Stat 136 (Bayesian Statistics)

Lesson 1.3: Odds and Bayes Factor

Odds

Another way of dealing with uncertain events that we are modeling as random is to form the odds of the event. The odds for an event C equals the probability of the event occurring divided by the probability of the event not occurring:

$$odds(C) = \frac{P(C)}{P(\overline{C})}$$

Since the probability of the event not occurring equals one minus the probability of the event, there is a one-to-one relationship between the odds of an event and its probability.

$$odds(C) = \frac{P(C)}{1 - P(C)}$$

If we are using prior probabilities, we get the *prior odds* - in other words, the ratio before we have analyzed the data. If we are using posterior probabilities, we get the *posterior odds*.

Solving the equation for the probability of event C we get

$$P(C) = \frac{odds(C)}{1 + odss(C)}$$

We see that there is a one-to-one correspondence between odds and probabilities.

Bayes Factor

The Bayes factor (BF) contains the evidence in the data D that occurred relevant to the question about C occurring. It is the factor by which the prior odds is changed to the posterior odds:

$$\text{prior odds}(C) \times BF = \text{posterior odds}(C)$$

We can solve this relationship for the Bayes factor to get

$$BF = \frac{\text{posterior odds}}{\text{prior odds}}$$

Thus, the Bayes factor is the ratio of the probability of getting the data which occurred given the event, to the probability of getting the data which occurred given the complement of the event. If the Bayes factor is greater than 1, then the data has made us believe that the event is more probable than we thought before. If the Bayes factor is less than 1, then the data has made us believe that the event is less probable than we originally thought.

The Bayes factor can be used in hypothesis testing or in model comparison. For example, in hypothesis testing, the Bayes factor quantifies how much more likely the data are to be observed under H_0 than under H_1. Therefore, the Bayes factor can be interpreted as the relative support in the observed data for H_0 versus H_1. If the Bayes factor equals 1, there is no preference for either hypotheses. If Bayes factor is larger than 1, H_0 is preferred. If Bayes factor is between 0 and 1, H_1 is preferred.

For example, we toss a coin $n = 10$ times and recorded $X = 6$ heads. Let p be the (unknown) probability of observing a head in each toss. Suppose we test $H_0 : p = 0.5$ against $H_1 : p = 0.9$. What is our decision on H_0?

To answer this question, we calculate the Bayes factor.

First, we calculate

$$P(X = 6|H_0) = \binom{10}{6} 0.5^6 \times 0.6^4 = 0.205$$

and

$$P(X = 6|H_1) = \binom{10}{6} 0.9^6 \times 0.1^4 = 0.011$$

Thus,

$$BF_{H_0} = BF_{01} = \frac{P(X=6|H_0)}{P(X=6|H_1)} = \frac{0.205}{0.011} \approx 19.$$

Therefore, the data ($X = 6$) is more likely coming from a Binomial distribution with $p = 0.5$. Hence, we do not reject H_0.

Equivalently, the Bayes factor is a measure of relative evidence, the comparison of the predictive performance of one model against another one. This comparison is a ratio of marginal likelihoods:

$$BF_{12} = \frac{P(\mathbf{y}|M_1)}{P(\mathbf{y}|M_2)}$$

BF_{12} indicates the extent to which the data are more likely under model M_1 over model M_2, or in other words, which of the two models is more likely to have generated the data, or the relative evidence that we have for M_1 over M_2. Values of BF_{12} larger than one indicate evidence in favor of M_1, smaller than one indicate evidence in favor of M_2, and values close to one indicate that the evidence is inconclusive. This model comparison does not depend on a specific parameter value. Instead, all possible prior parameter values are taken into account simultaneously.

For the Bayes factor, a scale has been proposed to interpret Bayes factors according to the strength of evidence in favor of one model (corresponding to some hypothesis) over another (Jeffreys 1939); but this scale should not be regarded as a hard and fast rule with clear boundaries.

BF_{12}	Interpretation
>100	Extreme evidence for M_1
30 - 100	Very strong evidence for M_1
10 - 30	Strong evidence for M_1
3 - 10	Moderate evidence for M_1
1 - 3	Anecdotal evidence for M_1
1	No evidence
$\frac{1}{1} - \frac{1}{3}$	Anecdotal evidence for M_2
$\frac{1}{3} - \frac{1}{10}$	Moderate evidence for M_2
$\frac{1}{10} - \frac{1}{30}$	Strong evidence for M_2
$\frac{1}{30} - \frac{1}{100}$	Very strong evidence for M_2
$< \frac{1}{100}$	Extreme evidence for M_2

Example

The incidence of a disease in the population is 1%. A medical test for the disease is 90% accurate in the sense that it produces a false reading 10% of the time, both: (a) when the test is applied to a person with the disease; and (b) when the test is applied to a person without the disease. A person is randomly selected from population and given the test. The test result is positive (i.e. it indicates that the person has the disease). Calculate the Bayes factor for testing that the person has the disease versus that they do not have the disease.

Let A be the event that the person has the disease, and let B be the event that he tests positive for the disease.

The relevant probabilities are:

- $P(A) = 0.01$
- $P(B|A) = 0.9$
- $P(\overline{B}|\overline{A}) = 0.9$

From the first example in Lesson 1.2, we obtained $P(A|B) = \frac{1}{12} \approx 0.0833$

We now wish to test $H_0 : A$ versus $H_1 : \overline{A}$. So we calculate:

- prior odds $= \frac{P(A)}{P(\overline{A})} = \frac{0.01}{0.99} = \frac{1}{99}$
- posterior odds $= \frac{P(A|B)}{P(\overline{A}|B)} = \frac{\frac{1}{12}}{\frac{11}{12}} = \frac{1}{11}$
- $BF = \frac{\frac{1}{11}}{\frac{1}{99}} = 9$

This means the positive test result has multiplied the odds of the person having the disease relative to not having it by a factor of 9 or 900%.

Another way to say this is that those odds have increased by 800%.