

# Lesson 2.2 Confounding in $2^k$ Factorial Experiment

## Introduction

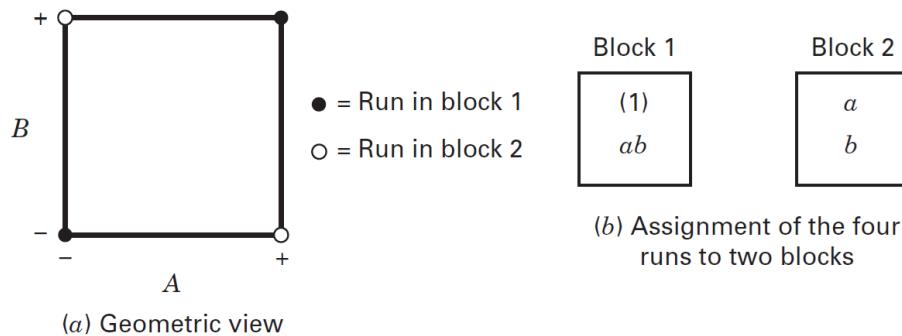
Recall that we use blocking strategy to remove or minimize the effect of a single nuisance factor. The design is referred to as the Randomized Complete Block Design, wherein we run a complete set of treatments in every block.

In many problems it is impossible to perform a complete replicate of a factorial design in one block. **Confounding** is a design technique for arranging a complete factorial experiment in blocks, where the block size is smaller than the number of treatment combinations in one replicate. The technique causes information about certain treatment effects (usually higher-order interactions) to be indistinguishable from, or confounded with, blocks.

## Confounding the $2^k$ factorial design in two blocks

Suppose that we wish to run a single replicate of the  $2^2 = 4$  treatment combinations. Each of the  $2^2 = 4$  treatment combinations requires a quantity of raw material, for example, and each batch of raw material is only large enough for two treatment combinations to be tested. Thus, two batches of raw material are required. If batches of raw material are considered as blocks, then we must assign two of the four treatment combinations to each block.

The figure below shows one possible design for this problem. The figure indicates that treatment combinations on opposing diagonals are assigned to different blocks. Notice that block 1 contains the treatment combinations  $(1)$  and  $ab$  and that block 2 contains  $a$  and  $b$ . Of course, the order in which the treatment combinations are run within a block is randomly determined. We would also randomly decide which block to run first.



From the previous lesson, we can estimate the main and interaction effects as follows.

$$\begin{aligned} A &= \frac{1}{2}\{ab + a - b - (1)\} \\ B &= \frac{1}{2}\{ab + b - a - (1)\} \\ AB &= \frac{1}{2}\{ab - a - b + (1)\} \end{aligned}$$

From the figure above, it is easy to see that the block effect is

$$Block = \frac{1}{2}\{ab + (1) - a - b\}$$

This means that the  $AB$  effect is identical to the  $Block$  effect. We also say that  $AB$  is **confounded** with blocks.

Below is the table of plus and minus signs for the  $2^2$  design.

Treatment	I	A	B	AB	Block
(1)	+	-	-	+	1
a	+	+	-	-	2
b	+	-	+	-	2
ab	+	+	+	+	1

From this table, we see that all treatment combinations that have a plus sign on  $AB$  are assigned to block 1, whereas all treatment combinations that have a minus sign on  $AB$  are assigned to block 2. This approach can be used to confound any effect ( $A$ ,  $B$ , or  $AB$ ) with blocks. For example, if (1) and b had been assigned to block 1 and a and ab to block 2, the main effect A would have been confounded with blocks. The usual practice is to confound the highest order interaction with blocks.

This scheme can be used to confound any  $2^k$  design in two blocks. As a second example, consider a  $2^3$  design run in two blocks. Suppose we wish to confound the three-factor interaction  $ABC$  with blocks. From the table of plus and minus signs shown below, we assign the treatment combinations that are minus on  $ABC$  to block 1 and those that are plus on  $ABC$  to block 2.

**Table of Plus and Minus Signs for the  $2^3$  Design**

Treatment Combination	Factorial Effect								Block
	<i>I</i>	<i>A</i>	<i>B</i>	<i>AB</i>	<i>C</i>	<i>AC</i>	<i>BC</i>	<i>ABC</i>	
(1)	+	-	-	+	-	+	+	-	1
<i>a</i>	+	+	-	-	-	-	+	+	2
<i>b</i>	+	-	+	-	-	+	-	+	2
<i>ab</i>	+	+	+	+	-	-	-	-	1
<i>c</i>	+	-	-	+	+	-	-	+	2
<i>ac</i>	+	+	-	-	+	+	-	-	1
<i>bc</i>	+	-	+	-	+	-	+	-	1
<i>abc</i>	+	+	+	+	+	+	+	+	2

## Other methods for constructing the blocks

There is another method for constructing these designs. The method uses the linear combination

$$L = a_1x_1 + a_2x_2 + \cdots + a_kx_k \equiv \text{defining contrast}$$

where  $x_i$  is the level of the  $i^{th}$  factor appearing in a particular treatment combination and  $\alpha_i$  is the exponent appearing on the  $i^{th}$  factor in the effect to be confounded. For the  $2^k$  system, we have  $i = 0\text{ or }1$  and  $x_i = 0$  (low level) or  $x_i = 1$  (high level). Treatment combinations that produce the same value of  $L(\text{mod}2)$  will be placed in the same block. Because the only possible values of  $L(\text{mod}2)$  are 0 and 1, this will assign the  $2^k$  treatment combinations to exactly two blocks.

To illustrate the approach, consider a  $2^3$  design with  $ABC$  confounded with blocks. Here  $x_1$  corresponds to A,  $x_2$  to B,  $x_3$  to C, and  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ . Thus, the defining contrast corresponding to  $ABC$  is

$$L = x_1 + x_2 + x_3$$

The treatment combination (1) is written 000 in the (0, 1) notation; therefore,

$$L = 1(0) + 1(0) + 1(0) = 0 = 0(\text{mod}2)$$

Similarly, the treatment combination *a* is 100, yielding

$$L = 1(1) + 1(0) + 1(0) = 1(\text{mod}2) = 1$$

Thus, (1) and *a* would be run in different blocks. For the remaining treatment combinations, we have

$$\begin{aligned}
b : L &= 1(0) + 1(1) + 1(0) = 1 \pmod{2} = 1 \\
ab : L &= 1(1) + 1(1) + 1(0) = 2 \pmod{2} = 0 \\
c : L &= 1(0) + 1(0) + 1(1) = 1 \pmod{2} = 1 \\
ac : L &= 1(1) + 1(0) + 1(1) = 2 \pmod{2} = 0 \\
bc : L &= 1(0) + 1(1) + 1(1) = 2 \pmod{2} = 0 \\
abc : L &= 1(1) + 1(1) + 1(1) = 3 \pmod{2} = 1
\end{aligned}$$

Thus, (1),  $ab$ ,  $ac$ , and  $bc$  are run in block 1 and  $a$ ,  $b$ ,  $c$ , and  $abc$  are run in block 2.

Another method may be used to construct these designs. The block containing the treatment combination (1) is called the **principal block**. The treatment combinations in this block have a useful group-theoretic property; namely, they form a group with respect to multiplication modulus 2. This implies that any element [except (1)] in the principal block may be generated by multiplying two other elements in the principal block modulus 2. For example, consider the principal block of the  $2^3$  design with  $ABC$  confounded.

Note that

$$\begin{aligned}
ab \times ac &= a^2bc = bc \\
ab \times bc &= ab^2c = ac \\
ac \times bc &= abc^2 = ab
\end{aligned}$$

Treatment combinations in the other block (or blocks) may be generated by multiplying one element in the new block by each element in the principal block modulus 2. For the  $2^3$  with  $ABC$  confounded, because the principal block is (1),  $ab$ ,  $ac$ , and  $bc$ , we know that  $b$  is in the other block. Thus, the elements of this second block are

$$\begin{aligned}
b \times (1) &= b \\
b \times ab &= ab^2 = a \\
b \times ac &= abc \\
b \times bc &= b^2c = c
\end{aligned}$$

## Estimation of error

When the number of variables is small, say  $k = 2$  or  $k = 3$ , it is usually necessary to replicate the experiment to obtain an estimate of error. For example, suppose that a  $2^3$  factorial must be run in two blocks with  $ABC$  confounded, and the experimenter decides to replicate the design four times. The resulting design is shown below. Note that  $ABC$  is confounded in each replicate.

Replicate I		Replicate II		Replicate III		Replicate IV	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
(1)	abc	(1)	abc	(1)	abc	(1)	abc
ac	a	ac	a	ac	a	ac	a
ab	b	ab	b	ab	b	ab	b
bc	c	bc	c	bc	c	bc	c

The analysis of variance for this design is shown below. There are 32 observations and 31 total degrees of freedom. Furthermore, because there are eight blocks, seven degrees of freedom must be associated with these blocks. The error sum of squares actually consists of the interactions between replicates and each of the effects ( $A$ ,  $B$ ,  $C$ ,  $AB$ ,  $AC$ ,  $BC$ ). It is usually safe to consider these interactions to be zero and to treat the resulting mean square as an estimate of error. Main effects and two-factor interactions are tested against the mean square error.

#### **Analysis of Variance for Four Replicates of a $2^3$ Design with ABC Confounded**

Source of Variation	Degrees of Freedom
Replicates	3
Blocks ( $ABC$ )	1
Error for $ABC$ (replicates $\times$ blocks)	3
$A$	1
$B$	1
$C$	1
$AB$	1
$AC$	1
$BC$	1
Error (or replicates $\times$ effects)	18
Total	31

### Complete vs Partial Confounding

If the replications are possible with confounding and blocking experiments, the confounding can be performed either completely or partially depending on the interest of the research questions or hypothesis. For an example, the  $ABC$  interaction is completely confounded with blocks in the figure below.

ABC Confounded	ABC Confounded	ABC Confounded	ABC Confounded
(1)	a	(1)	a
ab	b	ab	b
ac	c	ac	c
bc	abc	bc	abc
Replication I		Replication II	

In this situation, the three-way ABC interaction is not an interest of the experiment. In this design, no information can be retrieved for the ABC interaction. However, all the main effects and the second-order interaction can be obtained 100%.

However, if some information is useful for the ABC interaction, it could be partially confounded as shown in the figure below. In this situation, the ABC, AB, AC, and BC are confounded with blocks in the replication I, II, III, and IV, respectively. Therefore, 3/4th (75%) information can be retrieved for each of the interaction terms. For an example, the AB interaction effect can be obtained from replication I, III, and IV. This confounding process is known as **partial confounding**.

ABC Confounded	AB Confounded	AC Confounded	BC Confounded
(1)	a	(1)	a
ab	b	c	b
ac	c	ab	c
bc	abc	bc	ab
Replication I		Replication II	
Replication III		Replication IV	

The ANOVA table corresponding to the partially confounded design is given below.

Source of Variation	Degrees of Freedom
Replicates	3
Blocks within replicates [or ABC (rep. I) + AB (rep. II) + AC (rep. III) + BC (rep. IV)]	4
A	1
B	1
C	1
AB (from replicates I, III, and IV)	1
AC (from replicates I, II, and IV)	1
BC (from replicates I, II, and III)	1
ABC (from replicates II, III, and IV)	1
Error	17
Total	31

Nevertheless, three-way interaction ABC effect is rarely a practical interest. Therefore, complete confounding of higher-order interactions for the interest of the lower-order interactions would be preferable.

## Confounding the $2^k$ Factorial Design in Four Blocks

It is possible to construct  $2^k$  factorial designs confounded in four blocks of  $2^{k-2}$  observations each. These designs are particularly useful in situations where the number of factors is moderately large, say  $k \geq 4$ , and block sizes are relatively small.

As an example, consider the  $2^5$  design. If each block will hold only eight runs, then four blocks must be used. The construction of this design is relatively straightforward. Select two effects to be confounded with blocks, say ADE and BCE. These effects have the two defining contrasts

$$L_1 = x_1 + x_4 + x_5$$

$$L_2 = x_2 + x_3 + x_5$$

Now every treatment combination will yield a particular pair of values of  $L_1(mod2)$  and  $L_2(mod2)$ , that is, either  $(L_1, L_2) = (0, 0), (0, 1), (1, 0)$ , or  $(1, 1)$ . Treatment combinations yielding the same values of  $(L_1, L_2)$  are assigned to the same block. In this example the resulting design is given below

**Block 1**

$$L_1 = 0$$

$$L_2 = 0$$

(1)	abe
ad	ace
bc	cde
abcd	bde

**Block 2**

$$L_1 = 1$$

$$L_2 = 0$$

a	be
d	abde
abc	ce
bcd	acde

**Block 3**

$$L_1 = 0$$

$$L_2 = 1$$

b	abce
abd	ae
c	bcde
acd	de

**Block 4**

$$L_1 = 1$$

$$L_2 = 1$$

e	abcde
ade	bd
bce	ac
ab	cd