Stat 121 (Mathematical Statistics I)

Lesson 4.3 Conditional Distributions

Learning Outcomes

At the end of the lesson, the students must be able to derive conditional distributions of discrete and continuous random variables.

The discrete case

Definition

Suppose Y_1 and Y_2 are discrete random variables with joint PMF $p_{(Y_1.Y_2)}(y_1, y_2)$ and marginal PMFs $p_{Y_1}(y_1)$ and $p_{Y_2}(y_2)$, respectively. The **conditional probability mass function** of Y_1 given $Y_2 = y_2$ is given by

$$p_{(Y_1|Y_2)}(y_1|y_2) = \frac{p_{(Y_1.Y_2)}(y_1, y_2)}{p_{Y_2}(y_2)}, \text{ provided } p_{Y_2}(y_2) > 0$$

Similarly, The **conditional probability mass function** of Y_2 given $Y_1 = y_1$ is given by

$$p_{(Y_2|Y_1)}(y_2|y_1) = \frac{p_{(Y_1.Y_2)}(y_1, y_2)}{p_{Y_1}(y_1)}, \text{ provided } p_{Y_1}(y_1) > 0.$$

This means that the conditional PMF $p_{(Y_1|Y_2)}(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2)$ is a univariate PMF. It describes how the distribution of Y_1, that is how Y_1 varies when Y_2 is fixed at y_2. The same can be said about the conditional PMF $p_{(Y_2|Y_1)}(y_2|y_1) = P(Y_2 = y_2|Y_1 = y_1)$.

Example 4.3.1

Consider the joint PMF in *Example 4.1.1.* Find the

a. conditional distribution of Y_1 given $Y_2 = 1$.

b. conditional distribution of Y_2 given $Y_1 = 2$.

SOLUTION

Recall from *Example 4.1.1* that the joint PMF of Y_1 and Y_2 is given by

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.64	0.08	0.04
$y_1 = 1$	0.12	0.06	0.02
$y_1 = 2$	0.02	0.01	0.01

From *Example 4.2.1* the marginal PMFs of Y_1 and Y_2, respectively, are given by

Y_1	0	1	2
$P(Y_1 = y_1)$	0.76	0.20	0.04

and

Y_2	0	1	2
$P(Y_2 = y_2)$	0.78	0.15	0.07

a. The conditional distribution of Y_1 given $Y_2 = 1$ is obtained as follows:

$$p_{(Y_1|Y_2)}(y_1 = 0|y_2 = 1) = \frac{P(Y_1 = 0, Y_2 = 1)}{P(Y_2 = 1)} = \frac{0.08}{0.15} \approx 0.53$$

$$p_{(Y_1|Y_2)}(y_1 = 1|y_2 = 1) = \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_2 = 1)} = \frac{0.06}{0.15} = 0.40$$

$$p_{(Y_1|Y_2)}(y_1 = 2|y_2 = 1) = \frac{P(Y_1 = 2, Y_2 = 1)}{P(Y_2 = 1)} = \frac{0.01}{0.15} \approx 0.07$$

We can summarize the conditional PMF of Y_1 given $Y_2 = 1$ in the following table:

Y_1	0	1	2
$P(Y_1 \vert Y_2 = y_2)$	0.53	0.40	0.07

b. The determination of the conditional distribution of Y_2 given $Y_1 = 2$ is left as a classroom exercise.

The continuous case

Definition

Suppose Y_1 and Y_2 are discrete random variables with joint PDF $f_{(Y_1.Y_2)}(y_1, y_2)$ and marginal PMFs $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, respectively. The **conditional probability density function** of Y_1 given $Y_2 = y_2$ is given by

$$f_{(Y_1|Y_2)}(y_1|y_2) = \frac{f_{(Y_1.Y_2)}(y_1, y_2)}{f_{Y_2}(y_2)}, \text{ provided } f_{Y_2}(y_2) > 0.$$

Similarly, The **conditional probability density function** of Y_2 given $Y_1 = y_1$ is given by

$$f_{(Y_2|Y_1)}(y_2|y_1) = \frac{f_{(Y_1.Y_2)}(y_1, y_2)}{f_{Y_1}(y_1)}, \text{ provided } f_{Y_1}(y_1) > 0.$$

As in the discrete case, the conditional PDF of Y_1 given $Y_2 = y_2$ is a univariate PDF and it describes how the random variable Y_1 varies with Y_2 fixed at y_2. The same can be said about the conditional PDF of Y_2 given $Y_1 = y_1$.

Example 4.3.2

Consider *Example 4.2.3* where the joint PDF of two random variables Y_1 and Y_2 is given by

$$f_{(Y_1,Y_2)}(y_1, y_2) = \begin{cases} \frac{2y_1+2-y_2}{4}, & 0 \le y_1 \le 1; 0 \le y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

a. Find the conditional distribution of Y_1 given $Y_2 = 1$.

b. Find the conditional distribution of Y_2 given $Y_1 = 0.5$

SOLUTION

In *Example 4.2.3*, the marginal PDFs of Y_1 and Y_2, respectively, are

$$f_{Y_1}(y_1) = \begin{cases} y_1 + \frac{1}{2}, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_{Y_2}(y_2) = \begin{cases} \frac{1}{4}(3 - y_2), & 0 \le y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

a. When $Y_2 = 1$, we have $f_{(Y_1,Y_2)}(y_1, 1) = \frac{2y_1+1}{4}$ and $f_{Y_2}(1) = \frac{1}{4}(3 - 1) = 0.5$. Thus,

$$\begin{aligned} f_{(Y_1|Y_2)}(y_1|y_2=1) &= \frac{f_{(Y_1.Y_2)}(y_1,1)}{f_{Y_2}(1)} \\ &= \frac{\frac{2y_1+1}{4}}{0.5} \\ &= \frac{2y_1+1}{2} \end{aligned}$$

In summary, the conditional distribution of Y_1 given $Y_2 = 1$ is

$$f_{(Y_1|Y_2)}(y_1|y_2=1) = \begin{cases} \frac{2y_1+1}{2}, & 0 < y_1 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

b. When When $Y_1 = 0.5$, we have $f_{(Y_1,Y_2)}(0.5, y_2) = \frac{3-y_2}{4}$ and $f_{Y_1}(0.5) = 1$. Thus,

$$\begin{aligned} f_{(Y_2|Y_1)}(y_2|y_1=0.5) &= \frac{f_{(Y_1.Y_2)}(0.5,y_2)}{f_{Y_1}(0.5)} \\ &= \frac{\frac{3-y_2}{4}}{1} \\ &= \frac{3-y_2}{4} \end{aligned}$$

In summary, the conditional distribution of Y_2 given $Y_1 = 0.5$ is

$$f_{(Y_2|Y_1)}(y_2|y_1=0.5) = \begin{cases} \frac{3-y_2}{4}, & 0 < y_2 < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Learning Task

Instruction: Answer the following as indicated.

Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} e^{-y_2}, & 0 < y_1 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

1. Find the conditional distribution of Y_1 given Y_2.
2. Find the conditional distribution of Y_2 given Y_1.