Homework 5

Gram-Schmidt

1. Use the Gram-Schmidt algorithm to compute QR decompositions of the following matrices.

   (a) $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

2. Let $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$ be an orthonormal set in $\mathbb{R}^n$.

   (a) Show that
   $$P_1 = (I - q_1 q_1^T)(I - q_2 q_2^T)\ldots(I - q_k q_k^T)$$
   is a projection matrix.

   (b) Show that
   $$P_2 = I - q_1 q_1^T - q_2 q_2^T - \ldots - q_k q_k^T$$
   is a projection matrix.

   (c) Show that $P_1 = P_2$. Though mathematically equivalent, the first form of projection is more numerically stable and is used in modified Gram-Schmidt, whereas the second from is used in classical Gram-Schmidt.

In the next two problems, you will implement QR factorization by both the classical and the modified Gram-Schmidt algorithms, and study the instability of classical Gram-Schmidt. Note: You may do this assignment in Python if you prefer (in which case you should convert the code skeleton below to Python).

3. Write Matlab/Octave code to compute matrices Q and R such that $A = QR$ using the Gram-Schmidt process, or the modified Gram-Schmidt process, by filling in the function below. Include your code.

```
function [Q,R] = MyGS(A,modified)
% given a square real matrix A, compute:
%    an orthogonal matrix Q and
%    an upper triangle matrix R
% such that A = Q*R
% if modified == false, use the classical Gram-Schmidt process (known to be unstable)
% if modified == true,   use the modified Gram-Schmidt process

% TODO : write the function body

end
```

4. Study the instability of Classical Gram-Schmidt by running the function below.

```
function GSInstability(lo,hi)

mGS_orthogonality = [];
mGS_factorization = [];
cGS_orthogonality = [];
cGS_factorization = [];

for i = lo:hi
    A = hilb(i) + eye(i) * 1e-6;
    [Q,R] = MyGS(A,false);
    cGS_orthogonality(i-lo+1) = norm(Q'*Q-eye(i));
    cGS_factorization(i-lo+1) = norm(Q*R - A);
    [Q,R] = MyGS(A,true);
    mGS_orthogonality(i-lo+1) = norm(Q'*Q-eye(i));
    mGS_factorization(i-lo+1) = norm(Q*R - A);
end

figure
hold on
plot(lo:hi,log(cGS_orthogonality),'--k');
plot(lo:hi,log(mGS_orthogonality),'-b');
title('GS orthogonality');

figure
hold on
plot(lo:hi,cGS_factorization,'--k');
plot(lo:hi,mGS_factorization,'-b');
title('GS factorization');
end
```

(a) What is being plotted by the code?

(b) Include the plots generated.

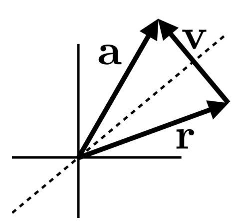
(c) Are Modified Gram-Schmidt and Classical Gram-Schmidt computing accurate factorizations? I.e., how close are A and $Q * R$? Explain based on the plot generated.

(d) Are Modified Gram-Schmidt and Classical Gram-Schmidt computing an orthogonal Q? Explain based on the plot generated.

Householder transformations

5. (Strang II.2 7) Find a Householder reflection matrix H such that $H \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \end{bmatrix}$, for some $r_1, r_2 \in \mathbb{R}$.

6. (Strang II.2 6) A Householder reflection matrix has the form $H = I - 2\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}$. Let $\mathbf{v} = \mathbf{a} - \mathbf{r}$, where $\|\mathbf{a}\|_2 = \|\mathbf{r}\|_2$, as illustrated in the figure. Confirm that $H\mathbf{a} = \mathbf{r}$. This shows how to construct a Householder reflection matrix that reflects one vector to another, as in the case of Householder QR, where $\mathbf{r} = \alpha\mathbf{e}_1$.



Singular Value Decomposition

7. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

8. Let A be an $m \times n$ singular matrix of rank r with SVD

$$A = U\Sigma V^T = \left(\begin{array}{c|c|c|c} & & & \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \\ & & & \end{array} \right) \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first r columns of U, $\tilde{U}$ consists of the remaining $m-r$ columns of U, $\hat{V}$ consists of the first r columns of V, and $\tilde{V}$ consists of the remaining $n-r$ columns of V. Give bases for the spaces range(A), null(A), range(A^T) and null(A^T) in terms of the components of the SVD of A, and a brief justification.

9. Use the SVD of A to show that for an $m \times n$ matrix of full column rank n, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A).

10. (adapted from Strang I.9 1) **(Eckart-Young theorem in ℓ^2)** Consider a matrix A with SVD $A = U\Sigma V^T$. The matrix A could have a large rank. Constructing **low rank** approximations to A can be very useful (e.g., data compression). One such low rank approximation, constructed from the SVD, is

$$A_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T,$$

for small k, i.e., the approximation we get by keeping the largest k terms in the SVD of A. The Eckart-Young Theorem says that A_k is a **best** rank-k approximation to A (when "best" is measured in ℓ^2). That is, any other rank-k approximation, B, will be no better: $\|A - B\|_2 \geq \|A - A_k\|_2 = \sigma_{k+1}$. What are the singular values (in descending order) of $A - A_k$? Omit any zeros.

11. (adapted from Strang I.9 2) Find a closest rank-1 approximation to these matrices (L^2 or Frobenius norm $\|A\|_F = \sqrt{\mathrm{tr}\,(A^TA)}$):

$$A_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$