Weekly Assignment 11

RC

2021-12-11

Question 1

Part A

$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} 2x - ydydx = \int_{0}^{1} 2x^{3/2} - \frac{x}{2} - \frac{3}{2}x^2dx = \frac{1}{20}$$

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## 1/20
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Part B

$$\int_{y=0}^{1}\int_{x=0}^{1-y} x^2 + 2ydxdy = \int_{0}^{1} \frac{1}{3}(1-y)^3 + 2y(1-y)dy = \frac{5}{12}$$

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## 5/12
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Part C

$$\int_{x=0}^{1}\int_{y=0}^{x}\int_{z=0}^{x-2y} x^2dzdydx = \int_{0}^{1}\int_{0}^{x} x^2(x-2y)dydx$$

$$= \int_{0}^{1} x^4 - x^4dx = 0$$

```
## 0
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Question 2

$$\int_{x=-2}^{2}\int_{y=x^2}^{4} x^2ydydx = \int_{y=0}^{4}\int_{x=-\sqrt{y}}^{\sqrt{y}} x^2ydxdy$$

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## 512/21
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## 512/21
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Question 3

Part A

$$y = x^2 + 1$$

$$y = 2x^2 - 3$$

Intersect at $x \pm 2$ and $y = 5$

$$A = \int_{x=-2}^{2} \int_{2x^2-3}^{x^2+1} dydx = \int_{-2}^{2} (x^2+1) - (2x^2-3) dy = \frac{32}{3}$$

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## 32/3
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Part B

Inner

$$r = 2sin\theta$$

Outer

$$r = 1$$

Intersect at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = \int_{\theta=\pi/6}^{5\pi/6} \int_{r=1}^{2sin\theta} rdrd\theta = \int_{\theta=\pi/6}^{5\pi/6} 2sin^2\theta - \frac{1}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

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## sqrt(3)/2 + pi/3
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Question 4

$$z = x^2 + 2y^2$$

$$z = 12 - 2x^2 - y^2$$

Intersect at $0 = 4 - x^2 - y^2$

$$\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 2x^2 - y^2) - (x^2 + 2y^2) dydx$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 12 - 3r^2 rdrd\theta = \int_{\theta=0}^{2\pi} 24 - 12d\theta = 24\pi$$

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## 24*pi
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Question 5

$$3x + y + 4z = 12, \quad x = 0, \quad y = 0, \quad z = 0$$

$$V = \int_{x=0}^{4} \int_{y=0}^{12-3x} \int_{z=0}^{12-3x-y)/4} dzdydx$$

$$= \int_{x=0}^{4} \int_{y=0}^{12-3x} (12 - 3x - y)/4dydz = \int_{0}^{4} \frac{9}{8}(4-x)^2 dx = 24$$

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## 24
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Question 6

$$y = x^3$$

$$y = 2x$$

$$x \geq 0$$

$$\rho(x, y) = 2y$$

$$M = \int_{x=0}^{\sqrt{2}} \int_{y=x^3}^{2x} 2y dy dx = \int_{x=0}^{\sqrt{2}} 4x^2 - x^6 dx = \frac{32}{21}\sqrt{2}$$

$$M_x = \int_{x=0}^{\sqrt{2}} \int_{y=x^3}^{2x} 2y^2 dy dx = \frac{16}{5}$$

$$M_y = \int_{x=0}^{\sqrt{2}} \int_{y=x^3}^{2x} 2xy dy dx = 2$$

$$\bar{x} = \frac{M_y}{M} = \frac{21}{32}\sqrt{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{21}{20}\sqrt{2}$$

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## 21*sqrt(2)/20
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## 21*sqrt(2)/32
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Question 7

$$\int_R \int \frac{1}{\sqrt{4 - x^2 - y^2}} dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{\sqrt{4 - r^2}} r dr d\theta$$

$$R : x^2 + y^2 = 1$$

$$= \int_{\theta=0}^{\pi/2} 2 - \sqrt{3} d\theta = (2 - \sqrt{3})\frac{\pi}{2}$$

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## pi*(2 - sqrt(3))/2
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Question 8

$$z = x^2 + y^2$$

$$x^2 + y^2 + z^2 = 20$$

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \int_{z=r^2}^{\sqrt{20-r^2}} rdzdrd\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} r(\sqrt{20-r^2} - r^2)drd\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{3}(-76 + 20^{3/2})d\theta = \frac{\pi}{6}(20^{3/2} - 76) = \frac{\pi}{6}(40\sqrt{5} - 76)$$

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## pi*(-76/3 + 40*sqrt(5)/3)/2
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Question 9

$$x^2 + y^2 + z^2 = 36$$

$z = 0$ to $z = 3$

$$\int_{\phi=\pi/3}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{6} \rho^2 sin\phi d\rho d\theta d\phi$$

$$= \int_{\phi=\pi/3}^{\pi/2} \int_{\theta=0}^{2\pi} 72 sin\phi \quad d\theta d\phi$$

$$= \int_{\phi=\pi/3}^{\pi/2} (2\pi)(72) sin\phi \quad d\phi = 72\pi$$

```
## 72*pi
```

Question 10

Part A

$$\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} x^2 dzdydx$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{\pi} \int_{z=0}^{4-r^2} r^3 cos^2\theta dzd\theta dr$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2} (4-r^2) r^3 cos^2\theta \quad drd\theta$$

$$= \int_{\theta=0}^{\pi} (16 - \frac{64}{6}) cos^2\theta \quad d\theta = \frac{32}{6}\frac{\pi}{2} = \frac{8\pi}{3}$$

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## 8*pi/3
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Part B

$$\int_{-\sqrt{1/2}}^{\sqrt{1/2}}\int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}dzdydx$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{1/2}}\int_{z=r}^{\sqrt{1-r^2}} r\sqrt{r^2+z^2}dzdrd\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=z}^{\sqrt{1-z^2}} r\sqrt{r^2+z^2}dzdrd\theta$$

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## 2*pi*(1/3 - sqrt(2)/6)
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Question 11

$$F = x^2yz\hat{i} + 3xyz^3\hat{j} + (x^2 - z^2)\hat{k}$$

A

$$2\mathbf{x_N}\mathbf{y_N}^2\mathbf{z_N} + 3\mathbf{x_N}\mathbf{z_N}^3 - 2\mathbf{z_N}$$

B

$$(-9\mathbf{x_N}\mathbf{y_N}\mathbf{z_N}^2)\hat{\mathbf{i}}_{\mathbf{N}} + (\mathbf{x_N}^2\mathbf{y_N}^2 - 2\mathbf{x_N})\hat{\mathbf{j}}_{\mathbf{N}} + (-2\mathbf{x_N}^2\mathbf{y_N}\mathbf{z_N} + 3\mathbf{y_N}\mathbf{z_N}^3)\hat{\mathbf{k}}_{\mathbf{N}}$$

Question 12

$$F = (2xy + 3x^2z)\hat{i} + (x^2 - 4z + 6y^2)\hat{j} + (x^3 - 4y)\hat{k}$$

F is a conservative field if and only if

$$\nabla f = F$$

F is defined everywhere in $\mathbb{R}^3$

If F is conservative, then the $curl(F) = 0$

$$\hat{\mathbf{0}}$$

Find the potential f

$$\frac{\partial f}{\partial x} = F \cdot \hat{i} = 2xy + 3x^2z$$

$$f = x^2y + x^3z + g(y, z)$$

$$\frac{\partial f}{\partial y} = F \cdot \hat{j} = x^2 - 4z + 6y^2$$

$$f = x^2y - 4yz + 2y^3 + h(x, z)$$

$$\frac{\partial f}{\partial z} = F \cdot \hat{k} = x^3 - 4y$$

$$f = x^3 z - 4yz + m(x, y)$$

Therefore

$$f = x^2 y + x^3 z - 4yz + 2y^3 + C$$

Since f can be found, F is conservative

Using python...

$$\mathbf{x_N}^3 \mathbf{z_N} + \mathbf{x_N}^2 \mathbf{y_N} + 2\mathbf{y_N}^3 - 4\mathbf{y_N}\mathbf{z_N}$$

Is F conservative? True

Question 13

$$\int_C x^3 y ds$$

$$C : r(t) = 3cost\hat{i} + 3sint\hat{j} \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\int_C f(r(t))ds = \int_a^b f(r(t))||r'(t)||dt$$

$$\int_C x^3 y \ ds = \int_0^{\pi/2} (3cost)^3 (3sint)\sqrt{9sin^2 t + 9cos^2 t} \quad dt$$

$$= 3^5 \frac{-1}{4} cos^4 t\Big|_0^{\pi/2} = \frac{3^5}{4}$$

Using python...

$$\frac{243}{4}$$

Question 14

$$\int_C (x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz$$

$$C : \ (0, 0, 0) \to (0, 4, 0) \to (2, 4, 0) \to (2, 4, 3)$$

$$\int_C F \cdot dr = \int_C P dx + Q dy + R dz = \int_a^b F(r(t)) \cdot r'(t)dt$$

$$F = (x + y + z)\hat{i} + (x - 2y + 3z)\hat{j} + (2x + y - z)\hat{k}$$

First segment

$$\int_0^4 (0-2y+0)dy = -16$$

$r(t) = t\hat{j}\;\; 0 \le t \le 4$ and $r'(t) = \hat{j}$ and $\int_0^4 -2tdt = -16$

Second segment

$$\int_0^2 (x+4+0)dx = 10$$

$r(t) = t\hat{i} + 4\hat{j}\;\; 0 \le t \le 2$ and $r'(t) = \hat{i}$ and $\int_0^2 (t+4)dt = 10$

Third segment

$$\int_0^3 (2(2)+4-z)dz = 8(3) - \frac{9}{2} = \frac{39}{2}$$

$r(t) = 2\hat{i} + 4\hat{j} + t\hat{k}\;\; 0 \le t \le 3$ and $r'(t) = \hat{k}$ and $\int_0^3 (8-t)dt = 24 - (9/2) = 39/2$

Total: $-16 + 10 + \frac{39}{2} = \frac{27}{2}$

Question 15

$$\int_C (x^2 - y^2)dx + (2xy)dy$$

$$C:\;\; r(t) = 2t\hat{i} + t^2\hat{j}\;\; 0 \le t \le 2$$

$$\int_C F \cdot dr = \int_C Pdx + Qdy + Rdz = \int_a^b F(r(t)) \cdot r'(t)dt$$

$$F = (x^2 - y^2)\hat{i} + (2xy)\hat{j}$$

$$r'(t) = 2\hat{i} + 2t\hat{j}$$

$$\int_0^2 (4t^2 - t^4)(2) + (2)(2t)(t^2)(2t)dt = \int_0^2 8t^2 + 6t^4 dt = (8)(8)/3 + 6/5(32) = 896/15$$

896/15

Question 16

$$\int_C F(x,y)\cdot dr$$

$$F = (y^2+2xy)\hat{\imath} + (x^2+2xy)\hat{\jmath}$$

$$C : (-1,2) \to (3,1)$$

C is a smooth curve given by $r(t)$ $a \leq t \leq b$

f is a function whose gradient vector ∇f is continuous on C then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$f = xy^2 + x^2y$$

$$\int_C F(x,y)\cdot dr = \int_C \nabla f \cdot dr = 3(1) + 9(1) - [(-4)+2] = 14$$

Question 17

$$\oint_C y^2dx + xydy$$

$y = 0$, $y = \sqrt{x}$, $x = 4$

Greens Theorem:

$$\oint_C F\cdot dr = \oint_C Pdx + Qdy = \int\int_D (Q_x - P_y)dA$$

$$= \int_0^4 \int_0^{\sqrt{x}} (y-2y)dydx = \int_0^4 \int_0^{\sqrt{x}} -ydydx$$

$$= \int_0^4 \frac{-x}{2}dx = -4$$