Facility Location - P.D.

Facility Location

Input: $P, Q \subseteq X$, (X,d) metric space, $f \in \mathbb{R}^+$

Goal: Open $Q' \subseteq Q$ s.t. to minimize $\sum_{i \in P} d(i, c(i)) + f \cdot |Q'|$.

$$c(i) = \arg\min_{j \in Q'} d(i,j)$$

LP. ($\{x_{ij}\}_{i \in P, j \in Q}$, $\{y_j\}_{j \in Q}$)

$$\min \sum_{ij} d_{ij} x_{ij} + \sum_j f \cdot y_j$$

s.t. $\sum_j x_{ij} \geq 1$. $\quad \forall i$. $\cdots \alpha_i$

$x_{ij} \leq y_j \Leftrightarrow$ $\quad \forall ij$. $\cdots \beta_{ij}$.

$y_j - x_{ij} \geq 0$

$x, y \geq 0$.

Dual ($\{\alpha_i\}_{i \in P}$, $\{\beta_{ij}\}_{i \in P, j \in Q}$)

$$\max \sum_i \alpha_i$$

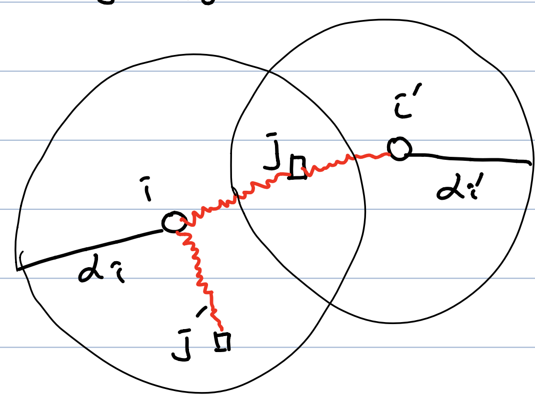
s.t. $\alpha_i - \beta_{ij} \leq d_{ij}$ $\quad \forall ij \cdots x_{ij}$ $\qquad \beta_{ij} = \max(0, \alpha_i - d_{ij})$

$\sum_i \beta_{ij} \leq f$. $\quad \forall j \cdots y_j$

$\alpha, \beta \geq 0$.

Primal-Dual

- Increase dual var.
- Choose primal vars corresponding to tight ineqs.



Phase 1.

S: connected clients

O: open facilities.

$S \leftarrow \emptyset$. $O \leftarrow \emptyset$. $\alpha \leftarrow 0$

While $S \neq P$

Increase all $\{\alpha_i\}_{i \in P \setminus S}$ at same rate until

- Some $j \in Q \setminus O$ gets "tight" — $\sum_i \beta_{ij} = f$.
- Some $i \in P \setminus S$ and $j \in O$ connected.

cli i - fac. j connected $\Leftrightarrow \alpha_i \geq d_{ij}$.

$O \leftarrow O \cup \{$tight facilities$\}$

$S \leftarrow S \cup \{$clients connected to $O\}$.

cli i "connected" if it's connected to $j \in O$

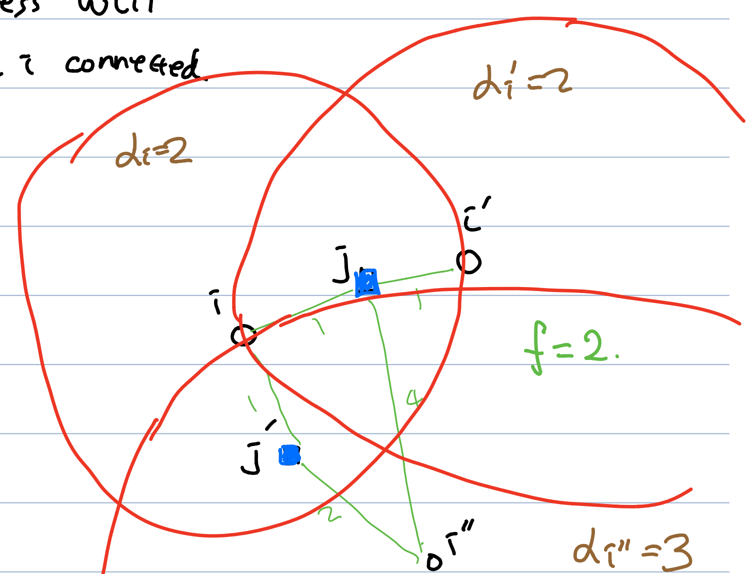
End

Claim, α is dual feasible.

Def., Connecting witness $w(i)$

= fac. that first made i connected

Claim, $d(i, w(i)) \leq \alpha_i$.



Analysis, (conn cost) + (fac cost) vs. $\sum_i \alpha_i$

$$\leq \sum \alpha_i + f \cdot |O| = \sum \alpha_i + \sum_{j \in O} \sum_i \beta_{ij}$$

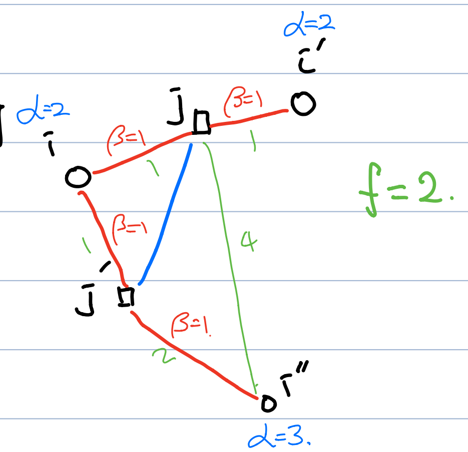
$$= \sum \alpha_i + \sum_i \sum_{j \in O} \beta_{ij}$$

Def. Call cli-fac pair (i,j) special if $\beta_{ij} > 0$.

Consider bipartite graph $G = (P \cup O, E)$ where $E = \{\text{special edges}\}$

Consider graph $G^2 = (O, E^2)$,

$E^2 = \{ (j, j') : \exists\, i \in P \text{ s.t. } (i,j), (i,j') \in E \}$



Phase 2.

Find maximal independent set $O' \subseteq O$.
Open only O'.

Say $i \in P$ is "directly connected" if $\exists\, j \in O'$ st $(i,j) \in E$.

- j is unique!
- $\alpha_i^f = \beta_{ij}$, $\alpha_i^e = d_{ij}$.

Otherwise, i is "indirectly connected" — then $\exists\, j \in O', i' \in P$ s.t.

$(\omega(i), j) \in E^2 \Rightarrow (i', \omega(i)), (i', j) \in E$

- $\alpha_i^f = 0$, $\alpha_i^e = \alpha_i$.

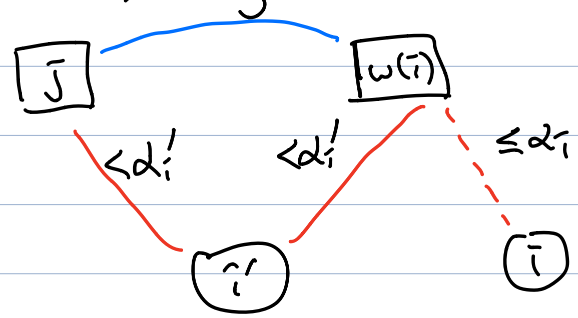
Final Analysis.

__Fac cost,__ $f \cdot |O'| = \sum_{j\in O'} f = \sum_{j \in O'} \sum_{i \text{ dir}} \beta_{ij} = \sum_{i \text{ dir}} \sum_{j\in O'} \beta_{ij} = \sum_{i \text{ dir}} \alpha_i^f.$

__Conn cost,__ dir. conn. i : $\text{dist} \le d_{ij} \le \alpha_i^e$

__Claim,__ For undir. i, $d_{ij} \le 3\alpha_i = 3\alpha_i^e$.

__Pf,__



$d_{iw(i)} \le \alpha_i$

$d_{i'w(i)} = \alpha_i' - \beta_{i'w(i)} < \alpha_i'$

$d_{i'j} < \alpha_i'$

WTS $\alpha_i' \le \alpha_i$.

If $d_{i'w(i)} \ge \alpha_i$, $\alpha_i' \le d_{i'w(i)} \Rightarrow \beta_{i'w(i)} = 0$, contradiction.

$d_{i'w(i)} < \alpha_i. \Rightarrow \alpha_i' \le \alpha_i$ $\square$

Final cost $\le \sum_i \alpha_i^f + 3\sum_i \alpha_i^e \le 3 \sum_i \alpha_i \le 3 LP$ $\square$.