Stat 131 (Mathematical Statistics III)

Lesson 3.3. Z test for the difference of two population means

Learning Outcomes

At the end of the lesson, students should be able to

1. derive the statistic for testing hypothesis on the difference of two population means; and
2. conduct the z test for the difference of two population means.

Introduction

In most applications of hypothesis testing, we are interested in comparing the parameters of two populations, such the mean or proportion. In this lesson, we extend the Z test to test hypothesis about the difference of two population means when the sample sizes are sufficiently large.

The Z test for the difference of two population means

Suppose that we have two independent samples; that is,

$$Y_{11}, Y_{12}, \cdots, Y_{1n_1} \overset{iid}{\sim} f_Y(y; \mu_1, \sigma_1^2)$$
$$Y_{21}, Y_{22}, \cdots, Y_{2n_2} \overset{iid}{\sim} f_Y(y; \mu_2, \sigma_2^2)$$

and that interest lies in testing

$$H_0 : \mu_1 - \mu_2 = d_0$$

versus

$$H_1 : \mu_1 - \mu_2 \neq d_0$$

(or any other appropriate alternative), where d_0 is a known constant. Most often $d_0 = 0$, hence, the problem reduces to testing the equality of μ_1 and μ_2. In other words, we wanted to test

$$H_0 : \mu_1 = \mu_2 \text{ versus } H_1 : \mu_1 \neq \mu_2$$

In this situation, we identify

$$\begin{aligned}
\theta &= \mu_1 - \mu_2 \\
\hat{\theta} &= \overline{Y}_1 - \overline{Y}_2 \\
\sigma_{\hat{\theta}} &= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \\
\hat{\sigma}_{\hat{\theta}} &= \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}
\end{aligned}$$

If σ_1^2 and σ_2^2 are both known (which would be *unlikely*), then we would use the test statistic

$$Z = \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Otherwise, we use

$$Z = \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

It can be shown via the MGF technique that both statistics have standard normal distributions.

Example 3.3.1

A psychological study was conducted to compare the reaction times of men and women to a stimulus. Independent random samples of 50 men and 50 women were employed in the experiment. The results are shown in the table below. Do the data present sufficient evidence to suggest a difference between true mean reaction times for men and women?

	Men	Women
n	50	50
Sample Mean	3.6	3.8
Sample Variance	0.18	0.14

SOLUTION

Let μ_1 and μ_2 be the true mean reaction times for men and women, respectively.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Test statistic

Notice that the population variances are unknown, hence, the appropriate test statistic is

$$Z = \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Given the summary statistics in the above table, the value of the test statistic is

$$\begin{aligned} Z &= \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \\ &= \frac{3.6 - 3.8}{\sqrt{\frac{0.18}{50} + \frac{0.14}{50}}} \\ &= -2.5 \end{aligned}$$

Rejection Region

For $\alpha = 0.05$, the rejection region would be $RR = \{z : |z| \geq z_{0.025} = 1.96\}$. In addition, the p-value associated with the test statistic is $2 \times P(Z \geq |-2.5|) = 0.0124$.

Decision on H_0

We reject H_0 since $|-2.5| > 1.96$; also, $p - value < \alpha = 0.05$.

Conclusion

At $\alpha = 0.05$ there is sufficient evidence to conclude that mean reaction times differ for men and women.

Example 3.3.2

An experiment is performed to determine whether the average nicotine content of brand A of cigarette exceeds that of brand B by 0.20 milligram. If a random sample of 50 cigarettes of brand A had an average nicotine content of 2.61 milligrams with a standard deviation of 0.12 milligram, whereas a random sample of 40 cigarettes of brand B had an average nicotine content of 2.38 milligrams with a standard deviation of 0.14 milligram. Formulate the appropriate hypotheses and test these at the 0.05 level of significance.

SOLUTION

Let μ_1 and μ_2 be the true mean nicotine contents of brands A and B of cigarette, respectively. We are interested in testing the following hypotheses,

$$H_0 : \mu_1 - \mu_2 = 0.2$$
$$H_1 : \mu_1 - \mu_2 > 0.2$$

Test statistic

Similar to *Example 3.3.1, the population variances are unknown, hence, the test statistic is

$$Z = \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Based on the data from the two samples, the value of the test statistic is

$$\begin{aligned} Z &= \frac{(\overline{Y}_1 - \overline{Y}_2) - d_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \\ &= \frac{(2.61 - 2.38) - 0.12}{\sqrt{\frac{0.12^2}{50} + \frac{0.14^2}{40}}} \\ &\approx 1.08 \end{aligned}$$

Rejection Region

At $\alpha = 0.05$, the rejection region would be $RR = \{z : z > z_{0.05} = 1.645\}$, and the corresponding p-value is $P(Z \geq 1.08) = 0.1401$.

Decision on H_0

We fail to reject H_0 since the computed value of Z does not fall in the rejection and the p-value is greater than the 0.05 significance level.

Conclusion

At the 0.05 significance level, there is no sufficient data to support the claim that the mean nicotine contents of two kinds of cigarettes differ by more than 0.20 milligrams.

Learning Task

Instruction: Answer the following as indicated.

To compare customer satisfaction levels of two competing cable television companies, 74 customers of Company 1 and 75 customers of Company 2 were randomly selected and were asked to rate their cable companies on a ten-point scale, with 1 being least satisfied and 10 most satisfied. The survey results are summarized in the following table:

	Company 1	Company 2
Mean	7.51	7.30
Standard deviation	0.55	0.52

Is the data sufficient to conclude that Company 1 has higher mean satisfaction rating than Company 2? Use $\alpha = 0.01$.