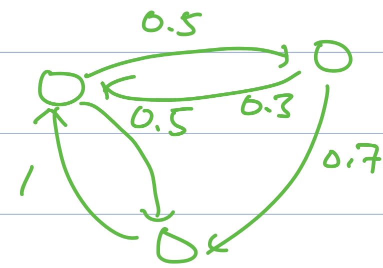
Markov Chains

(Finite) Markov Chains

allow self-loops

- Directed graph $G=(V,E)$ with edge weights $p: E \to \mathbb{R}_{\geq 0}$
- $\forall u \in V$, $\sum_{v \in V} p(u,v) = 1$.



Random walk on G.

- Start from $v_0 \in V$.
- Given $v_i \in V$, $\forall u \in V$, $\Pr[v_{i+1} = u] = p_{v_i u}$

intuitively, we choose a random edge out of v_i according to probabilities given by w.

distribution of v_{i+1} only depends on v_i! (not $v_0, \ldots, v_{i-1}$).

"Markovian".

Possible Questions:

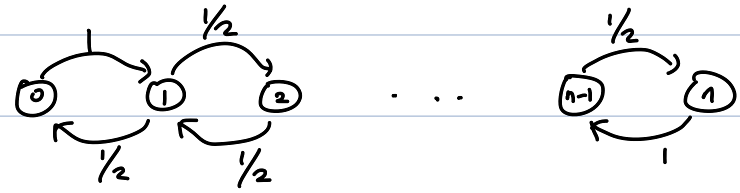
- For fixed $u, v \in V$, how long will random walk starting from u reach v (for the first time)? (this lecture)
- For fixed v_0, each v_i is a random vertex from some distribution P_i. Is there a fixed distribution P s.t. $P_i \approx P$ for all sufficiently large i? How fast can you reach P? (later, probably for undirected graphs)

Let $h_{u,v} = \mathbb{E}[\#\text{ steps random walk from } u \text{ reaches } v]$

Of course, $h_{u,u} = 0 \quad \forall u \in V$.

For $u \neq v \in V$, $h_{u,v} = 1 + \sum_{w \in V} p_{u,w} \cdot h_{w,v}$.

Example,



What is $h_i := h_{i,n}$?

- $h_n = 0$
- $i \in [n-1]$: $h_i = 1 + \frac{1}{2} h_{i-1} + \frac{1}{2} h_{i+1}$.
- $h_0 = 1 + h_1$.

Claim, $h_i = h_{i+1} + 2i + 1$ for $0 \leq i \leq n-1$.

Proof, True when $i=0$. If true for i,

$$h_{i+1} = 1 + \frac{1}{2} h_i + \frac{1}{2} h_{i+2} \Rightarrow h_{i+1} = 1 + \frac{1}{2}(h_{i+1} + 2i + 1) + \frac{1}{2} h_{i+2}$$

$$\Rightarrow h_{i+1} = h_{i+2} + 2(i+1) + 1$$

$\square$

So, $h_i = O(n^2)$ for all i!

2 SAT

k-CNF formula: Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ (and)
where each C_i is $\vee$ (or) of $\leq k$ literals.
(e.g. $\phi = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_3) \wedge \cdots$,
for simplicity, assume C_i has exactly k literals from
different variables)

k-SAT: Given k-CNF ϕ, is there an assignment
$\{x_1 \ldots x_n\} \to \{T, F\}$ that satisfies ϕ?

__Theorem.__ 3-SAT is NP-Hard!

We'll see a randomized (deterministic ones exist too) poly-time algo for 2SAT.

Algorithm.
$f \leftarrow$ arbitrary assignment.
While $\exists$ violated clause
  Choose such C_i (arbitrarily)
  Randomly choose one variable x in C_i (out of two)
  Flip $f(x)$ $\begin{cases} f(x) \leftarrow T & \text{if } f(x) = F \\ f(x) \leftarrow F & \text{o.w.} \end{cases}$
Output f.

Will output correct f.

What's $\mathbb{E}[\#\text{ iterations}]$?

Let f^* be a satisfying assignment.

For $j=0,1,\ldots$, let $f_j = f$ at (the end of) iteration j.

and $k_j = |\{i \in [n] : f_j(x_i) = f^*(x_i)\}|$.

We're done $k_j = n$.

If we're not done at j^{th} iteration, in $(j+1)^{th}$ iter,

we'll choose unsatisfied clause C and flip one variable.

Since f^* satisfies C and

f_j doesn't satisfy C, $\quad C = x_1 \vee \overline{x_2}$

① if f^* satisfies both literals of C. $\quad f^* = T, F$

$k_{j+1} = k_j + 1$. $\quad f_j = F, T$

② if f^* satisfies only one literal of C, $\quad f^* = F, F$

$$k_{j+1} = \begin{cases} k_j + 1 & \text{w.p. } 1/2 \\ k_j - 1 & \text{w.p. } 1/2. \end{cases} \quad f_j = F, T$$

Obviously, worst case is when ② happens whenever $k_j \in [1, n-1]$.

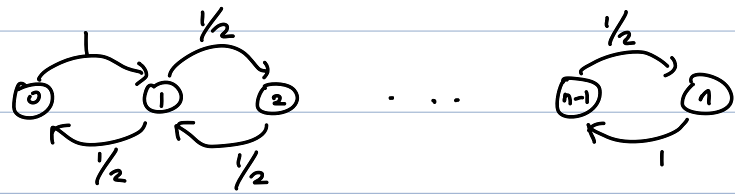
(formally, let $\{k'_j\}_{j=0,1,\ldots}$ be random vars s.t.

$k'_0 = k_0$, $(k'_j - k'_{j-1}) = (k_j - k_{j-1})$ for all j where ② happens,

and $k'_{j+1} = \begin{cases} k'_j + 1 & \text{w.p. } 1/2 \\ k'_j - 1 & \text{w.p. } 1/2 \end{cases}$ for all j where ① happens.

Then $k_j \geq k'_j$ for all j, so k_j reaches n no later than k'_j).

But this case is exactly random walk on



$\therefore$ For any starting f, $\mathbb{E}[\#\text{ iterations}] \leq O(n^2)$.