Iterative Methods (vs. direct)

LU $(Ax=b)$
QR
LL^T

$(Ax=b)$

- eigenvalue
  "spectral decomposition"
  $A = Q\Lambda Q^T$
- SVD $A = U\Sigma V^T$
- $Ax = b$
  - Jacobi ✓
  - Gauss-Seidel ✓

  Conjugate Gradients
  Krylov Subspace Methods

$$A = M - N$$

$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} = \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix} - \begin{pmatrix} 0 & -x & x \\ -\cdot & 0 & \cdot \\ -\cdot & \cdot & 0 \end{pmatrix}$$

$$\boxed{Ax = b}$$

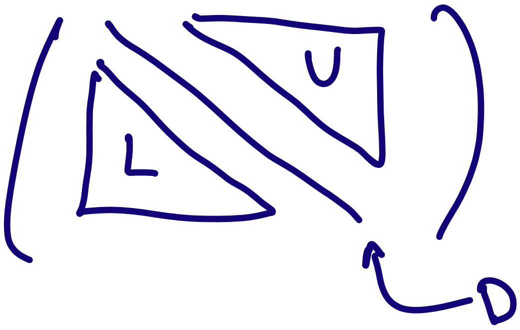
$$(M-N)x = b$$

$$\boxed{Mx = Nx + b}$$

$$\overset{x_0}{\underline{M}}\, \underline{x_{k+1}} = N \underline{x_k} + b$$

<u>Jacobi</u> $\quad M = D, \quad N = -(L+U)$

$$A = L + D + U$$



$$\underline{D}\, x_{k+1} = -\underline{(L+U)}\, \underline{x_k} + \underline{b}$$

easy to parallelize

<u>Gauss-Seidel</u>

$$M = D+L, \quad N = -U$$

$$(D+L)x_{k+1} = -U x_k + b$$

Convergence?

$$
\begin{aligned}
&\left(M x_{k+1} = N x_k + b\right) \\
-&\left(M x \phantom{{}_{k+1}} = N x + b\right) \\
\hline
&M\underbrace{(x_{k+1} - x)}_{e_{k+1}} = N\underbrace{(x_k - x)}_{e_k}
\end{aligned}
$$

$$M e_{k+1} = N e_k$$

converge $\iff$ $e_k \to 0$ as $k \to \infty$

$$e_{k+1} = M^{-1} N e_k$$

$$\underline{\| e_{k+1} \|} = \| M^{-1} N e_k \|$$

$$\leq \underbrace{\| M^{-1} N \|} \; \underline{\| e_k \|}$$

convergence when $\|M^{-1}N\| < 1$

Convergence Rate

$$\text{cost} = \boxed{\frac{\text{cost}}{\text{iter}}} \cdot \boxed{\#\text{ iter}}$$

conv. rate is r if

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C$$

$$\|e_{k+1}\| = 2\,\|e_k\|^2$$

$.0001$ $\quad 10^{-4}$

$.01$ $\quad 10^{-2}$

10^{-2}, $2\cdot 10^{-4}$, $4\,10^{-8}$

$.00000001$ $\quad 10^{-8}$

$$\|e_{k+1}\| = \boxed{.5}\,\|e_k\|$$

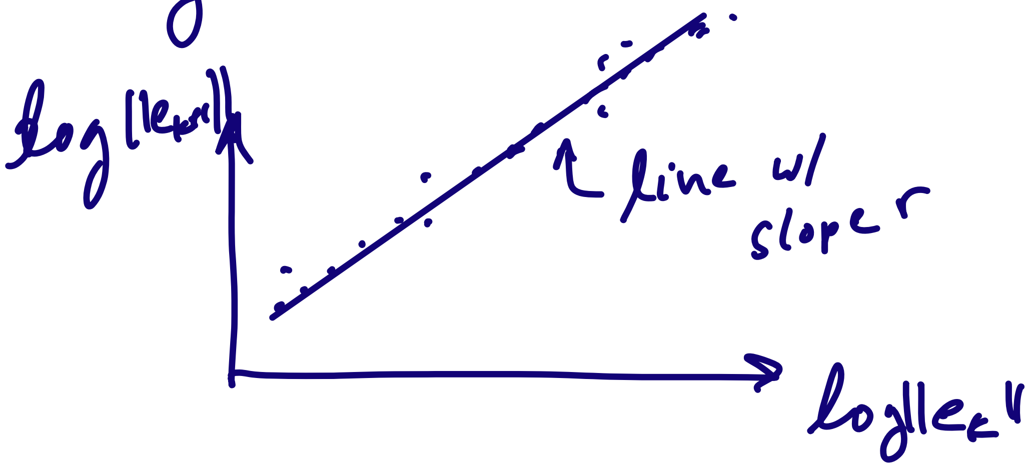
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots,$

- $r = 1$, $c < 1$ linear
- $r > 1$ super linear
- $r = 2$ quadratic
- $r = 3$ cubic

$\vdots$

$$\lim_{k\to\infty} \|e_{k+1}\| = C\|e_k\|^{r}$$

$$\log\|e_{k+1}\| = r\left(\boxed{\log\|e_k\|}\right) + \log C$$



$$\log\left(\|e_1\|, \|e_2\|, \ldots \|e_9\|, \cancel{\|e_{10}\|}\right)$$

$$\log\left(\|e_2\|, \|e_3\|, \ldots \|e_{10}\|\right)$$

Stopping Criteria

$$M x_{k+1} = N x_k + b$$

$$r_k = \underset{\substack{\shortparallel \\ Ax}}{\underline{b}} - A x_k = Ax - Ax_k$$

$$\frac{\| x_{k+1} - x_k \|}{\| x_k \|}$$

$$r_k = A(x - x_k)$$

$$r_k = A \underline{e_k}$$

Eigenvalue Problems

normalized power iteration

x_0

for $k = 1, 2, \ldots$

$$y_k = A x_{k-1}$$

$$x_k = y_k / \|y_k\|$$

end

conv. depends on $\dfrac{|\lambda_2|}{|\lambda_1|}$

$$A^k x_0 = \underline{\lambda_1^k} \left(\underline{\alpha_1 \vec{v}_1} + \alpha_2 \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^k \vec{v}_2 + \cdots + \alpha_n \left(\frac{|\lambda_n|}{|\lambda_1|}\right)^k \vec{v}_n \right)$$

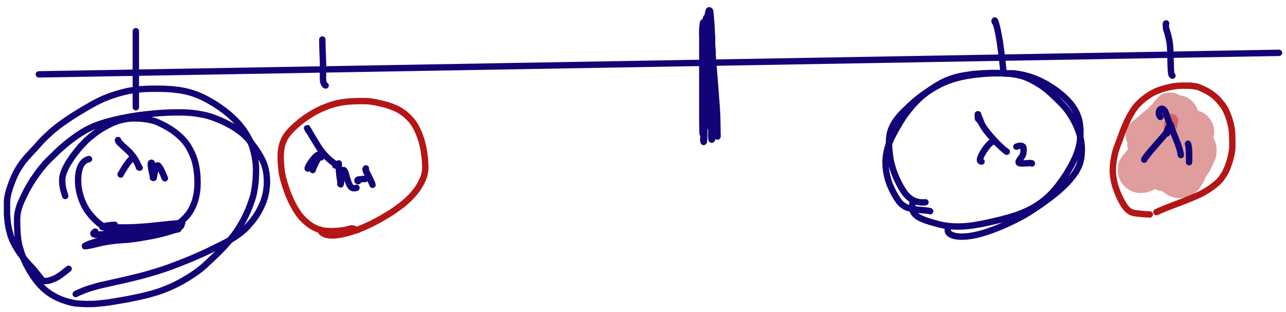
linear conv. w/

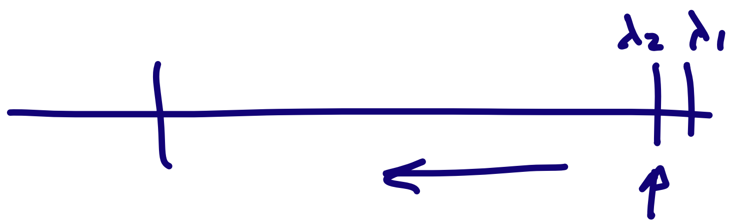
$$c = \frac{|\lambda_2|}{|\lambda_1|} < 1$$

shift to make $\frac{|\lambda_2|}{|\lambda_1|}$ as small as possible

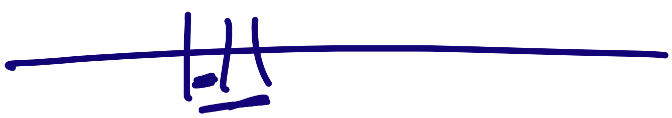
$$Av = \lambda v$$

$$\Rightarrow (A + sI)v = (\lambda + s)v$$





$\frac{\lambda_2}{\lambda_1} \approx .9$

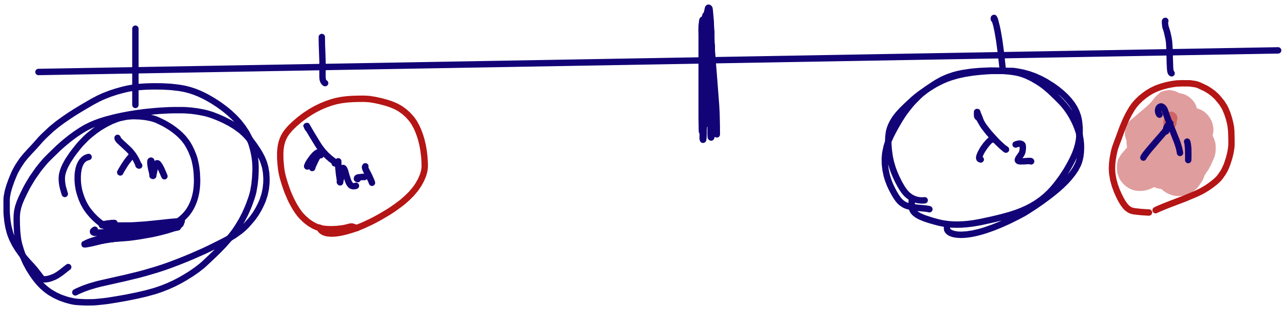


$\frac{\lambda_2}{\lambda_1} \approx .5$

to make $\frac{|\lambda_2|}{|\lambda_1|}$ small as possible:

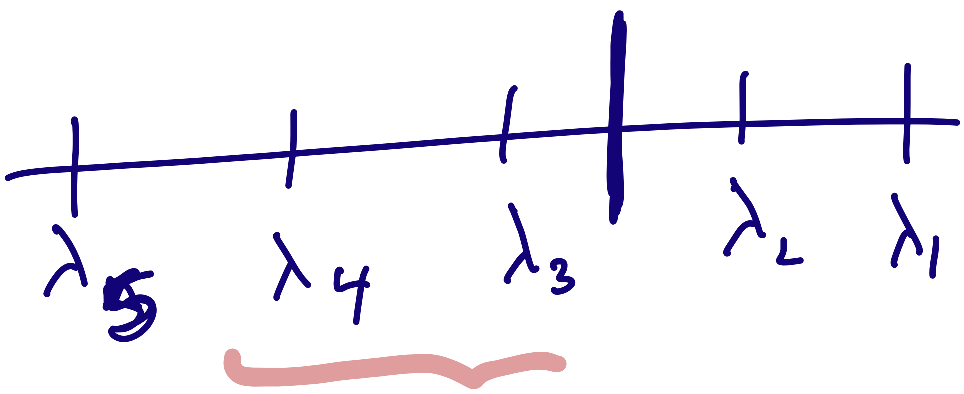
$$s = \frac{\lambda_2 + \lambda_n}{2}$$

Recover v_n by shifting to make $\lambda_n + s$ largest in magnitude.



$$s = \frac{\lambda_1 + \lambda_{n-1}}{2}$$

power method can recover the two eigenvectors associated w/ the extreme eigenvalue ($\vec{v}_1$ or $\vec{v}_n$)



Inverse Iteration

$$Av = \lambda v$$

$$A^{-1} v = \frac{1}{\lambda} v$$

algorithm:

x_0

for $k = 1, 2, \ldots$

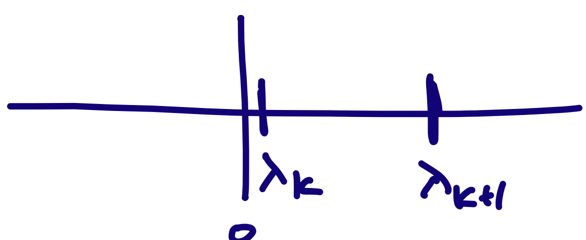
Solve: $A y_{k+1} = x_k$ $\quad (A^{-1} x_k = y_{k+1})$

$x_{k+1} = y_{k+1} / \| y_{k+1} \|$

end

- precompute $A = LU$
- useful if estimate of λ is available $\approx s$

$$A - sI$$



$$\frac{\left|\frac{1}{\lambda_{k+1}}\right|}{\left|\frac{1}{\lambda_k}\right|} \qquad \frac{|\lambda_k|}{|\lambda_{k+1}|} = C$$

Rayleigh Quotient Iteration

x approximate eigenvector

$$Ax \approx \lambda x$$

$$x\lambda \approx Ax$$

$$\boxed{Ax \approx b}$$ L.S.

normal equation

$$x^T x \lambda = x^T A x$$

$$\boxed{\lambda = \frac{x^T A x}{x^T x}}$$ Rayleigh Quotient

$$\boxed{Ax \approx b}$$
$$\boxed{A^T A x = A^T b}$$

Rayleigh Quotient Iteration

x_0

for $k = 1, 2, \ldots$

$$\sigma_{k+1} = \frac{x_k^T A x_k}{x_k^T x_k}$$

accelerate convergence

(inverse power iter w/ shifted matrix)

Solve $(A - \sigma_{k+1} I) y_{k+1} = x_k$

$x_{k+1} = y_{k+1} / \|y_{k+1}\|$

end

$$\text{cost} = \frac{\text{cost}}{\text{iter}} \; \# \text{iter}$$

$10^{-2}, 10^{-3}, 10^{-4}, \ldots, \quad 10^{-9}$

$10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$