Optimization

f objective

$$\min_{x} f(\vec{x})$$

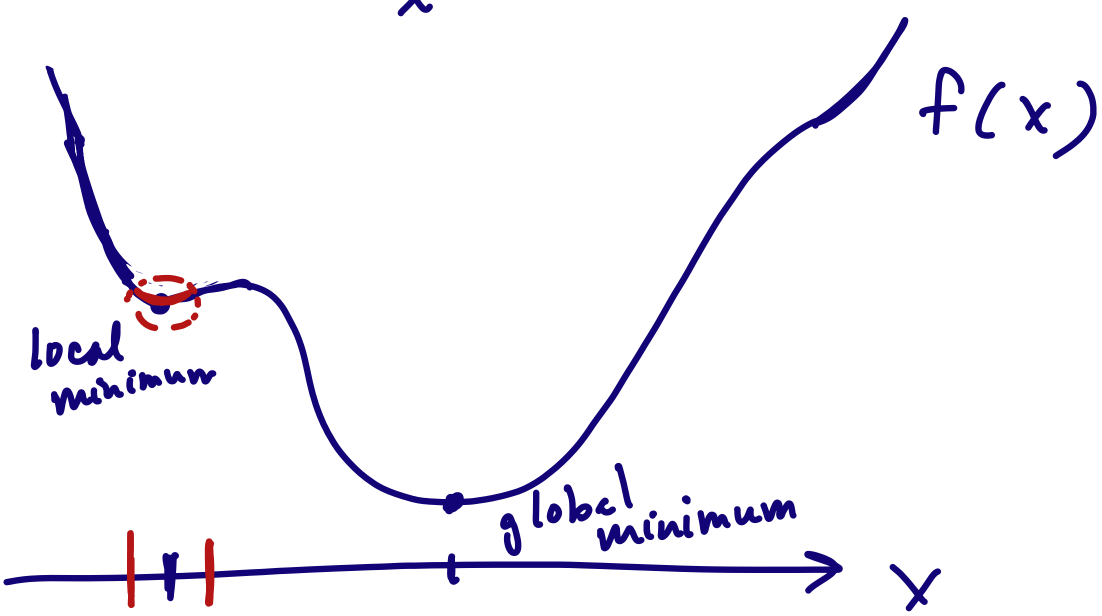
subject to $\begin{cases} \vec{g}(\vec{x}) = 0 & \text{equality constraint} \\ \vec{h}(\vec{x}) \le 0 & \text{inequality constraints} \end{cases}$

no constraints ...

" unconstrained optimization "

$$\min_{x} f(x)$$

$$x^* = \underset{x}{\operatorname{argmin}}\ f(x)$$



global min

$$f(x^*) \le f(x) \quad \forall \quad x \in \mathbb{R}^n$$

local

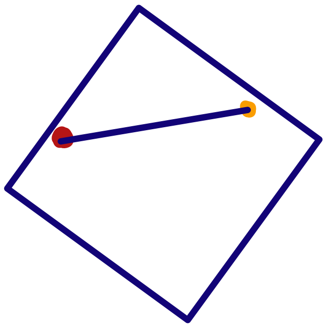
$$f(x^*) \le f(x) \quad \forall \quad x \in N(x^*)$$

neighborhood

Small Neigh

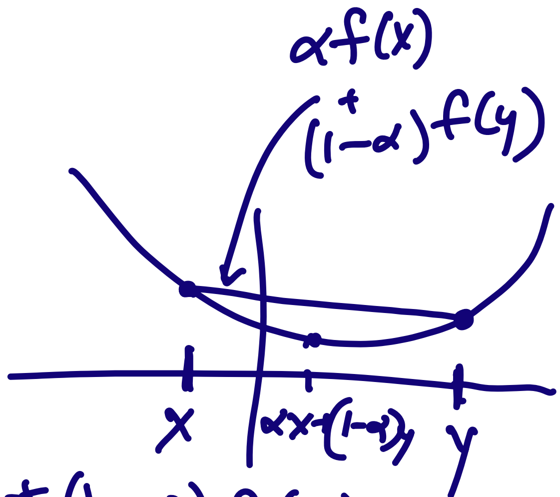
convexity

convex sets



$x, y \in S \Rightarrow$
$\alpha x + (1-\alpha) y \in S$

convex functions



$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$

Ex. $f(x) = \|x\|$

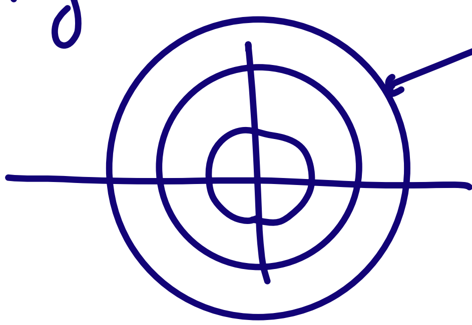
level set of f

iso contour of f

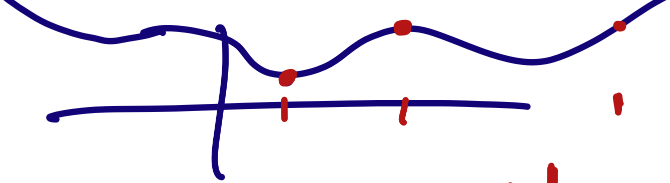
$$L_\gamma = \{ x \mid f(x) = \gamma \}$$

$f(x,y) = x^2 + y^2$

$f(x,y) = 3^2$



scalar case



$\min_x f(x)$

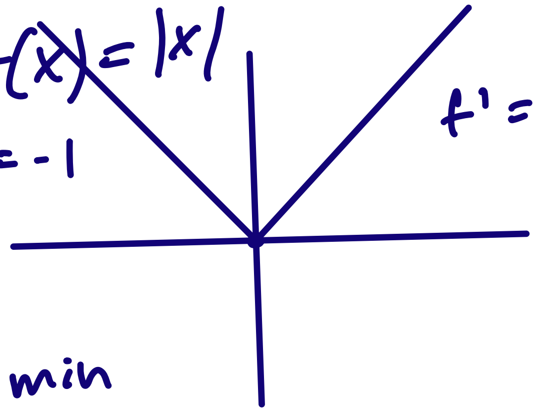
$f(x) = x^3$
$f'(x) = 3x^2$
$f'(0) = 0$

necessary cond. for a min.

$$f'(x) = 0$$

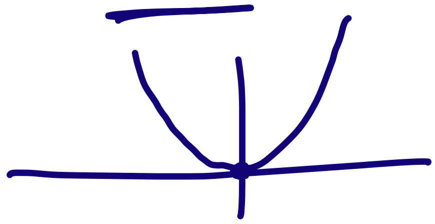
$f(x) = |x|$

$f' = -1$ $\quad$ $f' = 1$



+ sufficient cond

$$f''(x) > 0 \Rightarrow \text{min}$$

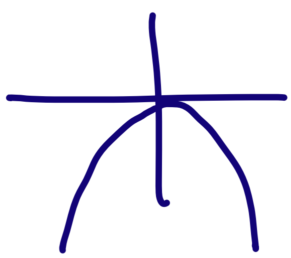


$f(x) = x^2$

$f'(x) = 2x \rightarrow f'(x) = 0 \Rightarrow \boxed{x = 0}$

$f''(x) = 2$ $\quad$ $\boxed{f''(0) = 2}$

$f(x) = -x^2$



$f'(x) = -2x$

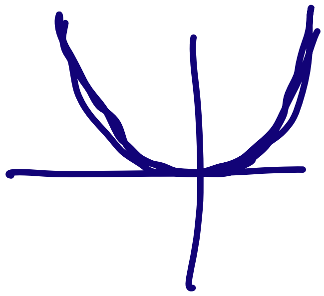
$f'(0) = 0$ ✓

$f''(x) = -2 < 0$ maximum

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$



minimizer $x=0$

$f'(0) = 0$ ✓

$f''(0) = 0$ ✗

$\min_{\vec{x}} f(\vec{x})$

"critical points"

1st order necessary cond:

$$\nabla f(\vec{x}) = \vec{0} \qquad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

2nd order sufficient cond

Hessian of f

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$\partial x_n \partial x_1$

if f is sufficiently smooth, H_f symmetric. $H_f = H_f^T$

eigenvalues of H_f

2nd order sufficient condition:

$$H_f(x) \succ 0$$

$\uparrow$ pos. def.

$H_f(x)$ pos. def:

$$\lambda_i > 0 \quad \forall\, i = 1, \dots, n$$

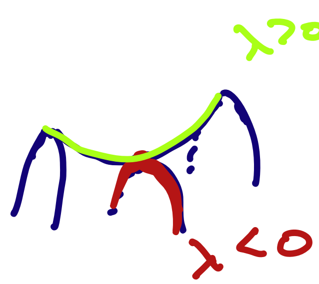
$$\vec{s}^T H_f(x) \vec{s} > 0 \quad \forall\, \vec{s} \neq 0$$

Classification critical pt. x:

$H_f(x) \succ 0 \Rightarrow$ min

$H_f(x) \prec 0 \Rightarrow$ max

$H_f(x)$ indef $\Rightarrow$ saddle



Taylor Series

$$f(\vec{x}+\boxed{\vec{s}}) = f(\vec{x}) + \boxed{\nabla f(\vec{x})^T \vec{s}} + \frac{1}{2}\vec{s}^T H_f(x)\vec{s}$$
$$+ O(\|s\|^3)$$

$$\nabla f(x) = 0$$

$$f(x+s) = f(x) + 0 + \frac{1}{2}\boxed{s^T \boxed{H_f(x)} s^T}$$
$$+ O(\|s\|^3)$$

maximum ascent

$$\begin{cases} \nabla f(x) & \text{direction} \\ -\nabla f(x) & \text{descent direction} \end{cases}$$

$$s = \alpha \vec{u}$$

θ = angle $(\nabla f, u)$

$$\nabla f(x)^T s = \alpha \underbrace{\nabla f(x)^T u}_{\|\nabla f(x)\| \cos\theta}$$

x^* $\quad \nabla f(x^*)$

test pos. def. of $H_f(x^*)$?

(1) find eigenvalues

(2) <u>Cholesky decomp.</u> $H_f(x^*)$

succeeds $\rightarrow$ pos def

fails $(\sqrt{-}\ ) \rightarrow$ not pos. def

LDL^T

Steepest Descent

$$f(\vec{x})$$

$$s = \underline{-\nabla f(x)}$$

$$\phi(\alpha) = f(x + \underline{\alpha}\vec{s})$$

x_0 = initial guess

for $k = 0, 1, 2, \ldots$

$$s_k = -\nabla f(x_k)$$

choose α_k

$$x_{k+1} = \underline{x_k + \alpha_k s_k}$$

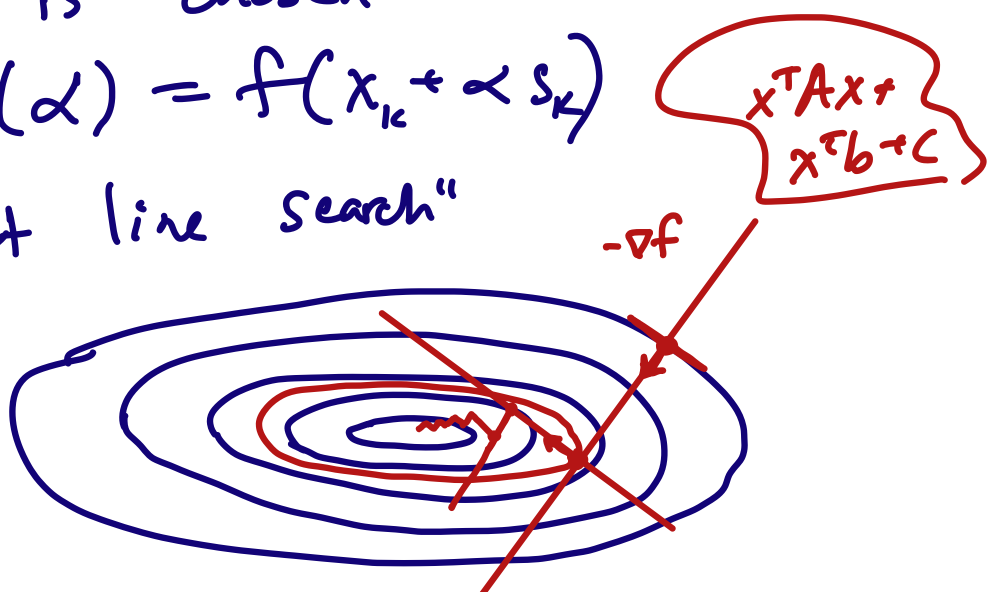
end

if α_k is chosen to minimize

$$\phi(\alpha) = f(x_k + \alpha s_k)$$

"exact line search"

Steepest Descent



Line Search approaches

Newton's Method for non. systems

$J_{\vec{f}}$

* $\boxed{\vec{f}(\vec{x}) = \vec{0}}$

$$\min_{\vec{x}} f(\vec{x}) \xrightarrow[\text{N.C.}]{\text{1st order}} \boxed{\nabla f(\vec{x}) = \vec{0}} \quad \boxed{\vec{g}(\vec{x}) = \vec{0}}$$

$$J_{\vec{g}} = H_f$$

$$X_{k+1} = X_k - \frac{f'(x_k)}{f''(x_k)}$$

Newton Dir:

$$\boxed{\underbrace{H_f(x)}\, s = -\nabla f(x)}$$

Steepest Desc.:

$$I\, s = -\nabla f(x)$$