

Lesson 1.1

The Cumulative Distribution Function Technique

Introduction

The objective of statistics is to make inferences about a population based on information contained in a sample taken from that population. The quantities used to estimate population parameters or to make decisions about a population are functions of the n random observations that appear in a sample.

To illustrate, consider the problem of estimating a population mean, μ . Intuitively, we draw a random sample of n observations, y_1, y_2, \dots, y_n , from the population and use the sample mean

$$\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$$

to estimate μ .

If the probability distribution of the Y_i 's is known what would be the distribution of the sample mean \bar{Y} . If Y_i 's are random variables then \bar{Y} is likewise a random variable.

In this module, we shall study methods in finding the distribution of functions of random variables. These methods include:

1. Cumulative Distribution Function (CDF) Technique
2. Transformation Technique
3. Moment Generation Function (MGF) Technique

The Cumulative Distribution Function (CDF) Technique

Suppose Y is a continuous random variable with cumulative distribution function $F_Y(y) = P(Y \leq y)$. The CDF technique is useful if $F_Y(y)$ has a closed form. The steps in the CDF method are:

1. Find a closed form of the CDF of Y .
2. Find the support of $U = g(Y)$.
3. Write $F_U(u) = P(U \leq u)$ in terms of $F_Y(y)$.
4. Differentiate $F_U(u)$ to obtain the PDF of U .

Example 1.1.1

Suppose $Y \sim U(0, 1)$. Find the distribution of $U = g(Y) = -\ln Y$.

SOLUTION

Verify that the CDF of Y is given by

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ y, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases}$$

Since the support for $Y \sim U(0, 1)$ is $R_Y = \{y : 0 < y < 1\}$, thus, it follows that the support for $U = -\ln Y$ is $R_U = \{u : u > 0\}$.

Now, we are going to find the CDF of U .

$$\begin{aligned} F_U(u) &= P(U \leq u) \\ &= P(-\ln Y \leq u) \\ &= P(\ln Y > -u) \\ &= P(Y > e^{-u}) \\ &= 1 - P(Y \leq e^{-u}) \\ &= 1 - F_Y(e^{-u}) \end{aligned}$$

Since $F_Y(y) = y$ for $0 < y < 1$, hence, for $u > 0$ we have

$$\begin{aligned} F_U(u) &= 1 - F_Y(e^{-u}) \\ &= 1 - e^{-u} \end{aligned}$$

To get the PDF of U , we have

$$\begin{aligned} f_U(u) &= \frac{d}{du} F_U(u) \\ &= \frac{d}{du} [1 - e^{-u}] \\ &= e^{-u} \end{aligned}$$

Therefore,

$$f_U(u) = \begin{cases} e^{-u}, & u > 0 \\ 0, & \text{elsewhere} \end{cases}$$

which shows that $U \sim \text{Exp}(1)$ which is an exponential distribution with mean $\beta = 1$.

Example 1.1.2

A process for refining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y , is a random variable because of machine breakdowns and other slowdowns. Suppose that Y has density function given by

$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

The company is paid at the rate of \$300 per ton for the refined sugar, but it also has a fixed overhead cost of \$100 per day. Thus the daily profit, in hundreds of dollars, is $U = 3Y - 1$. Find the probability density function for U .

SOLUTION

Verify that the CDF of Y is given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y^2, & 0 \leq y < 1 \\ 1, & y \geq 1 \end{cases}$$

Note that the support of $U = 3Y - 1$ is $R_U = \{u : -1 \leq u \leq 2\}$.

The CDF of U is obtained as follows:

$$\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(3Y - 1 \leq u) \\
&= P\left(Y \leq \frac{u+1}{3}\right) \\
&= F_Y\left(\frac{u+1}{3}\right)
\end{aligned}$$

That is, for $-1 \leq u \leq 2$,

$$\begin{aligned}
F_U(u) &= F_Y\left(\frac{u+1}{3}\right) \\
&= \left(\frac{u+1}{3}\right)^2
\end{aligned}$$

And the density function for U is

$$\begin{aligned}
f_U(u) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} \left[\left(\frac{u+1}{3}\right)^2 \right] \\
&= \frac{2}{9}(u+1)
\end{aligned}$$

Therefore,

$$f_U(u) = \begin{cases} \frac{2}{9}(u+1), & -1 \leq u \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Example 1.1.3

Let Y be a random variable with PDF

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2, & -1 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the PDF of $U = 3 - Y$.

SOLUTION [Left as a classroom exercise!]