Module 1
Introduction to Hypothesis Testing

Lesson 2: The p-value

Lesson Summary

When performing a hypothesis test, simply saying that we "reject H_0" or that we "do not reject H_0" is somewhat uninformative. In addition to the test statistic, we can compute the so-called p-value (probability value). It provides a numerical measure of how much support we have for H_0. P-values less than a pre-determined level of significance lead to the rejection of the null hypothesis.

Challenge/Motivation

In applied statistics, tests of hypothesis are implemented using a software. The output reports both the test statistic and the p-value. What is really a p-value and how do we use it to make a decision on the null hypothesis?

Learning Outcomes

At the end of the lesson, students should be able to compute the p-value and use it to make a decision on the null hypothesis.

Time Frame: 30 minutes

Instructional Materials and Equipment Needed

Learning module, calculator, PPT slides, Recorded Video of the Lecture

Teaching Strategies

Lecture/ Discussion, Demonstration

Discussion

The p-value associated with an observed test statistic is the probability of getting a value for that test statistic as extreme as or more extreme than what was actually observed (relative to H_a) given that H_0 is true (Larsen and Marx, 2012). Ramachandan and Tsokos (2009) defined the p-value as the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected. It is also referred to as attained significance level.

The *p*-value can be thought of as a measure of support for the null hypothesis: The lower its value, the lower the support. In other words, the smaller the *p*-value becomes, the more compelling is the evidence that the null hypothesis should be rejected (Wackerly, Mendenhall, and Scheaffer, 2008). Typically, one decides that the support for H_0 is insufficient when the *p*-value drops below a particular threshold, which is the significance level of the test (Ramachandan and Tsokos, 2009). Thus, if the probability value is less than (or equal to) α, we reject H_0. If the probability value is greater than α, we do not reject H_0.

Note that the p-value is a random variable that is dependent on the test statistic (Wackerly, Mendenhall, and Scheaffer, 2008). Moreover, p-values are computed in a manner consistent with the alternative hypothesis H_a. Since the probability value is viewed as a measure of how much evidence we have against H_0, *it is always computed under the assumption that H_0 is true* (Ramachandan and Tsokos, 2009). Therefore, if we let T as a test statistic and its value based on random sample of size n is t, then

$$p-value=\begin{cases}P\left(T<t\mid H_0 \text{ is true}\right), \text{if } H_a \text{ is lower tailed}\\ P\left(T>t\mid H_0 \text{ is true}\right), \text{if } H_a \text{ is upper tailed}\\ 2P\left(|T|>|t|\mid H_0 \text{ is true}\right), \text{if } H_a \text{ is two-tailed}\end{cases}$$

Example 1

Suppose that $Y_1, Y_2, ..., Y_{100}$ is an iid sample from a $N(\mu, 100)$ distribution and we would like to test $H_0: \mu = 75$ versus $H_a: \mu > 75$. Suppose that the sample yielded a mean $\bar{y} = 76.42$ and using the Z test statistic we obtain its value as

$$z=\frac{\bar{y}-\mu_0}{\sigma/\sqrt{n}}=\frac{76.42-75}{10/\sqrt{100}}=1.42 .$$

a) Compute the p-value associated with the observe test statistic.
b) Based on the p-value in (a) would you reject H_0 at α=0.05?

Solution:

a) Note that H_a is one-tailed (upper tailed), thus, the p-value associated with this test statistic value is

$$p-value = P\left(Z > 1.42\right) \cong 0.0778 .$$

You can obtain this probability using the R command *pnorm(1.42, 0, 1, lower.tail=F).*

b) No. Because the p-value = 0.0778 is greater than α = 0.05, one cannot reject H_0.

Example 2

It has been suggested that less than 20 percent of all individuals who sign up for an extended gym membership continue to use the gym regularly six months after joining. Suppose that Y denotes the number of members who use a certain gym regularly (i.e., at least 3 times per week on average) six months after joining, to be observed from a sample of n = 50 members. Assume that $Y \sim B(50, p)$ and that we are to test

$H_0: P = 0.20$

versus

$H_a: P < 0.20$

Can we reject H_0 at α=0.05 if we observe $Y = y = 6$?

Solution:

The exact probability value is

$$\begin{aligned} p-value &= P\left(Y \le 6\right)\\ &=\sum_{y=0}^{6}\binom{50}{y}0.20^{y}0.80^{50-y}\\ &\cong 0.1034\end{aligned}$$

You can calculate this probability using *pbinom(6, 50, 0.2)* or *pbinom(6, 50, 0.2, lower.tail=TRUE)* command in R. Similar to Example 1 we cannot reject H_0 since the p-value is greater than 0.05.

Evaluation

Answer the following as indicated.

1. An existing make of car is known to break down, on average, one and a half times per year. A new model is introduced and the manufacturer claims that this model is less likely to break down. Ten randomly selected cars break down a total of eight times within the first year.

a) Compute the p-value.
b) Is the data sufficient to reject H_0 at a 5% significance level?

Answer:
a)

b)

2. Historically, a factory has been able to produce a very specialized nano-technology component with 35% reliability, i.e. 35% of the components passed its quality assurance requirements. They have now changed their manufacturing process and hope that this has improved the reliability. To test this, they took a sample of 24 components produced using the new process and found that 14 components passed the quality assurance test. At $\alpha=0.05$, does this show a significant improvement over the old process?

   Answer:

References

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MOOD, A. M., GRAYBILL, F. A. and BOES, D. C. (1976). *Introduction to the Theory of Statistics*, 3rd Ed. McGraw-Hill, Inc.

RAMACHANDRAN, K. M. AND TSOKOS, C. P. (2009). *Mathematical Statistics with Applications*. Elsevier Inc.