Stat 113 (Introduction to Mathematical Statistics)

Lesson 1.5: Law of Total Probability and the Bayes' Theorem

2025-09-06

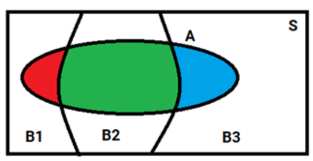
Learning Outcomes

At the end of the lesson, students must be able to

1. Determine the probability of an event by using a partition of the sample space S, and
2. Apply the Bayes' Theorem to find the conditional probability of an event when the "reverse" conditional probability is known.

Law of Total Probability

Consider a partition of the sample space S, say B_1, B_2, and B_3 (refer to the diagram below).



Then, for any set A in S, we have $A \cap B_1$ (red), $A \cap B_2$ (green), and $A \cap B_3$ (blue) also forms a partition of A. That is, $A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$. Therefore, by Axiom 3, we have

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

which by the Multiplication Rule becomes

$$P(A) = P(A|B_1) \times P(B_1) + P(A|B_2) \times P(B_2) + P(A|B_3) \times P(B_3).$$

This is called the **Law of Total Probability**.

In general, if $B_1, B_2, \cdots, B_k$ form a partition of S, then for any set A in S,

$$P(A) = \sum_{i=1}^{k} P(A|B_i) \times P(B_i)$$

Example 1.5.1:

Suppose there are three bags each containing 100 marbles: Bag 1 has 75 red and 25 blue marbles; Bag 2 has 60 red and 40 blue marbles; and Bag 3 has 45 red and 55 blue marbles. I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

SOLUTION:

There are two activities involved in here. The first one is selecting a bag at random and the second is selecting a red marble from the chosen bag.

Let B_1 be the event that the first bag is selected, B_2 be the event that the second bag is selected, and B_3 be the event that third bag is selected. Further, let R be the event that the selected marble is red. Assuming the three bags are equally likely to be selected, then

$$P(B_i) = \frac{1}{3}.$$

Now, the chance of selecting a red marble depends which bag is selected. For example, if the first bag is selected then $P(R|B_1) = \frac{75}{100}$. The probabilities of getting a red marble are $P(R|B_2) = \frac{60}{100}$ and $P(R|B_3) = \frac{45}{100}$, respectively for the second and third bags. Therefore, using the Law of Total Probability, the probability that a red marble is selected is

$$\begin{aligned} P(R) &= P(R|B_1) \times P(B_1) + P(R|B_2) \times P(B_2) + P(R|B_2) \times P(B_2) \\ &= \frac{75}{100} \times \frac{1}{3} + \frac{60}{100} \times \frac{1}{3} + \frac{45}{100} \times \frac{1}{3} \\ &= \frac{60}{100} \end{aligned}$$

Bayes Theorem

Now we are ready to state one of the most useful results in conditional probability: Bayes' rule. Suppose that we know $P(A|B)$, but we are interested in the probability $P(B|A)$. Using the definition of conditional probability, we have

$$P(A|B) \times P(B) = P(B|A) \times P(A)$$

which if we divide by $P(A)$ becomes

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A)}.$$

which is the famous **Bayes' Rule**. In this formulation, we call $P(B|A)$ as the *posterior* probability and $P(B)$ as the *prior* probability.

Often, in order to find $P(A)$ in Bayes' formula we need to use the law of total probability. We state formally the Bayes' theorem as follows.

Bayes Theorem

Suppose A is an event in S and suppose that $B_1, B_2, \cdots, B_k$ forms a partition of S. Then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\displaystyle\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

Example 1.5.2:

Suppose in *Example 1.5.1* we observed that the chosen marble is red. What is the probability that Bag 1 was chosen?

SOLUTION:

Using the same notation, we are interested to compute $P(B_1|R)$. It is shown below.

$$\begin{aligned} P(B_1|R) &= \frac{P(R|B_1)P(B_1)}{\sum_{i=1}^{3} P(R|B_i)P(B_i)} \\ &= \frac{\frac{75}{100} \times \frac{1}{3}}{\frac{75}{100} \times \frac{1}{3} + \frac{60}{100} \times \frac{1}{3} + \frac{45}{100} \times \frac{1}{3}} \\ &= \frac{5}{12} \end{aligned}$$

Example 1.5.3:

In screening for a certain disease, the probability that a healthy person wrongly gets a positive result is 0.05. The probability that a diseased person wrongly gets a negative result is 0.002. The overall rate of the disease in the population being screened is 1%. If Axel's test gives a positive result, what is the probability he actually has the disease?

SOLUTION:

Define the following events:

- $D = \{\text{a person has the disease}\}$;
- $T = \{\text{the test returns a positive result}\}$;

We are given these information:

- [*false positive*] $P(T|D^c) = 0.05 \implies$
- [*false negative*] $P(T^c|D) = 0.002 \implies P(T|D) = 1 - 0.002 = 0.998$
- [*disease incidence*] $P(D) = 0.01 \implies P(D^c) = 1 - 0.01 = 0.99$

We wanted to compute the probability that Axel actually has the disease given he gets a positive test result. The calculations are shown below

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} \\ &= \frac{0.998(0.01)}{0.998(0.01) + 0.05(0.99)} \\ &\approx 0.168 \end{aligned}$$

Learning Tasks

Answer the following as indicated.

1. A laboratory test is 95% effective at detecting a disease when it is present. It is 99% effective at declaring a subject negative when the subject is truly negative for the disease. Suppose 8% of the population has the disease.

a. What is the probability a randomly selected subject will test positively?

b. What is the probability a subject has the disease if his test is positive?

2. Suppose a statistics class contains 70% male and 30% female students. It is known that in a test, 5% of males and 10% of females got an "A" grade. If one student from this class is randomly selected and observed to have an "A" grade, what is the probability that this is a male student?

3. Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving on private planes and 90% of those arriving on other commercially owned planes are traveling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that the person

   a. is traveling on business?

   b. arrived on a privately owned plane, given that the person is traveling for business reasons?

   c. is traveling on business, given that the person is flying on a commercially owned plane?