# Stat 121 (Mathematical Statistics I)

## Lesson 4.4 Independent Random Variables

## Learning Outcomes

At the end of the lesson, the students must be able to show that two (or more) random variables are independent.

## Discrete Case

### Definition

Suppose $Y_1$ and $Y_2$ are discrete random variables with joint PMF $p_{(Y_1,Y_2)}(y_1, y_2)$ and marginal PMFs $p_{Y_1}(y_1)$ and $p_{Y_2}(y_2)$, respectively. We say that $Y_1$ and $Y_2$ are **independent** if

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

for all $(y_1, y_2) \in \mathbb{R}^2$. In other words, the joint PMF factors into the product of the marginal PMFs.

Example 4.4.1

Consider the random variables $Y_1$ and $Y_2$ with joint PMF in *Example 4.1.1.* Are the random variables independent?

*SOLUTION*

Recall from *Example 4.1.1* that the joint PMF of $Y_1$ and $Y_2$ is given by

| $p(y_1, y_2)$ | $y_2 = 0$ | $y_2 = 1$ | $y_2 = 2$ |
|---|---|---|---|
| $y_1 = 0$ | 0.64 | 0.08 | 0.04 |
| $y_1 = 1$ | 0.12 | 0.06 | 0.02 |
| $y_1 = 2$ | 0.02 | 0.01 | 0.01 |

While, from *Example 4.2.1* the marginal PMFs of $Y_1$ and $Y_2$, respectively, are given by

| $Y_1$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(Y_1 = y_1)$ | 0.76 | 0.20 | 0.04 |

and

| $Y_2$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(Y_2 = y_2)$ | 0.78 | 0.15 | 0.07 |

For $Y_1$ and $Y_2$ to be independent we would need

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

to hold for all $(y_1, y_2)$ in the support

$$R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

However, this condition does not even hold for the first value pair (0,0). Note that

- $p_{(Y_1,Y_2)}(0,0) = 0.64$ (from the joint PMF of $Y_1$ and $Y_2$
- $p_{Y_1}(0) = 0.76$ (from the marginal PMF of $Y_1$)
- $p_{Y_2}(0) = 0.78$ (from the marginal PMF of $Y_2$)

Thus, $p_{(Y_1,Y_2)}(0,0) \neq p_{Y_1}(0) \times p_{Y_2}(0)$ $[0.64 \neq 0.76 \times 0.78]$

Example 4.4.2

Let $\mathbf{Y} = (Y_1, Y_2)$ be a random vector with joint PMF

| $p_{(Y_1,Y_2)}(y_1, y_2)$ | $y_2 = 1$ | $y_2 = 2$ | $p_{Y_1}(y_1)$ |
|---|---|---|---|
| $y_1 = 1$ | $1/30$ | $4/30$ | $5/30$ |
| $y_1 = 2$ | $2/30$ | $8/30$ | $10/30$ |
| $y_1 = 3$ | $3/30$ | $12/30$ | $15/30$ |
| $p_{Y_2}(y_2)$ | $6/30$ | $24/30$ | |

Are $Y_1$ and $Y_2$ independent?

*SOLUTION*

It is easy to verify that

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

to hold for all $(y_1, y_2)$ in the support

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$$

For example,

$$p_{(Y_1,Y_2)}(1,1) = 1/30 = p_{Y_1}(1) \times p_{Y_2}(1) = 5/30 \times 6/30$$

It can be VERIFIED that the joint PMF can be expressed as the product of the marginal PMFs for the rest of the pairs in the support!

Therefore, $Y_1$ and $Y_2$ are independent.

## The continuous case

### Definition

Suppose $Y_1$ and $Y_2$ are discrete random variables with joint PDF $f_{(Y_1.Y_2)}(y_1, y_2)$ and marginal PDFs $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, respectively. We say that $Y_1$ and $Y_2$ are **independent** if

$$f_{(Y_1,Y_2)}(y_1, y_2) = f_{Y_1}(y_1) \times f_{Y_2}(y_2)$$

for all $(y_1, y_2) \in \mathbb{R}^2$. In other words, the joint PDF factors into the product of the marginal PDFs.

Example 4.4.3

This past year was an encouraging year for biodiversity discovery ,as scientists identified thousands of new species of life. For one newly discovered species, geneticists model

$$\begin{aligned} Y_1 &= \text{the percentage of the species possessing Trait 1} \\ Y_2 &= \text{the percentage of the species possessing Trait 2} \end{aligned}$$

using the joint pdf

$$f_{Y-1,Y_2}(y_1, y_2) = \begin{cases} 12y_1^3(1-y_2)^2, \ 0 < y_1 < 1, \ 0 < y_2 < 1 \\ 0, \ \text{otherwise} \end{cases}$$

Show that $Y_1$ and $Y_2$ are independent.

*SOLUTION*: Left as a classroom exercise!

### Remarks

1. Suppose $Y_1$ and $Y_2$ are random variables (discrete or continuous) with joint CDF $F_{Y_1,Y_2}(y_1, y_2)$. Then $Y_1$ and $Y_2$ are independent if and only if

$$F_{(Y_1,Y_2)}(y_1, y_2) = F_{Y_1}(y_1) \times F_{Y_2}(y_2)$$

2. If $Y_1$ and $Y_2$ are independent then

$$f_{(Y_2|Y_1)}(y_2|y_1) = f_{Y_2}(y_2)$$

and

$$f_{(Y_1|Y_2)}(y_1|y_2) = f_{Y_1}(y_1)$$

The discrete conclusion is analogous; simply replace PDFs with PMFs.

3. Suppose $Y_1$ and $Y_2$ are independent random variables (discrete or continuous). The random variables $U_1 = g(Y_1)$ and $U_2 = h(Y_2)$ are also independent. In other words, functions of independent random variables are independent.

## Learning Tasks:

Answer the following as indicated.

1. Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_2}, & 0 < y_1 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Are $Y_1$ and $Y_2$ independent?

2. Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{384} y_1^2 y_2^4 e^{y_1 - y_2/2}, & 0 < y_1 < \infty, 0 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Show that $Y_1$ and $Y_2$ are independent.