Stat 113 (Introduction to Mathematical Statistics)

Lesson 2.3: The Binomial Distribution

Learning Outcomes

At the end of the lesson, students must be able to

1. Describe the necessary conditions where a random variable can be modeled using a binomial distribution,
2. Derive the probability mass function of a binomial random variable,
3. Compute probabilities associated with a binomial random variable, and
4. Compute the mean and variance of a binomial random variable.

Introduction

Some experiments consist of the observation of a sequence of identical and independent trials, each of which can result in one of two outcomes. For example, an education graduate may *pass* or *fail* in the teacher's board examination, an item leaving a production line could be *defective* or *not defective*, the sex of a newborn can either be a *boy* or *girl*, and each of a random sample of 35 persons interviewed can be in *favor* or *not (favor)* to a new law.

In general, many experiments consist of a sequence of "trials," where:

1. each trial results in either a "success" or a "failure"
2. the probability of "success," denoted by $p, 0 \leq p \leq 1$, is the same on every trial, and
3. the trials are mutually independent.

Trials that obey these three properties are called **Bernoulli** trials.

In a sequence of Bernoulli trials, one is more interested in the total number of successes. The probability of observing exactly k successes in n independent Bernoulli trials yields the binomial probability distribution. In practice, the binomial probability distribution is used when, for example in a survey of 56 students, we may be interested in the number of them who has the propensity to make online transactions.

Definition:

A *binomial experiment* is one that has the following properties:

1. The experiment consists of n identical (Bernoulli) trials.
2. Each trial results in one of the two outcomes, called a success S and failure F.
3. The probability of success on a single trial is equal to p which remains the same from trial to trial. The probability of failure is $1 - p = q$.
4. The outcomes of the trials are independent.
5. The random variable Y is the number of successes in n trials.

Definition:

A random variable Y is said to have binomial probability distribution with parameters (n, p), written as $b(n, p)$, if and only if

$$\begin{aligned} P(Y = y) &=_n C_r \times p^y \times (1-p)^{n-y} \\ &= \binom{n}{y} p^y q^{n-y} \end{aligned}$$

Example 1:

Upon analyzing the cash register receipts of a large department store over an extended period of time, it is found that 30 percent of the customers pay for their purchases by credit card, 50 percent pay by cash, and 20 percent pay by check. Of the next five customers that make purchases at the store, what is the probability that

a. three of them will pay by credit card?
b. less than 2 will pay by cash?
c. at least 4 will pay by check?

SOLUTION:

a. Let X be the number of customers who pay by credit card. We have $n = 5,\ p = 0.30,$ and $y = 3$. Hence,

$$\begin{aligned} P(X = 3) &= \binom{5}{3} (0.3)^3 (1 - 0.3)^{5-3} \\ &= 0.1323 \end{aligned}$$

If you have access to the **R** programming language you can generate the answer using the code: **dbinom(3, 5, 0.3)**.

b. Let Y be the number of customers who pay by cash. This time we have $n = 5,\ p = 0.50,$ and $y = 0$ or 1. Thus,

$$
\begin{aligned}
P(Y < 2) &= P(Y = 0) + P(Y = 1) \\
&= \binom{5}{0}(0.5)^0(1 - 0.5)^{5-0} + \binom{5}{1}(0.5)^1(1 - 0.5)^{5-1} \\
&= 0.1875
\end{aligned}
$$

In R you can use either **dbinom(0,5,0.5) + dbinom(1,5,0.5)** or **pbinom(1, 5, 0.5, lower.tail=T)** to get the answer.

c. Let Z be the number of customers who pay by check. In this case $n = 5$, $p = 0.20$, and $y = 4$ or 5. Thus,

$$
\begin{aligned}
P(Z \geq 4) &= P(Y = 4) + P(Y = 5) \\
&= \binom{5}{4}(0.2)^4(1 - 0.2)^{5-4} + \binom{5}{5}(0.2)^5(1 - 0.2)^{5-5} \\
&= 0.00672
\end{aligned}
$$

In R you can use either **dbinom(4,5,0.2) + dbinom(5,5,0.2)**, **pbinom(3, 5, 0.2, lower.tail=F)**, or **1 - pbinom(3,5,0.2)** to get the answer.

Theorem:

If $Y \sim b(n, p)$, then its moment generating function is given by

$$m_Y(t) = (q + pe^t)^n, \text{ where } q = 1 - p$$

Theorem:

If $Y \sim b(n, p)$, then its mean and variance are, respectively, given by

$$
\begin{aligned}
E(Y) &= np \\
V(Y) &= npq
\end{aligned}
$$

<u>Example 2</u>:

Binge drinking is defined as having five or more drinks for male students and four or more drinks for female students at one drinking occasion. Suppose it was found out that the past two weeks approximately 40% of students are engaged in binge drinking. A random sample of 12 students were taken and interviewed.

1. What is the probability that

a. exactly seven binge drink?
b. at least 10 binge drink?
c. at most 4 binge drink?

2. Calculate the mean and standard deviation of the number of students who binge drink.

Example 3:

The probability that a patient recovers from a stomach disease is 0.8. Suppose 20 people are known to have contracted this disease. What is the probability that

a. exactly 14 recover?
b. at least 10 recover?
c. at least 14 but not more than 18 recover?
d. at most 16 recover?