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Given

$A = (n,n)$ matrix

$$B = \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}_{(n+1)\times(n+1)} \quad a \in \mathbb{R}^n$$

To Prove

$\alpha > 0$

A is positive definite

$$\exists\, v = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1} \text{ such that } v^T B v > 0$$

$$(v_1\ 0 \dots 0)\begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ 0 \end{pmatrix} = (v_1\alpha \quad v_1 a^T)_{1\times(n+1)}\begin{pmatrix} v_1 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1} \Rightarrow v_1^2\alpha > 0$$

Hence $\alpha > 0$

Now

$$\exists\, x = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}_{(n+1)\times 1} \text{ such that } x^T A x > 0 \quad x_i \in \mathbb{R}$$

Assume $\bar{x} = (x_1, \dots x_n)$

$$(0\ x_1 \dots x_n)\begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \cancel{(\bar{x}\, a_n)}\ \Big(\underbrace{\bar{x}a}_{\text{Scalar}},\ \bar{x}A\Big)\begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} > 0$$

$\Rightarrow \bar{x}A\bar{x}^T > 0 \rightarrow (1)$

where $\bar{x} \in \mathbb{R}^n$

Since B is symmetric

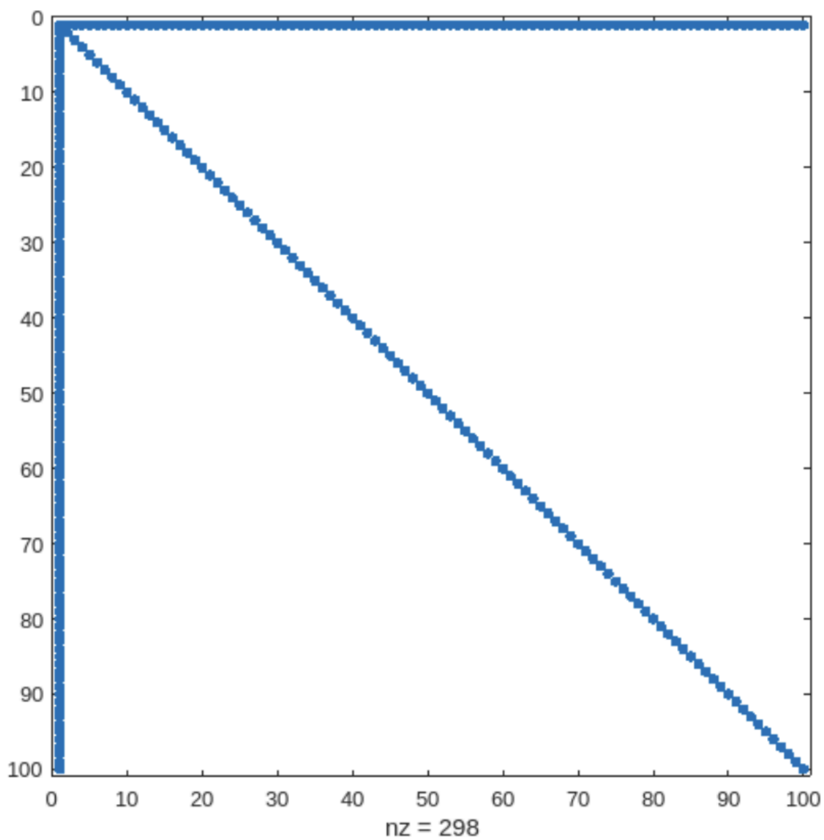
$B = B^T$

$$\begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix} = \begin{pmatrix} \alpha & a^T \\ a & A^T \end{pmatrix} \quad \therefore A = A^T \rightarrow (11)$$

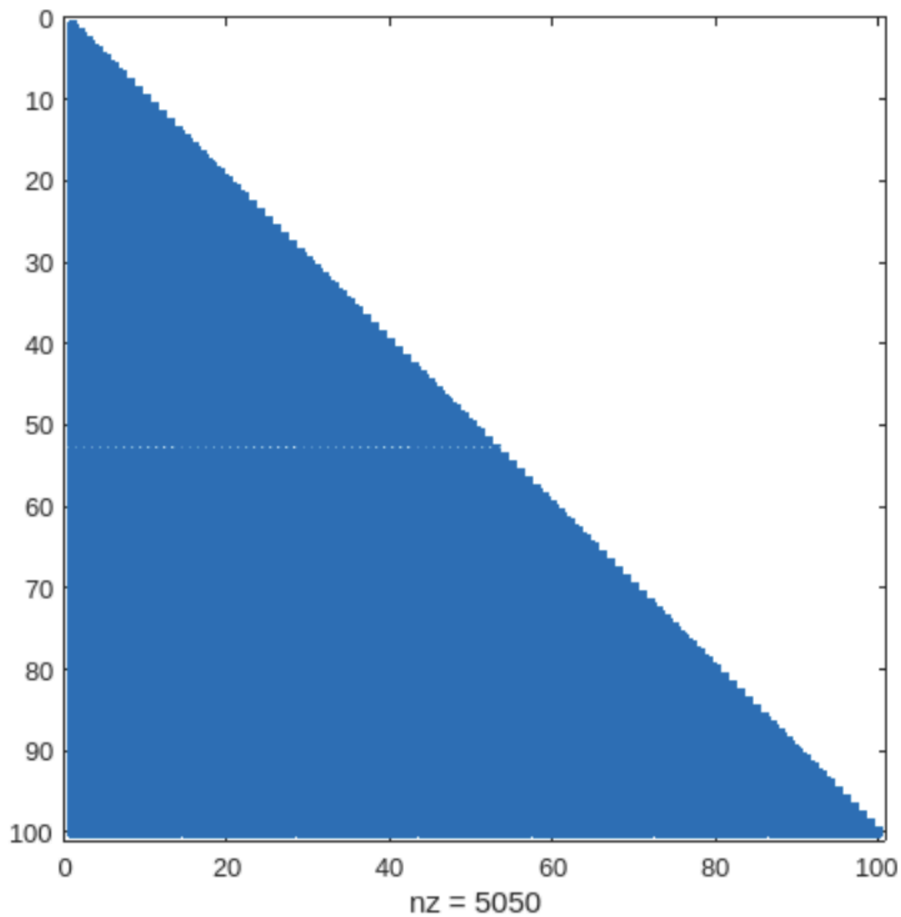
Combining (i) and (ii)

A is positive definite

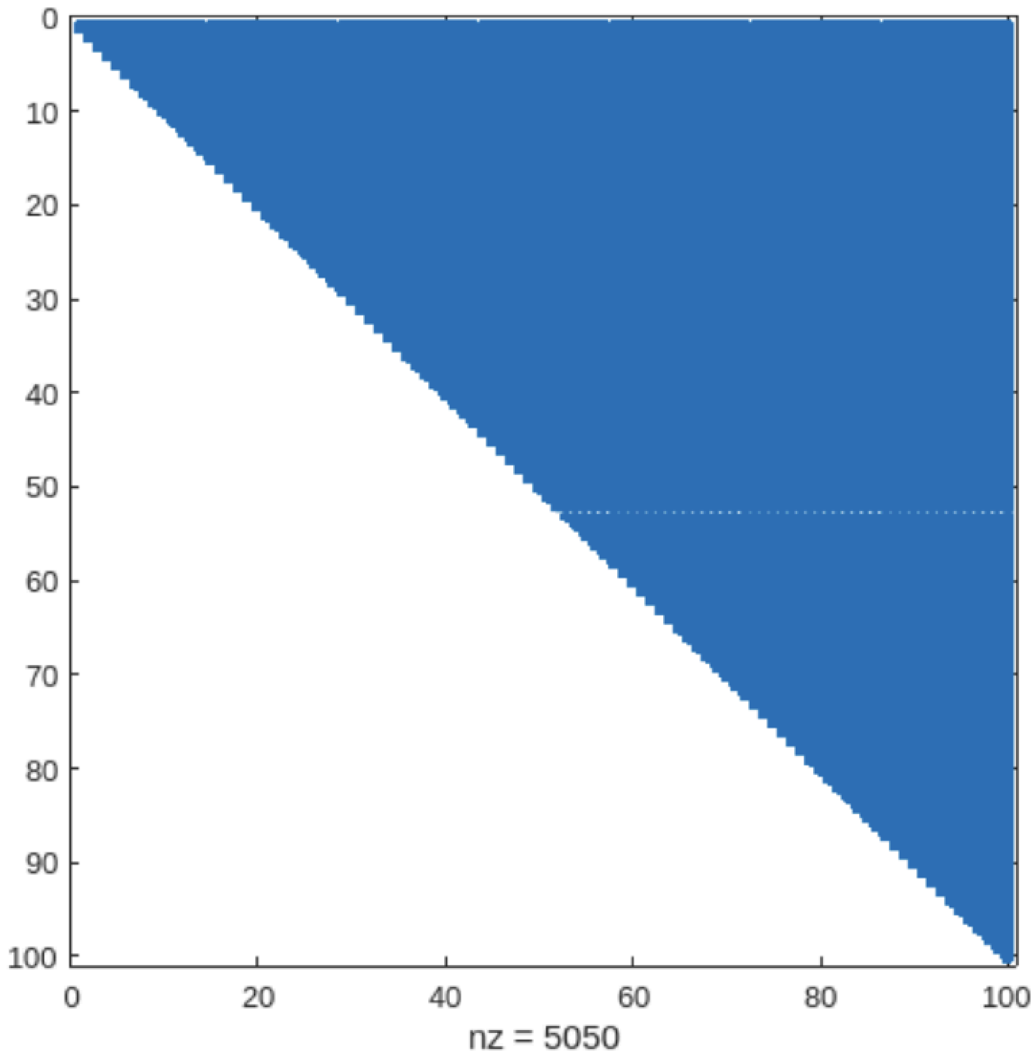
For Q2 (a) the matrix result after the spy command is



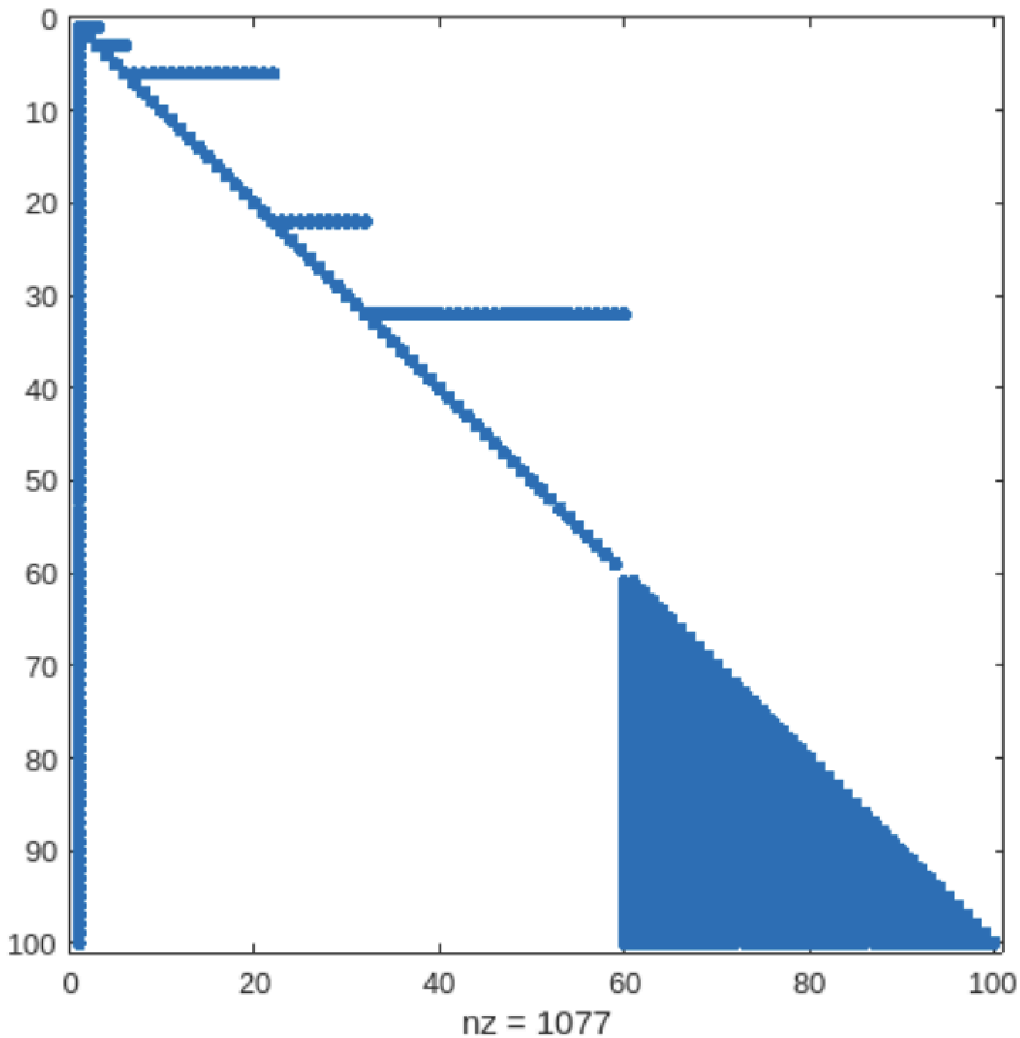
For Q2(b), spy(L) gives



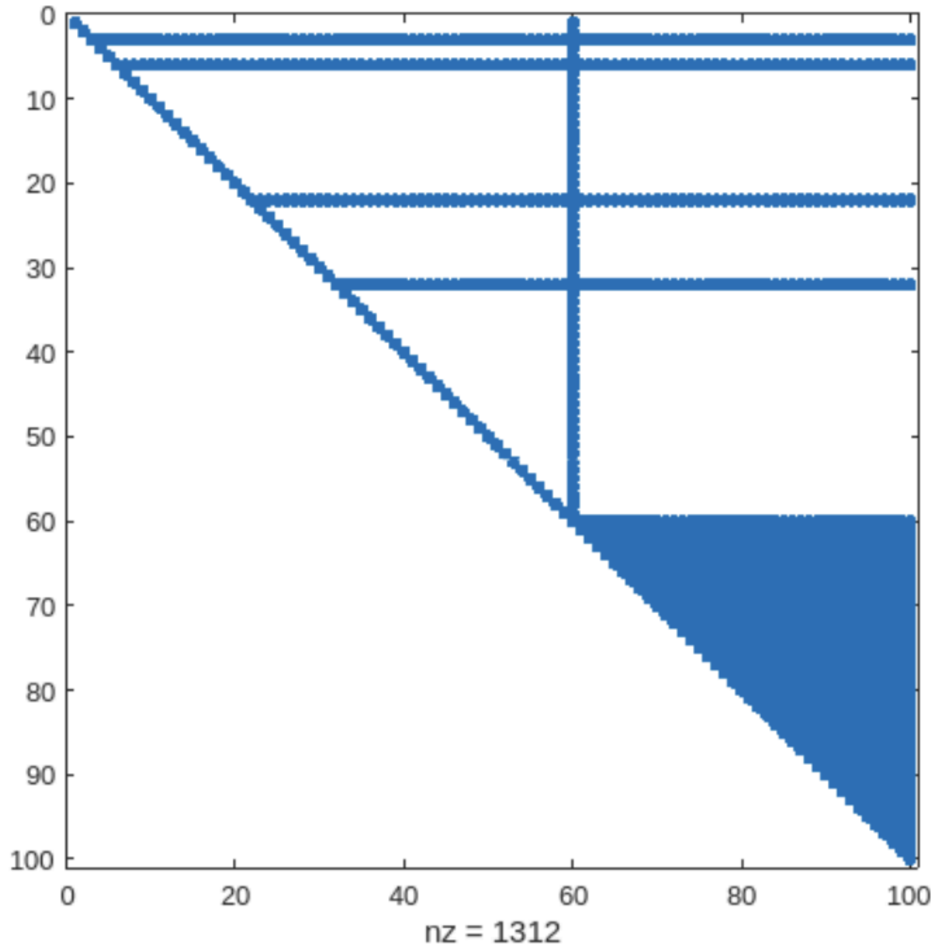
For Q2(b), spy(U), gives



For Q2(b) spy(L) for Matlab Lu function

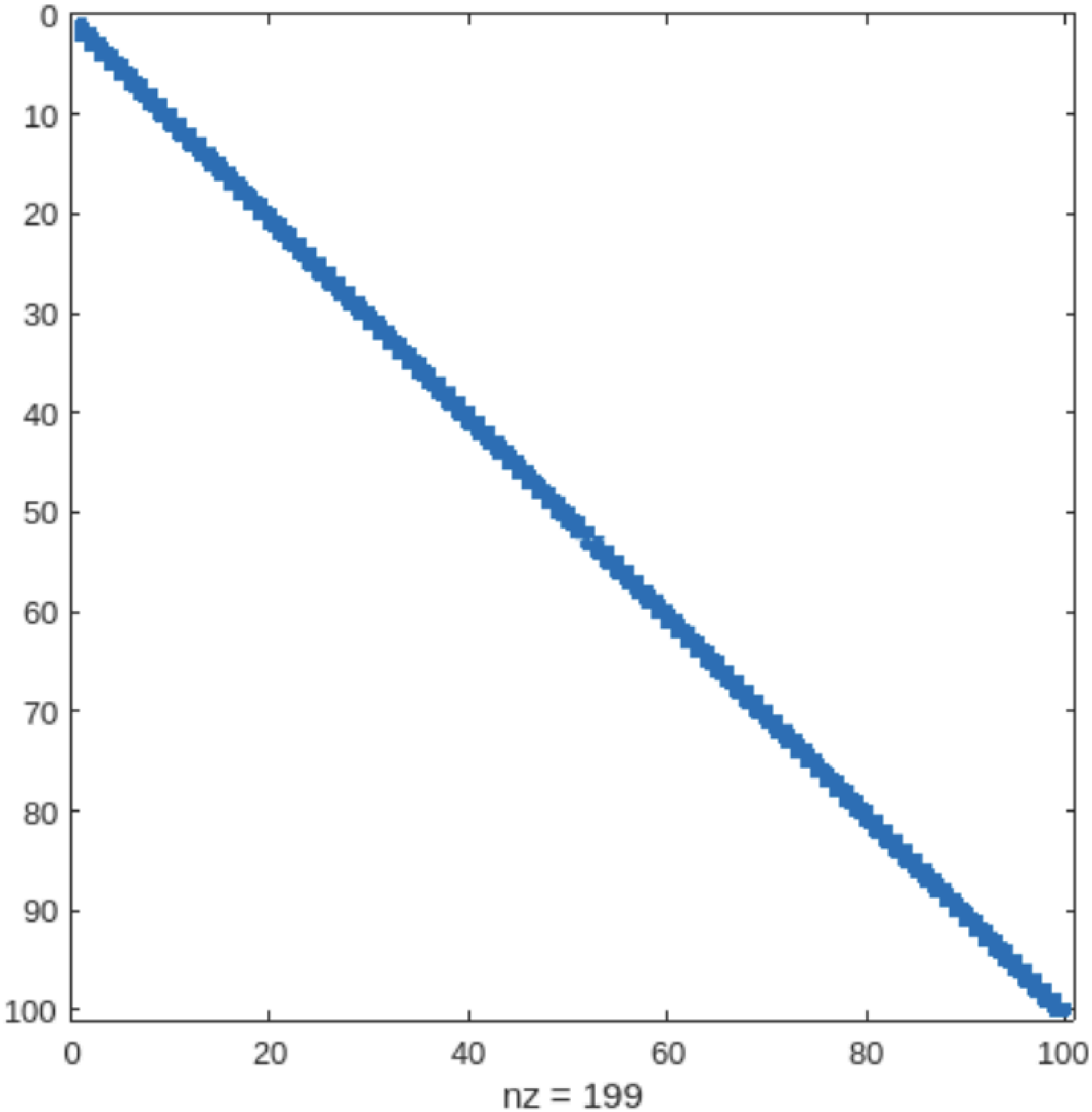


For Q2(b) spy(U) for Matlab lu function

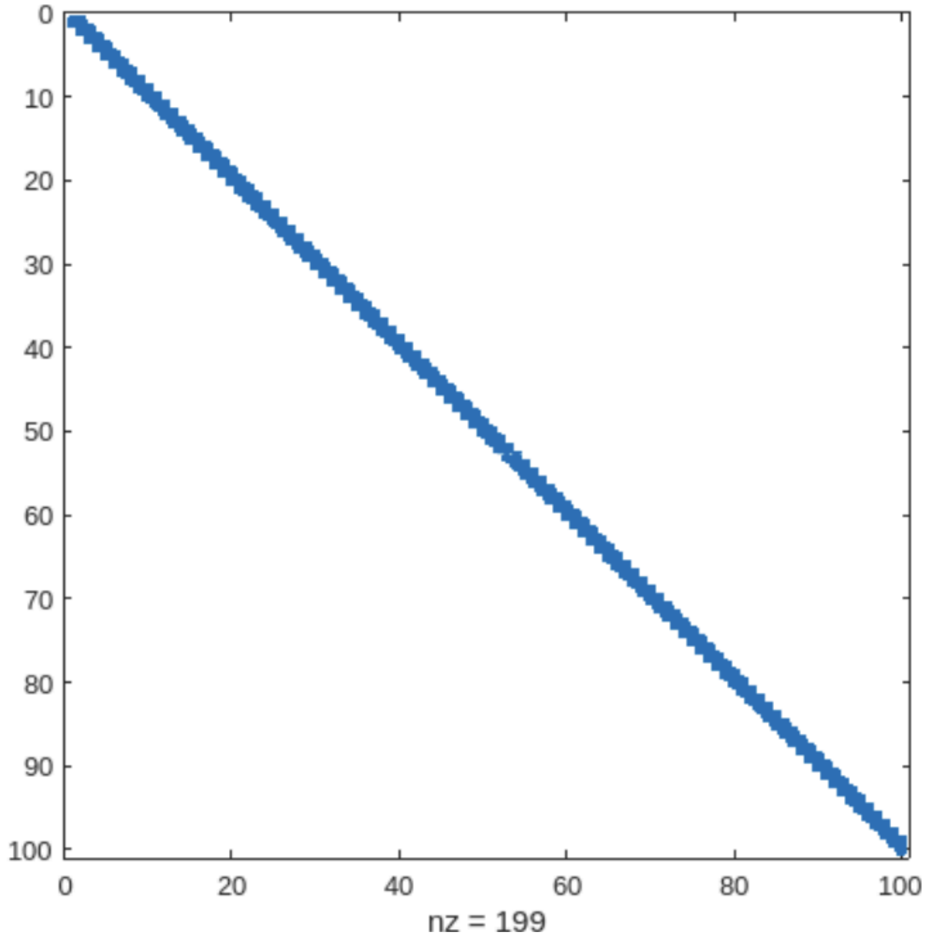


They are sparse.

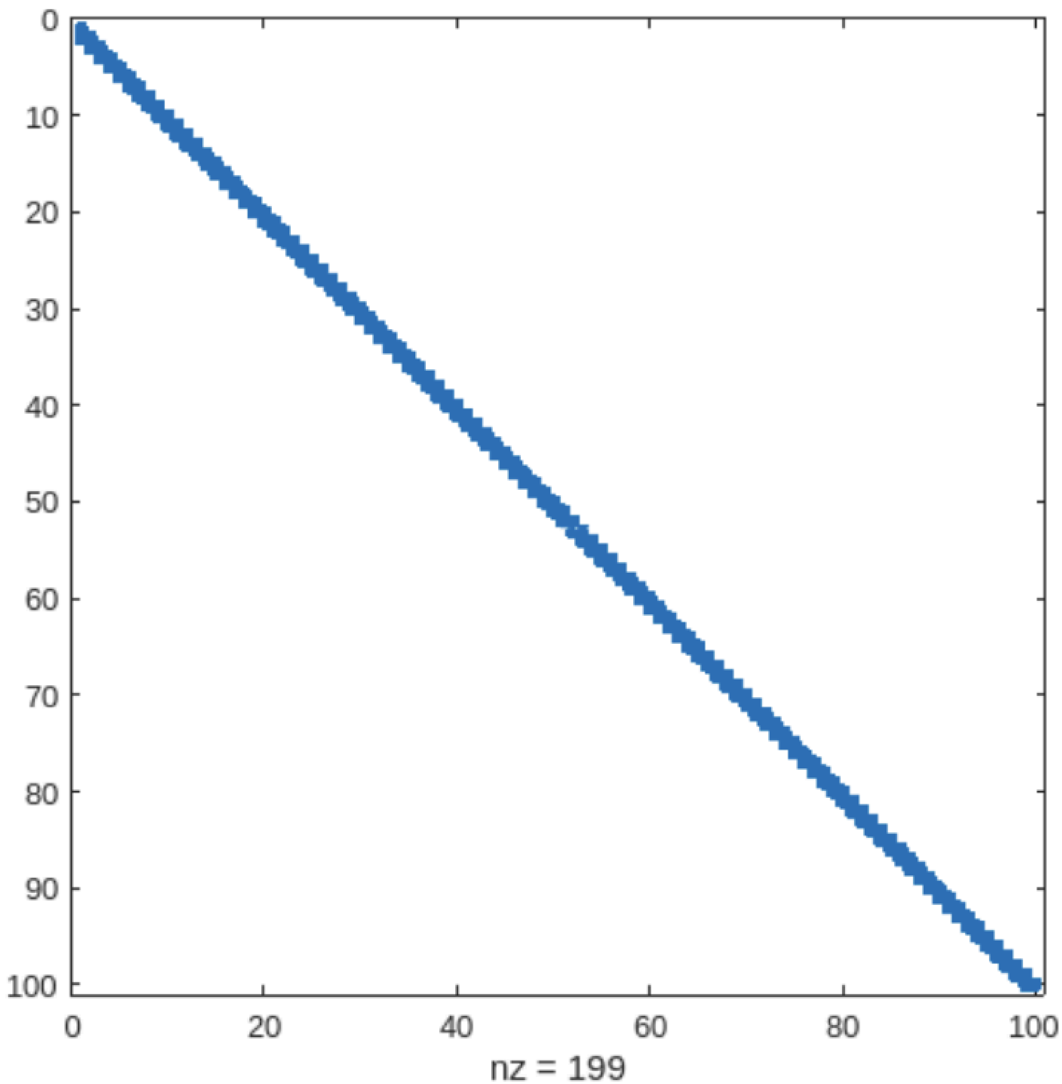
For Q2(c) spy(L)



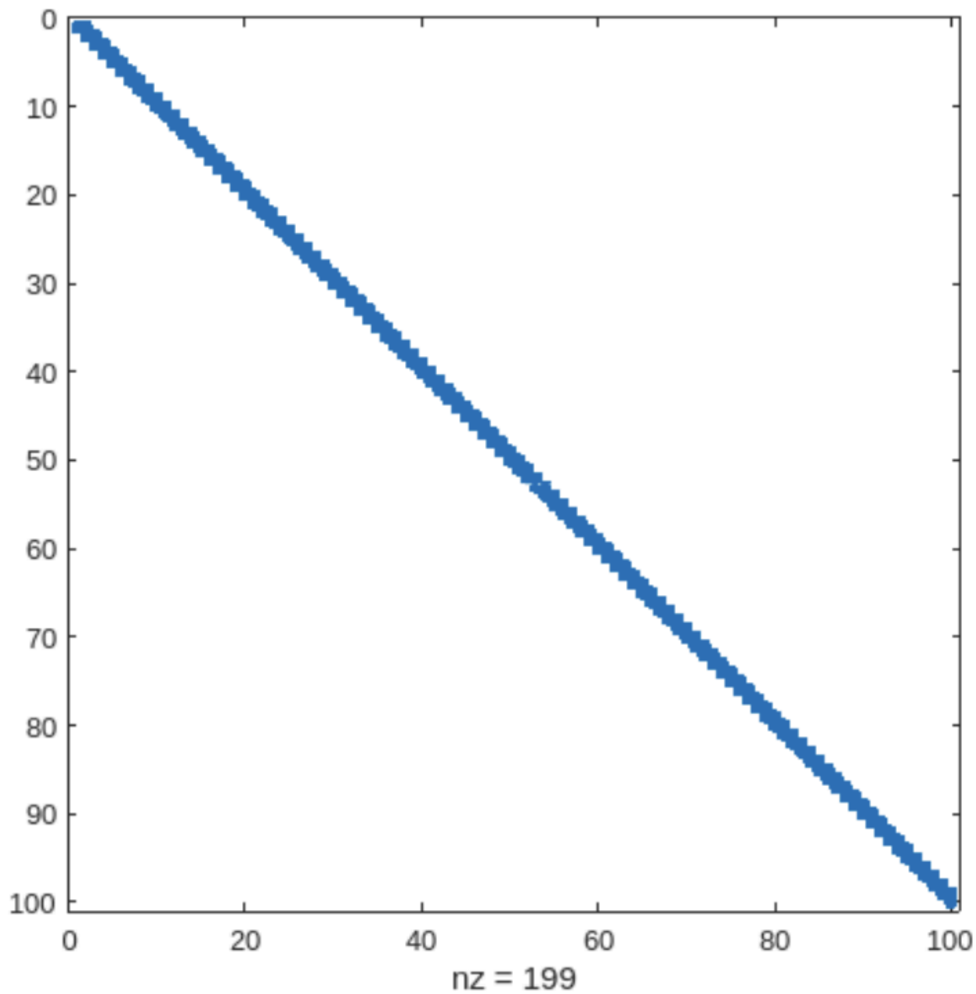
For Q2(c) spy(U)



For Q2(c) spy(L) for lu matlab function



For Q2(c) spy(U) for lu matlab function



For matrix B both lu and my_lu work give same sparse matrix

(b) $B = LL^*$ $L \to$ Lower Triangular, positive diagonal entries
$L^* \to$ Transpose Conjugate

$L \to (n+1) \times (n+1)$

$$\begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix} = \begin{pmatrix} l_{11} & \bar{0} \\ \bar{l} & c \end{pmatrix}_{n+1,\, n+1}$$

$$\therefore L^* = \begin{pmatrix} l_{11} & l^* \\ 0 & c^* \end{pmatrix}$$

$$LL^* = \begin{pmatrix} l_{11} & 0 \\ l & c \end{pmatrix} \begin{pmatrix} l_{11} & l^* \\ 0 & c^* \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} l_{11}^2 & l_{11} l^* \\ l l_{11} & l l^* + cc^* \end{pmatrix} = B = \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}$$

By comparing

$\alpha = l_{11}^2$ $\quad l_{11}(l) = a$

$l_{11} l^* = a^T$ $\quad l l^* + cc^* = A$

$\therefore l_{11}^2 (l l^*) = aa^T$

~~And~~ hence

$cc^* = A - \underbrace{\frac{1}{l_{11}^2}}_{l_{11}^2 = \alpha}(aa^T) \to$ The Cholesky factorization $A - \frac{1}{\alpha} aa^T$

$$B = LL^* = \begin{pmatrix} l_{11}^2 & l_{11} l^* \\ l_{11} l^* & l l^* + cc^* \end{pmatrix}$$

(4) (a) Condition number equal to 1 means the matrix is very far away from a singular matrix.

~~Aside from a constant~~

It means we can find an approximation of the solution having the same precision as the input data

∴ It is well condition

Hence TRUE

(c) To numerically solve a problem, we need an algorithm having very less forward error.

We know, Forward Error ≤ Condition number × backward error

(a) Well Conditioned

Algorithm should be backward stable (b)

Hence, ~~we need~~ if both (a) and (b) are less then we can guarantee that it will be solved accurately

~~Hence TRUE~~

However there is a chance that an unstable algorithm applied to a badly conditioned problem gives an accurate sol.
So its not both sides mapping, hence not iff ∴ $\not\Leftarrow$
FALSE

4 b Double Precision can reduce certain floating point calculation errors
But condition number is dependent upon input data, specifically changes in the input data.

However when we write an algorithm we give it an input which is represented in floating point. So it does carry the error.

Hence double precision can give us more accuracy despite a high condition number

Rules of thumb if $\kappa(A) = 10^{k}$, we lose upto k digits of accuracy
But we did not CHANGE the condition number!
~~TRUE~~

Hence FALSE

4 (a) TRUE

From the definition of ill conditioned

⑤ Given

$A_{n \times n}$

$cond(A) = 1 = ||A|| \cdot ||A^{-1}||$

(c) $k(cA) \Rightarrow ||cA|| \cdot ||(cA)^{-1}||$

$\Rightarrow |c| \times |c^{-1}| \times ||A|| \times ||A^{-1}||$

$\Rightarrow |c| \times \frac{1}{|c|} \times ||A|| \times ||A^{-1}||$

$\Rightarrow k(A) = 1$

(e) A^{-1}

$cond(A^{-1}) = ||A^{-1}|| \, ||A|| = cond(A) = 1$

(d) $cond(BA)$

Since B is non singular, the condition number now will be dependent upon B

Hence $cond(BA) \neq cond(A)$

Q6 Given

$u^T v = 0$ $\quad$ a, b, c, d are scalars $\begin{pmatrix} u_1 \\ u_2 \\ | \\ u_n \end{pmatrix}$ $\begin{pmatrix} v_1 \\ v_2 \\ | \\ v_n \end{pmatrix}$

$||u|| = ||v|| = 1$

$(au+bv)^T (cu+dv)^T = 1$

$\begin{pmatrix} au_1+bv_1 \\ au_2+bv_2 \\ | \end{pmatrix}^T \begin{pmatrix} cu_1+dv_1 \\ | \\ cu_n+dv_n \end{pmatrix} = 1$

$au.(cu+dv) + bv(cu+dv) = 0$

$ac + 0 + 0 + bd = 0$

$\therefore ac+bd = 0$

(8) Given Permutation Matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

Take another matrix of 4×4 $B = P^T$

$$PB = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$ Hence P is orthogonal

→ Each row of permutation matrix has exactly one entry = 1 rest all 0
→ " column " " " " " " " " " " "

$$(PP^T)_{ij} = \sum_{k=1}^{n} P_{ik} P_{jk} \rightarrow (1)$$

As There is only one k such that $P_{ik} \neq 0$ for each row i and each column j, the value of P_{ik} then becomes $P_{ik} = 1 \rightarrow (11)$

if $i=j$ (1) becomes

$$\sum_{k=1}^{n} P_{ik}^2 = 1 \text{ [From (11)]}$$

else if $i \neq j$

$$\sum_{k=1}^{n} P_{ik} P_{jk} = 0 \text{ [From (11)]}$$

Hence $(PP^T) = I$ ∴ P is orthogonal

(7) <u>To prove</u>

(a) $||Qx||_2^2 = ||x||_2^2$ (b) $(Qx)^T(Qy) = x^T y$

<u>Given</u>

$Q^TQ = QQ^T = I \rightarrow (1)$

$\Rightarrow \langle Qx, Qx \rangle = ||Qx||_2^2 \Rightarrow (Qx)^T(Qx) \Rightarrow x^T Q^T Q x \Rightarrow x^T I x \Rightarrow ||x||_2^2$

Similarly

$(Qx)^T(Qy) = x^T Q^T Q y \Rightarrow x^T y$ [Using (1)]

(9)

<u>Given</u> <u>To Prove</u>

$A \in \mathbb{R}^{m\times n}$ if $y \in range(A)$, $x \in nullspace(A^T)$ then $y^T x = 0$

$range(A) \subseteq colspan(A)$

$Ax = 0$ → Representing elements of A in the notation of column vectors

$c_{ij} \rightarrow$ column i, element no. j

$$\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ | & | & & | \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ | \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \therefore\ c_{11}x_1 + c_{21}x_2 + \cdots c_{m1}x_m = 0 \\ \quad | \\ c_{1n}x_1 + c_{2n}x_2 + \cdots c_{mn}x_m = 0 \end{array}\right\} \rightarrow (1)$$

Let $y = \alpha_1 C_1 + \alpha_2 C_2 \cdots + \alpha_m C_m$

$$y^T = [\alpha_1 c_{11} + \alpha_2 c_{21} + \alpha_3 c_{31} \cdots \alpha_m c_{m1},\ \alpha_1 c_{12} + \alpha_2 c_{22} + \cdots \alpha_m c_{m2} \cdots \cdots]$$

$y^T x =$

$$\begin{bmatrix} \alpha_1(c_{11}x_1 + c_{12}x_2 \cdots + c_{1n}x_n) \\ \alpha_2(c_{21}x_1 + c_{22}x_2 \cdots + c_{2n}x_n) \\ | \\ | \end{bmatrix} \rightarrow (11)$$

We see equations in (1) not matching equations in (11)

That is because ~~it~~ equation (1) is right Nullspace

The left Nullspace of A is $\{x : A^T x = 0 \mid x \neq 0\} = \text{nullspace}(A^T)$

Expanding it with same notation, it becomes

$$\left.\begin{array}{l} c_{11}x_1 + c_{12}x_2 + \ldots c_{1n}x_n = 0 \\ \qquad\qquad\qquad\qquad = 0 \\ \quad | \\ c_{n1}x_1 + c_{n2}x_2 + \ldots c_{nn}x_n = 0 \end{array}\right\} \rightarrow (111)$$

Using (111) in (11) we get

$$\begin{array}{l} (\alpha_1 \times 0) \\ (\alpha_2 \times 0) \\ \quad | \\ (\alpha_m \times 0) \end{array} = \begin{bmatrix} 0 \\ | \\ 0 \end{bmatrix}_{m\times 1}$$

Hence $y^T x = 0$ proved

(10) Given

$v \in \mathbb{R}^n$ $\{u_1, u_2 - u_k\} \to$ orthonormal $\in \mathbb{R}^n \to (1)$

To find

$\alpha_1, - \alpha_k \in \mathbb{R}$ such that $\|v - \sum_{i=1}^{k} \alpha_i u_i\|_2^2$ is minimum

~~Basically~~ INTUITION

Assume the correct answers are $\beta_1, - \beta_k$

$\therefore$ $\beta_1 u_1 + \beta_2 u_2 - \beta_k u_k$ is a unique representation of some vector $\vec{x}$

Hence we have to minimize $\|\vec{v} - \vec{x}\|_2^2$ where $x \in$ space of $\{u_1 - u_k\}$

$\therefore$ It will be the orthogonal projection of v onto the subspace

Hence we have to find $\beta_1 - \beta_k$ such that $\vec{x}$ is an orthogonal projection of v onto the subspace of $\{u_1, - u_k\}$

SOLUTION

We can write it as $\|\sum_{i=1}^{k} \alpha_i u_i - v\|_2^2$

$$\left\| \begin{bmatrix} | & | & & | \\ u_1 & u_2 & - & u_k \\ | & | & & | \end{bmatrix}_{n\times k} \begin{bmatrix} \alpha_1 \\ | \\ \alpha_k \end{bmatrix}_{k\times 1} - \begin{bmatrix} v_1 \\ | \\ v \end{bmatrix}_{n\times 1} \right\|_2^2$$

It is of the form $\|A\alpha - v\|_2^2$

We have to $\underset{\alpha}{\operatorname{argmin}} \|A\alpha - v\|_2^2$

Differentiating we get

$$A^T(A\alpha - v) = 0$$

$$A^T A\alpha = A^T v$$

$$\therefore \alpha = (A^T A)^{-1} A^T v \rightarrow (11)$$

$$A^T A = \begin{bmatrix} - u_1 - \\ - u_2 - \\ \vdots \\ - u_k - \end{bmatrix}_{k \times n} \begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix}_{n \times k}$$

From (1)

$$\|u_i\|^2 = 1 \qquad u_i^T u_j = 0 \text{ if } i \neq j$$

$$\therefore A^T A = I$$

Substituting in (11)

$$\alpha = A^T v$$

$$\therefore \alpha = \begin{bmatrix} u_1^T v \\ u_2^T v \\ \vdots \\ u_k^T v \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ | \\ \alpha_k \end{bmatrix} \text{ for minimum } \left\|v - \sum_{i=1}^{k} \alpha_i u_i\right\|_2^2$$

(1)

Given

$P^2 = P \rightarrow$ (1)

To Prove

$I-P$ is also a projection

if $P=P^T$ (11) then Px is orthogonal to $(I-P)y$ $\forall x,y \in \mathbb{R}^n$

$\Rightarrow (I-P)(I-P)$

$\Rightarrow I-P-P+P^2$

$\Rightarrow I-P$ [Using (1)]

$\therefore (I-P)^2 = I-P$

$\cancel{((I-P)y)^T(Px)}$

Now $(Px)^T((I-P)y)$

$\Rightarrow (x^TP^T)(I-P)y$

$(x^TP^T - x^TP^TP)y$

$\Rightarrow (x^TP^T - x^TP^T)y$ [From (1) and (11)]

$\Rightarrow 0$

Hence proved Px is orthogonal to $(I-P)y$

(3) $\|x_p\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$ $x \in \mathbb{R}^n$ $A \in \mathbb{R}^{n \times n}$

$\|x\|_\infty = \max_n x_n$

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

(i) $\|x\|_\infty \leq \|x\|_2 \rightarrow$ To Prove

$\max_{x_i}(\vec{x})$ $\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

Squaring both sides

$\left(\max_{x_i}(\vec{x})\right)^2$ $\sum_{i=1}^{n} |x_i|^2$

Assume the max element is at $i = \alpha$

$\therefore (x_\alpha)^2$ $x_1^2 + x_2^2 + \ldots x_\alpha^2 + x_{\alpha+1}^2 \ldots x_n^2 \rightarrow$ (ii)

$\downarrow$
(i)

Clearly

(i) $\leq$ (ii)

Equality case when $n = 1$

So $x = 2$ has $\|x\|_\infty = \|x\|_2$

(ii) $\|x\|_2 \le \sqrt{n}\,\|x\|_\infty \rightarrow$ To Prove $\} \rightarrow$ (III)

Doing same process as did in part(a) and reaching the same equations (I) and (II)

Squaring (III)

$\|x\|_2^2 \le n\|x\|_\infty^2 \rightarrow$ To prove

Using (I) and (II)

$$x_1^2 + x_2^2 + \_\_ \; x_\alpha^2 + x_{\alpha+1}^2 + \_\_ \; x_n^2 \le \underbrace{x_\alpha^2 + x_\alpha^2 \; \_\_ \; x_\alpha^2}_{n \text{ times}} = n x_\alpha^2$$

Hence (III) holds

Equality when all elements of x are equal

$\therefore$ If $x = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}_{n\times 1}$ then $\|x\|_2 = \sqrt{n}\,\|x\|_\infty$

(iii) $\|A\|_{\cancel{2}} \le \sqrt{n}\,\|A\|_\infty$

$$\|A\|_p = \max_{x \ne 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$\|A\|_2^2 =$ ~~max~~ [largest singular value] $= \lambda_{max}(A^T A)$

$$\|A_\infty\| = \max_i \sum_j |a_{ij}|$$

$\|A^TAx\|_\infty = \|\lambda_\alpha x\|_\infty$ [Assuming λ_α to be the max eigenvalue of A^TA
Now $A^TAx = \lambda x$]

$\therefore \|\lambda_\alpha x\|_\infty = \|A^TAx\|_\infty \leq \|A^TA\|_\infty \|x\|_\infty$ [Using property of matrix norm

$|\lambda_\alpha| \|x\|_\infty \leq \|A^TA\|_\infty \|\cancel{x}\|_\infty$ [Using property of vector norm]

$\Rightarrow |\lambda_\alpha| \leq \|A^TA\|_\infty \rightarrow$ (VIII)

<u>Now</u>

$\|A^TA\|_\infty = \max_i \sum_j |(A^TA)_{ij}| \leq \max_i \sum_j \sum_k |a_{ik} a_{kj}| \rightarrow$ (V)

$\Rightarrow \max_i \sum_j \sum_k |a_{ik} a_{kj}| \leq \max_{i,j} |a_{ij}| \max_i \sum_j \sum_k |a_{ik}|$

$\Rightarrow \max_{i,j} |a_{ij}| \leq \|A\|_\infty = \max_i \sum_j |a_{ij}| \rightarrow$ (IV)

$\Rightarrow \max_i \sum_j \sum_k |a_{ik}| \leq \sum_j \max_i \sum_k |a_{ik}| \Rightarrow n\|A\|_\infty \rightarrow$ (VI)

$\therefore$ Using (IV) and (VI), (V) becomes

$\|A^TA\|_\infty \leq n\|A\|_\infty^2 \rightarrow$ (VII)

From (VII) and (VIII)

$\|A\|_2^2 \leq \|A^TA\|_\infty \leq n\|A\|_\infty^2$

$\Rightarrow \|A\|_2 \leq \sqrt{n}\, \|A_\infty\|$

(iv) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$

If $x \in \mathbb{R}^n$

$\|y\|_\infty \le \|y\|_2 \le \sqrt{n}\,\|y\|_\infty \quad \forall y \in \mathbb{R}^n$ [Vector norm inequality] $\rightarrow$ ①

$\therefore \|Ax\|_\infty \le \|Ax\|_2$

$\le \|A\|_2 \|x\|_2 \quad [\|Ax\|_2 \le \|A\|_2 \|x\|_2]$

$\le \|A\|_2 \sqrt{n}\,\|x\|_\infty$ [Using ①]

$\therefore \|Ax\|_\infty \le \|A\|_2 \sqrt{n}\,\|x\|_\infty$

$\Rightarrow \|A\|_\infty \cancel{\|x\|_\infty} \le \sqrt{n}\,\|A\|_2 \cancel{\|x\|_\infty} \quad [\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty]$

$\Rightarrow \|A\|_\infty \le \sqrt{n}\,\|A\|_2$

For both (iii) and (iv) $A = 1\times1$ matrix with any value should satisfy the equality conditions

(b) <u>Given</u> | <u>To Prove</u>

$\exists c_1, d_1 \in \mathbb{R} \;\; c_1\|x\|_b \le \|x\|_a \le d_1\|x\|_b$ | $\exists c_2, d_2 \in \mathbb{R} \;\; c_2\|A\|_b \le \|A\|_a \le d_2\|A\|_b$

Ax is a vector

$\therefore c_1\|Ax\|_b \le \|Ax\|_a$

$\Rightarrow c_1\|Ax\|_b \le \|A\|_a \|x\|_a$

$$\Rightarrow c_1 \|Ax\|_b \le d_1 \|A\|_a \|x\|_b$$

$$\Rightarrow \frac{\|Ax\|_b}{\|x\|_b} \le \frac{d_1}{c_1} \|A\|_a$$

$$\Rightarrow \frac{c_1}{d_1} \|A\|_b \le \|A\|_a$$

Similarly, we get

$$\|Ax\|_a \le d_1 \|Ax\|_b$$

$$\Rightarrow \|Ax\|_a \le d_1 \|A\|_b \|x\|_b \quad [\because \|Ax\| \le \|A\|\|x\|]$$

$$\text{"} \quad \le d_1 \|A\|_b \frac{\|x\|_a}{c_1} \quad [\because c_1 \|x\|_b \le \|x\|_a]$$

$$\Rightarrow \frac{\|Ax\|_a}{\|x\|_a} \le \frac{d_1}{c_1} \|A\|_b$$

$$\Rightarrow \|A\|_a \le \frac{d_1}{c_1} \|A\|_b$$

$\therefore$ We have

$$\frac{c_1}{d_1} \|A\|_b \le \|A\|_a \le \frac{d_1}{c_1} \|A\|_b$$