Sampling Distribution

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2025-02-25

Let $Y \sim N(\mu, \sigma^2)$.

$$m_Y(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

Given an iid sample from $N(\mu, \sigma^2)$.

$$\begin{aligned}
m_{\overline{Y}}(t) &= E[e^{t\overline{Y}}] \\
&= E\Big[e^{t \times \frac{1}{n} \sum_{i=1}^{n} y_i}\Big] \\
&= E\Big[e^{\frac{t}{n}(Y_1 + Y_2 + \cdots + Y_n)}\Big] \\
&= E\Big[e^{(\frac{t}{n}Y_1 + \frac{t}{n}Y_2 + \cdots + \frac{t}{n}Y_n)}\Big] \\
&= E\Big[e^{\frac{t}{n}Y_1}\Big] \times E\Big[e^{\frac{t}{n}Y_2}\Big] \times \cdots \times E\Big[e^{\frac{t}{n}Y_n}\Big] \\
&= \Big(E\Big[e^{\frac{t}{n}Y}\Big]\Big)^n \\
&= \Big[e^{\mu t/n + \frac{(t/n)^2\sigma^2}{2}}\Big]^n \\
&= \Big[e^{n(\mu t/n) + n\frac{(t/n)^2\sigma^2}{2}}\Big] \\
&= e^{\mu t + \frac{t^2\sigma^2/n}{2}}
\end{aligned}$$

Therefore,

$$\overline{Y} \sim N(\mu, \sigma^2/n)$$

$$Y \sim N(48, 100)$$

a.

$$\begin{aligned}
P(Y > 50) &= 1 - P(Y \leq 50) \\
&= 1 - P(Z \leq \frac{50-48}{10}) \\
&= 1 - P(Z \leq 0.2) \\
&= 1 - 0.5793 \\
&= 0.4207
\end{aligned}$$

b.

$$
\begin{aligned}
P(\overline{Y} > 50) &= 1 - P(\overline{Y} \leq 50) \\
&= 1 - P(Z \leq \frac{50 - 48}{10/\sqrt{100}}) \\
&= 1 - P(Z \leq 2) \\
&= 1 - 0.9772 \\
&= 0.0228
\end{aligned}
$$

c.

$$
\begin{aligned}
P(\overline{Y} > 50) < 0.01 &\iff 1 - P(\overline{Y} \leq 50) < 0.01 \\
&\iff 1 - 0.01 < P(\overline{Y} \leq 50) \\
&\iff 0.99 < P\left(Z \leq \frac{50 - 48}{10/\sqrt{n}}\right) \\
&\iff P\left(Z \leq \frac{50 - 48}{10/\sqrt{n}}\right) \geq 0.99 \\
&\iff P(Z \leq 2.33) \geq 0.99
\end{aligned}
$$

$$
\begin{aligned}
\frac{50 - 48}{10/\sqrt{n}} = 2.333 &\iff \sqrt{n} = \frac{2.33 \times 10}{2} \\
&\iff \sqrt{n} = 11.65 \\
&\iff n = 11.65^2 \\
&\iff n = 135.7225 \approx 136
\end{aligned}
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