Linear shift-invariant image filtering

- Replace each pixel by a *linear* combination of its neighbors (and possibly itself).
- The combination is determined by the filter's *kernel*.
- The same kernel is *shifted* to all pixel locations so that all pixels use the same linear combination of their neighbors.

Motivation

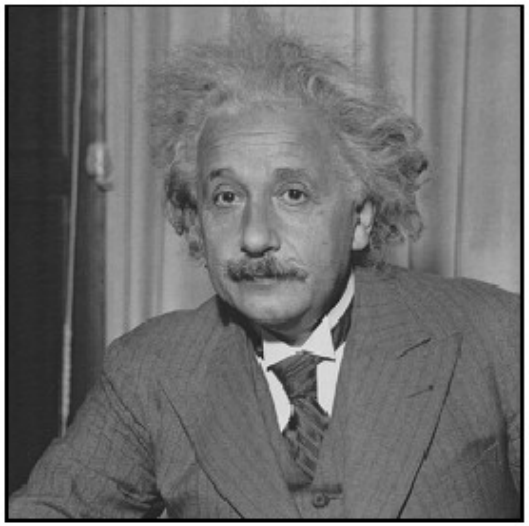
Filtering:

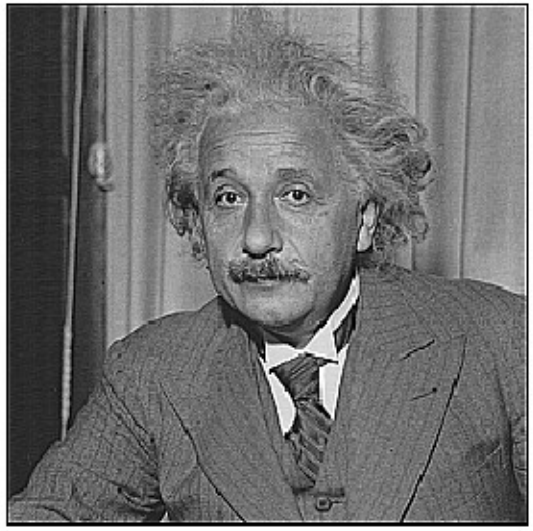
- Form a new image whose pixels are a combination original pixel values

Goals:

- Extract useful information from the images
  - Features (edges, corners, blobs…)
- Modify or enhance image properties:
  - super-resolution; in-painting; de-noising

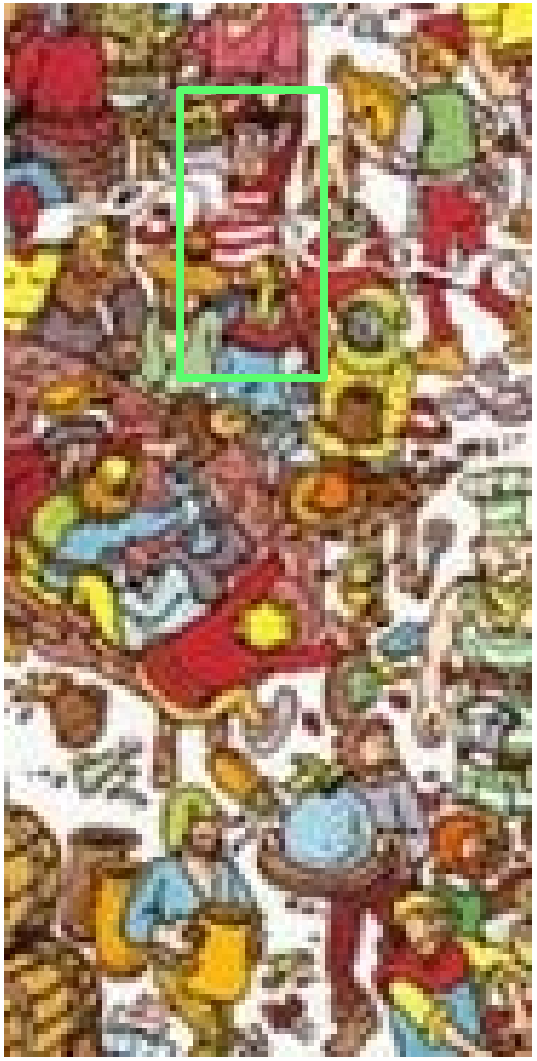
Image Filters





Smooth/Sharpen Images...

Find edges...

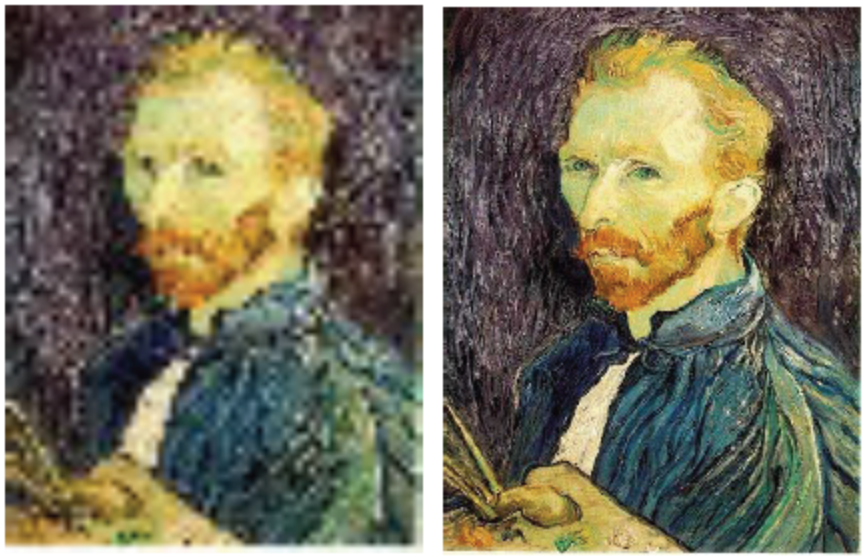


Find waldo…

De-noising

Salt and pepper noise

Super-resolution



In-painting

Bertamio et al

Source: F.F. Li

Convolution for 2D discrete signals

Definition of filtering as convolution:

notice the flip

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j) I(x - i, y - j)$$

filtered image

filter

input image

Convolution for 2D discrete signals

Definition of filtering as convolution:

notice the flip

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j) I(x - i, y - j)$$

filtered image

filter

input image

If the filter $f(i, j)$ is non-zero only within $-1 \leq i, j \leq 1$, then

$$(f * g)(x, y) = \sum_{i,j=-1}^{1} f(i, j) I(x - i, y - j)$$

The kernel we saw earlier is the 3x3 matrix representation of $f(i, j)$.

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 1, n = 1$

$k, l = [-1,0,1]$

Credit: S. Seitz

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10							

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 2, n = 1$

$k, l = [-1,0,1]$

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20						

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 3, n = 1$

$k, l = [-1,0,1]$

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30					

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 4, n = 1$

$k, l = [-1,0,1]$

Credit: S. Seitz

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30				

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 5, n = 1$

$k, l = [-1,0,1]$

Credit: S. Seitz

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30				
				?					

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$m = 4, n = 6$

$k, l = [-1,0,1]$

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30				
						?			
				50					

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

$$m = 6, n = 4$$
$$k, l = [-1,0,1]$$

Image filtering

$f[\cdot,\cdot]$ $\frac{1}{9}$

1	1	1
1	1	1
1	1	1

$I[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$h[m,n] = \sum_{k,l} f[k,l]\, I[m+k, n+l]$$

Credit: S. Seitz

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge

Source: S. Marschner

Properties of convolution

- Linear & shift invariant
- Commutative:

  $f * g = g * f$
- Associative

  $(f * g) * h = f * (g * h)$
- Identity:

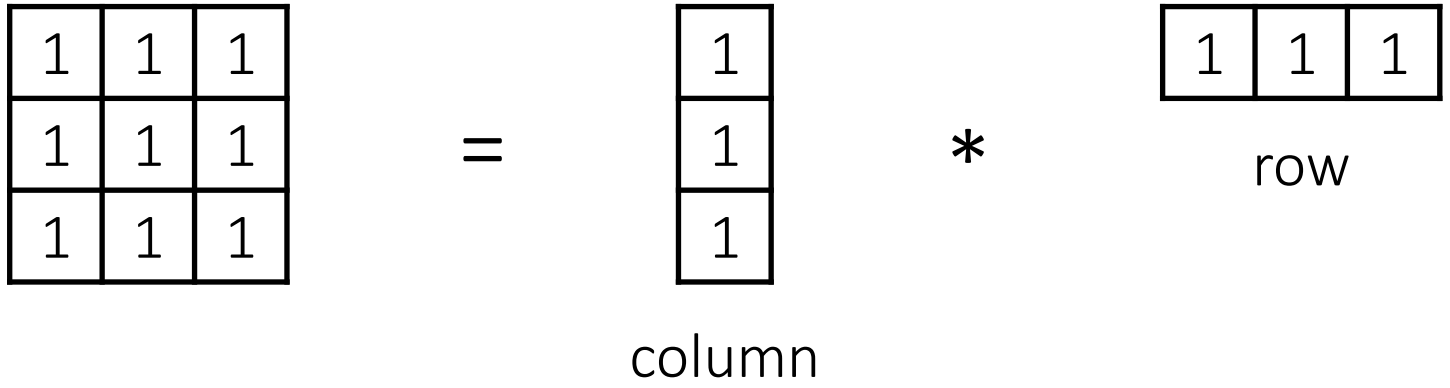
  unit impulse e = […, 0, 0, 1, 0, 0, …]. $f * e = f$
- Differentiation:

$$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$$

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter

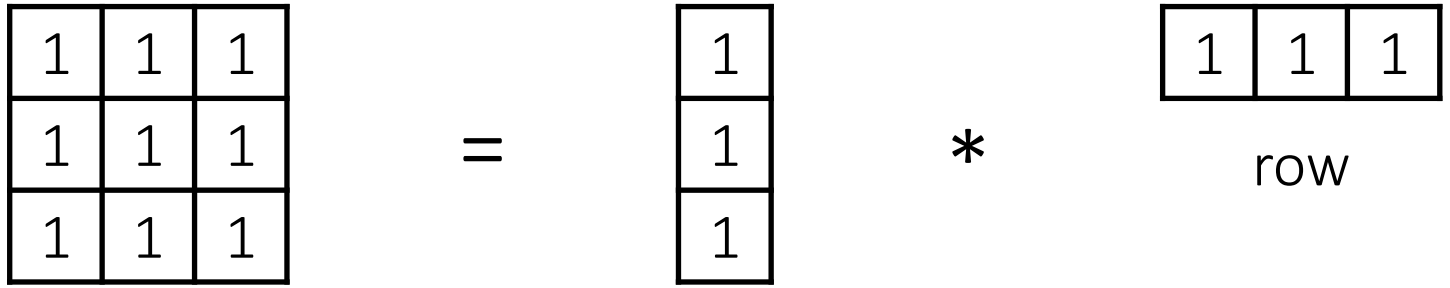


What is the rank of this filter matrix?

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter

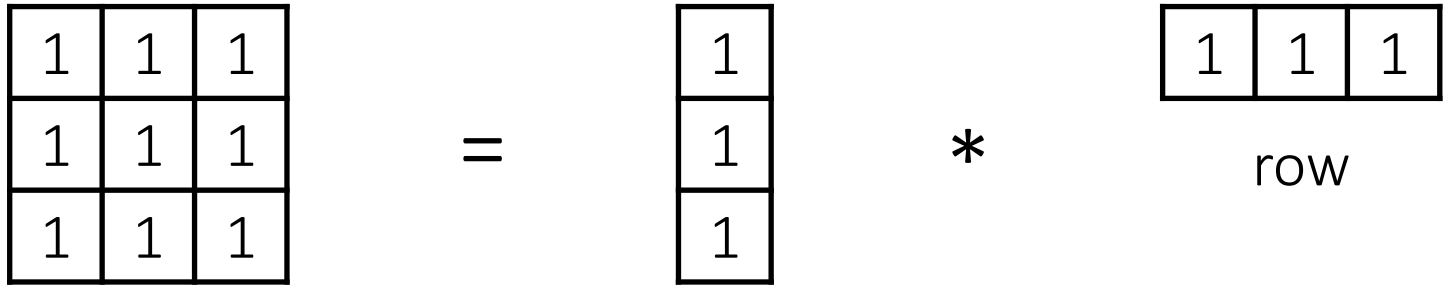


Why is this important?

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter

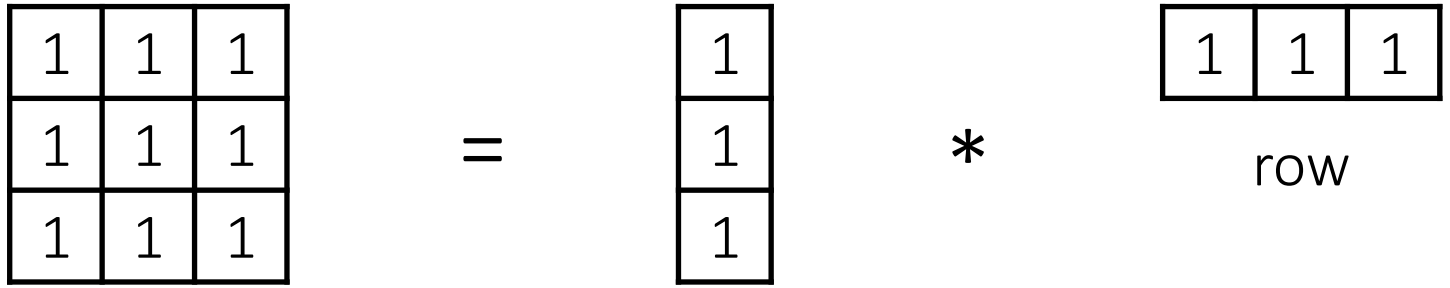


2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter



2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

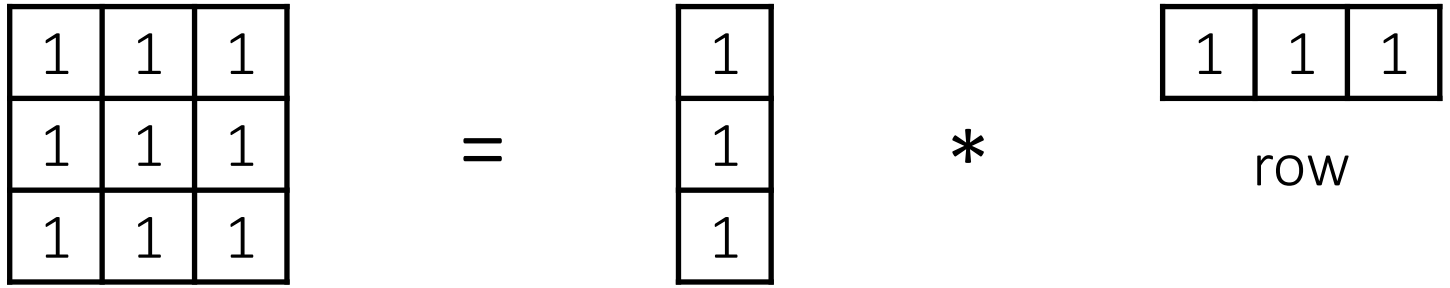
If the image has M x M pixels and the filter kernel has size N x N:

- What is the cost of convolution with a non-separable filter?

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter



2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

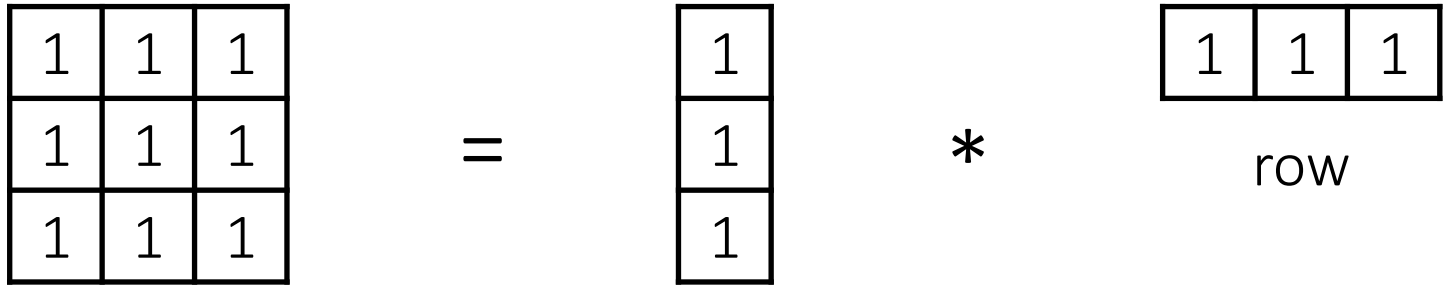
If the image has M x M pixels and the filter kernel has size N x N:

- What is the cost of convolution with a non-separable filter? ⟶ $M^2 \times N^2$
- What is the cost of convolution with a separable filter?

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter



2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

If the image has M x M pixels and the filter kernel has size N x N:

- What is the cost of convolution with a non-separable filter? ⟶ $M^2 \times N^2$
- What is the cost of convolution with a separable filter? ⟶ $2 \times N \times M^2$

Think-Pair-Share time

1.

0	0	0
0	1	0
0	0	0

2.

0	0	0
0	0	1
0	0	0

3.

1	0	-1
2	0	-2
1	0	-1

4.

0	0	0
0	2	0
0	0	0

$-$

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

1. Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

?

Source: D. Lowe

1. Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

Filtered
(no change)

Source: D. Lowe

2. Practice with linear filters

Original

0	0	0
0	0	1
0	0	0

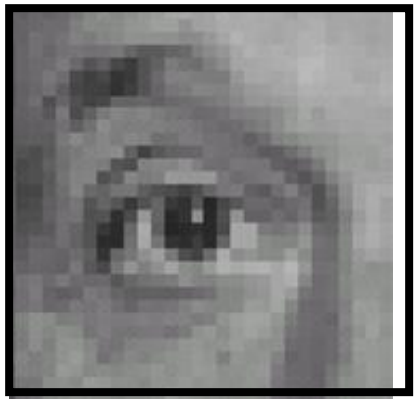
?

Source: D. Lowe

2. Practice with linear filters

Original

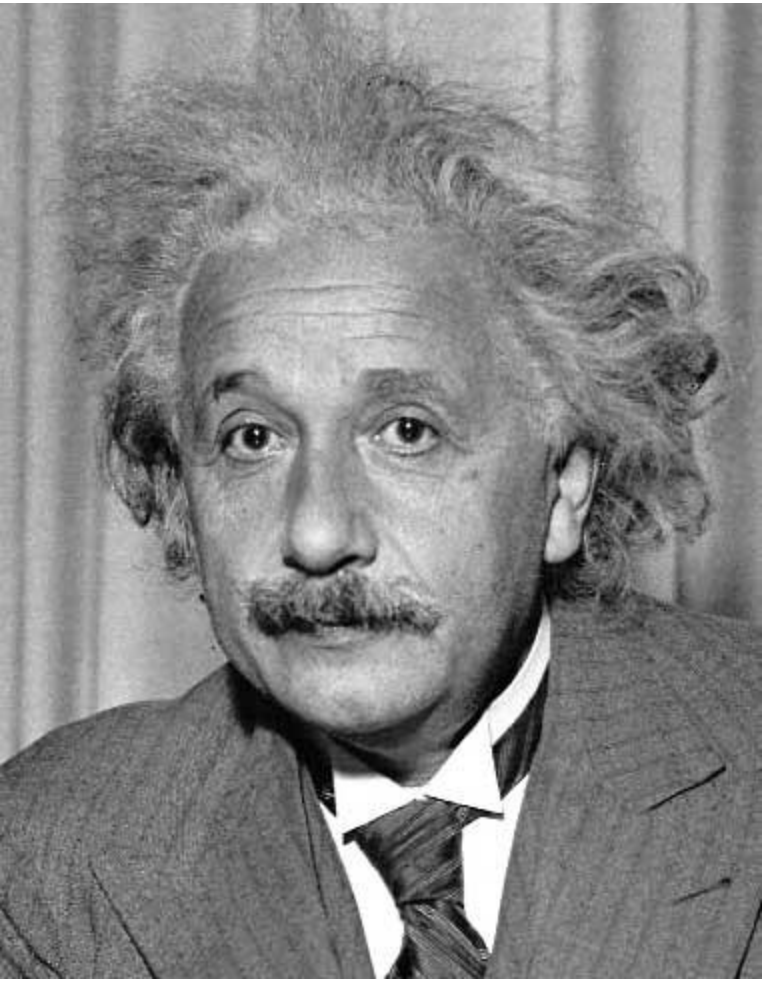
0	0	0
0	0	1
0	0	0



Shifted left
By 1 pixel

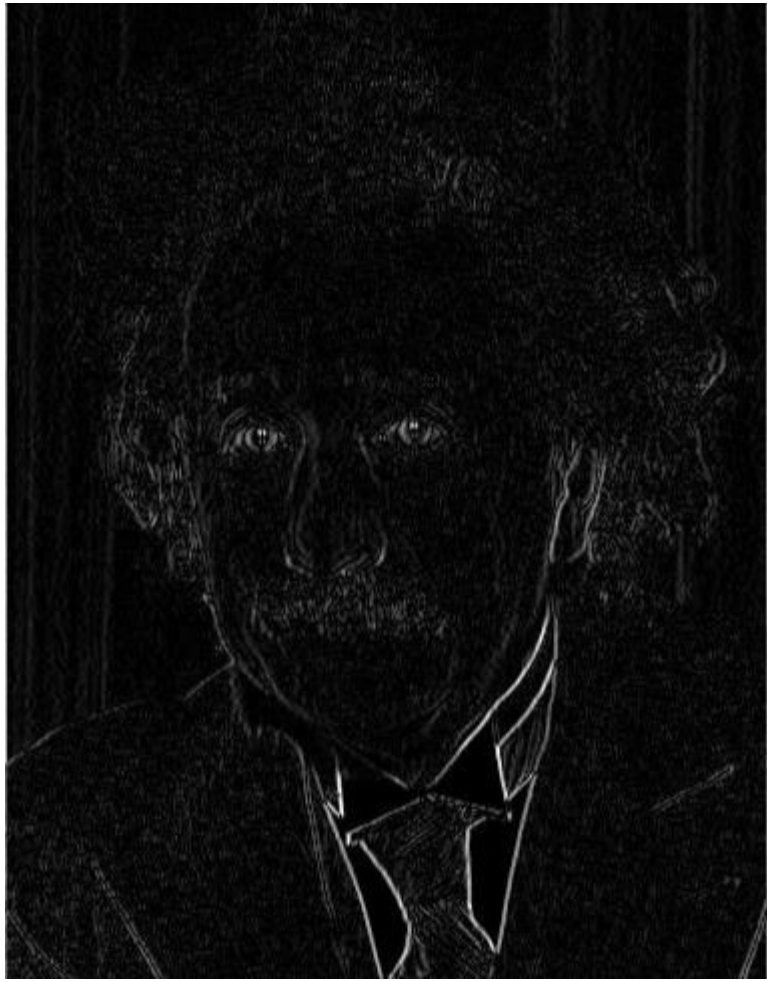
Source: D. Lowe

3. Practice with linear filters



1	0	-1
2	0	-2
1	0	-1

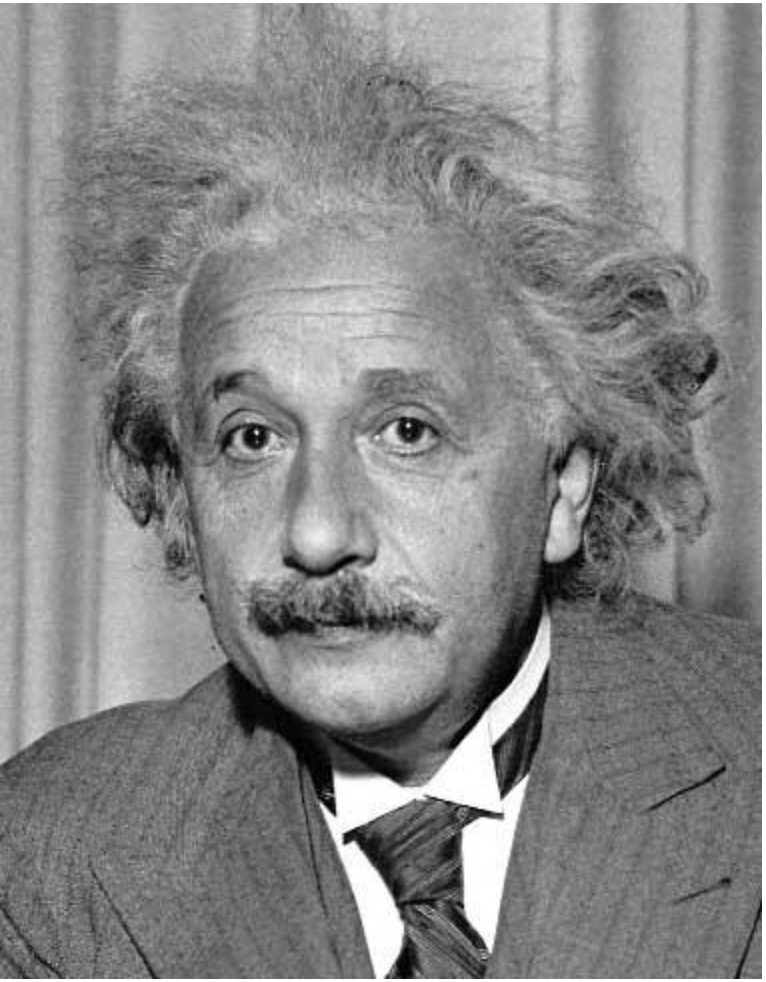
Sobel



Vertical Edge
(absolute value)

David Lowe

3. Practice with linear filters



1	2	1
0	0	0
-1	-2	-1

Sobel

Horizontal Edge
(absolute value)

David Lowe

4. Practice with linear filters

Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad ?$$

(Note that filter sums to 1)

Source: D. Lowe

4. Practice with linear filters

Original

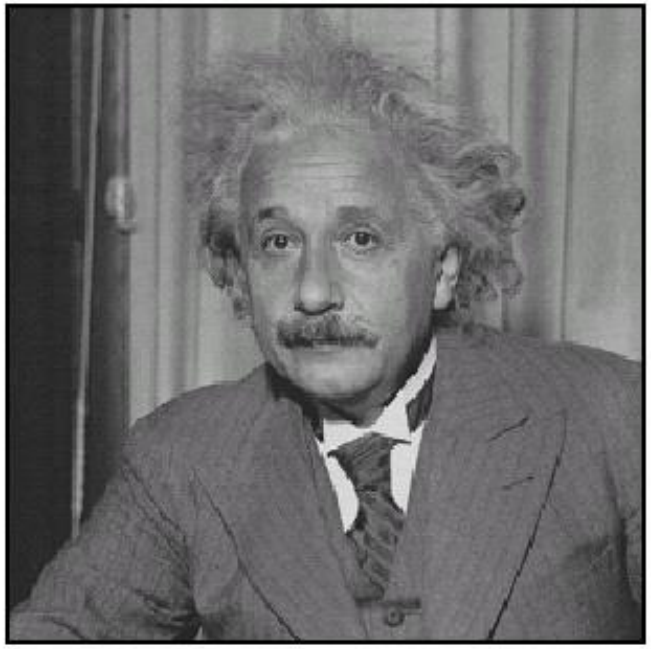
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sharpening filter

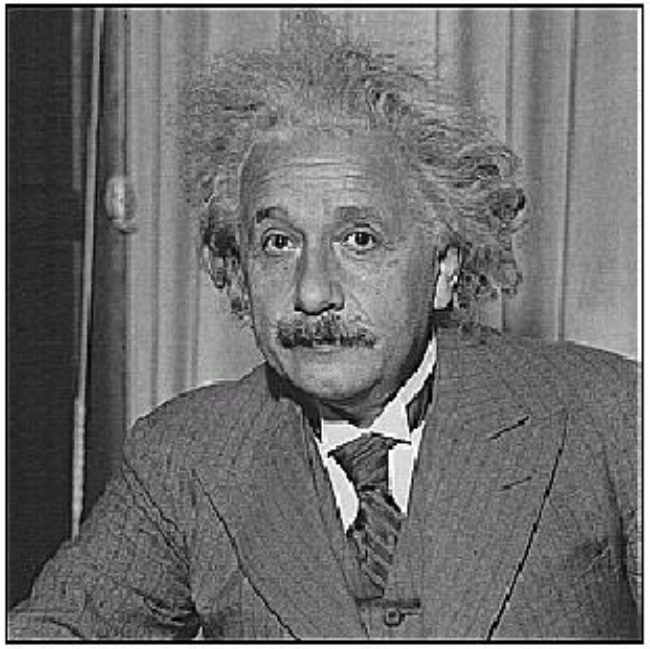
- Accentuates differences with local average

Source: D. Lowe

4. Practice with linear filters



before



after

Source: D. Lowe

Box Filter

What does it do?

- Replaces each pixel with an average of its neighborhood

- Achieve smoothing effect (remove sharp features)

$f[\cdot,\cdot]$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Slide credit: David Lowe (UBC)

Box Filter

What does it do?

- Replaces each pixel with an average of its neighborhood
- Achieve smoothing effect (remove sharp features)
- Why does it sum to one?

$f[\cdot,\cdot]$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Slide credit: David Lowe (UBC)

What does blurring remove?

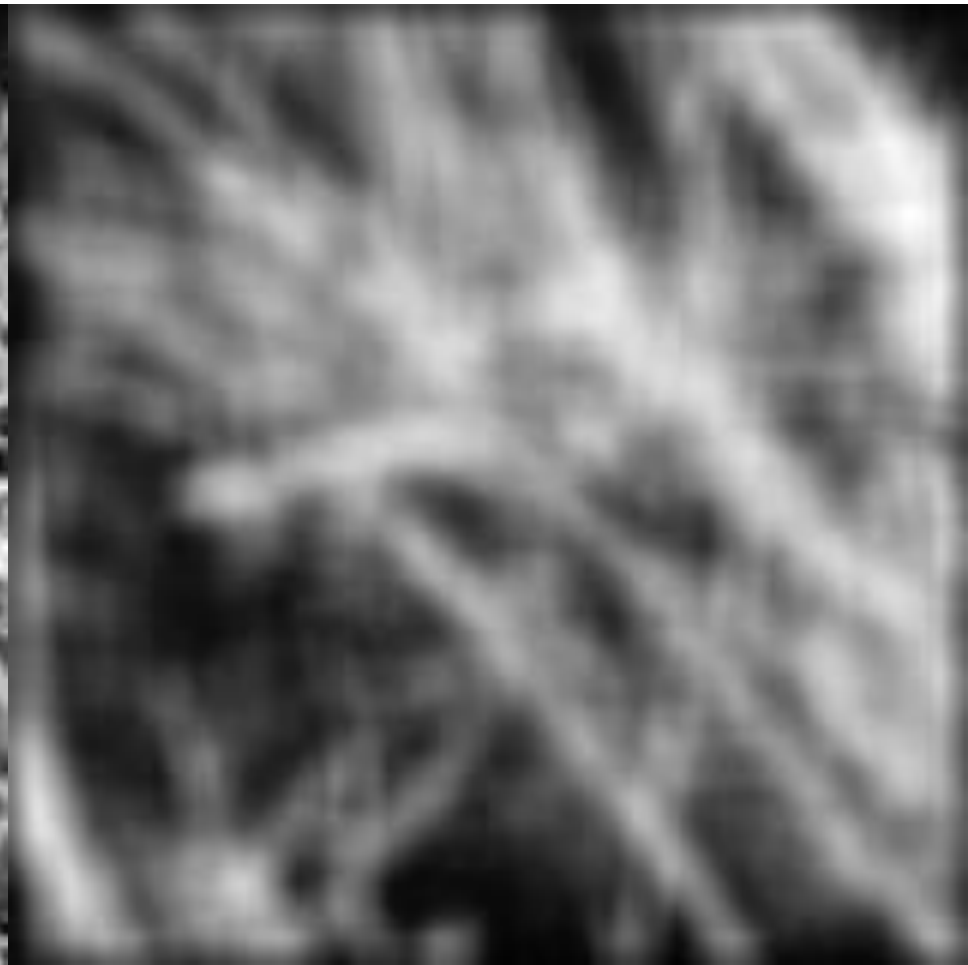
Now let's add it back……

Smoothing with box filter

David Lowe

$$f[\cdot,\cdot]$$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



A few more filters

original

3x3 box filter

do you see
any problems
in this image?

Example: Smoothing by Averaging

Gaussian filters

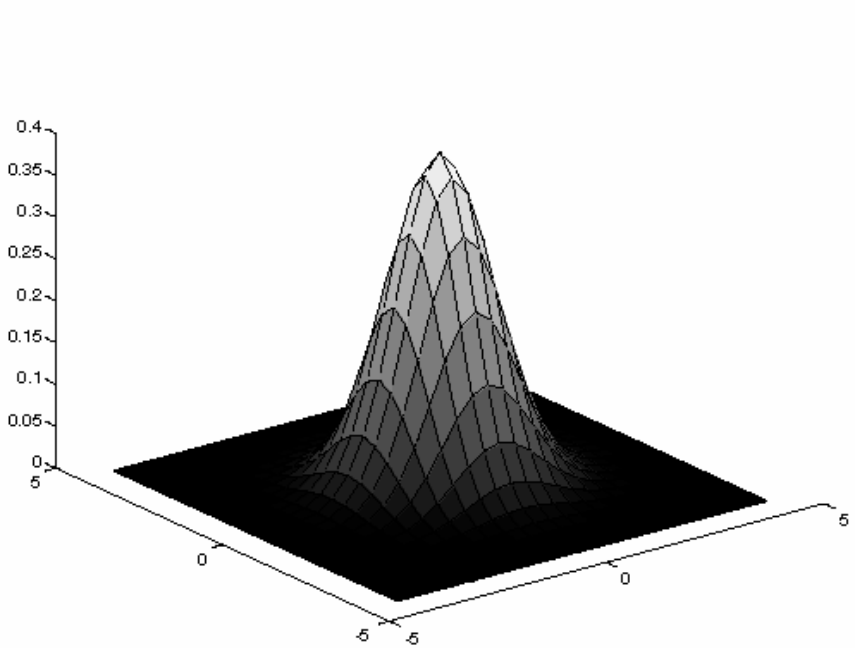
- Remove “high-frequency” components from the image (a low-pass filter)
  - Images become more smooth
- Gaussian convolved with Gaussian…

  …is another Gaussian
  - So can smooth with small-width kernel, repeat, and get same result as larger-width kernel would have
  - Convolving twice with Gaussian kernel of width σ is same as convolving once with kernel of width $\sigma\sqrt{2}$
- *Separable* kernel
  - Factors into product of two 1D Gaussians

Source: K. Grauman

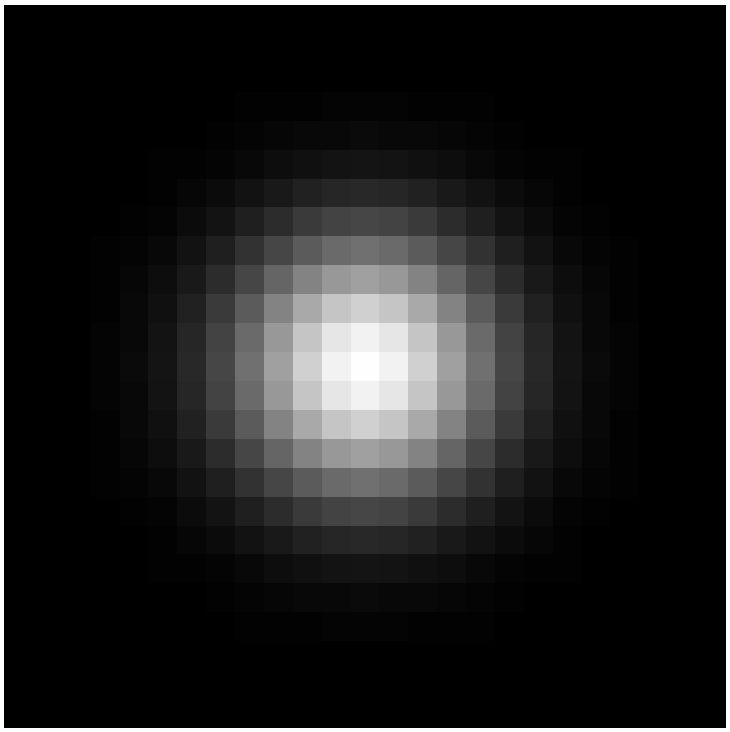
Gaussian Averaging

- Rotationally symmetric.
- Weights nearby pixels more than distant ones.
  - This makes sense as probabalistic inference.



- A Gaussian gives a good model of a fuzzy blob

An Isotropic Gaussian



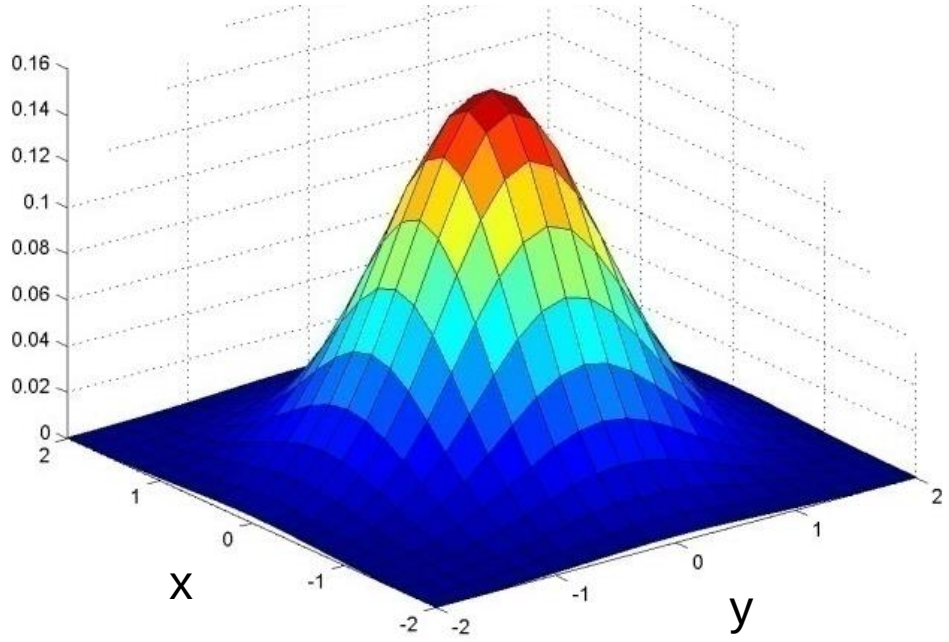
- The picture shows a smoothing kernel proportional to

$$\exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right)$$

(which is a reasonable model of a circularly symmetric fuzzy blob)

Important filter: Gaussian

Weight contributions of neighboring pixels by nearness

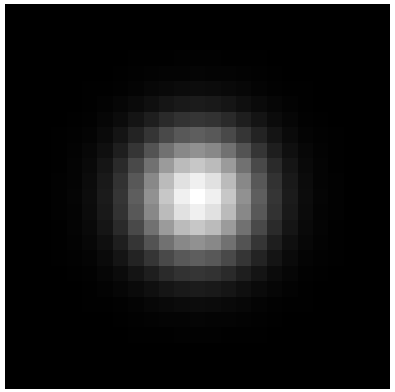


x

y					
	0.003	0.013	0.022	0.013	0.003
	0.013	0.059	0.097	0.059	0.013
	0.022	0.097	0.159	0.097	0.022
	0.013	0.059	0.097	0.059	0.013
	0.003	0.013	0.022	0.013	0.003

Kernel size 5 x 5,
Standard deviation $\sigma = 1$

Viewed from top



$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

Slide credit: Christopher Rasmussen

Separability of the Gaussian filter

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2}\exp^{-\frac{x^2+y^2}{2\sigma^2}}$$

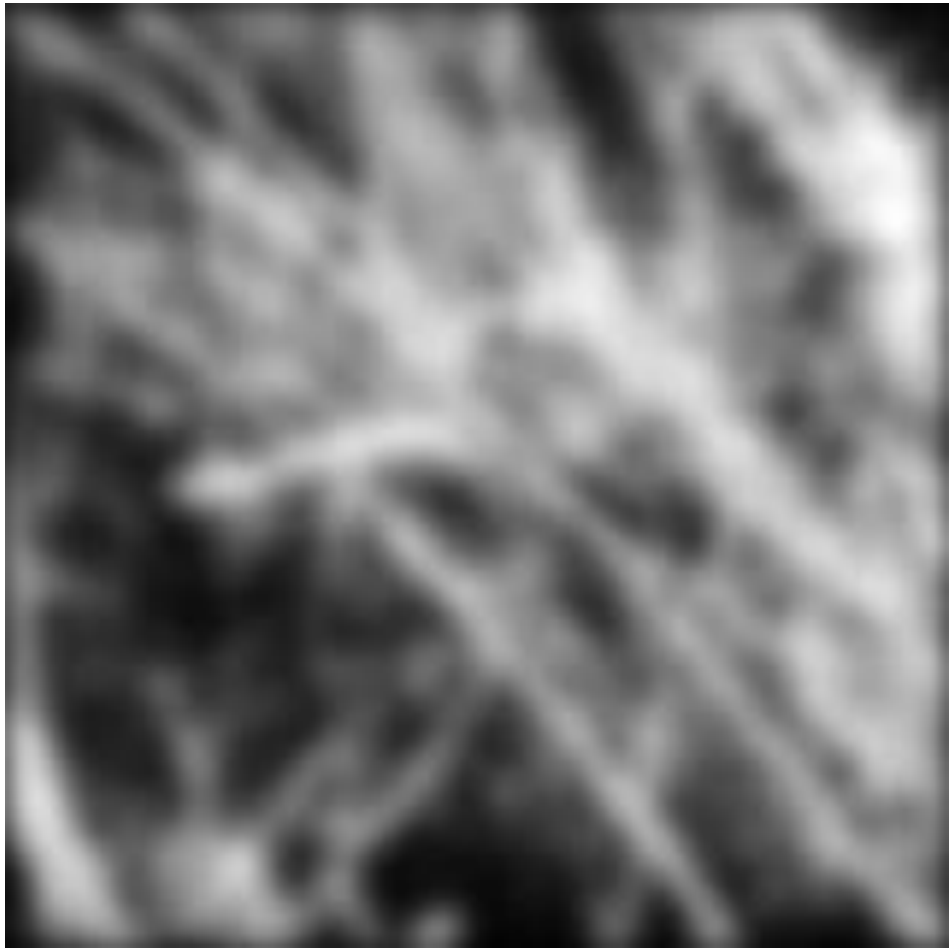
$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\exp^{-\frac{x^2}{2\sigma^2}}\right)\left(\frac{1}{\sqrt{2\pi}\sigma}\exp^{-\frac{y^2}{2\sigma^2}}\right)$$

The 2D Gaussian can be expressed as the product of two functions, one a function of x and the other a function of y

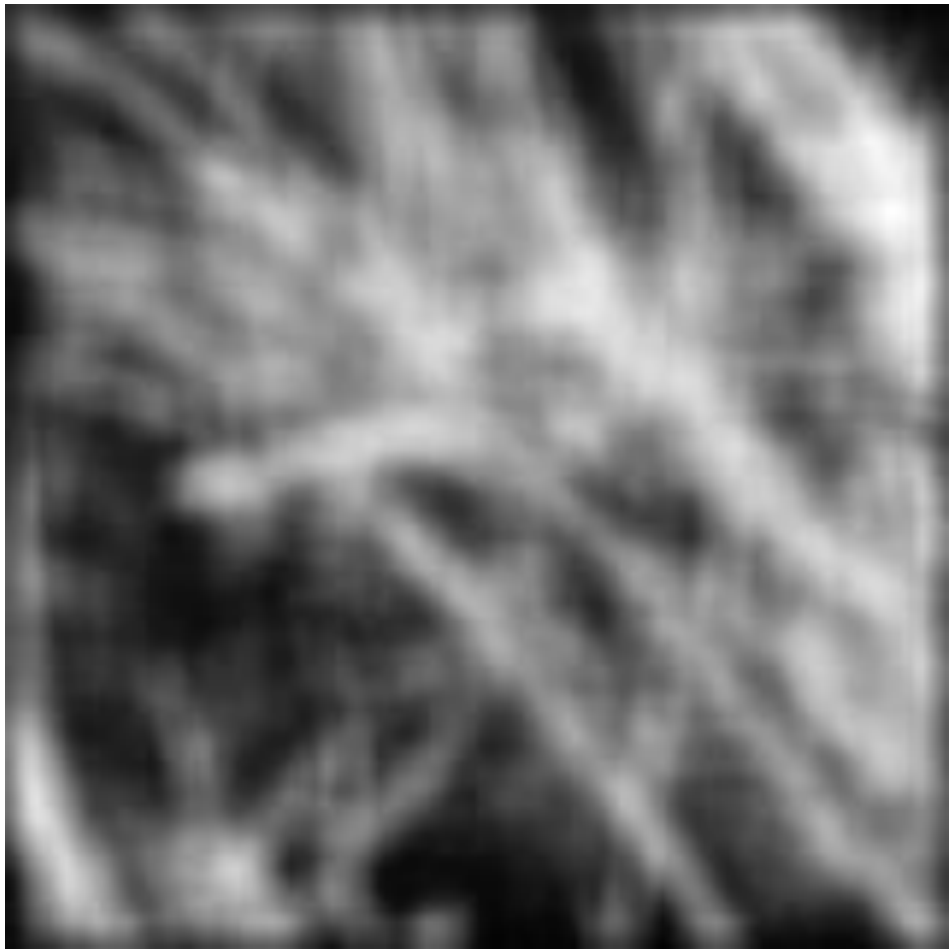
In this case, the two functions are the (identical) 1D Gaussian

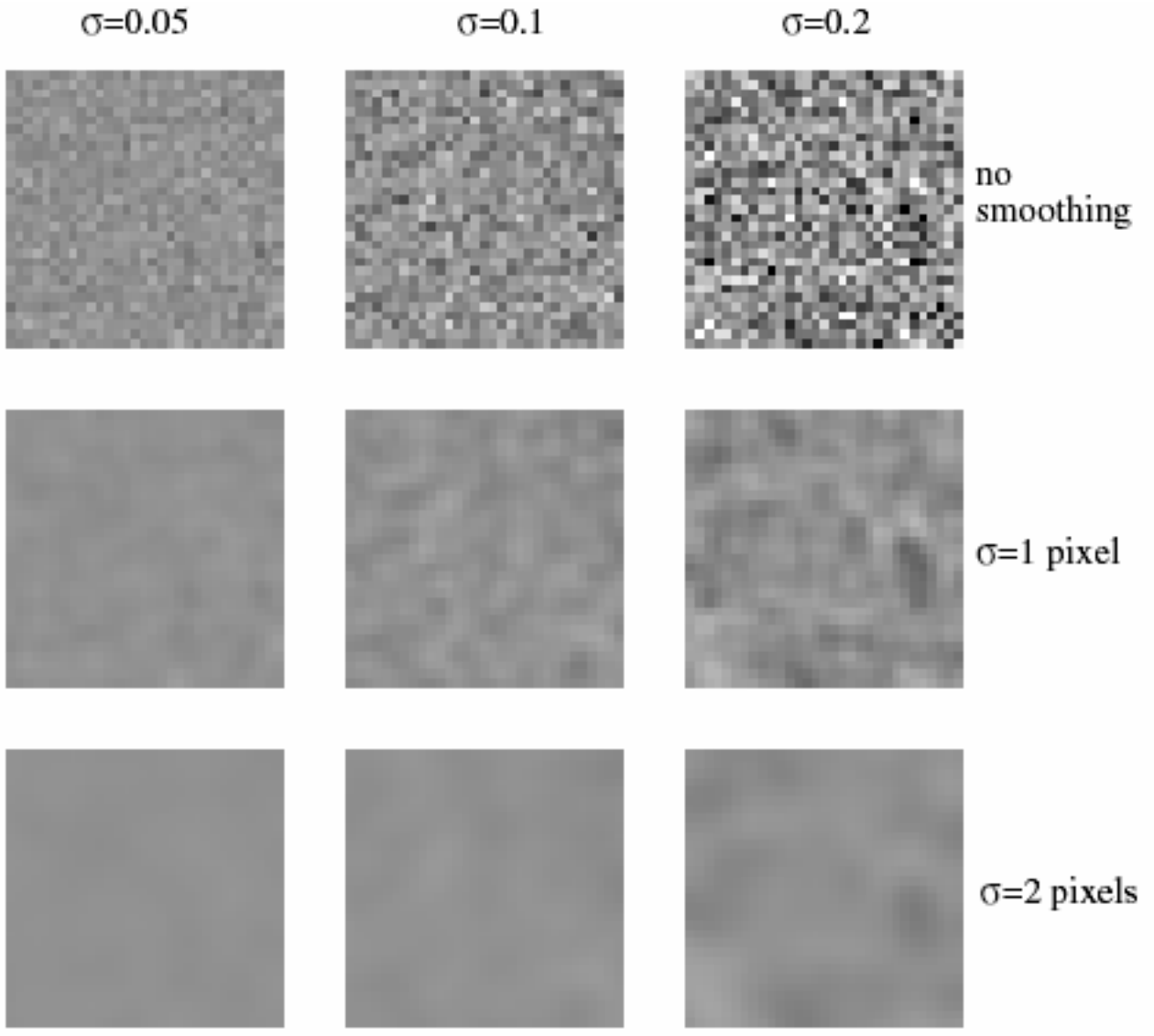
Source: D. Lowe

Smoothing with Gaussian filter



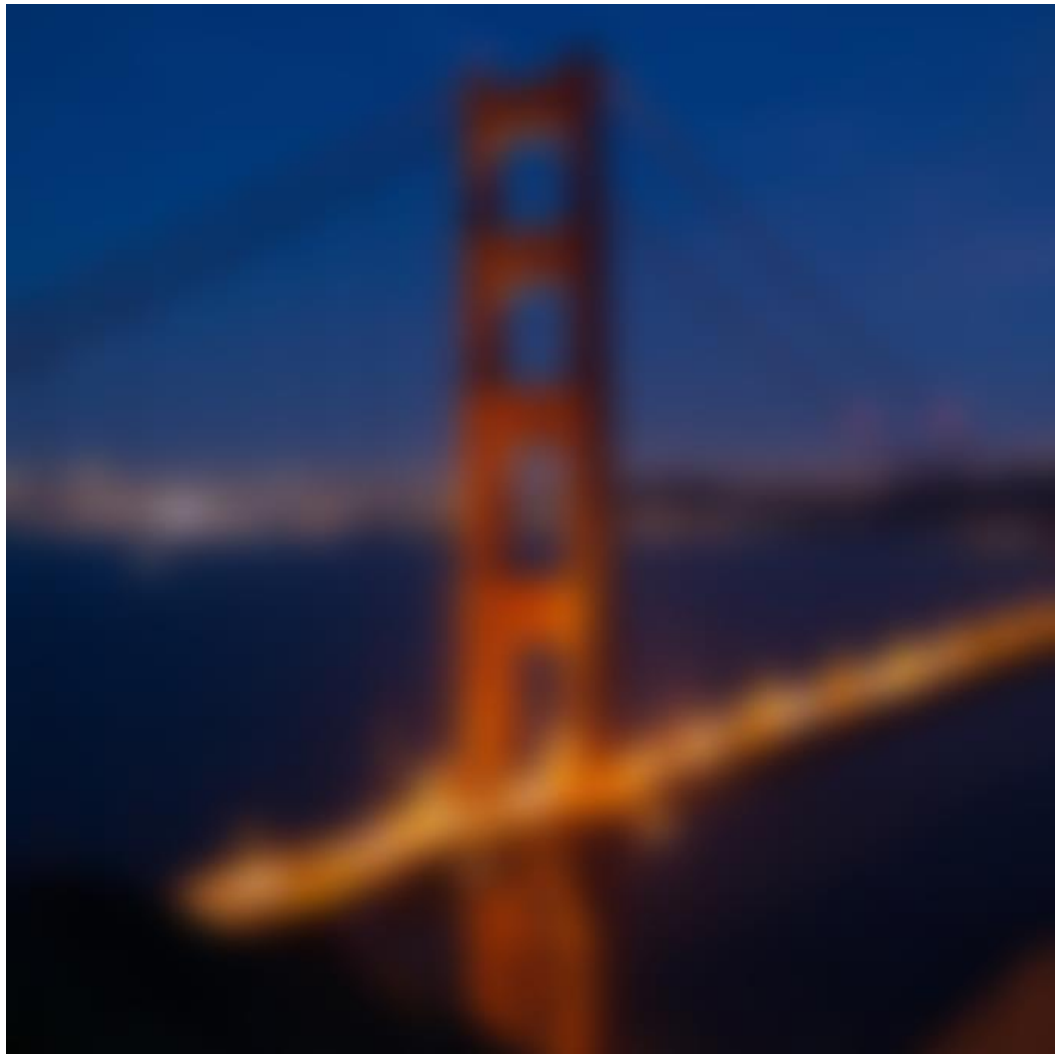
Smoothing with box filter



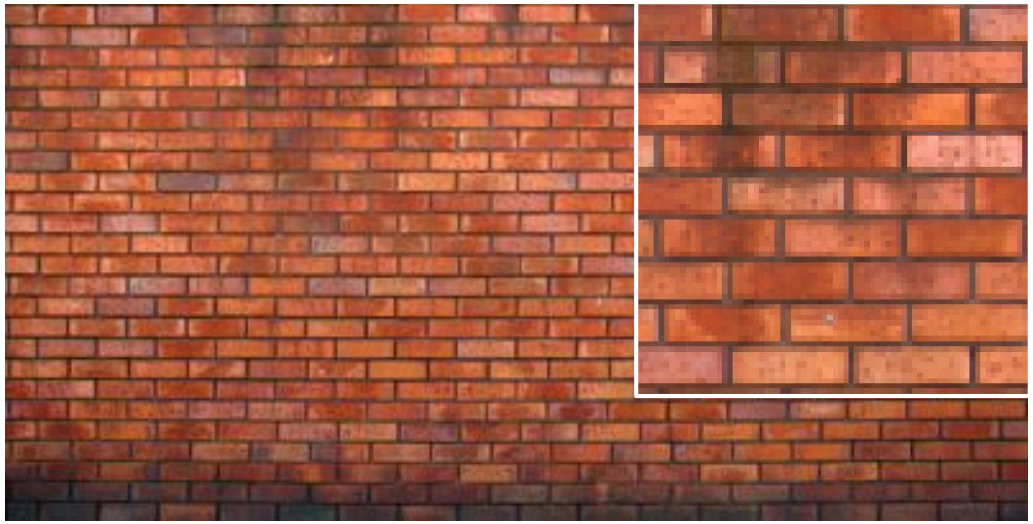


The effects of smoothing
Each row shows smoothing with gaussians of different width; each column shows different realizations of an image of gaussian noise.

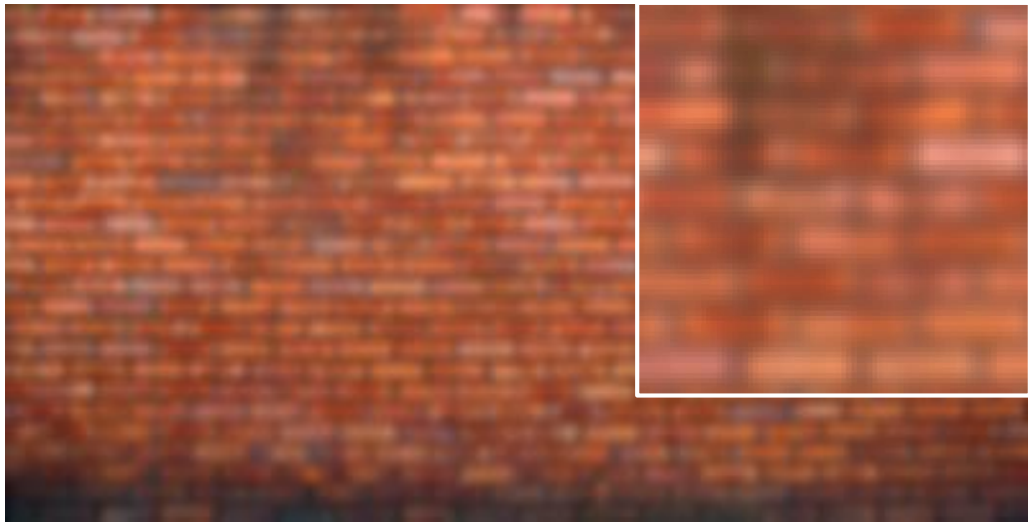
Gaussian filtering example



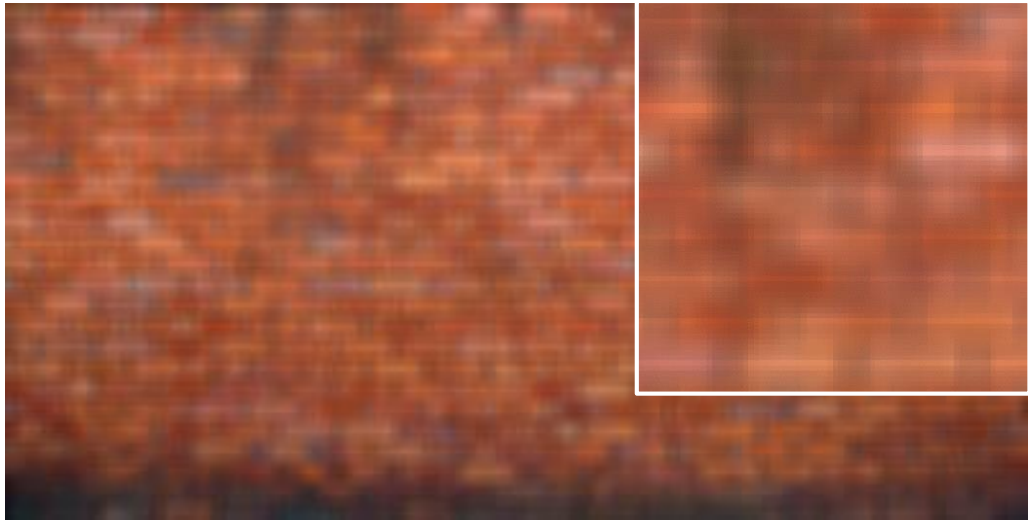
Gaussian vs box filtering



original



7x7 Gaussian



7x7 box

Which blur do you like better?

Edge Filters

- Sobel

$\frac{1}{8}$

-1	0	1
-2	0	2
-1	0	1

$\frac{1}{8}$

1	2	1
0	0	0
-1	-2	-1

- Prewitt

$\frac{1}{6}$

-1	0	1
-1	0	1
-1	0	1

$\frac{1}{6}$

1	1	1
0	0	0
-1	-1	-1

- Roberts

1	0
0	-1

0	1
-1	0

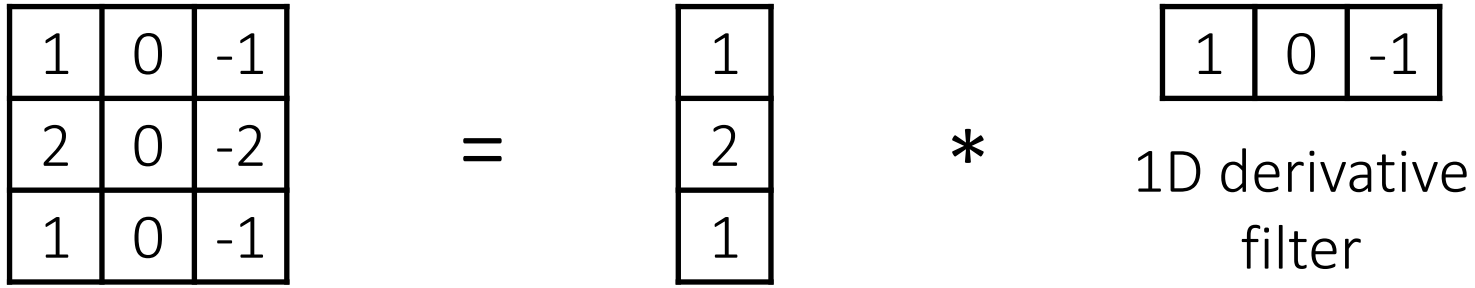
- Laplacian

0	1	0
1	-4	1
0	1	0

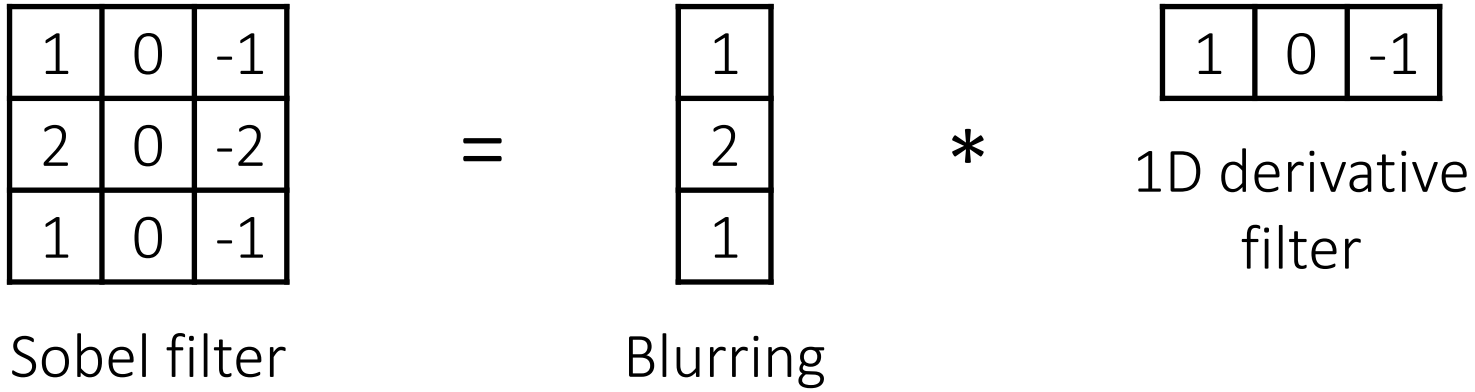
$\frac{1}{3}$

1	1	1
1	-8	1
1	1	1

The Sobel filter

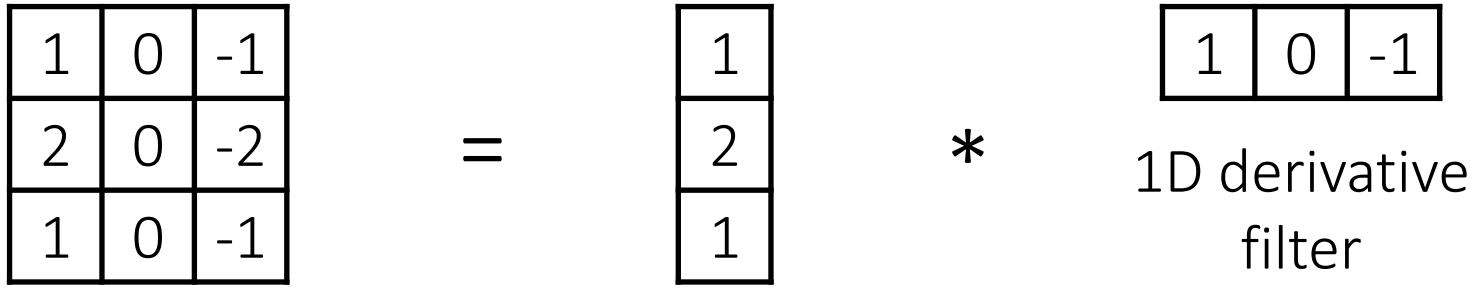


The Sobel filter



In a 2D image, does this filter responses along horizontal or vertical lines?

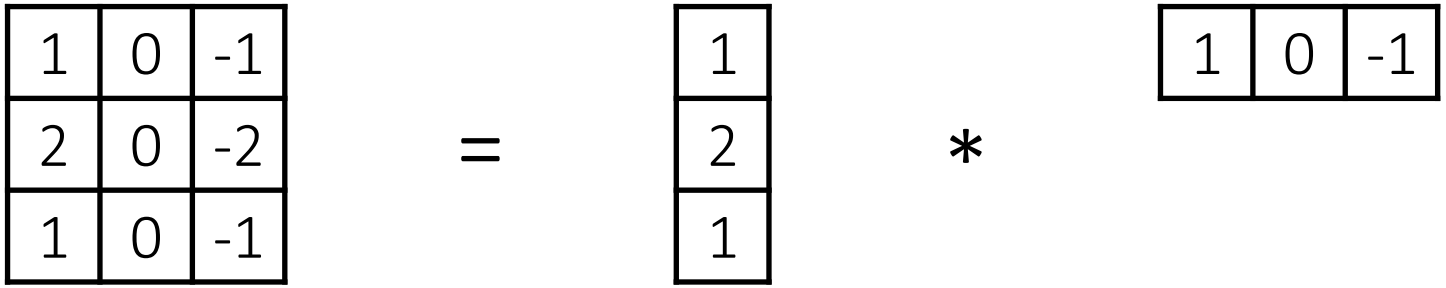
The Sobel filter



Does this filter return large responses on vertical or horizontal lines?

The Sobel filter

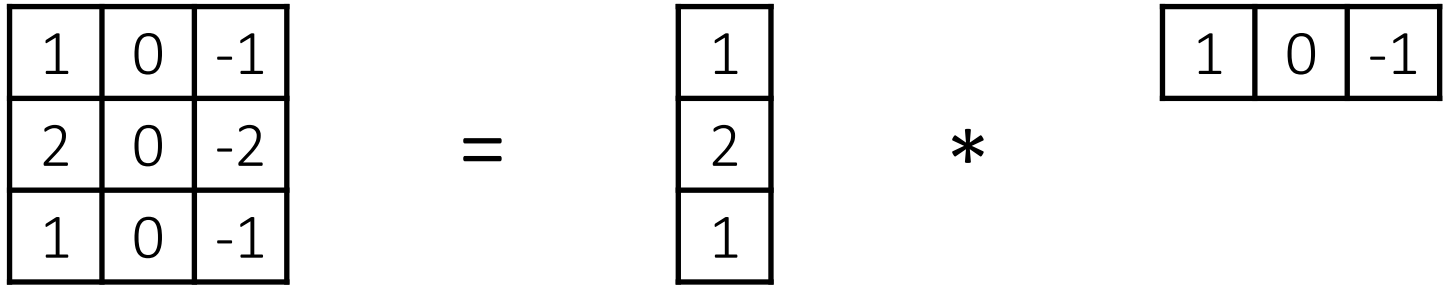
Horizontal Sober filter:



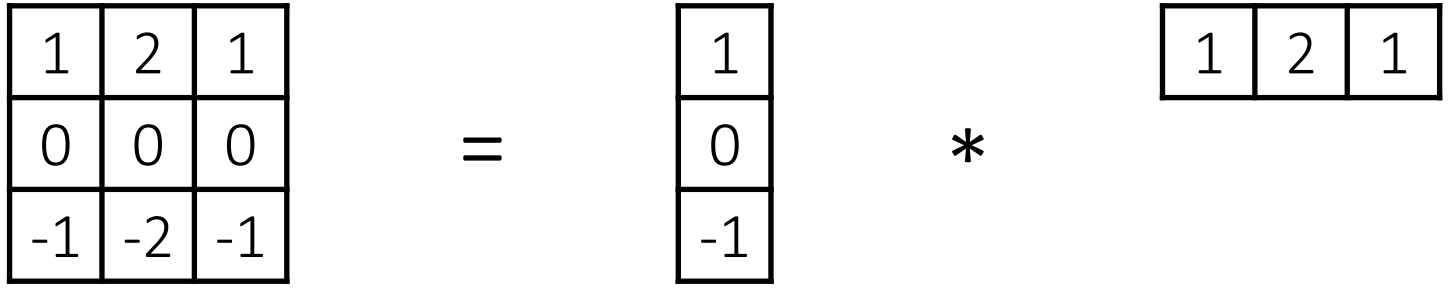
What does the vertical Sobel filter look like?

The Sobel filter

Horizontal Sober filter:

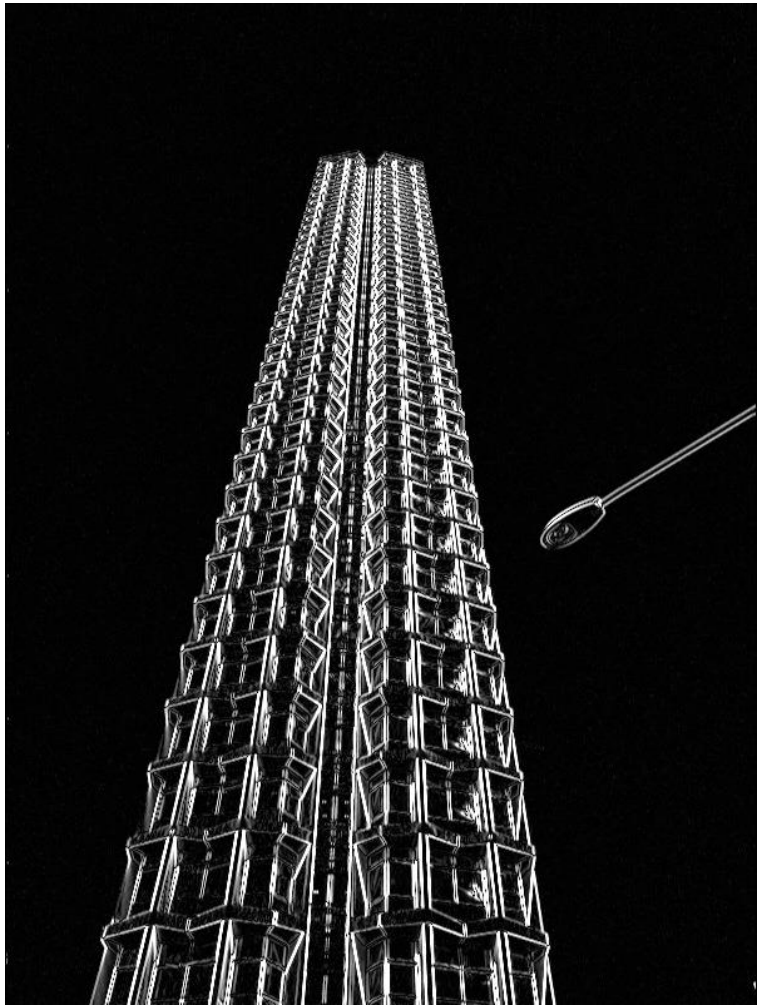


Vertical Sobel filter:

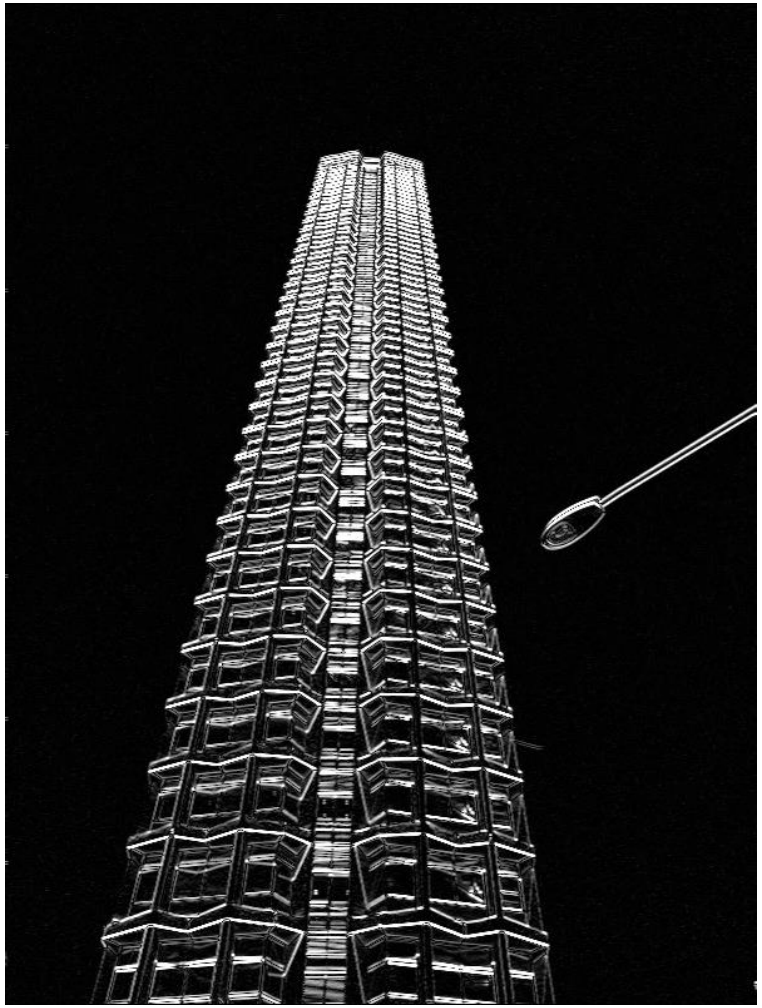


Sobel filter example

original



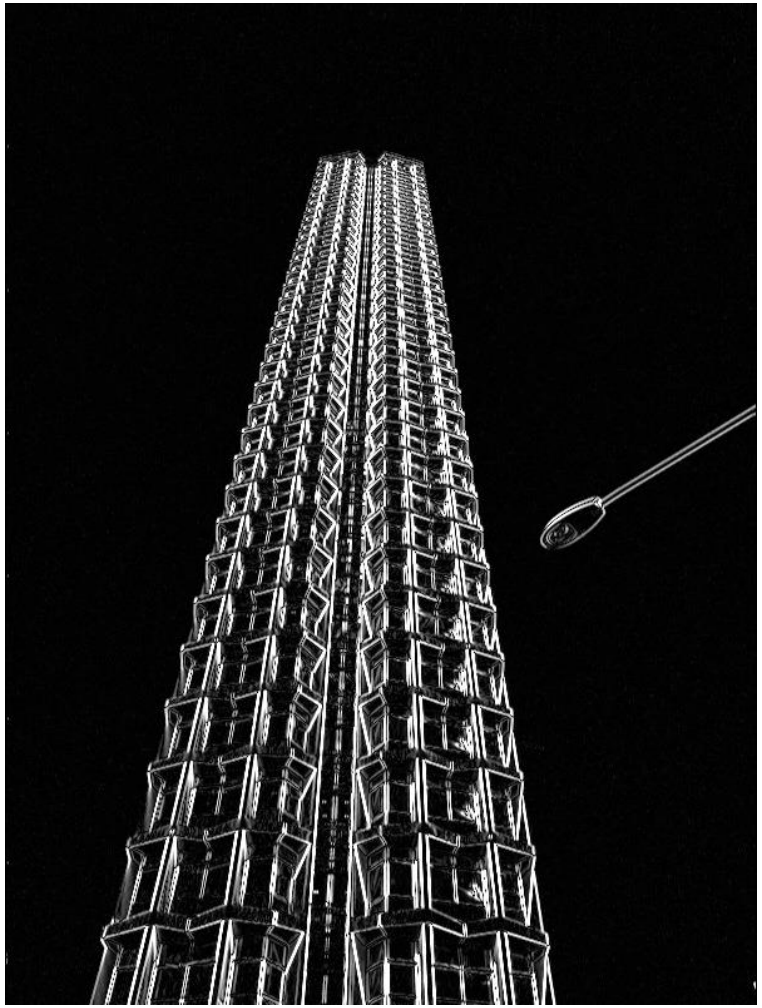
which Sobel filter?



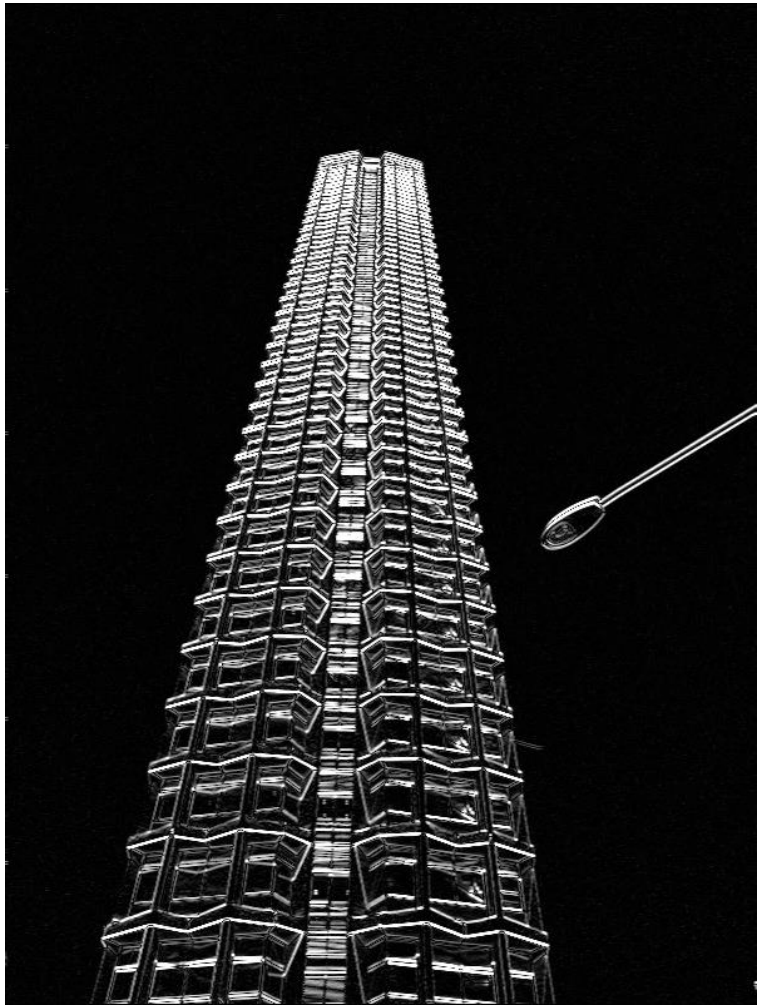
which Sobel filter?

Sobel filter example

original

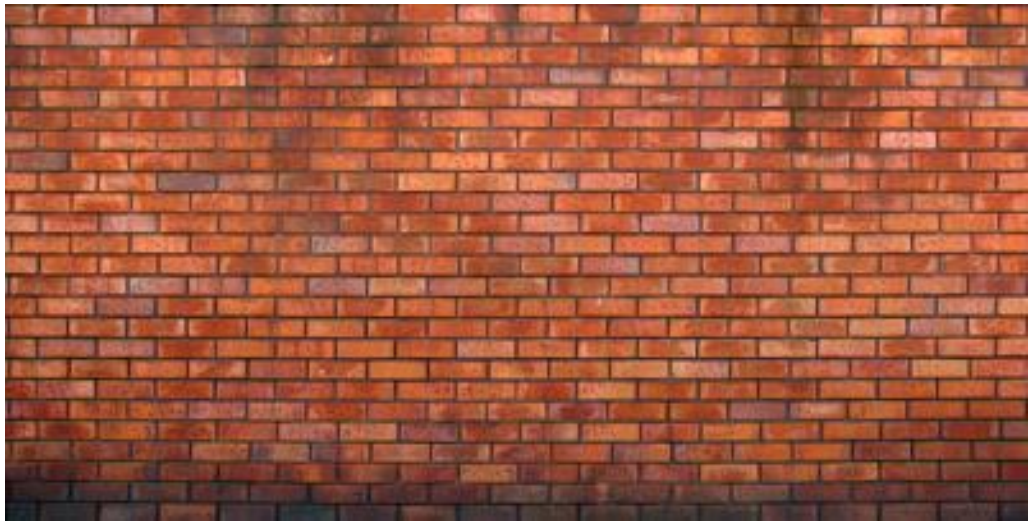


horizontal Sobel filter



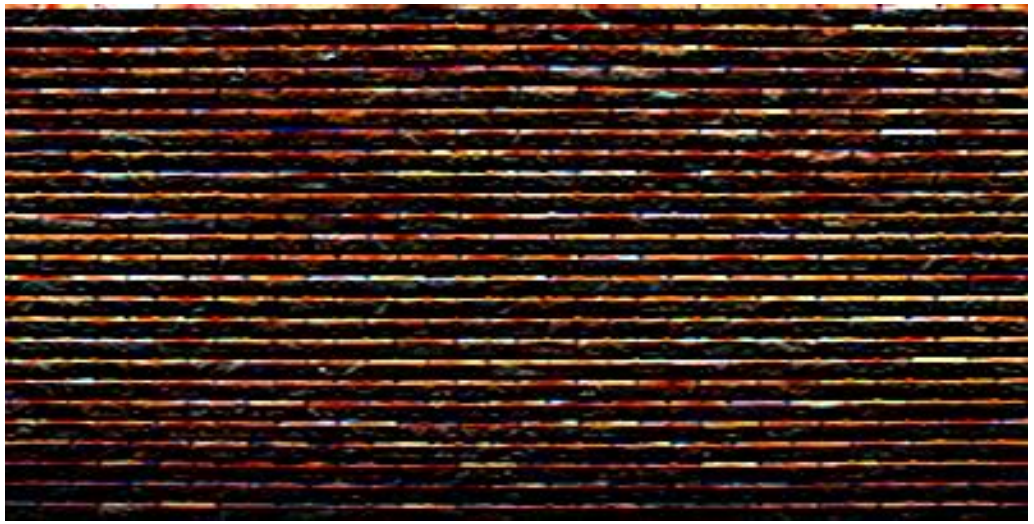
vertical Sobel filter

Sobel filter example



original

horizontal Sobel filter



vertical Sobel filter

Computing image gradients

1. Select your favorite derivative filters.

$$\boldsymbol{S}_x = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \qquad \boldsymbol{S}_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Computing image gradients

1. Select your favorite derivative filters.

$$\boldsymbol{S}_x = \begin{array}{|c|c|c|}\hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} \qquad \boldsymbol{S}_y = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -2 & -1 \\ \hline \end{array}$$

2. Convolve with the image to compute derivatives.

$$\frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{S}_x \otimes \boldsymbol{f} \qquad \frac{\partial \boldsymbol{f}}{\partial y} = \boldsymbol{S}_y \otimes \boldsymbol{f}$$

Computing image gradients

1. Select your favorite derivative filters.

$$\boldsymbol{S}_x = \begin{array}{|c|c|c|}\hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} \qquad \boldsymbol{S}_y = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -2 & -1 \\ \hline \end{array}$$

2. Convolve with the image to compute derivatives.

$$\frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{S}_x \otimes \boldsymbol{f} \qquad \frac{\partial \boldsymbol{f}}{\partial y} = \boldsymbol{S}_y \otimes \boldsymbol{f}$$

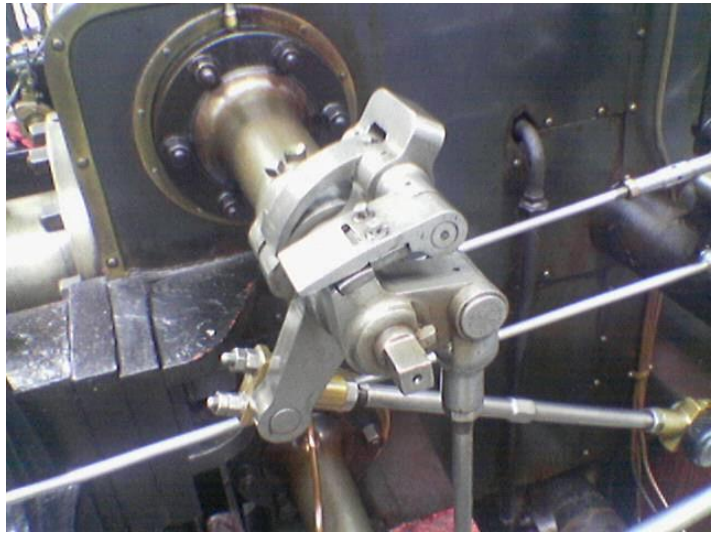
3. Form the image gradient, and compute its direction and amplitude.

$$\nabla \boldsymbol{f} = \left[\frac{\partial \boldsymbol{f}}{\partial x}, \frac{\partial \boldsymbol{f}}{\partial y}\right] \qquad \theta = \tan^{-1}\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right) \qquad \|\nabla f\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

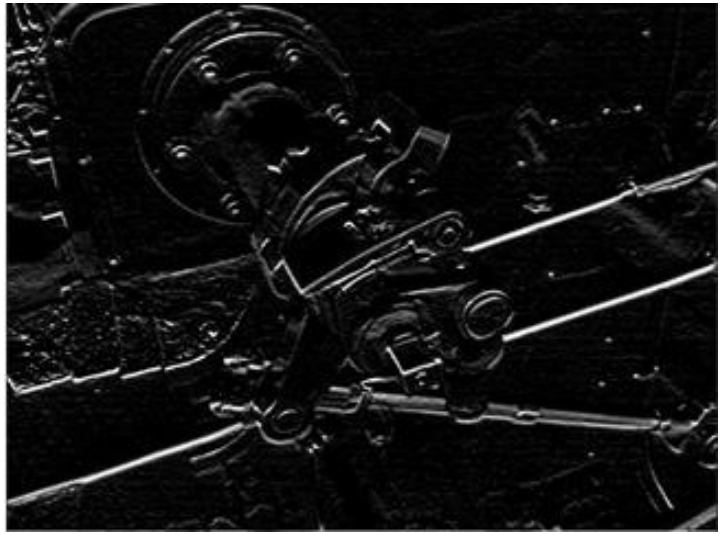
gradient direction amplitude

Image gradient example

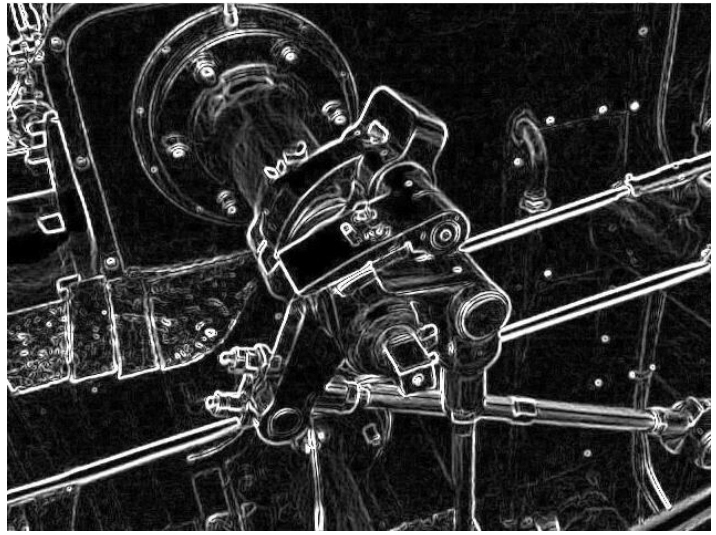
original



vertical derivative



gradient amplitude

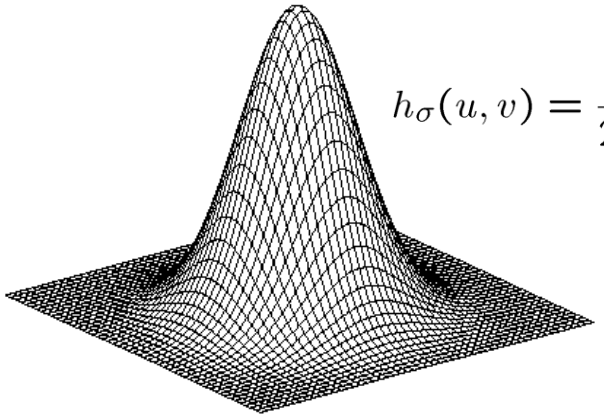


horizontal derivative

How does the gradient direction relate to these edges?

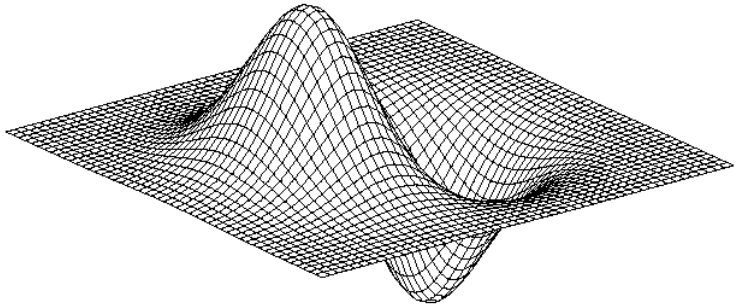
2D Gaussian filters

$$h_\sigma(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}}$$



Gaussian

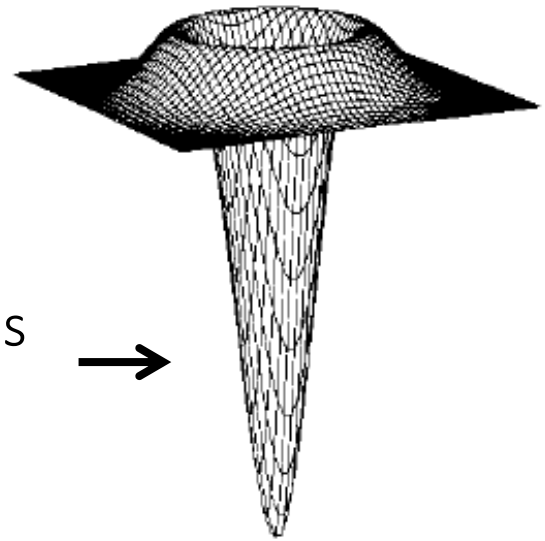
$$\frac{\partial}{\partial x} h_\sigma(u,v)$$



Derivative of Gaussian

$$\nabla^2 h_\sigma(u,v)$$

how does this relate to this lecture's cover picture?



Laplacian of Gaussian

Laplace and LoG filtering examples

Laplacian of Gaussian filtering

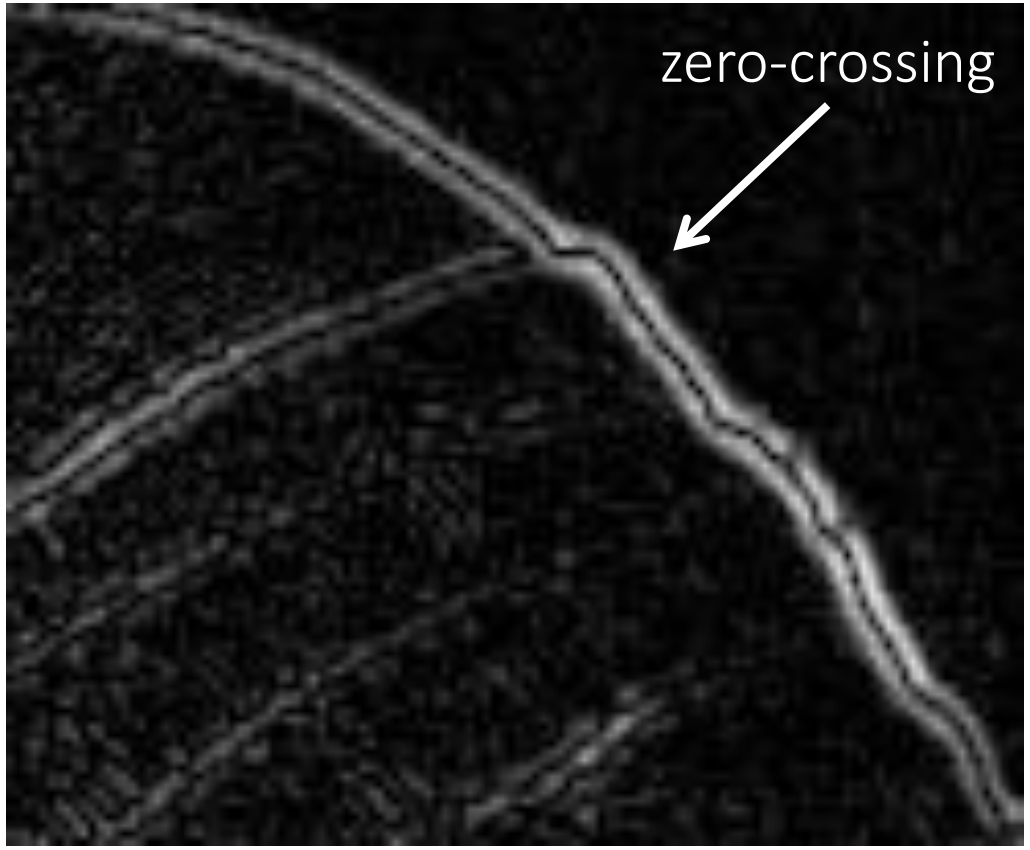
Laplace filtering

Laplacian of Gaussian vs Derivative of Gaussian

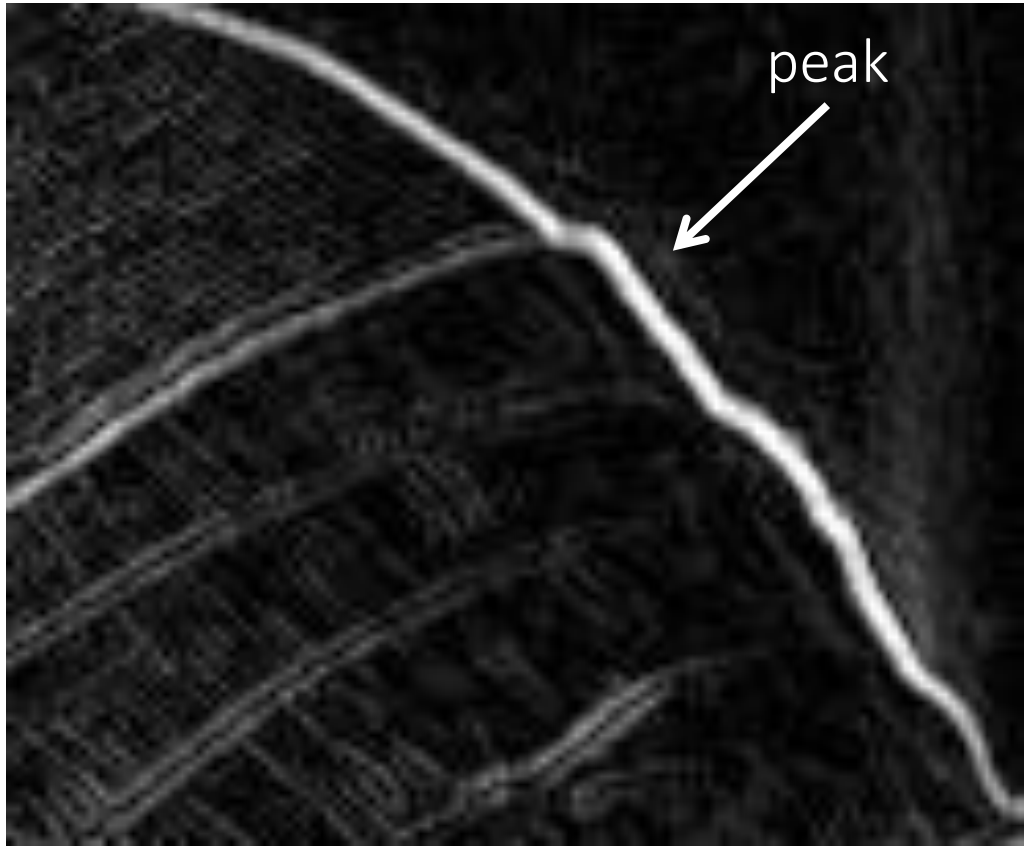
Laplacian of Gaussian filtering

Derivative of Gaussian filtering

Laplacian of Gaussian vs Derivative of Gaussian



Laplacian of Gaussian filtering



Derivative of Gaussian filtering

Zero crossings are more accurate at localizing edges (but not very convenient).