Stat 121 (Mathematical Statistics I)

Lesson 2.3 Moment Generating Function

Learning Outcomes

At the end of the lesson, students must be able to

1. Articulate the meaning of the moment generating function (MGF),
2. Determine the MGF of a discrete random variable,
3. Apply the MGF in finding the mean and variance of a discrete random variable, and
4. Identify the probability distribution of a random variable using the MGF.

Introduction

The parameters μ and σ are meaningful numerical descriptive measures that locate the center and describe the spread associated with the values of a random variable, say X. However, they do not provide a unique characterization of the distribution of X. Many different distributions possess the same means and standard deviations. We now consider a set of numerical descriptive measures that (at least under certain conditions) uniquely determine $p_X(x)$.

Definition

The k^{th} (raw) moment of a random variable X taken about the origin is defined to be $E[X^k]$ and is denoted by $\mu_k^{'}$.That is,

$$\mu_k^{'} = E[X^k] = \sum_{\forall x} x^k P(X = x)$$

Remark:

From the above definition, we have

1. First raw moment: $\mu_1^{'} = E[X] = \mu_X$
2. Second raw moment: $\mu_2^{'} = E[X^2] = \sigma_X^2 + \mu_X^2$
3. Third raw moment (used in determining skewness): $\mu_3^{'} = E[X^3]$

4. Fourth raw moment (used in determining kurtosis): $\mu_4^{'} = E[X^4]$

Definition

The k^{th} moment of a random variable X taken about its mean, known as the k^{th} central moment of X, is defined to be $E[(X - \mu_X)^k]$ and is denoted by μ_k. That is,

$$\mu_k = E[(X - \mu_X)^k] = \sum_{\forall x} (X - \mu_X)^k \; P(X = x)$$

Remark

From this definition, we have

1. First central moment: $E[(X - \mu_X)] = 0$
2. Second central moment: $E[(X - \mu_X)^2] = V(X) = \sigma_X^2$
3. Third central moment (measure of skewness): $E[(X - \mu_X)^3]$
4. Fourth central moment (measure of kurtosis): $E[(X - \mu_X)^4]$

Definition

Let Y be a discrete random variable with pmf $p_Y(y) = P(Y = y)$ and domain D. The **moment generating function (MGF)** for Y, denoted by $m_Y(t)$, is given by

$$m_Y(t) = E\left[e^{tY}\right] = \sum_{\forall y \in D} e^{ty} P(Y = y),$$

provided $E\left[e^{tY}\right] < \infty$ for all t in an open neighborhood about 0, that is, there exists some $h > 0$ such that $E\left[e^{tY}\right] < \infty$ for all $t \in (-h, h)$. If $E\left[e^{tY}\right]$ does not exist in an open neighborhood of 0, we say that the moment generating function exist.

Remark

The MGF can be used to generate moments. In fact, from the theory of Laplace transforms, it follows that if the MGF exists, it characterizes an infinite set of moments. So, how do we exactly generate moments using the MGF?

Theorem

Let Y denote a random variable (not necessarily a discrete random variable) with support U and MGF $m_Y(t)$. Then

$$E(Y^k) = \frac{d^k}{dt^k} m_Y(t)\bigg|_{t=0}.$$

Remark

This means that

$$E(Y) = \frac{d}{dt} m_Y(t)\bigg|_{t=0},$$

and

$$E(Y^2) = \frac{d^2}{dt^2} m_Y(t)\bigg|_{t=0}.$$

Example 2.3.1

Suppose a random variable Y has the following PMF

$$p_Y(y) = P(Y = y) = \begin{cases} \frac{1}{10}(5-y), & y = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

a. Derive the MGF of Y.

b. Compute the mean and variance of Y using the MGF.

SOLUTUION: Left as a classroom activity.

Remark

Another important application of the MGF is determining the distribution of a random variable based on its MGF. But we will explore this in the later lessons.

Learning Task

1. Let the random variable Y have PMF given by

$$P(Y = y) = \begin{cases} \frac{1}{6}(3 - y), & y = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Derive the MGF of Y and verify that the mean and variance of Y are $\frac{2}{3}$ and $\frac{5}{9}$, respectively.

2. Suppose the MGF of a random variable Y is given by:

$$m_Y(t) = \frac{1}{10}e^t + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}$$

What is the PMF of Y?