Lesson 3.3

The Continuous Uniform Distribution

Learning Outcomes

At the end of the lesson, students must be able to

1. Describe the key properties of a continuous uniform random variable, such as the mean, variance, and moment generating function, and
2. Compute probabilities associated with random variables having a uniform distribution.

Introduction

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur.

Definition:

A random variable Y is said to have a uniform distribution from θ_1 to θ_2 if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta_2-\theta_1}, \, \theta_1 < y < \theta_2 \\ 0, \text{ elsewhere} \end{cases}$$

Shorthand notation is $Y \sim U(\theta_1, \theta_2)$.

Remarks:

1. The constants that determine the specific form of a density function are called parameters of the density function.

2. An important uniform distribution is that for which $\theta_1 = 0$ and $\theta_1 = 0$, namely, $U(0,1)$. It is used in the generation of random numbers and in computer simulations.
3. The number of events, such as calls coming into a switchboard, that occur in the time interval $(0,t)$ has a Poisson distribution. If it is known that exactly one such event has occurred in the interval $(0,t)$, then the actual time of occurrence is distributed uniformly over this interval.

Definition:

If $Y \sim U(\theta_1, \theta_2)$, then the CDF of Y is given

$$F_Y(y) = \begin{cases} 0, \ y < \theta_1 \\ \frac{y-\theta_1}{\theta_2-\theta_1}, \ \theta_1 \le y < \theta_2 \\ 1, \ y \ge \theta_2 \end{cases} \qquad \text{(Verify!)}$$

Theorem:

If $Y \sim U(\theta_1, \theta_2)$, then its mean, variance, and moment generating function are given, respectively, by

$$E(Y) = \frac{\theta_1 + \theta_2}{2},$$

$$V(Y) = \frac{(\theta_2 - \theta_1)^2}{12},$$

and

$$m_Y(t) = \frac{e^{\theta_2 t} - e^{\theta_1 t}}{t(\theta_2 - \theta_1)}, \ t \neq 0$$

Proof: Left as a classroom exercise.

Example 1:

A continuous random variable Y is uniformly distributed over $[-a, a]$, where $a > 0$. Determine the following:

1. $f_Y(y)$
2. $F_Y(y)$
3. $P(0 \le Y \le a)$

ANSWERS: (Verify!)

1. PDF

$$f_Y(y) = \begin{cases} \frac{1}{2a},\ -a \leq y \leq a \\ 0,\ \text{elsewhere} \end{cases}$$

2. CDF

$$F_Y(y) = \begin{cases} 0,\ y < -a \\ \frac{1}{2a}(y+a),\ -a \leq y < a \\ 1,\ y \geq a \end{cases}$$

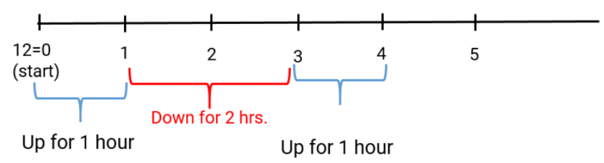
c. $P(0 \leq Y \leq a) = 0.5$

Example 2:

Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 A.M. What is the probability that the center is up when the person's call comes in?

SOLUTION:

Let Y = number of hours the call is made. Looking at the diagram below, it is easy to see that $Y \sim U(0,5)$.



Since $Y \sim U(0,5)$, then

$$f_Y(y) = \begin{cases} \frac{1}{5}, & 0 \le y \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Therefore, the probability that the center is up when the call comes in is

$$P(0 < Y < 1) + P(3 < Y < 4) = \int_0^1 \frac{1}{5}\, dy + \int_3^4 \frac{1}{5}\, dy = 0.4$$

Example 3:

The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

a. What is the probability that a person waits fewer than 12.5 minutes?

b. On the average, how long must a person wait?

c. Ninety percent of the time, the time a person must wait falls below what value?

SOLUTION: Left as a classroom exercise!

Example 4:

The cycle time for trucks hauling concrete to a highway construction site is uniformly distributed over the interval 50 to 70 minutes. What is the probability that the cycle time exceeds 65 minutes if it is known that the cycle time exceeds 55 minutes?

SOLUTION: Left as a classroom exercise!