Stat 131 (Mathematical Statistics III)

Lesson 2.3 Uniformly Most Powerful Test

Learning Outcomes

At the end of the lesson, students should be able to

1. explain what is a uniformly most powerful test; and
2. construct uniformly most powerful tests.

Introduction

For a simple-versus-simple test, the Neyman-Pearson Lemma shows us explicitly how to derive the most powerful level α rejection region. We now discuss simple-versus-composite tests; that is, testing the null hypothesis

$$H_0 : \theta = \theta_0$$

against

$$H_1 : \theta < \theta_0$$

or

$$H_1 : \theta > \theta_0$$

Uniformly Most Powerful Test

When a test maximizes the power for all θ in the alternative space; that is, for all $\theta \in H_1$, it is called the **uniformly most powerful** (UMP) level α test. In other words, if $K_U(\theta)$ denotes the power function for the UMP level α test of H_0 versus H_1, and if $K_U * (\theta)$ denotes the power function for some other level α test, then $K_U(\theta) > K_U * (\theta)$, $\forall \theta \in H_1$. Simply put, when we test a simple hypothesis against a composite alternative, we specify α, the probability of a type I error, and refer to one critical region of size α as uniformly more powerful than another if the values of its power function are always greater than or equal to those of the other, with the strict inequality holding for at least one value of the parameter under consideration (Miller and Miller, 2014).

Now, suppose we want to find the UMP level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. We do this by pretending to find a level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for an arbitrary $\theta_1 > \theta_0$. If we can then show that neither the test statistic nor the rejection region for the most powerful level α simple-versus-simple test depends on θ_1, then the test with the same rejection region will be UMP level α for the simple-versus-composite test $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

Essentially we are showing that for a given θ_1, the level α simple-versus-simple test is most powerful, by appealing to the Neyman-Pearson Lemma. However, since the value θ_1 is arbitrary and since the most powerful rejection region is free of θ_1, this same test must be most powerful level α for every value of $\theta_1 > \theta_0$; that is, it must be the uniformly most powerful (UMP) level α test for all $\theta > \theta_0$ (Mendenhall, Schaeffer, and Wackerly, 2008).

According to Mood, Graybill, and Boes (1974), a uniformly most powerful test does not exist for all testing problems, but when one does exist, we can see that it is quite a nice test since among all tests of size α or less it has the greatest chance of rejecting H_0 whenever it should.

Example 2.3.1

Suppose that $Y_1, Y_2, \cdots, Y_{15}$ is an iid sample from a Rayleigh distribution with pdf

$$f_Y(y) = \begin{cases} \frac{2y}{\theta} e^{-y^2/\theta}, \ y > 0 \\ 0, \ \text{elsewhere} \end{cases}$$

Find the UMP level $\alpha = 0.05$ test for

$$H_0 : \theta = 1$$

versus

$$H_1 : \theta > 1$$

Solution

We begin by using the Neyman-Pearson Lemma to find the most-powerful level $\alpha = 0.05$ test for

$$H_0 : \theta = 1$$

versus

$$H_1 : \theta = \theta_1, \text{ where } \theta_1 > 1$$

The likelihood function is given by

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{15} \frac{2y_i}{\theta} e^{-y_i^2/\theta} \\
&= \left(\frac{2}{\theta}\right)^{15} \prod_{i=1}^{15} y_i e^{-u/\theta}, \text{ where } u = \sum_{i=1}^{15} y_i^2
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{L(\theta_0)}{L(\theta_1)} &= \frac{L(1)}{L(\theta_1)} \\
&= \frac{(2)^{15} \prod_{i=1}^{15} y_i e^{-u}}{\left(\frac{2}{\theta_1}\right)^{15} \prod_{i=1}^{15} y_i e^{-u/\theta_1}} \\
&= \theta_1^{15} e^{-u(1-\frac{1}{\theta_1})}
\end{aligned}$$

Therefore, the Neyman-Pearson Lemma says that the most-powerful level $\alpha = 0.05$ test is created by choosing k such that

$$P\left[\theta_1^{15} e^{-U(1-\frac{1}{\theta_1})} < k | \theta = 1\right] = 0.05$$

where $U = \sum_{i=1}^{15} Y_i^2$.

Note that

$$\begin{aligned}
\theta_1^{15} e^{-U(1-\frac{1}{\theta_1})} < k &\iff e^{-U(1-\frac{1}{\theta_1})} < \frac{k}{\theta_1^{15}} \\
&\iff -U\left(1 - \frac{1}{\theta_1}\right) < ln\left(\frac{k}{\theta_1^{15}}\right) \\
&\iff U > -\frac{ln\left(\frac{k}{\theta_1^{15}}\right)}{1 - \frac{1}{\theta_1}} \\
&\iff U > k^*, \text{ where } k^* = -\frac{ln\left(\frac{k}{\theta_1^{15}}\right)}{1 - \frac{1}{\theta_1}}
\end{aligned}$$

Thus, the problem has now changed to choosing k^* so that

$$P(U > k^* | \theta = 1) = 0.05$$

Note that if $Y \sim Rayleigh(\theta)$, then $W = Y^2 \sim Exponential(\theta)$. Therefore,

$$U = \sum_{i=1}^{15} Y_i^2 = \sum_{i=1}^{15} W \sim Gamma(15, \theta).$$

When $H_0 : \theta = 1$ is true, then $U \sim Gamma(15, 1)$. Therefore, we choose k^* so that

$$\begin{aligned} 0.05 &= P(U > k^* | \theta = 1) \\ &= \int_{k^*}^{\infty} \frac{1}{\Gamma(15)} u^{14} e^{-u} \ du \end{aligned}$$

Using the R command **qgamma(0.05, 15, 1, lower.tail=FALSE)** we get $k^* \approx 21.886$. Thus, we conclude that using the Neyman-Pearson Lemma the most powerful level $\alpha = 0.05$ test of

$$H_0 : \theta = 1$$

versus

$$H_1 : \theta = \theta_1, \text{ where } \theta_1 > 1$$

uses the rejection region $RR = \{u : u \geq 21.886\}$.

Notice that neither the test statistic $U = \sum_{i=1}^{15} Y_i^2$ nor the rejection region $RR = \{u : u \geq 21.886\}$ depends on the specific value of θ_1 in this simple alternative. Therefore, this RR is the most powerful rejection region for any $\theta_1 > 1$, that is, the rejection region $RR = \{u : u \geq 21.886\}$ is the UMP level $\alpha = 0.05$ rejection region for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

Example 2.3.2

Suppose $Y_1, Y_2, \cdots, Y_{100}$ is a random sample from a normal distribution with a known mean of μ and an unknown variance of σ^2. Find the most powerful 0.01-level test for testing

$$H_0 : \sigma^2 = 1$$

versus

$$H_1 : \sigma^2 > 1$$

Is the test uniformly most powerful?

Solution

The likelihood function is given by

$$L(\sigma^2) = \prod_{i=1}^{100} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}$$

so that if $H_0 : \sigma^2 = 1$ is true, then

$$L(1) = \prod_{i=1}^{100} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\mu)^2}{2}} = \left(\frac{1}{\sqrt{2\pi}}\right)^{100} exp\Bigg[-\frac{1}{2}\sum_{i=1}^{100}(y_i-\mu)^2\Bigg].$$

If H_1 is true, then we can select an arbitrary value $\sigma_1^2 > 1$, and the likelihood function now becomes

$$L(\sigma_1^2) = \prod_{i=1}^{100} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(y_i-\mu)^2}{2\sigma_1^2}} = \left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^{100} exp\Bigg[-\frac{1}{2\sigma_1^2}\sum_{i=1}^{100}(y_i-\mu)^2\Bigg].$$

Thus,

$$\begin{aligned}
\frac{L(1)}{L(\sigma_1^2)} &= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^{100} exp\Bigg[-\frac{1}{2}\sum_{i=1}^{100}(y_i-\mu)^2\Bigg]}{\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^{100} exp\Bigg[-\frac{1}{2\sigma_1^2}\sum_{i=1}^{100}(y_i-\mu)^2\Bigg]} \\
&= (\sigma_1)^{100} exp\Bigg[-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{100}(y_i-\mu)^2\Bigg] \\
&= (\sigma_1)^{100} exp\Bigg[-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)u\Bigg], \text{ where } u = \sum_{i=1}^{100}(y_i-\mu)^2
\end{aligned}$$

Therefore, the Neyman-Pearson Lemma says that the most-powerful 0.01-level test is created by choosing k such that

$$P\left((\sigma_1)^{100} exp\Bigg[-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)U\Bigg] < k|\sigma^2 = 1 \right) = 0.01$$

But,

$$\begin{aligned}
(\sigma_1)^{100} exp\Bigg[-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)U\Bigg] < k &\iff 100ln(\sigma_1) - \frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)U < ln(k) \\
&\iff -\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)U < ln(k) - 100ln(\sigma_1) \\
&\iff U > \frac{ln(k) - 100ln(\sigma_1)}{-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)} \\
&\iff U > k^*, \text{ where } k^* = \frac{ln(k) - 100ln(\sigma_1)}{-\frac{1}{2}\left(1-\frac{1}{\sigma_1^2}\right)}
\end{aligned}$$

Thus, the problem has now changed to choosing k^* so that

$$P(U > k^*|\sigma^2 = 1) = 0.01, \text{ where } U = \sum_{i=1}^{100}(y_i - \mu)^2$$

Recall that

$$\frac{\sum_{i=1}^{n}(y_i - \mu)^2}{\sigma_0^2} \sim \chi^2_{(n)}.$$

Thus, when $H_0 : \sigma^2 = 1$ is true, then

$$\frac{\sum_{i=1}^{100}(y_i - \mu)^2}{1} = \sum_{i=1}^{100}(y_i - \mu)^2 = U \sim \chi^2_{(100)}.$$

Hence, using the command R command **qchisq(0.01, 100, lower.tail=FALSE)** we get $k^* \approx 135.81$ and the most powerful 0.01-level test for testing $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = \sigma_1^2 > 1$ has the rejection region

$$RR = \{u : u \geq 135.81\}$$

Since this rejection region is free of σ_1^2, then this is the same rejection region of the uniformly most powerful 0.01-level test for testing $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 > 1$.

Learning Task/Activity

Instruction: Answer the following as indicated.

Let $Y_1, Y_2, \cdots, Y_{150}$ be a random sample from a normal distribution with mean μ and variance 1. Find the uniformly most powerful $\alpha = 0.05$ level test for $H_0 : \mu = 50$ versus $H_1 : \mu > 50$.