Global Min-Cut

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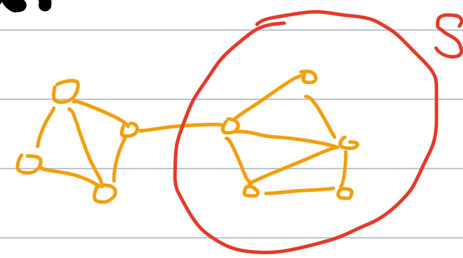
Input: Undirected graph $G=(V,E)$

Output: Set $\emptyset \subsetneq S \subsetneq V$.

Goal: Minimize $|\partial S|$



$\{(u,v)\in E : u\in S, v\notin S\}$

Poly-time algo: Choose an arbitrary $s\in V$. For every $t\in V\setminus\{s\}$, find the min s-t cut (Lec [redacted]). Output the cheapest one.
14-15

Karger

Input: $G_n=(V_n,E_n)$ $(n=|V|)$

Output: Global min-cut $S\subseteq V$.

Let $G=(V,E)$ be G_n.

While $|V|>2$.

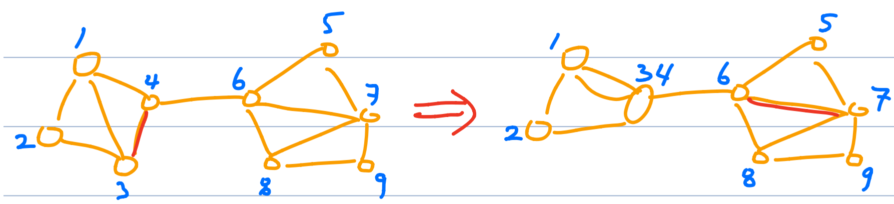
  Sample e from E.

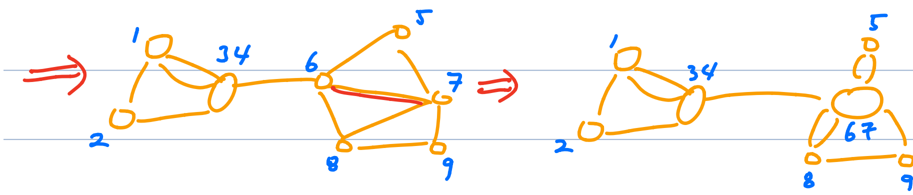
  "Contract" e.

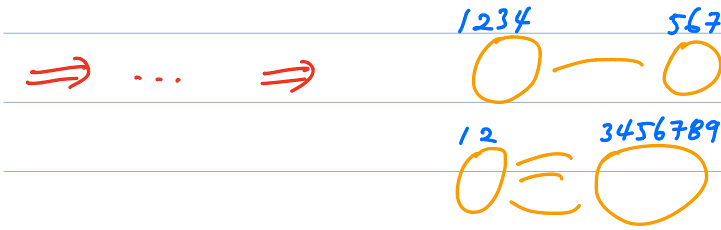
* merge two endpoints
* keep parallel edges
* remove self-loops

End.

Let $V=\{u,v\}$. Output $S\subseteq V_n$ contracted to u.







CORRECT!

INCORRECT!

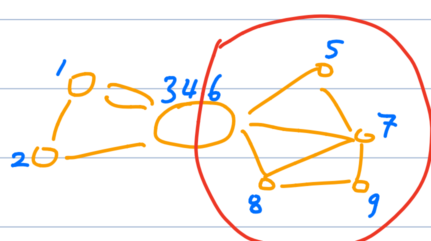
Pr[CORRECT=1]?

with probability

Theorem (Karger '94). Let $G_n=(V_n,E_n)$ $(n=|V_n|)$ be a graph and $S\subseteq V_n$ be a global mincut. Then **Karger** outputs S or $\bar{S}$ w.p. at least $1/\binom{n}{2}$.

Proof. G changes. Let $G_i=(V_i,E_i)$ be G when it had i vertices. For each $v\in V_i$, v is formed by contracting one or more vertices of V_n. Let $P(v)\subseteq V_n$ be the vertices contracted to v.

Say S is **destroyed** in G_i if $\exists v\in V_i$ s.t. $P(v)\not\subseteq S$ but $P(v)\cap S\neq\emptyset$.



(**alive** otherwise $\iff \forall v\in V_i$, $P(v)\subseteq S$ or $P(v)\cap S=\emptyset$)

If S is destroyed in G_i,

S is destroyed in $G_{i-1},\dots,G_2$.

CORRECT if and only if S is not destroyed in G_2!

Q. If S is alive in G_i, what is the prob. s.t. S is alive in G_{i-1}?

$\{(u,v) \in E_i : P(u) \subseteq S, P(v) \not\subseteq S\}$

When $G = G_i$,

$$\Pr[S \text{ destroyed in } G_{i-1}] = \Pr[\text{sampled } e \text{ belongs to } \partial_i S] = \frac{|\partial_i S|}{|E_i|}.$$

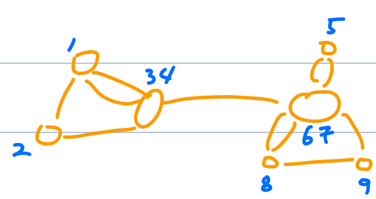
① Since S is alive in G_i, $|\partial_n S| = |\partial_i S|$

② $\forall v \in V_i$, $(\text{degree}(v) \text{ in } G_i) = |\partial_n P(v)|$
$\geq |\partial_n S| = |\partial_i S|$.

③ $|E_i| = \frac{1}{2}\sum_{v \in V_i} (\text{degree}(v) \text{ in } G_i) \geq \frac{i}{2} \cdot |\partial_i S|$.

$\therefore \Pr[S \text{ destroyed in } G_{i-1} \mid S \text{ alive in } G_i] \leq 2/i$.

conditional probability

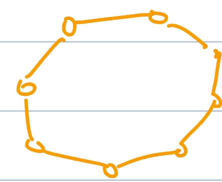


$|E_i| = 11$, $|E_i(S, V_i \setminus S)| = 1$

$\Pr[S \text{ alive in } G_2]$
$= \Pr[S \text{ alive in } G_{n-1}] \cdot \Pr[S \text{ alive in } G_{n-2} \mid S \text{ alive in } G_{n-1}]$
$\cdots \cdot \Pr[S \text{ alive in } G_2 \mid S \text{ alive in } G_3]$
$\geq (1-\frac{2}{n}) \cdot (1-\frac{2}{n-1}) \cdot \cdots \cdot (1-\frac{2}{3})$
$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \cdots \cdot \frac{2}{4} \cdot \frac{1}{3}$
$= \frac{2}{n(n-1)}$. $\square$

Corollary. For any graph with n vertices, there are at most $\binom{n}{2}$ global mincuts (counting S and $\bar{S}$ as one).

Proof. The above theorem is for "any mincut". The probabilities sum to 1. $\square$



$\binom{n}{2}$ mincuts

$\Pr[\text{CORRECT}=1]$ is only $\geq 1/n^2$!

Run T times "independently," and output the best solution.

$$\Pr[\text{INCORRECT overall}] = \Pr[\text{INCORRECT for each of } T \text{ tries}]$$

independence $\rightarrow$

$$\leq \left(1 - 1/n^2\right)^T$$

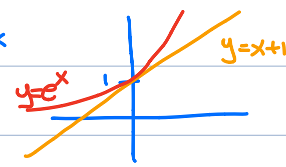
$$\leq e^{-T/n^2}$$

$$\leq e^{-\ln n} \text{ if } T \geq n^2 \ln n$$

$$= 1/n.$$

$(1-x) \leq e^{-x}$

$(1+x) \leq e^{x}$



Each try of **Karger**: Naively takes $O(nm)$ time

But, all T tries can run in time $O(n^2 \cdot \text{poly}(\log n))$ in total! [Karger–Stein 96].