PART A EXAMINATION OF ANALYSIS ACADEMIC YEAR 2012/2013
January 24th 2013

.

Theoretical question (10 pts).

1) Let $[a, b] \subset \mathbb{R}$ be a bounded and closed interval. Give the following definitions: a) definition of a partition of $[a, b]$; b) definition of a step function on $[a, b]$; c) definition of the integral of a step function on $[a, b]$; d) definition of integrability and of the integral of a bounded function on $[a, b]$.

2) State the extreme-value theorem for continuous functions (the Weierstrass theorem).

3) Prove the Weierstrass theorem.

Exercise 1 (10 pts)

Draw a qualitative picture of the graph of the function:

$$f(x) = \log \frac{2x - 8}{x - 3}$$

In particular, address the following features: domain, sign, interceptions with axes, limits, asymptotes, monotonicity, extremal points, convexity and inflection points.

Solution.

1. Dominio: $\frac{2x-8}{x-3} > 0$ Quindi $D = (-\infty, 3) \cup (4, +\infty)$
2. Studio del segno di f: $f(x) \geq 0 \iff \frac{2x-8}{x-3} \geq 1 \iff \frac{x-5}{x-3} \geq 0 \iff x < 3$ oppure $x \geq 5$. Quindi f positiva o nulla in $(-\infty, 3)$ e in $[5, +\infty)$ ed negativa in $(4, 5)$. Intersezione con l'asse x: $x = 5$, intersezione con l'asse y: $y = \log \frac{8}{3}$.
3. Limiti e asintoti:
$$\lim_{x \to -\infty} \log \frac{2x - 8}{x - 3} = \log 2,$$
$$\lim_{x \to +\infty} \log \frac{2x - 8}{x - 3} = \log 2$$
$$\lim_{x \to 3^-} \log \frac{2x - 8}{x - 3} = +\infty$$
$$\lim_{x \to 4^+} \log \frac{2x - 8}{x - 3} = -\infty$$
quindi
$$x = 3$$
e
$$x = 4$$
sono asintoti verticali e
$$y = \log 2$$
asintoto orizzontale.

4. Derivata prima, monotonia, massimi e minimi:

$$f'(x) = \frac{x-3}{2x-8} \cdot \frac{2(x-3)-(2x-8)}{(x-3)^2} = \frac{2}{(2x-8)(x-3)}$$

$f'(x) \geq 0 \iff (2x-8)(x-3) > 0$, sempre vera in D. Quindi f crescente in $(-\infty, 3)$ e in $(4, +\infty)$. f non ha massimi n minimi.

5. Derivata seconda, concavit, punti di flesso:

$$f''(x) = -\frac{2}{(2x-8)^2(x-3)^2} \cdot (4x-14) = \frac{-8x+28}{(2x-8)^2(x-3)^2}$$

$f''(x) \geq 0 \iff -8x+28 \geq 0 \iff x \leq \frac{7}{2}$. Quindi f convessa in $(-\infty, 3)$ ed concava in $(4, +\infty)$. f non ha punti di flesso.

Exercise 2 (5 pts) Compute the following limit

$$\lim_{x\to+\infty} \frac{\sin\left(\sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{1}{x}}\right)\log\left(1+\frac{1}{x}\right)}{\tan\left(\frac{3}{x^2}\right)}.$$

Solution.
Noting that

$$\lim_{x\to+\infty} \sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{1}{x}} = 0 \quad \lim_{x\to+\infty} \frac{3}{x^2} = 0, \lim_{x\to+\infty} \left(1+\frac{1}{x}\right) = 1,$$

we get (multiplying and dividing by suitable terms)

$$\begin{aligned} \lim\nolimits_{x\to+\infty} \quad & \frac{\sin\left(\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right)}{\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \frac{\frac{3}{x^2}}{\tan\left(\frac{3}{x^2}\right)} \frac{\left(\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right)\frac{1}{x}}{\frac{3}{x^2}} \\ & = \lim\nolimits_{x\to+\infty} \frac{\left(\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right)\frac{1}{x}}{\frac{3}{x^2}} \\ & = \lim\nolimits_{x\to+\infty} \frac{x}{3} \frac{1+\frac{1}{x}-1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{1}{3}\,, \end{aligned}$$

where, in the first passage, we have used the known limits for sine, logarithm, tangent (everyone of the first three factors converges to 1), and, in the second passage, we have multiplied and divided by $\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}$.

Exercise 3 (5 pts)
Find all the values $x \in \mathbb{R}$ such that the following series is convergent:

$$\sum_{n=0}^{+\infty} \left(\frac{x^2+1}{x^2+2x-1}\right)^n.$$

For $x \in \mathbb{R}$ such that the series converges, find the sum (expressed as function of x.).

Solution.

The series is a geometric series. It is convergent for

$$\left| \frac{x^2+1}{x^2+2x-1} \right| < 1 \tag{0.1}$$

and it does not converge everywhere else (neither absolutely nor simply (with or without absolute values)).

It is easy to verify that inequality

$$\frac{x^2+1}{x^2+2x-1} > -1$$

holds for

$$x < -1-\sqrt{2} \qquad \vee \qquad -1 < x < 0 \qquad \vee \qquad x > -1+\sqrt{2}\,;$$

analogously, inequality

$$\frac{x^2+1}{x^2+2x-1} < 1$$

holds for

$$-1-\sqrt{2} < x < -1+\sqrt{2} \qquad \vee \qquad x > 1\,.$$

Then, inequality (0.1) holds for

$$-1 < x < 0 \qquad \vee \qquad x > 1\,.$$

The sum of the series is

$$\frac{1}{1-\frac{x^2+1}{x^2+2x-1}} = \frac{x^2+2x-1}{2(x-1)}\,.$$