Lesson 1.2

The Transformation Technique

Introduction

The transformation method for finding the probability distribution of a function of random variables is an offshoot of the distribution function method. Through the distribution function approach, we can arrive at a simple method of writing down the density function of $U = g(Y)$, provided that $g(y)$ is either decreasing or increasing and one-to-one.

Suppose Y is a continuous random variable with CDF $F_Y(y)$ and support R_Y, and let $U = g(Y)$, where $g : R_Y \to \mathbb{R}$ is a continuous, **one-to-one** function defined over R_Y. Examples of scuh functions include continuous (strictly) **increasing/decreasing** functions. Recall from calculus that if g is one-to-one, it has a unique inverse g^{-1}. Also, recall that if g is increasing (decreasing), then so is g^{-1}.

Suppose that $g(y)$ is an increasing function of y defined over R_Y. Then it follows that $u = g(y) \iff g^{-1}(u) = y$ and

$$\begin{aligned} F_U(u) &= P[g(Y) \leq u] \\ &= P[Y \leq g^{-1}(u)] \\ &= F_Y[g^{-1}(u)] \end{aligned}$$

Differentiating $F_U(u)$ with respect to u, we get

$$\begin{aligned} f_U(u) &= \frac{d}{du} F_U(u) \\ &= \frac{d}{du} F_Y[g^{-1}(u)] \\ &= f_Y[g^{-1}(u)] \times \frac{d}{du}[g^{-1}(u)], \text{ by Chain Rule} \end{aligned}$$

Note that if g is increasing so is g^{-1}, thus, $\frac{d}{du}[g^{-1}(u)] > 0$.

Meanwhile, if $g(y)$ is strictly decreasing, then $F_U(u) = 1 - F_Y[g^{-1}(u)]$ and $\frac{d}{du}[g^{-1}(u)] < 0$, which gives

$$\begin{aligned} f_U(U) &= \frac{d}{du} F_U(u) \\ &= \frac{d}{du}\{1 - F_Y[g^{-1}(u)]\} \\ &= -f_Y[g^{-1}(u)] \times \frac{d}{du}[g^{-1}(u)] \end{aligned}$$

Combining both cases, we have shown taht the PDF of U, where nonzero, is given by

$$f_U(u) = f_Y[g^{-1}(u)] \times \left| \frac{d}{du}[g^{-1}(u)] \right| \tag{1}$$

The Transformation Technique

The steps to be followed in using the transformation method are:

1. Verify that the transformation $u = g(y)$ is continuous and one-to-one over R_Y.
2. Find the support of U.
3. Find the inverse transformation $y = g^{-1}(u)$ and its derivative with respect to u.
4. Use (1) to obtain $f_U(u)$.

Example 1.2.1

Let Y have the probability density function given by

$$f_Y(y) = \begin{cases} 2y, \ 0 \le y \le 1 \\ 0, \text{ elsewhere} \end{cases}$$

Find the distribution of $U = 3Y - 1$.

SOLUTION

It is clear that $U = 3Y - 1$ is continuous (increasing) and one-to-one in $R_Y = \{y : 0 \le y \le 1\}$. If $y \in (0, 1)$, then $u \in (-1, 2)$, in other words, $R_U = \{u : -1 \le u \le 2\}$.

If $u = 3y - 1$, then $y = g^{-1}(u) = \frac{u+1}{3}$ and

$$\begin{aligned}\frac{d}{du}g^{-1}(u) &= \frac{d}{du}\left[\frac{u+1}{3}\right]\\ &= \frac{1}{3}\end{aligned}$$

Therefore, usisng (1) the PDF of U is given by

$$\begin{aligned}f_U(u) &= f_Y[g^{-1}(u)] \times \left|\frac{d}{du}[g^{-1}(u)]\right|\\ &= 2 \times \frac{u+1}{3} \times \left|\frac{1}{3}\right|\\ &= \frac{2}{9}(u+1)\end{aligned}$$

That is,

$$f_U(u) = \begin{cases}\frac{2}{9}(u+1), & -1 \le u \le 2\\ 0, \text{ elsewhere}\end{cases}$$

Example 1.2.2

Suppose that Y has a $Beta(6,2)$ distribution. Find the distribution of $U = 1 - Y$.

SOLUTION

Since $Y \sim Beta(6,2)$, then

$$f_Y(y) = \begin{cases}42y^5(1-y), \; 0 < y < 1\\ 0, \text{ elsewhere}\end{cases}$$

It is clear that $u = g(y) = 1 - y$ is continuous (decreasing) and one-to-one over $R_Y = \{y : 0 < y < 1\}$. In addition, the support of U is given by $R_U = \{u : 0 < u < 1\}$.

The inverse transformation is

$$g(y) = u = 1 - y \iff y = g^{-1}(u) = 1 - u$$

and

$$\begin{aligned}\frac{d}{du}g^{-1}(u) &= \frac{d}{du}[1-u]\\ &= -1\end{aligned}$$

Thus, for $0 < u < 1$,

$$
\begin{aligned}
f_U(u) &= f_Y[g^{-1}(u)] \times \left|\frac{d}{du}[g^{-1}(u)]\right| \\
&= 42 \times (1-u)^5[1-(1-u)] \times |-1| \\
&= 42u(1-u)^5
\end{aligned}
$$

Therefore,

$$
f_Y(y) = \begin{cases} 42u(1-u)^5,\ 0 < y < 1 \\ 0, \text{ elsewhere} \end{cases}
$$

which means that $U \sim Beta(2, 6)$.

QUESTION: What happens if $u = g(y)$ is not a one-to-one transformation? In this case we can still use the method of transformation but we have to *"break up"* the transformation $g : R_Y \to R_U$ into disjoint regions where g is one-to-one.

Suppose Y is a continuous random variable with PDF $f_Y(y)$ and that $U = g(Y)$, not necessarily a one-to-one (but continuous) function of y over R_Y. Furthermore, suppose that we can partition R_Y into a finite collection of sets, say $B_1, B_2, \ldots, B_k$, where $P(Y_i \in B_i) > 0$ for all i, and $f_Y(y_)$ is continuous on each B_i. Furthermore, suppose that there exists functions $g_1(y), g_2(y), \ldots, g_k(y)$ such that $g_i(y)$ is defined on $B_i, i = 1, 2, \ldots, k$ and the $g_i(y)$ satisfy

(a) $g(y) = g_i(y),\ \forall y \in B_i$
(b) $g_i(y)$ is monotone on B_i, so that $g_i^{-1}(.)$ exists uniquely on B_i.

Then, the PDf of U is given by

$$
f_U(u) = \begin{cases} \sum_{i=1}^{k} f_Y[g_i^{-1}(u)]\left|\frac{d}{du}[g_i^{-1}(u)]\right|, u \in R_U \\ 0, \text{ elsewhere} \end{cases}
$$

Example 1.2.3

Suppose that $Y \sim N(0,1)$, that is Y has a standard normal distribution. Find the distribution of $U = Y^2$.

SOLUTION

The PDF of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \ -\infty < y < \infty \\ 0, \text{ elsewhere} \end{cases}$$

The transformation $U = g(Y) = Y^2$ is not one-to-one over $R_Y = \{y : -\infty < y < \infty\}$, but is one-to-one on $B_1 = (-\infty, 0)$ and $B_2 = [0, \infty)$. Also notice that $g(y)$ is increasing on B_1 and decreasing on B_2.

Clearly, $u = g(y) = y^2 > 0$, thus, the support of U is $R_U = \{u : u > 0\}$.

On B_1, $g_1(y) = y^2 = u$, hence, the inverse transformation is $g_1^{-1}(u) = y = -\sqrt{u}$. While on B_2, $g_2(y) = y^2 = u$, hence, the inverse transformation is $g_2^{-1}(u) = y = \sqrt{u}$.

Also notice that on both sets B_1 and B_2,

$$\Big|\frac{d}{du} g_i^{-1}(u)\Big| = \frac{1}{2\sqrt{u}}$$

For $u > 0$, the PDF of U is given by

$$\begin{aligned} f_U(u) &= f_Y[g_1^{-1}(u)]\Big|\frac{d}{du} g_1^{-1}(u)\Big| + f_Y[g_2^{-1}(u)]\Big|\frac{d}{du} g_2^{-1}(u)\Big| \\ &= f_Y[-\sqrt{u}] \times \frac{1}{2\sqrt{u}} + f_Y[\sqrt{u}] \times \frac{1}{2\sqrt{u}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{u})^2/2} \times \frac{1}{2\sqrt{u}} + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{u})^2/2} \times \frac{1}{2\sqrt{u}} \\ &= \frac{2}{\sqrt{2\pi}} e^{-u/2} \times \frac{1}{2\sqrt{u}} \\ &= \frac{1}{\sqrt{2\pi}} u^{\frac{1}{2}-1} e^{-u^2/2} \\ &= \frac{1}{\Gamma(1/2)2^{1/2}} u^{\frac{1}{2}-1} e^{-u^2/2}, \text{since}\, \Gamma(1/2) = \sqrt{2\pi} \end{aligned}$$

This density resembles the density of $Gamma(1/2, 2)$ which is the Chi-square distribution with 1 degree of freedom. Therefore, $U \sim \chi^2(1)$.

Example 1.2.4

The waiting time Y until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 5 days. The cost of this delay is given by $U = 2Y^2 + 3$. Find the probability density function for U.

SOLUTION [Left as classroom exercise!]