QR

$$A = QR$$
$n \times n$

Q $n \times n$ orthogonal matrix

R $n \times n$ upper triangular

- classical Gram-Schmidt algorithm
- modified Gram-Schmidt
- Householder

Householder matrix $(H^{-1} = H^T)$

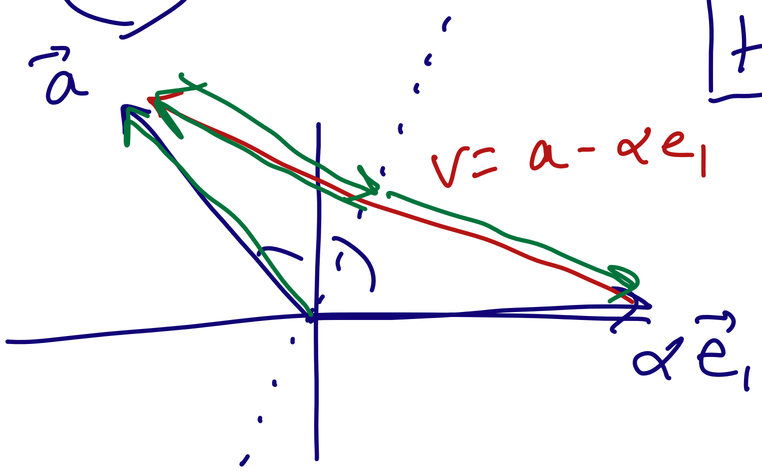
H $n \times n$ orthogonal matrix:

$$H^T H = H H^T = I$$

$\textcircled{H}$ reflection matrix $\det(H) = \underline{-1}$

$$\boxed{H\vec{a} = \alpha \vec{e}_1}$$

$$\alpha = \pm \|\vec{a}\|_2$$



$$Ha = a - \underline{2} \boxed{\frac{vv^T}{v^Tv}} a$$

$$Ha = \underbrace{\left(I - 2\frac{vv^T}{v^Tv}\right)}_{H} a$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \underline{H}\vec{a} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha \vec{e}_1$$

$$\vec{c} = \begin{pmatrix} c_1 \\ \boxed{\begin{matrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{matrix}} \end{pmatrix} \rightarrow \vec{c}_2$$

$$\underset{(n-1)\times(n-1)}{H_2}\,\vec{c}_2 = \begin{pmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad n-1$$

$$\left(\begin{array}{c|c} 1 & 0 \ 0 \cdots 0 \\ \hline 0 & \\ \vdots & H_2 \\ 0 & \end{array}\right) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ H_2 c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & H_2 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & H_2^T \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & H_2 H_2^T \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right)$$

$$= I_n$$

$$\begin{pmatrix} I_k & \\ & H_{k+1} \end{pmatrix}$$

orthogonal

$$\begin{pmatrix} 1 & \\ & H_2 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & H_3 \end{array}\right)$$

Householder QR

$$A = QR$$

$$Q^T A = R$$

$$H_1 \underbrace{\begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{matrix}} & a_2 & \cdots & a_n \end{pmatrix}}_{A} = \begin{pmatrix} \alpha_1 & \boxed{H_1 a_2} & \cdots & H_1 a_n \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

$$H_2 H_1 A = \begin{pmatrix} 1 & \\ & \hat{H}_2 \end{pmatrix} H_1 A = \begin{pmatrix} \alpha_1 & \bullet & \bullet & \cdots & \bullet \\ 0 & \times & \times & \cdots & \times \\ \vdots & \otimes & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \times & \cdots & \times \end{pmatrix}$$

$$H_3 H_2 H_1 A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \hat{H}_3 \end{pmatrix} H_2 H_1 A = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{pmatrix}$$

$$\underbrace{H_{n-1} \cdots H_1}\, A = \begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{pmatrix}$$

$$Q^T A = R$$

$$\boxed{A = QR}$$

$$Ha = \underbrace{\left(I - 2\frac{vv^T}{v^Tv}\right)}\,a \qquad \underline{O(n)}$$

cheaper than gen. matrix-vec

$$\boxed{Ax \qquad O(n^2)}$$

$$= a - 2\frac{v\boxed{v^Ta}}{v^Tv} \qquad O(n)$$

$$= a - \underbrace{\left(\frac{2}{v^Tv}\right)}(v^Ta)\,v \qquad O(n)$$

$$O(n)$$

$$a\,- \qquad O(n)$$

Q ?

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yet 1 more approach:

Givens rotations

SVD = Singular Value Decomposition

T&B: What if we take the SVD?

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

it exists for all matrices!

U : $m\times m$ orthogonal (unitary)

V : $n\times n$ orthogonal (unitary)

Σ : $m\times n$ diagonal with real positive entries s.t.

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$$

3×2

$$\begin{pmatrix} 100 & 0 \\ 0 & .01 \\ 0 & 0 \end{pmatrix}$$

$\Sigma_{ii} \geq 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$r = \operatorname{rank}(A)$

$$\underbrace{\sigma_1 \geq \sigma_2}_{} > \sigma_3 = 0$$

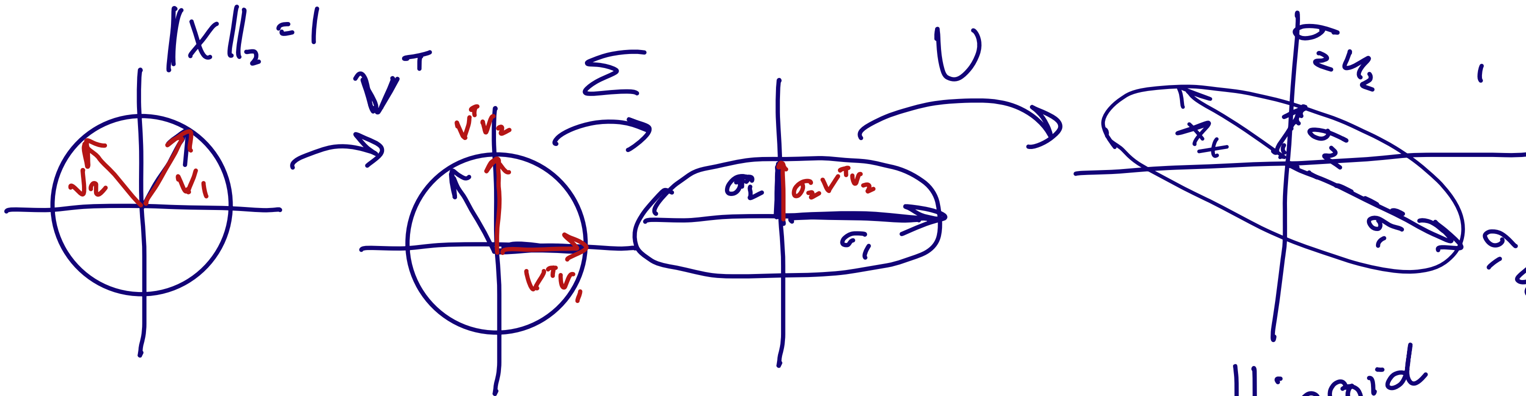
σ_i's singular values of A

$u_1, \dots, u_m$ left singular vectors

$v_1, \dots, v_n$ right " "

$$Ax = \underbrace{U \Sigma \underbrace{V^T x}}$$

$Ax \ \forall \ \|x\|_2 = 1$



hyper ellipsoid

$$\Sigma = \begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$V^T v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$V^T v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{aligned} Av_1 = U \Sigma V^T v_1 &= U \Sigma e_1 \\ &= U \sigma_1 e_1 \\ &= \sigma_1 \underbrace{U e_1} \\ &= \sigma_1 \vec{u}_1 \end{aligned}$$

$$A = U\Sigma V^T$$

$$\underbrace{\underset{m\times n}{A}\,\underset{n\times n}{V}}_{m\times n} = \underbrace{\underset{m\times m}{U}\,\underset{m\times n}{\Sigma}}_{m\times n}$$

$$\boxed{Av_i = \sigma_i u_i} \qquad i = 1, \dots, n$$

$n \le m$

$$\underset{m\times n}{A} = \underbrace{\begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}}\begin{pmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} | & | & & | \\ \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_n u_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}}$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \underbrace{\sigma_n u_n v_n^T}$$

$$\operatorname{rank}(A) = r \leq \min(m, n)$$

$$\boxed{A = \sum_{i=1}^{r} \sigma_i u_i v_i^T}$$

$r = \operatorname{rank}(A)$

reduced SVD

$$A = \underbrace{\begin{pmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{pmatrix}}_{m \times r} \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}}_{r \times r} \begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_r^T & - \end{pmatrix}$$