1a) $$f(x|\theta) = \frac{x}{\theta^2} e^{\{-\theta^3/2x^2\}}$$

$$\ell(\theta|x) = p(x|\theta) = \prod^{n} \frac{x_i}{\theta^2} e^{\{-\theta^3/2x_i^2\}}$$

$$= \left[\frac{x_1}{\theta^2} e^{\{-\theta^3/2x_1^2\}}\right]\left[\frac{x_2}{\theta^2} e^{\{-\theta^3/2x_2^2\}}\right] \cdots \left[\frac{x_n}{\theta^2} e^{\{-\theta^3/2x_n^2\}}\right]$$

$$= \frac{1}{\theta^{2n}} \prod^{n} x_i e^{\{-\frac{1}{2}\theta^3 \sum^{n} \frac{1}{x_i^2}\}}$$

Now, $\mathcal{L}(\theta|x) = \ln(\ell(\theta|x))$

$$= \ln\left[\frac{1}{\theta^{2n}} \prod^{n} x_i e^{\{-\frac{1}{2}\theta^3 \sum^{n} \frac{1}{x_i^2}\}}\right]$$

$$= \ln \frac{1}{\theta^{2n}} + \ln \prod^{n} x_i + \ln e^{\{-\frac{1}{2}\theta^3 \sum^{n} \frac{1}{x_i^2}\}}$$

$$= -2n \ln \theta + \ln \prod^{n} x_i - \frac{1}{2}\theta^3 \sum^{n} \frac{1}{x_i^2}$$

Then $$\frac{d\mathcal{L}}{d\theta} = -2n \frac{d\mathcal{L}}{d\theta}(\ln \theta) + \underbrace{\frac{d\mathcal{L}}{d\theta}\left(\ln \prod^{n} x_i\right)}_{0} - \frac{1}{2}\sum^{n} \frac{1}{x_i^2} \frac{d\mathcal{L}}{d\theta}(\theta^3)$$

$$= \frac{-2n}{\theta} - \frac{1}{2} \sum^{n} \frac{1}{x_i^2} \cdot 3\theta^2$$

1 a) cont'd

To find maximum:

$$\frac{-2n}{\theta} - \frac{1}{2}\sum^{n}\frac{1}{x_i^2}\cdot 3\theta^2 = 0$$

$$\Rightarrow -2n - \frac{3}{2}\theta^3\sum^{n}\frac{1}{x_i^2} = 0$$

$$\Rightarrow -\frac{3}{2}\theta^3\sum^{n}\frac{1}{x_i^2} = 2n$$

$$\Rightarrow \theta^3\sum^{n}\frac{1}{x_i^2} = \frac{-4}{3}n$$

$$\Rightarrow \theta^3 = \frac{-4}{3}n\cdot\frac{1}{\sum^{n}\frac{1}{x_i^2}}$$

$$\Rightarrow \theta^3 = \frac{-4n}{3\sum^{n} 1/x_i^2}$$

$$\boxed{\Rightarrow \theta = \sqrt[3]{\frac{-4n}{3\sum^{n} 1/x_i^2}}}$$

1.b $f(x|\alpha,\theta) = (\alpha\theta)^{\alpha} x^{-\theta}, \; x \geq 0, \; \alpha > 0, \; \theta > 0$

$$\ell(\alpha,\theta|x) = p(x|\alpha,\theta) = \prod^{n} (\alpha\theta)^{\alpha} x_i^{-\theta}$$

$$= \left[(\alpha\theta)^{\alpha} x_1^{-\theta}\right]\left[(\alpha\theta)^{\alpha} x_2^{-\theta}\right] \cdots \left[(\alpha\theta)^{\alpha} x_n^{-\theta}\right]$$

$$= (\alpha\theta)^{n\alpha}\left[\prod^{n} x_i\right]^{-\theta}$$

$$\mathcal{L}(\alpha,\theta|x) = \ln\left[(\alpha\theta)^{n\alpha}\left[\prod^{n} x_i\right]^{-\theta}\right]$$

$$= n\alpha \ln(\alpha\theta) + -\theta \ln\left(\prod^{n} x_i\right)$$

Then $\frac{d\mathcal{L}}{d\theta} = n\alpha \frac{d\mathcal{L}}{d\theta} \ln(\alpha\theta) - \ln\left(\prod^{n} x_i\right) \frac{d\mathcal{L}}{d\theta}\theta$

$$= \frac{n\alpha}{\theta} - \ln\left(\prod^{n} x_i\right)$$

$$= \frac{n\hat{\alpha}}{\theta} - \sum^{n} \ln(x_i)$$

Setting partial equal to 0 gives

$$\frac{n\alpha}{\theta} - \sum^{n}(\ln x_i) = 0 \Rightarrow \frac{n\alpha}{\theta} = \sum^{n}(\ln x_i)$$

$$\boxed{\Rightarrow \frac{n\alpha}{\sum^{n}(\ln x_i)} = \theta}$$

1.c Find MLE for $f(x|\theta) = \frac{2x}{\theta^2}$ $\quad 0 \le x \le \theta$, $\quad \theta > 0$

Then,

$$\ell(\theta|x) = p(x|\theta) = \prod^{n} \frac{2x_i}{\theta^2} \quad 0 \le x_i \le \theta$$

$$= \frac{2^n \prod^{n} x_i}{\theta^{2n}} \leftarrow \text{constant!}$$

$$= \begin{cases} \frac{2^n \prod^{n} x_i}{\theta^{2n}} & 0 \le x_i \le \theta, \ldots, 0 \le x_n \le \theta \\ 0 & \text{otherwise} \end{cases}$$

Thus, if any $x_i \in [0, \theta]$ then the likelihood would equal 0.

That is,

$$\ell(\theta|x) = 0 \text{ if } \theta < \max(x_1, \ldots x_n)$$

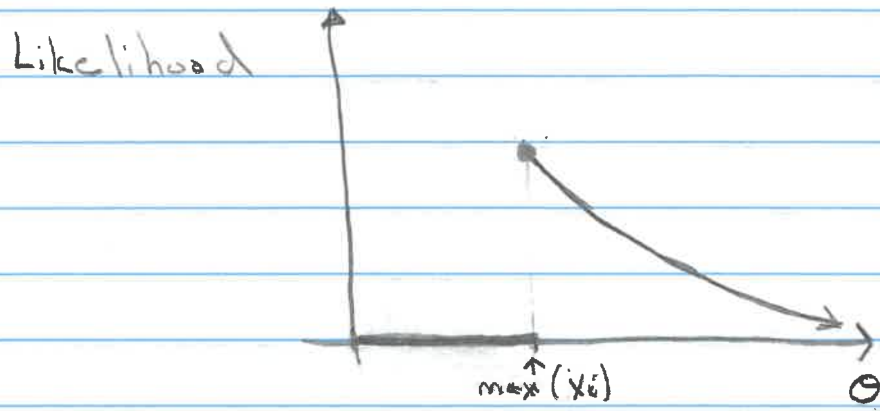
and

$$\ell(\theta|x) = \frac{2^n \prod^{n} x_i}{\theta^{2n}} \text{ if } \theta \ge \max(x_1, \ldots, x_n)$$

Therefore, taking $c = 2^n \prod^{n} x_i$ and looking at the graph of $\ell(\theta) = c/\theta^{2n}$, when $0 \le x_i \le \theta$ holds we find $\boxed{\hat{\theta} = \max(x_1, \ldots, x_n) \text{ is the MLE}}$

1.c) cont'd

Picture!



Likelihood function for problem 1.c $\quad \ell = \dfrac{c}{\theta^{2n}}$

where $c = 2^n \prod^n x_i$

2. a) Bayes Rule

$$P(C|x) = \frac{\overset{\text{prior}}{P(C)}\,\overset{\text{likelihood}}{P(x|C)}}{\underset{\text{Evidence}}{P(x)}}$$

posterior

We are given $P(x|C_1) = \begin{cases} 1/8 & \text{if } 1 \le x < 9 \\ 0 & \text{o.w.} \end{cases}$

and $P(x|C_2) = \begin{cases} 1/9\,(x-2) & \text{if } 2 \le x < 5 \\ 1/9\,(8-x) & \text{if } 5 \le x < 8 \\ 0 & \text{o.w.} \end{cases}$

a) Assuming equal priors, $P(C_1) = P(C_2) = 0.5$
Classify an object with the attribute value $x = 7$.

Well, $P(C_1|x) = \dfrac{P(C_1)P(x|C_1)}{\underset{P(x)=}{P(C_1)P(x|C_1) + P(C_2)P(x|C_2)}}$

and

$P(C_2|x) = \dfrac{P(C_2)P(x|C_2)}{\underset{P(x)=}{P(C_1)P(x|C_1) + P(C_2)P(x|C_2)}}$

Can ignore $P(x)$ for the classification since both divide by $P(x)$!

When $x = 7$

$P(x|C_1) = 1/8 = 0.125$
$P(x|C_2) = 1/9\,(8-7) = 1/9\,(1) = 1/9 = 0.\overline{11}$

Since $P(C_1) = P(C_2)$ and denominator in both formulas is $P(x)$ we simply compare $P(x|C_1)$ and $P(x|C_2)$.

$\boxed{\text{Since } 0.125 > 0.111,\ x \text{ is classified as } C_1}$

2. b)

when $x = 4$

$P(x|C_1) = 1/8 = 0.125$

$P(x|C_2) = 1/9(4-2) = 2/9 = 0.2\overline{2}$

But we need to take priors into account since $P(C_1) \neq P(C_2)$!

$P(C_1)P(x|C_1) = (0.6)(0.125) = 0.075$

$P(C_2)P(x|C_2) \approx (0.40)(0.222) = 0.0888$

$\boxed{\text{Since } 0.0888 > 0.075 \text{ we classify this object as } C_2.}$

2.c) $p(\text{mistake}) = p(X \in R_1, C_2) + p(X \in R_2, C_1)$

$$= \int_{R_1} p(x, C_2)\,dx + \int_{R_2} p(x, c_1)\,dx$$

Regions are symmetrical at $X=5$ so find probability and multiply by 2 to get prob. of misclassification at the end.

Also, from $[0, 2]$ the class C_2 has 0 probability so limits of integration for C_2 in R_1 start at 2, not 1. Thus,

$$p(\text{mistake})_{\frac{1}{2}} = \int_2^{5-\alpha} p(x, C_2)\,P(C_2)\,dx + \int_{5-\alpha}^{5} p(x, c_1)\,P(c_1)\,dx$$

$$= \int_2^{5-\alpha} 1/9(x-2)(1/2)\,dx + \int_{5-\alpha}^{5} 1/8\,(1/2)\,dx$$

$$= 1/18\left[\tfrac{1}{2}x^2 - 2x\right]_2^{5-\alpha} + 1/16\,x\Big|_{5-\alpha}^{5}$$

$$= 1/18\left[(1/2\,\alpha^2 - 5\alpha + 25/2) - 10 + 2\alpha - [-2]\right] + \alpha/16$$

$$= 1/18\left[1/2\,\alpha^2 - 3\alpha + 9/2\right] + \alpha/16$$

$$= 1/36\,\alpha^2 - 1/6\,\alpha + 1/4 + \alpha/16$$

★ Equation

$$= 1/36\,\alpha^2 - 5/48\,\alpha + 1/4$$

2.c Cont'd

Want to mimize, so take the derivative and set it equal to zero.

Can find α just using the equation I found for $x \le 5$

$$\frac{d}{dx}\left[\frac{1}{36}\alpha^2 - \frac{5}{48}\alpha + \frac{1}{4}\right] = 0$$

parabola opening up, min should be at vertex but derivative also works.

$$\Rightarrow \frac{1}{18}\alpha - \frac{5}{48} = 0$$

so, $\alpha = 15/8 = \boxed{1.875}$

To find probability put α in ☆ Equation and multiply by 2:

$$p(\text{mistake}) = 2\left[1/36\,(1.875)^2 - \frac{5}{48}(1.875) + 1/4\right]$$

$$\boxed{= 0.305}$$

3 a) If you replace the discrimant function of $C_2 + C_3$ with the discriminant function of C_1 the results will not change. The reason is that the discriminant function parameters change, not the discriminant. Actually, this is the way I coded mine up. The same discriminant function for C_1, C_2, and C_3 with different variances passed in.

b)

	Precision	Recall
C_1	0.802	0.929
C_2	0.915	0.929
C_3	0.937	0.843

Precision gives us a measure of relevent data points while recall gives us a measure of how accurate our model measures relevant data.

Ideally, we would like both values to be high. Therefore, I would choose model 2. Model 3 has low recall, comparitively and model 2 is a simpler model than model 1.