Stat 121 (Mathematical Statistics I)

Lesson 3.3 The Continuous Uniform Distribution

Learning Outcomes

At the end of the lesson, students must be able to

1. Describe the key properties of a continuous uniform random variable, such as the mean, variance, and moment generating function, and
2. Compute probabilities associated with random variables having a uniform distribution.

Introduction

The uniform distribution is very important for theoretical reasons. Simulation studies are valuable techniques for validating models in statistics. If we desire a set of observations on a random variable Y with distribution function $F_Y(y)$, we often can obtain the desired results by transforming a set of observations on a uniform random variable. For this reason, most computer systems contain a random number generator that generates observed values for a random variable that has a continuous uniform distribution.

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur.

Definition:

A random variable Y is said to have a uniform distribution from real numbers θ_1 to θ_2 if its PDF is given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, \ \theta_1 < y < \theta_2 \\ 0, \ \text{elsewhere} \end{cases}$$

We use the shorthand notation $Y \sim U(\theta_1, \theta_2)$ to denote that Y has a uniform distribution in the interval (θ_1, θ_2).

Remarks:

1. The constants $\theta_1, \theta_2 \in \mathbb{R}$ that determine the specific form of a uniform density function are called parameters of the density function.

2. An important uniform distribution is that for which $\theta_1 = 0$ and $\theta_2 = 1$, namely $U(0, 1)$. It is used in the generation of random numbers and in computer simulations.

3. The cumulative distribution function (CDF) of $Y \sim U(\theta_1, \theta_2)$ is given by

$$F_Y(y) = \begin{cases} 0, \ y < \theta_1 \\ \frac{y-\theta_1}{\theta_2-\theta_1}, \ \theta_1 \leq y < \theta_2 \\ 1, \ y \geq \theta_2 \end{cases}$$

Theorem:

If $Y \sim U(\theta_1, \theta_2)$, then its mean, variance, and moment generating function are, respectively, given as

$$E(Y) = \frac{\theta_1 + \theta_2}{2}$$
$$V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$$
$$m_Y(t) = \begin{cases} \frac{e^{\theta_2 t} - e^{\theta_1 t}}{t(\theta_2 - \theta_1)}, \ t \neq 0 \\ 1, \ t = 0 \end{cases}$$

PROOF: Left as a classroom exercise.

Example 3.3.1:

A continuous random variable Y is uniformly distributed over the interval $(-1, 2)$, Determine the following:

a. $f_Y(y)$

b. $F_Y(y)$

c. $P(0 < Y < 2)$

SOLUTION

a. If $Y \sim U(-1, 2)$, then its PDF is given

$$f_Y(y) = \begin{cases} \frac{1}{2-(-1)} = \frac{1}{3}, \ -1 < y < 2 \\ 0, \ \text{elsewhere} \end{cases}$$

b. The CDF of $Y \sim U(-1, 2)$ is

$$F_Y(y) = \begin{cases} 0, \ y < -1 \\ \frac{y-(-1)}{2-(-1)} = \frac{y+1}{3}, \ -1 \leq y < 2 \\ 1, \ y \geq 2 \end{cases}$$

c. The $P(0 < Y < 2)$ can be easily obtained using the CDF as follows

$$\begin{aligned} P(0 < Y < 2) &= F_Y(2) - F_Y(0) \\ &= \frac{1+2}{3} - \frac{0+1}{3} \\ &= \frac{2}{3} \end{aligned}$$

This probability is very easy to understand by just thinking of dividing the interval from -1 to 2 into 1-unit sub-intervals. There are 3 such 1-unit sub-intervals. Thus, the interval $0 < Y < 2$ consists of 2 1-unit sub-intervals. Hence, $P(0 < Y < 2) = \frac{2}{3}$.

Example 3.3.2:

The amount of time, in minutes, that a person must wait before he/she can be served by a teller in a bank is uniformly distributed between zero and 15 minutes, inclusive.

a. What is the probability that a person waits fewer than 12.5 minutes?

b. On the average, how long must a person wait?

SOLUTION

a. Let Y be the waiting time (in minutes). It is given that $Y \sim U(0, 15)$. Then

$$f_Y(y) = \begin{cases} \frac{1}{15}, \ 0 \leq Y \leq 15 \\ 0, \ \text{elsewhere} \end{cases}$$

Consequently,

$$\begin{aligned} P(Y < 12.5) &= \int_0^{12.5} f_Y(y) \ dy \\ &= \int_0^{12.5} \frac{1}{15} \ dy \\ &= \frac{1}{15} y \Big|_0^{12.5} \\ &= \frac{12.5 - 0}{15} \\ &\approx 0.83 \end{aligned}$$

This means that there is a 83% chance that the client will have to wait for at most 12.5 minutes before he/she is being served by a teller.

b. On average, the client will have to wait

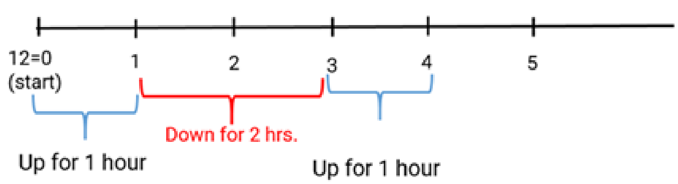
$$\begin{aligned} E(Y) &= \frac{a+b}{2} \\ &= \frac{0+15}{2} \\ &= 7.5 \text{ minutes} \end{aligned}$$

Example 3.3.3:

Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 A.M. What is the probability that the center is up when the person's call comes in?

SOLUTION

Let Y = number of hours the call is made. Looking at the diagram below, it is easy to see that $Y \sim U(0, 5)$.



Thus,

$$f_Y(y) = \begin{cases} \frac{1}{5}, & 0 < y < 5 \\ 0, & \text{elsewhere} \end{cases}$$

and the probability that the center is up when the call comes in is

$$P(0 < Y < 1) + P(3 < Y < 4) = \int_0^1 \frac{1}{5}\, dy + \int_3^4 \frac{1}{5}\, dy = 0.4.$$

Learning Tasks

Instruction: Answer the following as indicated.

1. The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

   a) What is the probability that a person waits fewer than 12.5 minutes?

   b) On the average, how long must a person wait?

   c) Ninety percent of the time, the time a person must wait falls below what value?

2. The cycle time for trucks hauling concrete to a highway construction site is uniformly distributed over the interval 50 to 70 minutes. What is the probability that the cycle time exceeds 65 minutes if it is known that the cycle time exceeds 55 minutes?