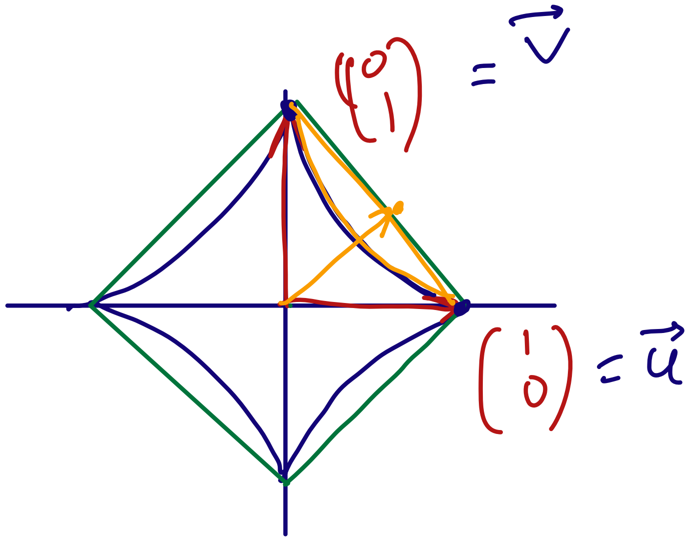


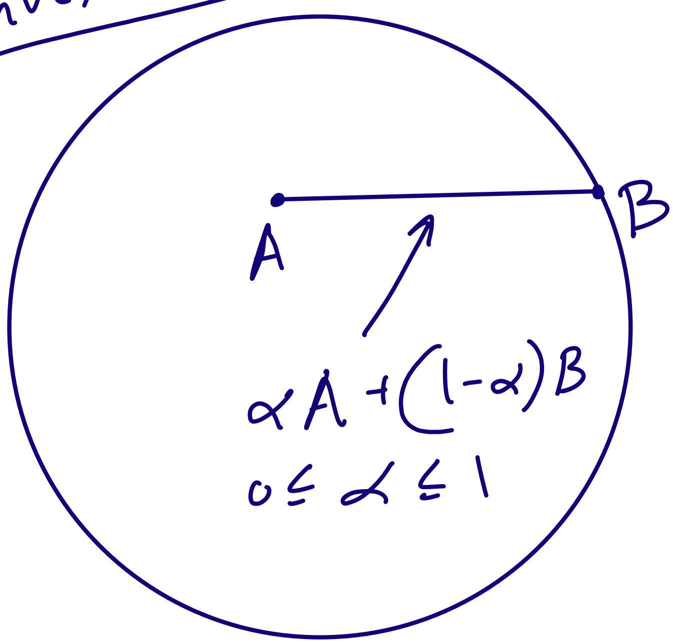
$p=\frac{1}{2}$

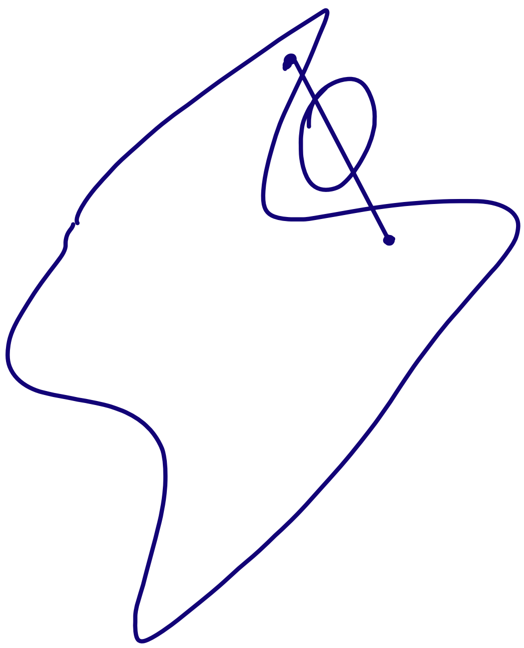
not a norm

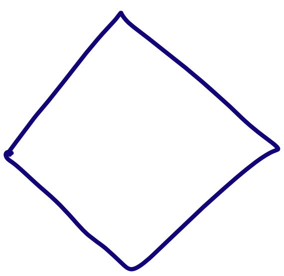
convex combinations $\underbrace{0\le\alpha\le1}$

$$\underbrace{\|\alpha\vec u+(1-\alpha)\vec v\|}_{>1}\le\|\alpha\vec u\|+\|(1-\alpha)\vec v\|$$
$$=\alpha\|\vec u\|+(1-\alpha)\|\vec v\|$$
$$=\alpha\cdot1+(1-\alpha)\cdot1$$
$$=\alpha+1-\alpha=1$$

convex set







important prop. of 2-norm

① $\vec{v} \cdot \vec{v} = \vec{v}^T \vec{v} = \|v\|_2^2$

$\|v\|_2 = \sqrt{v \cdot v}$

② $\boxed{v^T u = \underbrace{\|v\|_2 \|u\|_2}\ \underbrace{\cos\theta}}$

Cauchy-Schwartz inequality

$|v^T u| \leq \|v\|_2 \|u\|_2$

1-norm $\quad \|\cdot\|_1 \quad \|\cdot\|_2 \quad \|\cdot\|_\infty$

S-norm $\|\cdot\|_S$

matrix S that is s.p.d

symmetric ~~positive~~ definite :

$\boxed{\text{for } x \neq 0, \ x^T S x > 0}$

For any $\vec{v}$,

$\boxed{\|\vec{v}\|_S = (v^T S v)^{1/2}}$

<u>Cholesky factorization</u> of S

$S = L L^T$

$$\boxed{v^T S v} = v^T (L L^T) v$$

$$= \underline{(v^T L)}\,\underline{(L^T v)}$$

$$\langle\, y = L^T v \,\rangle \Rightarrow y^T = v^T L$$

$$= y^T y > 0 \quad \text{for } y \neq 0$$

if $v \neq 0$, then $y \neq 0$

Matrix norms $\quad A$

$\|A\|$

1. $\|A\| > 0 \quad$ for $A \neq 0$

   $\|A\| = 0 \quad$ for $A = 0$

2. $\|\alpha A\| = |\alpha|\,\|A\|$

3. $\|A + B\| \leq \|A\| + \|B\|$

④ For p-norms, & Frobenius norm

$$\| AB \| \leq \| A \| \| B \|$$

Frobenius norm:

$$\left\| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{pmatrix} \right\|_F = \Big[|a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2 + |a_{21}|^2 + \cdots + \cdots \cdots + |a_{nn}|^2 \Big]^{1/2}$$

$$\| A \|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

Matrix p norms:

"induced" norms

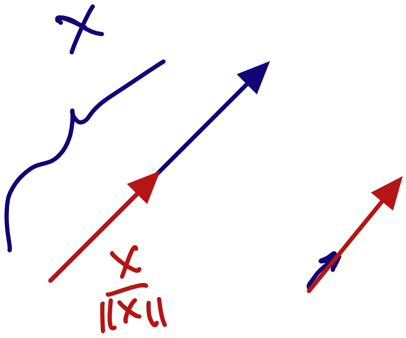
$$\| A \|_2 = \max_{\vec{x} \neq \vec{0}} \boxed{\frac{\| Ax \|_2}{\| x \|_2}}$$

amount of "stretch" under A

$$\frac{1}{\| x \|_2}$$

$$= \max_{\vec{x} \neq \vec{0}} \left\| \frac{1}{\|x\|_2} A x \right\|_2$$

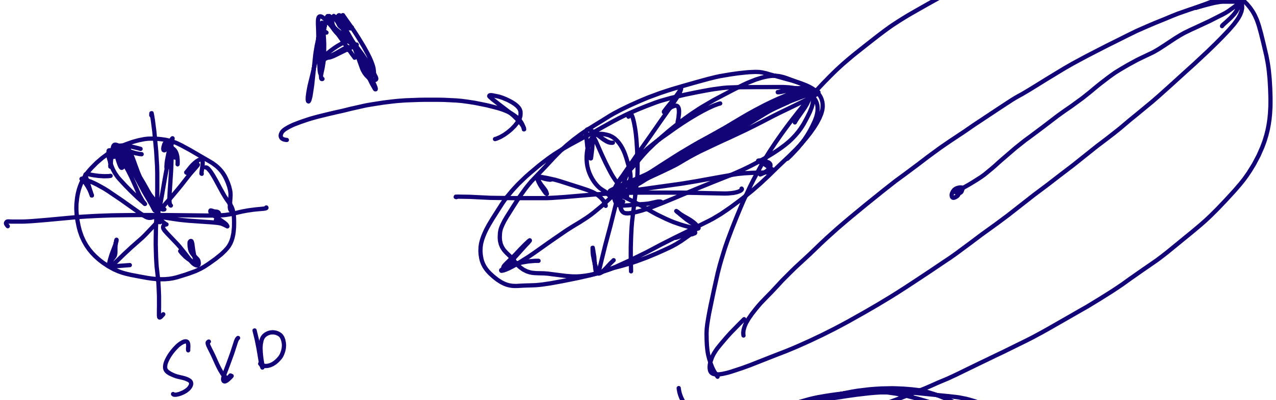
$$= \max_{\vec{x} \neq \vec{0}} \left\| A\left(\frac{x}{\|x\|_2}\right) \right\|_2$$



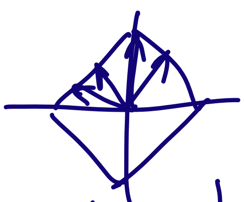
$$y = \frac{x}{\|x\|_2}$$

$$\|A\|_2 = \max_{\|y\|_2 = 1} \|Ay\|_2$$

$$\|y\|_2 = \frac{\|x\|_2}{\|x\|_2} = 1$$



$$\|A\|_1 = \max_{\|y\|_1 = 1} \boxed{\|Ay\|_1}$$

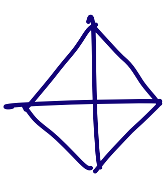


$$\overset{?}{=} \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|$$

$$\begin{array}{ccc} \multicolumn{3}{c}{\begin{pmatrix} 1 & -4 & -1 \\ 3 & 1 & -2 \\ 2 & 1 & 0 \end{pmatrix}} \\ 6 & 6 & 3 \end{array}$$

$$\|A\| = 6$$

$$\|\overset{z}{Ay}\|_1 = \sum_{i=1}^{n} |z_i|$$



$$y = \hat{e}_j \qquad Ay = a_j$$

$$\|Ay\|_1 = \|a_j\|$$

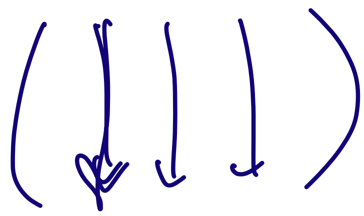
$$y_2 = \frac{1}{n}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \qquad Ay_2 = \begin{pmatrix} \frac{1}{n}(a_{11}+a_{12}+ \dots a_{1n}) \\ \vdots \\ \frac{1}{n}(a_{n1}+ \dots a_{nn}) \end{pmatrix}$$

---

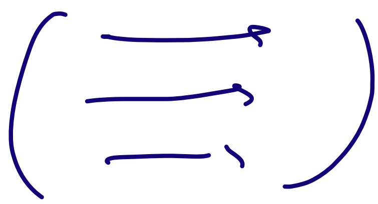
induced norm:

$$\|A\| = \max_{\|y\|=1} \|Ay\|$$

$$\boxed{\|A\|_1} = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|$$



$$\boxed{\|A\|_\infty} = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|$$



$$\|A\|_2 = \sigma_1$$

largest singular value (SVD)

$$A = \begin{array}{cc} \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix} & \begin{array}{c} 4 \\ 2 \\ 8 \end{array} \end{array}$$

$$\boxed{\begin{aligned} \|A\|_1 &= 6 \\ \|A\|_\infty &= 8 \end{aligned}}$$

6 2 6

$$\boxed{C}\|A\|_1 \le \underbrace{\|A\|_\infty} \le \boxed{D}\|A\|_1$$

C, D indep. of A
depend on n (dim)

$$\boxed{\frac{1}{D}}\|A\|_\infty \le \underline{\|A\|_1} \le \boxed{\frac{1}{C}}\|A\|_\infty$$

Ax