Stat 121 (Mathematical Statistics I)

Lesson 2.7: The Poisson Distribution

Learning Outcomes

At the end of the lesson, students must be able to

1. Describe the properties of a Poisson process,
2. Derive the probability distribution of a random variable having a Poisson distribution,
3. Compute probabilities associated with a random variable with a Poisson distribution, and
4. Compute the mean and variance of a random variable with a Poisson distribution.

Introduction

In this lesson, we shall study the distribution of a random variable which counts the number of times an event occurs in an interval of time (or space) such as the number of customers at an ATM in 10-minute intervals, the number of typing errors on a page, or the number of earthquakes per month.

Suppose we count the number of "occurrences" in a continuous interval of time (or space). A **Poisson process** enjoys the following properties:

1. the number of occurrences in non-overlapping intervals are independent random variables,

2. the probability of an occurrence in a sufficiently short interval is proportional to the length of the interval, and

3. the probability of 2 or more occurrences in a sufficiently short interval is zero.

Suppose a counting process satisfies the three conditions above. Define Y = the number of occurrences in a unit interval of time (or space). Our goal is to find an expression for $p_Y(y) = P(Y = y)$, the PMF of Y.

Suppose we partition the unit interval $[0, 1]$ into n sub-intervals, each of size $\frac{1}{n}$. Observe that if n is sufficiently large (i.e., much larger than y), then we can approximate the probability y events occur in the unit interval by finding the probability that exactly one event (occurrence) occurs in exactly y of the sub-intervals. By Property (2), we know that the probability of one event in any one subinterval is proportional to the sub-interval's length, say, $\frac{\lambda}{n}$, where λ is the proportionality constant. By Property (3), the probability of more than one occurrence in any sub-interval is zero (for large n).

Consider the occurrence/non-occurrence of an event in each sub-interval as a Bernoulli trial. By Property (1), we have a sequence of n Bernoulli trials, each with probability of "success" $\frac{\lambda}{n}$. Thus, a binomial (approximate) calculation gives

$$P(Y = y) \approx \binom{n}{y} \left(\frac{\lambda}{n}\right)^y \left(1 - \frac{\lambda}{n}\right)^{n-y}$$

To improve the approximation for $P(Y = y)$,we let n grow large without bound, that is, we let $n \to \infty$. We have

$$\begin{aligned}
\lim_{n\to\infty} P(Y = y) &= \lim_{n\to\infty} \binom{n}{y} \left(\frac{\lambda}{n}\right)^y \left(1 - \frac{\lambda}{n}\right)^{n-y} \\
&= \lim_{n\to\infty} \frac{n!}{y!(n-y)!} \lambda^y \left(\frac{1}{n}\right)^y \left(1 - \frac{\lambda}{n}\right)^n \left(\frac{1}{1-\frac{\lambda}{n}}\right)^y \\
&= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-[y-1])(n-y)!}{(n-y)!n^y} \frac{\lambda^y}{y!} \left(1 - \frac{\lambda}{n}\right)^n \left(\frac{1}{1-\frac{\lambda}{n}}\right)^y \\
&= \lim_{n\to\infty} \underbrace{\frac{n(n-1)(n-2)\cdots(n-y+1)}{n^y}}_{a_n} \underbrace{\frac{\lambda^y}{y!}}_{b_n} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{c_n} \underbrace{\left(\frac{1}{1-\frac{\lambda}{n}}\right)^y}_{d_n}
\end{aligned}$$

Recall that the limit of the product is the product of the limits. Thus,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-y+1)}{n^y} = 1 \\
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{\lambda^y}{y!} = \frac{\lambda^y}{y!} \\
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda} \\
\lim_{n\to\infty} d_n &= \lim_{n\to\infty} \left(\frac{1}{1-\frac{\lambda}{n}}\right)^y = 1
\end{aligned}$$

Therefore, we have shown that

$$\lim_{n\to\infty} P(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!}$$

Definition

We say that Y follows a Poisson distribution with parameter λ if its probability mass function is given by

$$P(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!},\ y = 0, 1, 2, \cdots$$

If Y follows a Poisson distribution with parameter λ, then we write $Y \sim Poi(\lambda)$

<u>Example 2.7.1</u>

The number of typing errors made by a typist has a Poisson distribution with an average of four errors per page. If more than four errors appear on a given page, the typist must retype the whole page. What is the probability that a randomly selected page does not need to be retyped?

SOLUTION

Let Y denote the number of typing errors per page. Then $Y \sim Pois(4)$. A page does not need to be retyped if four or less errors appear on it. Thus,

$$\begin{aligned}
P(\text{do not retype the page}) &= P(Y \leq 4) \\
&= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) \\
&= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} \\
&\approx 0.6288
\end{aligned}$$

This probability can be calculated also using the R command **ppois(4,4)**.

Theorem

If $Y \sim Pois(\lambda)$, then its moment generating function is given by

$$m_Y(t) = e^{\lambda(e^t - 1)}$$

PROOF

From the definition of the moment generating function, we have

$$
\begin{aligned}
m_Y(t) = E[e^{tY}] &= \sum_{y=0}^{\infty} e^{ty} P(Y=y) \\
&= \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\lambda}\lambda^y}{y!} \\
&= e^{-\lambda} \underbrace{\sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!}}_{e^{\lambda e^t}} \\
&= e^{-\lambda} e^{\lambda e^t} \\
&= e^{\lambda(e^t-1)}
\end{aligned}
$$

Theorem

If $Y \sim Pois(\lambda)$, then its mean and variance are given by

1. $E(Y) = \lambda$, and
2. $V(Y) = \lambda$.

PROOF: Left as a classroom exercise!

Example 2.7.2

Flaws in a large plate of glass occur on the average, one per 20 sq. feet. Using the Poisson distribution, find the probability that a 3ft by 10ft sheet will contain

a. no flaw
b. at least one flaw

SOLUTION

If on average, we observe one flaw on a 20 square feet glass plate, so on a 30 square feet (3x10) plate glass we expect to observe 1.5 flaws. That is, if we let Y be the number of flaws on a 30 square feet glass plate, then $Y \sim Pois(.5)$.

a. $P(Y=0) = \frac{e^{-1.5}1.5^0}{0!} \approx 0.2231$
b. $P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y=0) \approx 1 - 0.2231 = 0.7769$

Poisson approximation of the binomial distribution

For the binomial distribution with large n, calculating the mass function is pretty nasty. For example if $n = 1000$ and $y = 500$. Evaluating $P(Y = 500)$ would be a formidable task for many scientific calculators, even today.

When the value of n in a binomial distribution is large and the value of p is very small, the binomial distribution can be approximated by a Poisson distribution. In general, the approximation works well if $n \geq 20$ and $p \leq 0.05$.

Example 2.7.3

The probability that a mouse inoculated with a serum will contract a certain disease is 0.2.

a. Find the probability that at most 3 of 30 inoculated mice will contract the disease.

b. Use the Poisson distribution to approximate the probability that at most 3 of 30 inoculated mice will contract the disease.

SOLUTION

1. Let Y be the number of inoculated mice that contract the disease, $Y \sim B(30, 0.2$. Thus,

$$\begin{aligned} P(Y \leq 3) &= \sum_{y=0}^{3} \binom{30}{y} 0.2^y 0.8^{30-y} \\ &\approx 0.1227 \end{aligned}$$

You can use the R command **pbinom(3,30,0.2)** to calculate this probability.

2. To approximate the above probability using the Poisson distribution, we first need to calculate $\lambda = np = 30(0.2) = 6$. Thus, $Y \dot{\sim} Poi(6)$. Hence,

$$\begin{aligned} P(Y \leq 3) &\approx \sum_{y=0}^{3} \frac{e^{-6} 6^y}{y!} \\ &\approx 0.1512 \end{aligned}$$

You can use the R command **ppois(3,6)* to calculate this probability.

Learning Tasks

Instruction: Answer the following as indicated.

1. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. During a given hour, what are the probabilities that

    a. no more than three customers arrive?

    b. at least two customers arrive?

    c. exactly five customers arrive?

2. Five percent (5%) of Christmas tree light bulbs manufactured by a company are defective. The company's Quality Control Manager is quite concerned and therefore randomly samples 100 bulbs coming off of the assembly line. Compute the exact and approximate probabilities that the sample contains at most three defective bulbs?