Stat 131 (Mathematical Statistics III)

Lesson 2.2 Neyman-Pearson Lemma and Most Powerful Test

Learning Outcome

At the end of the lesson students should be able to

1. explain the importance of the Neyman-Pearson Lemma in constructing tests; and
2. construct most powerful tests using the Neyman-Pearson Lemma.

Introduction

Let us begin by considering a *sharp* or *simple* null hypothesis where there is just one value of θ possible under H_0. The alternative may be simple or composite. Here is an example of a simple-versus-simple test:

$$H_0 : \mu = 5 \text{ versus } H_1 : \mu = 6$$

Here is an example of a simple-versus-composite test.

$$H_0 : \mu = 5 \text{ versus } H_1 : \mu > 5$$

Note that there are an infinite number of values of θ specified in a composite alternative hypothesis. In this example, H_1 consists of any value of θ larger than 5.

For a level α simple-versus-simple test, our main goal is to find the most powerful rejection region, that is, the rejection region that maximizes the probability of rejecting H_0 when H_0 is false (or H_1 is true). The Neyman-Pearson Lemma tells us how to find this **most powerful test**.

Neyman-Pearson Lemma

Suppose that $Y_1, Y_2, \cdots, Y_n$ is an iid sample from $f_Y(y;\theta)$, and let $L(\theta)$ denote the likelihood function. Consider the following simple-versus-simple hypothesis test:

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1$$

The level α test that maximizes the power when $H_1 : \theta = \theta_1$ is true uses the rejection region RR

$$RR = \left\{ \mathbf{y} : \frac{L(\theta_0)}{L(\theta_1)} < k \right\}$$

where k is so chosen so that

$$P(Reject\ H_0 | H_0 \text{ is true}) = \alpha$$

This is called the **most-powerful** level α test for H_0 versus H_1.

Example 2.2.1

Suppose that Y is a single observation (i.e., an iid sample of size n = 1) from an exponential distribution with mean θ. Using this single observation, we would like to test

$$H_0 : \theta = 2$$

versus

$$H_1 : \theta = 3$$

Use the Neyman-Pearson Lemma to find the most powerful level $\alpha = 0.10$ test.

Solution

Because sample size is $n = 1$, the likelihood function $L(\theta)$ is simply

$$L(\theta) = f_Y(y;\theta) = \frac{1}{\theta} e^{-y/\theta}.$$

To use the Neyman-Pearson Lemma, we first form the ratio

$$\begin{aligned} \frac{L(\theta_0)}{L(\theta_1)} &= \frac{L(2)}{L(3)} \\ &= \frac{\frac{1}{2}e^{-y/2}}{\frac{1}{3}e^{-y/3}} \\ &= \frac{3}{2}e^{-y/6} \end{aligned}$$

Therefore, the Neyman-Pearson Lemma says that the most-powerful level $\alpha = 0.10$ test is created by choosing k such that

$$P\left(\frac{3}{2}e^{-Y/6} < k|\theta = 2\right) = 0.10$$

Now,

$$\begin{aligned}
\frac{3}{2}e^{-Y/6} < k &\iff e^{-Y/6} < \frac{2}{3}k \\
&\iff -\frac{Y}{6} < ln\left(\frac{2}{3}k\right) \\
&\iff Y > -6ln\left(\frac{2}{3}k\right) \\
&\iff Y > k^*, \text{ where } k^* = -6ln\left(\frac{2}{3}k\right)
\end{aligned}$$

Thus, we have changed the problem to now choosing k^* so that

$$P(Y > k^*) = 0.10$$

Recall that when $\theta = 2$ (that is when H_0 is true), then $Y \sim Exp(2)$ and therefore we need to solve the following integral to find k^*.

$$0.10 = P(Y > k^*) = \int_{k^*}^{\infty} \frac{1}{2}e^{-y/2}\, dy$$

Using the R code **qexp(0.10,1/2,lower.tail=FALSE)** we obtain $k^* = 4.60517$.

Therefore, the most powerful level $\alpha = 0.10$ test uses the rejection region

$$RR = \{y : y > 4.60517\}$$

That is, we reject $H_0 : \theta = 2$ in favor of $H_1 : \theta = 3$ whenever $Y > 4.60517$.

Question: What is the power this test when H_1 is true?

<u>Answer</u>

$$\begin{aligned}
K(3) &= P(Y > 4.60517|\theta = 3) \\
&= \int_{4.60517}^{\infty} \frac{1}{3}e^{-y/3}\, dy \\
&\approx 0.215
\end{aligned}$$

The following R command **pexp(4.605, 1/3, lower.tail=FALSE)** was used to get this probability.

Remark

Note that even though we have found the most powerful level $\alpha = 0.10$ test of H_0 versus H_1, the test is not all that powerful- we have only about a 21.5 percent chance of correcting rejecting H_0 when H_1 is true. Of course, this should not be surprising, given that we have just a single observation Y. We are trying to make a decision with very little information about θ.

Example 2.2.2

Suppose that $Y_1, Y_2, \cdots, Y_{10}$ is an iid sample of Poisson(θ) observations and that we want to test

$$H_0 : \theta = 1$$

versus

$$H_1 : \theta = 2$$

Find the most-powerful level $\alpha = 0.05$ test.

Solution

The likelihood function for θ is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{10} \frac{\theta^{y_i} e^{-\theta}}{y_i!} \\ &= \frac{\theta^u e^{-10\theta}}{\prod_{i=1}^{10} y_i!}, \text{ where } u = \sum_{i=1}^{10} y_i \end{aligned}$$

Now,

$$\begin{aligned} \frac{L(\theta_0)}{L(\theta_1)} &= \frac{L(1)}{L(2)} \\ &= \frac{\frac{1^u e^{-10(1)}}{\prod_{i=1}^{10} y_i!}}{\frac{2^u e^{-10(2)}}{\prod_{i=1}^{10} y_i!}} \\ &= \frac{1^u e^{-10(1)}}{2^u e^{-10(2)}} \\ &= \frac{1}{2^u e^{-10}} \end{aligned}$$

The Neyman-Pearson Lemma says that the most-powerful level $\alpha = 0.05$ test is created by choosing k such that

$$P\left(\frac{1}{2^U e^{-10}} < k|\theta = 1\right) = 0.05$$

Note that

$$\begin{aligned}
\frac{1}{2^U e^{-10}} < k &\iff 2^U e^{-10} > \frac{1}{k} \\
&\iff 2^U > \frac{e^{10}}{k} \\
&\iff U ln(2) > 10 - ln(k) \\
&\iff U > \frac{10 - ln(k)}{ln(2)} \\
&\iff U > k^*, \text{ where } k^* = \frac{10 - ln(k)}{ln(2)}
\end{aligned}$$

Thus, we have changed the problem to now choosing k^* so that

$$P(U > k^*|\theta = 1) = 0.05$$

Note that when $H_0 : \theta = 1$ is true, the sufficient statistic $U = \sum_{i=1}^{10} Y_i \sim Poisson(10)$. Further, note that k^* is not an integer, we need to solve the equation

$$\alpha = P(U > k^*|\theta = 1) = P(U > m|\theta = 1)$$

where $m = [k^*] + 1$ and [] denotes the greatest integer function. The entries in the following table was obtained using the R code **ppois(m-1, 10, lower.tail=FALSE)**.

m	α
14	0.1355
15	0.0835
16	0.0487
17	0.0270
18	0.0143

Thus, an approximate level $\alpha = 0.05$ most powerful test based on the Neyman-Pearson Lemma has the following rejection region $RR = \{u : u \geq 16\}$.

Question: What is the power of the approximate level $\alpha = 0.05$ test when H_1 is true?

Answer:

When $H_1 : \theta = 2$ is true, then $U \sim Poisson(20)$. Therefore,

$$\begin{aligned} K(2) &= P(U \geq 16|\theta = 2) \\ &= \sum_{j=16}^{\infty} \frac{20^j e^{-20}}{j!} \\ &\approx 0.7789 \end{aligned}$$

This probability was obtained using the R function **ppois(16, 20, lower.tail=FALSE)**.

Remark:

Notice that in *Example 2.2.2* the rejection region for the most powerful level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ always depends on a sufficient statistic U. This is generally true.

Learning Task/Activity

Instruction: Answer the following as indicated.

A random sample of size 20 from an exponential distribution with parameter θ is used to test the null hypothesis $H_0 : \theta = 2$ against the alternative hypothesis $H_1 : \theta = 3$.

a. Use the Neyman–Pearson lemma to find the most powerful critical region of size approximately 0.05.

b. Find the power of the test in (a).