Smoothing Methods for Time Series with Trend and Seasonality

Lesson 2.4

Holt-Winters' seasonal method

Holt (1957) and Winters (1960) extended Holt's method to capture seasonality. The Holt-Winters seasonal method comprises the forecast equation and three smoothing equations — one for the level l_t, one for the trend b_t, and one for the seasonal component s_t, and with corresponding smoothing parameters α, β^*, and γ. We use m to denote the frequency of the seasonality, i.e., the number of seasons in a year. For example, for quarterly data $m = 4$ and for monthly data $m = 12$.

There are two variations to this method that differ in the nature of the seasonal component: *additive method*, *multiplicative method*.

The *additive method* is preferred when the seasonal variations are roughly constant through the series, while the *multiplicative method* is preferred when the seasonal variations are changing proportional to the level of the series. With the additive method, the seasonal component is expressed in absolute terms in the scale of the observed series, and in the level equation the series is seasonally adjusted by subtracting the seasonal component. Within each year, the seasonal component will add up to approximately zero.

While with the multiplicative method, the seasonal component is expressed in relative terms (percentages), and the series is seasonally adjusted by dividing through by the seasonal component. Within each year, the seasonal component will sum up to approximately m.

The component form for the additive method is:

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + hb_t + s_{t+h-m(k+1)} \\
\ell_t &= \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma(y_t + \ell_{t-1} - b_{t-1}) + (1-\gamma)(s_{t-m})
\end{aligned}$$

where k is the integer part of $\frac{h-1}{m}$ which ensures that the estimates of the seasonal indices used for forecasting come from the final year of the sample.

The level equation shows a weighted average between the seasonally adjusted observation $(y_t - s_{t-m})$ and the non-seasonal forecast $(l_{t-1} + b_{t-1})$. The trend equation is identical to Holt's linear method and the seasonal equation shows a weighted average between the current seasonal index $(y_t + \ell_{t-1} - b_{t-1})$ and the seasonal index of the same season last year (i.e.,m time periods ago)

Holt-Winters' additive method

The equation for the seasonal component is often expressed as

$$s_t = \gamma^*(y_t - \ell_t) + (1 - \gamma^*)s_{t-m}$$

- If we substitute ℓ_t from the smoothing equation for the level of the component form above, we get

$$s_t = \gamma^*(1 - \alpha)(y_t - \ell_{t-1} - b_{t-1}) + [1 - \gamma^*(1 - \alpha)]s_{t-m}$$

This is identical to the smoothing equation for the seasonal component we specify here, with $\gamma = \gamma^*(1 - \alpha)$. The usual parameter restriction is $0 \leq \gamma^* \leq 1$, which translate to $0 \leq \gamma \leq 1 - \alpha$.

As an example, let us apply Holt-Winters' method with both additive and seasonality to forecast quarterly visitor nights in Australia spent by international tourists

```r
library(tidyverse)
library(fpp2)
library(forecast)
aust <- window(austourists,start=2005)
fit1 <- hw(aust,seasonal="additive",h=8)#additive seasonality
summary(fit1)
```

```
Forecast method: Holt-Winters' additive method

Model Information:
Holt-Winters' additive method

Call:
hw(y = aust, h = 8, seasonal = "additive")

  Smoothing parameters:
```

```
    alpha = 0.3063
    beta  = 1e-04
    gamma = 0.4263

  Initial states:
    l = 32.2597
    b = 0.7014
    s = 1.3106 -1.6935 -9.3132 9.6962

  sigma:  1.9494

     AIC     AICc      BIC
234.4171 239.7112 250.4748

Error measures:
                      ME     RMSE      MAE        MPE     MAPE      MASE
Training set 0.008115785 1.763305 1.374062 -0.2860248 2.973922 0.4502579
                    ACF1
Training set -0.06272507

Forecasts:
        Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
2016 Q1       76.09837 73.60011 78.59664 72.27761 79.91914
2016 Q2       51.60333 48.99039 54.21626 47.60718 55.59947
2016 Q3       63.96867 61.24582 66.69153 59.80443 68.13292
2016 Q4       68.37170 65.54313 71.20027 64.04578 72.69762
2017 Q1       78.90404 75.53440 82.27369 73.75061 84.05747
2017 Q2       54.40899 50.95325 57.86473 49.12389 59.69409
2017 Q3       66.77434 63.23454 70.31414 61.36069 72.18799
2017 Q4       71.17737 67.55541 74.79933 65.63806 76.71667
```

Holt-Winters' multiplicative method

The component form for the multiplicative method is:

$$
\begin{aligned}
\hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t+h-m(k+1)} \\
\ell_t &= \alpha\frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma\frac{y_t}{(\ell_{t-1} + b_{t-1}}) + (1-\gamma)s_{t-m}
\end{aligned}
$$

Let us apply Holt-Winters' method with multiplicative seasonality to the Australian international tourists data.

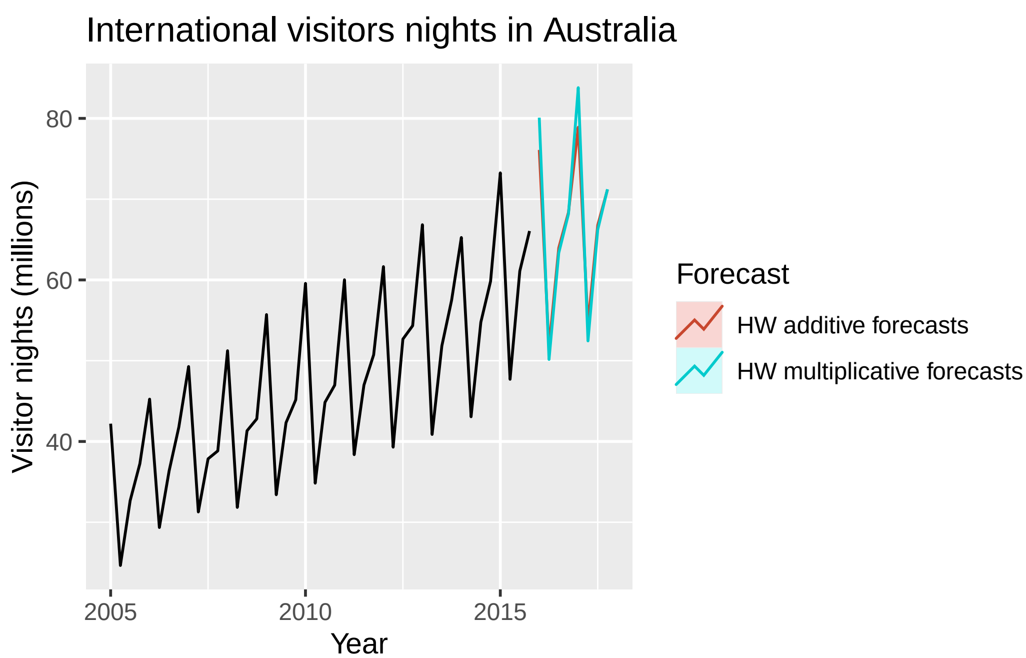
```r
fit2 <- hw(aust,seasonal="multiplicative",h=8)#multiplicative seasonality
summary(fit2)
```

```
Forecast method: Holt-Winters' multiplicative method

Model Information:
Holt-Winters' multiplicative method

Call:
hw(y = aust, h = 8, seasonal = "multiplicative")

  Smoothing parameters:
    alpha = 0.4406
    beta  = 0.0134
    gamma = 0.0023

  Initial states:
    l = 32.4875
    b = 0.6974
    s = 1.0237 0.9618 0.7704 1.2442

  sigma:  0.0367

     AIC     AICc      BIC
221.1313 226.4254 237.1890

Error measures:
                     ME     RMSE     MAE           MPE    MAPE      MASE
Training set 0.09206228 1.575631 1.25496 -0.0006505533 2.70539 0.4112302
                    ACF1
Training set -0.07955726

Forecasts:
        Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
2016 Q1       80.08894 76.31865 83.85922 74.32278 85.85509
2016 Q2       50.15482 47.56655 52.74309 46.19640 54.11324
2016 Q3       63.34322 59.80143 66.88502 57.92652 68.75993
2016 Q4       68.17810 64.08399 72.27221 61.91670 74.43950
```

```
2017 Q1       83.80112 78.43079 89.17146 75.58790 92.01434
2017 Q2       52.45291 48.88795 56.01787 47.00077 57.90504
2017 Q3       66.21274 61.46194 70.96353 58.94702 73.47845
2017 Q4       71.23205 65.85721 76.60690 63.01194 79.45217
```

The code chunk below will plot the series and the forecast for the 2 methods

```
autoplot(aust) +
  autolayer(fit1, series="HW additive forecasts", PI=FALSE) +
  autolayer(fit2, series="HW multiplicative forecasts", PI=FALSE) +
  xlab("Year") +
  ylab("Visitor nights (millions)") +
  ggtitle("International visitors nights in Australia") +
  guides(colour=guide_legend(title="Forecast"))
```



Holt-Winters' damped method

Damping is possible with both additive and multiplicative Holt-Winters' methods. Holt-Winters method with a damped trend and multiplicative seasonality often provides accurate and robust forecasts for seasonal data.

$$
\begin{aligned}
\hat{y}_{t+h|t} &= \left[\ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t\right] s_{t+h-m(k+1)} \\
\ell_t &= \alpha \left(\frac{y_t}{s_{t-m}}\right) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1} \\
s_t &= \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1-\gamma)s_{t-m}
\end{aligned}
$$

- The Holt-Winters method with a damped trend and multiplicative seasonality can be implemented using the following code chunk

```
hw(y, damped=TRUE, seasonal="multiplicative")
```

As an example, consider the *hyndsight* data which contains the daily pageviews on the *Hyndsight* blog of Prof. Rob Hyndman. It is a daily time series from April 30, 2014 to April 30, 2015

```
hs.ts <- subset(hyndsight,end=length(hyndsight)-35)
fc <- hw(hs.ts, damped = TRUE, seasonal="multiplicative", h=35)
summary(fc)
```

```
Forecast method: Damped Holt-Winters' multiplicative method

Model Information:
Damped Holt-Winters' multiplicative method

Call:
hw(y = hs.ts, h = 35, seasonal = "multiplicative", damped = TRUE)

  Smoothing parameters:
    alpha = 0.4189
    beta  = 1e-04
    gamma = 0.0964
    phi   = 0.9533

  Initial states:
    l = 1169.6237
    b = 0.3841
    s = 1.2235 1.0607 0.6176 0.7694 1.0574 1.1257
           1.1455
```

```
  sigma:  0.1984

     AIC     AICc      BIC 
5529.690 5530.842 5579.078 

Error measures:
                   ME     RMSE      MAE       MPE     MAPE      MASE      ACF1
Training set 2.262107 222.5991 156.6953 -2.139331 12.86201 0.7069489 0.1305779

Forecasts:
         Point Forecast     Lo 80    Hi 80      Lo 95    Hi 95
48.57143       2035.252 1517.6913 2552.813 1243.71112 2826.793
48.71429       1810.067 1309.5271 2310.608 1044.55716 2575.578
48.85714       1353.964  951.4964 1756.431  738.44305 1969.485
49.00000       1015.333  693.7201 1336.945  523.46870 1507.196
49.14286       1523.818 1012.9460 2034.689  742.50688 2305.129
49.28571       2009.983 1300.6290 2719.337  925.11963 3094.847
49.42857       2044.262 1288.1910 2800.333  887.95102 3200.573
49.57143       2038.604 1235.0056 2842.203  809.60616 3267.602
49.71429       1813.044 1070.2535 2555.834  677.04402 2949.044
49.85714       1356.187  780.1671 1932.207  475.24057 2237.133
50.00000       1016.997  570.1636 1463.831  333.62415 1700.370
50.14286       1526.313  833.9390 2218.687  467.41837 2585.208
50.28571       2013.270 1071.9700 2954.570  573.67575 3452.865
50.42857       2047.601 1062.3701 3032.833  540.82019 3554.382
50.57143       2041.933 1018.6513 3065.215  476.95855 3606.908
50.71429       1816.001  882.4209 2749.582  388.21331 3243.789
50.85714       1358.397  642.7819 2074.012  263.95814 2452.836
51.00000       1018.653  469.2741 1568.032  178.45058 1858.855
51.14286       1528.795  685.4631 2372.128  239.02995 2818.561
51.28571       2016.542  879.6943 3153.389  277.88373 3755.199
51.42857       2050.925  870.1707 3231.680  245.11690 3856.734
51.57143       2045.248  831.5772 3258.919  189.09882 3901.397
51.71429       1818.947  718.5171 2919.377  135.98477 3501.909
51.85714       1360.599  521.9036 2199.294   77.92525 2643.272
52.00000       1020.303  379.8370 1660.768   40.79500 1999.810
52.14286       1531.270  552.9350 2509.604   35.03593 3027.503
52.28571       2019.803  706.9877 3332.618   12.02521 4027.581
52.42857       2054.240  696.5298 3411.951  -22.19881 4130.680
52.57143       2048.553  661.2225 3435.884  -73.18613 4170.293
52.71429       1821.885  568.5476 3075.222  -94.92900 3738.699
52.85714       1362.795  410.8094 2314.781  -93.14132 2818.732
53.00000       1021.949  297.2978 1746.600  -86.30930 2130.207
```

```
53.14286       1533.739  430.1533 2637.325 -154.04978 3221.528
53.28571       2023.059  546.3999 3499.718 -235.29612 4281.414
53.42857       2057.550  534.5215 3580.579 -271.72132 4386.822
```

```r
hs.ts <- subset(hyndsight,end=length(hyndsight)-35)
fc <- hw(hs.ts, damped = TRUE, seasonal="multiplicative", h=35)
```

```r
autoplot(hyndsight) +
  autolayer(fc, series="HW multi damped", PI=FALSE)+
  guides(colour=guide_legend(title="Daily forecasts"))
```