Lesson 2.2 Confounding in 2^k Factorial Experiment

Introduction

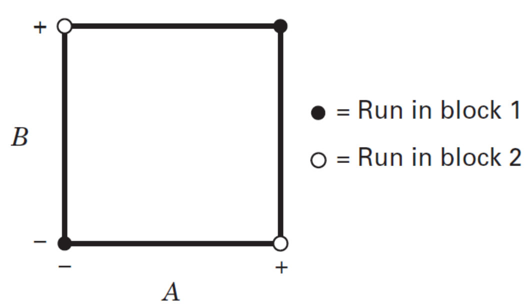
Recall that we use blocking strategy to remove or minimize the effect of a single nuisance factor. The design is referred to as the Randomized Complete Block Design, wherein we run a complete set of treatments in every block.

In many problems it is impossible to perform a complete replicate of a factorial design in one block. **Confounding** is a design technique for arranging a complete factorial experiment in blocks, where the block size is smaller than the number of treatment combinations in one replicate. The technique causes information about certain treatment effects (usually higher-order interactions) to be indistinguishable from, or confounded with, blocks.

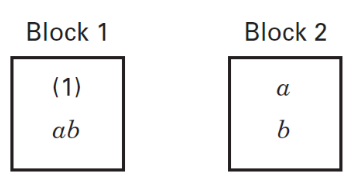
Confounding the 2^k factorial design in two blocks

Suppose that we wish to run a single replicate of the 2^2 design. Each of the $2^2 = 4$ treatment combinations requires a quantity of raw material, for example, and each batch of raw material is only large enough for two treatment combinations to be tested. Thus, two batches of raw material are required. If batches of raw material are considered as blocks, then we must assign two of the four treatment combinations to each block.

The figure below shows one possible design for this problem. The figure indicates that treatment combinations on opposing diagonals are assigned to different blocks. Notice that block 1 contains the treatment combinations (1) and ab and that block 2 contains a and b. Of course, the order in which the treatment combinations are run within a block is randomly determined. We would also randomly decide which block to run first.



(a) Geometric view



(b) Assignment of the four runs to two blocks

From the previous lesson, we can estimate the main and interaction effects as follows.

$$A = \frac{1}{2}\{ab + a - b - (1)\}$$
$$B = \frac{1}{2}\{ab + b - a - (1)\}$$
$$AB = \frac{1}{2}\{ab - a - b + (1)\}$$

From the figure above, it is easy to see that the block effect is

$$Block = \frac{1}{2}\{ab + (1) - a - b\}$$

This means that the AB effect is identical to the $Block$ effect. We also say that AB is **confounded** with blocks.

Below is the table of plus and minus signs for the 2^2 design.

Treatment	I	A	B	AB	Block
(1)	+	-	-	+	1
a	+	+	-	-	2
b	+	-	+	-	2
ab	+	+	+	+	1

From this table, we see that all treatment combinations that have a plus sign on AB are assigned to block 1, whereas all treatment combinations that have a minus sign on AB are assigned to block 2. This approach can be used to confound any effect (A, B, or AB) with blocks. For example, if (1) and b had been assigned to block 1 and a and ab to block 2, the main effect A would have been confounded with blocks. The usual practice is to confound the highest order interaction with blocks.

This scheme can be used to confound any 2^k design in two blocks. As a second example, consider a 2^3 design run in two blocks. Suppose we wish to confound the three-factor interaction ABC with blocks. From the table of plus and minus signs shown below, we assign the treatment combinations that are minus on ABC to block 1 and those that are plus on ABC to block 2.

Table of Plus and Minus Signs for the 2^3 Design

Treatment Combination	Factorial Effect								
	I	*A*	*B*	*AB*	*C*	*AC*	*BC*	*ABC*	**Block**
(1)	+	−	−	+	−	+	+	−	1
a	+	+	−	−	−	−	+	+	2
b	+	−	+	−	−	+	−	+	2
ab	+	+	+	+	−	−	−	−	1
c	+	−	−	+	+	−	−	+	2
ac	+	+	−	−	+	+	−	−	1
bc	+	−	+	−	+	−	+	−	1
abc	+	+	+	+	+	+	+	+	2

Other methods for constructing the blocks

There is another method for constructing these designs. The method uses the linear combination

$$L = a_1x_1 + a_2x_2 + \cdots + a_kx_k \quad \equiv \textbf{defining contrast}$$

where x_i is the level of the i^{th} factor appearing in a particular treatment combination and α_i is the exponent appearing on the i^{th} factor in the effect to be confounded. For the 2^k system, we have $i = 0 or 1$ and $x_i = 0$ (low level) or $x_i = 1$ (high level). Treatment combinations that produce the same value of $L(mod 2)$ will be placed in the same block. Because the only possible values of $L(mod 2)$ are 0 and 1, this will assign the 2^k treatment combinations to exactly two blocks.

To illustrate the approach, consider a 2^3 design with ABC confounded with blocks. Here x_1 corresponds to A, x_2 to B, x_3 to C, and $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Thus, the defining contrast corresponding to ABC is

$$L = x_1 + x_2 + x_3$$

The treatment combination (1) is written 000 in the (0, 1) notation; therefore,

$$L = 1(0) + 1(0) + 1(0) = 0(mod 2) = 0$$

Similarly, the treatment combination a is 100, yielding

$$L = 1(1) + 1(0) + 1(0) = 1(mod 2) = 1$$

Thus, (1) and a would be run in different blocks. For the remaining treatment combinations, we have

$$
\begin{aligned}
b : L &= 1(0) + 1(1) + 1(0) = 1(mod2) = 1 \\
ab : L &= 1(1) + 1(1) + 1(0) = 2(mod2) = 0 \\
c : L &= 1(0) + 1(0) + 1(1) = 1(mod2) = 1 \\
ac : L &= 1(1) + 1(0) + 1(1) = 2(mod2) = 0 \\
bc : L &= 1(0) + 1(1) + 1(1) = 2(mod2) = 0 \\
abc : L &= 1(1) + 1(1) + 1(1) = 3(mod2) = 1
\end{aligned}
$$

Thus, (1), ab, ac, and bc are run in block 1 and a, b, c, and abc are run in block 2.

Another method may be used to construct these designs. The block containing the treatment combination (1) is called the **principal block**. The treatment combinations in this block have a useful group-theoretic property; namely, they form a group with respect to multiplication modulus 2. This implies that any element [except (1)] in the principal block may be generated by multiplying two other elements in the principal block modulus 2. For example, consider the principal block of the 2^3 design with ABC confounded.

Note that

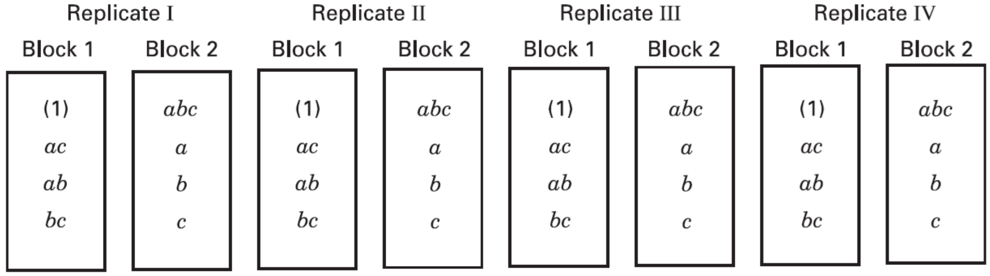
$$
\begin{aligned}
ab \times ac &= a^2bc = bc \\
ab \times bc &= ab^2c = ac \\
ac \times bc &= abc^2 = ab
\end{aligned}
$$

Treatment combinations in the other block (or blocks) may be generated by multiplying one element in the new block by each element in the principal block modulus 2. For the 2^3 with ABC confounded, because the principal block is (1), ab, ac, and bc, we know that b is in the other block. Thus, the elements of this second block are

$$
\begin{aligned}
b \times (1) &= b \\
b \times ab = ab^2 &= a \\
b \times ac &= abc \\
b \times bc = b^2c &= c
\end{aligned}
$$

Estimation of error

When the number of variables is small, say $k = 2$ or $k = 3$, it is usually necessary to replicate the experiment to obtain an estimate of error. For example, suppose that a 2^3 factorial must be run in two blocks with ABC confounded, and the experimenter decides to replicate the design four times. The resulting design is shown below. Note that ABC is confounded in each replicate.



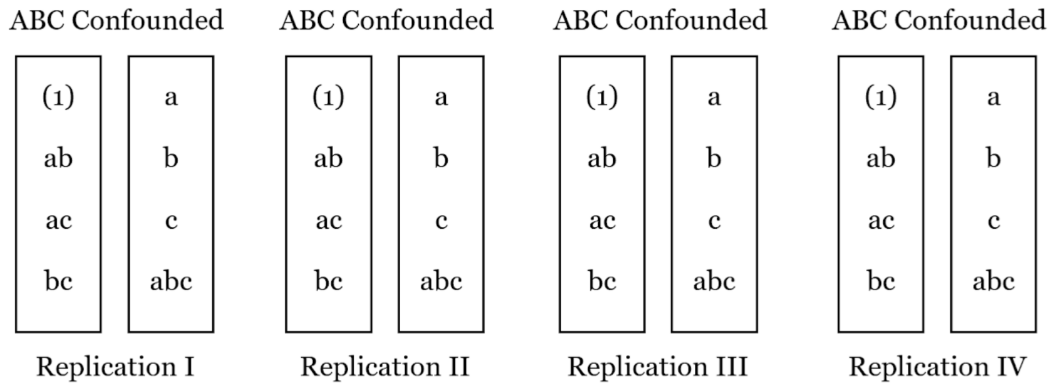
The analysis of variance for this design is shown below. There are 32 observations and 31 total degrees of freedom. Furthermore, because there are eight blocks, seven degrees of freedom must be associated with these blocks. The error sum of squares actually consists of the interactions between replicates and each of the effects (A, B, C, AB, AC, BC). It is usually safe to consider these interactions to be zero and to treat the resulting mean square as an estimate of error. Main effects and two-factor interactions are tested against the mean square error.

Analysis of Variance for Four Replicates of a 2^3 Design with ABC Confounded

Source of Variation	Degrees of Freedom
Replicates	3
Blocks (*ABC*)	1
Error for *ABC* (replicates × blocks)	3
A	1
B	1
C	1
AB	1
AC	1
BC	1
Error (or replicates × effects)	18
Total	31

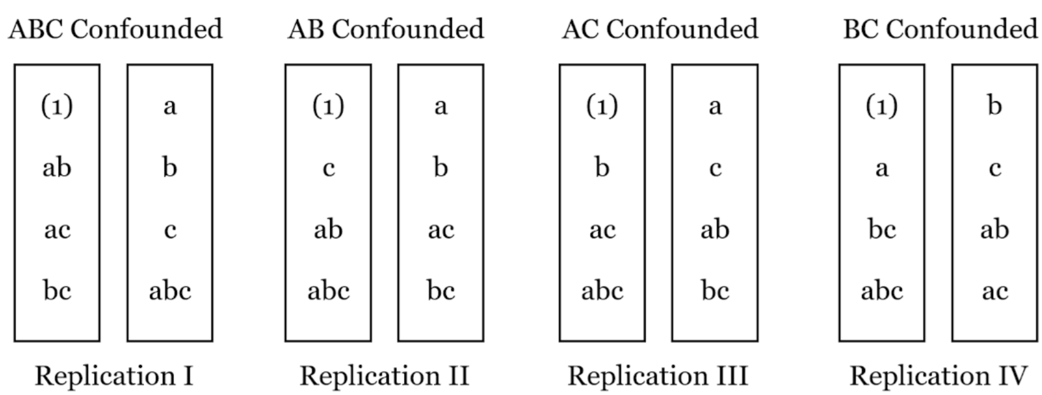
Complete vs Partial Confounding

If the replications are possible with confounding and blocking experiments, the confounding can be performed either completely or partially depending on the interest of the research questions or hypothesis. For an example, the ABC interaction is completely confounded with blocks in the figure below.



In this situation, the three-way ABC interaction is not an interest of the experiment. In this design, no information can be retrieved for the ABC interaction. However, all the main effects and the second-order interaction can be obtained 100%.

However, if some information is useful for the ABC interaction, it could be partially confounded as shown in the figure below. In this situation, the ABC, AB, AC, and BC are confounded with blocks in the replication I, II, III, and IV, respectively. Therefore, 3/4th (75%) information can be retrieved for each of the interaction terms. For an example, the AB interaction effect can be obtained from replication I, III, and IV. This confounding process is known as **partial confounding**.



The ANOVA table corresponding to the partially confounded design is given below.

Source of Variation	Degrees of Freedom
Replicates	3
Blocks within replicates [or ABC (rep. I) + AB (rep. II) + AC (rep. III) + BC (rep. IV)]	4
A	1
B	1
C	1
AB (from replicates I, III, and IV)	1
AC (from replicates I, II, and IV)	1
BC (from replicates I, II, and III)	1
ABC (from replicates II, III, and IV)	1
Error	17
Total	31

Nevertheless, three-way interaction ABC effect is rarely a practical interest. Therefore, complete confounding of higher-order interactions for the interest of the lower-order interactions would be preferable.

Complete confounding (single replicate): An example

Consider the experiment in Lesson 2.1 wherein four factors—temperature (A), pressure (B), concentration of formaldehyde (C), and stirring rate (D) are studied in a pilot plant to determine their effect on product filtration rate. We will use this experiment to illustrate the ideas of blocking and confounding in an unreplicated design. We will make two modifications to the original experiment. First, suppose that the $2^4 = 16$ treatment combinations cannot all be run using one batch of raw material. The experimenter can run eight treatment combinations from a single batch of material, so a 2^4 design confounded in two blocks seems appropriate. It is logical to confound the highest order interaction ABCD with blocks. It can be verified using the matrix of (-) and (+) signs that treatments (1), *ab*, *ac*, *bc*, *ad*, *bd*, *cd*, and *abcd* are in the principal block. The second modification that we will make is to introduce a block effect so that the utility of blocking can be demonstrated. Suppose that when we select the two batches of raw material required to run the experiment, one of them is of much poorer quality and, as a result, all responses will be 20 units lower in this material batch than in the other. poor quality batch becomes block 1 and the good quality batch becomes block 2. Now all the tests in block 1 are performed first (the eight runs in the block are, of course, performed in random order), but the responses are 20 units lower than they would have been if good quality material had been used.

```
library(tidyverse)
library(FrF2)

#generates the design matrix
design <- expand.grid(A = c(-1, 1),
                      B = c(-1, 1),
                      C = c(-1, 1),
```

```r
                       D = c(-1, 1))


adj.rate <- c(25, 71, 48, 45, 68, 40, 60, 65,
              43, 80, 25, 104, 55, 86, 70, 76)

blocks <- c(1, 2, 2, 1, 2, 1, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1)

#Add the response column
df <- design %>%
  mutate(Y = adj.rate,
         A = factor(A),
         B = factor(B),
         C = factor(C),
         D = factor(D),
         Blk = factor(blocks))

#prints the design matrix with the response
df
```

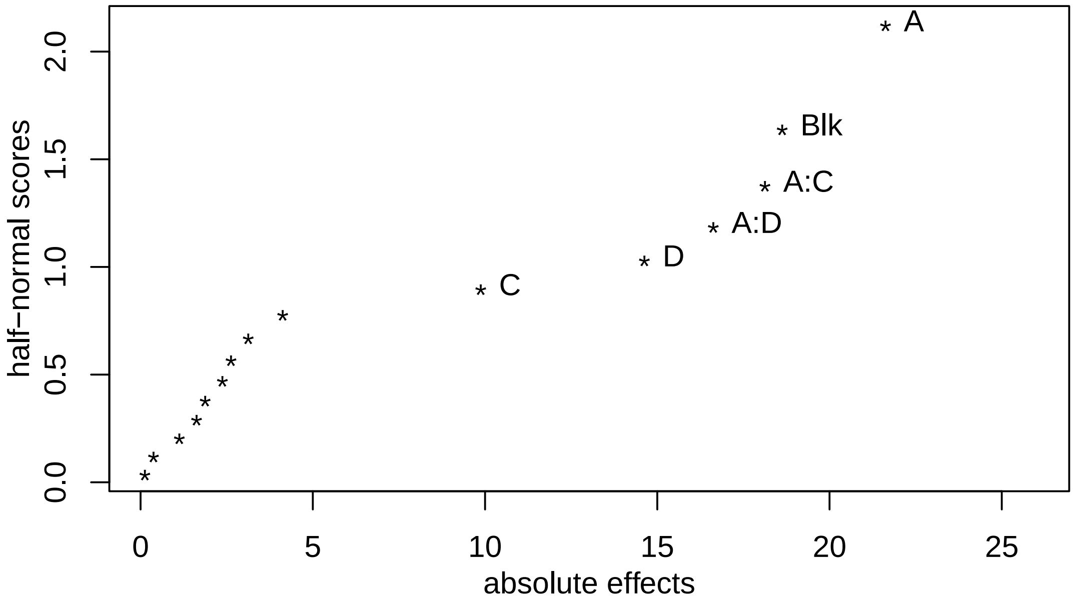
```
##     A  B  C  D   Y Blk
## 1  -1 -1 -1 -1  25   1
## 2   1 -1 -1 -1  71   2
## 3  -1  1 -1 -1  48   2
## 4   1  1 -1 -1  45   1
## 5  -1 -1  1 -1  68   2
## 6   1 -1  1 -1  40   1
## 7  -1  1  1 -1  60   1
## 8   1  1  1 -1  65   2
## 9  -1 -1 -1  1  43   2
## 10  1 -1 -1  1  80   1
## 11 -1  1 -1  1  25   1
## 12  1  1 -1  1 104   2
## 13 -1 -1  1  1  55   1
## 14  1 -1  1  1  86   2
## 15 -1  1  1  1  70   2
## 16  1  1  1  1  76   1
```

```r
#fits the model using lm() function
model <- lm(Y ~ Blk + A*B*C*D, data = df)

#prints the estimates
summary(model)
```

```
##
## Call:
## lm.default(formula = Y ~ Blk + A * B * C * D, data = df)
##
## Residuals:
## ALL 16 residuals are 0: no residual degrees of freedom!
##
## Coefficients: (1 not defined because of singularities)
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   25.000        NaN     NaN      NaN
## Blk2          18.625        NaN     NaN      NaN
## A1            27.375        NaN     NaN      NaN
## B1             4.375        NaN     NaN      NaN
## C1            24.375        NaN     NaN      NaN
## D1            -0.625        NaN     NaN      NaN
## A1:B1        -11.750        NaN     NaN      NaN
## A1:C1        -36.750        NaN     NaN      NaN
## B1:C1          6.250        NaN     NaN      NaN
## A1:D1         28.250        NaN     NaN      NaN
## B1:D1         -3.750        NaN     NaN      NaN
## C1:D1          6.250        NaN     NaN      NaN
## A1:B1:C1       7.500        NaN     NaN      NaN
## A1:B1:D1      16.500        NaN     NaN      NaN
## A1:C1:D1      -6.500        NaN     NaN      NaN
## B1:C1:D1     -10.500        NaN     NaN      NaN
## A1:B1:C1:D1       NA         NA      NA       NA
##
## Residual standard error: NaN on 0 degrees of freedom
## Multiple R-squared:      1,  Adjusted R-squared:    NaN
## F-statistic:   NaN on 15 and 0 DF,  p-value: NA
```

```r
#generates the normal probability plot of the effects
DanielPlot(model, half = TRUE, main = "")
```




```r
options(digits = 10)
anova(lm(Y ~ Blk + A + C + D +A*C + A*D, data = df))
```

```
## Analysis of Variance Table
##
## Response: Y
##           Df    Sum Sq    Mean Sq  F value     Pr(>F)
## Blk        1 1387.5625 1387.56250 66.58081 1.8895e-05 ***
## A          1 1870.5625 1870.56250 89.75708 5.5998e-06 ***
## C          1  390.0625  390.06250 18.71676 0.00191547 **
## D          1  855.5625  855.56250 41.05332 0.00012421 ***
## A:C        1 1314.0625 1314.06250 63.05398 2.3490e-05 ***
## A:D        1 1105.5625 1105.56250 53.04932 4.6461e-05 ***
## Residuals  9  187.5625   20.84028
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Partial confounding (replicated): an example

An experiment was conducted to develop a plasma etching process. There were three factors, A = gap, B = gas flow, and C = RF power, and the response variable is the etch rate. Suppose that only four treatment combinations can be tested during a shift, and because there could be shift-to-shift differences in etching tool performance, the experimenters decide to use shifts as a blocking factor. Thus, each replicate of the 2^3 design must be run in two blocks. Two replicates are run, with ABC confounded in replicate I and AB confounded in replicate II. The data are as follows:

Replicate I *ABC* Confounded		Replicate II *AB* Confounded	
(1) = 550	*a* = 669	(1) = 604	*a* = 650
ab = 642	*b* = 633	*c* = 1052	*b* = 601
ac = 749	*c* = 1037	*ab* = 635	*ac* = 868
bc = 1075	*abc* = 729	*abc* = 860	*bc* = 1063

```r
library(FrF2)
library(tidyverse)
design <- expand.grid(A = c(-1,1),
                      B = c(-1,1),
                      C = c(-1,1))
design1 <- data.frame(rbind(design,design))

rep <- c(rep(1,8),rep(2,8))
block <- c(1, 2, 2, 1, 2, 1, 1, 2,
           1, 2, 2, 1, 1, 2, 2, 1)
y <- c(550, 669, 633, 642, 1037, 749, 1075, 729,
       604, 650, 601, 635, 1052, 868, 1063, 860)

df <- cbind(design1,rep, block, y)
df
```

```
##     A  B  C rep block    y
## 1  -1 -1 -1   1     1  550
## 2   1 -1 -1   1     2  669
## 3  -1  1 -1   1     2  633
## 4   1  1 -1   1     1  642
## 5  -1 -1  1   1     2 1037
## 6   1 -1  1   1     1  749
## 7  -1  1  1   1     1 1075
## 8   1  1  1   1     2  729
```

```
## 9   -1 -1 -1    2      1  604
## 10   1 -1 -1    2      2  650
## 11 -1   1 -1    2      2  601
## 12   1   1 -1    2      1  635
## 13 -1 -1   1    2      1 1052
## 14   1 -1   1    2      2  868
## 15 -1   1   1    2      2 1063
## 16   1   1   1    2      1  860
```

```
df <- df %>%
  mutate(A = factor(A),
         B = factor(B),
         C = factor(C),
         Rep = factor(rep),
         Block = factor(block))

fit <- lm(y ~ Rep + Block:Rep + A*B*C, data=df)
options(digits = 10)
anova(fit)
```

```
## Analysis of Variance Table
##
## Response: y
##           Df    Sum Sq   Mean Sq   F value     Pr(>F)
## Rep        1   3875.06   3875.06   1.51906  0.2725512
## A          1  41310.56  41310.56  16.19411  0.0100789 *
## B          1    217.56    217.56   0.08529  0.7819866
## C          1 374850.06 374850.06 146.94456 6.7494e-05 ***
## Rep:Block  2    458.13    229.06   0.08979  0.9155599
## A:B        1   3528.00   3528.00   1.38301  0.2925288
## A:C        1  94402.56  94402.56  37.00664  0.0017355 **
## B:C        1     18.06     18.06   0.00708  0.9362050
## A:B:C      1      6.12      6.12   0.00240  0.9628160
## Residuals  5  12754.81   2550.96
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Confounding the 2^k Factorial Design in Four Blocks

It is possible to construct 2^k factorial designs confounded in four blocks of 2^{k-2} observations each. These designs are particularly useful in situations where the number of factors is moderately large, say $k \geq 4$, and block sizes are relatively small.

As an example, consider the 2^5 design. If each block will hold only eight runs, then four blocks must be used. The construction of this design is relatively straightforward. Select two effects to be confounded with blocks, say ADE and BCE. These effects have the two defining contrasts

$$L_1 = x_1 + x_4 + x_5$$
$$L_2 = x_2 + x_3 + x_5$$

Now every treatment combination will yield a particular pair of values of $L_1(mod2)$ and $L_2(mod2)$, that is, either $(L_1, L_2) = (0,0), (0,1), (1,0),$ or $(1,1)$. Treatment combinations yielding the same values of (L_1, L_2) are assigned to the same block. In this example the resulting design is given below

Block 1		Block 2		Block 3		Block 4	
$L_1 = 0$		$L_1 = 1$		$L_1 = 0$		$L_1 = 1$	
$L_2 = 0$		$L_2 = 0$		$L_2 = 1$		$L_2 = 1$	
(1)	*abe*	*a*	*be*	*b*	*abce*	*e*	*abcde*
ad	*ace*	*d*	*abde*	*abd*	*ae*	*ade*	*bd*
bc	*cde*	*abc*	*ce*	*c*	*bcde*	*bce*	*ac*
abcd	*bde*	*bcd*	*acde*	*acd*	*de*	*ab*	*cd*

Since there are four blocks with three degrees of freedom between them, and because ADE and BCE have only one degree of freedom each, clearly an additional effect with one degree of freedom must be confounded. This effect is the **generalized interaction** of ADE and BCE, which is defined as the product of ADE and BCE modulus 2. Thus, the generalized interaction $(ADE)(BCE) = ABCDE^2 = ABCD$ must also be confounded with blocks.

The group-theoretic properties of the principal block mentioned still hold. For example, we see that the product of two treatment combinations in the principal block yields another element of the principal block. To construct another block, select a treatment combination that is not in the principal block (e.g., b) and multiply b by all the treatment combinations in the principal block. This will produce the eight treatment combinations in block 3. In practice, the principal block can be obtained from the defining contrasts and the group-theoretic property, and the remaining blocks can be determined from these treatment combinations by this method.

The general procedure for constructing a 2^k design confounded in four blocks is to choose two effects to generate the blocks, automatically confounding a third effect that is the generalized interaction of the first two. Then, the design is constructed by using the two defining contrasts (L_1, L_2) and the group-theoretic properties of the principal block. In selecting effects to be confounded with blocks, care must be exercised to obtain a design that does not confound effects that may be of interest. For example, in a 2^5 design we might choose to confound $ABCDE$ and ABD, which automatically confounds CE, an effect that is probably of interest. A better choice is to confound ADE and BCE, which automatically confounds $ABCD$. It is preferable to sacrifice information on the three-factor interactions ADE and BCE instead of the two-factor interaction CE.

Confounding the 2^k Factorial Design in 2^p Blocks

The methods described above may be extended to the construction of a 2^k factorial design confounded in 2^p blocks ($p \leq k$), where each block contains exactly 2^{k-p} runs. We select p independent effects to be confounded, where by "independent" we mean that no effect chosen is the generalized interaction of the others. The blocks may be generated by use of the p defining contrasts $L_1, L_2, \cdots, L_p$ associated with these effects. In addition, exactly 2^p-p-1 other effects will be confounded with blocks, these being the generalized interactions of those p independent effects initially chosen. Care should be exercised in selecting effects to be confounded so that information on effects that may be of potential interest is not sacrificed.

The choice of the p effects used to generate the block is critical because the confounding structure of the design directly depends on them. The table below presents a list of useful designs.

Suggested Blocking Arrangements for the 2^k Factorial Design

Number of Factors, k	Number of Blocks, 2^p	Block Size, 2^{k-p}	Effects Chosen to Generate the Blocks	Interactions Confounded with Blocks
3	2	4	*ABC*	*ABC*
	4	2	*AB, AC*	*AB, AC, BC*
4	2	8	*ABCD*	*ABCD*
	4	4	*ABC, ACD*	*ABC, ACD, BD*
	8	2	*AB, BC, CD*	*AB, BC, CD, AC, BD, AD, ABCD*
5	2	16	*ABCDE*	*ABCDE*
	4	8	*ABC, CDE*	*ABC, CDE, ABDE*
	8	4	*ABE, BCE, CDE*	*ABE, BCE, CDE, AC, ABCD, BD, ADE*
	16	2	*AB, AC, CD, DE*	All two- and four-factor interactions (15 effects)
6	2	32	*ABCDEF*	*ABCDEF*
	4	16	*ABCF, CDEF*	*ABCF, CDEF, ABDE*
	8	8	*ABEF, ABCD, ACE*	*ABEF, ABCD, ACE, BCF, BDE, CDEF, ADF*
	16	4	*ABF, ACF, BDF, DEF*	*ABF, ACF, BDF, DEF, BC, ABCD, ABDE, AD, ACDE, CE, CDF, BCDEF, ABCEF, AEF, BE*
	32	2	*AB, BC, CD, DE, EF*	All two-, four-, and six-factor interactions (31 effects)
7	2	64	*ABCDEFG*	*ABCDEFG*
	4	32	*ABCFG, CDEFG*	*ABCFG, CDEFG, ABDE*
	8	16	*ABCD, CDEF, ADFG*	*ABC, DEF, AFG, ABCDEF, BCFG, ADEG, BCDEG*
	16	8	*ABCD, EFG, CDE, ADG*	*ABCD, EFG, CDE, ADG, ABCDEFG, ABE, BCG, CDFG, ADEF, ACEG, ABFG, BCEF, BDEG, ACF, BDF*
	32	4	*ABG, BCG, CDG, DEG, EFG*	*ABG, BCG, CDG, DEG, EFG, AC, BD, CE, DF, AE, BF, ABCD, ABDE, ABEF, BCDE, BCEF, CDEF, ABCDEFG, ADG, ACDEG, ACEFG, ABDFG, ABCEG, BEG, BDEFG, CFG, ADEF, ACDF, ABCF, AFG, BCDFG*
	64	2	*AB, BC, CD, DE, EF, FG*	All two-, four-, and six-factor interactions (63 effects)

To illustrate the use of this table, suppose we wish to construct a 2^6 design confounded in $2^3 = 8$ blocks of $2^3 = 8$ runs each. The table indicates that we would choose $ABEF$, $ABCD$, and ACE as the $p = 3$ independent effects to generate the blocks. The remaining $2^p - p - 1 = 2^3 - 3 - 1 = 4$ effects that are confounded are the generalized interactions of these three; that is,

$$\begin{aligned} ABEF \times ABCD &= A^2B^2CDEF = CDEF \\ ABEF \times ACE &= A^2BCE^2F = BCF \\ ABCD \times ACE &= A^2BC^2DE = BDE \\ ABEF \times ABCD \times ACE &= A^3B^2C^2DE^2F = ADF \end{aligned}$$