Floating Point

$$x \text{ flop } y = fl(x \text{ op } y)$$

$$+$$
$$-$$
$$\vdots$$
$$\times$$

$$x \otimes (y \oplus z)$$

$$y \oplus z = fl(y+z) = (1+\delta_1)(y+z), \quad |\delta_1| \le \boxed{\frac{\varepsilon_{mac}}{2}}$$

$$x \otimes (y \oplus z) = fl(x \times (1+\delta_1)(y+z))$$

$$= x(1+\delta_1)(y+z)(1+\delta_2) \quad |\delta_2| \le \frac{\varepsilon}{2}$$

$$\circledast \quad = x(y+z)\underbrace{(1+\delta_1)(1+\delta_2)}$$

$$(1+\delta_1)(1+\delta_2) = 1 + \delta_1 + \boxed{\delta_2} + \delta_1\delta_2 \approx \underline{1 + \delta_1 + \delta_2}$$

$$\circledast \quad \approx \underline{x(y+z)(1+\delta_1+\delta_2)} \quad \underline{\underline{|\delta_i| \le \frac{\varepsilon}{2}}}$$

Cancellation error

$$x = 1.9\underline{2}403\boxed{} \qquad \times 10^2$$
$$-y = 1.9\underline{2}275 \qquad \times 10^2$$

$$0.00\boxed{128} \therefore \qquad \times 10^2$$

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots$$

① $e^1 = e = 1 + 1 + \frac{1}{2} + \ldots + \frac{1}{n!} + \ldots$

② $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

$$\lim_{n \to \infty} 1 + \frac{1}{n} \to 1 \qquad 1^n = 1$$

e^{-5}

e^{-20}

$$e^{-x} = \frac{1}{e^x}$$
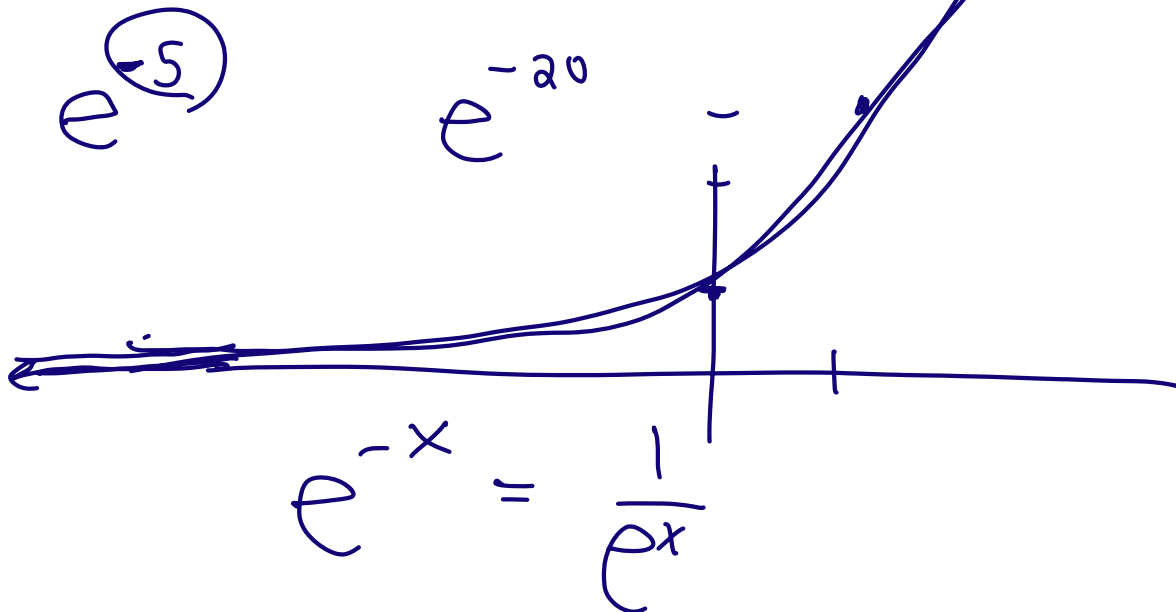
$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots$$

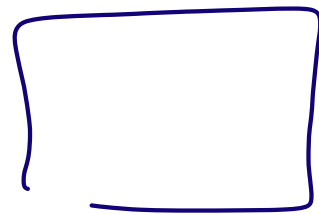$$\boxed{e^{-5}} = 1 - 5 + \frac{25}{2} - \frac{125}{6} + \ldots + \frac{x^n}{n!} \ldots$$

$$\begin{cases} e^5 = 1 + 5 + \frac{25}{2} + \ldots + \ldots \\ \\ e^{-5} = \frac{1}{e^5} \end{cases}$$
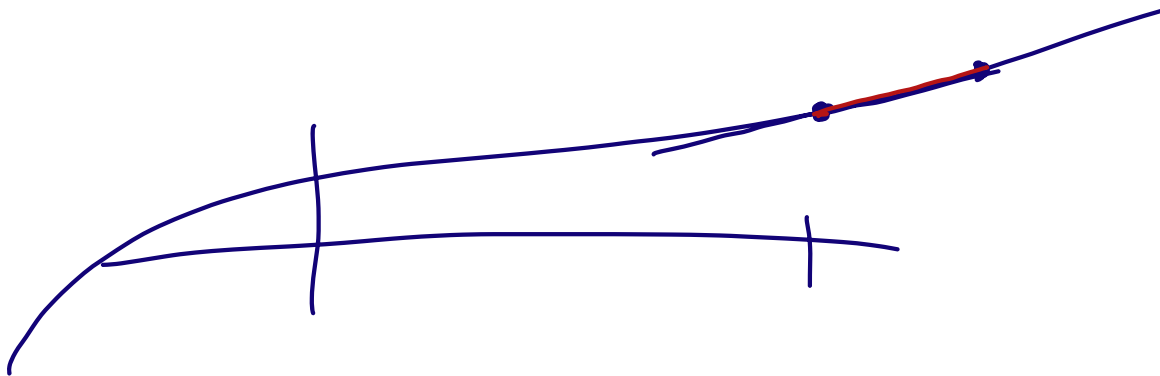
Sources of error

(1) floating pt. errors

(2) discretization errors

Cont.
problem $\xrightarrow{\text{discretize}}$

finite diff.
finite elements

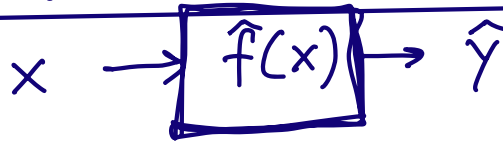$$f'(x) \cong \frac{f(x+h) - f(x)}{h}$$

③ modeling error

④ input data

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absolute $\quad |x - \hat{x}|$

relative $\quad \dfrac{|x - \hat{x}|}{|x|}$

forward error + backward error

$$x \longrightarrow \boxed{\hat{f}(x)} \longrightarrow \hat{y} \qquad y = f(x)$$
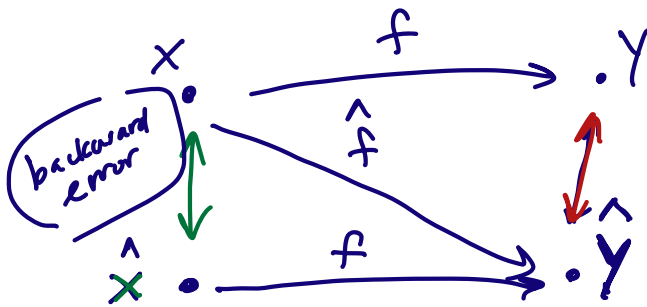

(relative)
forward error

$$\frac{|\hat{y} - y|}{|y|} \quad ??$$

relative
backward error

$$\frac{|\hat{x} - x|}{|x|}$$

$$A\vec{y} = \vec{b}$$

$$\vec{y} = A^{-1}\vec{b} = f(\vec{b})$$

algorithm $\qquad \hat{y}$

$$\frac{|\hat{y} - y|}{|y|} = ?$$

$$A\hat{y} = \hat{b}$$

$$\hat{y} = A^{-1}\hat{b}$$

$$\frac{|\hat{b} - b|}{|b|} \quad (residual)$$

$$Ay = b \longrightarrow \hat{y}$$

$$\underbrace{|A\hat{y} - b|}$$

backward error