Multiplicative Weight Update (MWU)

Warmup: prediction with expert advice

- For $t=1,2,3,\dots T$, we want to predict $e_t \in \{\pm 1\}$
  (e.g., will it rain on day t? will stock market go up on day t?)
- There are N "experts" that make their prediction on e_t
  (imagine $N = \text{poly}(T)$ or less).
- At the beginning of day t, we see experts' predictions, and need to choose our own prediction.
- After our prediction, e_t is revealed (by nature/adversary)
- Goal: make as many correct predictions as possible!
  (it is unfair — adversary can choose e_t so that our prediction is wrong for every t!)
  Then, do as well as any expert!
  <u>**Ideal Thm**</u>, $\forall i \in [N]$, (# my mistakes) $\leq O$(# expert i's mistakes) + (some function of N)

Then, for fixed N, as $T \to \infty$, we're doing as well as the best expert up to a constant factor.

Algorithm.

$w_i^{(1)} \leftarrow 1 \quad \forall i \in [N].$

For $t = 1, \ldots, T$,

Our prediction $\leftarrow$ "weighted majority of experts"

compare total w of experts who predict $e_t = 1$ / $e_t = -1$.

e_t is revealed.

For $i \in [N]$, $w_i^{(t+1)} \leftarrow \begin{cases} w_i^{(t)}/2 & \text{if } i \text{ made wrong prediction about } e_t, \\ w_i^{(t)} & \text{o.w.} \end{cases}$

Thm. $\forall i \in [N]$, (# my mistakes) ≤ 2.41(# expert i's mistakes $+ \log_2 N$).

Pf. Let $\Phi^{(t)} := \sum_{i \in [N]} w_i^{(t)}$. Then

- $\Phi^{(1)} = N$.
- $\forall i \in [N]$, $\Phi^{(T+1)} \geq w_i^{(T+1)} = 2^{-(\# \text{ } i\text{'s mistakes})}$.
- If we made a mistake on day t,

$$\Phi^{(t+1)} = \frac{1}{2} \sum_{i:\text{ wrong about } e_t} w_i^{(t)} + \sum_{i:\text{ correct about } e_t} w_i^{(t)} \leq \frac{3}{4} \cdot \Phi^{(t)}$$

(we were wrong since was larger than).

$$\Rightarrow \Phi^{(T+1)} \leq \left(\tfrac{3}{4}\right)^{(\#\text{ our mistakes})} \cdot \Phi^{(1)}.$$

Then $\forall i \in [N]$, $\left(\tfrac{1}{2}\right)^{(\#\text{ } i\text{'s mistakes})} \leq \left(\tfrac{3}{4}\right)^{(\#\text{ my mistakes})} \cdot N$.

Taking logs, $-(\#\text{ } i\text{'s mistakes}) \leq \log_2(3/4)(\#\text{ my mistakes}) + \log_2(N)$

$$\Rightarrow (\#\text{ my mistakes}) \leq \frac{1}{\log_2(4/3)} (\#\text{ } i\text{'s mistakes} + \log_2(N))$$

≈ 2.41. $\square$

Full MWU

Full Setting

- For $t=1,2,3,\dots T$, we want to predict "Something"
- There are N "experts" that make their prediction on "Something"
  (imagine $N=\text{poly}(T)$ or less).
- At the beginning of day t, we see experts' predictions, and need to choose our own prediction.
  (Our prediction is described by "distribution" $p^{(t)}=(p_1^{(t)},\dots,p_N^{(t)})$. ($p_i^{(t)}\geq 0$, $\sum_i p_i^{(t)}=1$)
  It can be interpreted as "hedging" or "randomization")
- After our prediction, adversary reveals how well experts did represented by $m_i^{(t)}\in[-1,1]$: a.k.a "loss" - large value indicates mistakes.
  Then (our loss on day t) $=\sum_{i\in[N]} p_i^{(t)}\cdot m_i^{(t)}$.

- Goal: do as well as any expert!

<u>**Ideal Thm,**</u> $\forall i\in[N]$, (# my total loss) $\leq O$(# expert i's total loss)
+ (Some function of N)
+ (small function of T)

Algorithm (ε: parameter TBD)

$w_i^{(1)} \leftarrow 1 \quad \forall i \in [N]$.

For $t=1,\dots,T$,

Our prediction $p^{(t)} = w^{(t)} / \Phi^{(t)}$ — still defined as $\sum_{i\in[N]} w_i^{(t)}$.

$m^{(t)}$ is revealed.

For $i \in [N]$, $w_i^{(t+1)} \leftarrow w_i^{(t)} \cdot \exp(-\varepsilon \cdot m_i^{(t)})$. $\quad (\exp(a) := e^a)$

Thm. Let $\varepsilon \le 1$. For any $i \in [N]$, $\underbrace{\sum_{t\in[T]} \langle p^{(t)}, m^{(t)} \rangle}_{\text{my loss}} \le \underbrace{\sum_{t\in[T]} m_i^{(t)}}_{i\text{'s loss}} + \frac{\ln N}{\varepsilon} + \varepsilon T$.

Pf. $-\ \Phi^{(1)} = N$.

$-\ \Phi^{(T+1)} \ge w_i^{(T+1)} = \exp(-\varepsilon \sum_t m_i^{(t)})$.

$-\ \Phi^{(t+1)} = \sum_{i\in[N]} w_i^{(t)} \cdot \exp(-\varepsilon\, m_i^{(t)})$.

$$\le \sum_i w_i^{(t)} \cdot (1 - \varepsilon m_i^{(t)} + (\varepsilon m_i^{(t)})^2) \qquad e^x \le 1+x+x^2 \quad \forall x \in [-1,+1]$$

$$\le \sum_i w_i^{(t)} \cdot (1 - \varepsilon m_i^{(t)} + \varepsilon^2)$$

$$= \sum_i w_i^{(t)} (1+\varepsilon^2) - \sum_i w_i^{(t)} \cdot \varepsilon \cdot m_i^{(t)}$$

$$= \Phi^{(t)} (1+\varepsilon^2) - \sum_i \Phi^{(t)} \cdot \varepsilon \cdot p_i^{(t)} m_i^{(t)}$$

$$= \Phi^{(t)} (1 + \varepsilon^2 - \varepsilon \langle p^{(t)}, m^{(t)} \rangle)$$

$$\le \Phi^{(t)} \exp(\varepsilon^2 - \varepsilon \langle p^{(t)}, m^{(t)} \rangle) \qquad e^x \ge 1+x \quad \forall x \in \mathbb{R}$$

So, $\exp(-\varepsilon \sum_t m_i^{(t)}) \le \Phi^{(1)} \cdot \exp(\varepsilon^2 T - \varepsilon \sum_t \langle m^{(t)}, p^{(t)} \rangle)$.

$\Rightarrow\ -\varepsilon \sum_t m_i^{(t)} \le \ln \Phi^{(1)} + \varepsilon^2 T - \varepsilon \sum_t \langle m^{(t)}, p^{(t)} \rangle$ $\square$