# Stat 113 (Intro. to Mathematical Statistics)

**Lesson 1.2: Counting Methods**

## Learning Outcomes

1. **Explain** the fundamental principle of counting and relate it to permutation,
2. **Explain** the difference between permutation and combination, and
3. **Apply** permutation and combination in assigning probabilities to events

## Introduction

In the previous lesson, we learned that the classical approach to assigning probability to an event involves determining the number of elements in the event and the sample space. There are many situations in which it would be too difficult and/or too tedious to list all of the possible outcomes in a sample space. In this lesson, we will learn various ways of counting the number of elements in a sample space without actually having to identify or list down the specific outcomes. The specific counting techniques we will explore include the *multiplication rule*, *permutations*, and *combinations*.

## Fundamental Principle of Counting

**Definition**:

If a certain experiment can be performed in m ways and, corresponding to each of these ways, another experiment can be performed in n ways, then the combined experiment can be performed in mn ways. We call this process as the **fundamental principle of counting** or the **multiplication rule of counting**.

To understand this principle, suppose the outcomes of an experiment are written as $\mathbf{A} = \{a_1, a_2, \cdots, a_m\}$ and those of the second experiment as $\mathbf{B} = \{b_1, b_2, \cdots, b_n\}$ . Then the outcomes of the combined experiment can be represented in a rectangular array as ordered pairs $(a_i, b_j)$.

In other words, the outcomes of the combined experiment can be represented as the Cartesian product $A \times B$. Clearly, $n(A \times B) = n(A) \times n(B) = n \times m$.

Example 1.2.1

Suppose we roll a die and toss a coin. The die has $m = 6$ outcomes. The coin has $n = 2$ outcomes. Hence, there are $mn = 12$ paired outcomes. The 12 paired outcomes are: $\{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)\}$.

Example 1.2.2

If a person has 8 different shirts, 6 different ties, and 5 different jackets, then he can get dressed for an occasion in $8 \times 6 \times 5 = 240$ ways.

Example 1.2.3

Suppose license plates are formed with three distinct letters followed by three distinct digits. Then there 26 choices for the first letter, 25 for the second, and 24 for the third. Also, there are 10 choices for the first digit, 9 for the second, and 8 for the third. Therefore, there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$ different license plates that can be made.

## Factorial Notation

Before going any further, let me introduce the factorial notation ($n!$).

**Definition**

The factorial of a positive integer $n$, denoted by $n!$, is the product of all integers less than or equal to $n$. That is, $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$

Example 1.2.4

The factorial of 4 is $4! = 4 \times 3 \times 2 \times 1 = 24$ and the factorial of 6 is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

*Remark*: It is convenient to define $0! = 1$.

## Permutation

Suppose we have a collection of objects say $\{a, b, c\}$ and we pick two objects. The different pairs of objects are: $ab, ba, ac, ca, bc, cb$.

The order in which the letters are written is important. For example, $ab$ is a different arrangement from $ba$. Any particular arrangement is called a **permutation**. In the above list there are 6 permutations in all. The reason for this is simple and is based on the fundamental principle of counting. Since we are going to select 2 objects (letters) from 3 available objects (letters), we have 3 ways or choices for the first letter and 2 ways or choices for the second letter. Therefore, there are $3 \times 2 = 6$ permutations or arrangements.

Now let us turn to the general case via the following definition.

**Definition**

A **permutation** is an arrangement of distinct objects in a particular order. Order is important. The number of permutations of $r$ objects selected from a collection of $n$ distinct objects is

$$P(n, r) = n \times (n-1) \times (n-2) \times \cdots \times (n-(r-1)), \;\; r \leq n$$

*Remarks*

a. Another notation used to refer to the number of permutations of $r$ objects from a collection of $n$ distinct objects is ${}_nP_r$. In other words. $P(n,r) =_n P_r$.

b. Using the factorial notation, we have

$$_nP_r = \frac{n!}{(n-r)!}$$

3. Note that $r \leq n$ and if $r = n$, then ${}_nP_r = n!$.

Example 1.2.5

An artist has 9 paintings. How many ways can he hang 4 paintings side-by-side on a gallery wall?

Answer:

We can arrive at the answer in two ways:

1. Using the Fundamental Principle of Counting: $9 \times 8 \times 7 \times 6 = 3,024$ adifferent arragements.

2. We can get the same answer using the permutations formula:

$$_nP_r = \frac{n!}{(n-r)!} = \frac{9!}{(9-4)!} = 3,024$$

Example 1.2.6

In how many ways can a president, a treasurer and a secretary be chosen from among 7 candidates?

Answer:

a. using FPC: $7 \times 6 \times 5 = 210$.

b. Using the formula of permutation:

$$_nP_r = \frac{n!}{(n-r)!} = \frac{7!}{(7-3)!} = 210$$

In the definition of permutation, we emphasized that the $n$ objects are distinct or unique. There are instances where this assumption is not true. For example, we may be interested on the number of rearrangements of the letter in the word STATISTICS. The letters are no longer unique- there are 3 S's, 3 T's, 2 I's, 1 A, and 1 C. How do we determine the number of different rearrangements of these letters?

Another permutation formula is required to solve this type of problem and it is given below.

**Definition**

The number of ways to arrange $n$ objects, $n_1$ being of one kind, $n_2$ of a second kind, . . . , and $n_k$ of the $k_{th}$ kind, is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}, \text{ where } n = n_1 + n_2 + \cdots + n_k$$

Example 1.2.7

The number of permutations or (re)arrangements of the letters of the word STATISTICS is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!} = \frac{10!}{3! \times 3! \times 2! \times 1! \times 1!} = \frac{3,628,800}{72} = 50,400$$

## Combination

Maria has three tickets for a concert. She'd like to use one of the tickets herself. She could then offer the other two tickets to any of four friends (Ann, Beth, Chris, Dave). How many ways can 2 people be selected from 4 to go to a concert?

Let us denote Maria's friends as A for Ann, B for Beth, C for Chris, and D for Dave. If order matters, then these are the pairs of people: (A,B), (B,A), (A,C), (C,A), (A,D), (D,A), (B,C), (C,B), (B,D), (D,B), (C,D), (D,C).

Obviously, order doesn't matter here, so (A, B) is the same pair of people as (B, A), (A, C) is the same pair as (C, A), and so on. Hence, instead of 12 there must only be 6 possible pairs.

The arrangement of objects without regard to order is called **combination**.

**Definition**

Suppose from a collection of $n$ distinct objects we choose $r$ of them ($rleqn$) without regard to the order in which the objects are chosen. The number of ways to do this is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

*Remarks*

1. The symbol $C(n, r)$ is read as "the combination of $n$ things taken $r$ at a time." Sometimes, we use the notation ${}_nC_r$ or $\binom{n}{k}$ in place of ${}_nC_r$.
2. Combination is most often used in problems which involves creating groups or committees of people from a given set of people and the order of selection is not important.
3. The number of combinations of $n$ objects taken $r$ at a time is less than or equal to the number of permutations of $n$ objects taken $r$ at a time since ${}_nC_r = \frac{{}_nP_r}{r!}$.
4. You may apply more than one method in solving a counting problem. Say combination with fundamental principle of counting or permutation with fundamental principle of counting.

Example 1.2.8

How many different ways can the admissions committee of a statistics department choose four foreign graduate students from 20 foreign applicants and three Filipino students from 10 Filipino applicants?

Answer:

The 4 foreign students can be selected from the 20 foreign students in $\binom{20}{4} = 4845$ ways and the 3 Filipino students can be selected from 10 Filipino students in $\binom{10}{3} = 120$ ways. Therefore, according to the Fundamental Principle of Counting, the whole selection of 4 foreign students and 3 Filipino students can be done in $4845 \times 120 = 581400$ ways.

Let us now use these counting rules in assigning probabilities to events.

<u>Example 1.2.9</u>

In a tank containing 10 fishes, there are three yellow and seven black fishes. We select three fishes at random.

a. What is the probability that exactly one yellow fish gets selected?

b. What is the probability that at most one yellow fish gets selected?

c. What is the probability that at least one yellow fish gets selected?

<u>Answer</u>

There are $n(S) = \binom{10}{3} = 120$ ways to select three fishes from 10.

a. Let A be the event that exactly one yellow fish gets selected. If one yellow fish is to be selected, then the other two must be black fishes to complete the 3 sample fishes. Now, there are $\binom{3}{1} = 3$ ways of selecting a yellow fish from 3 yellow fishes and $\binom{7}{2} = 21$ ways of selecting 2 black fishes. Thus, by the fundamental principle of counting, the number of ways of selecting a yellow fish and 2 black fishes is $n(A) = 3 \times 21 = 63$ ways. Therefore, the probability that exactly one yellow fish gets selected is

$$P(A) = \frac{n(A)}{n(S)} = \frac{63}{120} = 0.525.$$

b. Let B be the event that at most one yellow. When we say at most one, it means 0 or 1. If 0 yellow fish is selected that means all 3 fishes are black and if 1 fish is yellow then other 2 must be black. In (a) above, the number of ways of selecting 1 yellow and 2 black fishes is $\binom{3}{1} \times \binom{7}{2} = 63$ . The number of ways of selecting no yellow fish (all 3 black fishes is selected) is $\binom{3}{0} \times \binom{7}{3} = 35$. Therefore, the total number of ways of selecting 0 or 1 yellow fish is $n(B) = 63 + 35 = 98$ and the probability that at most one yellow fish gets selected is

$$P(B) = \frac{n(B)}{n(S)} = \frac{98}{120} \approx 0.817.$$

*Remark*: Recall that the disjunction "or" implies addition, that is why we added 63 and 35.

## Learning Tasks

Answer the following as indicated.

1. If repetitions are not allowed, how many three-digit numbers can be formed from the digits 2, 3, 5, 6, 7, 9?
2. How many permutations of the digits in the number 1234567 will result in an even number?
3. From a group of 5 swimmers and 8 runners, an athletic contingent of 7 is to be formed. How many teams are possible if there are to be

   a. 2 swimmers and 5 runners

   b. at least 3 swimmers

4. How many permutations are possible from the letters in the word PHILIPPINES?
5. A package of 15 apples contains two defective apples. Four apples are selected at random.

   a. Find the probability that none of the selected apples is defective.

   b. Find the probability that at least one of the selected apples is defective.