# Stat 131 (Mathematical Statistics III)

Lesson 4.2. F test for the difference of two population variances

## Learning Outcomes

At the end of the lesson, students should be able to

1. explain the procedure for testing hypothesis about the difference of two population variances; and
2. apply the F test statistic to test hypothesis about the difference of two population variances.

## Discussion

Sometimes we wish to compare the variances of two normal distributions, particularly by testing to determine whether they are equal. These problems are encountered in comparing the precision of two measuring instruments, the variation in quality characteristics of a manufactured product, or the variation in scores for two testing procedures. Now, suppose $Y_{11}, Y_{12}, \cdots, Y_{1n_1}$ and $Y_{21}, Y_{22}, \cdots, Y_{2n_2}$ are independent random samples from normal distributions with unknown means $\mu_1$ and $\mu_2$, and unknown variances $\sigma_1^2$ and $\sigma_2^2$, respectively.

Suppose further that we want to test $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative $H_1 : \sigma_1^2 > \sigma_2^2$. When the null hypothesis is true, we say that the variances are homogeneous or equal. Because the sample variances $S_1^2$ and $S_2^2$ estimate the respective population variances, we reject $H_0$ in favor of $H_1$ if $S_1^2$ is much larger than $S_2^2$. That is, we use a rejection region of the form

$$RR = \left\{ \frac{S_1^2}{S_2^2} \geq k \right\}$$

where $k$ is so chosen so that the probability of Type I error is $\alpha$.

The appropriate value of $k$ depends on the probability distribution of the statistic $\frac{S_1^2}{S_2^2}$. From Stat 122, recall that

$$\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2(n_1-1)$$

and

$$\frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2(n_2-1)$$

Also the ratio of two independent Chi-square-distributed random variables divided by their respective degrees of freedom has an F distribution. Hence,

$$F = \frac{\left[\frac{(n_1-1)S_1^2}{\sigma_1^2}\right]/(n_1-1)}{\left[\frac{(n_2-1)S_2^2}{\sigma_2^2}\right]/(n_2-1)} = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} \sim F(n_1-1, n_2-1)$$

If the null hypothesis were true, that is $\sigma_1^2 = \sigma_2^2$, then it follows that

$$F = \frac{S_1^2}{S_2^2}$$

and the rejection region given earlier is equivalent to $RR = \{F \geq F_\alpha\}$, where $F_\alpha\}$ is the $\alpha$-upper quantile of the F distribution with numerator and denominator degrees of freedom $n_1 - 1$ and $n_2 - 1$, respectively. The $\alpha$-upper quantile of the F distribution can be obtained from an F table or using R.

Now, suppose want to test $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative $H_1 : \sigma_1^2 < \sigma_2^2$. We are at liberty to identify either population as population 1. That is, if the research hypothesis is that the variance of one population is larger than the variance of another population, we identify the population with the hypothesized larger variance as population 1, and proceed as before.

On the other hand, if we want to test $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative $H_1 : \sigma_1^2 \neq \sigma_2^2$, we form the test statistic as before. That is, we use the larger sample variance in the numerator and the smaller sample variance in the denominator to calculate the F statistic. The rejection region would be

$$RR = \{F \geq F_{\alpha/2}\}$$

where $\alpha$-upper quantile of the F distribution with degrees of freedom $\nu_L$ and $\nu_S$. The degrees of freedom $\nu_L$ is associated with the sample with larger sample variance and $\nu_S$ is the degrees of freedom associated with the sample with smaller sample variance.

Example 4.2.1

An experiment to explore the pain thresholds to electrical shocks for males and females resulted in the data summary given below. Do the data provide sufficient evidence to indicate a significant difference in the variability of pain thresholds for men and women?

| | Males | Females |
|---|---|---|
| n | 14 | 10 |
| $\bar{y}$ | 16.2 | 14.9 |
| $s^2$ | 12.7 | 26.4 |

*SOLUTION*

Let $Y_m$ and $Y_f$ denote the pain threshold to electric shock of males and females, respectively and assume that $Y_m \sim N(\mu_m, \sigma_m^2)$ and $Y_f \sim N(\mu_f, \sigma_f^2)$. We desire to test

$$H_0 : \sigma_m^2 = \sigma_f^2$$

versus

$$H_1 : \sigma_m^2 \neq \sigma_f^2$$

The females have a higher sample variance, hence,

$$F = \frac{S_f^2}{S_m^2} = \frac{26.4}{12.7} \approx 2.08.$$

The degrees of freedom associated with the sample of females is $\nu_l = 10 - 1 = 9$ and the degrees of freedom for the sample of males is $\nu_S = 14 - 1 = 13$. Consequently, the rejection region at the 5% level of significance is

$$RR = \{F \geq F_{0.025,(9,13)} = 3.312\}.$$

Meanwhile, the p-value associated with $F = 2.08$ is 0.1116.

Using the rejection region approach, we fail to reject $H_0$ since $F = 2.08 < F_{0.025,(9,13)} = 3.312$. Likewise, based on the p-value we do not reject $H_0$ since $p - value = 0.1116 > \alpha = 0.05$.

Therefore, at $\alpha = 0.05$ , the data do not provide sufficient evidence to indicate a significant difference in the variability of pain thresholds for men and women. In other words, the variances of pain thresholds for men and women are homogeneous.

Example 4.2.2

Suppose that we wish to compare the variation in diameters of parts produced by company A with the variation in diameters of parts produced by a competitor, company B. The sample variance for company A, based on a random sample 10 diameters was 0.0003. In contrast, the sample variance of the diameter measurements for 20 of the competitor's parts was 0.0001. Do the data provide sufficient information to indicate a smaller variation in diameters for the competitor?

*SOLUTION*: Left as a classroom exercise.

## Learning Task

Instruction: Answer the following as indicated.

One of the quality measures of blood glucose meter strips is the consistency of the test results on the same sample of blood. The consistency is measured by the variance of the readings in repeated testing. Suppose two types of strips, A and B, are compared for their respective consistencies. Suppose 16 Type A strips were tested with blood drops from a well-shaken vial and 21 Type B strips were tested with the blood from the same vial. The results are summarized below. Assume the glucose readings using Type A strips follow a normal distribution with variance and those using Type B strips follow a normal distribution with variance with . Is the above data sufficient to conclude that Type B strips are more consistent than Type A strips? Use $\alpha = 0.01$.

| Type of Strip | Sample size | Sample variance |
|---|---|---|
| A | 16 | 2.09 |
| B | 21 | 1.10 |