Gdańsk University of Technology

Random Processes and Stochastic Control – Part 1 of 2

Exercises

by

Maciej Niedźwiecki

Gdańsk University of Technology

Departament of Automatic Control

Gdańsk, Poland

Exercise 1

Using MATLAB generate and plot the sequence of 100 consecutive realizations of 3 random variables: A with uniform PDF, B with Gaussian PDF and C with Laplace PDF. All random variables should have the same mean:

$$m_A = m_B = m_C = 2$$

and the same variance:

$$\sigma_A^2 = \sigma_B^2 = \sigma_C^2 = 2$$

Use the same scale on the x and y axes

$$x \in [1, 100], \quad y \in [-3, 7]$$

Exercise 2

Using MATLAB generate and plot the sequence of 100 consecutive realizations of the Cauchy random variable $D \sim \mathcal{C}(2, 1)$.

Exercise 3

The mean and variance of a random variable X can be estimated using the following relationships

$$\widehat{m}_X(N) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\widehat{\sigma}_X^2(N) = \frac{1}{N}\sum_{i=1}^{N}[x_i - \widehat{m}_X(N)]^2$$

Estimate the mean and the variance of the random variables A, B, C and D defined in Exercises 1 and 2, based on $N =$ 100, 1,000, 10,000 and 100,000 samples.

Exercise 4

Using MATLAB plot normalized histograms obtained for 10,000 samples of random variables C and D (defined in exercises 1 and 2) and compare them with the corresponding PDF plots. In both cases divide the interval [-3, 7] (subset of the observation space) into 20 bins of equal width 0.5.

Exercise 5

Write MATLAB procedure generating samples of a random variable E with the following PDF

$$\text{a)} \qquad p_E(e) = \begin{cases} 0.50 & e \in [-1, 0] \\ 0.25 & e \in (0, 2] \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{b)} \qquad p_E(e) = \begin{cases} -e & e \in [-1, 0] \\ 0.5 & e \in (0, 1] \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate the mean, median and standard deviation of E.

$$\int_{-1}^{0} e p_E(e) de = 0.5 \left. \frac{e^2}{2} \right|_{-1}^{0} = -\frac{1}{4}$$

$$\int_{0}^{2} e p_E(e) de = 0.25 \left. \frac{e^2}{2} \right|_{0}^{2} = \frac{1}{2}$$

$$\int_{-1}^{0} e^2 p_E(e) de = 0.5 \left. \frac{e^3}{3} \right|_{-1}^{0} = -\frac{1}{6}$$

$$\int_{0}^{2} e^2 p_E(e) de = 0.25 \left. \frac{e^3}{3} \right|_{0}^{2} = \frac{2}{3}$$

$$m_X = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\sigma_X^2 = \frac{1}{6} + \frac{2}{3} - \left(\frac{1}{4}\right)^2 = \frac{37}{48}$$

$$\alpha_X = 0$$

Exercise 6

Denote by $E_1, \ldots, E_{16}$ the sequence of 16 independent random variables with PDF specified in Exercise 5. Let

$$F = \frac{1}{4} \sum_{i=1}^{16} \frac{E_i - m_E}{\sigma_E}$$

Using MATLAB plot the normalized histogram obtained for 10,000 samples of F. Compare it with an appropriately scaled plot of a standard Gaussian PDF $\mathcal{N}(0, 1)$. Divide the interval [-5, 5] (subset of the observation space) into 20 bins of equal width 0.5.

Exercise 7

Let X be a random variable that is uniformly distributed over the interval (1,100). Form a new random variable Y by rounding X to the nearest integer. In MATLAB code, this could be represented by $\texttt{Y} = \texttt{round}(\texttt{X})$. Finally, form the roundoff error variable according to $Z = X - Y$. Using MATLAB generate 10,000 realizations of Z and estimate its mean and variance. Based on the available data create a normalized histogram. Find the analytical model that seems to fit your estimated PDF.

Exercise 8

Which of the following mathematical functions could be the cumulative distribution function of some random variable?

1. $F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$
2. $F_X(x) = [1 - e^{-x}]\,1(x)$
3. $F_X(x) = e^{-x^2}$
4. $F_X(x) = x^2 1(x)$

Note: $1(x)$ is the unit step function.

Exercise 9

Suppose a random variable has a cumulative distribution function given by $F_X(x) = [1 - e^{-x}]\,1(x)$. Find the following quantities:

1. $P(X > 5)$
2. $P(X < 5)$
3. $P(3 < X < 7)$
4. $P(X > 5 | X < 7)$

Exercise 10

Suppose a random variable has a cumulative distribution function given by

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 0.04x & 0 < x \leq 5 \\ 0.6 + 0.04x & 5 < x \leq 10 \\ 1 & x > 10 \end{cases}$$

Find the following quantities:

1. $P(X > 5)$
2. $P(X < 5)$
3. $P(3 < X < 7)$
4. $P(X > 5 | X < 7)$

Exercise 11

Which of the following are valid probability density functions?

1. $p_X(x) = e^{-x}1(x)$

2. $p_X(x) = e^{-|x|}$

3. $p_X(x) = \begin{cases} \frac{3}{4}(x^2 - 1) & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$

4. $p_X(x) = 2xe^{-x^2}1(x)$

Exercise 12

The conditional cumulative distribution function of random variable X, conditioned on the event A having occurred is

$$F_{X|A}(x) = P(X \leq x|A) = \frac{P(X \leq x, A)}{P(A)}$$

(assuming that $P(A) \neq 0$).

Suppose that $X \sim \mathcal{U}(0, 1)$. Find the mathematical expression for $F_{X|X<1/2}(x)$.

Exercise 13

Consider the Laplace random variable with a PDF given by

$$p_X(x) = \frac{b}{2}\exp(-b|x|)$$

Evaluate the mean, the variance, the coefficient of skewness and the coefficient of kurtosis of X.

Note:

$$\int x^n e^{ax} dx = \frac{1}{a}\, x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

Exercise 14

Consider two independent normally distributed random variables

$$X \sim \mathcal{N}(0,1), \quad Z \sim \mathcal{N}(0,1)$$

and let

$$Y = aX + bZ, \qquad a, b \in R$$

Determine the coefficients a and b in such a way that $Y \sim \mathcal{N}(0,1)$ and ρ_{XY} takes any prescribed value from $[-1, 1]$.

Using MATLAB generate 100 realizations of the pair (X, Y) for $\rho_{XY} = 0$, $\rho_{XY} = 0.5$, $\rho_{XY} = -0.5$, $\rho_{XY} = 0.9$, $\rho_{XY} = -0.9$, $\rho_{XY} = 1$ and $\rho_{XY} = -1$.

Display the obtained results using scatter plots.

Facts about matrices

Fact 1

Given a square matrix $\mathbf{A}_{n\times n}$, an eigenvalue λ and its associated eigenvector $\mathbf{v}$ are a pair obeying the relation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

The eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ are the roots of the characteristic polynomial of $\mathbf{A}$

$$\det(\mathbf{A} - \lambda\mathbf{I})$$

For a positive definite matrix $\mathbf{A}$ it holds that $\lambda_1, \ldots, \lambda_n > 0$, i.e., all eigenvalues are positive real.

Fact 2

Let $\mathbf{A}$ be a matrix with linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then $\mathbf{A}$ can be factorized as follows

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

where $\boldsymbol{\Lambda}_{n\times n} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ and $\mathbf{V}_{n\times n} = [\mathbf{v}_1|\ldots|\mathbf{v}_n]$.

Fact 3

Let $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ be a real symmetric matrix. Such matrix has n linearly independent real eigenvectors. Moreover, these eigenvectors can be chosen such that they are orthogonal to each other and have norm one

$$\mathbf{w}_i^{\mathrm{T}}\mathbf{w}_j = 0, \ \forall i \neq j$$
$$\mathbf{w}_i^{\mathrm{T}}\mathbf{w}_i = ||\mathbf{w}_i||^2 = 1, \ \forall i$$

A real symmetrix matrix $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{\mathrm{T}}$$

where $\mathbf{W}$ is an orthogonal matrix: $\mathbf{W}_{n\times n} = [\mathbf{w}_1|\ldots|\mathbf{w}_n]$, $\mathbf{W}\mathbf{W}^{\mathrm{T}} = \mathbf{W}^{\mathrm{T}}\mathbf{W} = \mathbf{I}$.

The eigenvectors $\mathbf{w}_i$ can be obtained by normalizing the eigenvectors $\mathbf{v}_i$

$$\mathbf{w}_i = \frac{\mathbf{v}_i}{||\mathbf{v}_i||^2}, \quad i = 1, \ldots, n$$

Exercise 15

Consider a vector random variable $\mathbf{X} = [X_1, X_2]^{\mathrm{T}}$ with the mean and covariance matrix given by

$$\mathbf{m}_X = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{\Sigma}_X = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

1. Perform eigendecomposition of the matrix $\mathbf{\Sigma}_X$

$$\mathbf{\Sigma}_X = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{\mathrm{T}}$$

where $\mathbf{\Lambda}$ is a diagonal matrix made up of eigenvalues of $\mathbf{\Sigma}_X$ and $\mathbf{W}$ is an orthogonal matrix.

Sketch the covariance ellipse of $\mathbf{X}$. What is the correlation coefficient of X_1 and X_2?

2. Suppose that $\mathbf{Z} = [Z_1, Z_2]^{\mathrm{T}}$ is made up of two uncorrelated random variables with zero mean and unit variance. Consider the following linear transformation

$$\mathbf{Y} = \mathbf{W}\mathbf{\Lambda}^{1/2}\mathbf{Z}$$

where $\mathbf{W}$ and $\mathbf{\Lambda}$ are the matrices determined in point 1 above. Show that

$$\mathbf{\Sigma}_Y = \mathbf{\Sigma}_X.$$

3. Use a linear transformation, defined in point 2, to generate two-dimensional Gaussian random variable $\mathbf{X}$ with the mean and covariance matrix specified above. Create a scatter plot based on 200 realizations of $\mathbf{X}$.

<u>Exercise 16</u>

Consider a zero-mean vector random variable $\mathbf{X}$ with covariance matrix

$$\mathbf{\Sigma}_X = \mathrm{E}[\mathbf{X}\mathbf{X}^{\mathrm{T}}] = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{\mathrm{T}}$$

where $\mathbf{\Lambda}$ is a diagonal matrix made up of eigenvalues of $\mathbf{\Sigma}_X$ and $\mathbf{W}$ is an orthogonal matrix.

Let

$$\mathbf{Y} = \mathbf{W}\mathbf{\Lambda}^{-1/2}\mathbf{W}^{\mathrm{T}}\mathbf{X}$$

Show that

$$\mathbf{\Sigma}_Y = \mathbf{I}$$

i.e., $\mathbf{Y}$ is the vector random variable made up of uncorrelated components.

Project 1

Record and mix signals obtained from two independent speech sources (30 second long recordings, 2 linear mixtures with different mixing coefficients).

Perform blind source separation using the FastICA algorithm:

1. Center the data to make its mean zero.
2. Whiten the data to obtain uncorrelated mixture signals.
3. Initialize $\mathbf{w}_0 = [w_{1,0}, w_{2,0}]^{\mathrm{T}}$, $||\mathbf{w}_0|| = 1$ (e.g. randomly)
4. Perform an iteration of a one-source extraction algorithm

$$\widetilde{\mathbf{w}}_i = \operatorname{avg}\left\{\mathbf{z}(t)\left[\mathbf{w}_{i-1}^{\mathrm{T}}\mathbf{z}(t)\right]^3\right\} - 3\mathbf{w}_{i-1}$$

where $\mathbf{z}(t) = [z_1(t), z_2(t)]^{\mathrm{T}}$ denotes the vector of whitened mixture signals and $\operatorname{avg}(\cdot)$ denotes time averaging

$$\operatorname{avg}\{\mathrm{x(t)}\} = \frac{1}{N}\sum_{t=1}^{N} x(t)$$

5. Normalize $\widetilde{\mathbf{w}}_i$ by dividing it by its norm

$$\mathbf{w}_i = \frac{\widetilde{\mathbf{w}}_i}{||\widetilde{\mathbf{w}}_i||}$$

6. Determine a unit-norm vector $\mathbf{v}_i$ orthogonal to $\mathbf{w}_i$.

$$\mathbf{w}_i^{\mathrm{T}}\mathbf{v}_i = 0, \quad ||\mathbf{v}_i|| = 1$$

7. Display and listen to the current results of source separation

$$\begin{bmatrix} \widehat{x}_1(t) \\ \widehat{x}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{w}_i^{\mathrm{T}} \\ \mathbf{v}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad t = 1, \ldots, N$$

8. If $\mathbf{w}_i^{\mathrm{T}}\mathbf{w}_{i-1}$ is not close enough to 1, go back to step 4. Otherwise stop.

Literature: A. Hyvärinen, E. Oja. "A fast fixed-point algorithm for independent component analysis", *Neural Computation*, vol. 9, pp. 1483-1492, 1997.

A special bonus will be granted for writing a procedure that extracts $n > 2$ source signals from n mixtures.

Project 2

Create a noisy audio recording (add artificially generated noise to a clean music or speech signal). The first second of the recording should contain noise only. Then denoise recording using the method of spectral subtraction.

Exercise 17

Consider a sinusoid with a random frequency

$$X(t) = \cos(2\pi f t)$$

where f is a random variable uniformly distributed over the interval $[0, f_0]$.

1. Show that the theoretical mean function of $X(t)$ is given by
$$m_X(t) = \mathrm{E}[X(t)] = \frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$$

2. Estimate the mean function of $X(t)$ using ensemble averaging in the case where $f_0 = 0.01$
$$\widehat{m}_X(t) = \frac{1}{1000}\sum_{i=1}^{1000} X(t, \xi_i), \quad t = -300, \ldots, 300$$

3. Plot the obtained estimates and compare them with the theoretical mean function.

Exercise 18

Generate 400 samples of the first-order autoregressive process governed by

$$Y(t) = 0.99Y(t-1) + X(t), \quad t = 1, 2, \ldots$$

where $Y(0) = 0$ and $X(t)$ denotes Gaussian white noise with mean 0 and variance $\sigma_X^2 = 1$.

1. Depict on one plot 100 realizations of $Y(t), t \in [1, 400]$. Starting from what time instant the process can be regarded as wide-sense stationary?

2. Determine analytically the mean function of $Y(t)$ for any initial condition $Y(0) = y_0$.

3. Estimate the steady-state autocorrelation function of $Y(t)$ using time averaging

$$\widehat{R}_X(\tau) = \frac{1}{100} \sum_{t=201}^{300} Y(t)Y(t+\tau), \quad \tau = 0, \ldots, 100$$

4. Estimate the steady-state autocorrelation function of $Y(t)$ using ensemble averaging

$$\widehat{R}_X(\tau) = \frac{1}{100} \sum_{i=1}^{100} Y(t, \xi_i)Y(t+\tau, \xi_i)$$

$$t = 201, \quad \tau = 0, \ldots, 100$$

5. Plot the obtained estimates and compare them with the theoretical autocorrelation function

$$R_X(\tau) = \frac{(0.99)^{|\tau|}}{1 - (0.99)^2}$$

Exercise 19

Consider random telegraph signal governed by

$$X(t) \in \{-1, 1\}$$

$$P(X(t+1) = 1 | X(t) = 1) = 0.99$$
$$P(X(t+1) = -1 | X(t) = 1) = 0.1$$
$$P(X(t+1) = 1 | X(t) = -1) = 0.1$$
$$P(X(t+1) = -1 | X(t) = -1) = 0.99$$

$$t = 1, \ldots, 200$$

$$P(X(0) = 1) = P(X(0) = -1) = 0.5$$

Estimate and plot $\widehat{R}_X(\tau), \tau = 0, \ldots, 100$. To obtain the estimates use a) time averaging b) ensemble averaging. Plot and compare the obtained results.

Exercise 20

Consider a zero-mean Gausiian white noise $X(t)$ with variance $\sigma_X^2 = 1$.

1. Evaluate spectral density function $S_X(\omega)$ of this process.

2. Compute and plot periodogram-based estimates of $S_X(\omega)$ using datasets consisting of $N = 100, 1000$ and 10000 samples

$$P_X(\omega_i, N) = \frac{1}{N}\left|\sum_{t=1}^{N} X(t)e^{-j\omega_i t}\right|^2$$

$$\omega_i = 2\pi i/100\ , \quad i = -50, \ldots, 50$$

   Compare the obtained results with the true spectral density function of $X(t)$.

3. Compute time-averaged periodogram-based estimates

$$\bar{P}_X(\omega_i, N) = \frac{1}{M}\sum_{j=1}^{M} P_X^j(\omega_i, N)$$

$$\omega_i = 2\pi i/100\ , \quad i = -50, \ldots, 50$$

   where $P_X^j(\omega_i, N)$ denotes periodogram obtained for the j-th segment of $X(t)$ of length N. Plot the results obtained for $N = 100$.

Exercise 21

$Y(t)$ is a random process observed at the output of the second-order filter

$$H(z^{-1}) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

excited with white Gaussian noise $X(t) \sim \mathcal{N}(0,1)$ under zero initial conditions. The poles of the filter are given in the form

$$z_1 = re^{j\phi}, \quad z_1 = re^{-j\phi}$$

where

$$\begin{array}{lll} \text{A)} & r = 0.9, & \phi = \pi/100 \\ \text{B)} & r = 0.9, & \phi = \pi/10 \\ \text{C)} & r = 0.99, & \phi = \pi/100 \\ \text{D)} & r = 0.99, & \phi = \pi/10 \end{array}$$

1. Compute the coefficients a_1 and a_2 in each of the four cases indicated above.

2. Plot the amplitude characteristics of $H(z^{-1})$

$$A(\omega_i) = \left| H(e^{-j\omega_i}) \right|, \ \omega_i = \pi i/1000, \ i = 1, \ldots, 1000$$

What is rhe influence of r and ϕ on the shape of the amplitude characteristic of $H(z^{-1})$?