Weekly Assignment 7

RC

2021-10-16

Question 1

$$f(x, y) = x^2 + xy$$

bounded by $y = x^2$ and $y = x + 2$

Find the intersection of the boundaries:

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$x = 2, -1$ and $y = 4, 1$

on $-1 \leq x \leq 2$, $x^2 \leq y \leq x + 2$

$$f_x(x, y) = 2x + y$$

$$f_y(x, y) = x$$

$$f_{xx}(x, y) = 2$$

$$f_{yy}(x, y) = 0$$

$$f_{xy}(x, y) = 1$$

Solve for critial points simultaneously

$$f_x(x, y) = 0, \quad f_y(x, y) = 0$$

$$2x + y = 0x = 0$$

Therefore, $f(x, y)$ has a critical point at $(0, 0, 0)$

Second derivative test:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2 * 0 - 1 = -1$$

which shows it is a saddle point

Check the boundaries:

$$f(x, y) = x^2 + x^3, \quad y = x^2$$

$$f_x(x, y) = 2x + 3x^2 = x(2 + 3x) = 0, \quad y = x^2$$

$f_{xx}(x, y) = 2 + 6x$ is positive at $x = 0$

therfore there is a minimum along the boundary at $x = 0, x = -3/2$. $f(0, 0) = 0$ and $x = -3/2$ is outside the region

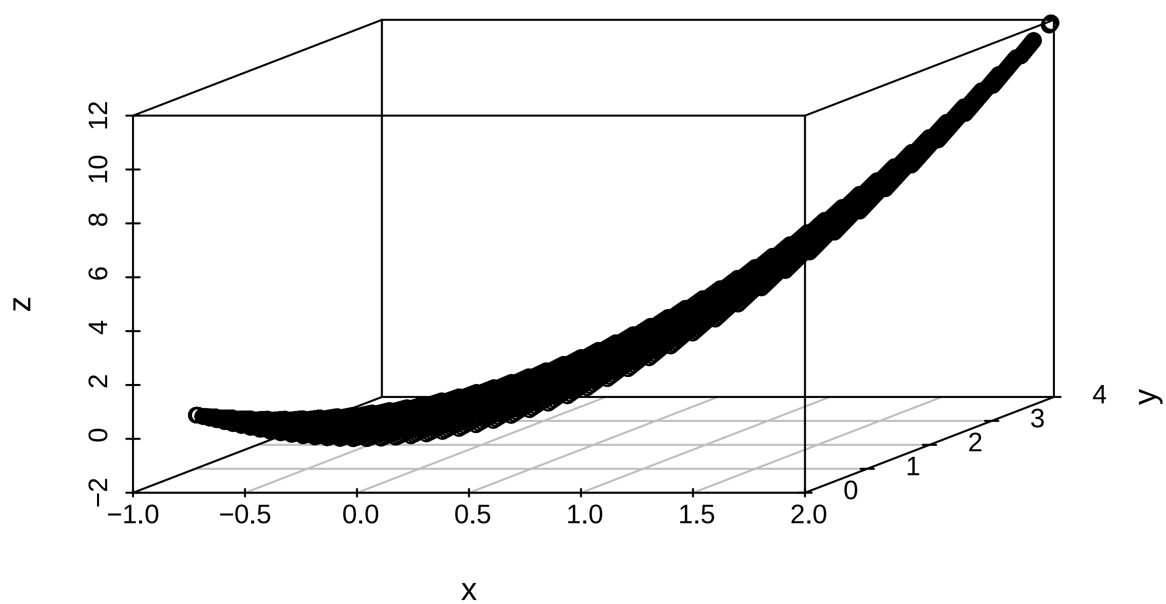
$$f(x, y) = x^2 + x * (x + 2) = 2x^2 + 2x, \quad y = x + 2$$

$$f_x(x, y) = 4x + 2 = 0, \quad y = x + 2$$

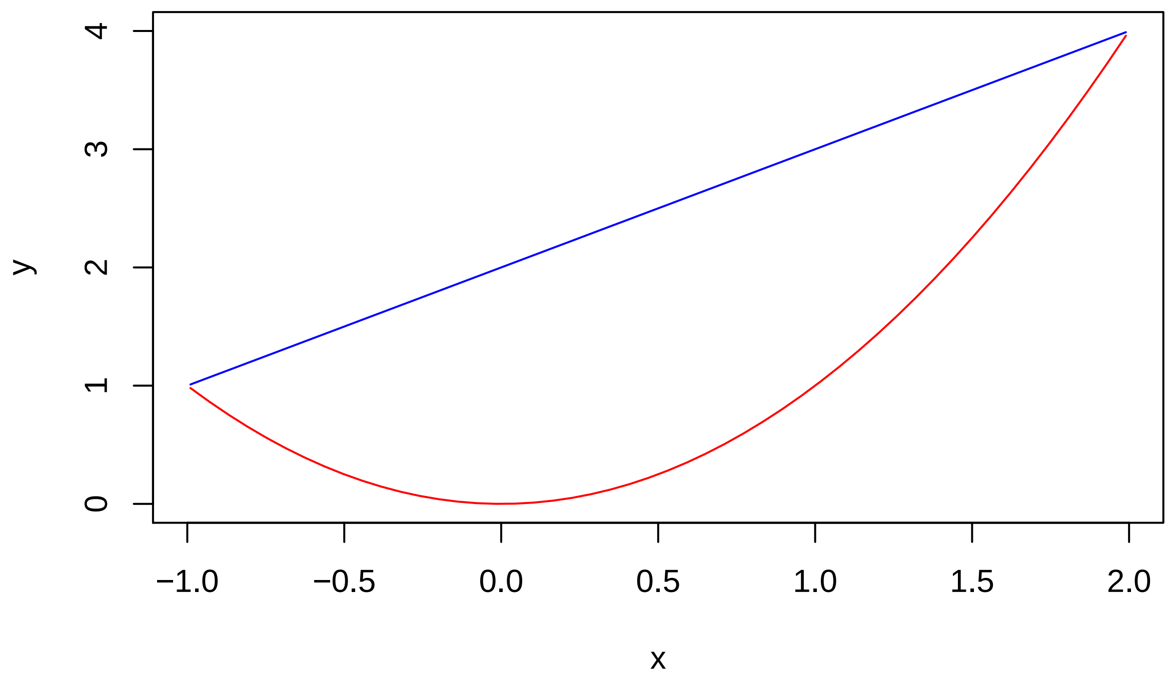
$f_{xx}(x, y) = 4$ is positive

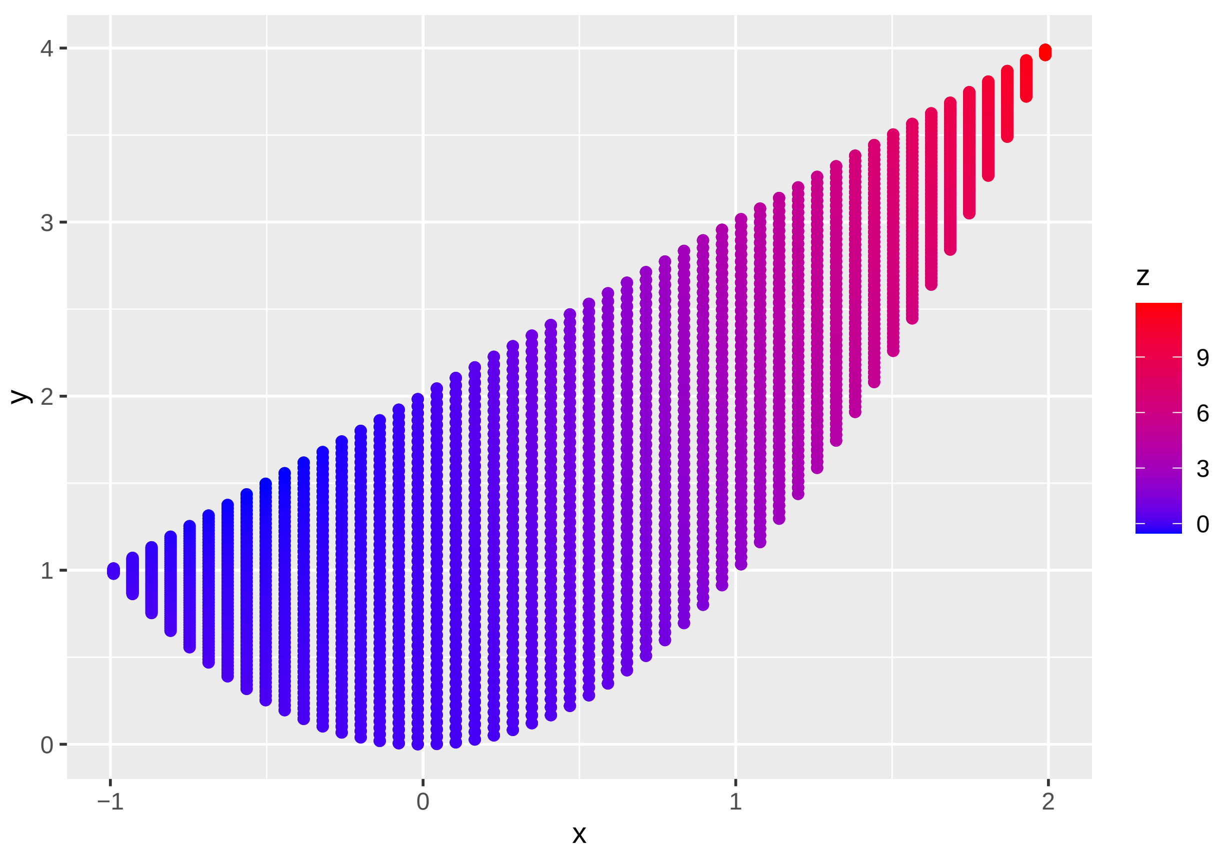
Therefore there is a minimum along the boundary at $x = -1/2, y = 3/2, z = -1/2$

$x = -1/2, y = 3/2, z = -1/2$ is the global minimum in the region.

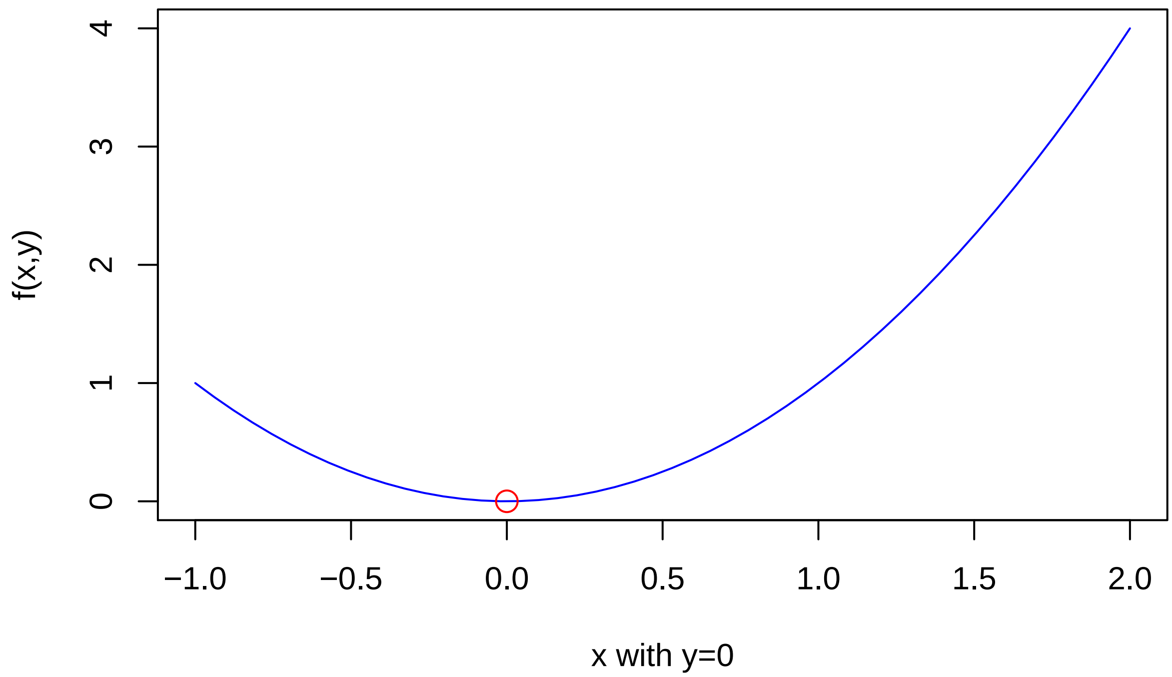


Checking the minimum over the region: -0.503469387755102, 1.4965306122449, -0.499975926697209

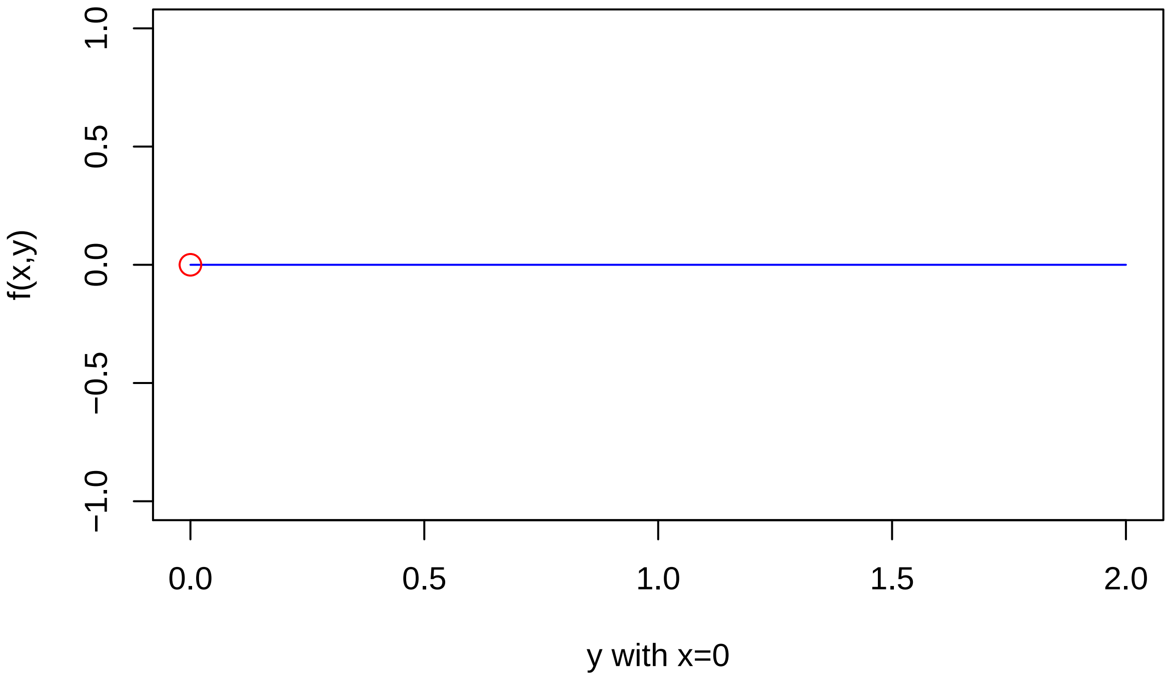


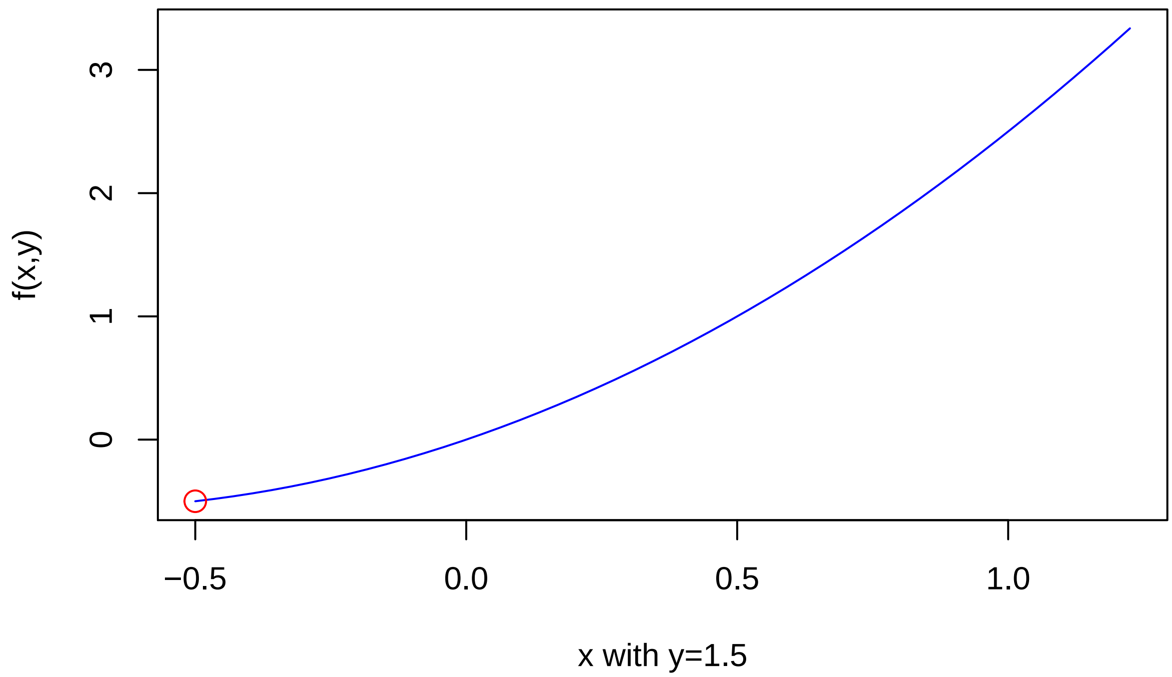


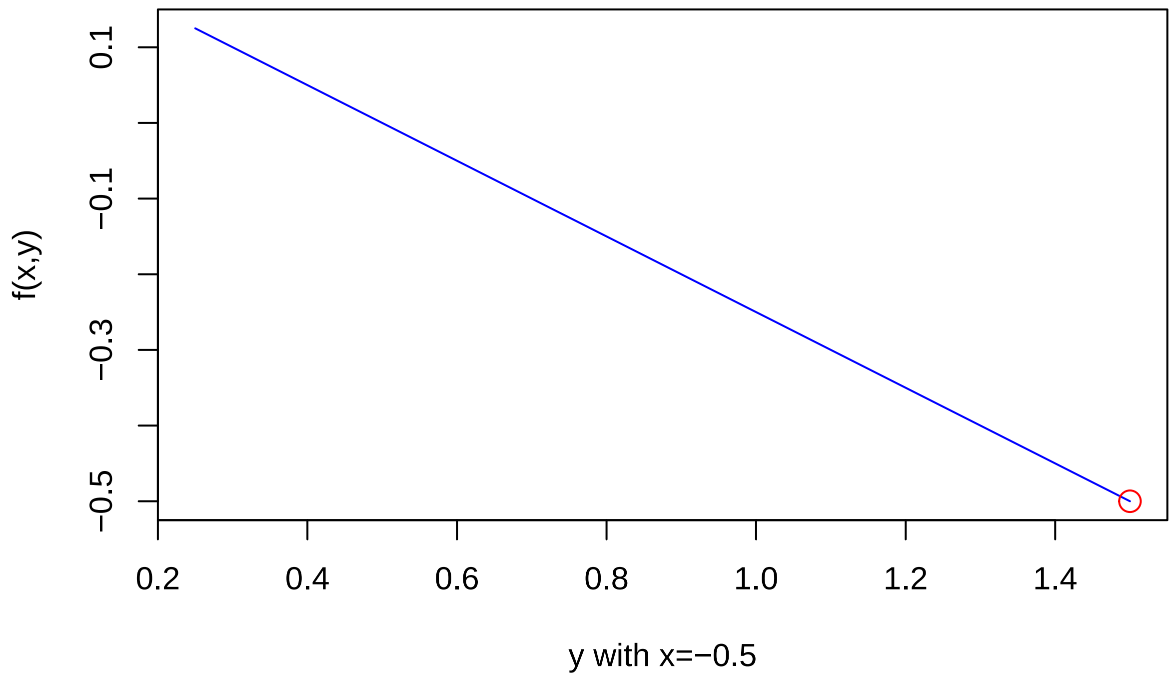
4
3
2
1
0
y
−1
0
1
2
x
z
9
6
3
0



f(x,y)
0
1
2
3
4
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x with y=0


1.0
0.5
0.0
−0.5
−1.0
f(x,y)
0.0
0.5
1.0
1.5
2.0
y with x=0


f(x,y)
0
1
2
3
−0.5
0.0
0.5
1.0
x with y=1.5



Question 2

Part A

$$f(x,y) = \frac{1}{4}x^4 - x^3 - 6xy + y^2$$

$$f_x = x^3 - 3x^2 - 6y$$

$$f_y = -6x + 2y$$

$$f_{xx} = 3x^2 - 6x$$

$$f_{yy} = 2$$

$$f_{xy} = -6$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 6x^2 - 12x - 36$$

Solve:

$$x^3 - 3x^2 - 6y = 0 - 6x + 2y = 0$$

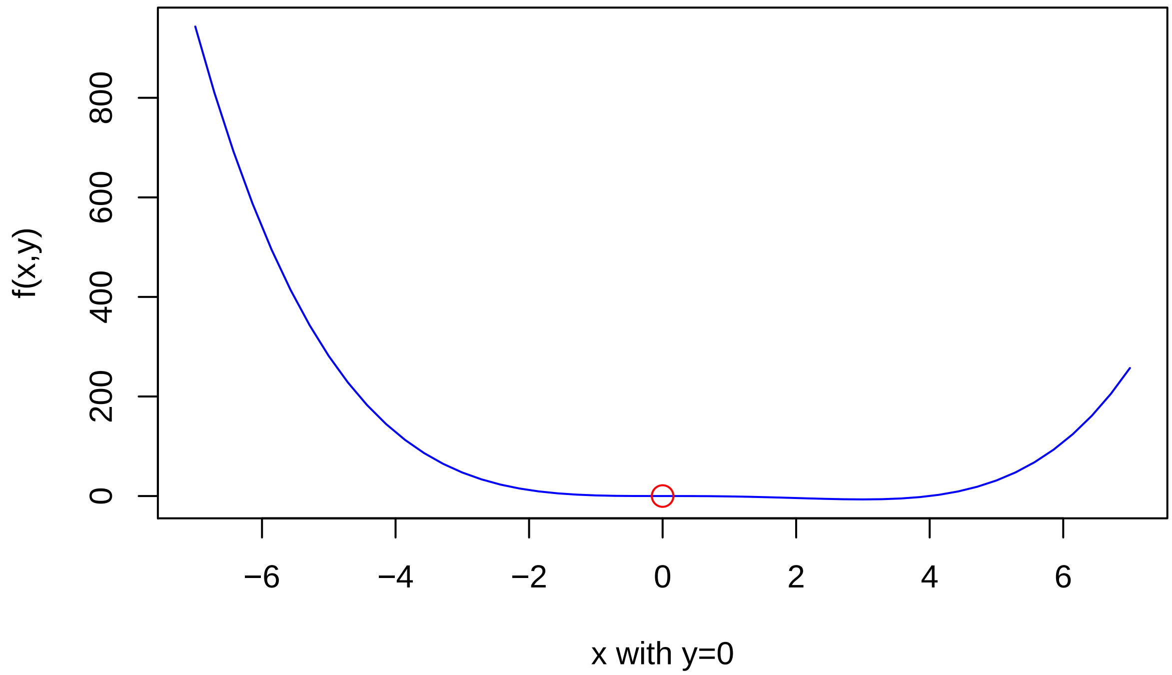
$$x^3 - 3x^2 - (6)(3x) = 0$$

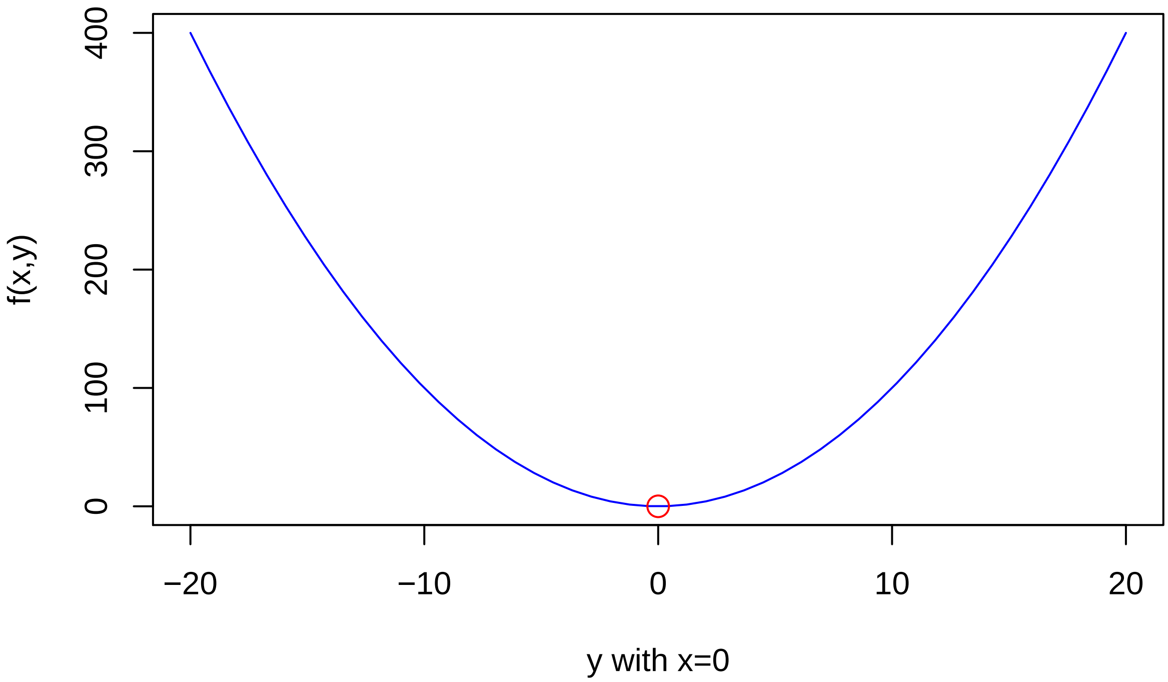
$x = 0, 6, -3$ and $y = 0, 18, -9$

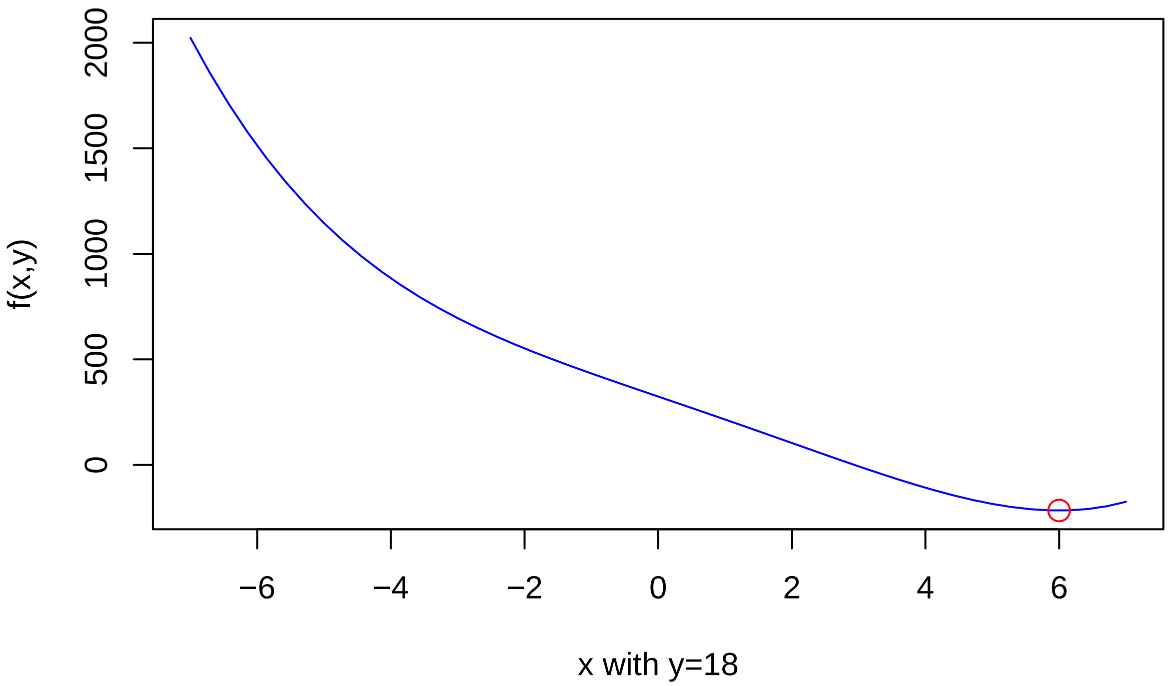
$(0, 0)$: $D < 0$, therefore saddle point

$(6, 18)$: $D > 0$ and $f_{xx} > 0$, therefore local minimum

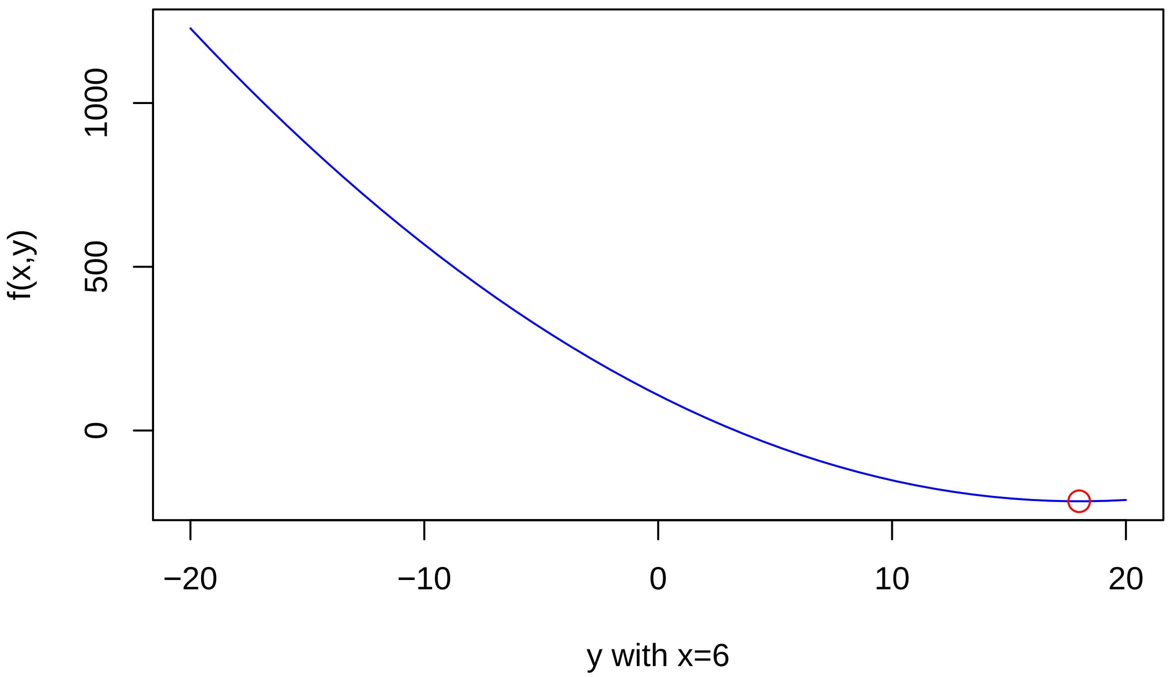
$(-3, -9)$: $D > 0$ and $f_{xx} > 0$, therefore local minimum



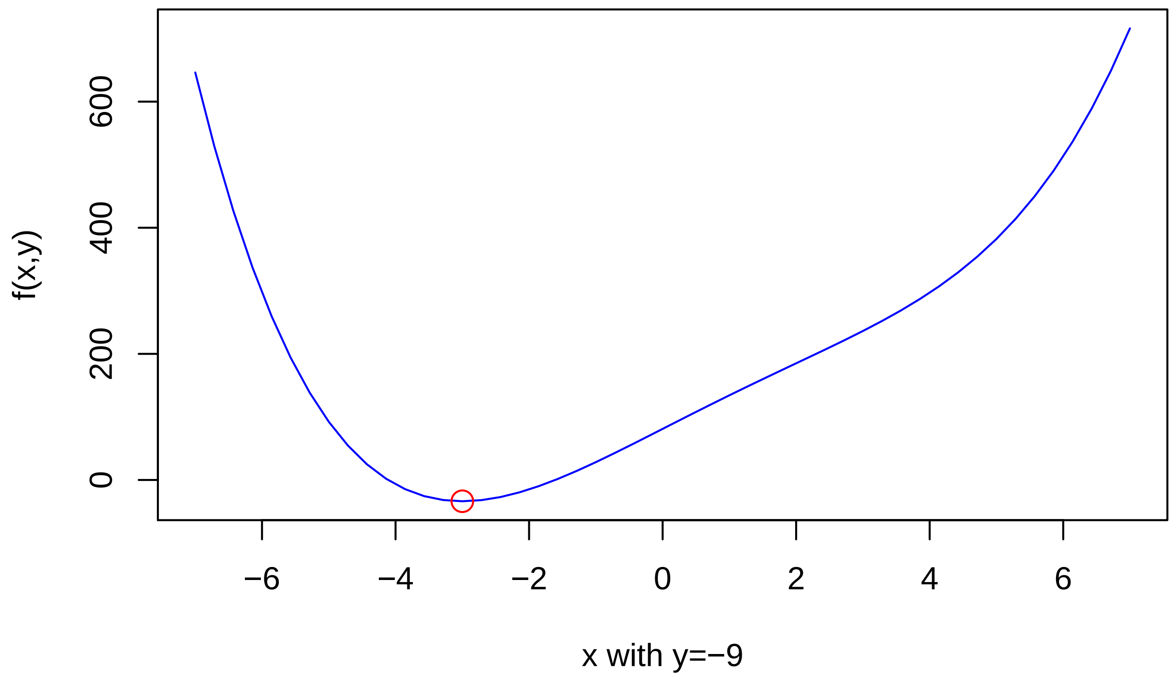

f(x,y)
0
100
200
300
400
−20
−10
0
10
20
y with x=0

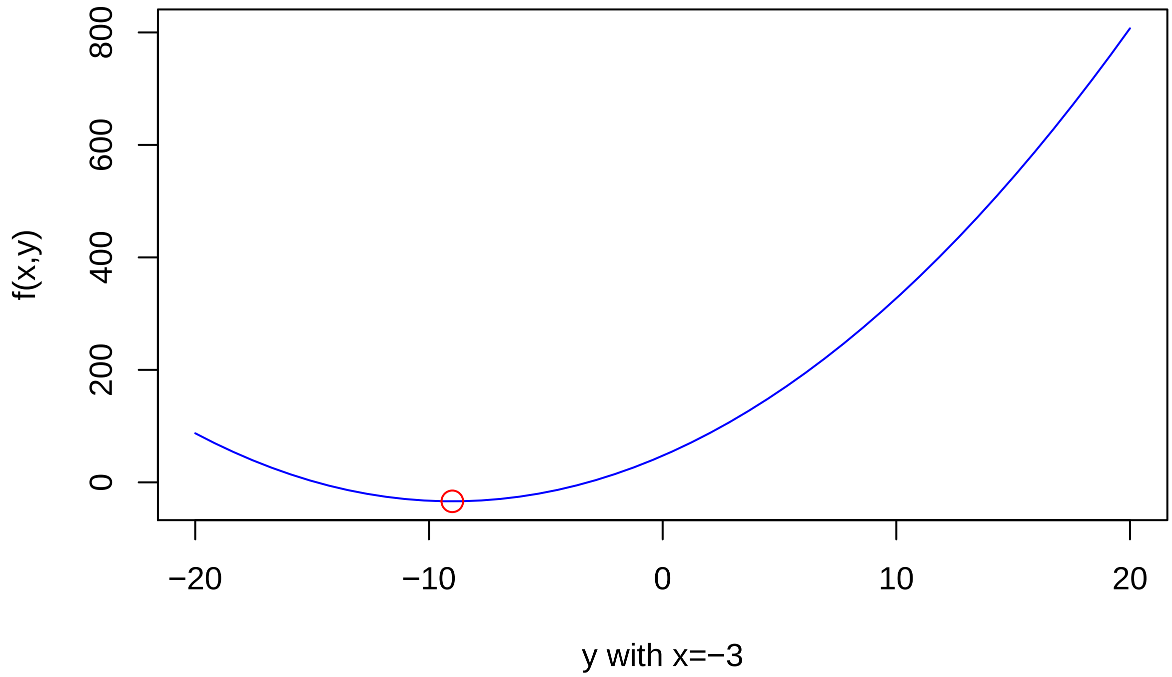


f(x,y)
2000
1500
1000
500
0
−6
−4
−2
0
2
4
6
x with y=18



f(x,y)
1000
500
0
−20
−10
0
10
20
y with x=6


600
400
200
0
f(x,y)
−6
−4
−2
0
2
4
6
x with y=−9



Part B

$$f(x, y) = x^2y + 2y^3 + x^2 + 5y^2$$

$$f_x = 2xy + 2x$$

$$f_y = x^2 + 6y^2 + 10y$$

$$f_{xx} = 2y + 2$$

$$f_{yy} = 12y + 10$$

$$f_{xy} = 2x$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2y + 2)(12y + 10) - 4x^2 = 4(y + 1)(6y + 5) - 4x^2$$

Solve:

$$2xy + 2x = 0x^2 + 6y^2 + 10y = 0$$

Solutions to the first equation are $x = 0 \quad \forall y$ and $y = -1 \quad \forall x$

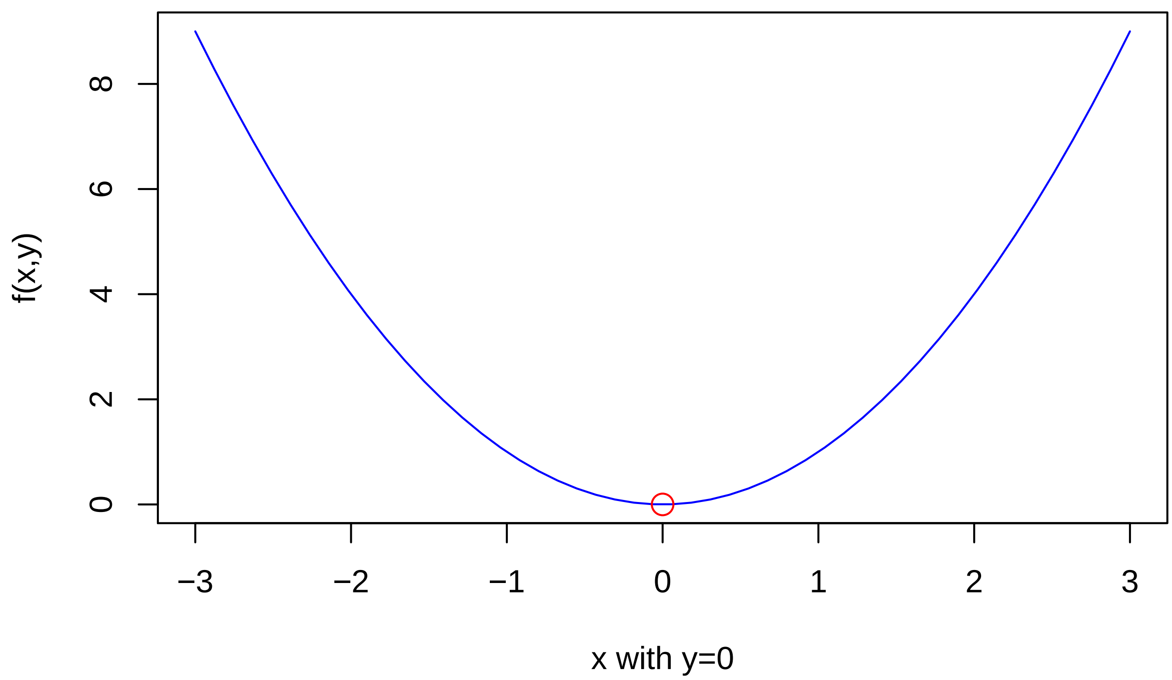
Substituting each into the second equation

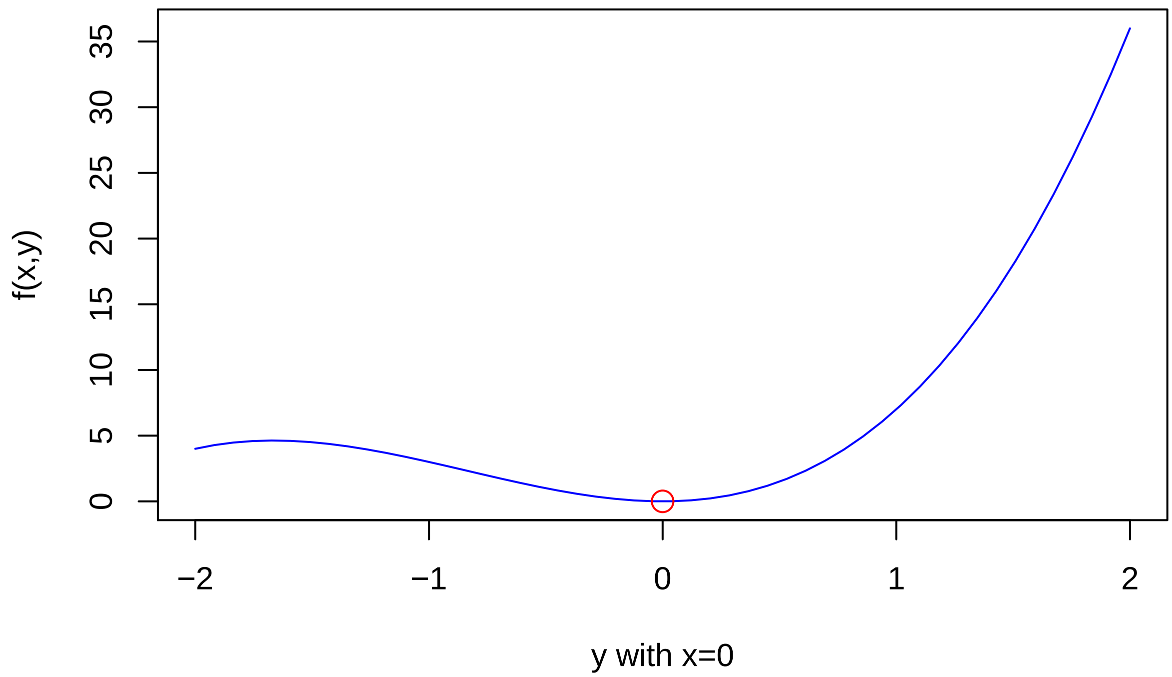
$x = 0$ therefore $6y^2 + 10y = 0$ and $y = 0, -10/6$

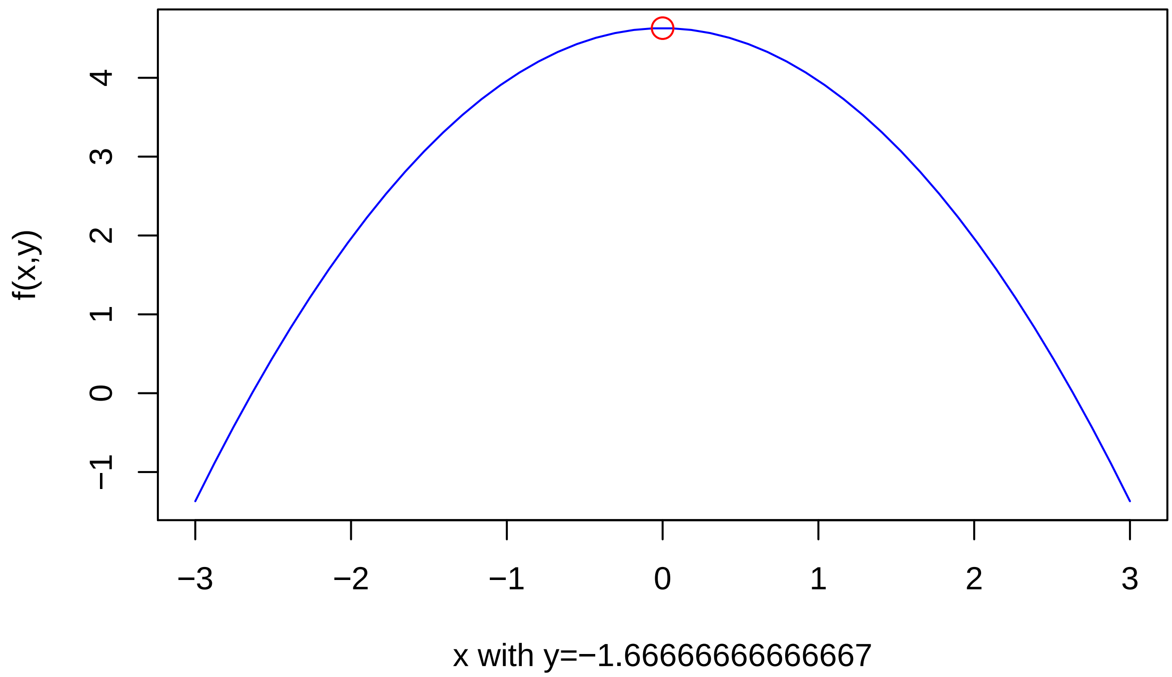
$y = -1$ therefore $x^2 - 4 = 0$ $x = \pm 2$

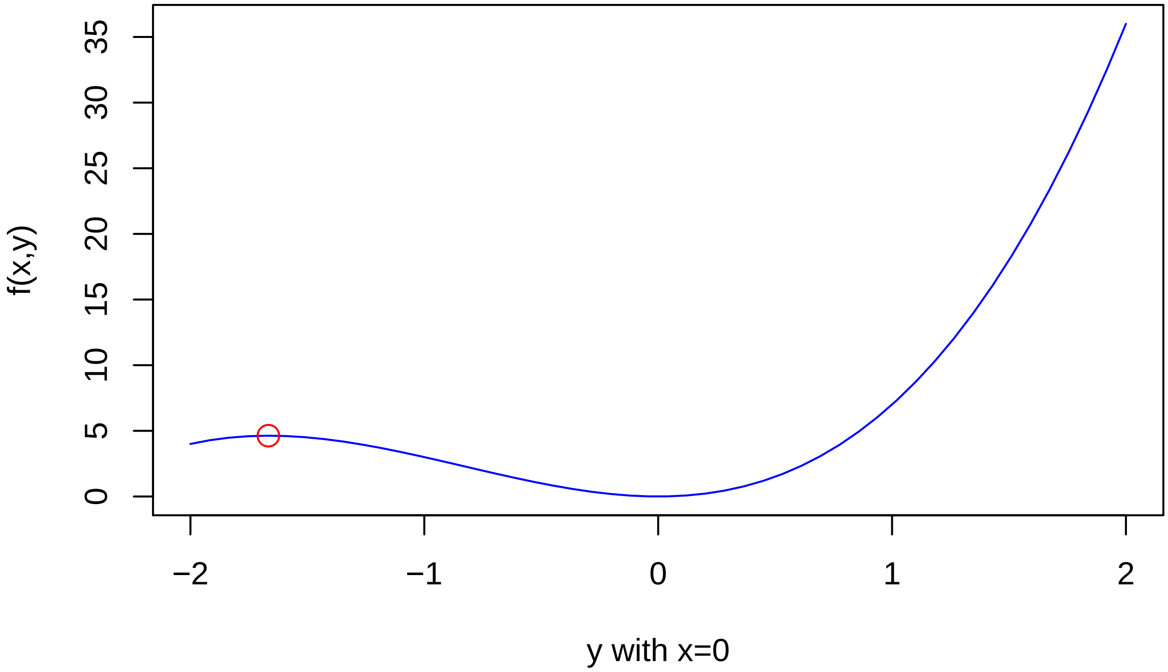
Critical Points

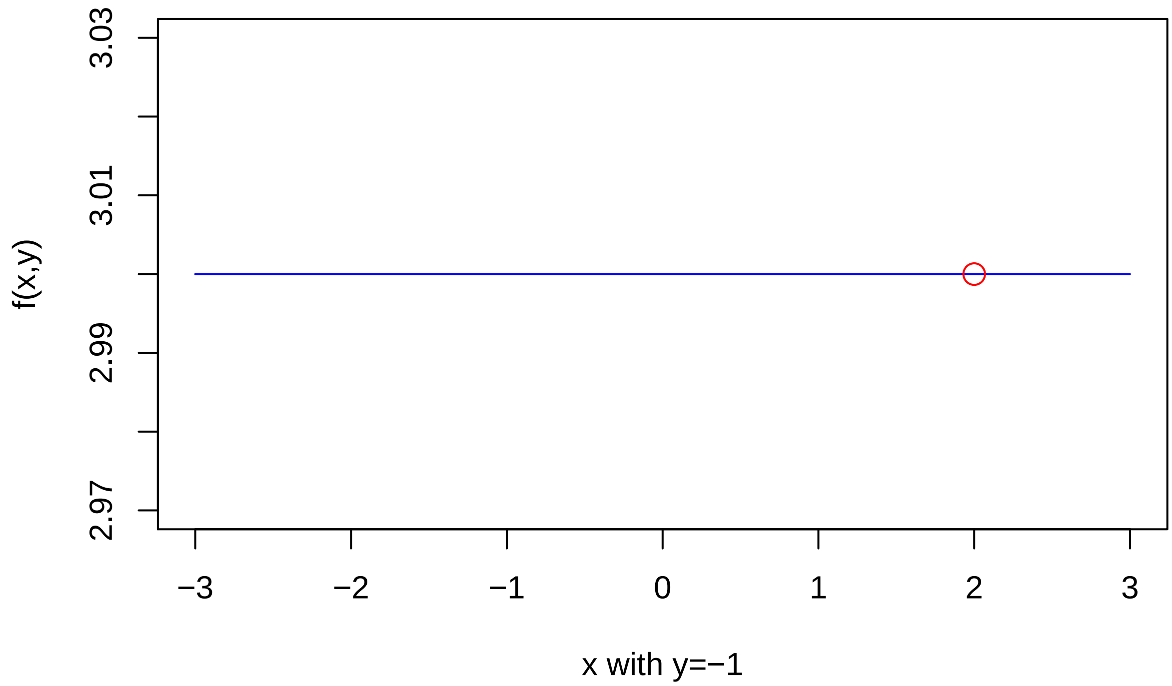
- $(0, 0)$: $D > 0$ $f_{xx} > 0$ therefore it is a local minimum
- $(0, -5/3)$: $D = (4)(-2/3)(-5) > 0$ $f_{xx} < 0$ therefore it is a maximum
- $(2, -1)$: $D < 0$ therefore it is a saddle point
- $(-2, -1)$: $D < 0$ therefore it is a saddle point

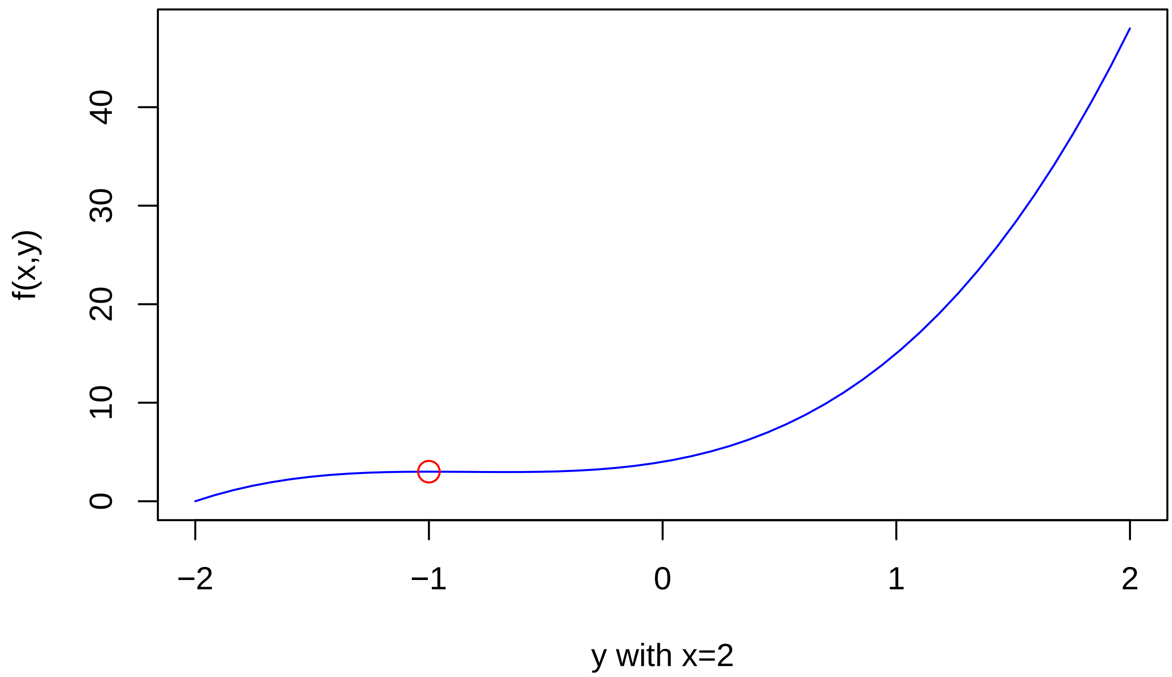


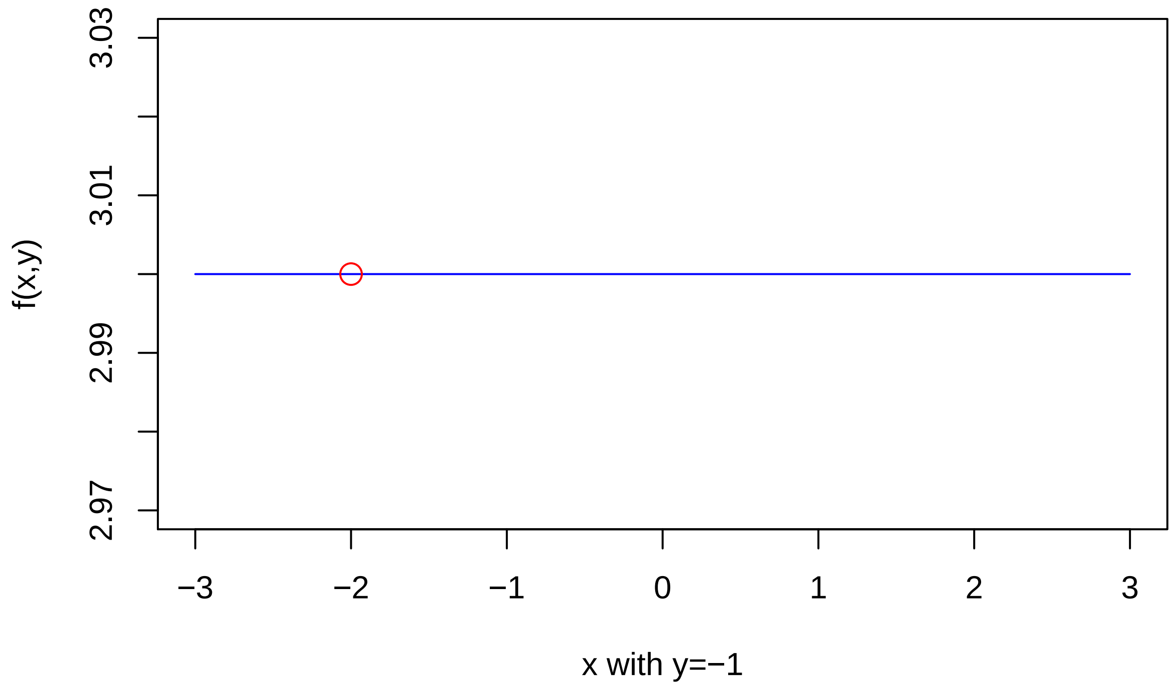

0
5
10
15
20
25
30
35
f(x,y)
−2
−1
0
1
2
y with x=0


f(x,y)
−1
0
1
2
3
4
−3
−2
−1
0
1
2
3
x with y=−1.66666666666667

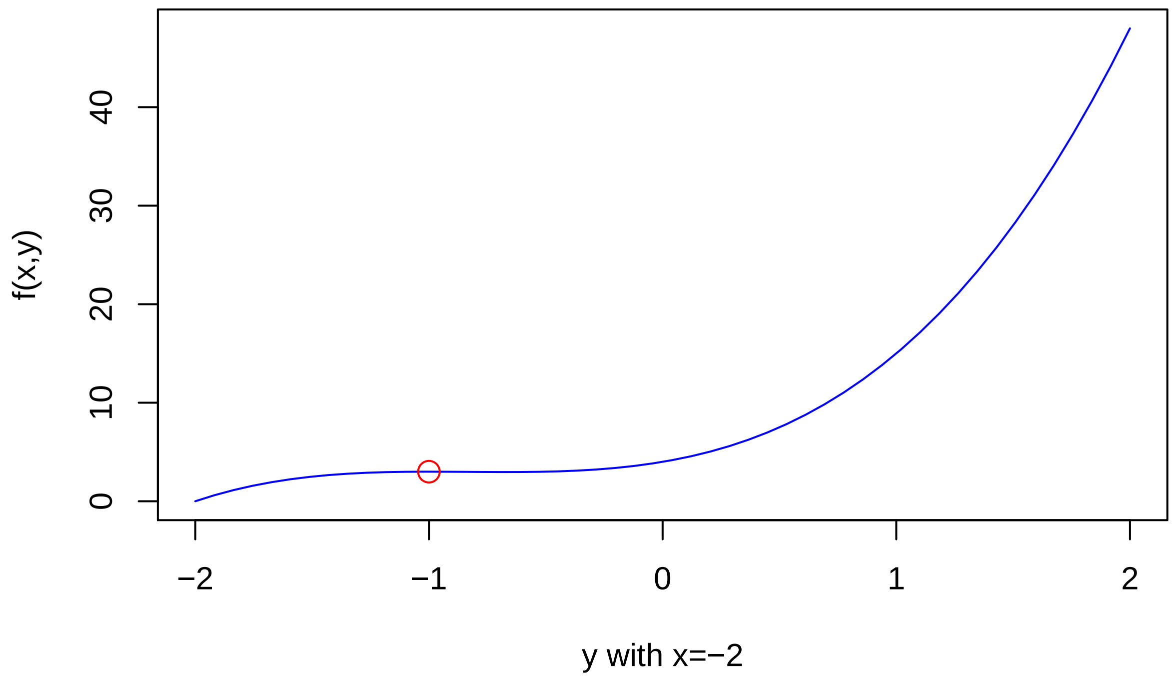

f(x,y)
0
5
10
15
20
25
30
35
−2
−1
0
1
2
y with x=0


3.03
3.01
2.99
2.97
f(x,y)
−3
−2
−1
0
1
2
3
x with y=−1


f(x,y)
0
10
20
30
40
−2
−1
0
1
2
y with x=2



3.03
3.01
2.99
2.97
f(x,y)
−3
−2
−1
0
1
2
3
x with y=−1



Question 3

$$V_0 = xyz$$

$$A_{top} = xy$$

$$A_{bottom} = xy$$

$$A_{sides} = 2xz + 2yz$$

$$C = f(x, y, x) = xy\alpha + xy\beta + (2xz + 2yz)\gamma$$

$$g(x, y, z) = xyz - V_0 = 0$$

$$\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

finding $\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_z = 0$

$$\mathcal{L}_x = y(\alpha + \beta) + 2\gamma z - \lambda yz = 0$$

$$\mathcal{L}_y = x(\alpha+\beta) + 2\gamma z - \lambda xz = 0$$

$$\mathcal{L}_z = 2\gamma(x+y) - \lambda xy = 0$$

and $g(x, y, z) = V_0 - xzy = 0$

$$y = \frac{-2\gamma z}{(\alpha+\beta) - \lambda z}$$

$$x = \frac{-2\gamma z}{(\alpha+\beta) - \lambda z}$$

$$-4\gamma^2 z \frac{2}{(\alpha+\beta) - \lambda z} - \frac{4\lambda\gamma^2 z^2}{((\alpha+\beta) - \lambda z)^2} = 0$$

$$2z[(\alpha+\beta) - \lambda z] + \lambda z^2 = 0$$

$$2z(\alpha+\beta) - \lambda z^2 = 0$$

$z = 0$ or $z = \frac{2(\alpha+\beta)}{\lambda}$

$$x = y = \frac{-2\gamma 2(\alpha+\beta)}{\lambda} \frac{1}{-(\alpha+\beta)} = \frac{4\gamma}{\lambda}$$

$$\frac{16\gamma^2}{\lambda^2} \frac{2(\alpha+\beta)}{\lambda} = V_0$$

$$\frac{32\gamma^2(\alpha+\beta)}{V_0} = \lambda^3$$

$$\lambda = (\frac{32(10)^2(12+4)}{8})^{1/3} = e^{log(6400)/3}$$

$$x = y = \frac{40}{e^{log(6400)/3}} = 2.1544$$

$$z = \frac{2(12+4)}{e^{log(6400)/3}} = 1.72355$$

```
##    x y z v   f
## 74 2 2 2 8 224
```

```
## [1] 222.7744
```

Question 4

$$f(x, y) = x^2y^2$$

$$x^2 + y^2 = 8$$

$$\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

finding $\mathcal{L}_x = \mathcal{L}_y = 0$

$$\mathcal{L} = x^2y^2 - \lambda(x^2 + y^2 - 8)$$

$$\mathcal{L}_x = 2xy^2 - 2\lambda x = 2x(y^2 - \lambda) = 0$$

$$\mathcal{L}_y = 2x^2y - 2\lambda y = 2y(x^2 - \lambda) = 0$$

$$x^2 + y^2 - 8 = 0$$

$x = 0$ or $y = \pm\sqrt{\lambda}$

$y = 0$ or $x = \pm\sqrt{\lambda}$

Therefore for x and y greater than 0, $2\lambda - 8 = 0$ and $\lambda = 4$

The critical points are therefore: $(2, 2)$, $(2, -2)$, $(-2, 2)$, $(-2, -2)$

$x = 0$ and $y = 0$ are not simultaneous solutions because of the constraints.

plugging the constraint into f: $f(x) = x^2(8 - x^2)$, $f_x = 16x - 4x^3$, $f_{xx} = 16 - 12x^2$ which is $f_{xx} < 0$ at the critical points, therfore the critical points are maxima within the constraint. $f(\pm 2, \pm 2) = 16$

Notice that $x = 0$ is a solution when plugging in the boundary condition into $f(x, y)$. Therefore $x = 0$ and $y = \pm\sqrt{8}$ is a solution, and $f_{xx} > 0$ so it is a minimum. Similarly, we could have plugged in the boundary condition and created $f(y) = y^2(8 - y^2)$. Therefore, $y = 0$ and $x = \pm\sqrt{(8)}$ is a solution, and $f_{yy} > 0$ so it is also a mimimum. $f(0, \pm\sqrt{8}) = 0$ and $f(\pm\sqrt{8}, 0) = 0$

Alternatively

$$f(r, \theta) = r^4 cos^2\theta sin^2\theta$$

$$r^2 - 8 = 0$$

$$\mathcal{L}(r, \theta) = r^4 cos^2\theta sin^2\theta - \lambda(r^2 - 8)$$

$$\mathcal{L}_r = 4r^3 cos^2\theta sin^2\theta - 2\lambda r = 0$$

$$\mathcal{L}_\theta = r^4[-2cos\theta sin^3\theta + 2cos^3\theta sin\theta] = 0$$

therefore $r = \sqrt{8}$ and $\lambda = cos^2\theta sin^2\theta$ and $cos\theta sin^3\theta = cos^3\theta sin\theta$ or $cos^2\theta = sin^2\theta$ which is true at $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$. $f(r,\theta) = 8^2(1/2)(1/2) = 16$

$\lambda = 0$ is also a solution when $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$. $f(r,\theta) = 0$

Question 5

$$f(x,y,z) = x^2 + 2y^2 + 4z^2$$

$$xyz = 16\sqrt{2}$$

$$\mathcal{L} = x^2 + 2y^2 + 4z^2 - \lambda(xyz - 16\sqrt{2})$$

$$\mathcal{L}_x = 2x - \lambda yz = 0$$

$$\mathcal{L}_y = 4y - \lambda xz = 0$$

$$\mathcal{L}_z = 8z - \lambda xy = 0$$

$$4y - \lambda(\frac{\lambda}{2}yz)z = y(4 - \frac{\lambda^2 z^2}{2}) = 0$$

$$8z - \lambda(\frac{\lambda}{2}yz)y = z(8 - \frac{\lambda^2 y^2}{2}) = 0$$

$z = \sqrt{\frac{8}{\lambda^2}}$ and $y = \sqrt{\frac{16}{\lambda^2}}$

$x\frac{\sqrt{8}}{\lambda}\frac{4}{\lambda} = 16\sqrt{2}$ therefore $x = 2\lambda^2$

$2x - \lambda yz = 0 = 4\lambda^2 - \frac{4\sqrt{8}}{\lambda}$

$\lambda^3 = \sqrt{8}$ and $\lambda = 8^{1/6}$

- $x = (2)8^{1/3} = 4$
- $y = \frac{4}{8^{1/6}} = 2\sqrt{2}$
- $z = \frac{2\sqrt{2}}{8^{1/6}} = 2$

Question 6

$$x^2 + 6y^2 + 3xy = 90$$

since we want the greatest x coordinate, maximize $f(x,y) = x$

$$\mathcal{L} = x - \lambda(90 - x^2 - 6y^2 - 3xy)$$

$$\mathcal{L}_x = 1 + 2\lambda x + 3\lambda y = 0$$

$$\mathcal{L}_y = 12\lambda y + 3\lambda x = 0$$

$x = -4y$

$$1 - 8\lambda y + 3\lambda y = 1 - 5\lambda y = 0$$

$y = \frac{1}{5\lambda}$

$$90 - \frac{16}{25\lambda^2} - \frac{6}{25\lambda^2} + \frac{12}{25\lambda^2} = 90 - \frac{10}{25\lambda^2} = 0$$

$\lambda^2 = \frac{1}{(9)(25)}$ and $\lambda = \pm\frac{1}{15}$

$y = \pm 3$ but choose $y = -3$ to maximize x

$x = 12$