

Stat 131 (Mathematical Statistics III)

Lesson 4.5. Paired-samples t test

Learning Outcomes

At the end of the lesson, students should be able to

1. explain the scenarios where a paired-samples t test is appropriate to use; and
2. apply the paired-samples t test to compare the means of two dependent populations

Discussion

In comparing parameters of two populations based on random samples taken from these populations, it is equally important to distinguish if the populations (samples) are independent or not. This distinction is another important consideration in choosing the statistical test to use to analyze the data.

Dependent or paired samples arise according to the following situations:

1. Same variable is measured on at least two occasions using similar instruments on a group of subjects. We call this in the literature as a repeated measures design. If the measurements or observations were taken on two occasions, before and after an intervention or treatment, then this is popularly known as the **pretest-posttest** design.
2. Pairs of comparable units or subjects are used and within each pair, units or subjects are randomly assigned to two treatment conditions. This referred to as the **matched-pairs** design.

Suppose we have data generated using either from a pre-test-post-test design or a matched-pairs design and we are interested in testing $H_0 : \mu_d = 0$, where $\mu_d = \mu_1 - \mu_2$, against any suitable alternative hypothesis, then the appropriate test statistic is given by

$$t = \frac{\bar{d}}{s_d / \sqrt{n}}$$

where

- $\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i$, $i = 1, 2, \dots, n$
- $d_i = X_i - Y_i$ are the differences between pairs of X_i and Y_i
- s_d is the standard deviation of d_i
- n is the number of pairs of observations

This test statistic is called the **paired-samples t test statistic** and follows a t distribution with $n - 1$ degrees of freedom. The test assumes that the observations X_i and Y_i are interval or ratio scaled and the pairwise differences d_i are normally distributed, that is $d_i \sim N(\mu_d, \sigma_d^2)$.

The rejection regions for this test are similar to the independent samples t test. That is,

Alternative Hypothesis	Rejection Region
$H_1: \mu_1 > \mu_2$	$RR = \{t : t \geq t_{\alpha, df}\}$
$H_1: \mu_1 < \mu_2$	$RR = \{t : t \leq -t_{\alpha, df}\}$
$H_1: \mu_1 \neq \mu_2$	$RR = \{t : t \geq t_{\alpha/2, df}\}$

Example 4.5.1

With the COVID-19 pandemic, faculty members of SUCs and HEIs in the country are required to prepare course modules for flexible learning delivery when classes resume in August. One of the important parts of the module is a pretest and a posttest. Suppose the pretest and posttest scores of a class of 20 students in Elementary Statistics are given below. Based on the data, can we conclude that the module is effective?

Student	Pre test	Post test
1	18	22
2	21	25
3	16	17
4	22	24
5	19	16
6	24	29
7	17	20
8	21	23
9	23	19
10	18	20
11	14	15
12	16	15
13	16	18
14	19	26

Student	Pre test	Post test
15	18	18
16	20	24
17	12	18
18	22	25
19	15	19
20	17	16

SOLUTION

Let μ_1 and μ_2 be the mean score of students in the pre-test and posttest, respectively. We want to test

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 < \mu_2$$

To compute the value of the test statistic we need to determine the pairwise differences first and these are given in the table below.

Student	Pre test	Post test	d_1
1	18	22	-4
2	21	25	-4
3	16	17	-1
4	22	24	-2
5	19	16	3
6	24	29	-5
7	17	20	-3
8	21	23	-2
9	23	19	4
10	18	20	-2
11	14	15	-1
12	16	15	1
13	16	18	-2
14	19	26	-7
15	18	18	0
16	20	24	-4
17	12	18	-6
18	22	25	-3
19	15	19	-4
20	17	16	1

It can be verified that $\bar{d} = -2.05$ and $s_d = 2.84$. Thus, assuming the pairwise differences to be distributed normally, the test statistic is

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{-2.05}{2.84/\sqrt{20}} = -3.23$$

At a 5% level of significance, the rejection region is given by $RR = \{t : t < -t_{0.05,19} = -1.729\}$ and the p-value associated with the test statistic is 0.0022.

Therefore, based on either the rejection region or the p-value, we reject H_0 and conclude that the 5% level of significance the module is effective.

Learning Tasks

Instruction: Answer the following as indicated.

1. A new diet and exercise program has been advertised as remarkable way to reduce blood glucose levels in diabetic patients. Ten randomly selected diabetic patients are put on the program, and the results after 1 month are given by the following table. Do the data provide sufficient evidence to support the claim that the new program reduces blood glucose level in diabetic patients?

Before	268	225	252	192	307	228	246	298	231	185
After	106	186	223	110	203	101	211	176	194	203

2. An aquaculture farm takes water from a stream and returns it after it has circulated through the fish tanks. In order to find out how much organic matter is left in the waste water after the circulation, some samples of the water are taken at the intake and other samples are taken at the downstream outlet and tested for biochemical oxygen demand (BOD). BOD is a common environmental measure of the quantity of oxygen consumed by microorganisms during the decomposition of organic matter. If BOD increases, it can be said that the waste matter contains more organic matter than the stream can handle. The following table gives data for this problem. Test that the mean BOD for the downstream samples is less than for the upstream samples.

Upstream	9.0	6.8	6.5	8.0	7.7	8.6	6.8	8.9	7.2	7.0
Downstream	10.2	10.2	9.9	11.1	9.6	8.7	9.6	9.7	10.4	8.1