Estimation of Variance Components in Random Effects Model

Norberto E. Milla, Jr.

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Estimation of σ_ϵ^2

Recall:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}, i = 1, 2, \dots, t; j = 1, 2, \dots, r_i; \epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$$

- $\tau_i \sim N(0, \sigma_\tau^2)$
- $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$
- τ_i and ϵ_{ij} are independent
- $V(Y_{ij}) = \sigma_\tau^2 + \sigma_\epsilon^2$
- Note that we can rewrite SSE algebraically as

$$SSE = \sum_{i=1}^{t} \sum_{j=1}^{r_i} Y_{ij}^2 - \sum_{i=1}^{r_i} r_i \overline{Y}_{i.}^2$$

- Also,

$$E[Y_{ij}^2] = V(Y_{ij}) + [E(Y_{ij})]^2 = \sigma_\tau^2 + \sigma_\epsilon^2 + \mu^2$$

- It can be shown also algebraically that

$$\overline{Y}_{i.} = \mu + \tau_i + \frac{1}{r_i} \sum_{i=1}^{t} \epsilon_{ij}$$

- Thus,

$$V(\overline{Y}_{i.}) = \sigma_\tau^2 + \frac{\sigma_\epsilon^2}{r_i}$$

- Consequently,

$$
\begin{aligned}
E(SSE) &= \sum_{i=1}^{t}\sum_{j=1}^{r_i}(\sigma_\tau^2 + \sigma_\epsilon^2 + \mu^2) - \sum_{i=1}^{t} r_i(\sigma_\tau^2 + \frac{\sigma_\epsilon^2}{r_i} + \mu^2) \\
&= n\sigma_\epsilon^2 - t\sigma_\epsilon^2 \\
&= (n-t)\sigma_\epsilon^2
\end{aligned}
$$

- Hence,

$$
E(MSE) = E\left(\frac{SSE}{n-t}\right) = \sigma_\epsilon^2
$$

Estimation of σ_τ^2

- We can rewrite SSTr as follows:

$$
SSTr = \sum_{i=1}^{r_i} r_i \overline{Y}_{i.}^2 - n\overline{Y}_{..}^2
$$

- Using the same approach earlier, we have

$$
\overline{Y}_{..} = \mu + \frac{1}{n}\sum_{i=1}^{t} r_i\tau_i + \frac{1}{n}\sum_{i=1}^{t}\sum_{j=1}^{r_i}\epsilon_{ij}
$$

- It follows that

$$
V(\overline{Y}_{..}) = \frac{\sum_{i=1}^{t} r_i^2}{n^2}\sigma_\tau^2 + \frac{\sigma_\epsilon^2}{n}
$$

- Therefore,

$$
\begin{aligned}
E(SSTr) &= \sum_{i=1}^{t} r_i\left(\sigma_\tau^2 + \frac{\sigma_\epsilon^2}{r_i} + \mu^2\right) - n\left(\frac{\sum_{i=1}^{t} r_i^2\sigma_\tau^2}{n^2} + \frac{\sigma_\epsilon^2}{n} + \mu^2\right) \\
&= \left(n - \frac{\sum_{i=1}^{t} r_i^2}{n}\right)\sigma_\tau^2 + (t-1)\sigma_\epsilon^2
\end{aligned}
$$