

Stat 122 (Mathematical Statistics II)

Lesson 1.2: The Transformation Technique

Introduction

The transformation method for finding the probability distribution of a function of random variables is an offshoot of the distribution function method. Through the distribution function approach, we can arrive at a simple method of writing down the density function of $U = g(Y)$, provided that $g(y)$ is either decreasing or increasing and one-to-one.

Suppose Y is a continuous random variable with CDF $F_Y(y)$ and support R_Y , and let $U = g(Y)$, where $g : R_Y \rightarrow \mathbb{R}$ is a continuous, **one-to-one** function defined over R_Y . Examples of such functions include continuous (strictly) **increasing/decreasing** functions. Recall from calculus that if g is one-to-one, it has a unique inverse g^{-1} . Also, recall that if g is increasing (decreasing), then so is g^{-1} .

Suppose that $g(y)$ is an increasing function of y defined over R_Y . Then it follows that $u = g(y) \iff g^{-1}(u) = y$ and

$$\begin{aligned} F_U(u) &= P[g(Y) \leq u] \\ &= P[Y \leq g^{-1}(u)] \\ &= F_Y[g^{-1}(u)] \end{aligned}$$

Differentiating $F_U(u)$ with respect to u , we get

$$\begin{aligned} f_U(u) &= \frac{d}{du} F_U(u) \\ &= \frac{d}{du} F_Y[g^{-1}(u)] \\ &= f_Y[g^{-1}(u)] \times \frac{d}{du} [g^{-1}(u)], \text{ by Chain Rule} \end{aligned}$$

Note that if g is increasing so is g^{-1} , thus, $\frac{d}{du} [g^{-1}(u)] > 0$.

Meanwhile, if $g(y)$ is strictly decreasing, then $F_U(u) = 1 - F_Y[g^{-1}(u)]$ and $\frac{d}{du} [g^{-1}(u)] < 0$, which gives

$$\begin{aligned}
f_U(U) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} \{1 - F_Y[g^{-1}(u)]\} \\
&= -f_Y[g^{-1}(u)] \times \frac{d}{du} [g^{-1}(u)]
\end{aligned}$$

Combining both cases, we have shown that the PDF of U , where nonzero, is given by

$$f_U(u) = f_Y[g^{-1}(u)] \times \left| \frac{d}{du} [g^{-1}(u)] \right| \quad (1)$$

The Transformation Technique

The steps to be followed in using the transformation method are:

1. Verify that the transformation $u = g(y)$ is continuous and one-to-one over R_Y .
2. Find the support of U .
3. Find the inverse transformation $y = g^{-1}(u)$ and its derivative with respect to u .
4. Use (1) to obtain $f_U(u)$.

Example 1.2.1

Let Y have the probability density function given by

$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the distribution of $U = 3Y - 1$.

SOLUTION

It is clear that $U = 3Y - 1$ is continuous (increasing) and one-to-one in $R_Y = \{y : 0 \leq y \leq 1\}$. If $y \in (0, 1)$, then $u \in (-1, 2)$, in other words, $R_U = \{u : -1 \leq u \leq 2\}$.

If $u = 3y - 1$, then $y = g^{-1}(u) = \frac{u+1}{3}$ and

$$\begin{aligned}
\frac{d}{du} g^{-1}(u) &= \frac{d}{du} \left[\frac{u+1}{3} \right] \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, using (1) the PDF of U is given by

$$\begin{aligned}
f_U(u) &= f_Y[g^{-1}(u)] \times \left| \frac{d}{du}[g^{-1}(u)] \right| \\
&= 2 \times \frac{u+1}{3} \times \left| \frac{1}{3} \right| \\
&= \frac{2}{9}(u+1)
\end{aligned}$$

That is,

$$f_U(u) = \begin{cases} \frac{2}{9}(u+1), & -1 \leq u \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Example 1.2.2

Suppose that Y has a $Beta(6, 2)$ distribution. Find the distribution of $U = 1 - Y$.

SOLUTION

Since $Y \sim Beta(6, 2)$, then

$$f_Y(y) = \begin{cases} 42y^5(1-y), & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

It is clear that $u = g(y) = 1 - y$ is continuous (decreasing) and one-to-one over $R_Y = \{y : 0 < y < 1\}$. In addition, the support of U is given by $R_U = \{u : 0 < u < 1\}$.

The inverse transformation is

$$g(y) = u = 1 - y \iff y = g^{-1}(u) = 1 - u$$

and

$$\begin{aligned}
\frac{d}{du}g^{-1}(u) &= \frac{d}{du}[1 - u] \\
&= -1
\end{aligned}$$

Thus, for $0 < u < 1$,

$$\begin{aligned}
f_U(u) &= f_Y[g^{-1}(u)] \times \left| \frac{d}{du}[g^{-1}(u)] \right| \\
&= 42 \times (1-u)^5 [1 - (1-u)] \times |-1| \\
&= 42u(1-u)^5
\end{aligned}$$

Therefore,

$$f_Y(y) = \begin{cases} 42u(1-u)^5, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

which means that $U \sim \text{Beta}(2, 6)$.

QUESTION: What happens if $u = g(y)$ is not a one-to-one transformation? In this case we can still use the method of transformation but we have to “*break up*” the transformation $g : R_Y \rightarrow R_U$ into disjoint regions where g is one-to-one.

Suppose Y is a continuous random variable with PDF $f_Y(y)$ and that $U = g(Y)$, not necessarily a one-to-one (but continuous) function of y over R_Y . Furthermore, suppose that we can partition R_Y into a finite collection of sets, say B_1, B_2, \dots, B_k , where $P(Y_i \in B_i) > 0$ for all i , and $f_Y(y)$ is continuous on each B_i . Furthermore, suppose that there exists functions $g_1(y), g_2(y), \dots, g_k(y)$ such that $g_i(y)$ is defined on $B_i, i = 1, 2, \dots, k$ and the $g_i(y)$ satisfy

- (a) $g(y) = g_i(y), \forall y \in B_i$
- (b) $g_i(y)$ is monotone on B_i , so that $g_i^{-1}(\cdot)$ exists uniquely on B_i .

Then, the PDF of U is given by

$$f_U(u) = \begin{cases} \sum_{i=1}^k f_Y[g_i^{-1}(u)] \left| \frac{d}{du} [g_i^{-1}(u)] \right|, & u \in R_U \\ 0, & \text{elsewhere} \end{cases}$$

Example 1.2.3

Suppose that $Y \sim N(0, 1)$, that is Y has a standard normal distribution. Find the distribution of $U = Y^2$.

SOLUTION

The PDF of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, & -\infty < y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

The transformation $U = g(Y) = Y^2$ is not one-to-one over $R_Y = \{y : -\infty < y < \infty\}$, but is one-to-one on $B_1 = (-\infty, 0)$ and $B_2 = [0, \infty)$. Also notice that $g(y)$ is increasing on B_1 and decreasing on B_2 .

Clearly, $u = g(y) = y^2 > 0$, thus, the support of U is $R_U = \{u : u > 0\}$.

On B_1 , define $g_1(y) = y^2 = u$, hence, the inverse transformation is $g_1^{-1}(u) = y = -\sqrt{u}$. While on B_2 , let $g_2(y) = y^2 = u$, hence, the inverse transformation is $g_2^{-1}(u) = y = \sqrt{u}$.

Also notice that on both sets B_1 and B_2 ,

$$\left| \frac{d}{du} g_i^{-1}(u) \right| = \frac{1}{2\sqrt{u}}$$

For $u > 0$, the PDF of U is given by

$$\begin{aligned}
f_U(u) &= f_Y[g_1^{-1}(u)] \left| \frac{d}{du} g_1^{-1}(u) \right| + f_Y[g_2^{-1}(u)] \left| \frac{d}{du} g_2^{-1}(u) \right| \\
&= f_Y[-\sqrt{u}] \times \frac{1}{2\sqrt{u}} + f_Y[\sqrt{u}] \times \frac{1}{2\sqrt{u}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{u})^2/2} \times \frac{1}{2\sqrt{u}} + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{u})^2/2} \times \frac{1}{2\sqrt{u}} \\
&= \frac{2}{\sqrt{2\pi}} e^{-u/2} \times \frac{1}{2\sqrt{u}} \\
&= \frac{1}{\sqrt{2\pi}} u^{\frac{1}{2}-1} e^{-u^2/2} \\
&= \frac{1}{\Gamma(1/2)2^{1/2}} u^{\frac{1}{2}-1} e^{-u^2/2}, \text{ since } \Gamma(1/2) = \sqrt{2\pi}
\end{aligned}$$

This density resembles the density of $Gamma(1/2, 2)$ which is the Chi-square distribution with 1 degree of freedom. Therefore, $U \sim \chi^2(1)$.

Example 1.2.4

The waiting time Y until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 5 days. The cost of this delay is given by $U = 2Y^2 + 3$. Find the probability density function for U .

SOLUTION [Left as classroom exercise!]