Lesson 1.4

Bivariate Transformation

Introduction

So far in this module, we have talked about transformations involving a single random variable Y. It is sometimes of interest to consider **bivariate transformation** such as

$$U_1 = g_1(Y_1, Y_2)$$
$$U_2 = g_1(Y_1, Y_2)$$

To discuss such transformations, we will assume that Y_1 and Y_2 are jointly **continuous** random variables. Furthermore, for the following methods to apply, the transformation needs to be **one-to-one**. We start with the joint distribution of $\mathbf{Y} = (Y_1, Y_2)$. Our first goal is to find the joint distribution of $\mathbf{U} = (U_1, U_2)$.

Bivariate Transformation

Suppose that $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF $f_{Y_1,Y_2}(y_1, y_2)$. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous one-to-one vector-valued mapping from R_{Y_1,Y_2} to R_{U_1,U_2}, where $U_1 = g_1(Y_1, Y_2)$ and $U_2 = g_2(Y_1, Y_2)$, and where R_{Y_1,Y_2} and R_{U_1,U_2} denote two-dimensional supports of $\mathbf{Y} = (Y_1, Y_2)$ and $\mathbf{U} = (U_1, U_2)$, respectively.

If $g_1^{-1}(u_1, u_2)$ and $g_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to both u_1 and u_2 and the Jacobian, J, where "det" denote "determinant",

$$J = det \begin{bmatrix} \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_2} \end{bmatrix} \neq 0$$

then

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} f_{Y_1,Y_2}[g_1^{-1}(u_1,u_2), g_2^{-1}(u_1,u_2)] \times |J|, \ (u_1,u_2) \in R_{U_1,U_2} \\ 0, \text{ elsewhere} \end{cases} \tag{1}$$

where $|J|$ denotes the absolute value of J.

To summarize, the steps involved in bivariate trabsformation are:

1. Find the joint distribution $f_{Y_1,Y_2}(y_1,y_2)$ of Y_1 and Y_2.
2. Find the R_{U_1,U_2} the support of $\mathbf{U} = (U_1,U_2)$.
3. Find the inverse transformations $y_1 = g_1^{-1}(u_1,u_2)$ and $y_2 = g_2^{-1}(u_1,u_2)$.
4. Find the Jacobian, J, of the inverse transformation.
5. Use (1) to find $f_{U_1,U_2}(u_1,u_2)$ the joint distribution of U_1 and U_2.

Example 1.4.1

Suppose $Y_1 \sim Gamma(\alpha,1)$ and $Y_2 \sim Gamma(\beta,1)$ abd that Y_1 and Y_2 are independent. Define the transformation

$$U_1 = g_1(Y_1,Y_2) = Y_1 + Y_2$$
$$U_2 = g_2(Y_1,Y_2) = \frac{Y_1}{Y_1+Y_2}$$

Find each of the following distributions.

a. $f_{U_1,U_2}(u_1,u_2)$
b. $f_{U_1}(u_1)$
c. $f_{U_2}(u_2)$

SOLUTION

a. Since Y_1 and Y_2 are independent, then

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1}(y_1) \times f_{Y_1}(y_1) \\
&= \frac{1}{\Gamma(\alpha)} y_1^{\alpha-1} e^{-y_1} \times \frac{1}{\Gamma(\beta)} y_2^{\beta-1} e^{-y_2} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1} y_2^{\beta-1} e^{-(y_1+y_2)}
\end{aligned}
$$

for $y_1 > 0, y_2 > 0$, and 0, otherwise. That is, $R_{Y_1,Y_2} = \{(y_1,y_2) : y_1 > 0, y_2 > 0\}$.

By inspection, $u_1 = y_1 + y_2 > 0$ and $u_2 = \frac{y_1}{y_1+y_2}$ must fall between 0 and 1. Thus, $R_{U_1,U_2} = \{(u_1,u_2) : u_1 > 0, 0 < u_2 < 1\}$.

Next we derive the inverse transformation. Verify that,

$$
\begin{aligned}
y_1 &= g_1^{-1}(u_1,u_2) = u_1 u_2 \\
y_2 &= g_2^{-1}(u_1,u_2) = u_1 - u_1 u_2
\end{aligned}
$$

Hence, the Jacobian is

$$
J = det \begin{bmatrix} \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_2} \end{bmatrix} = det \begin{bmatrix} u_2 & u_1 \\ 1-u_2 & -u_1 \end{bmatrix} = -u_1
$$

Hence, the joint distribution of U_1 and U_2 is

$$
\begin{aligned}
f_{U_1,U_2}(u_1,u_2) &= f_{Y_1,Y_2}[g_1^{-1}(u_1,u_2), g_2^{-1}(u_1,u_2)] \times |J| \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (u_1u_2)^{\alpha-1}(u_1 - u_1u_2)^{\beta-1} e^{-[u_1u_2+(u_1-u_1u_2)]} |-u_1|
\end{aligned}
$$

After some algebraic simplification, we get

$$
f_{U_1,U_2}(u_1,u_2) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} u_2^{\alpha-1}(1-u_2)^{\beta-1} u_1^{\alpha+\beta-1} e^{-u_1}, \ u_1 > 0, 0 < u_2 < 1 \\ 0, \ \text{elsewhere} \end{cases}
$$

b. To determine the marginal density of U_1, we integrate the joint PDF $f_{U_1,U_2}(u_1,u_2)$ with respect to u_2. That us,

$$
\begin{aligned}
f_{U_1}(u_1) &= \int_0^1 f_{U_1,U_2}(u_1,u_2)\, du_2 \\
&= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} u_2^{\alpha-1}(1-u_2)^{\beta-1} u_1^{\alpha+\beta-1} e^{-u_1}\, du_2 \\
&\vdots \\
&= \frac{1}{\Gamma\alpha+\beta} u_1^{\alpha+\beta-1} e^{-u_1}
\end{aligned}
$$

Summarizing,

$$f_{U_1}(u_1) = \begin{cases} \frac{1}{\Gamma\alpha+\beta} u_1^{\alpha+\beta-1} e^{-u_1},\ u_1 > 0 \\ 0,\ \text{elsewhere} \end{cases}$$

This PDF resembles that of $Gamma(\alpha+\beta, 1)$, that is, $U_1 \sim Gamma(\alpha+\beta, 1)$.

c. Left as classroom exercise!

Suppose $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF $f_{Y_1,Y_2}(y_1, y_2)$ and suppose we would like to find the distribution of a single random variable

$$U_1 = g_1(Y_1, Y_2)$$

Even if there is no U_2 we can still use the bivariate transformation technique by defining a *dummy* variable $U_2 = g_2(Y_1, Y_2)$ that is of no interest to us. While the choice of the dummy variable U_2 is arbitrary, there are certainly bad choices, hence, we suggest to use a simple and easy one: $U_2 = g_2(Y_1, Y_2) = Y_2$.

Example 1.4.2

Suppose Y_1 and Y_2 are random variables with joint PDF

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2,\ 0 < y_1 < y_2 < 1 \\ 0,\ \text{elsewhere} \end{cases}$$

Find the distribution of $U_1 = \frac{Y_1}{Y_2}$.

SOLUTION: Left as a classroom exercise!