

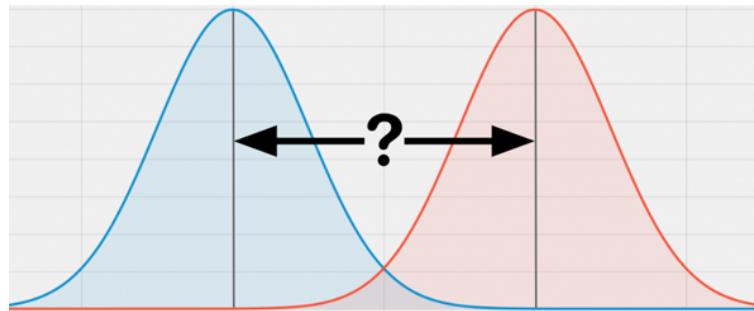
# Stat 136 (Bayesian Statistics)

## Lesson 2.2: Bayesian Inference for the Difference Between Two Normal Means

### Introduction

Oftentimes the objective of a study is comparison of two or more groups or populations based on measures such as the means or variances.

- Treated vs Control
- Independent samples or paired samples



### Likelihoods

Let  $Y_{11}, Y_{12}, \dots, Y_{1n_1}$  be a random sample of size  $n_1$  from  $N(\mu_1, \sigma^2)$ . Also, let  $Y_{21}, Y_{22}, \dots, Y_{2n_2}$  be a random sample of size  $n_2$  from  $N(\mu_2, \sigma^2)$ . Further, suppose that the 2 random samples are independent and the common variance,  $\sigma^2$ , is known. Then,

$$L(y_{11}, y_{12}, \dots, y_{1n_1} | \mu_1, \sigma^2) \propto \exp\left(-\frac{n_1(\bar{y}_1 - \mu_1)^2}{2\sigma^2}\right)$$

and

$$L(y_{21}, y_{22}, \dots, y_{2n_2} | \mu_2, \sigma^2) \propto \exp\left(-\frac{n_2(\bar{y}_2 - \mu_2)^2}{2\sigma^2}\right)$$

## Normal priors

Let the priors be:  $N(\mu_{1p}, \sigma_{1p}^2)$  and  $N(\mu_{2p}, \sigma_{2p}^2)$ . That is,

$$\pi(\mu_1) \propto \exp\left(-\frac{(\mu_1 - \mu_{1p})^2}{2\sigma_{1p}^2}\right)$$

and

$$\pi(\mu_2) \propto \exp\left(-\frac{(\mu_2 - \mu_{2p})^2}{2\sigma_{2p}^2}\right)$$

## Posterior densities

Based on the previous lesson, it can be easily shown that the posterior densities for  $\mu_1$  and  $\mu_2$  are, respectively

$$\pi(\mu_1 | y_{11}, y_{12}, \dots, y_{1n_1}) \propto \exp\left[-\frac{1}{2\left(\frac{\sigma_{1p}^2 \sigma^2}{\sigma^2 + n_1 \sigma_{1p}^2}\right)} \left(\mu_1 - \frac{\sigma^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{\sigma^2 + n_1 \sigma_{1p}^2}\right)\right]$$

and

$$\pi(\mu_2 | y_{21}, y_{22}, \dots, y_{2n_2}) \propto \exp\left[-\frac{1}{2\left(\frac{\sigma_{2p}^2 \sigma^2}{\sigma^2 + n_2 \sigma_{2p}^2}\right)} \left(\mu_2 - \frac{\sigma^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{\sigma^2 + n_2 \sigma_{2p}^2}\right)\right]$$

Thus, the posterior densities for  $\mu_1$  and  $\mu_2$  are, respectively,  $N(\mu'_1, \sigma'^2_1)$  and  $N(\mu'_2, \sigma'^2_2)$ , where

$$\mu'_1 = \frac{\sigma^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{\sigma^2 + n_1 \sigma_{1p}^2},$$

$$\sigma'^2_1 = \frac{\sigma^2 \sigma_{1p}^2}{\sigma^2 + n_1 \sigma_{1p}^2},$$

$$\mu'_2 = \frac{\sigma^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{\sigma^2 + n_2 \sigma_{2p}^2},$$

and

$$\sigma'^2_2 = \frac{\sigma^2 \sigma_{2p}^2}{\sigma^2 + n_2 \sigma_{2p}^2}$$

## Posterior density of the mean difference $\mu_d$

Recall that if the sampling distributions (likelihood) and the prior distributions are independent, then the posterior distributions are independent as well.

Let  $\mu_d = \mu_1 - \mu_2$ . Then

$$\mu_d \sim N(\mu'_d, \sigma_d^{2'})$$

where

$$\mu'_d = \mu'_1 - \mu'_2$$

and

$$\sigma_d^{2'} = \sigma_1^{2'} + \sigma_2^{2'}$$

## Credible interval for the mean difference $\mu_d$

A  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  is given by

$$\mu'_d \pm z_{\frac{\alpha}{2}} \sigma'_d$$

where:

$$\sigma'_d = \sqrt{\sigma_1^{2'} + \sigma_2^{2'}}$$

## Bayesian hypothesis test for $\mu_d$

To test the hypothesis  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$  we calculate the posterior probability of the null hypothesis  $\implies P(H_0|data) = P(\mu_d \leq 0|data)$

- If this probability is less than some  $\alpha$ -level of significance, we reject  $H_0$

To test the hypothesis  $H_0 : \mu_d = 0$  versus  $H_1 : \mu_d \neq 0$ , we construct the  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$

- If the  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  includes zero, we fail to reject  $H_0$

## Example 1

A thermal power station discharges its cooling water into a river. An environmental scientist wants to determine if this has adversely affected the dissolved oxygen level. She takes samples of water one kilometer upstream from the power station, and one kilometer downstream from the power station, and measures the dissolved oxygen level. The data are given below. Is the data sufficient to conclude that dissolved oxygen level is higher upstream than downstream?

Upstream	Downstream
10.1	9.7
10.2	10.3
13.4	6.4
8.2	7.3
9.8	11.7
	8.9

- Assume that the observations come from  $N(\mu_1, 4)$  and  $N(\mu_2, 4)$ . Use the independent  $N(10, 4.5)$  and  $N(9, 4)$  as prior distributions for  $\mu_1$  and  $\mu_2$ , respectively. Find the posterior distributions of  $\mu_1$  and  $\mu_2$ , respectively.
- Find the posterior distribution of  $\mu_d = \mu_1 - \mu_2$ .
- Find a 95% credible interval of  $\mu_d = \mu_1 - \mu_2$ .
- At the 5% level of significance, perform a Bayesian test of the hypothesis  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$

```
u <- c(10.1, 10.2, 13.4, 8.2, 9.8)
d <- c(9.7, 10.3, 6.4, 7.3, 11.7, 8.9)
n1 <- length(u)
ybar1 <- mean(u)
n2 <- length(d)
ybar2 <- mean(d)
print(cbind(n1,ybar1,n2,ybar2))
```

```
##      n1 ybar1 n2 ybar2
## [1,]  5 10.34  6  9.05
```

- Upstream sample:  $n_1 = 5$  and  $\bar{y}_1 = 10.34$
- Downstream sample:  $n_2 = 6$  and  $\bar{y}_1 = 9.05$

- Parameters of the normal priors are:  $\mu_{1p} = 10$ ,  $\sigma_{1p}^2 = 4.5$ ,  $\mu_{2p} = 9$ , and  $\sigma_{2p}^2 = 4.0$
- The posterior distributions of  $\mu_1$  and  $\mu_2$ , respectively, are:  $N(\mu'_1, \sigma'^2_1)$  and  $N(\mu'_2, \sigma'^2_2)$ , where

$$\mu'_1 = \frac{\sigma^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{\sigma^2 + n_1 \sigma_{1p}^2} = \frac{4(10) + 5(4.5)(10.34)}{4 + 5(4.5)} \approx 10.2887,$$

$$\sigma'^2_1 = \frac{\sigma^2 \sigma_{1p}^2}{\sigma^2 + n_1 \sigma_{1p}^2} = \frac{4(4.5)}{4 + 5(4.5)} \approx 0.6792,$$

$$\mu'_2 = \frac{\sigma^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{\sigma^2 + n_2 \sigma_{2p}^2} = \frac{4(9) + 6(4)(9.05)}{4 + 6(4)} \approx 9.0429,$$

and

$$\sigma'^2_2 = \frac{\sigma^2 \sigma_{2p}^2}{\sigma^2 + n_2 \sigma_{2p}^2} = \frac{4(4)}{4 + 6(4)} \approx 0.5714$$

- The posterior distribution for  $\mu_d = \mu_1 - \mu_2$  is  $N(\mu'_d, \sigma'^2_d)$ , where:

$$\mu'_d = \mu'_1 - \mu'_2 = 10.2887 - 9.0429 = 1.2458,$$

and

$$\sigma'^2_d = \sigma'^2_1 + \sigma'^2_2 = 0.6792 + 0.5714 = 1.2506$$

- A 95% credible interval for  $\mu_d = \mu_1 - \mu_2$  is

$$\mu'_d \pm z_{\alpha/2} \sigma'_d \implies 1.2458 \pm 1.96(\sqrt{1.2506}) \implies (-0.9461, 3.4377)$$

- To test the hypothesis  $H_0 : \mu_d \leq 0$  against  $H_1 : \mu_d > 0$ , we compute  $P(\mu_d \leq 0 | data)$

```
round(pnorm(0, 1.2458, sqrt(1.2506)), 4)
```

```
## [1] 0.1326
```

- Since this probability is greater than the 5% level of significance, we fail to reject  $H_0$ .
- Therefore, the data is not sufficient to conclude that the amount of dissolved oxygen upstream is significantly greater than the amount of dissolved oxygen downstream

## When the variance $\sigma^2$ is unknown

When the common variance  $\sigma^2$  is unknown, we estimate it using the pooled sample variance given by

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Suppose we use independent “flat” priors for  $\mu_1$  and  $\mu_2$ , that is,  $\pi(\mu_1) = \pi(\mu_2) = 1$ . Just like before, the posterior distributions of  $\mu_1$  and  $\mu_2$  will be identical to the likelihoods of the sampling distributions. That is,  $\mu_1 \sim N(\mu'_1, \sigma'^2_1)$  and  $\mu_2 \sim N(\mu'_2, \sigma'^2_2)$ , where:

$$\begin{aligned}\mu'_1 &= \bar{y}_1 \\ \sigma'^2_1 &= \frac{s_p^2}{n_1} \\ \mu'_2 &= \bar{y}_2 \\ \sigma'^2_2 &= \frac{s_p^2}{n_2}\end{aligned}$$

As in the case where  $\sigma^2$  is known,  $\mu_d \sim N(\mu'_d, \sigma'^2_d)$ , where the parameters  $\mu'_d$  and  $\sigma'^2_d$  are defined as before.

A  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  is constructed based on the  $t$  distribution, and is given by

$$\mu'_d \pm t_{\frac{\alpha}{2}, df}(\sigma'_d), \text{ where: } df = n_1 + n_2 - 2$$

To test the hypothesis  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$ , we calculate the posterior probability of the null hypothesis using the  $t$  distribution with  $n_1 + n_2 - 2$  degrees of freedom.

To test the hypothesis  $H_0 : \mu_d = 0$  versus  $H_1 : \mu_d \neq 0$ , we construct the  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  based on the  $t$  distribution with  $n_1 + n_2 - 2$  degrees of freedom.

## When the variance $\sigma^2$ is unknown: an example

Let us revisit the previous example. This time we assume that the observations are a random sample from  $N(\mu_1, \sigma^2)$  and  $N(\mu_2, \sigma^2)$ , where  $\sigma^2$  is not known

Consider “flat” prior distributions for  $\mu_1$  and  $\mu_2$ .

- Find the posterior distributions of  $\mu_1$  and  $\mu_2$ , respectively.
- Find the posterior distribution of  $\mu_d = \mu_1 - \mu_2$ .
- Find a 95% credible interval of  $\mu_d = \mu_1 - \mu_2$ .

d. At the 5% level of significance, perform a Bayesian test of the hypothesis  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$

```
sp <- sqrt(((n1-1)*var(u)+(n2-1)*var(d))/(n1+n2-2))
print(sp^2)
```

```
## [1] 3.714111
```

When “flat” priors are used then the posterior distributions for  $\mu_1$  and  $\mu_2$  are, respectively,  $N(\mu'_1, \sigma'^2_1)$  and  $N(\mu'_2, \sigma'^2_2)$ , where

$$\mu'_1 = \bar{y}_1 = 10.34$$

$$\sigma'^2_1 = \frac{s_p^2}{n_1} \approx 0.743$$

$$\mu'_2 = \bar{y}_2 = 9.05$$

$$\sigma'^2_2 = \frac{s_p^2}{n_2} \approx 0.619$$

We also learned that  $\mu_d \sim N(\mu'_d, \sigma'^2_d)$ , where

$$\mu'_d = \bar{y}_1 - \bar{y}_2 = 1.29$$

and

$$\sigma'^2_d = \sigma'^2_1 + \sigma'^2_2 = 1.362$$

A  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  is given by

$$\begin{aligned} \mu'_d &\pm t_{\frac{\alpha}{2}, df} \sqrt{\sigma'^2_d \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \\ &\implies 1.29 \pm 2.262 \times \sqrt{1.362 \left( \frac{1}{5} + \frac{1}{6} \right)} \\ &\implies (-0.309, 2.889) \end{aligned}$$

To test the hypothesis  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$ , we compute  $P(\mu_d \leq 0 | data)$  using the code chunk below

```
print(pt(0,9))
```

```
## [1] 0.5
```

Since this probability is greater than the 5% level of significance, we fail to reject  $H_0$ . Therefore, the data is not sufficient to conclude that the amount of dissolved oxygen upstream is significantly greater than the amount of dissolved oxygen downstream.

## Unequal but known variances

Suppose we draw independent random samples from two normal distributions  $N(\mu_1, \sigma_1^2)$  and  $N(\mu_2, \sigma_2^2)$ , where  $\sigma_1^2 \neq \sigma_2^2$  but both are known. Then the likelihood of the sample observations are, respectively, given by

$$L(y_{11}, y_{12}, \dots, y_{1n_1} | \mu_1, \sigma_1^2) \propto \exp \left[ -\frac{n_1(\bar{y}_1 - \mu_1)^2}{2\sigma_1^2} \right]$$

and

$$L(y_{21}, y_{22}, \dots, y_{2n_2} | \mu_2, \sigma_2^2) \propto \exp \left[ -\frac{n_2(\bar{y}_2 - \mu_2)^2}{2\sigma_2^2} \right]$$

Let the priors be  $N(\mu_{1p}, \sigma_{1p}^2)$  and  $N(\mu_{2p}, \sigma_{2p}^2)$ , i. e.

$$\pi(\mu_1) \propto \exp \left[ -\frac{(\mu_1 - \mu_{1p})^2}{2\sigma_{1p}^2} \right]$$

and

$$\pi(\mu_2) \propto \exp \left[ -\frac{(\mu_2 - \mu_{2p})^2}{2\sigma_{2p}^2} \right]$$

Therefore, using the Bayes' theorem the posterior distributions of  $\mu_1$  and  $\mu_2$  are

$$\pi(\mu_1 | \bar{y}_1) \propto \exp \left[ -\frac{1}{2 \left( \frac{\sigma_1^2 \sigma_{1p}^2}{\sigma_1^2 + n_1 \sigma_{1p}^2} \right)} \left( \mu_1 - \frac{\sigma_1^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{\sigma_1^2 + n_1 \sigma_{1p}^2} \right) \right]$$

and

$$\pi(\mu_2 | \bar{y}_2) \propto \exp \left[ -\frac{1}{2 \left( \frac{\sigma_2^2 \sigma_{2p}^2}{\sigma_2^2 + n_2 \sigma_{2p}^2} \right)} \left( \mu_2 - \frac{\sigma_2^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{\sigma_2^2 + n_2 \sigma_{2p}^2} \right) \right]$$

## Unequal and unknown variances

The unknown variances  $\sigma_1^2$  and  $\sigma_2^2$  will be estimated by the sample variances  $S_1^2$  and  $S_2^2$ , respectively. Thus, the likelihood of the sample observations become

$$L(y_{11}, y_{12}, \dots, y_{1n_1} | \mu_1, S_1^2) \propto \exp \left[ -\frac{n_1(\bar{y}_1 - \mu_1)^2}{2S_1^2} \right]$$

and

$$L(y_{21}, y_{22}, \dots, y_{2n_2} | \mu_2, S_2^2) \propto \exp \left[ -\frac{n_2(\bar{y}_2 - \mu_2)^2}{2S_2^2} \right]$$

Assuming the same prior densities for  $\mu_1$  and  $\mu_2$  then the posterior densities of  $\mu_1$  and  $\mu_2$  are, respectively:

$$\pi(\mu_1 | \bar{y}_1) \propto \exp \left[ -\frac{1}{2 \left( \frac{S_1^2 \sigma_{1p}^2}{S_1^2 + n_1 \sigma_{1p}^2} \right)} \left( \mu_1 - \frac{S_1^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{S_1^2 + n_1 \sigma_{1p}^2} \right) \right]$$

and

$$\pi(\mu_2 | \bar{y}_2) \propto \exp \left[ -\frac{1}{2 \left( \frac{S_2^2 \sigma_{2p}^2}{S_2^2 + n_2 \sigma_{2p}^2} \right)} \left( \mu_2 - \frac{S_2^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{S_2^2 + n_2 \sigma_{2p}^2} \right) \right]$$

Test of one-sided hypotheses on  $\mu_d = \mu_1 - \mu_2$  will be based on the posterior distribution just like before. However, When the variances are estimated by the sample variances, the critical value used in constructing the credible interval is now based on the t distribution with degrees of freedom (Satterthwaite's approximation).

$$df \approx \frac{\left( \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right)^2}{\frac{\left( \frac{S_1^2}{n_1} \right)^2}{n_1-1} + \frac{\left( \frac{S_2^2}{n_2} \right)^2}{n_2-1}}$$

## Unequal and unknown variances: an example

Independent random samples of ceramic produced by two different processes (P1 & P2) were tested for hardness. The results were:

```
library(kableExtra)
P1 <- c(8.8, 9.6, 8.9, 9.2, 9.9, 9.4, 9.2, 10.1)
P2 <- c(9.2, 9.5, 10.2, 9.5, 9.8, 9.5, 9.3, 9.2)
kable(cbind(P1,P2))
```

P1	P2
8.8	9.2
9.6	9.5
8.9	10.2
9.2	9.5
9.9	9.8
9.4	9.5
9.2	9.3
10.1	9.2

```
print(cbind(m1=mean(P1), v1=var(P1), m2=mean(P2), v2=var(P2)))
```

```
##          m1      v1      m2      v2
## [1,] 9.3875 0.2098214 9.525 0.1135714
```

Let these observations be distributed as  $N(\mu_1, \sigma_1^2)$  and  $N(\mu_2, \sigma_2^2)$ , respectively, for Process 1 and Process 2. Further, suppose that  $\sigma_1^2 \neq \sigma_2^2$  and are both unknown. Let us use the following priors for  $\mu_1$  and  $\mu_2$ :  $N(10, 1)$  and  $N(11, 2)$ .

Consequently, the posterior distribution of  $\mu_1$  is  $N(\mu'_1, \sigma'^2_1)$ , where

$$\mu'_1 = \frac{S_1^2 \mu_{1p} + n_1 \sigma_{1p}^2 \bar{y}_1}{S_1^2 + n_1 \sigma_{1p}^2} = \frac{0.21(10) + 8(1)(9.39)}{0.21 + 8(1)} \approx 9.41$$

and

$$\sigma'^2_1 = \frac{S_1^2 \sigma_{1p}^2}{S_1^2 + n_1 \sigma_{1p}^2} = \frac{0.21(1)}{0.21 + 8(1)} \approx 0.026$$

The posterior distribution of  $\mu_2$  is  $N(\mu'_2, \sigma'^2_2)$ , where

$$\mu'_2 = \frac{S_2^2 \mu_{2p} + n_2 \sigma_{2p}^2 \bar{y}_2}{S_2^2 + n_2 \sigma_{2p}^2} = \frac{0.11(11) + 8(2)(9.53)}{0.11 + 8(2)} \approx 9.54$$

and

$$\sigma'^2_2 = \frac{S_2^2 \sigma_{2p}^2}{S_2^2 + n_2 \sigma_{2p}^2} = \frac{0.11(2)}{0.11 + 8(2)} \approx 0.014$$

Finally, the posterior distribution of  $\mu_d$  is  $N(\mu'_d, \sigma'^2_d)$ , where

$$\mu'_d = \mu'_1 - \mu'_2 = 9.41 - 9.54 = -0.13$$

and

$$\sigma_d^{2'} = \sigma_1^{2'} + \sigma_2^{2'} = 0.026 + 0.014 = 0.04$$

A  $(1 - \alpha) \times 100\%$  credible interval for  $\mu_d$  is

$$\mu_d' \pm t_{\frac{0.05}{2}, 13} \sqrt{\sigma_d^{2'}} \implies (-0.13 \pm 2.16\sqrt{0.04}) \implies (-0.562, 0.302)$$

Suppose we would like to test  $H_0 : \mu_1 = \mu_2$  versus  $H_1 : \mu_1 > \mu_2$ . This is equivalent to testing  $H_0 : \mu_d \leq 0$  versus  $H_1 : \mu_d > 0$

$$P(\mu_d \leq 0) \approx 0.5$$

```
pt(0,13)
```

```
## [1] 0.5
```

We fail to reject  $H_0$ , therefore, the data is not sufficient to indicate that  $\mu_1 > \mu_2$

### Comparing two means based on paired observations

- *Matched-pairs design:* experimental units are matched into pairs of similar units.
  - One of the units from each pair is assigned at random to the 1st treatment, and the other to the 2nd treatment
- *Repeated-measures design:* one set of experimental subjects were observed (data is collected) at two or more time periods.
  - The simplest case is the pre-test-post-test design wherein a pre evaluation is administered before a treatment (e. g. training) is conducted and the same (similar) evaluation is administered after the treatment
- We compute the pairwise differences  $d_i = y_{1i} - y_{2i}$  and assume that  $d_i \sim N(\mu_d, \sigma_d^2)$  with  $\sigma_d^2$  known
- We can treat the differences  $d_i$  as a sample from a normal distribution and do Bayesian inference using techniques in the previous lessons.

## Comparing two means based on paired observations: an example

An experiment was designed to determine whether a mineral supplement was effective in increasing annual yield in milk. Fifteen pairs of identical twin dairy cows were used as the experimental units. One cow from each pair was randomly assigned to the treatment group that received the supplement. The other cow from the pair was assigned to the control group that did not receive the supplement. The data is shown below.

Twin Set	Milk Yield: Control (liters)	Milk Yield: Treatment (liters)
1	3525	3340
2	4321	4279
3	4763	4910
4	4899	4866
5	3234	3125
6	3469	3680
7	3439	3965
8	3658	3849
9	3385	3297
10	3226	3124
11	3671	3218
12	3501	3246
13	3842	4245
14	3998	4186
15	4004	3711

```
ctrl <- c(3525, 4321, 4763, 4899, 3234, 3469, 3439,
       3658, 3385, 3226, 3671, 3501, 3842, 3998, 4004)
trt <- c(3340, 4279, 4910, 4866, 3125, 3680, 3965,
       3849, 3297, 3124, 3218, 3246, 4245, 4186, 3711)
d <- trt - ctrl
print(round(mean(d), 3))
```

## [1] 7.067

Suppose  $\sigma_d^2 = 270^2$ . Let the normal prior be  $N(0, 200^2)$ .

The posterior distribution is  $N(\mu', \sigma'^2)$ , where

$$\mu' = \frac{\sigma_d^2 \mu_p + n \sigma_p^2 \bar{y}_d}{\sigma_d^2 + n \sigma_p^2} = \frac{270^2(0) + 15(200^2)(7.067)}{270^2 + 15(200^2)} \approx 6.30$$

and

$$\sigma'^2 = \frac{\sigma_d^2 \sigma_p^2}{\sigma_d^2 + n \sigma_p^2} = \frac{270^2(200^2)}{270^2 + 15(200^2)} \approx 4333.482$$

NOTE:

When  $\sigma_d^2$  is unknown, we estimate it using the sample variance of the pairwise differences, consequently, the  $t$  distribution ( $df = n - 1$ ) will be used in creating the credible interval.