Lesson 3.3

Smoothing Methods for Time Series with Trend and Seasonality

Introduction

Holt (1957) and Winters (1960) extended Holt's method to capture seasonality. The Holt-Winters seasonal method comprises the forecast equation and three smoothing equations — one for the level l_t, one for the trend b_t, and one for the seasonal component s_t, and with corresponding smoothing parameters α, β^*, and γ. We use m to denote the frequency of the seasonality, i.e., the number of seasons in a year. For example, for quarterly data $m = 4$ and for monthly data $m = 12$.

There are two variations to this method that differ in the nature of the seasonal component: *additive method*, *multiplicative method*.

The *additive method* is preferred when the seasonal variations are roughly constant through the series, while the *multiplicative method* is preferred when the seasonal variations are changing proportional to the level of the series. With the additive method, the seasonal component is expressed in absolute terms in the scale of the observed series, and in the level equation the series is seasonally adjusted by subtracting the seasonal component. Within each year, the seasonal component will add up to approximately zero.

While with the multiplicative method, the seasonal component is expressed in relative terms (percentages), and the series is seasonally adjusted by dividing through by the seasonal component. Within each year, the seasonal component will sum up to approximately m.

Holt-Winters' additive method

The component form for the additive method is:

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + hb_t + s_{t+h-m(k+1)} \\
\ell_t &= \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma(y_t + \ell_{t-1} - b_{t-1}) + (1-\gamma)(s_{t-m})
\end{aligned}$$

where k is the integer part of $\frac{h-1}{m}$ which ensures that the estimates of the seasonal indices used for forecasting come from the final year of the sample.

The level equation shows a weighted average between the seasonally adjusted observation $(y_t - s_{t-m})$ and the non-seasonal forecast $(l_{t-1} + b_{t-1})$. The trend equation is identical to

Holt's linear method and the seasonal equation shows a weighted average between the current seasonal index $(y_t + \ell_{t-1} - b_{t-1})$ and the seasonal index of the same season last year (i.e.,m time periods ago)

The equation for the seasonal component is often expressed as

$$s_t = \gamma^*(y_t - \ell_t) + (1 - \gamma^*)s_{t-m}$$

If we substitute ℓ_t from the smoothing equation for the level of the component form above, we get

$$s_t = \gamma^*(1 - \alpha)(y_t - \ell_{t-1} - b_{t-1}) + [1 - \gamma^*(1 - \alpha)]s_{t-m}$$

This is identical to the smoothing equation for the seasonal component we specify here, with $\gamma = \gamma^*(1 - \alpha)$. The usual parameter restriction is $0 \leq \gamma^* \leq 1$, which translate to $0 \leq \gamma \leq 1 - \alpha$.

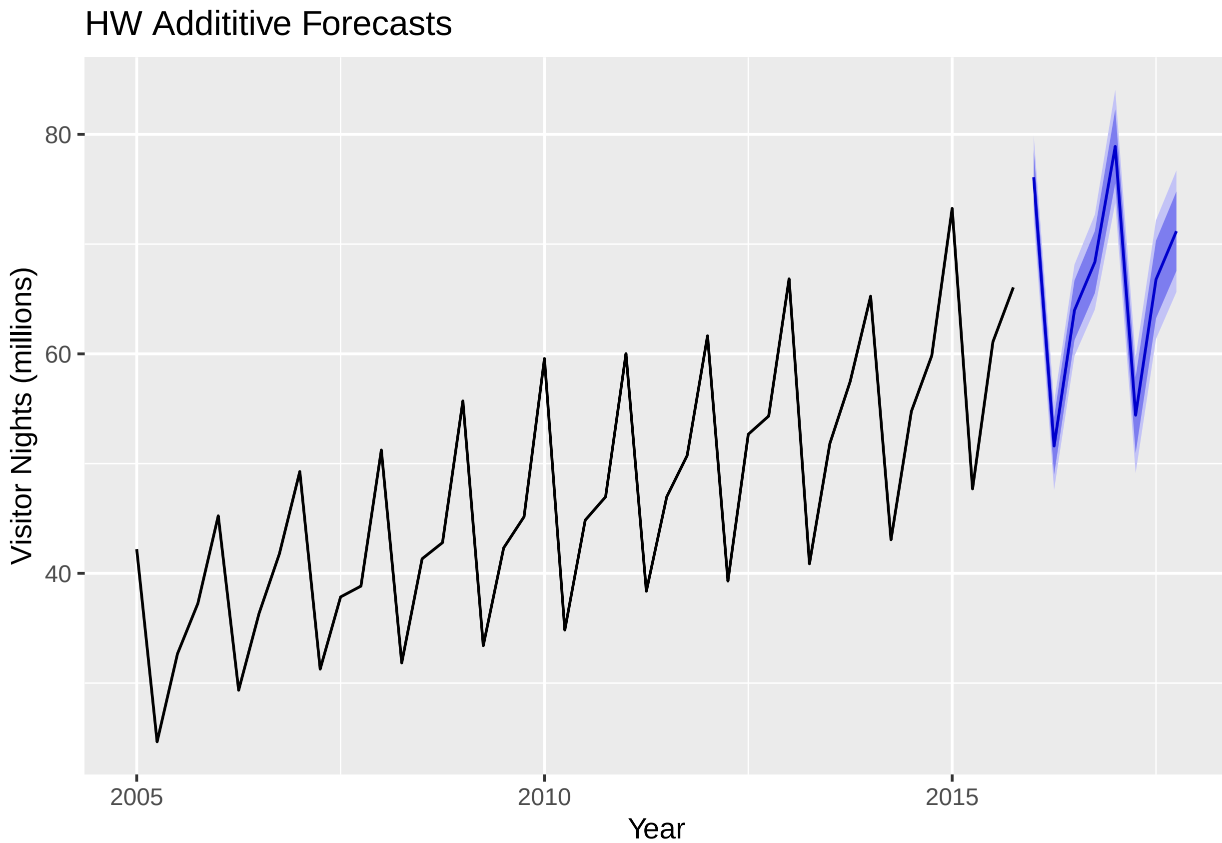
As an example, let us apply Holt-Winters' method with both additive and seasonality to forecast quarterly visitor nights in Australia spent by international tourists.

```
library(tidyverse)
library(fpp2)
library(forecast)
aust <- window(austourists,start=2005)
fit1 <- hw(aust,seasonal="additive",h=8)#additive seasonality
summary(fit1)
```

```
##
## Forecast method: Holt-Winters' additive method
##
## Model Information:
## Holt-Winters' additive method
##
## Call:
## hw(y = aust, h = 8, seasonal = "additive")
##
##   Smoothing parameters:
##     alpha = 0.3063
##     beta  = 1e-04
##     gamma = 0.4263
##
##   Initial states:
##     l = 32.2597
##     b = 0.7014
##     s = 1.3106 -1.6935 -9.3132 9.6962
```

```
##
##   sigma:  1.9494
##
##      AIC     AICc      BIC
## 234.4171 239.7112 250.4748
##
## Error measures:
##                       ME     RMSE      MAE        MPE     MAPE      MASE
## Training set 0.008115785 1.763305 1.374062 -0.2860248 2.973922 0.4502579
##                     ACF1
## Training set -0.06272507
##
## Forecasts:
##         Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
## 2016 Q1       76.09837 73.60011 78.59664 72.27761 79.91914
## 2016 Q2       51.60333 48.99039 54.21626 47.60718 55.59947
## 2016 Q3       63.96867 61.24582 66.69153 59.80443 68.13292
## 2016 Q4       68.37170 65.54313 71.20027 64.04578 72.69762
## 2017 Q1       78.90404 75.53440 82.27369 73.75061 84.05747
## 2017 Q2       54.40899 50.95325 57.86473 49.12389 59.69409
## 2017 Q3       66.77434 63.23454 70.31414 61.36069 72.18799
## 2017 Q4       71.17737 67.55541 74.79933 65.63806 76.71667
```

```r
autoplot(aust) +
  autolayer(fit1, PI = TRUE) +
  labs(x = "Year", y = "Visitor Nights (millions)") +
  ggtitle("HW Additive Forecasts")
```



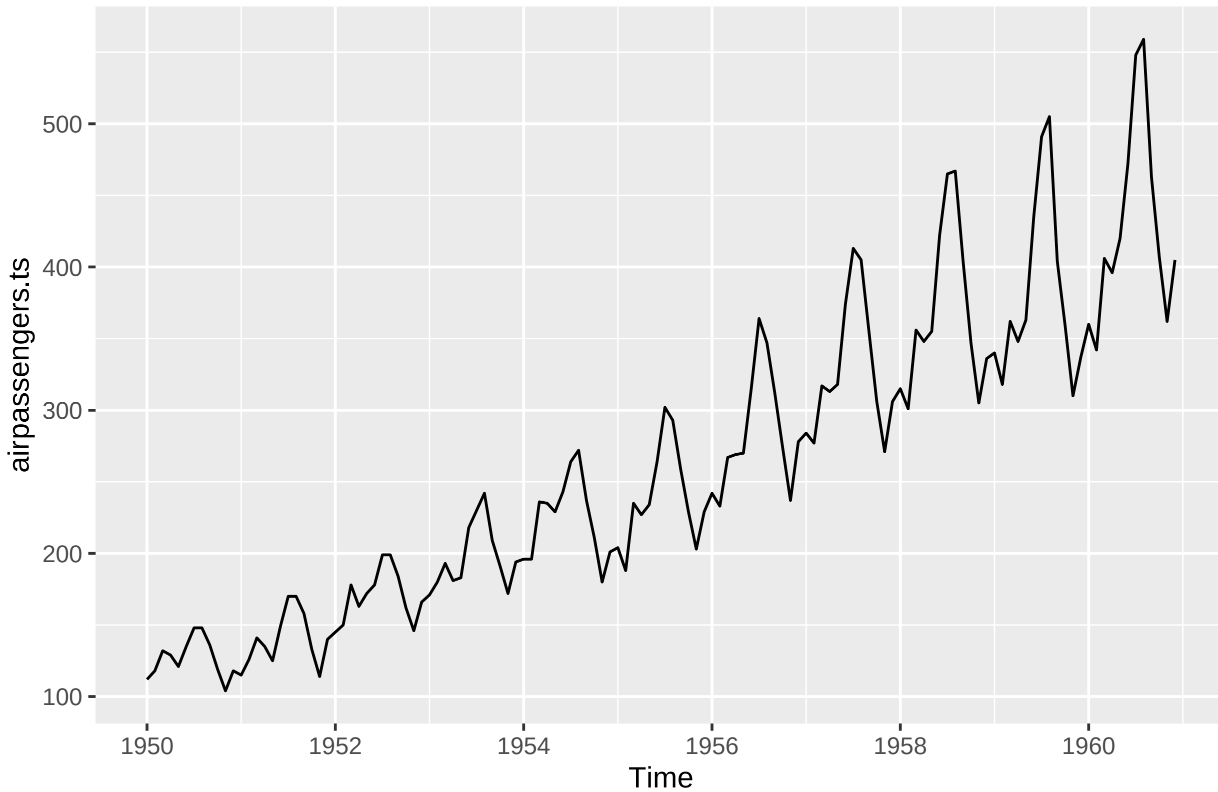
Holt-Winters' multiplicative method

The component form for the multiplicative method is:

$$
\begin{aligned}
\hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t+h-m(k+1)} \\
\ell_t &= \alpha \frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1}})) + (1-\gamma)s_{t-m}
\end{aligned}
$$

Let us apply Holt-Winters' method with multiplicative seasonality to air passengers data.

```
airpassengers <- read.csv("AirPassengers.csv")
airpassengers.ts <- ts(airpassengers$Passengers,
                       frequency = 12,
                       start = c(1950, 1),
                       end = c(1960, 12))
autoplot(airpassengers.ts)
```



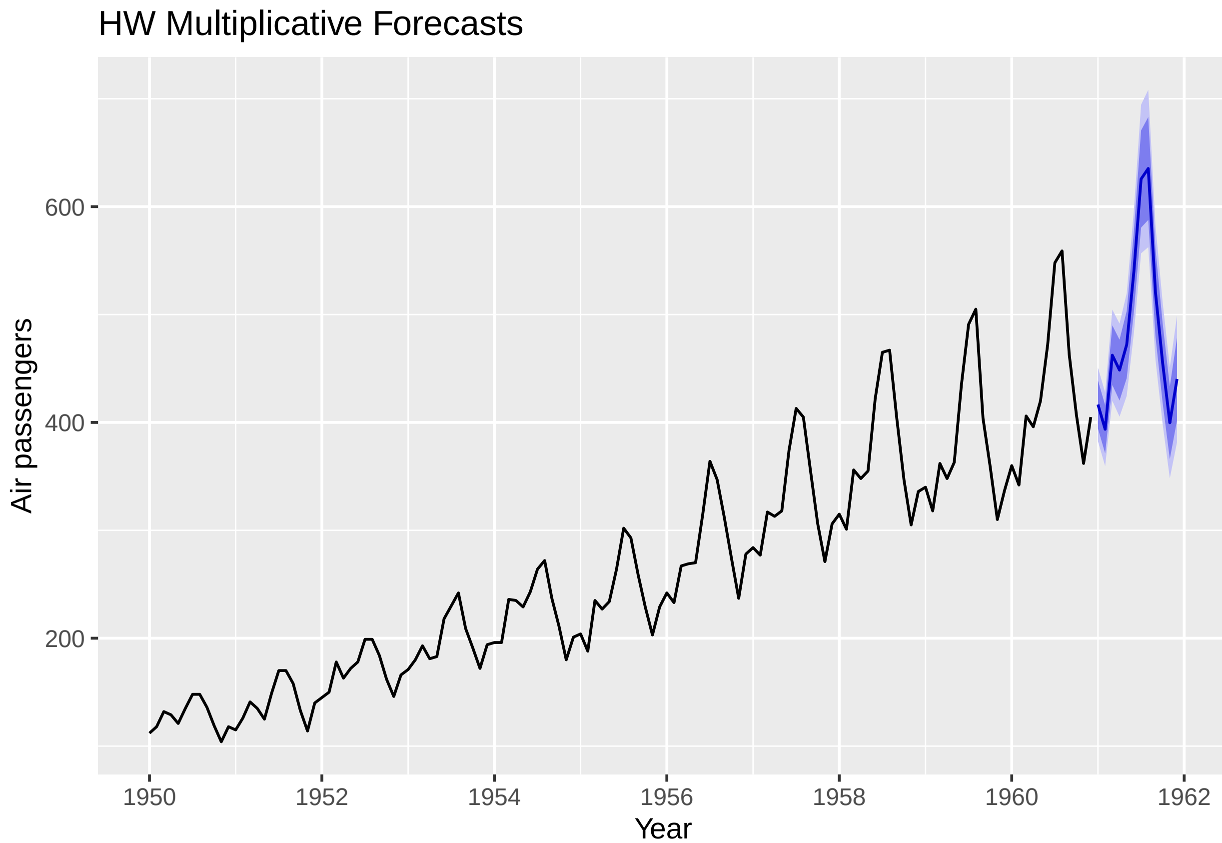
The code chunk below will plot the series and the forecast for the 2 methods

```
fit2 <- hw(airpassengers.ts,seasonal="multiplicative",h=12)#multiplicative seasonality
summary(fit2)
```

```
##
## Forecast method: Holt-Winters' multiplicative method
##
## Model Information:
## Holt-Winters' multiplicative method
##
## Call:
## hw(y = airpassengers.ts, h = 12, seasonal = "multiplicative")
##
##   Smoothing parameters:
##     alpha = 0.3392
##     beta  = 0.0105
##     gamma = 0.6534
##
##   Initial states:
##     l = 122.569
```

```
##     b = 1.51
##     s = 0.9296 0.7992 0.9096 1.0615 1.1326 1.177
##            1.0374 0.9334 1.0075 1.0984 0.9852 0.9288
##
##   sigma:  0.0418
##
##      AIC     AICc      BIC
## 1270.460 1275.829 1319.468
##
## Error measures:
##                    ME     RMSE      MAE       MPE    MAPE      MASE      ACF1
## Training set 1.369382 9.949946 7.533284 0.2993092 2.99775 0.2473985 0.3047973
##
## Forecasts:
##          Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
## Jan 1961       416.6188 394.3272 438.9105 382.5267 450.7110
## Feb 1961       393.6628 371.3633 415.9624 359.5587 427.7670
## Mar 1961       462.3467 434.7187 489.9747 420.0934 504.6001
## Apr 1961       448.5228 420.3370 476.7085 405.4164 491.6291
## May 1961       472.2089 441.0871 503.3307 424.6123 519.8055
## Jun 1961       540.0026 502.7660 577.2392 483.0542 596.9510
## Jul 1961       625.6443 580.6024 670.6862 556.7586 694.5300
## Aug 1961       635.3948 587.7281 683.0615 562.4948 708.2947
## Sep 1961       520.6261 479.9980 561.2542 458.4908 582.7614
## Oct 1961       455.1924 418.2998 492.0850 398.7701 511.6147
## Nov 1961       399.6811 366.0861 433.2762 348.3020 451.0603
## Dec 1961       440.2986 401.9675 478.6297 381.6763 498.9209
```

```
autoplot(airpassengers.ts) +
  autolayer(fit2, PI=TRUE) +
    labs(x = "Year", y = "Air passengers") +
  ggtitle("HW Multiplicative Forecasts")
```



Holt-Winters' damped method

Damping is possible with both additive and multiplicative Holt-Winters' methods. Holt-Winters method with a damped trend and multiplicative seasonality often provides accurate and robust forecasts for seasonal data.

$$\hat{y}_{t+h|t} = \left[\ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t\right] s_{t+h-m(k+1)}$$
$$\ell_t = \alpha\left(\frac{y_t}{s_{t-m}}\right) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$$
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$$
$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1-\gamma) s_{t-m}$$

The Holt-Winters method with a damped trend and multiplicative seasonality can be implemented using the following code chunk

```
hw(y, damped=TRUE, seasonal="multiplicative")
```

As an example we apply the Holt-Winters' damped method to the air passengers data.

```r
fit3 <- hw(airpassengers.ts, damped = TRUE, seasonal="multiplicative", h=12)
summary(fit3)
```

```
##
## Forecast method: Damped Holt-Winters' multiplicative method
##
## Model Information:
## Damped Holt-Winters' multiplicative method
##
## Call:
## hw(y = airpassengers.ts, h = 12, seasonal = "multiplicative",
##     damped = TRUE)
##
##   Smoothing parameters:
##     alpha = 0.8096
##     beta  = 0.0186
##     gamma = 1e-04
##     phi   = 0.9772
##
##   Initial states:
##     l = 120.9009
##     b = 1.8207
##     s = 0.8897 0.8008 0.921 1.0614 1.245 1.2484
##            1.1155 0.9764 0.9671 1.0027 0.868 0.9039
##
##   sigma:  0.0401
##
##      AIC     AICc      BIC
## 1259.896 1265.949 1311.786
##
## Error measures:
##                    ME     RMSE      MAE       MPE     MAPE      MASE
## Training set 1.431481 8.482344 6.764148 0.4725379 2.857529 0.2221395
##                     ACF1
## Training set -0.03712534
##
## Forecasts:
##          Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
## Jan 1961       412.4977 391.3256 433.6698 380.1177 444.8776
## Feb 1961       397.6365 371.1686 424.1045 357.1573 438.1158
## Mar 1961       461.0355 424.4562 497.6148 405.0923 516.9787
## Apr 1961       446.2415 405.7653 486.7178 384.3384 508.1446
## May 1961       452.0866 406.3673 497.8058 382.1650 522.0081
## Jun 1961       518.2883 460.8212 575.7555 430.3999 606.1767
```

```
## Jul 1961       581.9212 512.0233 651.8191 475.0216 688.8209
## Aug 1961       582.2161 507.1399 657.2923 467.3969 697.0353
## Sep 1961       497.9653 429.5175 566.4132 393.2834 602.6473
## Oct 1961       433.4040 370.2616 496.5465 336.8360 529.9721
## Nov 1961       377.9462 319.8578 436.0346 289.1076 466.7847
## Dec 1961       421.1556 353.1374 489.1739 317.1307 525.1805
```

```r
autoplot(airpassengers.ts)+
  autolayer(fit2, PI=TRUE) +
    labs(x = "Year", y = "Air passengers")+
  ggtitle("HW Multiplicative with damp")
```