Stat 121 (Mathematical Statistics I)

Lesson 3.2 Mathematical Expectation of Continuous Random Variables

Learning Outcomes

At the end of the lesson, students must be able to

1. Determine expected value of continuous random variables, and
2. Compute special expectations such as the mean, variance, and moment generating function of continuous random variables.

Introduction

Mathematical expectations, such as the mean, variance, and moment generating function for continuous random variables are just a straightforward extension of those of the discrete case. All we need to do is replace the summations with integrals.

Definition:

Let Y be a continuous random variable with PDF $f_Y(y)$ and domain D. The expected value of Y is given by

$$E(Y) = \int_{\mathbb{R}} y f_Y(y)\, dy$$

NOTE:

For $E(Y)$ to exist, we need the integral to converge absolutely, that is

$$\int_{\mathbb{R}} |y| f_Y(y)\, dy < \infty$$

Otherwise, $E(Y)$ does not exist.

Remarks:

1. Basic properties of expectation that were introduced in *Lesson 2.2* apply to continuous random variables.

2. The expected value of any real-valued function of Y, say $g(Y)$, is given by

$$E[g(Y)] = \int_{\mathbb{R}} g(y) f_Y(y)\, dy$$

provided that this integral converges absolutely. Otherwise, we say that $E[g(y)]$ does not exist.

3. The variance of a continuous random variable Y with mean $E(Y) = \mu_Y$ is given by

$$\sigma_Y^2 = V(Y) = E[(Y - \mu_Y)^2] = \int_{\mathbb{R}} (y - \mu_Y)^2 f_Y(y)\, dy$$

4. The variance computing formula still applies, that is,

$$V(Y) = E(Y^2) - \mu_Y^2$$

Example 3.2.1:

Consider a continuous random variable with PDF

$$f_Y(y) = \begin{cases} \frac{3}{8}y^2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the mean and standard deviation of Y.

SOLUTION

We have

$$\begin{aligned} E(Y) = \mu_Y &= \int_{\mathbb{R}} y f_Y(y)\, dy, \text{ by definition} \\ &= \int_0^2 y \frac{3}{8} y^2\, dy \\ &= \frac{3}{8} \int_0^2 y^3\, dy \\ &= \frac{3}{8} \left[\frac{y^4}{4}\right]_0^2 \\ &= \frac{3}{8} \left[\frac{2^4}{4} - \frac{0^4}{4}\right] \\ &= 1.5 \end{aligned}$$

Meanwhile,

$$\begin{aligned} E(Y^2) &= \int_{\mathbb{R}} y^2 f_Y(y)\ dy \\ &= \int_0^2 y^2 \frac{3}{8} y^2\ dy \\ &= \frac{3}{8}\int_0^2 y^4\ dy \\ &= \frac{3}{8}\left[\frac{y^5}{5}\right]_0^2 \\ &= \frac{3}{8}\left[\frac{2^5}{5} - \frac{0^5}{5}\right] \\ &= 2.4 \end{aligned}$$

Therefore,

$$\begin{aligned} V(Y) &= E(Y^2) - \mu_Y^2 \\ &= 2.4 - 1.5^2 \\ &= 0.15 \end{aligned}$$

$$\implies \sigma_Y = \sqrt{V(Y)} = \sqrt{0.15} = 0.3872983$$

<u>Example 3.2.2</u>:

Suppose X has the following PDF

$$f_X(x) = \begin{cases} 2x,\ 0 < x < 1 \\ 0,\ \text{elsewhere} \end{cases}$$

Find $E(U)$, where $U = -2X + 3$.

SOLUTION

$$\begin{aligned} E(U) &= \int_0^1 u f_X(x)\ dx \\ &= \int_0^1 (-2x+3)(2x)\ dx \\ &= \int_0^1 (-4x^2 + 6x)\ dx \\ &= \left[-4\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right)\right]_0^1 \\ &= \frac{5}{3},\ \text{(Verify!)} \end{aligned}$$

Definition

Suppose Y is a continuous random variable with PDF $f_Y(y)$. The moment generating function (MGF) of Y is given by

$$m_Y(t) = E[e^{tY}] = \int_{\mathbb{R}} e^{ty} f_Y(y)\, dy$$

provided this integral exists.

Remark:

Recall that we can use the moment generating function to obtain special moments such the mean and the variance. In general,

$$E[Y^k] = \frac{d^k}{dt^k} m_Y(t)\bigg|_{t=0}$$

Hence,

$$E(Y) = \frac{d}{dt} m_Y(t)\bigg|_{t=0}$$

and

$$E(Y^2) = \frac{d^2}{dt^2} m_Y(t)\bigg|_{t=0}$$

Example 3.2.3:

The PDF of a continuous random variable Y is given by

$$f_Y(y) = \begin{cases} 3e^{-3y},\ y \geq 0 \\ 0,\ \text{elsewhere} \end{cases}$$

Find the moment generating function of Y and use this to find the mean of Y.

SOLUTION:

$$\begin{aligned}
m_Y(t) &= E[e^{tY}] \\
&= \int_0^\infty e^{ty}\left(3e^{-3y}\right)\,dy \\
&= \int_0^\infty 3e^{(t-3)y}\,dy \\
&= 3\int_0^\infty e^{(t-3)y}\left(\frac{t-3}{t-3}\right)\,dy, \text{ HINT: Use u-substitution} \\
&= \frac{3}{t-3}\int_0^\infty e^{(t-3)y}(t-3)\,dy \\
&= \frac{3}{t-3}\left[e^{(t-3)y}\right]_0^\infty \\
&= \frac{3}{t-3}\left[0-1\right], \text{ [Verify!]} \\
&= \frac{-3}{t-3} \\
&= \frac{3}{3-t}
\end{aligned}$$

Using the moment generating function of Y, its mean is given by

$$\begin{aligned}
E(Y) &= \frac{d}{dt}m_Y(t)\bigg|_{t=0} \\
&= \frac{d}{dt}\left(\frac{3}{3-t}\right)\bigg|_{t=0} \\
&= \frac{d}{dt}\left[3(3-t)^{-1}\right]\bigg|_{t=0} \\
&= 3(-1)(3-t)^{-2}(-1)\bigg|_{t=0}, \text{ by Chain Rule} \\
&= \left[\frac{3}{(3-t)^2}\right]_{t=0} \\
&= \frac{1}{3}
\end{aligned}$$

Learning Tasks

Instruction: Answer the following as indicated.

1. The PDF of a continuous random variable Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{\beta} e^{-\frac{y}{\beta}}, \ y > 0 \\ 0, \ \text{elsewhere} \end{cases}$$

Find the MGF of Y.

2. Consider the function

$$f_Y(y) = \begin{cases} kye^{-2y}, \ \ y \geq 0 \\ 0, \ \text{otherwise} \end{cases}$$

a. Find the value of k that makes a $f_Y(y)$ a valid probability density function.

b. Derive the moment-generating function for Y.

c. Compute the mean and variance for Y using the MGF.