Stat 131 (Mathematical Statistics III)

Lesson 3.4. Z test for the difference of two population proportions

Learning Outcomes

At the end of the lesson, students should be able to

1. derive the statistic for testing hypothesis on the difference of two population proportions; and
2. conduct the z test for the difference of two population proportions.

Introduction

In some applications of hypothesis testing, we are interested in comparing two independent populations with respect to binary responses. In this lesson, we extend the Z test to test hypothesis about the difference of two population proportions when the sample sizes are sufficiently large.

The Z test for the difference of two population proportions

Now we present the test procedure for testing the difference of two proportions, inherent in two binomial populations. Here, again we assume that the binomial distribution is approximated by the normal distribution and thus, it is an approximate test. The approximation works best if $n_i\hat{p}_i \geq 5$ and $n_i(1-\hat{p}_i) \geq 5$, for $i = 1, 2$ (Ramachandan and Tsokos, 2009).

Suppose that we have two independent samples; that is,

$$Y_{11}, Y_{12}, \cdots, Y_{1n_1} \overset{iid}{\sim} Bernoulli(p_1)$$
$$Y_{21}, Y_{22}, \cdots, Y_{2n_2} \overset{iid}{\sim} Bernoulli(p_2)$$

and that interest lies in testing

$$H_0 : p_1 - p_2 = d_0$$

versus

$$H_1 : p_1 - p_2 \neq d_0$$

(or any other appropriate alternative), where d_0 is a known constant. Note that taking $d_0 = 0$ allows one to test the equality of p_1 and p_2. In this situation, we identify

$$\begin{aligned}
\theta &= p_1 - p_2 \\
\hat{\theta} &= \hat{p}_1 - \hat{p}_2 \\
\sigma_{\hat{\theta}} &= \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \\
\hat{\sigma}_{\hat{\theta}} &= \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}
\end{aligned}$$

where $\hat{p}_1 = \frac{\sum_{i=1}^{n_1} y_{1i}}{n_1}$ and $\hat{p}_2 = \frac{\sum_{i=1}^{n_2} y_{2i}}{n_2}$ are the proportions of *successes* in each sample, respectively.

There are two test statistics that one can consider when testing hypothesis on the difference between two proportions: Wald's, and Score statistics. The Wald's statistic is given by

$$Z_W = \frac{(\hat{p}_1 - \hat{p}_2) - d_0}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

while when $d_0 = 0$ the Score statistic is

$$Z_S = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

is the pooled sample proportion, as it estimates the common proportion p under H_0.

It can be shown that both statistics Z_W and Z_S have approximate standard normal distribution when H_0 is true. As in the one-sample problem, the Score statistic performs better in small samples.

Example 3.4.1

For developing countries in Africa and the Asia, let p_1 and p_2 be the respective proportions of babies with a low birth weight (below 2500 grams). A random sample of 900 babies from developing countries in Africa yielded 135 babies with a low birth weight and a random sample 700 babies from developing countries in Asia yielded 77 babies with a low birth weight. Is this data sufficient to conclude that there is a higher proportion of babies with low birth weight in developing countries in Africa than in Asia? Use $\alpha = 0.01$.

SOLUTION

Let p_1 and p_2 be the true proportions of babies with a low birth weight in developing countries in Africa and Asia, respectively. We wanted to test the following hypotheses.

$$\begin{aligned} H_0 &: p_1 = p_2 \\ H_1 &: p_1 > p_2 \end{aligned}$$

Test statistic From the given information for both samples, we have

$$\hat{p}_1 = \frac{135}{900} = 0.15,$$

$$\hat{p}_2 = \frac{77}{700} = 0.11$$

and the pooled estimate of the proportion is

$$\begin{aligned} \hat{p} &= \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} \\ &= \frac{135 + 77}{900 + 700} \\ &= 0.1325 \end{aligned}$$

Notice that both $n_1\hat{p}_1 = 135 > 5$ and $n_2\hat{p}_2 = 77 > 5$, thus, the normal approximation can be used. Since $d_0 = 0$, we shall use the Score statistic.

$$\begin{aligned} Z_S &= \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \\ &= \frac{0.15 - 0.11}{\sqrt{0.1325(1 - 0.1325)\left(\frac{1}{900} + \frac{1}{700}\right)}} \\ &\approx 2.34 \end{aligned}$$

Rejection Region

At $\alpha = 0.01$, the rejection region would be $RR = \{z : z \geq z_{0.01} = 2.33\}$ and the corresponding p-value is $P(Z > 2.34) = 0.0096$.

Decision on H_0

The computed value of Z falls in the rejection and the p-value is less than the 0.01 significance level, thus, we reject H_0.

Conclusion

At the 0.01 significance level, the data is sufficient to conclude that there is a higher proportion of babies with low birth weight in developing countries in Africa than in Asia.

Example 3.4.2

A country is testing two COVID-19 vaccines (A and B) on humans. Vaccine A works on 401 people out of a sample of 595. Vaccine B works on 359 people in a sample of 605. Are the two vaccines comparable?

SOLUTION: Left as a classroom exercise!

Learning Task

Instruction: Answer the following as indicated.

Researchers conducted a study of smartphone use among adults. A cell phone company claimed that iPhone smartphones are more popular with whites (non-Hispanic) than with African Americans. The results of the survey indicate that of the 232 African American cell phone owners randomly sampled, 5% have an iPhone. Of the 1,343 white cell phone owners randomly sampled, 10% own an iPhone. Is the proportion of white iPhone owners greater than the proportion of African American iPhone owners? Use a 5% level of significance.