Lesson 1.4

Bivariate Transformation

Introduction

So far in this module, we have talked about transformations involving a single random variable Y. It is sometimes of interest to consider **bivariate transformation** such as

$$U_1 = g_1(Y_1, Y_2)$$
$$U_2 = g_1(Y_1, Y_2)$$

To discuss such transformations, we will assume that Y_1 and Y_2 are jointly **continuous** random variables. Furthermore, for the following methods to apply, the transformation needs to be **one-to-one**. We start with the joint distribution of $\mathbf{Y} = (Y_1, Y_2)$. Our first goal is to find the joint distribution of $\mathbf{U} = (U_1, U_2)$.

Bivariate Transformation

Suppose that $\mathbf{Y} = (Y_1, Y_2)$ is a continuous random vector with joint PDF $f_{Y_1,Y_2}(y_1, y_2)$. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous one-to-one vector-valued mapping from R_{Y_1,Y_2} to R_{U_1,U_2}, where $U_1 = g_1(Y_1, Y_2)$ and $U_2 = g_2(Y_1, Y_2)$, and where R_{Y_1,Y_2} and R_{U_1,U_2} denote two-dimensional supports of $\mathbf{Y} = (Y_1, Y_2)$ and $\mathbf{U} = (U_1, U_2)$, respectively.

If $g_1^{-1}(u_1, u_2)$ and $g_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to both u_1 and u_2 and the Jacobian, J, where "det" denote "determinant",

$$J = det \begin{bmatrix} \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial g_2^{-1}(u_1,u_2)}{\partial u_2} \end{bmatrix} \neq 0$$

then

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} f_{Y_1,Y_2}[g_1^{-1}(u_1,u_2), g_2^{-1}(u_1,u_2)] \times |J|, \ (u_1,u_2) \in R_{U_1,U_2} \\ 0, \text{ elsewhere} \end{cases} \tag{1}$$

where $|J|$ denotes the avbsolute value of J.

To summarize, the steps involved in bivariate trabsformation are:

1. Find the joint distribution $f_{Y_1,Y_2}(y_1,y_2)$ of Y_1 and Y_2.
2. Find the R_{U_1,U_2} the support of $\mathbf{U} = (U_1,U_2)$.
3. Find the inverse transformations $y_1 = g_1^{-1}(u_1,u_2)$ and $y_2 = g_2^{-1}(u_1,u_2)$.
4. Find the Jacobian, J, of the inverse transformation.
5. Use (1) to find $f_{U_1,U_2}(u_1,u_2)$ the joint distribution of U_1 and U_2.