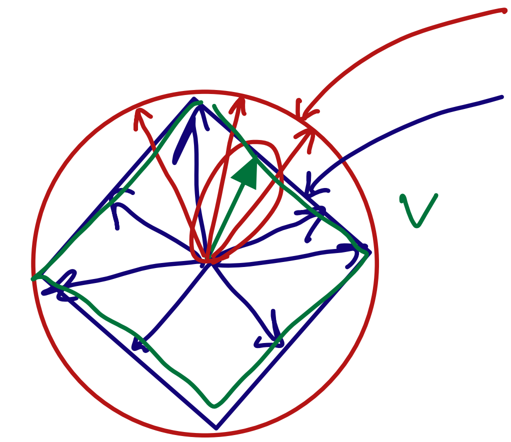


$\|v\|_2 < 1$

$\|v\|_1 = 1$

Orthogonality

orthogonal matrices $Q \in \mathbb{R}^{n \times n}$

has orthonormal columns:

$$\begin{cases} v_i^T v_j = 0 & i \neq j \\ v_i^T v_i = 1 & \end{cases}$$

$$\begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$Q^T Q$

$$\underbrace{\begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}}_{Q}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix} = I$$

$$Q^T Q = I = \cancel{Q}Q^T$$

$$Q^{-1} = Q^T$$

Q also has orthonormal rows

Ex. Q rotation matrix

$$Q^T Q = I$$

lengths: $x^T x$

angles x, y $\quad x^T y$

Unitary matrix $\quad Q^* Q = I$

$$Q^H Q = I$$

---

det(Q) $= \pm 1$

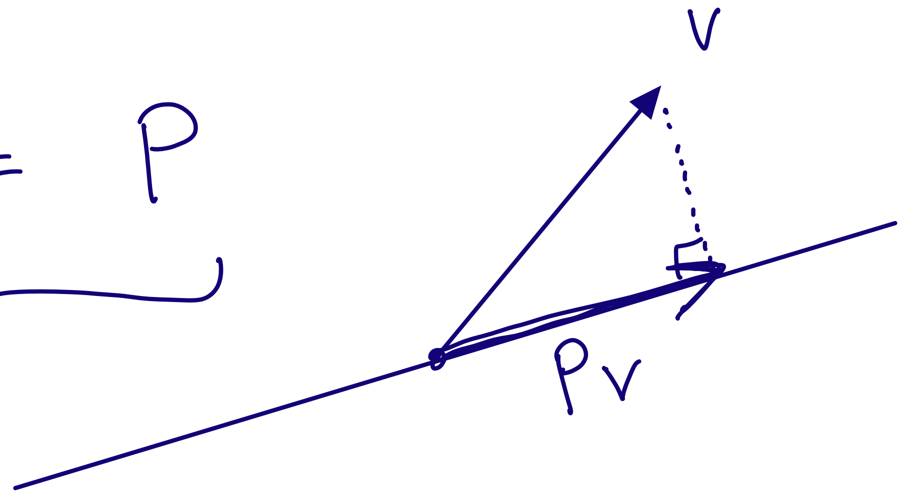
$$\underbrace{Q^T Q}_{} = I$$

$$q_i^T q_j = \boxed{\delta_{ij}} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Kronecker delta

Projector Matrices

$$\underbrace{P^2 = P}$$

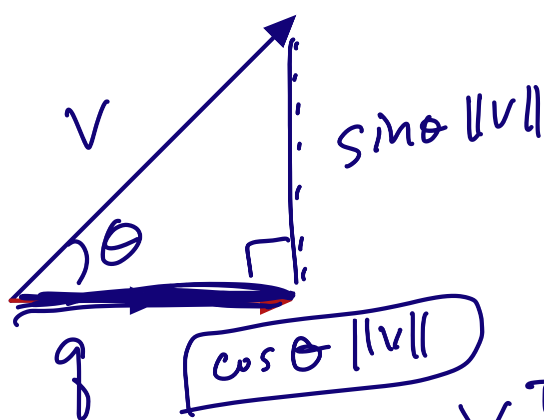


$$P(Pv) = Pv$$

$P^2 = P$ P is "idempotent"

$\|q\|_2 = 1$

project $\vec{v}$ onto the direction $\vec{q}$



$$v^T q = \|v\|\,\|\cancelto{1}{q}\|\cos\theta$$

$$P\vec{v} = \vec{v}_q = \underbrace{(\vec{v}^T\vec{q})}_{\text{length}}\underbrace{\vec{q}}_{\text{direction}}$$

$$\cos^2\theta \|v\|^2 + \sin^2\theta \|v\|^2 = \|v\|^2$$

$$P\vec{v} = \boxed{(\vec{v}^T\vec{q})}\vec{q}$$

$$= (\vec{q}^T\vec{v})\vec{q}$$

$$= \underbrace{q(q^Tv)}$$

$$P\vec{v} = \underbrace{(qq^T)}_{P}\vec{v}$$

$$q\boxed{q^T v}$$

$$P = qq^T$$

projector? $\quad P^2 = P$

$$q^Tq = \|q\|_2^2 = 1$$

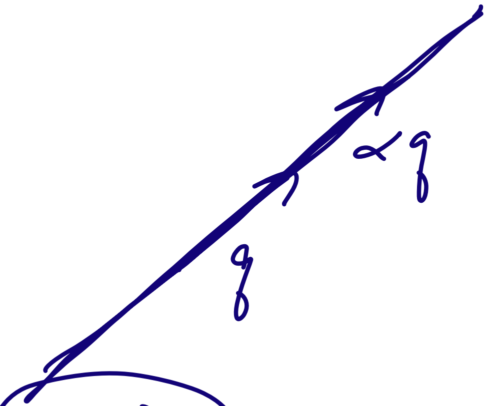
$$P^2 = (qq^T)(qq^T)$$

$$= q(q^Tq)q^T = qq^T = P$$

$$\text{rank}(qq^T) = 1$$

$q \quad q^T$

$n\times 1 \quad 1\times n$



$$\vec{q}\,\vec{q}^{\,T} w = \alpha \vec{q}$$

$$\alpha = \vec{q}\cdot\vec{w}$$

$$A = u_1 v_1^T + u_2 v_2^T$$

SVD

$$\boxed{A = \sum_{i=1}^{r} \sigma_i u_i v_i^T}$$

$$A = u w^T + u v^T$$

$$Ax = (w^T x + v^T x)\,\underline{u}$$

$$v w^T$$

$$P = \boxed{q\, q^T}$$

$$Q \quad \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$$Q = \begin{bmatrix} | \\ q_1 \\ | \end{bmatrix} \qquad QQ^T = q_1 q_1^T$$

$$Q_2 = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \qquad \underbrace{Q_2 Q_2^T \, v}$$

$$\parallel$$

$$\left[(q_1^T v) q_1 + (q_2^T v) q_2 \right]$$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad QQ^T = I$$

$$I^2 = I \quad \checkmark$$

$$Q_r = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{bmatrix} \qquad P = Q_r Q_r^T$$

$$P^2 = Q_r \underbrace{Q_r^T Q_r}_{I_r} Q_r^T$$

$$\underline{\underline{q_i^T q_j = \delta_{ij}}}$$

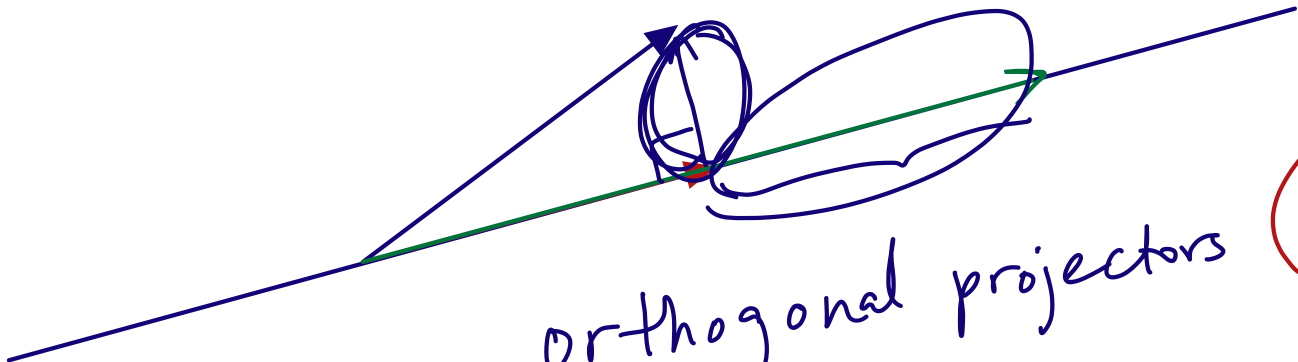
$$\underset{r \times n}{\begin{bmatrix} — & q_1^T & — \\ & \vdots & \\ — & q_r^T & — \end{bmatrix}} \underset{n \times r}{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{bmatrix}} = \underset{r \times r}{\begin{bmatrix} I_r \end{bmatrix}}$$

$$\boxed{Q_r Q_r^T = P} \qquad n \times n \qquad \operatorname{rank}(P) = r < n$$

$$\boxed{Q_r^T Q_r = I_r} \qquad r \times r \qquad \operatorname{rank}(I_r) = r$$

$$\underbrace{I_r^2} = I_r$$

$$Q_r = \begin{bmatrix} \quad \end{bmatrix}$$



orthogonal projectors $\quad P = P^T$

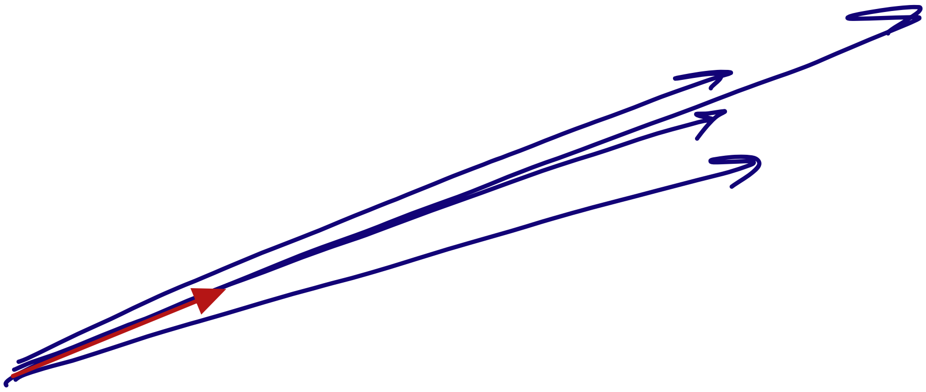
$$P = q q^T \qquad \| q \|_2 = 1$$

$$\vec{v} \longrightarrow \hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} \hat{v}^T = \frac{v}{\|v\|} \frac{v^T}{\|v\|} = \frac{1}{\|v\|^2} v v^T = \frac{v v^T}{v^T v}$$

$$P = \frac{v v^T}{v^T v}$$

qq^T



$(qq^T)^T = (q^T)^T q^T = qq^T$ symmetric

symmetric projector = orthogonal projector

vector length direction

$$(vv^T)(vv^T) = v\underbrace{(v^Tv)}v^T$$

$$= \|v\|^2 vv^T$$