Lesson 3.2

Mathematical Expectation of Continuous Random Variables

Learning Outcomes

At the end of the lesson, students must be able to

1. Determine expected value of continuous random variables, and
2. Compute special expectations such as the mean, variance, and moment generating function of continuous random variables.

Introduction

The special expectations, such as the mean, variance, and moment generating function, for continuous random variables are just a straightforward extension of those of the discrete case. Again, all we need to do is replace the summations with integrals.

Definition:

Suppose Y is a continuous random variable with PDF $f_Y(y)$. Then the expected value (or mean) of Y is

$$E(Y) = \mu = \int_{\mathbb{R}} y \times f_Y(y)\, dy$$

Remarks:

1. We interpret $E(Y)$ in the same way as we did when Y is discrete.
2. For $E(Y)$ to exist, we need the integral to converge absolutely, that is,

$$E(Y) = \int_{\mathbb{R}} |y| \times f_Y(y)\, dy < \infty,$$

otherwise, $E(Y)$ does not exist.

Definition:

Suppose Y is a continuous random variable with PDF $f_Y(y)$. The expected value of a function of Y, say $g(Y)$, is

$$E[g(Y)] = \int_{\mathbb{R}} g(y) \times f_Y(y)\, dy,$$

provided that this integral converges absolutely. Otherwise, we say that $E[g(Y)]$ does not exist.

Remarks:

1. The expectation operator $E()$ enjoys the same properties as in the discrete case.
2. The variance of Y is $V(Y) = \sigma_Y^2 = \int_{\mathbb{R}} (y-\mu)^2 \times f_y(y)\, dy$, provided the integral exists. Alternatively, $V(Y) = E(Y^2) - \mu^2$, where $E(Y^2) = \int_{\mathbb{R}} y^2 \times f_Y(y)\, dy$.

Example 1:

Consider a continuous random variable with PDF

$$f_Y(y) = \begin{cases} \frac{3}{8}y^2,\ 0 < y < 2 \\ 0, \text{ elsewhere} \end{cases}$$

Compute the mean and variance of Y.

SOLUTION:

We have

$$\mu = E(Y) = \int_0^2 y \times \frac{3}{8}y^2\, dy = 1.5$$

and

$$E(Y^2) = \int_0^2 y^2 \times \frac{3}{8}y^2\, dy = 2.4.$$

Hence,

$$V(Y) = E(Y^2) - \mu^2 = 2.4 - 1.5^2 = 0.15.$$

Example 2:

Suppose X has the following PDF

$$f_X(x) = \begin{cases} 2x, 0 < x < 1 \\ 0, \text{ elsewhere} \end{cases}$$

Find $E(U)$, where $U = -2x + 3$.

SOLUTION:

$$\begin{aligned} E(U) &= \int_0^1 u \times f_X(x)\, dx \\ &= \int_0^1 (-2x+3) \times 2x\, dx \\ &= \frac{5}{3} \end{aligned}$$

Definition:

The moment generating function (MGF) of a continuous random variable Y with PDF $f_Y(y)$ is given by

$$m_Y(t) = E(e^{tY}) = \int_{\mathbb{R}} e^{ty} \times f_Y(y)\, dy$$

provided this expectation is finite for all t in an open neighborhood about $t = 0$, that is, there exists a $b > 0$ such that $E(e^{tY}) < \infty$, for all $t \in (-b, b)$. If no such $b > 0$ exists, then the moment generating function of Y does not exist.

Remarks

Recall that we can use the moment generating function to obtain special moments such the mean and the variance. That is,

$$E(Y^k) = \frac{d^k}{dt} [m_Y(t)] \Big|_{t=0}$$

So,

$$E(Y) = \frac{d}{dt} [m_Y(t)] \Big|_{t=0}$$

and

$$E(Y^2) = \frac{d^2}{dt} \left[m_Y(t) \right] \Big|_{t=0}$$

<u>Example 3</u>:

The PDF of a continuous random variable Y is given by

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, y > 0 \\ 0, \text{ elsewehre} \end{cases}$$

a. Show that the MGF of Y is $m_Y(t) = \frac{\lambda}{\lambda - t}$.
b. Find the mean and variance of Y using the MGF.

SOLUTION: Left as a classroom exercise!

<u>Example 4</u>:

A random variable Y has a PDF

$$f_Y(y) = \begin{cases} 1 - \frac{y}{2},\ 0 \leq y \leq 2 \\ 0,\ \text{elsewehere} \end{cases}$$

Find the MGF of Y and calculate its mean and standard deviation.

SOLUTION: Left as a classroom exercise!