Lesson 1.3

The Moment Generating Function Technique

Introduction

The moment-generating function method for finding the probability distribution of a function of random variables $Y_1, Y_2, \ldots, Y_n$ is based on the following uniqueness theorem.

Theorem (Uniqueness Property of MGFs)

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Remark:

The MGF completely determines the distribution of a random variable. Once the moment-generating function for $U = g(Y)$ has been found, it is compared with the moment-generating function for random variables with well-known distributions.

The Moment Generating Function Technique

The steps involve in using the MGF techniques are:

1. Derive the MGF of $U = g(Y)$, which is defined as $m_U(t) = E[e^{tU}]$.
2. Try to compare $m_U(t)$ with the MGF of existing probability distributions.
3. Because of the Uniqueness Theorem, U must have the distribution as the one whose MGF you have recognized in (2).

Example 1.3.1

Suppose $Y \sim Gamma(\alpha, \beta)$. Derive the distribution of $U = g(Y) = 2Y/\beta$ using the MGF technique.

SOLUTION

Since $Y \sim Gamma(\alpha, \beta)$, then

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$$

Now,

$$\begin{aligned}
m_U(t) &= E[e^{tU}] \\
&= E[e^{t(2Y/\beta)}] \\
&= E[e^{(2t/\beta)Y}] \\
&= E[e^{t'Y}], \text{ where: } t' = \frac{2t}{\beta} \\
&= m_Y(t') \\
&= \left(\frac{1}{1-\beta t'}\right)^{\alpha} \\
&= \left(\frac{1}{1-\beta(\frac{2t}{\beta})}\right)^{\alpha} \\
&= \left(\frac{1}{1-2t}\right)^{\alpha}
\end{aligned}$$

The resulting MGF is the MGF of a Chi-square distribution with degrees of freedom equal to 2α. Therefore, $U \sim \chi^2(2\alpha)$.

Remark:

The MGF techniques i very useful when we have a independent random variables $Y_1, Y_2, \dots, Y_n$ and we wanted to derive the distribution of the sum $U = Y_1 + Y_2 + \cdots + Y_n$, for example.

$$
\begin{aligned}
m_U(t) &= E[e^{tU}] \\
&= E[e^{t(Y_1+Y_2+\cdots+Y_n)}] \\
&= E[e^{tY_1+tY_2+\cdots+tY_n}] \\
&= E[e^{tY_1} \times e^{tY_2} \times \cdots \times e^{tY_n}] \\
&= E[e^{tY_1}] \times E[e^{tY_2}] \times \cdots \times E[e^{tY_n}], \text{because of independence} \\
&= m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t) \\
&= [m_Y(t)]^n, \text{ if iid}
\end{aligned}
$$

Example 1.3.2

Suppose we have an iid sample $Y_1, Y_2, \dots, Y_n$ from a Bernoulli distribution. What is the distribution of the sum $U = Y_1 + Y_2 + \dots + Y_n$?

SOLUTION

Recall that the MGF of Bernoulli distributed random variable Y is given by

$$m_Y(t) = q + pe^t, \text{ where: } q = 1 - p$$

Using the above result, the MGF of U is

$$m_Y(t) = [m_Y(t)]^n = [q + pe^t]^n$$

The resulting MGF resembles the MGF of a binomial distribution with parameter p. In other words, $U \sim Bi(n, p)$.

Example 1.3.3

Suppose we have a random sample $Y_1, Y_2, \dots, Y_n$ from an exponential distribution with mean β. What is the distribution of the sum $U = Y_1 + Y_2 + \dots + Y_n$?

SOLUTION [Left as a classroom exercise!]