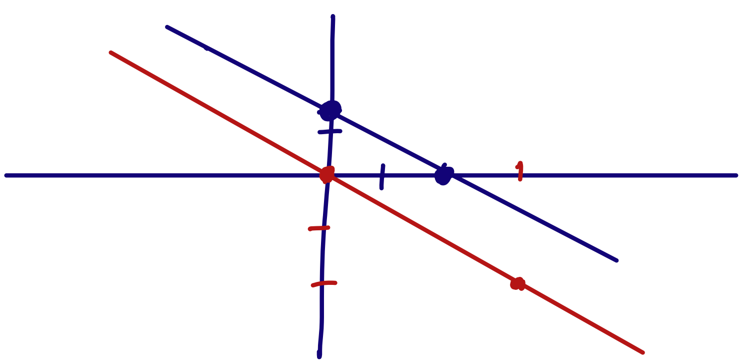
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{4-2\alpha}{3} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$2x + 3y = 4$

$\frac{4x + 6y}{2} = \frac{0}{2}$

$2x + 3y = 0$



soln's? 0

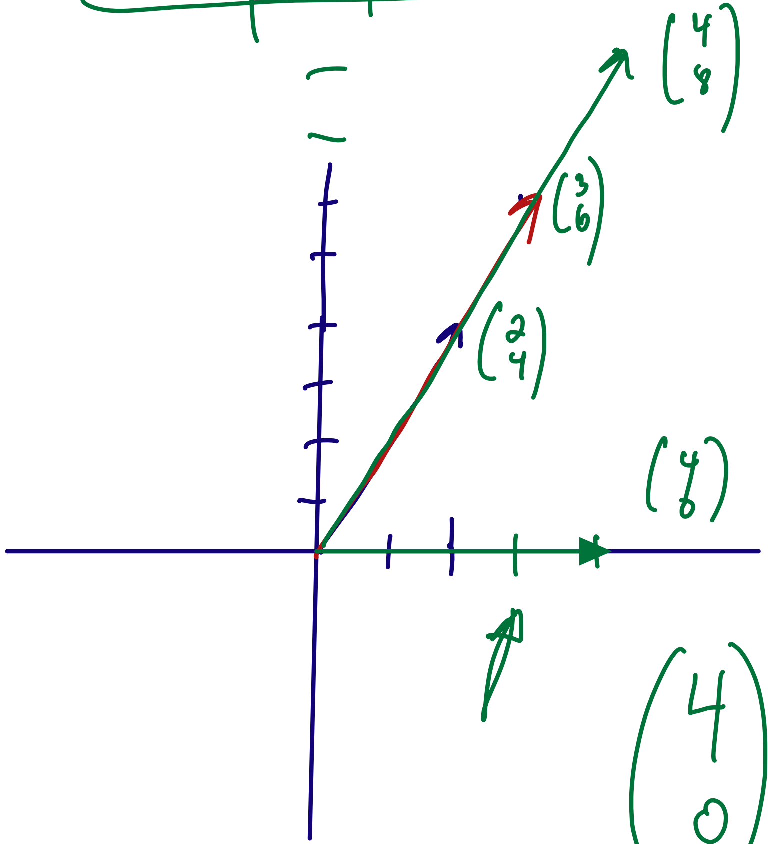
$$\boxed{\left(\begin{pmatrix}2\\4\end{pmatrix}\begin{pmatrix}3\\6\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}}$$

∞ many soln's

$$\begin{pmatrix}4\\8\end{pmatrix} = x\begin{pmatrix}2\\4\end{pmatrix} + y\begin{pmatrix}3\\6\end{pmatrix}$$

no soln:

$$\begin{pmatrix}4\\0\end{pmatrix} = \underbrace{x\begin{pmatrix}2\\4\end{pmatrix} + y\begin{pmatrix}3\\6\end{pmatrix}}$$



$$\begin{pmatrix}4\\0\end{pmatrix}$$

$$\begin{pmatrix}2 & 3\\4 & 6\end{pmatrix}$$

$$3\alpha\begin{pmatrix}2\\4\end{pmatrix} + (-2\alpha)\begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}2 & 3\\4 & 6\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\alpha$$

$z \neq 0$

$A\vec{z} = \vec{0}$

$$\boxed{Ax = b}$$

A $n \times n$ real matrix

b $n \times 1$ real vector

A nonsingular if any of these hold (and then they all hold)

1. A has an inverse A^{-1}

   $AA^{-1} = A^{-1}A = I$

2. $\det(A) \neq 0$

3. $\operatorname{rank}(A) = n$

4. $\operatorname{null}(A) = \{\vec{0}\}$

   $\vec{z} \neq 0 \qquad A\vec{z} \neq \vec{0}$

$$Ax = b$$

Existence & Uniqueness of solutions:

1. A is nonsingular

$\exists\ A^{-1}$

$$\underbrace{A^{-1}A}x = A^{-1}b$$

$$Ix = A^{-1}b$$

$$\boxed{x = A^{-1}b}$$

there is a solution for $Ax=b$ & it is unique

$$\begin{array}{r} Ax = b \\ -\quad Ay = b \\ \hline A\underbrace{(x-y)} = 0 \end{array}$$

A nonsingular. $\therefore\ x-y = 0$

$\Rightarrow x = y$

$\vec{v}_1, \dots, \vec{v}_n$ basis for $\mathbb{R}^n$

$\vec{b}$

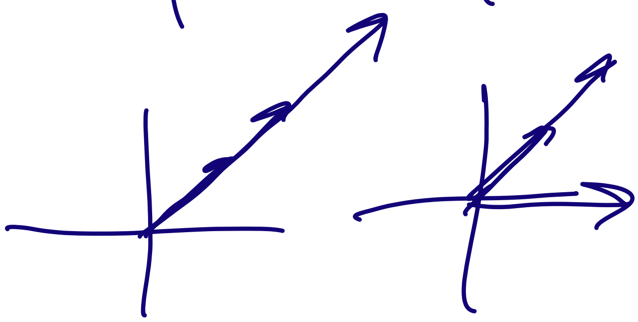
$b = \alpha_1\vec{v}_1 + \dots \alpha_n\vec{v}_2$

$$b = \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

A singular

① ∞ # soln's

$\vec{b} \in \mathrm{col}(A)$



② no solutions

$\vec{b} \notin \mathrm{col}(A)$

$$Ax = b$$

A singular $\rightarrow$ $\exists\ \underline{z \neq 0}$ $Az = 0$

$$\underline{A}\left(x + \alpha \vec{z}\right)$$

$$= Ax + \alpha \cancel{A\vec{z}}^{\,0} = b$$

$$Ax = b$$

smallest $\|x\|$

$$\vec{x} + \alpha \vec{z}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{array}{cc}\vec{x} & \vec{z}\end{array}$$

$1+\alpha$

$2+\alpha$

$3+\alpha$

$$(1+\alpha)\vec{a}_1 + (2+\alpha)\vec{a}_2 + (3+\alpha)\vec{a}_3 = \vec{b}$$

Solving $Ax = b$

transform the problem int something easier to solve

(or do iteration that converges ...)

direct methods

easy to solve:

diagonal

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

$a_{11}x_1 + a_{1j} \cdot x_j = b_1$

$$\begin{cases} a_{11} x_1 = b_1 \\ a_{22} x_2 = b_2 \\ \vdots \\ a_{nn} x_n = b_n \end{cases}$$

decoupled!

$$\Rightarrow \boxed{x_i = \frac{b_i}{a_{ii}}}$$

triangular system

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10^{-5} & 1 & 2 \end{pmatrix}$$

lower tri.

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$$ upper tri

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10^{-5} & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

forward substitution $x_1 = \frac{b_1}{a_{11}}$ $x_1 = 2$

$$3 \cdot 2 - x_2 = 3$$

$$x_2 = -3$$

$$\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

backward substitution

backwa

A nonsingular, $n \times n$

$$Ax = b$$

$$A = LU$$

LU factorization of A

$$\boxed{Ax = b}$$

$$L\underbrace{\boxed{Ux}}_{=y} = b$$

$$Ly = b$$ forward subst.

$$Ux = y$$ back subst.

$\rightarrow x$