Stat 131 (Mathematical Statistics III)

Lesson 2.1: Power of a Statistical Test

Learning Outcomes

At the end of the lesson, students must be able to

1. articulate the power of a statistical test; and
2. compute and interpret the power of a statistical test.

Introduction

Statistical power, or sensitivity, is the likelihood of a significance test detecting an effect when there actually is one. A true effect is a real, non-zero relationship between variables in a population. An effect is usually indicated by a real difference between groups or a correlation between variables. High power in a study indicates a large chance of a test detecting a true effect. Low power means that your test only has a small chance of detecting a true effect or that the results are likely to be distorted by random and systematic error.

Statistical Power

Definition

Suppose that $Y_1, Y_2, \cdots, Y_n$ is an iid sample from $f_Y(y)$ and that we use a level α rejection region to test $H_0 : \theta = \theta_0$ versus a suitable alternative. The **power function** of the test, denoted by $K(\theta)$, is given by

$$K(\theta) = P(Reject\ H_0 | \theta \in H_1)$$

This means that the power function gives the probability of rejecting H_0 as a function of θ.

In view of the definition of the power function the following remarks are noteworthy.

1. If H_0 is true, that is $\theta = \theta_0$, then $K(\theta_0) = \alpha$.

2. For values of θ that are "close" to θ_0, one would expect the power to be smaller, than, say, when θ is far away from θ_0. This makes sense intuitively; namely, it is more difficult to detect a small departure from H_0 (i.e., to reject H_0) than it is to detect a large departure from H_0.

3. The shape of the power function always depends on the alternative hypothesis.

4. If θ_1 is a value of θ in the alternative space; that is, if a $\theta_1 \in H_1$, then $K(\theta_1) = 1-\beta(\theta_1)$.

Proof:

This follows directly from the complement rule; that is,

$$\begin{aligned} K(\theta_1) &= P(Reject\ H_0|\theta = \theta_1) \\ &= 1 - P(\text{Do not reject } H_0|\theta = \theta_1) \\ &= 1 - \beta(\theta_1) \end{aligned}$$

5. A good test is one with high power.

Example 2.1.1

Suppose that $Y_1, Y_2, \cdots, Y_n$ is an iid sample from $N(\mu, \sigma^2)$ and that we would like to test the following hypotheses:

$$H_0 : \mu = \mu_0 H_1 : \mu > \mu_0$$

Suppose that we use the level α rejection region $RR = \{z : z > z_\alpha\}$ where

$$z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$$

and z_α denotes the upper α quantile of the standard normal distribution.

The power function for the test, for $\mu > \mu_0$, is given by

$$\begin{aligned}
K(\theta) &= P(Reject\ H_0|\theta) \\
&= P(Z > z_\alpha|\theta) \\
&= P\left(\frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha \middle| \theta\right) \\
&= P\left(\overline{Y} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \middle| \theta\right) \\
&= P\left(Z > \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \theta}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 1 - P\left(Z \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \theta}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 1 - F_Z(z*),\ \ z* = \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \theta}{\frac{\sigma}{\sqrt{n}}}
\end{aligned}$$

where $F_Z(z)$ denotes the $N(0,1)$ cumulative distribution function.

Note that the power when $H_0 : \mu = \mu_0$ is true, then $K(\mu_0) = 1 - F_Z(z_\alpha) = 1 - (1-\alpha) = \alpha$.

Now, suppose $n = 10$, $\mu_0 = 6$, and $\sigma^2 = 4$. For $\alpha = 0.05$, $z_{0.05} = 1.645$, thus,

$$K(6) = 1 - F_Z\left(\frac{6 + 1.645\frac{2}{\sqrt{10}} - 6}{\frac{2}{\sqrt{10}}}\right) = 1 - F_Z(1.645) = 1 - P(Z < 1.645) = 0.05,$$

$$K(6.5) = 1 - F_Z\left(\frac{6 + 1.645\frac{2}{\sqrt{10}} - 6.5}{\frac{2}{\sqrt{10}}}\right) = 1 - F_Z(0.854) = 1 - P(Z < 0.854) \approx 0.197,$$

$$K(7) = 1 - F_Z\left(\frac{6 + 1.645\frac{2}{\sqrt{10}} - 7}{\frac{2}{\sqrt{10}}}\right) = 1 - F_Z(0.064) = 1 - P(Z < 0.064) \approx 0.475,$$

$$K(8) = 1 - F_Z\left(\frac{6 + 1.645\frac{2}{\sqrt{10}} - 8}{\frac{2}{\sqrt{10}}}\right) = 1 - F_Z(-1.517) = 1 - P(Z < -1.517) \approx 0.935,$$

and

$$K(9) = 1 - F_Z\left(\frac{6 + 1.645\frac{2}{\sqrt{10}} - 9}{\frac{2}{\sqrt{10}}}\right) = 1 - F_Z(-3.098) = 1 - P(Z < -3.098) \approx 0.999,$$

Observe that $K(\theta)$ is an increasing function of θ. Therefore, the probability of rejecting H_0 increases as θ increases.

Example 2.1.2

An experimenter has prepared a drug dosage level that she claims will induce sleep for 80% of people suffering from insomnia. After examining the dosage, we feel that her claims regarding the effectiveness of the dosage are inflated. In an attempt to disprove her claim, we administer her prescribed dosage to 20 insomniacs and we observe Y, the number for whom the drug dose induces sleep. We wish to test the hypothesis

$$H_0 : p = 0.8 \text{ versus } H_1 : p < 0.8$$

Assume that the rejection region $RR = \{y : y \leq 12\}$ is used. What is the power of the test when $p = 0.6$?

Solution

Here, the random variable Y is distribution as $Bi(20, p)$. Thus, the power of the test can be computed as

$$\begin{aligned} K(0.6) &= P(Reject\ H_0|p = 0.6) \\ &= P(Y \leq 12|p = 0.6) \\ &= \sum_{y=0}^{12} 0.6^y(1 - 0.6)^{20-y} \\ &\approx 0.5841 \end{aligned}$$

This probability was obtained using the R command *pbinom(12, 20, 0.6)*.

Learning Task

1. Refer to *Example 2.1.2* and compute and interpret the power of the test for each of the following values of p:

   a. 0.4
   b. 0.5
   c. 0.7

Together with the power obtained in *Example 2.1.2* sketch the power function against the values of p. what generalization can you make about the power of the test?

2. Let Y denote the IQ of a randomly selected adults. Assume that Y is normally distributed with unknown mean μ and standard deviation 16. Take a random sample of $n = 16$ students, so that, after setting the probability of committing a Type I error at $\alpha = 0.01$, we can test the null hypothesis $H_0 : \mu = 100$ against the alternative hypothesis that $H_1 : \mu > 100$. What is the power of the hypothesis test if the true population mean were $\mu = 108$?