Stat 131 (Mathematical Statistics III)

Lesson 4.3. T test for a population mean

Learning Outcomes

At the end of the lesson, students should be able to perform hypothesis testing for a population mean based on data from a small sample, and make a statistical decision using the p-value approach and the rejection region approach based on the T distribution.

Discussion

Suppose we have a random sample $Y_1, Y_2, \cdots, Y_n$ from $N(\mu, \sigma^2)$ and we want to test $H_0 : \mu = \mu_0$ against any suitable alternative. We know that when H_0 is true,

$$Z = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}} \sim N(0, 1)$$

as $n \to \infty$ by the Central Limit Theorem and Slutsky's Theorem. Therefore, Z can be used as a large sample test statistic to test $H_0 : \mu = \mu_0$. However, the large sample $N(0, 1)$ distribution may be inaccurate when n is small. This occurs when the underlying distribution is highly skewed and/or when S is not a good estimator of σ.

The cut-off size between large and small sample is 30. A sample is considered large if its size is at least 30 and the sample is small if its size is less than 30. One reason for this is that the t distribution is approximately normal when . Simulation studies show that the difference in the cumulative distribution function (CDF) of the standard normal distribution and the CDF of the t distribution with v degrees of freedom gets smaller and smaller as n gets larger and larger. In particular, the maximum absolute error begins at 0.1256 for $\nu = 1$ and declines steadily to 0.005244 for $\nu = 30$ as illustrated in Figure 1.

Now, if $H_0 : \mu = \mu_0$, it can be shown that

$$T = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}} \sim t(n - 1)$$

Since the t distribution is symmetric and mound-shaped, the rejection region for a small-sample test of hypothesis $H_0 : \mu = \mu_0$ must be located on the tails of the t distribution and

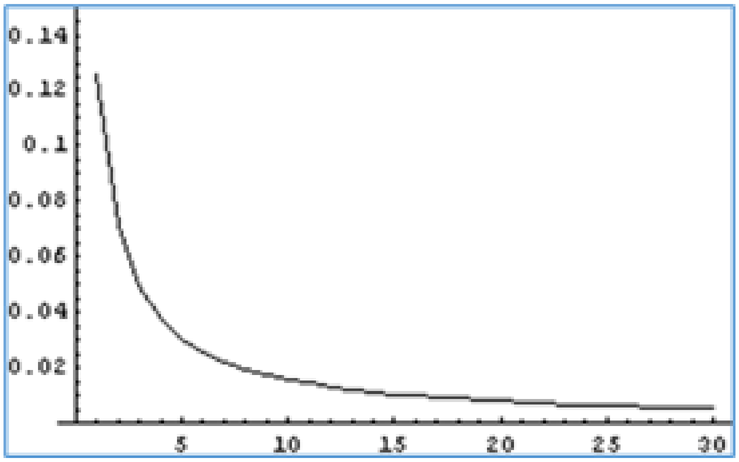

Figure 1: Difference between N(0,1) and t(v)

be determined in a manner similar to that used with the large-sample Z test statistic. For example, if we test $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$, then

$$RR = \{t : t \geq t_{\alpha,n-1}\}$$

where $t_{\alpha,n-1}$ is such that $P(T \geq t_{\alpha,n-1} = \alpha$ for a t distribution with $n-1$ degrees of freedom. Meanwhile, for a lower-tail alternative $H_1 : \mu < \mu_0$, we have

$$RR = \{t : t \leq -t_{\alpha,n-1}\}$$

and for a two-tailed alternative $H_1 : \mu \neq \mu_0$, the rejection region is

$$RR = \{t : |t| \geq t_{\alpha/2,n-1}\}$$

The critical value $t_{\alpha/2,n-1}$ is determined from the so-called T table or R. A sample table is given in Figure 2.

Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

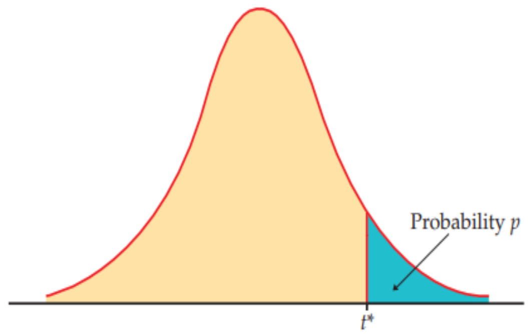

TABLE D

t distribution critical values

df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
	Upper-tail probability p											
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
z^*	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

Figure 2: T Table

Example 4.3.1

It is assumed that the mean systolic blood pressure is $\mu = 120$ mm Hg. In one study, a sample of 25 people had an average systolic blood pressure of 130.1 mm Hg with a standard deviation of 21.21 mm Hg. Is the group significantly different (with respect to systolic blood pressure!) from the regular population?

SOLUTION

Let Y denote systolic blood pressure. Assume $Y \sim N(\mu, \sigma^2)$. The null and alternative hypotheses for this situation are:

$$H_0 : \mu = 120$$
$$H_1 : \mu \neq 120$$

The population standard deviation is unknown and we have a small sample size ($n = 25$), so the appropriate test is T. For this sample, we have $\bar{Y} = 130.1$ and $S = 21.21$, thus, the observed value of the test statistic is

$$T = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}} = \frac{130.1 - 120}{21.21/\sqrt{25}} \approx 2.38$$

The rejection region is

$$RR = \{t : |t| \geq t_{0.025,24} = 2.064\}$$

The p-value associated with $T = 2.38$ is 0.

Thus, we reject the H_0 using the rejection region or the p-value.

Therefore, at the 5% level of significance, the data is sufficient to conclude that the group's mean systolic blood pressure is different from the regular population.

Example 4.3.2

In order to find out whether children with chronic diarrhea have the same average hemoglobin level (Hb) that is normally seen in healthy children in the same area, a random sample of 10 children with chronic diarrhea are selected and their Hb levels (g/dL) are obtained as follows:

12.3, 11.4, 14.2, 15.3, 14.8, 13.8, 11.1, 15.1, 15.8, 13.2

Do the data provide sufficient evidence to indicate that the mean Hb level for children with chronic diarrhea is less than that of the normal value of 14.6 g/dL? Test the appropriate hypothesis using $\alpha = 0.01$.

SOLUTION: Left as a classroom.

Learning Task

Instruction: Answer the following as indicated.

A consumer group is investigating a producer of diet meals to examine if its prepackaged meals actually contain the advertised 6 ounces of protein in each package. Based on the following data, is there any evidence that the meals contain less than the advertised amount of protein? Use a 1% level of significance.

5.1 4.9 6.0 5.1 5.7 5.5 4.9 6.1 6.0 5.8 5.2 4.8 4.7 4.2 4.9 5.5 5.6 5.8 6.0 6.1