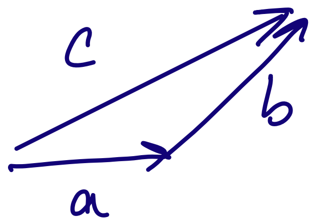
$$\vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_i b_i \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$\vec{a}, \vec{b} \in \mathbb{R}^n$

$$\left(\mathbb{C}^n \quad \vec{a}\cdot\vec{b} = \sum_{i=1}^{n} \overline{a_i}\, b_i \right)$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\| \cos\theta$$

Ex.



$$c = a + b$$

$$\|c\|^2 = c\cdot c = (a+b)\cdot(a+b)$$
$$= a\cdot a + \underbrace{a\cdot b + b\cdot a} + b\cdot b$$
$$= \|a\|^2 + 2\|a\|\,\|b\|\cos\theta + \|b\|^2$$

---

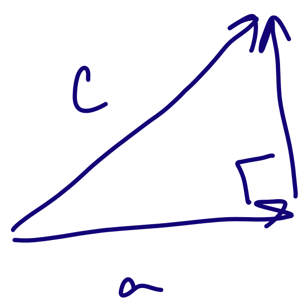
for $a, b \in \mathbb{R}^n$ $\quad a\cdot b = b\cdot a$

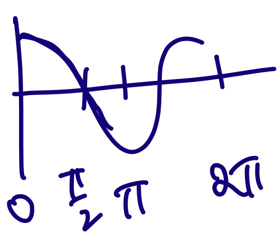
$a, b \in \mathbb{C}^n$ $\quad a\cdot b = \overline{(b\cdot a)}$

$$\boxed{\|c\|^2 = \|a\|^2 + \|b\|^2 + 2\|a\|\,\|b\|\cos\theta}$$

if $\vec{a} \perp \vec{b} \Rightarrow \theta = 90^\circ = \frac{\pi}{2}$

$$\|c\|^2 = \|a\|^2 + \|b\|^2$$



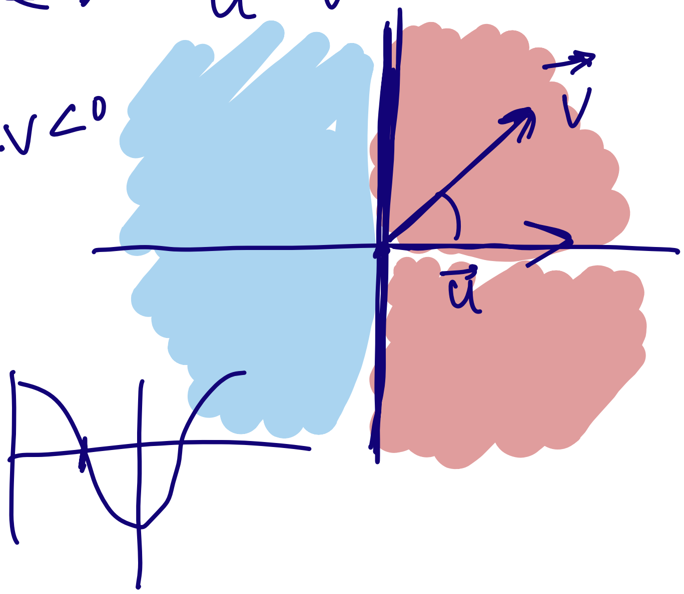


vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are "orthogonal"

$\iff \quad \vec{u} \cdot \vec{v} = 0$

$$\vec{u} \cdot \vec{v} = \underbrace{\|u\|}\,\underbrace{\|v\|}\cos\theta$$

$u \cdot v < 0$



$u \cdot v > 0$

Linearity

linear map
function
operator

$L : V \to V'$ is linear

$$L[\vec{u} + \vec{v}] = L[u] + L[v]$$

$$L[c\vec{u}] = c\,L[\vec{u}]$$

Ex.

$f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x \\ 2x+y \\ -y \end{pmatrix}$$

$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$

check linear?

$$f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 3(x_1 + x_2) \\ 2(x_1 + x_2) + (y_1 + y_2) \\ -(y_1 + y_2) \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 + 3x_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \\ -y_1 - y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 \\ 2x_1 + y_1 \\ -y_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ 2x_2 + y_2 \\ -y_2 \end{pmatrix}$$

$$= f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + f\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

$$f\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = \quad \cdots \quad = c\, f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \checkmark$$

linear operator $\mathcal{L} : V \to V'$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx}\left(c f(x)\right) = c \frac{d}{dx} f(x)$$

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underline{\underline{xy}}$$

$$g\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} cx \\ cy \end{pmatrix}\right) = (cx)(cy) = c^2xy$$

$$\neq$$

$$c\, g\begin{pmatrix} x \\ y \end{pmatrix} = c\,xy$$

$$\vec{a} = \sum_{i=1}^{n} c_i \vec{u}_i$$

basis vectors $\{\vec{u}_i \mid i = 1, \ldots, n\}$

$$\mathcal{L}(\vec{a}) = \mathcal{L}\left(\sum \quad\right)$$

$$= \sum_{i=1}^{n} c_i \mathcal{L}(\vec{u}_i)$$

linear op is determined by its action on the basis vectors

<u>matrix - vector multiplication:</u>

$$\begin{bmatrix} | & | & & | \\ \mathcal{L}(\vec{u}_1) & \mathcal{L}(\vec{u}_2) & \cdots & \mathcal{L}(\vec{u}_n) \\ | & | & & | \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \sum_{i=1}^{n} c_i \mathcal{L}(\vec{u}_i)$$

Matrices

"column view"

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

$M \in \mathbb{R}^{m \times n}$

$m \times n$ matrix

m rows

n columns

$$\begin{pmatrix} \text{—} & u_1^T & \text{—} \\ \text{—} & u_2^T & \text{—} \\ & \vdots & \\ \text{—} & u_m^T & \text{—} \end{pmatrix}$$

$1 \times n$

block matrices

$k \times p$ blocks

more generally ...

$$\left\{ \begin{pmatrix} (\cdot) & (\cdot) \\ (\cdot) & (\;) \end{pmatrix} \right.$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & .1 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ .1 \end{pmatrix} \right) \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} (1 \;\; 2) \\ (-1 \;\; 0) \\ (3 \;\; .1) \end{pmatrix} = \begin{pmatrix} \text{—} & u_1^T & \text{—} \\ \text{—} & u_2^T & \text{—} \\ \text{—} & u_3^T & \text{—} \end{pmatrix}$$

Identity Matrix

$\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ row

$I\vec{v} = \vec{v}$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} - & e_1^T & - \\ & \vdots & \\ - & e_n^T & - \end{pmatrix}$$

Matrix-vector mult

$$\underset{m \times \underline{n}}{M} \; \underset{\underline{n} \times 1}{\vec{v}} = \underset{m \times 1}{\vec{p}}$$

$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}$

$$\boxed{p_i = \sum_{j=1}^{n} M_{ij} \vec{v}_j}$$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} m_{11}v_1 + m_{12}v_2 + m_{13}v_3 \\ m_{21}v_1 + m_{22}v_2 + m_{23}v_3 \end{pmatrix}$$

$$= v_1 \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} + v_2 \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} + v_3 \begin{pmatrix} m_{13} \\ m_{23} \end{pmatrix}$$

$$= \begin{pmatrix} v_1 m_{11} + v_2 m_{12} + v_3 m_{13} \\ v_1 m_{21} + v_2 m_{22} + v_3 m_{23} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Matrix transpose

$$A \in \mathbb{R}^{m \times n}$$

transpose of A : $A^T \in \mathbb{R}^{n \times m}$

$$(A^T)_{ij} = A_{ji}$$

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 0 \\ 3 & -1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \end{pmatrix}$$

$A \in \mathbb{C}^{m \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad A^H = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \end{pmatrix}$$

$$\vec{a} \cdot \vec{b} = \underbrace{\vec{a}}^T \vec{b} = (a_1 \cdots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\vec{a} \in \mathbb{R}^n$, $\vec{a} \in \mathbb{R}^{n \times 1}$

$1 \times n \qquad n \times 1$

$$\boxed{\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \sum_{i=1}^{n} a_i b_i}$$

Identities

$$(A+B)^T = A^T + B^T$$

$$(A^T)^T = A$$

$$(AB)^T = B^T A^T$$

$$C = AB$$

$$\uparrow$$

$$C\vec{x} = AB\vec{x}$$

$$C^T = B^T A^T$$

$$\underset{1\times m}{\vec{x}^T}\ \underset{m\times n}{A}\ \underset{n\times 1}{\vec{b}} = \underset{1\times 1}{\boxed{}}$$

$$(x^T A b)^T = (b^T A^T x)$$

"columnspace" of a matrix $A \in \mathbb{R}^{m\times n}$

$$\text{col}(A) = \text{span}(\vec{a}_1, \dots, \vec{a}_n)$$

$$= \left\{ \vec{v} \,\middle|\, \vec{v} = \sum_{i=1}^{n} c_i \vec{a}_i \right\}$$

$$A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$$

$$\vec{y} = A\vec{x}$$

$$\vec{y} \in \text{col}(A)$$

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row(A) = span rows of A
