Lesson 1.2: Counting Methods

Learning Outcomes

At the end of the lesson, students must be able to

1. Explain the fundamental principle of counting and relate it to permutation,
2. Explain the difference between permutation and combination, and
3. Apply permutation and combination in assigning probabilities to events.

Discussion

Fundamental Principle of Counting

Definition:

If a certain experiment can be performed in *m* ways and, corresponding to each of these ways, another experiment can be performed in *n* ways, then the combined experiment can be performed in *mn* ways. We call this process as the **fundamental principle of counting** or the **multiplication rule of counting**.

To understand this principle, suppose the outcomes of experiment A are written as $A=\{a_1,a_2,a_3,\cdots,a_m\}$ and those of the second experiment as $B=\{b_1,b_2,b_3,\cdots,b_n\}$. Then the outcomes of the combined experiment can be represented in a rectangular array as ordered pairs (a_i,b_j). In other words, the outcomes of the combined experiment can be represented as the Cartesian product $A\times B$. Clearly, $n(A\times B)=n(A)\times n(B)=m\times n$.

$$
\begin{array}{c|cccccc}
 & b_1 & b_2 & \cdots & b_j & \cdots & b_n \\
\hline
a_1 & (a_1,b_1) & (a_1,b_2) & \cdots & (a_1,b_j) & \cdots & (a_1,b_n) \\
a_2 & (a_2,b_1) & (a_2,b_2) & \cdots & (a_2,b_j) & \cdots & (a_2,b_n) \\
\vdots & \vdots & & & & & \\
a_i & (a_i,b_1) & (a_i,b_1) & \cdots & (a_i,b_j) & \cdots & (a_i,b_n) \\
\vdots & \vdots & & & & & \\
a_m & (a_m,b_1) & (a_m,b_2) & \cdots & (a_m,b_j) & \cdots & (a_m,b_n) \\
\hline
\end{array}
$$

The fundamental principle of counting can be extended to any number of experiments in an obvious way. We shall now give some examples.

Example 2.1:

An experiment consists of rolling a die (with faces 1, 2, ..., 6) and tossing a coin (with sides H and T). The die has m =6 outcomes. The coin has n =2 outcomes. Hence, there are mn =12 paired outcomes. These 12 paired outcomes are:

{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)}.

Example 2.2:

If a person has 8 different shirts, 6 different ties, and 5 different jackets, then he can get dressed for an occasion in $8 \times 6 \times 5 = 240$ ways.

Example 2.3:

Suppose license plates are formed with three distinct letters followed by three distinct digits. Then there 26 choices for the first letter, 25 for the second, and 24 for the third. Also, there are 10 choices for the first digit, 9 for the second, and 8 for the third. Therefore, there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11{,}232{,}000$ different license plates that can be made.

Factorial Notation

Before going any further, let me introduce the factorial notation (!).

Definition:

The **factorial** of a positive integer n, denoted by $n!$, is the product of all integers less than or equal to n. That is, $n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 1$.

Example 2.4:

The factorial of 4 is $4! = 4 \times 3 \times 2 \times 1 = 24$ and the factorial of 6 is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Remark:

It is convenient to define $0! = 1$.

Permutation

Suppose we have a collection of objects say $\{a, b, c\}$ and we pick two objects. The different pairs of objects are:

ab *ba*

ac *ca*

bc *cb*

The order in which the letters are written is important. For example, *ab* is a different arrangement from *ba*. Any particular arrangement is called a permutation. In the above listing there are 6 permutations in all. The reason for this is simple and is based on the fundamental principle of counting. Since we are going to select 2 objects (letters) from 3 available objects (letters), we have 3 ways or choices for the first letter and 2 ways or choices for the second letter. Therefore, $3 \times 2 = 6$ permutations or arrangements.

Now let us turn to the general case via the following definition.

Definition:

A **permutation** is an arrangement of distinct objects in a particular order. *Order is important*. The number of permutations of r objects selected from a collection of n distinct objects is $P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$.

Remarks:

1. Another notation used to refer to the number of permutations of *r* objects from a collection of *n* distinct objects is ${}_nP_r$.
2. Using the factorial notation, we have
$${}_nP_r = \frac{n!}{(n-r)!}.$$
3. Note that $r \leq n$ and if *r* = *n*, then ${}_nP_n = n!$.

Let us look at some illustrations.

Example 2.5:

An artist has 9 paintings. How many ways can he hang 4 paintings side-by-side on a gallery wall?

Answer:

We can arrive at the answer in two ways:

1. Consider these boxes as the positions of the four paintings on the wall:

1^{st}	2^{nd}	3^{rd}	4^{th}

There are 9 paintings to choose from for the 1^{st} position, 8 remaining paintings for the 2^{nd} position, 7 remaining paintings for the 3^{rd} position, and 6 paintings for the 4^{th} position. Therefore, by the fundamental principle of counting, the number of permutations is $9 \times 8 \times 7 \times 6 = 3024$.

2. We can get the same answer using the permutations formula:
$${}_9P_4 = \frac{9!}{(9-4)!} = 3024.$$

Example 2.6:

In how many ways can a president, a treasurer and a secretary be chosen from among 7 candidates?

Answer:

Similar to Example 2.5, we can get the answer using either the fundamental principle of counting or the permutation formula. Based on the fundamental principle of counting, we have 7 choices for the president, 6 for the treasurer, and 5 for secretary, thus, we have $7 \times 6 \times 5 = 210$ ways of selecting a president, a treasurer, and secretary from 7 candidates.

Using the permutation formula, we get the same answer:

$$_7P_3 = \frac{7!}{(7-3)!} = 210 .$$

Example 2.7:

In how many ways can 7 different books be arranged on a shelf?

Answer:

In this case, all 7 books will be rearranged, thus, *r* = *n*. Therefore, there are $_7P_7 = 7! = 5040$ ways of arranging the 7 books.

In the definition of permutation, we emphasized that the *n* objects are distinct or unique. There are instances where this assumption is not true. For example, we may be interested on the number of rearrangements of the letter in the word PEPPER. The letters are no longer unique-there are 3 P's, 2 E's, and 1 R. How do we determine the number of different rearrangements of these letters?

Another permutation formula is required to solve this type of problem and it is given below.

The number of ways to arrange n objects, n_1 being of one kind, n_2 of a second kind, . . . , and n_r of an r^{th} kind, is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$$

where $n = n_1 + n_2 + \cdots + n_r$.

Example 2.8:

There are *n = 6* letters in the word PEPPER of which there are 3 P's, 2 E's, and 1 R. Thus, the number of permutations of the letter in the word PEPPER is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!} = \frac{6!}{3! \times 2! \times 1!} = 60 .$$

Combination

Maria has three tickets for a concert. She'd like to use one of the tickets herself. She could then offer the other two tickets to any of four friends (Ann, Beth, Chris, Dave). How many ways can 2 people be selected from 4 to go to a concert?

Let us denote Maria's friends as A for Ann, B for Beth, C for Chris, and D for Dave. If order matters, then these are the pairs of people:

(A, B) (B, A)
(A, C) (C, A)
(A, D) (D, A)
(B, C) (C, B)
(B, D) (D, B)
(C, D) (D, C)

Obviously, order doesn't matter here, so (A, B) is the same pair of people as (B, A), (A, C) is the same pair as (C, A), and so on. Hence, instead of 12 there must only be 6 possible pairs.

The arrangement of objects without regard to order is called **combination.**

Definition:

From a collection of n distinct objects, we choose r of them ($r \leq n$) *without regard to the order in which the objects are chosen*. The number of ways to do this is

$$C(n,r) = \frac{n!}{(n-r)! \times r!}.$$

The symbol $C(n,r)$ is read as "the combination of n things taken r at a time." Sometimes, we use the notation ${}_nC_r$ or $\binom{n}{r}$ in place of $C(n,r)$.

Remarks:

1. Combination is most often used in problems which involves creating groups or committees of people from a given set of people and order of selection is not important.
2. The number of combinations of n objects taken r at a time is less than or equal to the number of permutations of n objects taken r at a time since ${}_nC_r = \frac{{}_nP_r}{r!}$.
3. You may apply more than one method in solving a counting problem, say combination with fundamental principle of counting or permutation with fundamental principle of counting.

Example 2.9:

How many different ways can the admissions committee of a statistics department choose four foreign graduate students from 20 foreign applicants and three Filipino students from 10 Filipino applicants?

Answer:

The 4 foreign students can be selected from the 20 foreign students in ${}_{20}C_4 = \frac{20!}{16!4!} = 4845$ ways and the 3 Filipino students can be selected from 10 Filipino students in ${}_{10}C_3 = \frac{10!}{7!3!} = 120$ ways. Therefore, according to the Fundamental Principle of Counting, the whole selection of 4 foreign students and 3 Filipino students can be done in $4845 \times 120 = 581400$ ways.

Let us now use these counting rules in assigning probabilities to events.

Example 2.10

In a tank containing 10 fishes, there are three yellow and seven black fishes. We select three fishes at random.

a. *What is the probability that exactly one yellow fish gets selected?*
b. *What is the probability that at most one yellow fish gets selected?*
c. *What is the probability that at least one yellow fish gets selected?*

Answers:

There are ${}_{10}C_3 = 120$ *ways to select three fishes from 10. That is,* $n(S) = 120$.

1. Let *A be the event that exactly one yellow fish gets selected. If one yellow fish is to be selected, then the other two must be black fishes to complete the 3 sample fishes. Now, there are* ${}_3C_1 = 3$ *ways of selecting a yellow fish from 3 yellow fishes and* ${}_7C_2 = 21$ *ways of selecting 2 black fishes. Thus, by the fundamental principle of counting, the number of ways of selecting a yellow fish and 2 black fishes is* $3 \times 21 = 63$ *ways or* $n(A) = 63$*. Therefore, the probability that exactly one yellow fish gets selected is*

$$P(A) = \frac{n(A)}{n(S)} = \frac{63}{120} = 0.525.$$

2. *Let B be the event that at most one yellow. When we say at most one, it means 0 or 1. If 0 yellow fish is selected that means all 3 fishes are black and if 1 fish is yellow then other 2 must be black. In (a) above, the number of ways of selecting 1 yellow and 2 black fishes is* ${}_3C_1 \times {}_7C_2 = 3 \times 21 = 63$*. The number of ways of selecting no yellow fish (all 3 black fishes is selected) is* ${}_3C_0 \times {}_7C_3 = 1 \times 35 = 35$*. Therefore, the total number of ways of selecting 0 or 1 yellow fish is* $n(B) = 63 + 35 = 98$ *and the probability that at most one yellow fish gets selected is*

$$P(B)=\frac{n(B)}{n(S)}=\frac{98}{120}\cong 0.817\,.$$

Remark:

Recall that the disjunction "or" implies addition, that is why we added 63 and 35.

3. *Let C be the event that at least one yellow fish gets selected. When we say at least 1 it means 1, 2, or 3 yellow fishes. Further, if 1 yellow fish is selected, then the other 2 must be black; if 2 are yellow, then the other 1 is yellow; and if all 3 are yellow, then there is no black fish selected. Using the same principle as in (a) and (b), the probability of selecting at least one yellow fish is*

$$\begin{aligned}P(C)=\frac{n(C)}{n(S)}&=\frac{{}_3C_1\times{}_7C_2+{}_3C_2\times{}_7C_1+{}_3C_3\times{}_7C_0}{{}_{10}C_3}\\&=\frac{3\times 21+3\times 7+1\times 1}{120}\\&=\frac{85}{120}\\&\cong 0.708\end{aligned}$$

Alternatively, we can use the complement rule to get the same answer. The complement of at least 1 yellow fish is 0 yellow fish. Therefore,

$$\begin{aligned}P(C)=1-P(C')&=1-\frac{n(C')}{n(S)}\\&=1-\frac{{}_3C_0\times{}_7C_3}{120}\\&=1-\frac{1\times 35}{120}\\&\cong 1-0.282\\&=0.708\end{aligned}$$

Learning Tasks/Activities

Answer the following as indicated.

1. If repetitions are not allowed, how many three-digit numbers can be formed from the digits 2, 3, 5, 6, 7, 9?
2. How many permutations of the digits in the number 1234567 will result in an even number?
3. From a group of 5 swimmers and 8 runners, an athletic contingent of 7 is to be formed. How many teams are possible if there are to be
    a. 2 swimmers and 5 runners
    b. at least 3 swimmers
4. How many permutations are possible from the letters in the word PHILIPPINES?

5. A package of 15 apples contains two defective apples. Four apples are selected at random.
    a. Find the probability that none of the selected apples is defective.
    b. Find the probability that at least one of the selected apples is defective.

Assessment

Answer the following as indicated.

1. *An examination is designed where the students are required to answer any 20 questions from a group of 25 questions. How many ways can a student choose the 20 questions?*
2. *How many different six-place license plates are possible if the first three places and the last place are to be occupied by letters and the fourth and fifth places are to be occupied by numbers?*
3. A student prepares for an exam by studying a list of ten problems. She can solve six of them. For the exam, the instructor selects five problems at random from the ten on the list given to the students. What is the probability that the student can solve all five problems on the exam?