MTH 347. Practice Problems for Test 1.

Problem 1. *Verify that the given function is a solution of the given differential equation on some interval I, and specify that interval.*
$(y-x)y' = y - x + 8, \quad y = x + 4\sqrt{x+2}$

Problem 2. *Verify that the given relation defines an implicit solution of the given differential equation*
$\dfrac{dx}{dt} = (x-1)(1-2x), \quad \ln\left(\dfrac{2x-1}{x-1}\right) = t.$

Problem 3. *Solve the following differential equations:*

$$\textbf{1.}\quad x\frac{dy}{dx} = 4y, \qquad \textbf{2.}\quad \frac{dy}{dx} + 2xy^2 = 0, \qquad \textbf{3.}\quad \frac{dy}{dx} = e^{3x+2y}$$

$$\textbf{4.}\quad e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}, \qquad \textbf{5.}\quad y\ln x\frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2, \qquad \textbf{6.}\quad \frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$$

$$\textbf{8.}\quad (e^y+1)^2 e^{-y}dx + (e^x+1)^3 e^{-x}dy = 0$$

$$\textbf{9.}\quad x(1+y^2)^{1/2}dx = y(1+x^2)^{1/2}dy, \qquad \textbf{10.}\quad \frac{dS}{dr} = kS, \qquad \textbf{11.}\quad \frac{dQ}{dt} = k(Q-70)$$

$$\textbf{12.}\quad \frac{dP}{dt} = P - P^2, \qquad \textbf{13.}\quad \frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$$

Problem 4. *Solve the following initial-value problems*:

$$\textbf{1.}\ \frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \quad y(2)=2 \qquad \textbf{2.}\quad x^2\frac{dy}{dx} = y - xy, \quad y(-1) = -1 \qquad \textbf{3.}\quad \frac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$$

$$\textbf{4.}\quad x\frac{dy}{dx} = y^2 - y, \quad y\left(\frac{1}{2}\right) = \frac{1}{2} \qquad \textbf{5.}\quad \frac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1$$

Problem 5. *Solve the following problems*:

a. A thermometer is removed from a room where the temperature is 70°F and is taken outside where the air temperature is 10°F. After one-half minute the thermometer reads 50°F. What is the reading of the thermometer at $t = 1$ minute? How long will it take the thermometer to reach 15°F?

b. A thermometer is taken from an inside room to the outside where the air temperature is 5°F. After one minute the thermometer reads 55°F, and after five minutes it reads 30°F. What is the initial temperature of the inside room?

Problem 6. *A: Solve the following differential equations*:

$$\textbf{1.}\quad \frac{dy}{dx} + y = e^{3x}, \qquad \textbf{2.}\quad 3\frac{dy}{dx} + 12y = 4, \qquad \textbf{3.}\quad \frac{dy}{dx} + 3x^2y = x^2$$

$$\textbf{4.}\quad \frac{dy}{dx} = x^3 - 2xy, \qquad \textbf{5.}\quad x^2\frac{dy}{dx} = 1 - xy, \qquad \textbf{6.}\quad \frac{dy}{dx} = 2y + x^2 + 5$$

7. $x\dfrac{dy}{dx} - y - x^2 \sin x = 0,$ **8.** $x^2\dfrac{dy}{dx} + x(x+2)y - e^x = 0$

9. $ydx - 4(x + y^6)dy = 0$ **10.** $ydx = (ye^y - 2x)dy,$

12. $(x+1)\dfrac{dy}{dx} + (x+2)y = 2xe^{-x},$ **13.** $(x+2)^2\dfrac{dy}{dx} = 5 - 8y - 4xy.$

14. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$ **15.** $(x^2-1)\dfrac{dy}{dx} = (x+1)^2 - 2y$

B. Solve the following initial-value problems

16. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5.$

18. $\dfrac{dT}{dt} = k(T - T_m), \quad T(0) = T_0,$ where k, T_m, T_0 are constants

19. $t\dfrac{dy}{dt} + (t+1)y = t, \quad y(\ln 2) = 1 \quad (t > 0).$

C. Solve the following initial-value problems and sketch a graph of the solution indicating horizontal or vertical asymptotes if appropriate

20. $\dfrac{dx}{dt} = 7x(x-6), \quad x(0) = 8$ **21.** $\dfrac{dx}{dt} = 7x(6-x), \quad x(0) = 10$

22. $\dfrac{dx}{dt} = 3x^2 - 6x, \quad x(0) = 1$ **23.** $\dfrac{dx}{dt} = 2x^2 - 10x, \quad x(0) = 7$

Problem 7. *Set up a differential equation and additional conditions (if any) for the following problems. Do NOT solve the problem*

1. A tank contains 300 gallons of water in which 50 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and then the well-stirred solution (so there is an even distribution of salt in the water) leaves the tank at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time t. What is $A(0)$?

2. A tank contains 300 gallons of water in which 50 pounds of salt have been dissolved. Brine containing 2 lb/gal of salt is pumped into the tank at a rate of 3 gal/min, and then the well-stirred solution (so there is an even distribution of salt in the water) leaves the tank at the rate of 2 gal/min. Determine a differential equation for the amount of salt $A(t)$ in the tank at time t. What is $A(0)$?

3. Write an equation for the previous problem (Problem # 2) assuming that the solution is pumped out at the rate of 3.5 gal/min. When is the tank empty?

Problem 8. *Solve the following problems*:

1. A tank contains 30 grams of salt dissolved in 200 liters of water. Pure water is then pumped into the tank at a rate of 4 L/min; the well-stirred solution (so there is an even distribution of salt in the water) leaves the tank at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.
2. A tank contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is then pumped into the tank at a rate of 5 gal/min. The well-stirred solution (so there is an even distribution of salt in the water) leaves the tank at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

3. Solve the previous part of the problem under the assumption that the solution is pumped out of the tank at a faster rate of 10 gal/min. When is the tank empty?

4. A tank contains 100 gallons of fluid in which 10 pounds of salt dissolved. Brine containing 1/2 pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-stirred solution is then pumped out of the tank at a rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

Problem 9. *Solve the following differential equations by using an appropriate substitution.*

a. $\frac{dy}{dx} = (x+y+1)^2$ **b.** $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$ **c.** $\frac{dy}{dx} = \frac{3x+2y}{3x+2y+2}$

1. $(x+y)dx + xdy = 0$ **2.** $(y^2+yx)dx - x^2dy = 0$ **3.** $(y^2+yx)dx + x^2dy = 0$

4. $xy^2\frac{dy}{dx} = y^3 - x^3$ **5.** $-ydx + (x + \sqrt{xy})dy = 0$ **6.** $x\frac{dy}{dx} + y = \frac{1}{y^2}$

7. $\frac{dy}{dx} - y = e^x y^2$ **8.** $\frac{dy}{dx} = y(xy^3 - 1)$ **9.** $x\frac{dy}{dx} = y + xy + xy^2$

10. $\frac{dy}{dx} = 1 + e^{y-x+5}$ **11.** $t^2\frac{dy}{dt} + y^2 = ty$ **12.** $3(1+t^2)\frac{dy}{dt} = 2ty(y^3 - 1)$

13. $x\frac{dy}{dx} + 2y = 6x^2\sqrt{y}$

14. *Solve the following initial–value problem:* $(x^2+2y^2)\frac{dx}{dy} = xy, \qquad y(-1) = 1$

Problem 10. *Determine whether the following differential equation is exact. If it is exact, solve it:*

1. $(2x-1)dx + (3y+7)dy = 0,$ **2.** $(2x+y)dx - (x+6y)dy = 0,$

3. $(5x+4y)dx + (4x-8y^3)dy = 0,$ **4.** $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0,$

5. $(2xy^2-3)dx + (2x^2y+4)dy = 0,$ **6.** $(x^2-y^2)dx + (x^2-2xy)dy = 0$

7. $\left(1 + \ln x + \frac{y}{x}\right)dx = (1 - \ln x)\,dy,$ **8.** $(x - y^3 + y^2\sin x)dx = (3xy^2 + 2y\cos x)dy,$

9. $(x^3+y^3)dx + 3xy^2dy = 0,$ **10.** $(3x^2y + e^y)dx + (x^3 + xe^y - 2y)dy = 0$

11. $x\frac{dy}{dx} = 2xe^x - y + 6x^2,$ **12.** $\left(1 - \frac{3}{y} + x\right)\frac{dy}{dx} + y = \frac{3}{x} - 1$

Solve the following initial-value problems

13. $(x+y)^2dx + (2xy + x^2 - 1)dy = 0, \ y(1) = 1$

14. $(e^x + y)dx + (2 + x + ye^y)dy = 0,\ y(0) = 1$

15. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0,\ y(1) = 1$

16. $(y^2 \cos x - 3x^2 y - 2x)dx + (2y \sin x - x^3 + \ln y)dy = 0, \quad y(0) = e$

Problem 11.

1. Suppose that a community contains 15,000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$ the number $N(t)$ of people who have developed Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and of those who have not. How long will it take for another 5000 people to develop Michaud's syndrome?

2. Consider an animal population $P(t)$ with constant death rate $\delta = 0.01$ (deaths per animal per month) and with birth rate β proportional to P. Suppose that $P(0) = 200$ and $P'(0) = 2$ (a) When is $P = 1000$? (b) When does doomsday occur?

3. Suppose that the population $P(t)$ (in millions) of a country satisfies the differential equation $\dfrac{dP}{dt} = kP(300 - P)$ with k constant. Its population in 1960 was 100 million and was then growing at the rate of 1 million per year. (a) Predict this country's population for the year 2020. (b) What is the level of population that this country will never exceed if this equation remains the correct model?

Answers

Problem 3. **1.** $\ln|y| = 4\ln|x| + C \quad \rightarrow \quad y = C_1x^4$ **3.** $-\dfrac{1}{2}e^{-2y} = \dfrac{1}{3}e^{3x} + C$

4. $ye^y - e^y = -e^{-x} - \dfrac{1}{3}e^{-3x} + C,$ **6.** $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + C.$

8. $\dfrac{1}{2}(e^x + 1)^{-2} = -(1 + e^y)^{-1} + C$

10. $\ln|S| = kr + C \quad \rightarrow \quad S = C_1e^{kr}$ **11.** $\ln|Q - 70| = kt + C \quad \rightarrow \quad Q = 70 + C_1e^{kt}$

12. $\ln\left|\dfrac{P}{1-P}\right| = t + C \quad \rightarrow \quad P = \dfrac{C_1e^t}{1 + C_1e^t}$ **13.** $y - 5\ln|y + 3| = x - 5\ln|x + 4| + C.$

Problem 4.

2. $\ln|y| = -\dfrac{1}{x} - \ln|x| - 1$ **3.** $-\dfrac{1}{2}\ln|1 - 2y| = t - \dfrac{1}{2}\ln 4$ or $1 - 2y = -4e^{-2t}$ or $y = 2e^{-2t} + \dfrac{1}{2}$

4. $y = \dfrac{1}{1+2x}$ or $\ln\left|\dfrac{y-1}{y}\right| = \ln|x| - \ln\dfrac{1}{2}$ **5.** $y(x) = \dfrac{1}{1 - x^2 - x^3}.$

Problem 5. a) $T(1) = 36.67°$, $t = 3.06$ min b) $T(0) = 64.46°$

Problem 6. **3.** $y = \dfrac{1}{3} + Ce^{-x^3}$ **4.** $y = \dfrac{1}{2}x^2 - \dfrac{1}{2} + Ce^{-x^2},$ **5.** $y = \dfrac{\ln x}{x} + \dfrac{C}{x}$ $(x > 0)$

6. $y = -\dfrac{1}{2}x^2 - \dfrac{1}{2}x - \dfrac{11}{4} + Ce^{2x}$ **8.** $y = \dfrac{1}{2}e^x\dfrac{1}{x^2} + Ce^{-x}\dfrac{1}{x^2}$ $(x > 0)$

Notice that in # 9 and # 10, x is a function of y:

9. $x = 2y^6 + Cy^4$ $(y > 0)$ **10.** $x = e^y - \dfrac{2}{y}e^y + \dfrac{2}{y^2}e^y + \dfrac{C}{y^2}$ $(y > 0)$

12. $y = \dfrac{x^2}{x+1}e^{-x} + \dfrac{C}{x+1}e^{-x}$ $(x > -1)$ **14.** $P = 2 + Ce^{t-t^2}$

15. $y = \dfrac{x(x+1)}{x-1} + C\dfrac{x+1}{x-1}$ $(-1 < x < 1)$

16. $x = 2y^2 - \dfrac{49}{5}y.$ **18.** $T = T_m + (T_0 - T_m)e^{kt}$ **19.** $y = (t - 1 + 2e^{-t})/t \quad (t > 0).$

Problem 7. **1.** $\dfrac{dA}{dt} = -\dfrac{3A}{300},\quad A(0) = 50.$ **2.** $\dfrac{dA}{dt} = (2)(3) - \dfrac{2A}{300+t},\quad A(0) = 50.$

3. $\dfrac{dA}{dt} = (2)(3) - \dfrac{3.5A}{300 - 0.5t},\quad A(0) = 50.$ The tank is empty in 600 minutes.

Problem 8. **1.** A model is $\dfrac{dA}{dt} = 0 - \dfrac{4A}{200} = -\dfrac{A}{50},\quad A(0) = 30.$ The answer is $A(t) = 30e^{-t/50}.$

2. A model is $\dfrac{dA}{dt} = 10 - \dfrac{5A}{500} = 10 - \dfrac{A}{100},\quad A(0) = 0.$ The answer is $A(t) = 1000 - 1000e^{-t/100}.$

3. A model is $\dfrac{dA}{dt} = 10 - \dfrac{2A}{100-t},\quad A(0) = 0.$ The answer is $A(t) = 1000 - 10t - 1/10(100 - t)^2.$

4. A model is $\dfrac{dA}{dt} = 3 - \dfrac{4A}{100+(6-4)t} = 3 - \dfrac{2A}{50+t}, \quad A(0) = 10.$

The solution is $A(t) = 50 + t - 100000(50+t)^{-2}$. The answer is $A(30) = 64.38$ pounds.

Problem 9.

a. $y = \tan(x+C) - x - 1$. **b.** $2\sqrt{y-2x+3} = x+C$ **c.** $\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x+50y+30| = x+C$

1. Homogeneous, use $u = \frac{y}{x}$. Answer: $\frac{1}{2}\ln\left|1+2\frac{y}{x}\right| = -\ln|x| + C$. The answer can be simplified to $x^2 + 2xy = C_1$

2. Homogeneous, use $u = \frac{y}{x}$. Answer: $\ln|x| = -\frac{x}{y} + C$.

3. Homogeneous, use $u = \frac{y}{x}$. Answer: $\frac{1}{2}\ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|\frac{y}{x}+2\right| = -\ln|x| + C$. The answer can be simplified to $x^2y = C_1(y+2x)$

4. Homogeneous, use $u = \frac{y}{x}$. Answer: $\ln|x| = -\frac{1}{3}\frac{y^3}{x^3} + C$.

5. Homogeneous, use $u = \frac{y}{x}$. Answer: $-2\sqrt{x/y} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C$.

6. Bernoulli, substitution: $u = y^3$. Answer: $y^3 = 1 + Cx^{-3}$

7. Bernoulli, substitution: $u = y^{-1}$. Answer: $y^{-1} = -\frac{1}{2}e^x + Ce^{-x}$

8. Bernoulli **9.** Bernoulli **10.** Use the substitution $u = y - x + 5$. Answer: $-e^{y-x+5} = x + C$

11. Bernoulli, substitution: $u = y^{-1}$. Answer: $y^{-1} = \frac{1}{t}\ln t + Ct^{-1}$. This equation is also homogeneous so it can be solved in any of the two ways.

12. Bernoulli, substitution: $u = y^{-3}$. Answer: $y^{-3} = 1 + C(1+t^2)$.

13. Bernoulli, substitution: $u = y^{1/2}$. Answer: $\sqrt{y} = x^2 + C/x$.

14. Homogeneous, use $u = \frac{y}{x}$. Answer: $\ln|x| = \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) - \frac{1}{2}\ln(2)$. The answer can be simplified to $2x^4 = y^2 + x^2$.

Problem 10. **1.** $x^2 - x + \frac{3}{2}y^2 + 7y = C$ **2.** Not exact. **3.** $\frac{5}{2}x^2 + 4xy - 2y^4 = C$

5. $x^2y^2 - 3x + 4y = C$ **6.** Not exact **7.** $-y + y\ln x + x\ln x = C$

8. $xy^3 + y^2\cos x - \frac{1}{2}x^2 = C.$ **9.** $\frac{1}{4}x^4 + xy^3 = C.$ **10.** $x^3y + xe^y - y^2 = C$

11. $xy - 2x^3 - 2xe^x + 2e^x = C$ **12.** $x + y + xy - 3\ln|xy| = C$

13. $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$ **14.** $e^x + xy + 2y + ye^y - e^y = 3$

15. $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\dfrac{5}{4}$ **16.** $y^2\sin x - x^3y - x^2 + y\ln y - y = 0$