Stat 121 (Mathematical Statistics I)

Lesson 4.2 Marginal Distributions

Learning Outcomes

At the end of the lesson, students must be able to derive the marginal distributions from the joint distribution.

Introduction

A joint probability distribution describes how two random variables are jointly distributed. In this lesson we shall learn how to derive the marginal (univariate) distribution of a random variable from the joint probability distribution.

Marginal distribution of a discrete random variable

In order to obtain the marginal probability mass function (PMF) of one random variable, one takes the joint PMF and sum over the possible values of the other variable.

Definition

Let Y_1 and Y_2 be jointly discrete random variables with probability function mass function $p_{(Y_1,Y_2)}(y_1, y_2)$. Then the marginal PMF of Y_1 is

$$p_{Y_1}(y_1) = \sum_{\forall y_2} p_{(Y_1,Y_2)}(y_1, y_2).$$

Similarly,

$$p_{Y_2}(y_2) = \sum_{\forall y_1} p_{(Y_1,Y_2)}(y_1, y_2).$$

Example 4.2.1

Consider the joint PMF in *Example 4.1.1.* Find the marginal PMFs of Y_1 and Y_2.

SOLUTION

We have

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.64	0.08	0.04
$y_1 = 1$	0.12	0.06	0.02
$y_1 = 2$	0.02	0.01	0.01

The marginal PMF of Y_1 is given by

Y_1	0	1	2
$P(Y_1 = y_1)$	0.76	0.20	0.04

and the marginal PMF of Y_2 is

Y_2	0	1	2
$P(Y_2 = y_2)$	0.78	0.15	0.07

Example 4.2.2

Consider the joint PMF in *Example 4.1.2*. Find the marginal distributions of Y_1 and Y_2.

SOLUTION

We have

	$Y_2 = 0$	$Y_2 = 1$	$Y_2 = 2$
$Y_1 = 0$	1/9	2/9	1/9
$Y_1 = 1$	2/9	2/9	0
$Y_1 = 2$	1/9	0	0

The marginal PMF of Y_1 is given by

Y_1	0	1	2
$P(Y_1 = y_1)$	4/9	4/9	1/9

and the marginal PMF of Y_2 is

Y_2	0	1	2
$P(Y_2 = y_2)$	4/9	4/9	1/9

Marginal distribution of a continuous random variable

In the case of continuous random variables, the marginal probability function (PDF) of one random variable is obtained by integrating the joint PDF over the other variable.

Definition

Suppose Y_1 and Y_2 are continuous random variables with joint PDF $f_{(Y_1,Y_2)}(y_1,y_2)$. The marginal PDF of Y_1 is

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{(Y_1,Y_2)}(y_1,y_2)dy_2.$$

and the marginal PDF of Y_2 is

$$f_{Y_2}(y_2) = \int_{\mathbb{R}} f_{(Y_1,Y_2)}(y_1,y_2)dy_1.$$

Example 4.2.3

An insurance company insures a large number of drivers. Let Y_1 denote the company's losses under collision insurance and let Y_2 denote the company's losses under liability insurance. The joint PDF of Y_1 and Y_2 is given by

$$f_{(Y_1,Y_2)}(y_1,y_2) = \begin{cases} \frac{2y_1+2-y_2}{4}, & 0 \le y_1 \le 1; 0 \le y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the marginal PDFs of Y_1 and Y_2.

SOLUTION

The marginal PDF of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^2 f_{(Y_1,Y_2)}(y_1,y_2)\,dy_2 \\ &= \int_0^2 \frac{2y_1+2-y_2}{4}\,dy_2 \\ &= \frac{1}{4}\left[\left(2y_1y_2 + 2y_2 - \frac{y_2^2}{2}\right)\Bigg|_0^2\right] \\ &= \frac{1}{4}(4y_1+4-2) \\ &= y_1 + \frac{1}{2}. \end{aligned}$$

Thus,

$$f_{Y_1}(y_1) = \begin{cases} y_1 + \frac{1}{2}, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Meanwhile, (VERIFY!) the marginal PDF of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} \frac{1}{4}(3 - y_2), & 0 \leq y_2 \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Learning Tasks

Instruction: Answer the following as indicated.

1. Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the marginal mass functions of Y_1 and Y_2.

2. A bank operates a drive-up and a walk-up window. Let $Y_1 be the proportion of time the drive-up facility is in use, and Y_2 the proportion of time the walk-up window in use. Say the manager has given us the joint PDF based on his experience:

$$f_{(Y_1,Y_2)}(y_1, y_2) = \begin{cases} \frac{6}{5}(y_1 + y_2^2), & 0 \leq y_1 \leq 1; 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the marginal PDFs of Y_1 and Y_2.