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Homework 5

$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$q_1' = a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $a_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$q_1 = \frac{1}{\|q_1'\|} q_1' \Rightarrow \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\rightarrow q_2' = a_2 - (a_2 \cdot q_1) q_1$

$\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \frac{(2\times1+4\times3)}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \frac{14}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore q_2 = \frac{q_2'}{\|q_2'\|} \Rightarrow \frac{1}{0.6325} \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}$

$Q = [q_1, q_2] = \begin{bmatrix} 1/\sqrt{10} & \frac{3}{5} \times \sqrt{\frac{10}{25}} \\ 3/\sqrt{10} & -1/5 \sqrt{\frac{10}{25}} \end{bmatrix}$

$r_{11} = \|q_1'\|$

$r_{12} = a_2 \cdot q_1$

$r_{22} = \|q_2'\|$

$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$

$\therefore R = \begin{bmatrix} \sqrt{10} & 14/\sqrt{10} \\ 0 & \sqrt{\frac{10}{25}} \end{bmatrix}$

(b) $A_2 = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix} \end{array}$

$q_1' = a_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ $\quad r_{11} = \|q_1'\| = 5 \quad q_1 = \frac{1}{\|q_1'\|} = \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$

$r_{12} = a_2 \cdot q_1 = 0 \quad q_2' = a_2 - (a_2 q_1) q_1 \Rightarrow a_2 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \frac{1}{\|q_2'\|} q_2' \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$r_{22} = \|q_2'\| = 1$

$r_{13} = a_3 \cdot q_1 \Rightarrow 4/5$

$r_{23} = a_3 \cdot q_2 \Rightarrow 1 \quad q_3' = a_3 - r_{13} \cdot q_1 - r_{23} \cdot q_2 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 \\ -12/25 \\ 9/25 \end{bmatrix}$

$r_{33} = \|q_3'\| \Rightarrow 3/5$

$q_3 = \frac{q_3'}{\|q_3'\|} \Rightarrow \begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix}$

$\therefore \ Q = [q_1, q_2, q_3] \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & -4/5 \\ 4/5 & 0 & 3/5 \end{bmatrix}$

$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 & 4/5 \\ 0 & 1 & 1 \\ 0 & 0 & 3/5 \end{bmatrix}$

GIVEN

② $\{q_1, q_2, \ldots q_k\}$ is orthonormal $q_i \in \mathbb{R}^n \ \forall\ i\ 1..k$

(a) $P_1 = (I - q_1 q_1^T)(I - q_2 q_2^T) \ldots (I - q_k q_k^T)$

To Prove

P_1 is a projection matrix $\Leftrightarrow P_1^2 = P_1$

$$P_1 \cdot P_1 \Rightarrow \underbrace{(I - q_1 q_1^T)(I - q_2 q_2^T) \ldots (I - q_k q_k^T)}_{k \text{ pairs}} \underbrace{(I - q_1 q_1^T) \ldots (I - q_k q_k^T)}_{k \text{ pairs}}$$

$(I - q_1 q_1^T)(I - q_1 q_1^T)$ {Taking only 1 pair}

$(I - 2q_1 q_1^T + q_1 q_1^T q_1 q_1^T)$

$\Rightarrow I - 2q_1 q_1^T + q_1 q_1^T$

$\Rightarrow (I - q_1 q_1^T)$

Similarly doing this for every pair, we get

$P_1 \cdot P_1 = (I - q_1 q_1^T)(I - q_2 q_2^T) \ldots (I - q_k q_k^T) \Rightarrow P_1$

$\therefore P_1^2 = P_1$

Hence projection matrix proved

(b) $P_2 = I - q_1 q_1^T - q_2 q_2^T - \cdots - q_k q_k^T$

To Prove

P_2 is a projection matrix $\Leftrightarrow P_2^2 = P_2$

$P_2 \cdot P_2 \Rightarrow P_2^2 = \left(I - \sum_{i=1}^{k} q_i q_i^T\right)\left(I - \sum_{i=1}^{k} q_i q_i^T\right)$

$\Rightarrow I - 2\sum_{i=1}^{k} q_i q_i^T + \underbrace{\left(\sum_{i=1}^{k} q_i q_i^T\right)^2}$ $\rightarrow$ (1)

Only those terms will survive which will be of the form
$(q_i q_i^T) \cdot (q_i q_i^T) \Rightarrow q_i q_i^T$

The other terms of the form

$(q_i q_i^T) \cdot (q_j q_j^T) \Rightarrow 0$ {orthonormal}

$\therefore \left(\sum_{i=1}^{k} q_i q_i^T\right)^2 \Rightarrow \left(\sum_{i=1}^{k} q_i q_i^T\right)$

Plugging into (1)

$\therefore P_2 \cdot P_2 = P_2^2 \Rightarrow I - 2\sum_{i=1}^{k} q_i q_i^T + \left(\sum_{i=1}^{k} q_i q_i^T\right)$

$\Rightarrow I - \sum_{i=1}^{k} q_i q_i^T \Rightarrow P_2$

Hence it is a projection matrix

(c) $P_1 = \prod_{i=1}^{k} (I - q_i q_i^T)$

$P_2 = I - \sum_{i=1}^{k} q_i q_i^T$

<u>To Prove</u>

$P_1 = P_2$

If we expand P_1 we have certain possibilities

I, $-q_i q_i^T$, some multiplicative combination of $I, q_i q_i^T, q_j q_j^T \ldots$ {any number}

0 {orthonormal}

$\therefore$ P_1 gives us $I - \sum_{i=1}^{k} q_i q_i^T = P_2$

<u>Hence Proved</u>

Q3

Matlab Code for MY_GS

```
function [Q, R] = MyGS(A, modified)
[m, n] = size(A);
Q = zeros(m, n);
for j = 1:n
   v = A(:, j);
   for i = 1:j-1
       if modified == false
           R(i, j) = Q(:,i)' * A(:, j);
       end
       if modified == true
           R(i, j) = Q(:,i)' * v;
       end
       v = v - R(i, j) * Q(:, i);
   end
   R(j, j) = norm(v);
   Q(:, j) = v / R(j,j);
end
```

Q4

Code for GS Instability

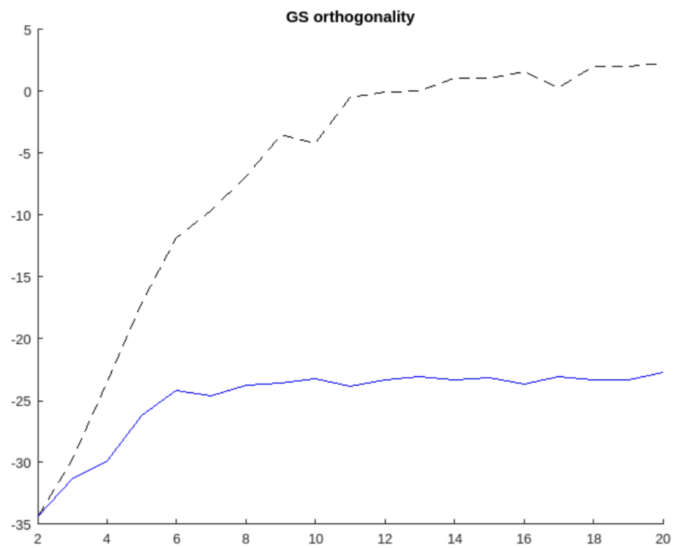
```
function GSInstability(lo,hi)
mGS_orthogonality = [];
mGS_factorization = [];
cGS_orthogonality = [];
cGS_factorization = [];
for i = lo:hi
   A = hilb(i) + eye(i) * 1e-6;
   [Q, R] = MyGS(A, false);
   cGS_orthogonality(i-lo+1) = norm(Q'*Q-eye(i));
   cGS_factorization(i-lo+1) = norm(Q*R - A);
   [Q, R] = MyGS(A, true);
   mGS_orthogonality(i-lo+1) = norm(Q'*Q-eye(i));
   mGS_factorization(i-lo+1) = norm(Q*R - A);
end
figure
hold on
plot(lo:hi, log(cGS_orthogonality), '--k');
plot(lo:hi, log(mGS_orthogonality), '-b');
title('GS orthogonality');
figure
hold on
plot(lo:hi, cGS_factorization, '--k');
plot(lo:hi, mGS_factorization, '-b');
title('GS factorizatin')
end
```

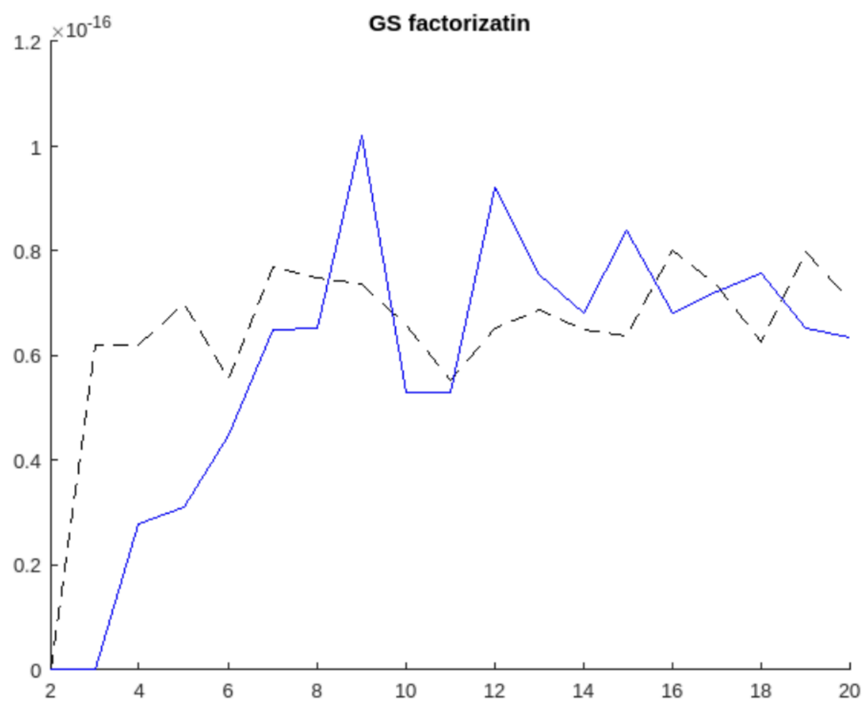
For lo = 2 and hi = 20

So there are 2 metrics

- Orthogonality metric does Q^2 - I to check the error (ideal error is 0)
- Factorization metric is error plot of QR - A (again ideal error is 0)

- For both these error metrics both CGS and MGS are plotted.





- We can see that in terms of factorization as size of matrix increases, the error for both CGS and MGS increases with sort of convergence, the modified gram schmidt show more spiky behaviour.
- We can see that in terms of orthogonality the modified gram schmidt represents much better orthogonal Q as its error is less.

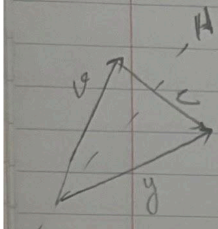
(5) **Given**

$$\underbrace{H\begin{bmatrix}4\\2\\1\\-2\end{bmatrix}}_{v} = \underbrace{\begin{bmatrix}r_1\\r_2\\0\\0\end{bmatrix}}_{y} \quad r_1, r_2 \in \mathbb{R}$$

To Find

H

H is a householder reflection matrix



$H = I - \frac{2vv^*}{\|v\|^2}$ where v^* is the conjugate transpose

The direction $c = \left(\begin{bmatrix}4\\2\\1\\-2\end{bmatrix} - \begin{bmatrix}r_1\\r_2\\0\\0\end{bmatrix}\right) \Big/ \|v-y\|$

$$\begin{bmatrix}4-r\\2-r_2\\1\\-2\end{bmatrix} \Big/ \sqrt{(4-r_1)^2+(2-r_2)^2+1+4}$$

$H = I - 2cc^T$

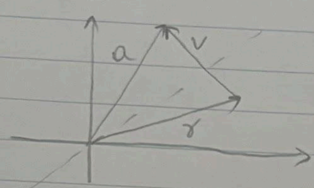
$$\Rightarrow I - 2\begin{bmatrix}4-r_1\\2-r_2\\1\\-2\end{bmatrix}\left[4-r_1, \ 2-r_2, \ 1 \ -2\right] \Big/ (4-r_1)^2+(2-r_2)^2+1+4$$

⑥ Given

$H = I - \frac{2vv^T}{\|v\|^2}$

$v = a - r$

$\|a\|_2 = \|r\|_2$



To prove

$Ha = r$

Let $u = v/\|v\|$

$\left(I - \frac{2vv^T}{\|v\|^2}\right)a$

$a - \frac{2vv^Ta}{\|v\|^2}$ $\left\{\begin{array}{l} v^Ta \text{ is scalar} \\ \|v\|^2 \text{ is scalar} \end{array}\right\}$

$\Rightarrow a - \frac{2v^Ta}{\|v\|^2} v$

$\Rightarrow a - \frac{2(a-r)^Ta}{\|a-r\|^2}(a-r)$

$\Rightarrow a - \frac{2(a-r)^Ta}{a^Ta - a^Tr - r^Ta + r^Tr}(a-r)$ $\{\|a-r\|^2 = (a-r)^T(a-r)\}$

$\rightarrow a - \frac{2(a-r)^Ta}{2\|a\|_2 - 2a^Tr}(a-r)$

Because $a^Ta = \|a\|_2$ and

$r^Tr = \|r\|_2$ which are equal

Also $a^Tr = r^Ta$

$\Rightarrow a - \frac{\cancel{2}(\|a\|_2 - r^Ta)}{\cancel{2}(\|a\|_2 - a^Tr)}(a-r)$

$= a - \cancel{a} + r$

$\Rightarrow r$

Hence proved

(7) (a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = A$

To Do

SVD!

$X = AA^T = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

Eigenvalues and Eigenvectors of X

$\det(X-\lambda) = \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$

$\therefore \lambda_1 = 9, \lambda_2 = 4 \Rightarrow$ Eigenvalues

Respectively eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore \Sigma$ matrix $= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{9}} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad v_2 = \frac{1}{\sqrt{4}} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} u_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Hence we have values of V, U, Σ

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{V^T}$$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = A$

$X = AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

Eigenvalues and eigenvectors

$\det(X - \lambda) = \begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix} = 0$

$\therefore \lambda_1 = 4, \lambda_2 = 9$ Respectively eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} u_1 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{9}} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} u_2 \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence V is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U\Sigma V^T$

1c

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = A$$

$$X = AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

$$\det(X - \lambda) = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0$$

$\therefore \lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 0$

Respectively the eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$v_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

λ_1 was the only non-zero vector, we can take one orthogonal vector to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and normalize it. Best choice is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_2$

$$\therefore \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \overset{U}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} \overset{V^T}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$

$X = AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\det(X - \lambda) = 0 = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \lambda_1 = 2, \lambda_2 = 0$

Respectively eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^T u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Similarly as in (c), we take v_2 to be orthogonal to v_1 and normalized, hence $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & +1/\sqrt{2} \\ -1/\sqrt{2} & +1/\sqrt{2} \end{bmatrix}}_{V^T}$$

(e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A$

$X = AA^T \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

Eigenvalues and Eigenvectors

$\det(X - \lambda) = \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$

$\lambda_1 = 4, \lambda_2 = 0$

Respectively eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Similarly as (d) and (e), we take v_2 as orthogonal to v_1 and normalized it

$v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

(8)

Given

$$A = U\Sigma V^T = \begin{pmatrix} u_1 \,|\, u_2 \,|\, \cdots \,|\, u_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_n^T \end{pmatrix}$$

Rank$(A) = r$

$$\Rightarrow \begin{pmatrix} \hat{U} & \bar{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \bar{V}^T \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$

$\hat{U} \to$ First r columns of U $\quad$ $\hat{V} \to$ First r columns of V

$\bar{U} \to$ remaining $m-r$ columns $\quad$ $\bar{V} \to$ remaining $m-r$ columns

To Find

Basis for range(A), null(A), range(A^T), null(A^T)

null$(\Sigma) = y \mid \Sigma y = 0$

If $\Sigma y = 0$ then $U\Sigma y = 0 \Rightarrow$ y is in nullspace of $U\Sigma$

null$(U\Sigma) = x \mid U\Sigma x = 0$

If $U\Sigma x = 0$ then $U^* U \Sigma x = 0 \Rightarrow I\Sigma x = 0$ {U is unitary}

$\Rightarrow \Sigma x = 0$

$\Rightarrow$ x is in nullspace of Σ

$\therefore$ null(Σ) = null$(U\Sigma) \to$ (11)

$\text{Rank}(\Sigma) = \{e_1, \_ e_r\}$ $\left[\text{where } e_i = [0, \_ i \_ 0]^T\right] \rightarrow$ (IV)
$\text{Null}(\Sigma) = \{e_{r+1} \_ e_n\}$

This is because Σ is a $m \times n$ diagonal matrix

From (II)

$x \mid x \in \text{null}(U\Sigma) \rightarrow$ (III)

$U\Sigma V^* V x \Rightarrow U\Sigma x \Rightarrow 0$ {From (III)}

V is unitary $\therefore$ If $x \in \text{null}(U\Sigma)$ then $Vx \in \text{null}(U\Sigma V^*)$

$\therefore$ $Ve_{r+1} \_ Ve_n \in \text{null}(U\Sigma V^*)$

$\text{Rank}(A) = r \Rightarrow \text{null}(A) = n - r$

$Z = \{Ve_{r+1} \_ Ve_n\}$ are also orthogonal to each other, since size of Z is $n - r = \text{null}(A)$, there are no more extra vectors required to be added into Z and form the basis set

$\therefore$ Z is basis set for $\text{null}(U\Sigma V^*)$

where $Ve_i = v_i$

$\Rightarrow Z = \{v_{r+1} \_ v_n\} = \bar{V}$

Similarly $\text{null}(A^T) = \hat{V}$

★ Here "span" means the colspan/range

range(Σ) = {y | $y \in$ span(Σ)} [Redundant Definition but written just for formality]

if $y \in$ span(Σ)

y can be represented as Σw for some vector w [This representation is possible because Σ is a diagonal matrix]

Hence

$\Sigma w = y$

$\Sigma (V^* V) w = y$

$(\Sigma V^*)(Vw) = y$

Hence $y \in$ span(ΣV^*) for some vector Vw

Now lets take z | $z \in$ span(ΣV^*)

z = some combination of $\Sigma V^* = \Sigma V^* x = z$

$\Sigma (V^* x) = z$

So z can be represented as some combination of eigenvectors of Σ

$\therefore z \in$ span(Σ)

Hence span(Σ) = span(ΣV^*) = $r \Rightarrow \{e_1, \_ e_r\}$

$\therefore$ if $x \in$ range(ΣV^*)

$\Sigma V^* y = x$ {For some y}

$\Rightarrow U \Sigma V^* y = Ux$

$\therefore Ux \in$ range($U \Sigma V^*$)

Hence $\underbrace{\{Ue_1, \_ Ue_r\}}_{\text{Orthonormal}} \subset range(U\Sigma_r V^*)$

We know that $range(U\Sigma_r V^*) = r$

Assume $C = \{Ue_1, \_ Ue_r\}$, $size(C) = r$

We do not need need any more independent vector in C to make C a basis for $range(U\Sigma V^*)$

$\therefore$ C is a basis set for $range(U\Sigma_r V^*) = range($

where $Ue_i = u_i$

$\Rightarrow C = \{u_1, \_ u_r\} = \hat{U}$

Similarly $range(C^T) = \hat{U}$

(Q)

Given

$A \rightarrow m \times n$ matrix
column rank $= n$

To Prove

$A(A^TA)^{-1}A^T$ is orthogonal projector onto range (A)

(A^TA) is an invertible matrix {Because A has full column rank}

$\Rightarrow A^TAx$, assume $Ax = b \in \mathbb{R}^m$

$\Rightarrow A^TAx = A^Tb$

Orthogonal Projector means $P^2 = P$ and $P = P^T$ $\{P = A(A^TA)A^T\}$

$P = P^T$ [To Show]

$\Rightarrow (A(A^TA)^{-1}A^T)^T$

$(A^T)^T((A^TA)^{-1})^TA^T$

$A((A^TA)^T)^{-1}A^T$

$\Rightarrow A(A^TA)^{-1}A^{-1}$

$\Rightarrow P \rightarrow$ (1)

$P^2 = P$ [To Show]

$\Rightarrow A(A^TA)^{-1}A^TA(A^TA)^{-1}A$

$\Rightarrow A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A$

$\Rightarrow A(A^TA)^{-1}A$

$\Rightarrow P \rightarrow$ (11)

From (1) and (11)

$\therefore$ P is an orthogonal projector

USING SVD

$$A = U\Sigma V^T \quad \begin{Bmatrix} U \to m\times m & V \to n\times n \\ \Sigma \to m\times n & \end{Bmatrix} \quad \begin{Bmatrix} UU^T = I \\ VV^T = I \end{Bmatrix}$$

$$\Rightarrow A\left((A^TA)^{-1}A^T\right)$$

$$\Rightarrow U\Sigma V^T\left(\left(V\Sigma^TU^TU\Sigma V^T\right)^{-1}V\Sigma^TU^T\right.$$

$$\Rightarrow U\Sigma V^T\left(\left(V\Sigma^T\Sigma V^T\right)^{-1}V\Sigma^TU^T\right.$$

$$\Rightarrow U\Sigma V^T\ (V^T)^{-1}(\Sigma^T\Sigma)^{-1}V^{-1}V\Sigma^TU^T$$

$$\Rightarrow U\Sigma V^TV\ (\Sigma^T\Sigma)^{-1}\Sigma^TU^T$$

$$\Rightarrow U\Sigma\ (\Sigma^T\Sigma^{-1})\Sigma^TU^T$$

Since P is of the form UDU^T [Where D is a diagonal matrix such that $D = \Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T$]

So SVD of $P = UDU^T$

[We know that $\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T$ will result in a diagonal matrix because Σ is a diagonal]

Range(P) = Vectors in U → ⓐ

SVD of $A = U\Sigma V^T$

Range(A) = Vectors in the same matrix U → ⓑ

Since R.H.S of ⓐ and ⓑ are same ∴ LHS are also the same

Hence

Range(P) = Range(A)

Alternative method for Q9

(9) Given

$A \to m\times n$ matrix
column rank $= n$

To Prove

$A(A^TA)^{-1}A^T$ is orthogonal projector onto range (A)

(A^TA) is an invertible matrix {Because A has full column rank}

$\Rightarrow A^TAx$, assume $Ax = b \in \mathbb{R}^m$

$\Rightarrow A^TAx = A^Tb$

Orthogonal Projector means $P^2 = P$ and $P = P^T$ $\{P = A(A^TA)A^T\}$

$P = P^T$ [To Show]

$\Rightarrow (A(A^TA)^{-1}A^T)^T$

$(A^T)^T((A^TA)^{-1})^TA^T$

$A((A^TA)^T)^{-1}A^T$

$\Rightarrow A(A^TA)^{-1}A^{-1}$

$\Rightarrow P \to$ (I)

$P^2 = P$ [To Show]

$\Rightarrow A(A^TA)^{-1}A^TA(A^TA)^{-1}A$

$\Rightarrow A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A$

$\Rightarrow A(A^TA)^{-1}A$

$\Rightarrow P \to$ (II)

From (I) and (II)

$\therefore$ P is an orthogonal projector

Now, to show that P projects on range(A)

$P = A(A^TA)^{-1}A^T$

$Px = A(A^TA)^{-1}A^Tx$ for some x

$\Rightarrow A\left((A^TA)^{-1}A^Tx\right)$

$\Rightarrow Ay$ for some y where $y = (A^TA)^{-1}A^Tx$

$\therefore$ P is acting on some vector x and Px transforms into the form Ay hence the projection is onto the range of $A \rightarrow$ (vi)

⑨ [Part 2]

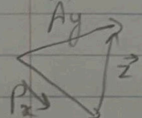
We have shown that $Px \in Range(A)$ for some vector $x \in \mathbb{R}^m$ {P is a m×m matrix}

Now to show that $Ay \in Range(P)$ for some $y \in \mathbb{R}^n$

Assume some projection of vector Ay exists of the form Px.

{If $Ay \in Range(P)$ then this projection will be ~~its~~ equal to itself} → Our hint!

↳ (IV)



$\vec{z}$ is orthogonal to $\vec{Px}$, z is actually orthogonal to $Range(P)$

$\vec{z}$ is $\vec{Ay} - \vec{Px}$

$\Rightarrow (Ay - Px)^T P = 0$

$(Ay - A(A^TA)^{-1}A^Tx)^T A(A^TA)^{-1}A^T = 0$

$(A(y - (A^TA)^{-1}A^Tx))^T A(A^TA)^{-1}A^T = 0$

$(y - (A^TA)^{-1}A^Tx)^T \underbrace{A^TA(A^TA)^{-1}}_{I} A^T = 0$

$(y - (A^TA)^{-1}A^Tx)^T A^T = 0$

$(A(y - (A^TA)^{-1}A^Tx))^T = 0$

$\therefore Ay = A(A^TA)^{-1}A^Tx$

Hence condition (IV) holds true

⇓

(V)

→ ∴ From (V) and (VI)

$Rang(P) = Range(A)$

HENCE PROVED

(10) Given

$A = U\Sigma V^T$ [SVD]

Eckart-Young Theorem

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

To Find

Singular Values in decreasing order of $A - A_k$

$X = (A - A_k)(A - A_k)^T$

Singular values are the sq. root of +ve eigenvalues of X

$$A - A_k = \sigma_{k+1} u_{k+1} v_{k+1}^T \ldots \sigma_r u_r v_r^T$$

$$\Rightarrow \sum_{i=k+1}^{r} \sigma_i u_i v_i^T \rightarrow (11)$$

So from (11), we can clearly infer that singular values are $\{\sigma_{k+1}, \ldots \sigma_r\}$ in the order written

(11) (a) $A_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SVD

$$A = U\Sigma V^T \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Largest Singular Value = 3

$$\therefore A_1 = \sigma_1 u_1 v_1^T \Rightarrow 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $A_2 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$

SVD

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow U \Sigma V^T$$

Largest Singular Value is 3

$$\therefore A_1 = \sigma_1 u_1 v_1^T \Rightarrow 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$

SVD

$$A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \Rightarrow U \Sigma V^T$$

Largest Singular Value is $3\sqrt{5}$

$$\therefore A_1 = \sigma_1 u_1 v_1^T \Rightarrow 3\sqrt{5} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 & 9/2 \\ 3/2 & 9/2 \end{bmatrix}$$

,