Stat 121 (Mathematical Statistics I)

Lesson 4.4 Independent Random Variables

Learning Outcomes

At the end of the lesson, the students must be able to show that two (or more) random variables are independent.

Discrete Case

Definition

Suppose Y_1 and Y_2 are discrete random variables with joint PMF $p_{(Y_1,Y_2)}(y_1, y_2)$ and marginal PMFs $p_{Y_1}(y_1)$ and $p_{Y_2}(y_2)$, respectively. We say that Y_1 and Y_2 are **independent** if

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

for all $(y_1, y_2) \in \mathbb{R}^2$. In other words, the joint PMF factors into the product of the marginal PMFs.

Example 4.4.1

Consider the random variables Y_1 and Y_2 with joint PMF in *Example 4.1.1.* Are the random variables independent?

SOLUTION

Recall from *Example 4.1.1* that the joint PMF of Y_1 and Y_2 is given by

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.64	0.08	0.04
$y_1 = 1$	0.12	0.06	0.02
$y_1 = 2$	0.02	0.01	0.01

While, from *Example 4.2.1* the marginal PMFs of Y_1 and Y_2, respectively, are given by

Y_1	0	1	2
$P(Y_1 = y_1)$	0.76	0.20	0.04

and

Y_2	0	1	2
$P(Y_2 = y_2)$	0.78	0.15	0.07

For Y_1 and Y_2 to be independent we would need

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

to hold for all (y_1, y_2) in the support

$$R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

However, this condition does not even hold for the first value pair (0,0). Note that

- $p_{(Y_1,Y_2)}(0,0) = 0.64$ (from the joint PMF of Y_1 and Y_2
- $p_{Y_1}(0) = 0.76$ (from the marginal PMF of Y_1)
- $p_{Y_2}(0) = 0.78$ (from the marginal PMF of Y_2)

Thus, $p_{(Y_1,Y_2)}(0,0) \neq p_{Y_1}(0) \times p_{Y_2}(0)$ $[0.64 \neq 0.76 \times 0.78]$

Example 4.4.2

Let $\mathbf{Y} = (Y_1, Y_2)$ be a random vector with joint PMF

$p_{(Y_1,Y_2)}(y_1, y_2)$	$y_2 = 1$	$y_2 = 2$	$p_{Y_1}(y_1)$
$y_1 = 1$	$1/30$	$4/30$	$5/30$
$y_1 = 2$	$2/30$	$8/30$	$10/30$
$y_1 = 3$	$3/30$	$12/30$	$15/30$
$p_{Y_2}(y_2)$	$6/30$	$24/30$	

Are Y_1 and Y_2 independent?

SOLUTION

It is easy to verify that

$$p_{(Y_1,Y_2)}(y_1, y_2) = p_{Y_1}(y_1) \times p_{Y_2}(y_2)$$

to hold for all (y_1, y_2) in the support

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$$

For example,

$$p_{(Y_1,Y_2)}(1,1) = 1/30 = p_{Y_1}(1) \times p_{Y_2}(1) = 5/30 \times 6/30$$

It can be VERIFIED that the joint PMF can be expressed as the product of the marginal PMFs for the rest of the pairs in the support!

Therefore, Y_1 and Y_2 are independent.

The continuous case

Definition

Suppose Y_1 and Y_2 are continuous random variables with joint PDF $f_{(Y_1.Y_2)}(y_1, y_2)$ and marginal PDFs $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, respectively. We say that Y_1 and Y_2 are **independent** if

$$f_{(Y_1,Y_2)}(y_1, y_2) = f_{Y_1}(y_1) \times f_{Y_2}(y_2)$$

for all $(y_1, y_2) \in \mathbb{R}^2$. In other words, the joint PDF factors into the product of the marginal PDFs.

Example 4.4.3

This past year was an encouraging year for biodiversity discovery ,as scientists identified thousands of new species of life. For one newly discovered species, geneticists model

$$\begin{aligned} Y_1 &= \text{the percentage of the species possessing Trait 1} \\ Y_2 &= \text{the percentage of the species possessing Trait 2} \end{aligned}$$

using the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 12y_1^3(1-y_2)^2, \ 0 < y_1 < 1, \ 0 < y_2 < 1 \\ 0, \ \text{otherwise} \end{cases}$$

Show that Y_1 and Y_2 are independent.

SOLUTION: Left as a classroom exercise!

Remarks

1. Suppose Y_1 and Y_2 are random variables (discrete or continuous) with joint CDF $F_{Y_1,Y_2}(y_1, y_2)$. Then Y_1 and Y_2 are independent if and only if

$$F_{(Y_1,Y_2)}(y_1, y_2) = F_{Y_1}(y_1) \times F_{Y_2}(y_2)$$

2. If Y_1 and Y_2 are independent then

$$f_{(Y_2|Y_1)}(y_2|y_1) = f_{Y_2}(y_2)$$

and

$$f_{(Y_1|Y_2)}(y_1|y_2) = f_{Y_1}(y_1)$$

The discrete conclusion is analogous; simply replace PDFs with PMFs.

3. Suppose Y_1 and Y_2 are independent random variables (discrete or continuous). The random variables $U_1 = g(Y_1)$ and $U_2 = h(Y_2)$ are also independent. In other words, functions of independent random variables are independent.

Learning Tasks:

Answer the following as indicated.

1. Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_2}, & 0 < y_1 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Are Y_1 and Y_2 independent?

2. Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{384} y_1^2 y_2^4 e^{y_1 - y_2/2}, & 0 < y_1 < \infty, 0 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Show that Y_1 and Y_2 are independent.