Name: Hamed Vaheb

High Dimensional Statistics: Exercise Sheet 2

The exercises that I ask to be corrected are: 1, 4

1. Show: If $Z \sim \mathcal{N}_n(0, \Sigma)$ and Σ is invertible, then $Z^T\Sigma^{-1}Z \sim \chi^2_{(n)}$.

Solution. In general, we can decompose an arbitrary matrix P in the form $P = U^{-1}DU$, where D is the diagonal matrix containing eigenvalues of P, i.e., $D = diag(\lambda_1, ..., \lambda_n)$. If P is symmetric, $U^{-1} = U^T$ and hence $P = U^TDU$.

We first prove that Σ is symmetric:

$$\forall i, j \in \{1, ..n\}, (\Sigma)_{ij} = cov(Z_i, Z_j) = cov(Z_j, Z_i) = (\Sigma)_{ji}$$

Therefore, we can decompose Σ^{-1} as $\Sigma^{-1} = U^TDU$ and hence we can state the following:

$$\begin{aligned} Z^T\Sigma^{-1}Z &= Z^T(U^TDU)Z \\ &= (Z^TU^T)D(UZ) \\ &= (UZ)^TD(UZ) \end{aligned}$$

Now we note that since $Z \sim \mathcal{N}_n(0, \Sigma)$ and $U^TU = I$, then $Y = UZ \sim \mathcal{N}_n(0, \Sigma)$

Hence, we can state that

$$\begin{aligned} (UZ)^TD(UZ) &= Y^TDY \\ &= Y^Tdiag(\lambda_1, ..., \lambda_n)Y \\ &= \sum_{i=1}^{n} \lambda_i {Y_i}^2 \end{aligned}$$

Let $A = \sum_{i=1}^{n} \lambda_i {Y_i}^2$. We know that $Y_i \sim \mathcal{N}_n(0, \sigma I_n)$, and therefore $\frac{Y_i}{\sigma^2} \sim \mathcal{N}_n(0, I_n)$. Then, $\frac{1}{\sigma^2}A = \frac{1}{\sigma^2}\sum_{i=1}^{n} \lambda_i {Y_i}^2$ will be summation of squares of independent normal random variables, which follow a Chi-squared distribution, i.e., $\frac{A}{\sigma^2} \sim \chi^2_{(n)}$, and since σ^2 is a constant, we can deduce that $A = \sum_{i=1}^{n} \lambda_i {Y_i}^2 \sim \chi^2_{(n)}$