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Camera Models

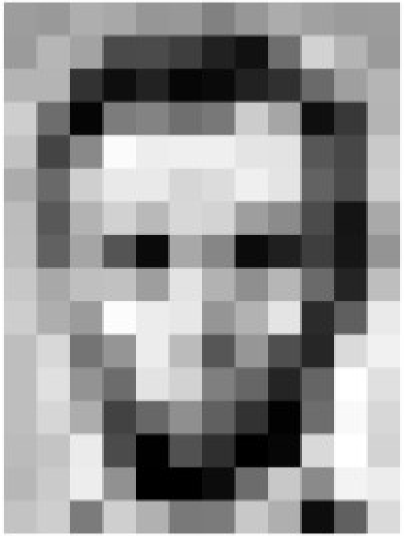
Summary

- Pin Hole Camera
- Lenses
- Pin-Hole Camera Geometry

Courtesy of *CS231A · Computer Vision: from 3D reconstruction to recognition*, Prof. Silvio Savarese – Stanford University

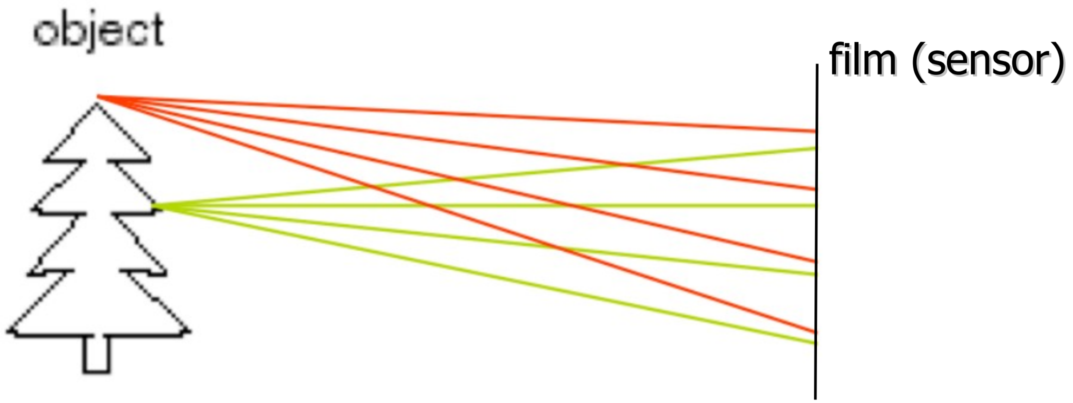
Camera Model

- Since now we discussed image processing
- Namely, we saw fundamental techniques to process a 2D matrix…
- How that image is created?
- What is the relation, if any, to the 3D world?
- During this lesson we will try to answer to those questions



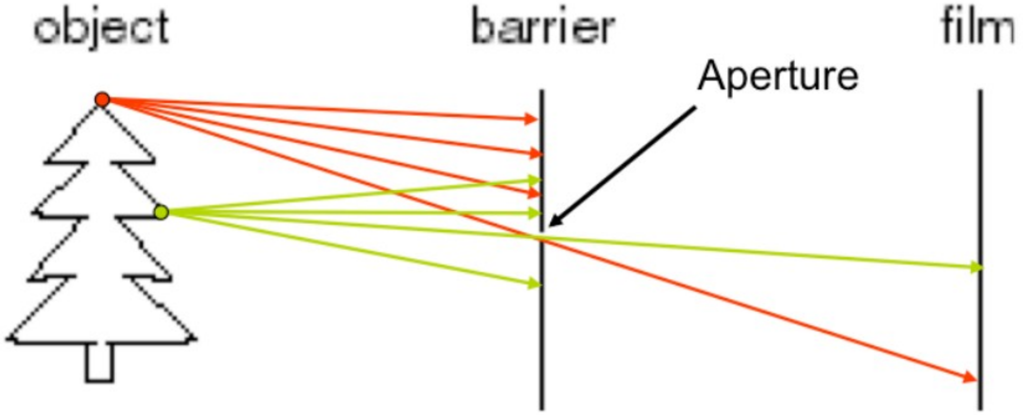
World and Camera

- Too much simple camera
  - A sensitive film in front of an object
  - What is the result?



Pin-Hole Camera

- We can add a barrier with a very small hole
  - So-called pin-hole or aperture
  - Now, only one ray hits the film in a given position



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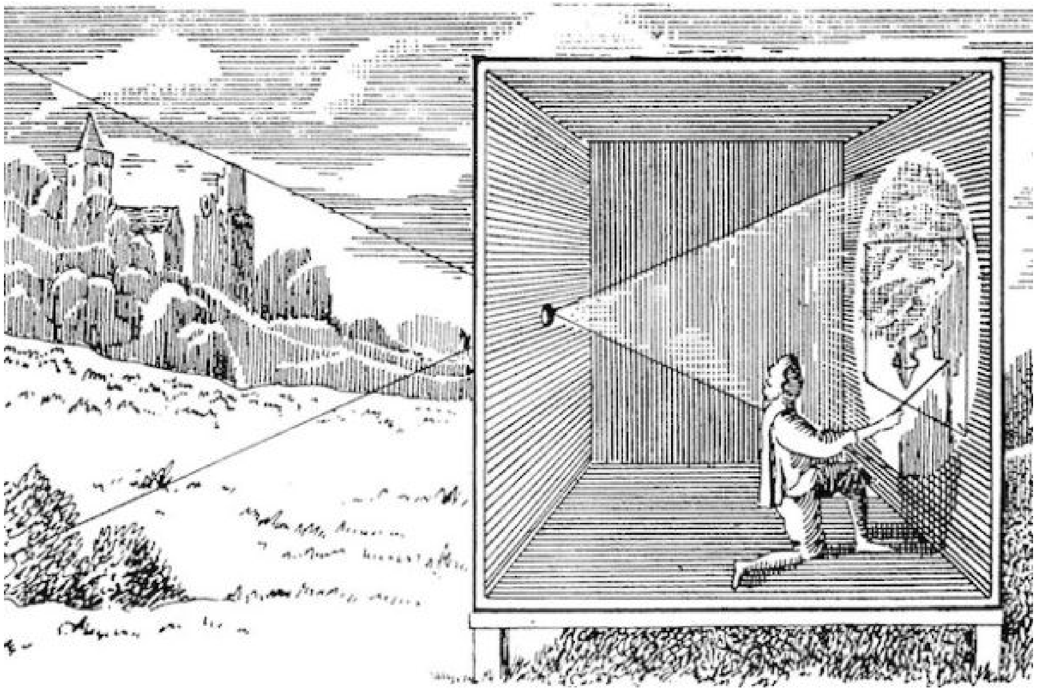
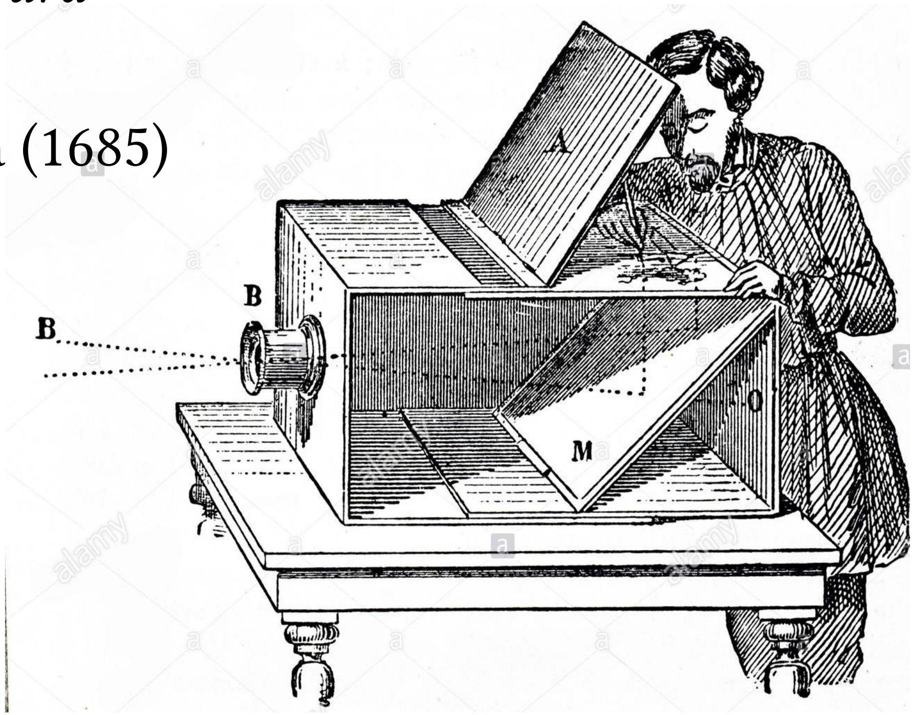
History

- Milestones
  - Leonardo da Vinci's *Camera Obscura* (1502)



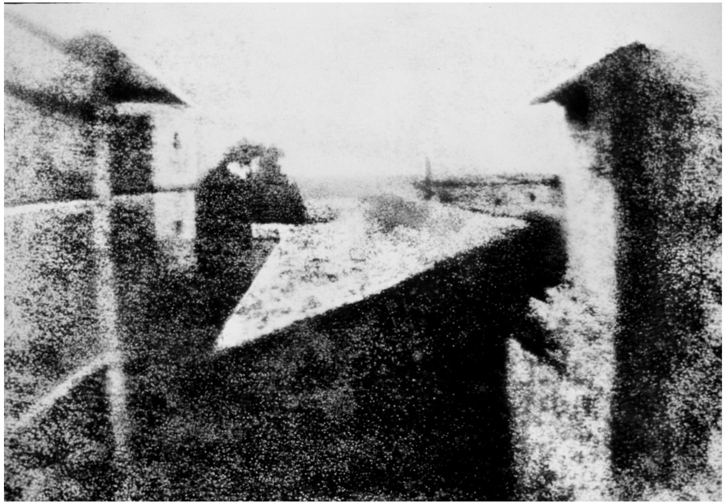
History

- Milestones
  - Leonardo da Vinci's *Camera Obscura* (1502)
  - Johan Zahn: first portable camera (1685)



History

- Milestones
  - Leonardo da Vinci's *Camera Obscura* (1502)
  - Johan Zahn: first portable camera (1685)
  - Joseph Nicéphore Niépce: first photo (1822)



Pin-Hole Camera

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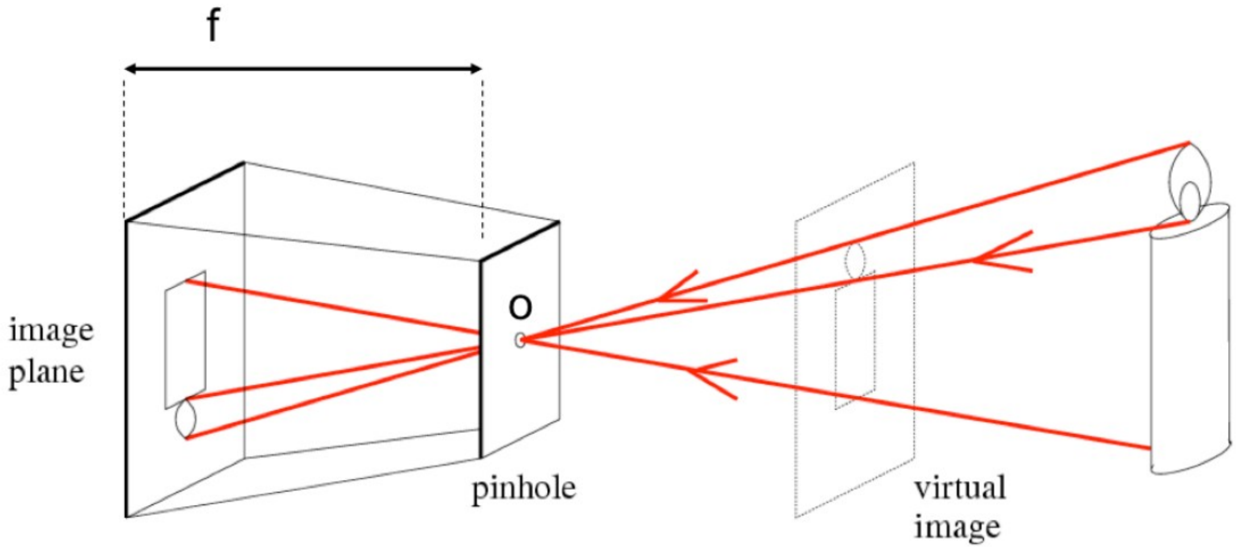
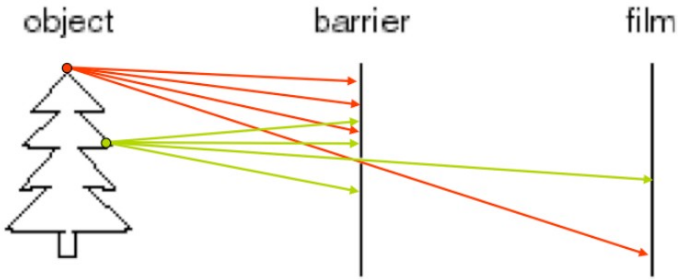
- f $\longrightarrow$ focal lenght
- o $\longrightarrow$ pin-hole, aperture (center of the lens)



Camera Aperture size

- The larger the pin-hole, the greater the number of rays
  - More rays ⟶ More light energy
  - More rays ⟶ Blurred image



Camera Aperture size

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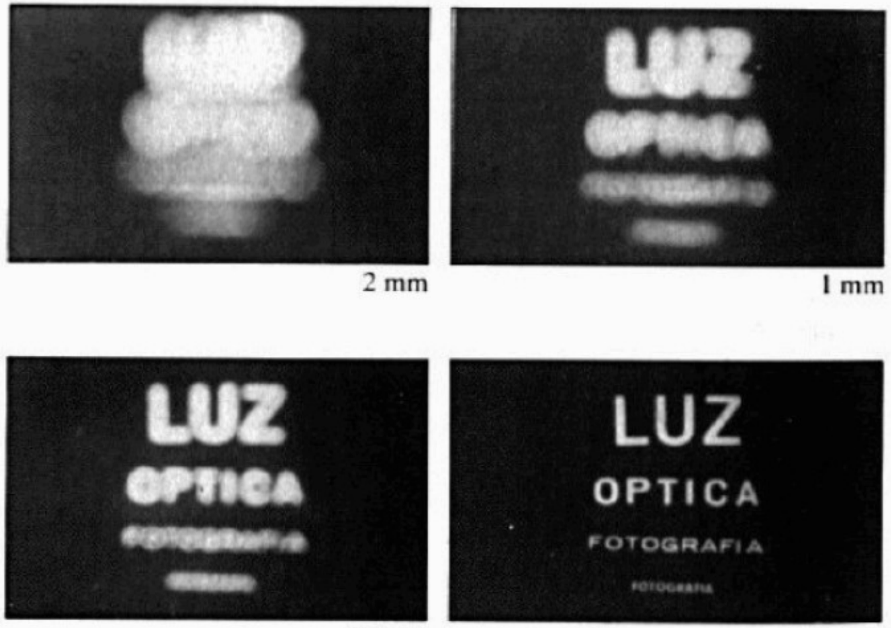
Aperture Size decrease



2 mm

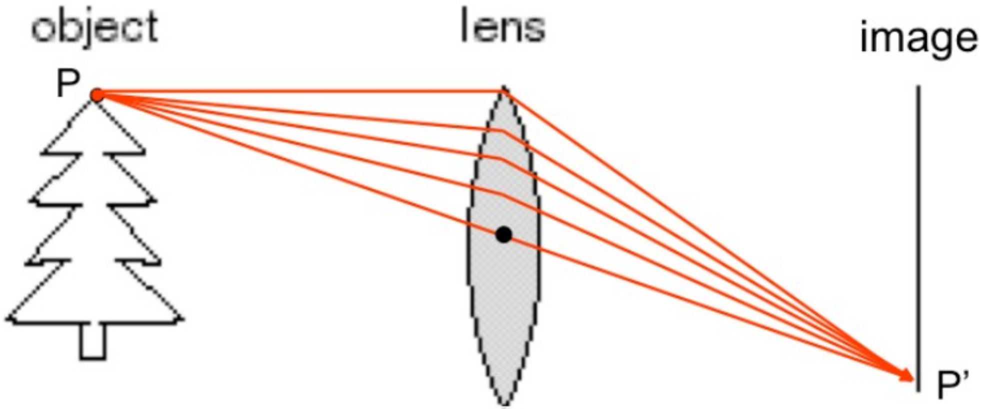
1 mm

0.6mm

0.35 mm

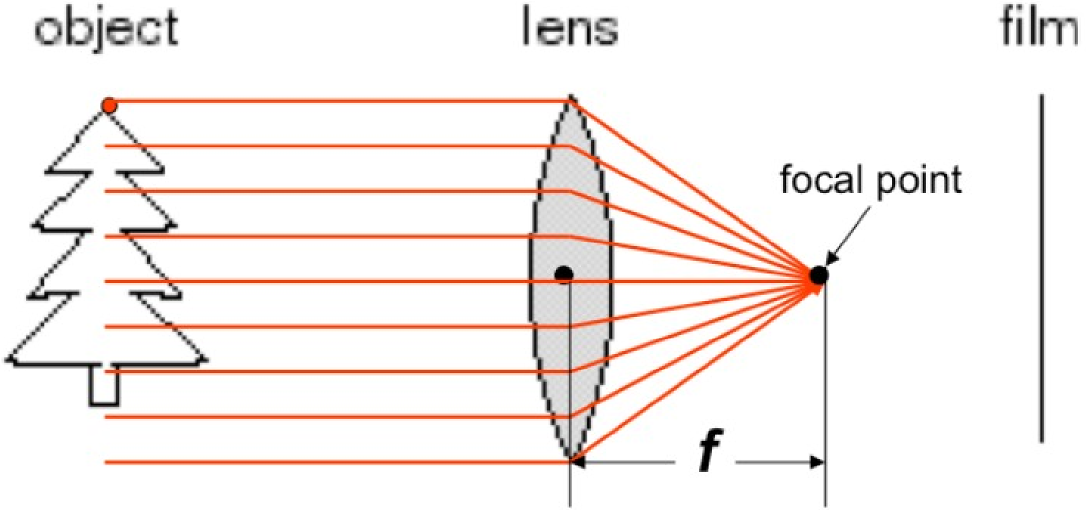
Lenses

- Actual pin-holes are barely used
- Lenses are much more confortable (some issues anyway)
  - We can intercept more rays coming from same 3D point



Model with Lenses

- Parallel light rays converge in a specific point
  - Focal point at distance f from the center of the lens
- Only the ray passing through the center of the lens is not deviated

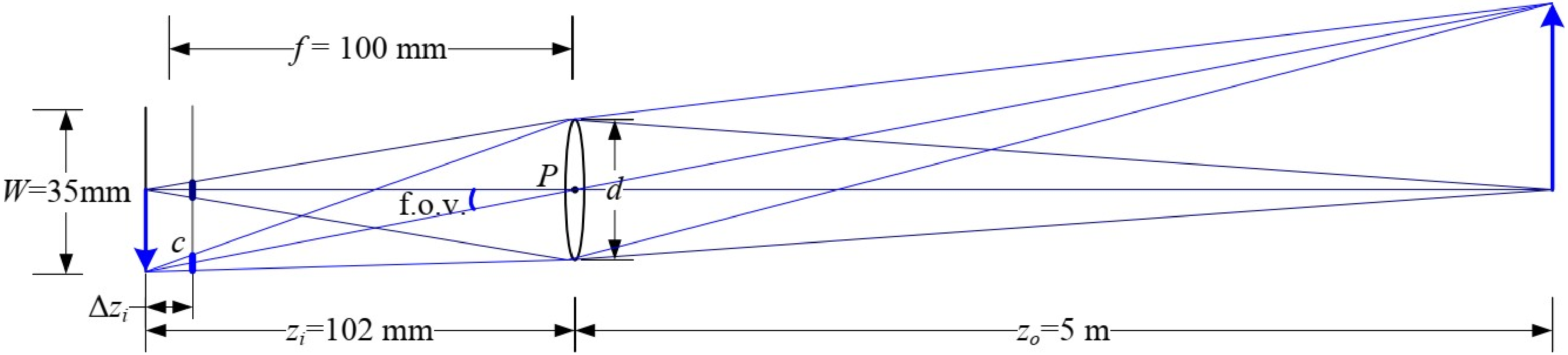


Model with Lenses

- Rays emitted from a given plane points in the world converge to points that lie on a specific plane
- The following formula can be used:

$$\frac{1}{f}=\frac{1}{z_i}+\frac{1}{z_0}$$

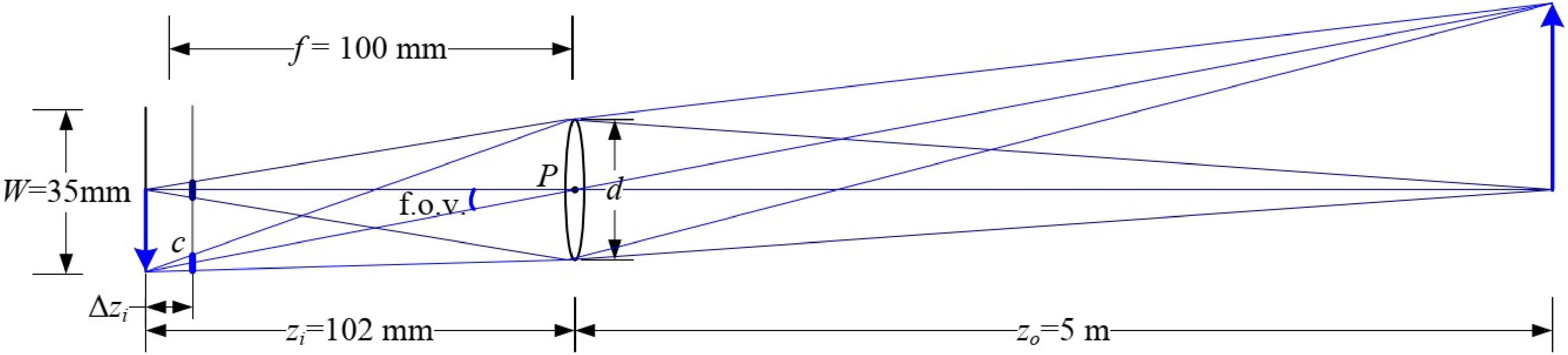
  - Varying z_0 means that also z_i is modified



Model with Lenses

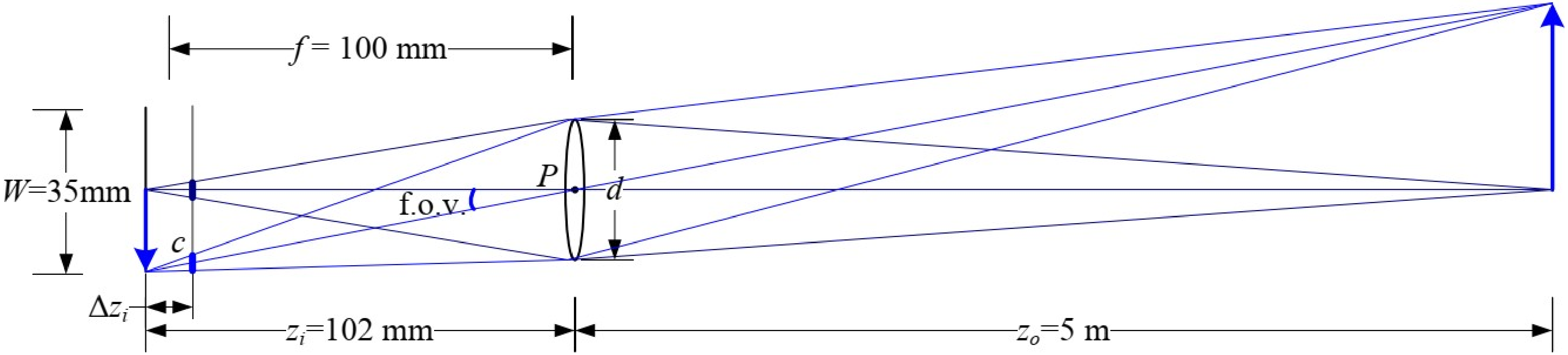
- Therefore the distance between lens and sensor give us the *perfect* focus distance
- For other points we have a **circle of confusion**

$$\frac{1}{f} = \frac{1}{z_i} + \frac{1}{z_0}$$



Model with Lenses

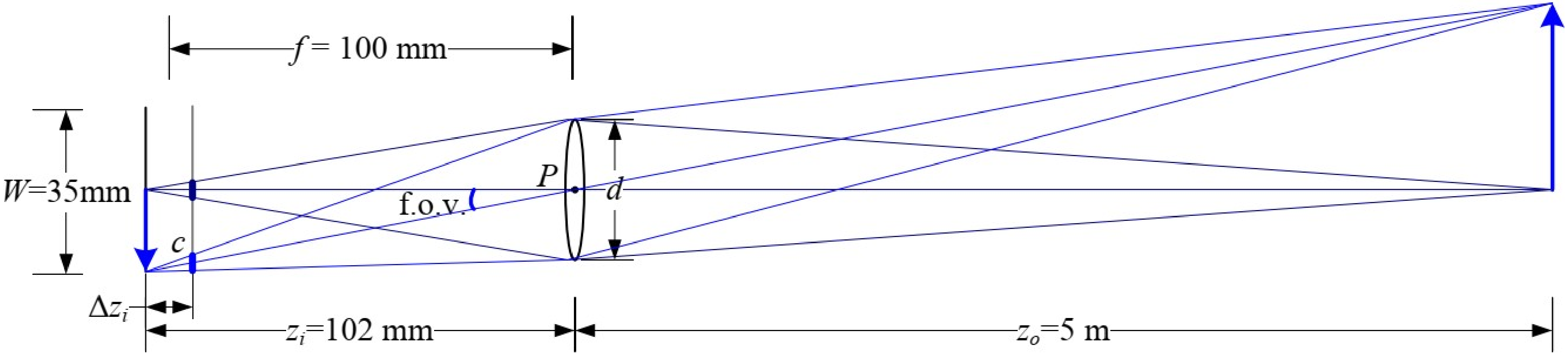
- Anyway in the real world sensor elements have a finite size
- Then we can consider a small portion of the world to be sufficiently in focus: **Shallow Depth of Field**



Model with Lenses

- The depth of field depends on the f-number or focal ratio (f/#)
- The higher the f-number, the larger the depth of field

$$f/\# = \frac{f}{d}$$

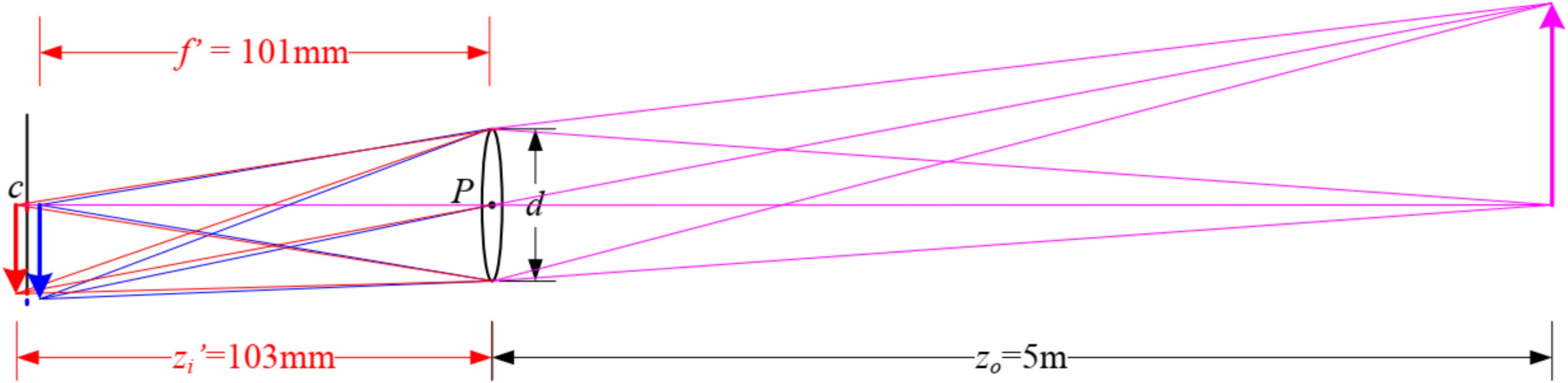


Model with Lenses

- Example of shallow depth of field

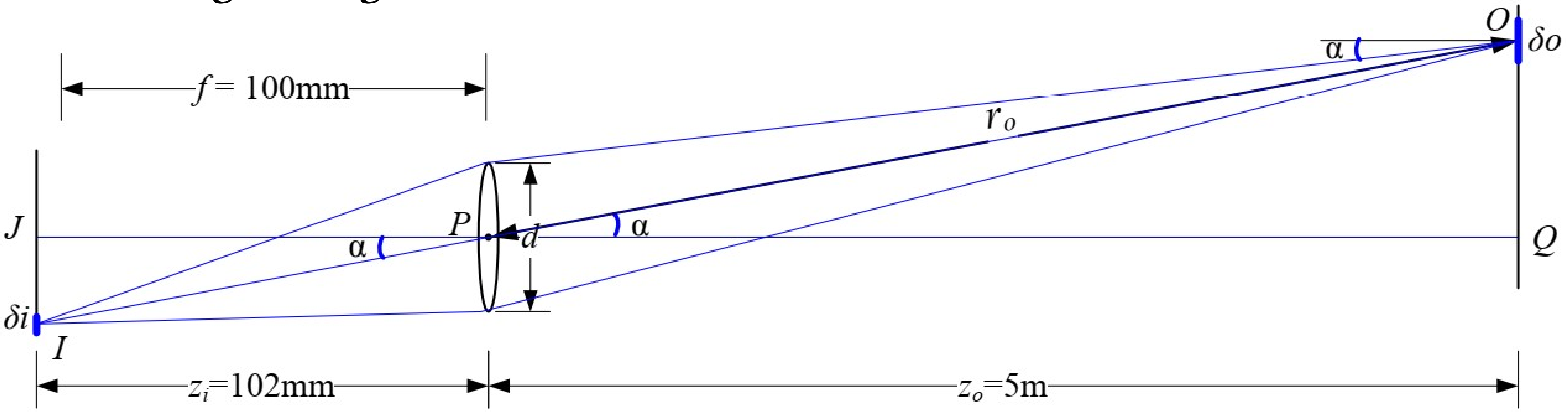
Model with Lenses

- Refractive index depends on wavelenght
- Different colors are then projected in different positions
- **Chromatic Aberration**



Model with Lenses

- Areas in the borders/corners are typically darker…
- Distance from optical axis affects energy
  - Vignetting



Distortion

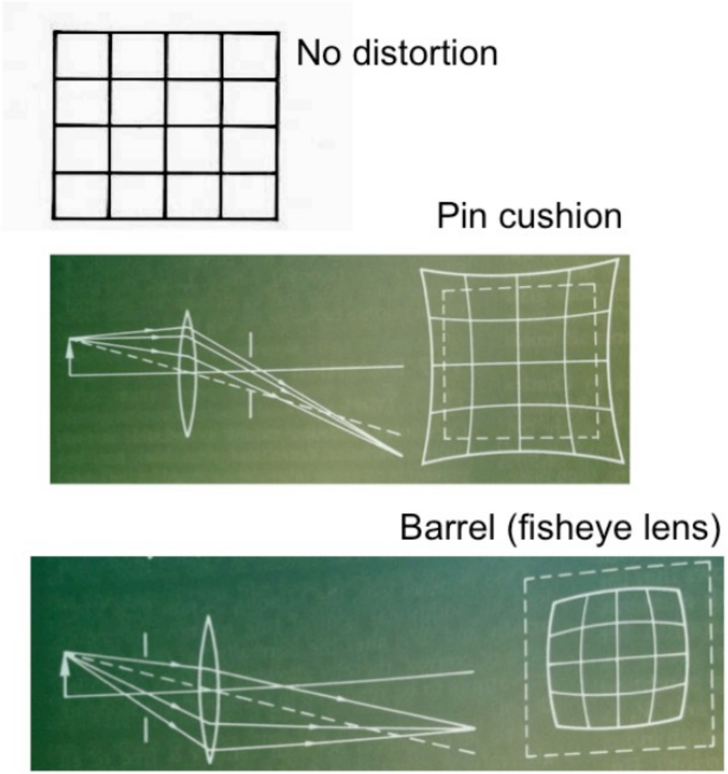
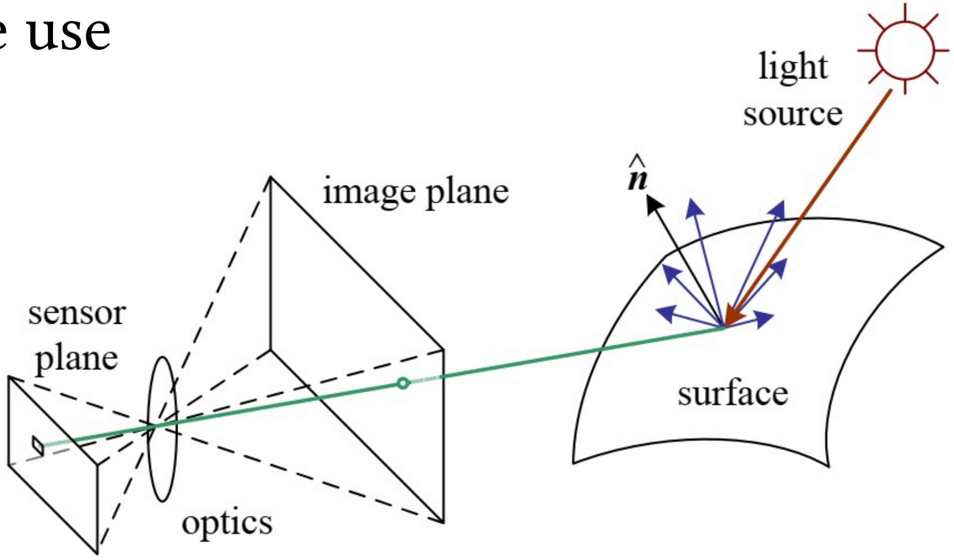


Image magnification decreases with distance from the optical axis

- Distortion effect is much more evident in lateral areas

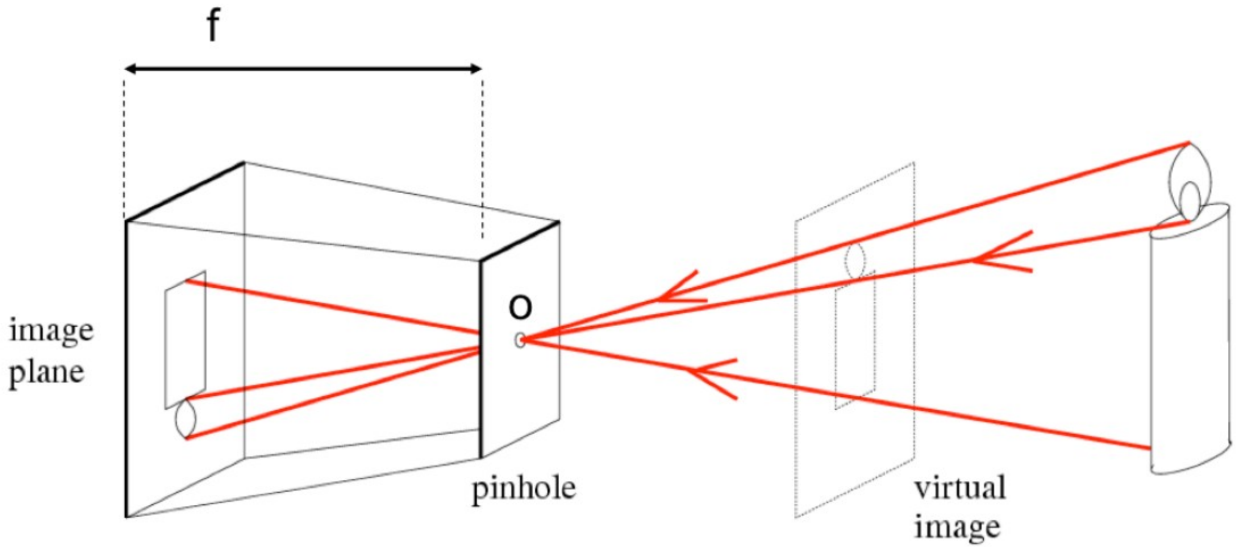
Pin-Hole Camera

- We saw many issues related to the use of a lens
- This also affects the camera model
- Anyway in the following we will simply assume to have:
  - Thin lenses
  - Small angles of view
  - No or compensated chromatic aberration
  - No or rectified distortion effects

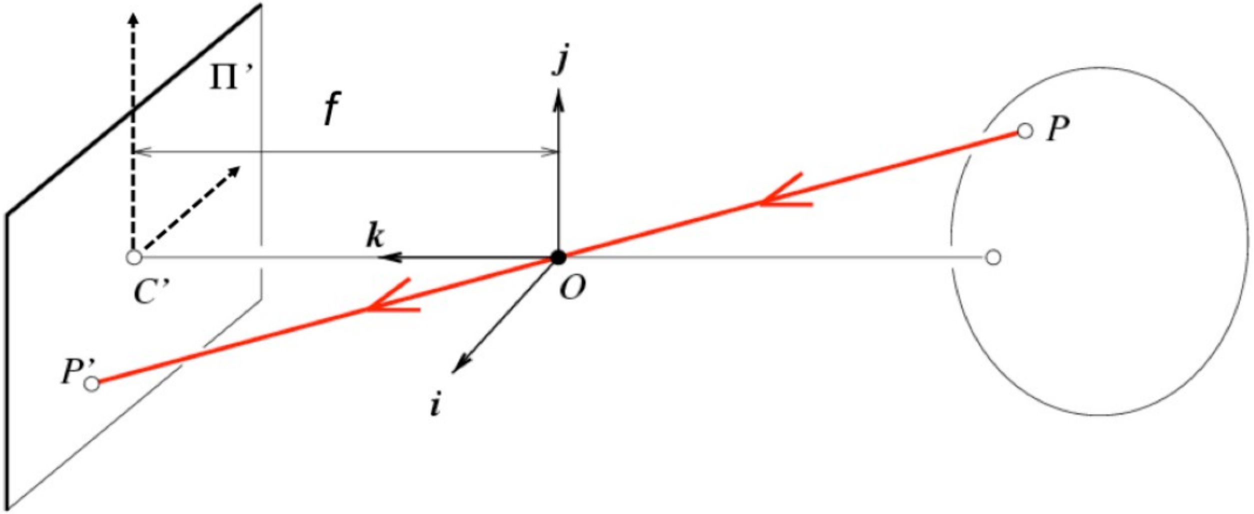


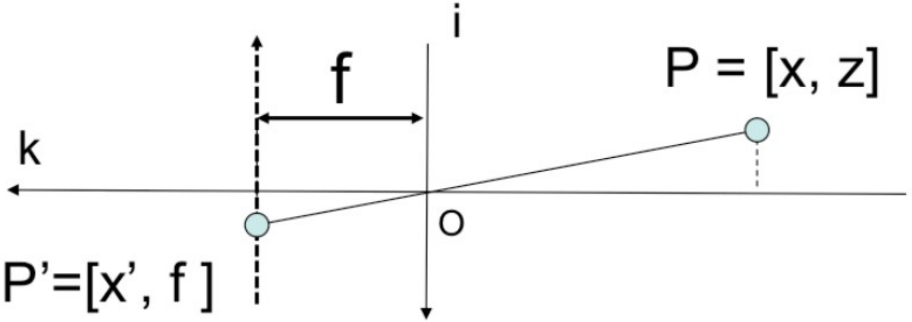
Pin-Hole Camera

- f ⟶ focal lenght
- o ⟶ pin-hole, aperture (center of the lens)



Pin-Hole Camera





[Eq. 2]

$$\frac{x'}{f} = \frac{x}{z}$$

Courtesy: Silvio Savarese

Pin-Hole Camera: perspective tranformation

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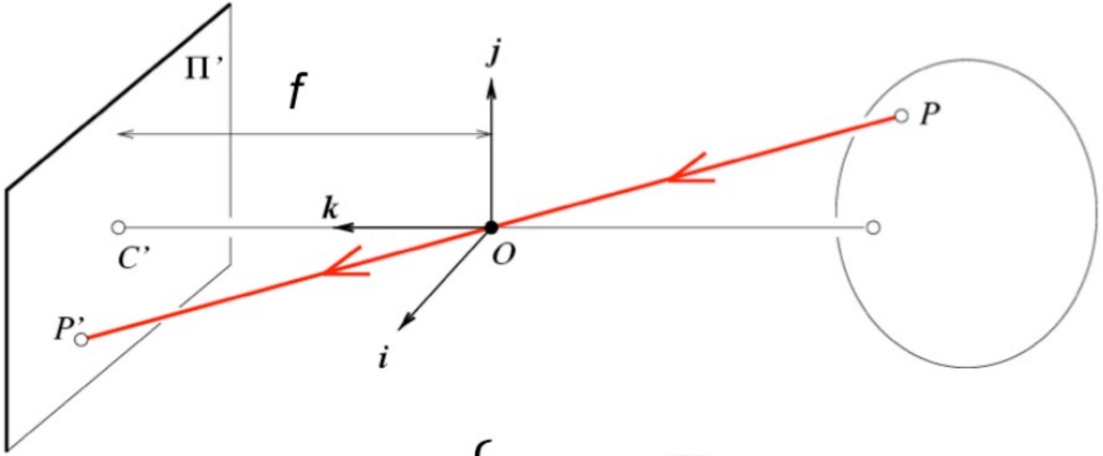


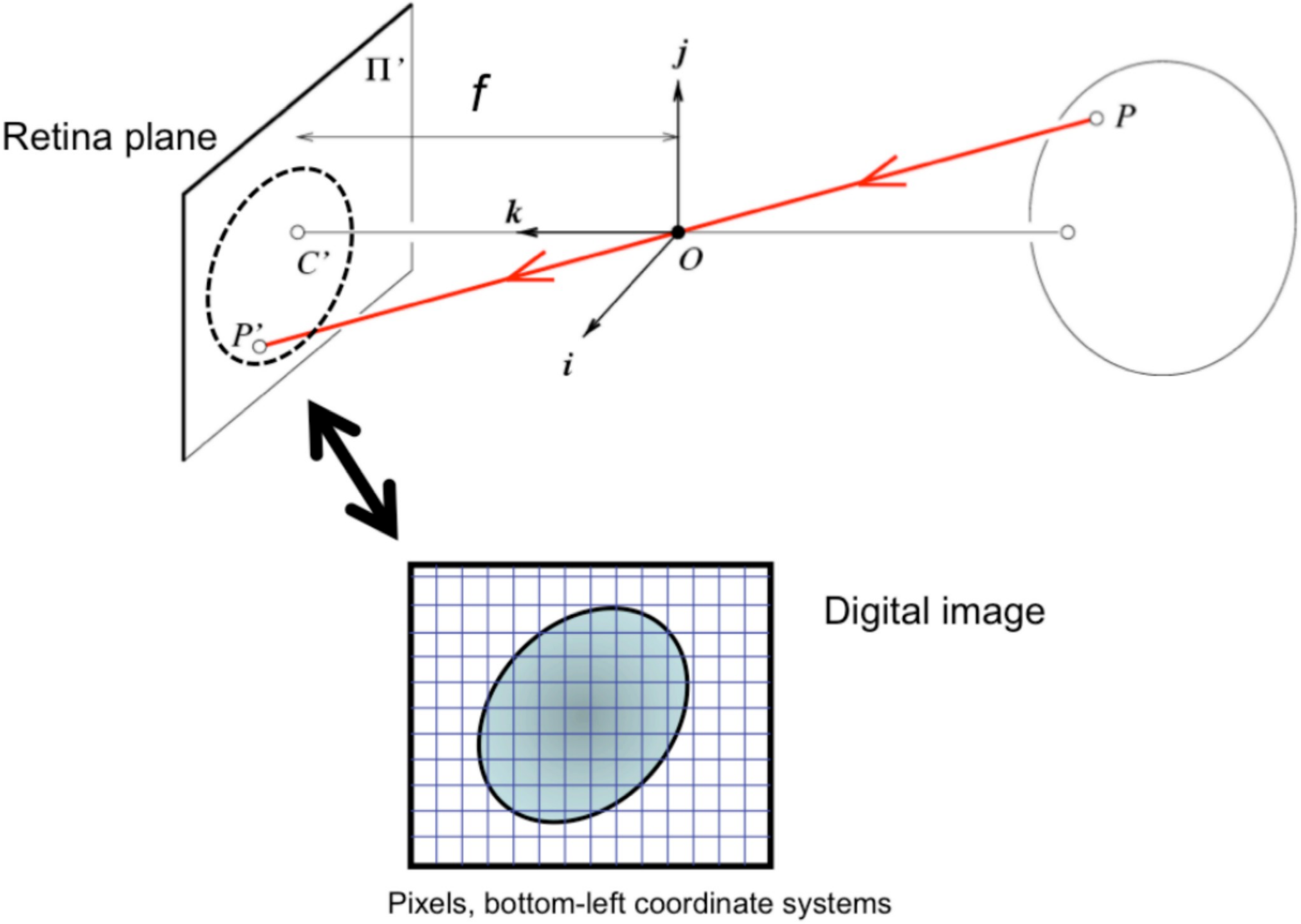
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad \begin{cases} x' = f \dfrac{x}{z} \\ y' = f \dfrac{y}{z} \end{cases} \qquad \Re^3 \xrightarrow{E} \Re^2$$

[Eq. 1]

f = focal length
o = center of the camera

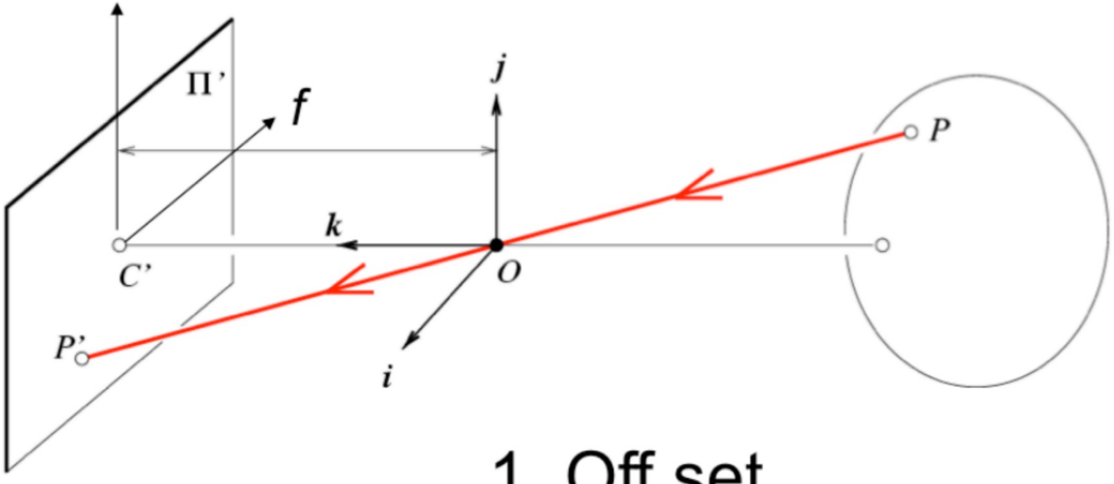
Courtesy: Silvio Savarese

Image & Sensor Planes

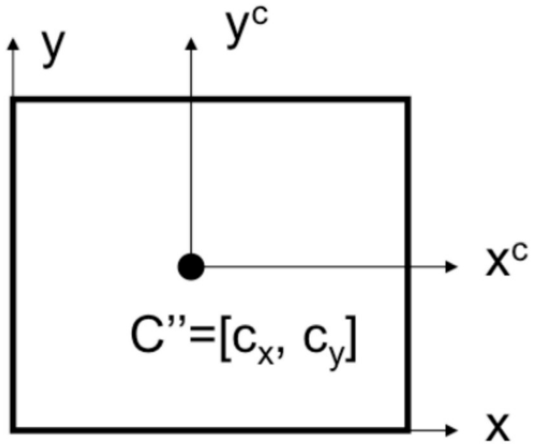


Courtesy: Silvio Savarese

Image & Sensor Planes → Origin



1. Off set

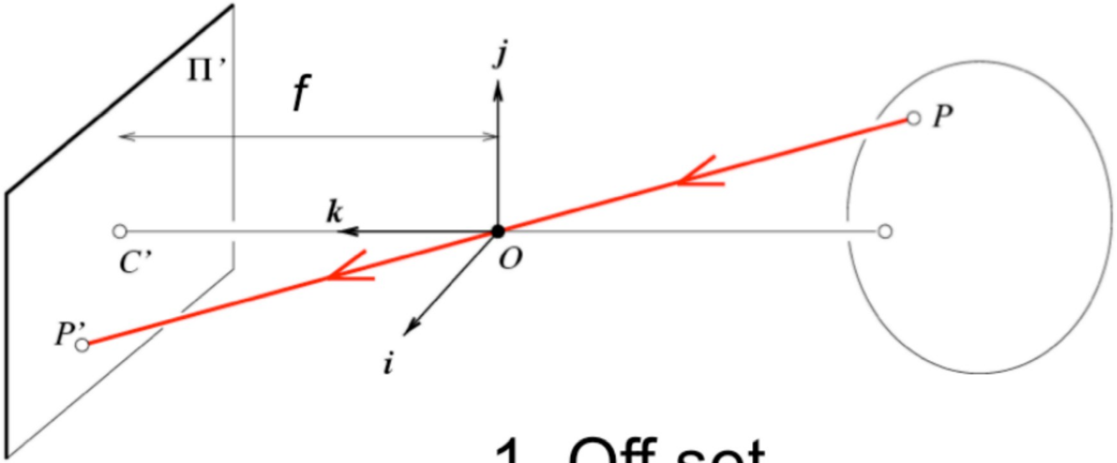


$$(x, y, z) \rightarrow (f\frac{x}{z} + c_x, f\frac{y}{z} + c_y)$$

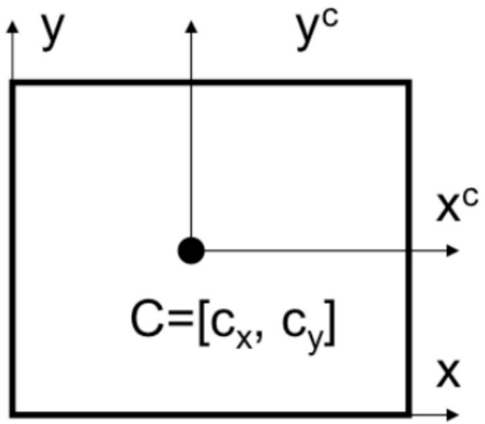
[Eq. 5]

Courtesy: Silvio Savarese

Image & Sensor Planes → Pixel size



1. Off set
2. From metric to pixels



$$(x, y, z) \rightarrow (\underbrace{f\,k}_{\alpha}\frac{x}{z} + c_x, \underbrace{f\,l}_{\beta}\frac{y}{z} + c_y)$$

[Eq. 6]

Units: k,l : pixel/m

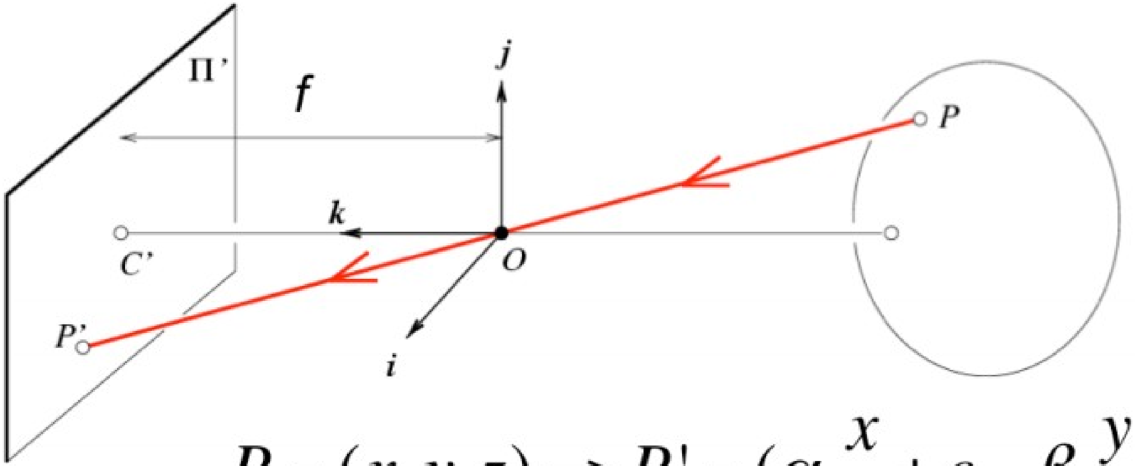
f : m

Non-square pixels

α, β : pixel

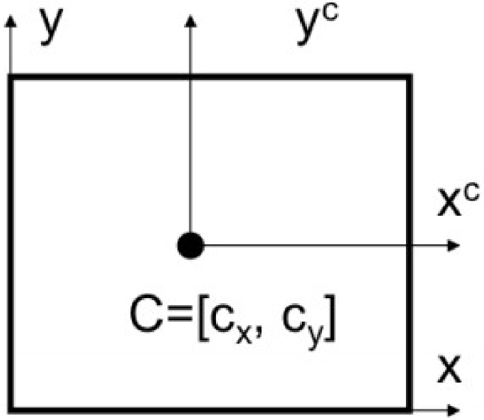
Courtesy: Silvio Savarese

Non Linear Transformation



$$P = (x, y, z) \rightarrow P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

[Eq. 7]



- Is this a linear transformation?
  No — division by z is nonlinear

- Can we express it in a matrix form?

Courtesy: Silvio Savarese

Homogeneous Coordinates

- The non linearity can be solved using Homogeneous Coordinates
- HC are an augmented representation of points
- We add another "coordinate", i.e. $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$
- In 2D space P=(x, y) can be represented as P=(x, y, 1)
  - Or more generally as (kx, ky, k)
  - The third value can be considered as a scale factor

Homogeneous vs Euclidean

- Conversions are simple:
- Euclidean $\longrightarrow$ Homogeneous

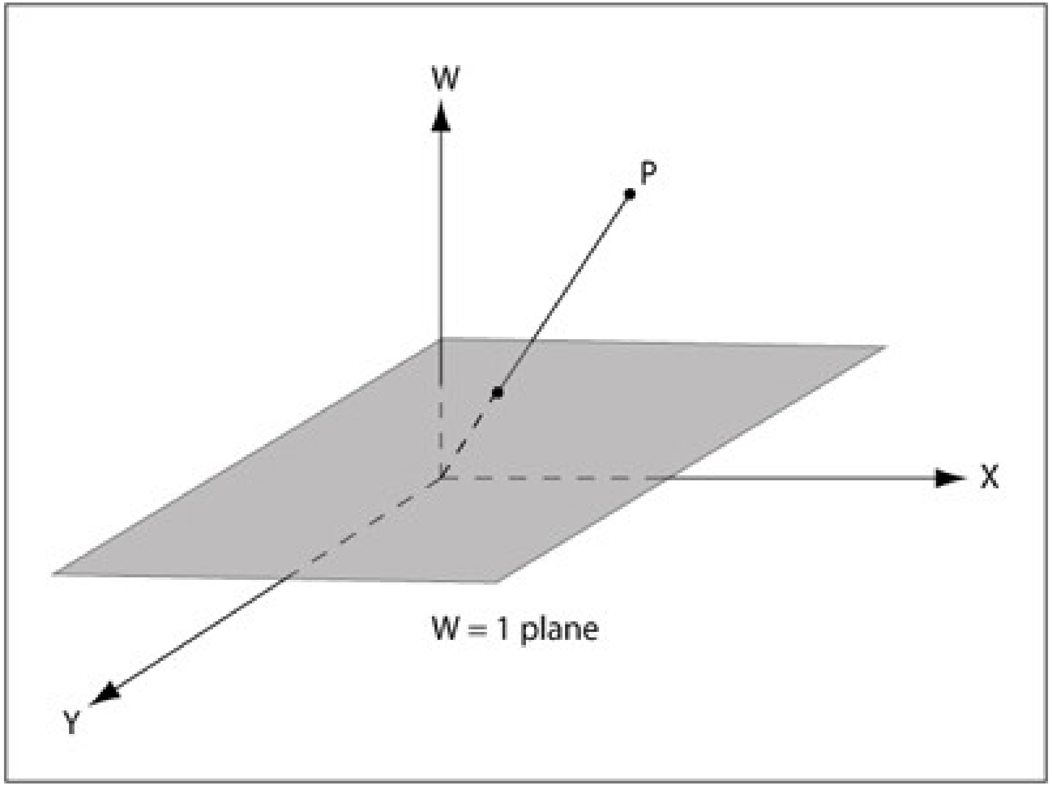
$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Homogeneous $\longrightarrow$ Euclidean

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Projective Space

- A geometric interpretation for HC can be given as follows



Homogeneous Coordinates

- Three main reasons to use homogeneous coordinates
  - Simple representation of points and lines (no special cases)
    - Homogenous space contains more points than Euclidean one!
    - (x,y,0)
  - Simple representation of Euclidean Transformations
    - Traslation
    - Scale
    - Rotation
  - Simple representation of perspective projections

Non Linear Transformation (again)

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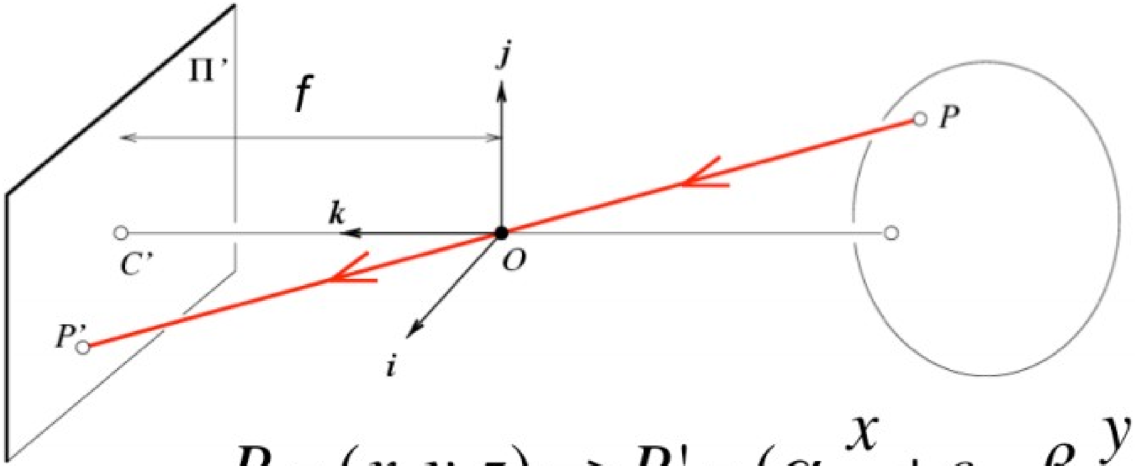
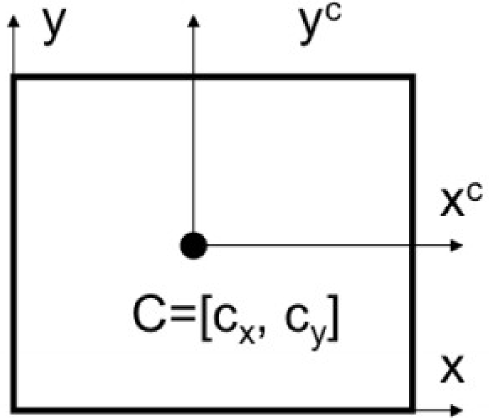


$$P = (x, y, z) \rightarrow P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

[Eq. 7]



- Is this a linear transformation?
  No — division by z is nonlinear
- Can we express it in a matrix form?

Courtesy: Silvio Savarese

Perspective Transformation

- $P \longrightarrow P'$ projection becomes $P_h \longrightarrow P_h'$
- The $P=[x\ y\ z]$ in the 3D space is $P_h=[x\ y\ z\ 1]$ in Homogeneous reference system
- P' was computed as $[\alpha(x/z)+c_x \quad \beta(y/z)+c_y]$
- P_h' can be then $[\alpha x+c_x z \quad \beta y+c_y z \quad z]$
- Do you see how to express $P_h \longrightarrow P_h'$ as matrix product?

Perspective Linear Transformation

$$P_h' = \begin{bmatrix} \alpha x + c_x z \\ \beta y + c_y z \\ z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{P_h}$$

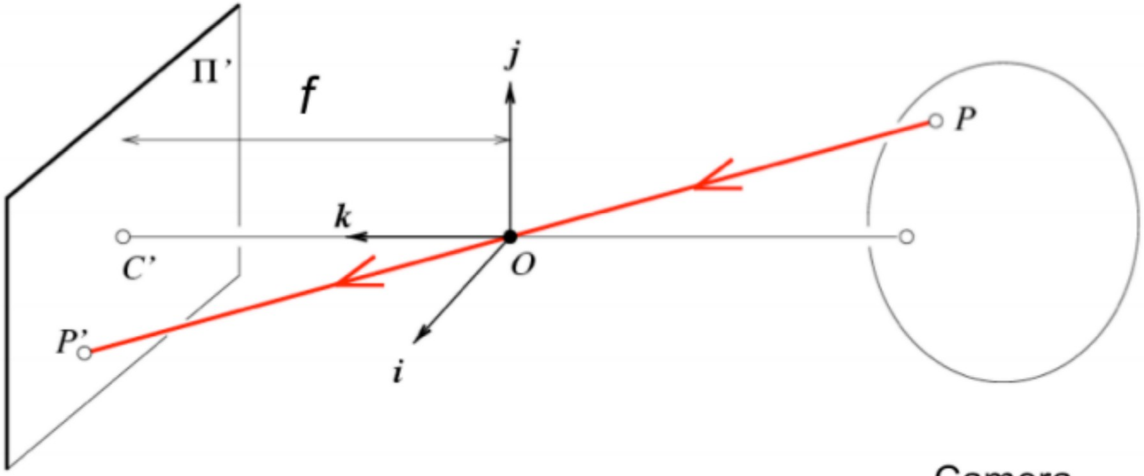
[Eq.8]

$$M = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\overbrace{P_h'}^{\text{Homogenous}} \rightarrow \overbrace{P' = \left(\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y\right)}^{\text{Euclidian}}$$

Courtesy: Silvio Savarese

The Intrinsic Matrix



Camera matrix K

[Eq.9]

$$P' = M\,P$$

$$= K\begin{bmatrix} I & 0 \end{bmatrix} P$$

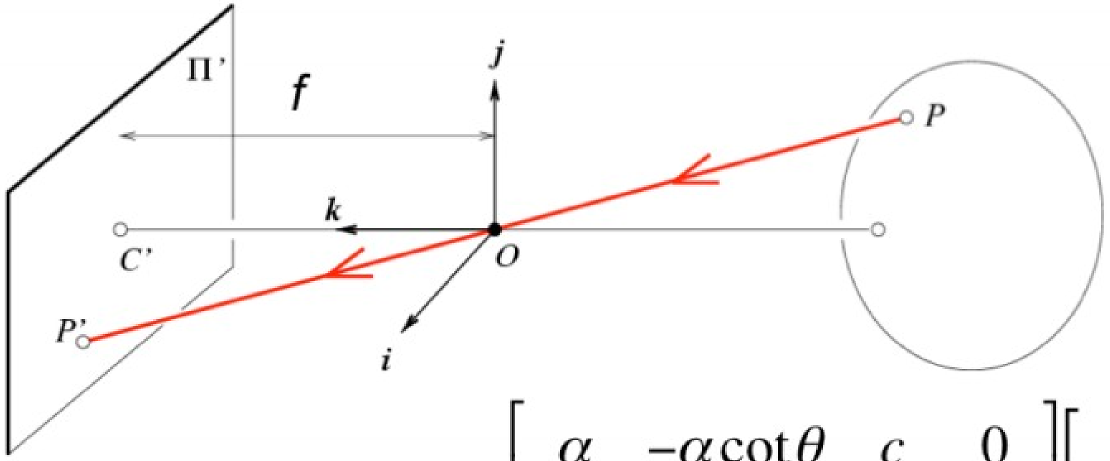
$$P' = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

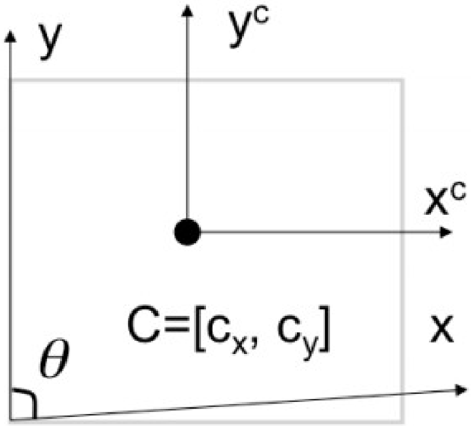
Courtesy: Silvio Savarese

Skewness

- Not enough, we have to consider skeweness!
- Sometimes, the 2D image plane is not a rectangle but rather is skewed
  - i.e. the angle between the image axis is not 90 degrees.
- Another transformation needs to be carried out to go from the rectangular plane to the skewed plane
- No demonstration

Skewness





$$P' = \begin{bmatrix} \alpha & -\alpha\cot\theta & c_x & 0 \\ 0 & \dfrac{\beta}{\sin\theta} & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many degree does K have?
5 degrees of freedom!

Courtesy: Silvio Savarese

Intrinsic Camera Parameters

- So far we adopted a camera centric approach
  - Reference system centered on the pin-hole
  - All parameters are camera dependent only
  - No external world
- K $\longrightarrow$ **Camera Intrinsic Matrix**

Introducing the external world…

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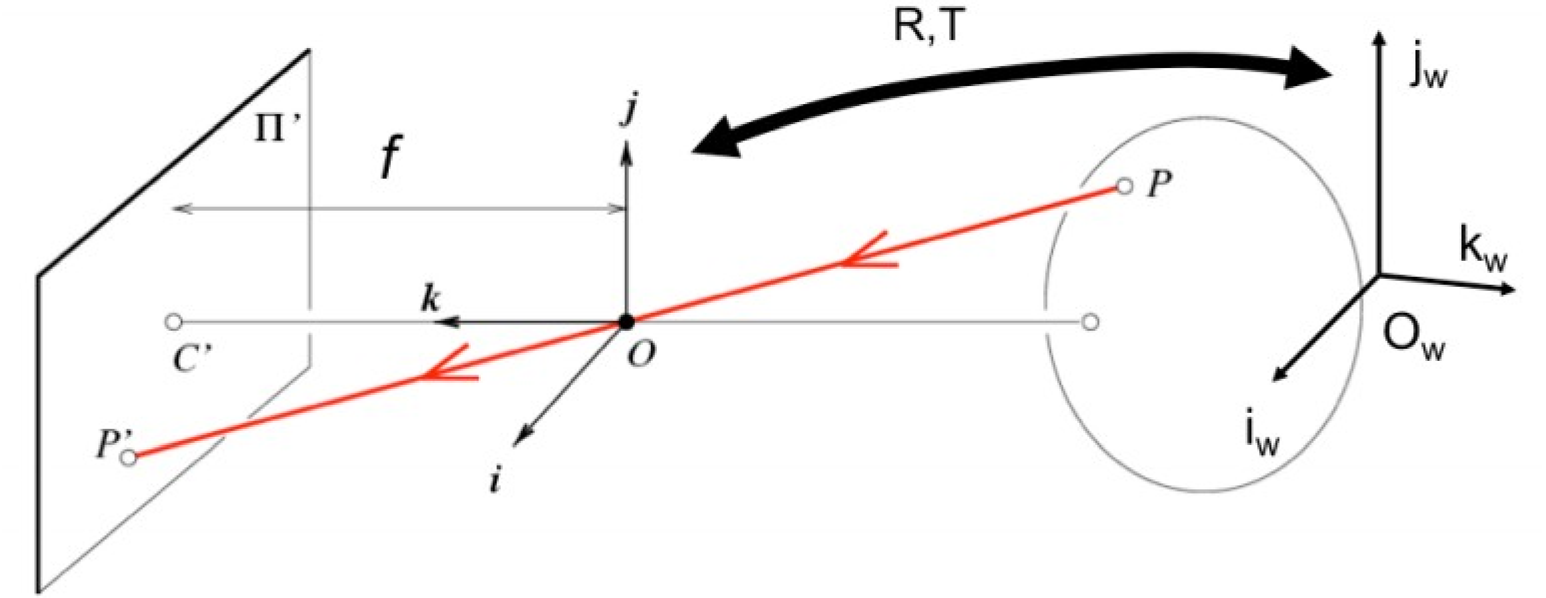
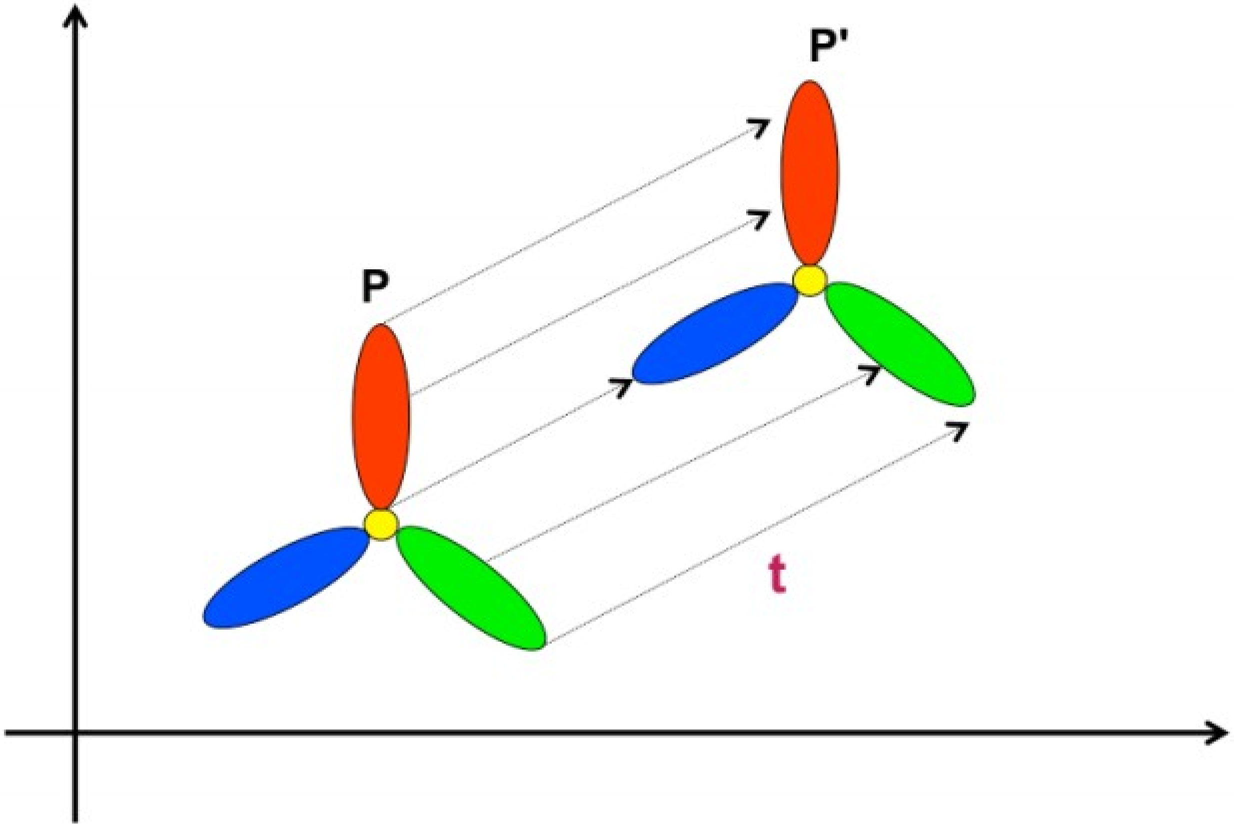
- Having a camera reference system is a bit limiting
- Usually a different reference system is used
- We need an additional transformation



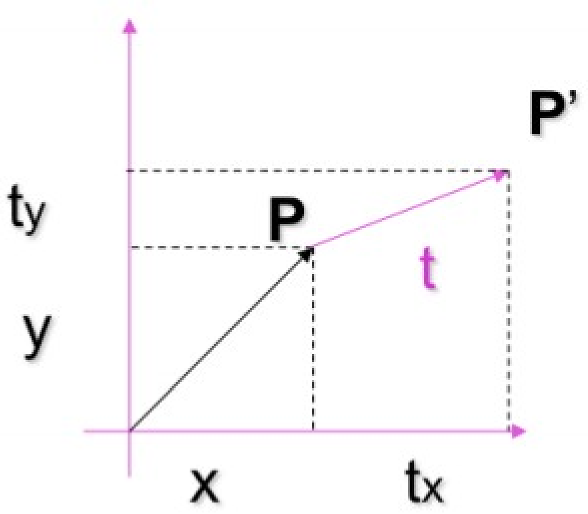
Courtesy: Silvio Savarese

Review: 2D translation



Courtesy: Silvio Savarese

Review: 2D translation



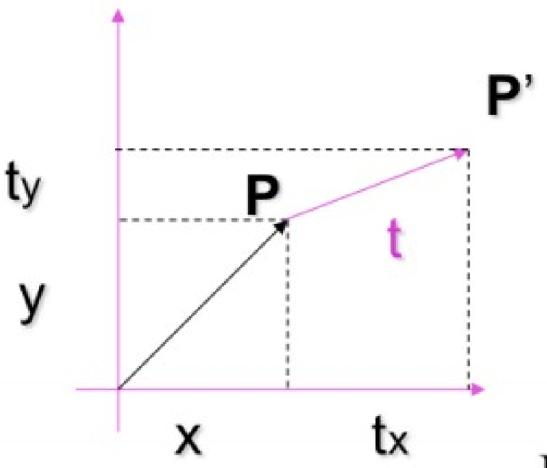
$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

Courtesy: Silvio Savarese

Review: homogeneous 2D translation



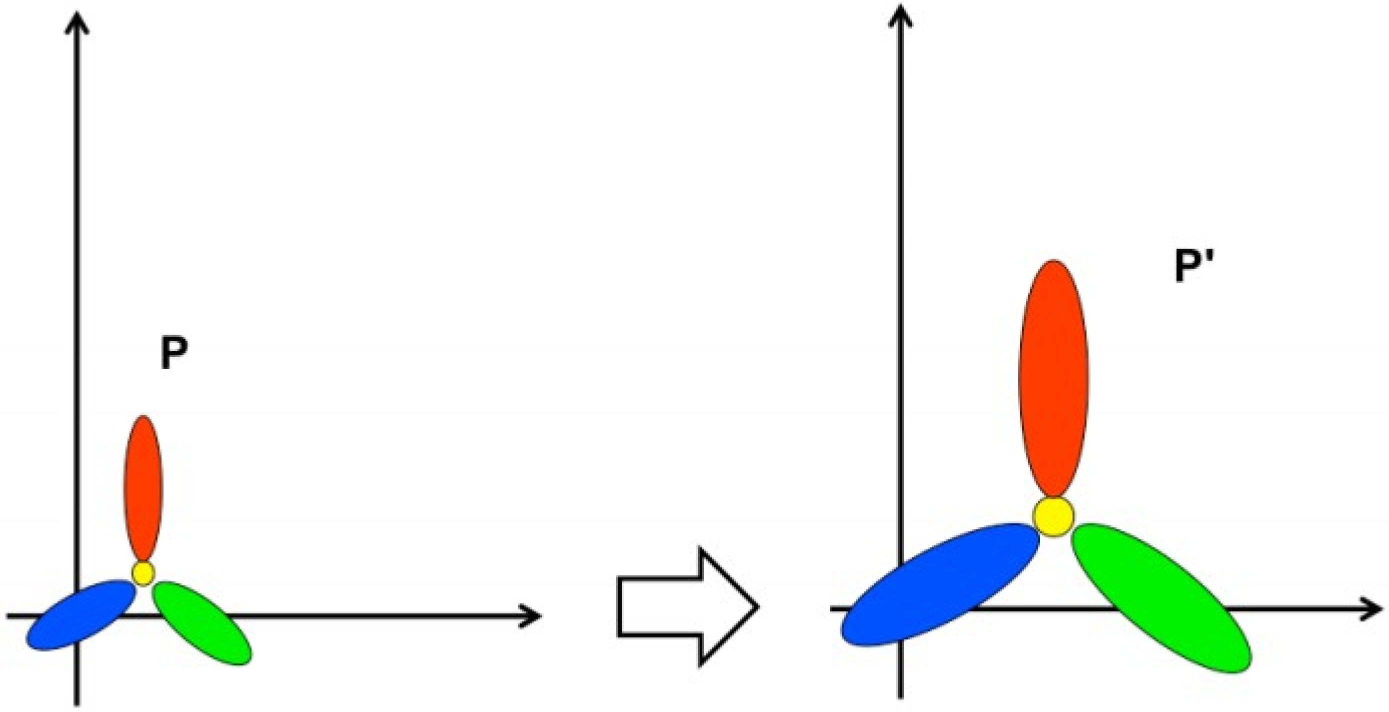
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{T} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Courtesy: Silvio Savarese

Review: 2D scaling



Courtesy: Silvio Savarese

Review: homogeneous 2D scaling

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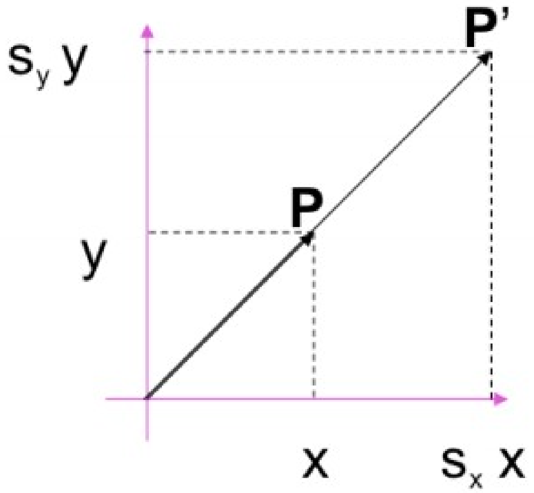


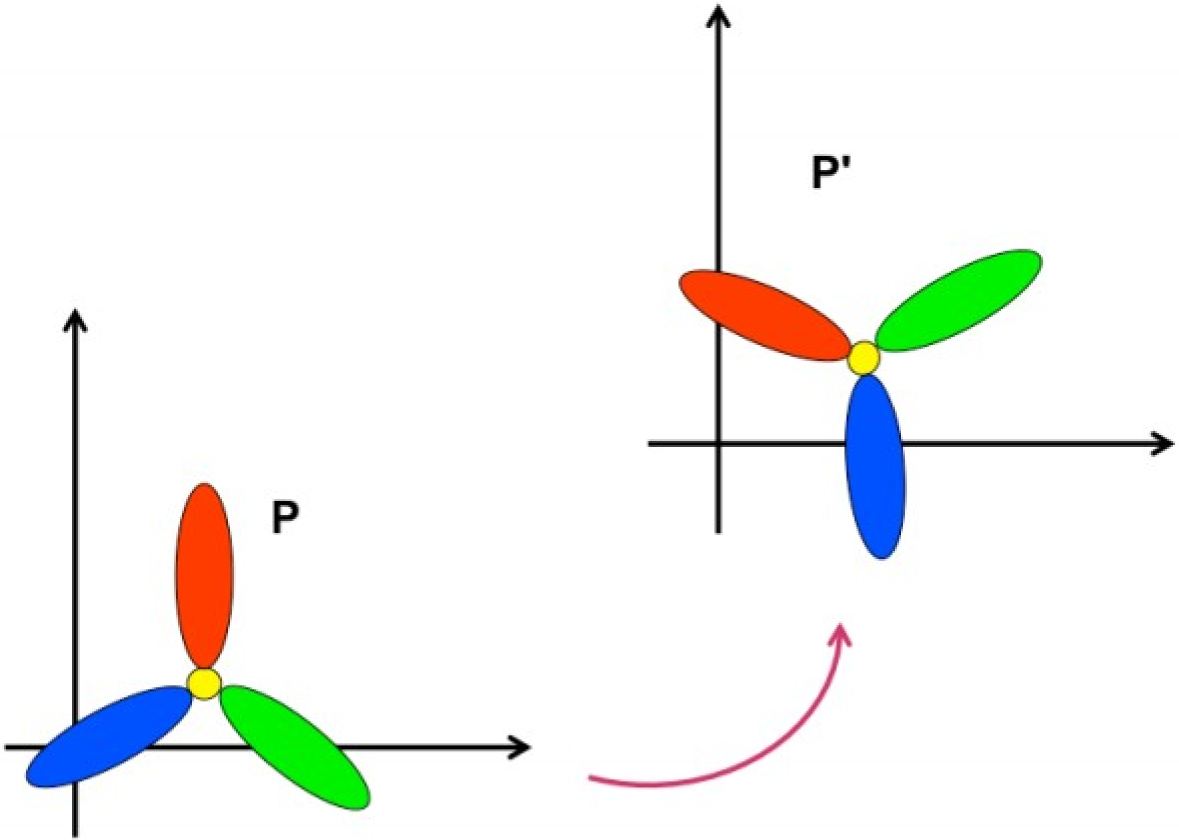
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{S} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Courtesy: Silvio Savarese

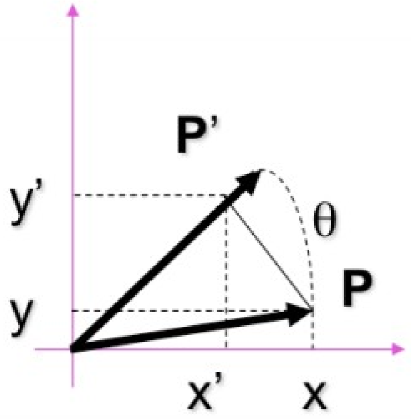
Review: 2D rotation



Courtesy: Silvio Savarese

Review: 2D rotation

- Rotate around the *z* axis by Θ



$$x' = \cos\theta\, x - \sin\theta\, y$$

$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{P'} = \mathbf{R}\ \mathbf{P}$$

How many degrees of freedom? 1

$$\mathbf{P'} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Courtesy: Silvio Savarese

Review: put everything togheter

$$\mathbf{P}' \rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

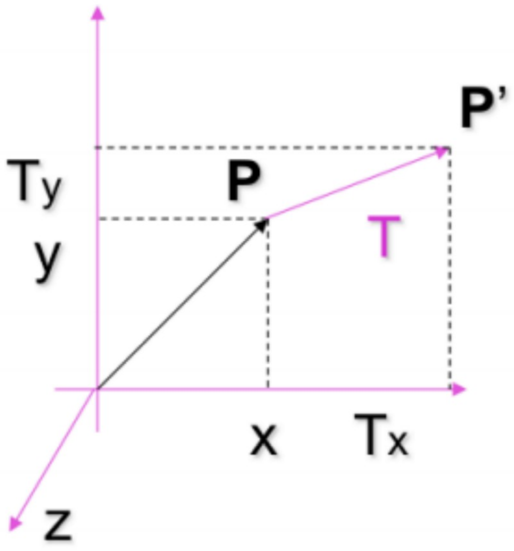
$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} \mathbf{R}\,\mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{matrix}} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x = s_y$, this is a similarity transformation

Courtesy: Silvio Savarese

Review: 3D translation



$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} \boldsymbol{I} & T \\ 0 & 1 \end{bmatrix}_{4\times4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A translation vector in 3D has 3 degrees of freedom

Courtesy: Silvio Savarese

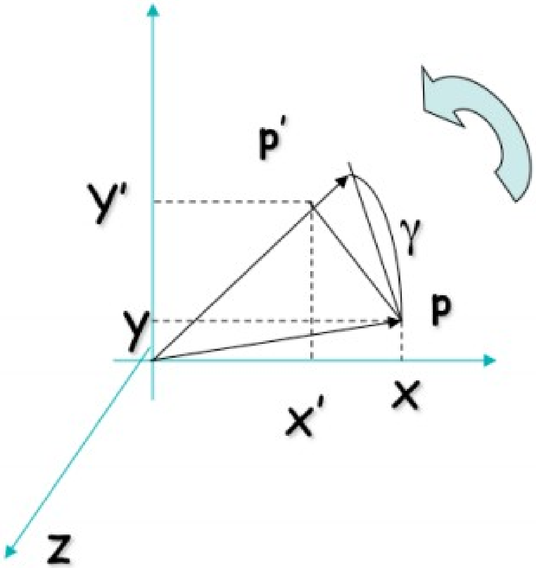
Review: 3D rotation

Rotation around the coordinate axes, counter-clockwise:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



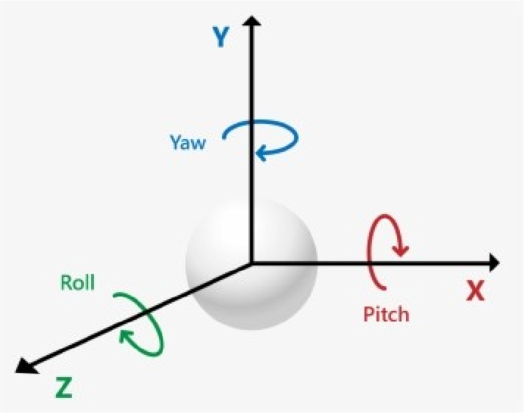
$$P' \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}_{4\times4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

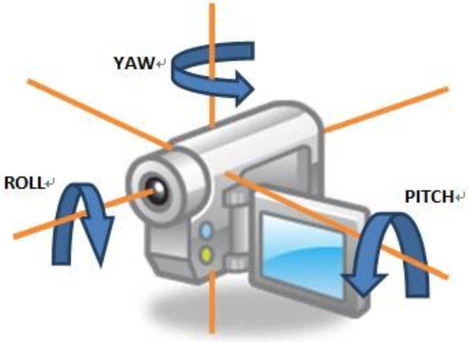
Courtesy: Silvio Savarese

A rotation matrix in 3D has 3 degree of freedom

Digression: rotation matrix properties

- Each rotation matrix is orthogonal
  - Proof: try to complement the rotation angle
  - Remember that for an orthogonal matrix $M^{-1}=M^{T}$
- Given that, the product among orthogonal matrices is still an orthogonal matrix





Review: 3D rotation & translation

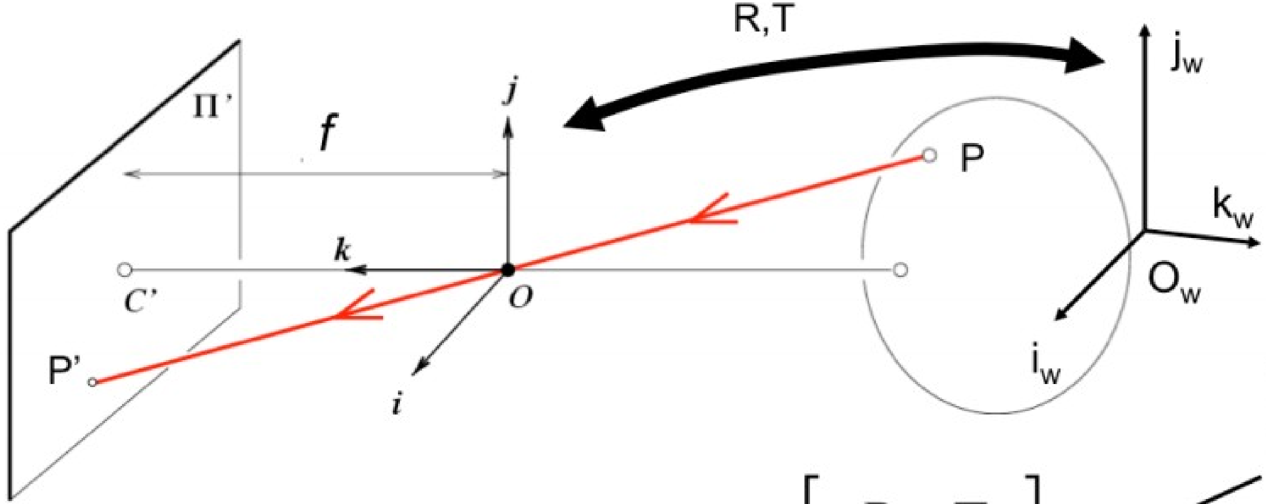
- Rotation order matters!

$$R = R_x(\alpha)\ R_y(\beta)\ R_z(\gamma) \qquad T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Courtesy: Silvio Savarese

World Reference System



In 4D homogeneous coordinates: $P = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} P_w$

$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Internal parameters

External parameters

$$P' = K\begin{bmatrix} I & 0 \end{bmatrix} P = K\begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} P_w \qquad P' = \underbrace{\boxed{K\begin{bmatrix} R & T \end{bmatrix}}}_{M} P_w$$

[Eq.11]

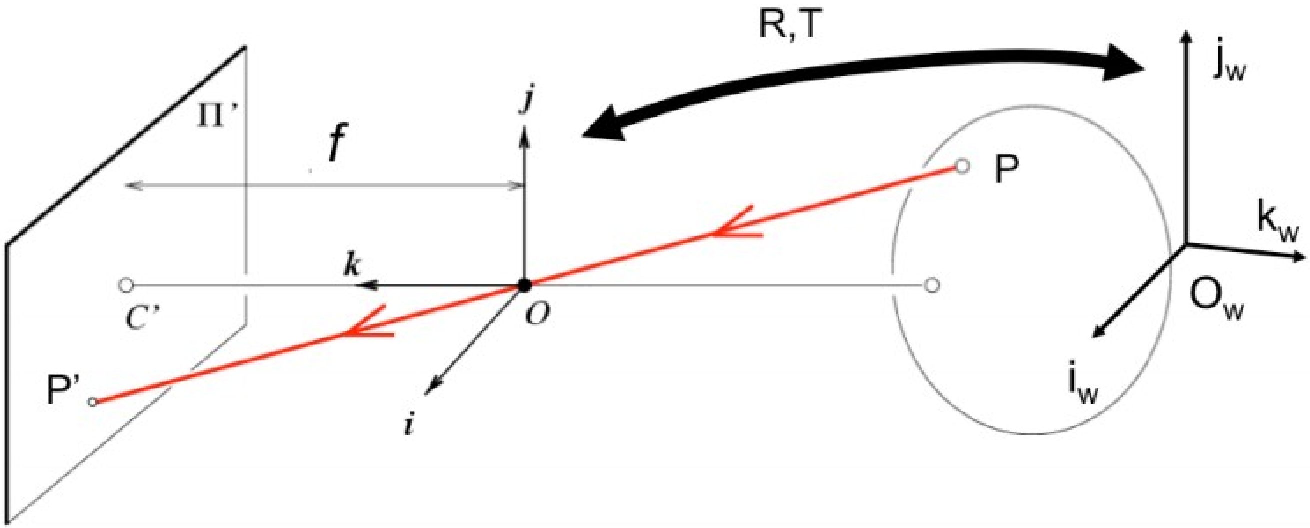
Courtesy: Silvio Savarese

Extrinsic Camera Parameters

- We added world information
  - Transformation from world reference system to the camera one
  - Extrinsic parameters of a camera only depend on its location and orientation
    - If I “move” the camera they need to be computed again
- [RT] $\rightarrow$ **Camera Extrinsic Matrix**

World Reference System

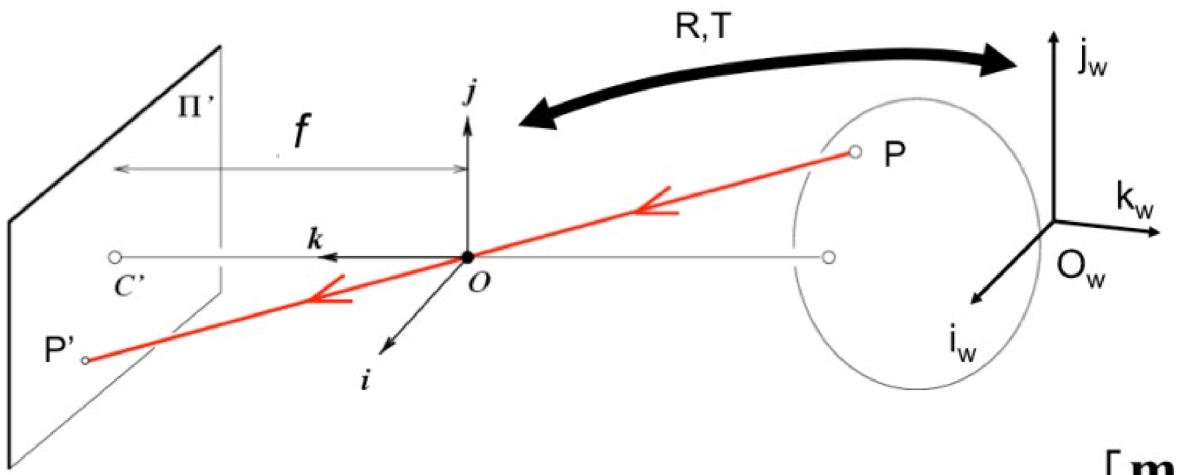
- 11 degrees of freedom (5 + 3 + 3)



$$P'_{3\times1} = M_{3\times4}\, P_w = K_{3\times3} \begin{bmatrix} R & T \end{bmatrix}_{3\times4} P_{w\,4\times1}$$

Courtesy: Silvio Savarese

Back to Euclidean Coordinates



$$P'_{3\times1} = M\,P_w = K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4} P_{w\,4\times1} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_W = \begin{bmatrix} \mathbf{m}_1 P_W \\ \mathbf{m}_2 P_W \\ \mathbf{m}_3 P_W \end{bmatrix} \xrightarrow{\mathbf{E}} \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right) \quad \text{[Eq.12]}$$

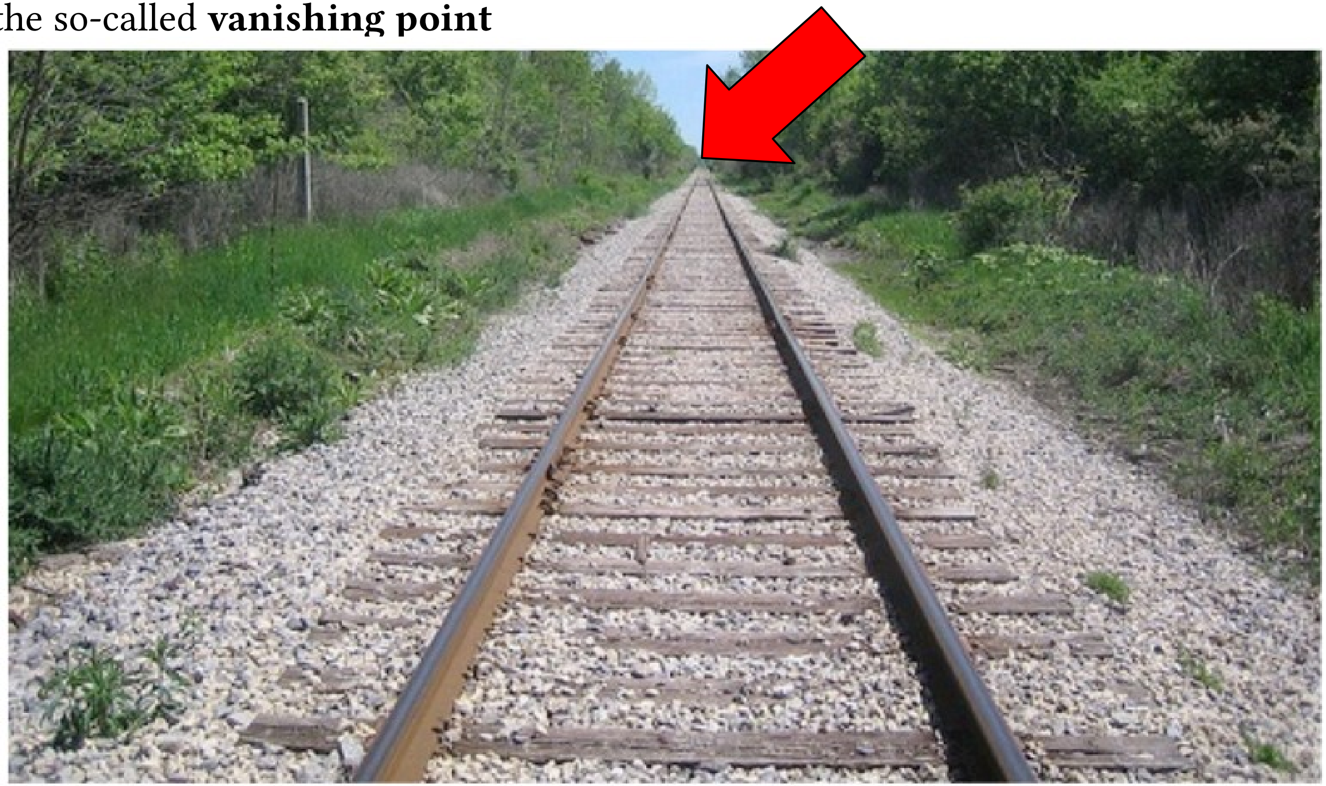
Courtesy: Silvio Savarese

Perspective transformation properties

- Points become…. Points!
- Lines become… Lines!
- Far away objects are smaller (divide by z)

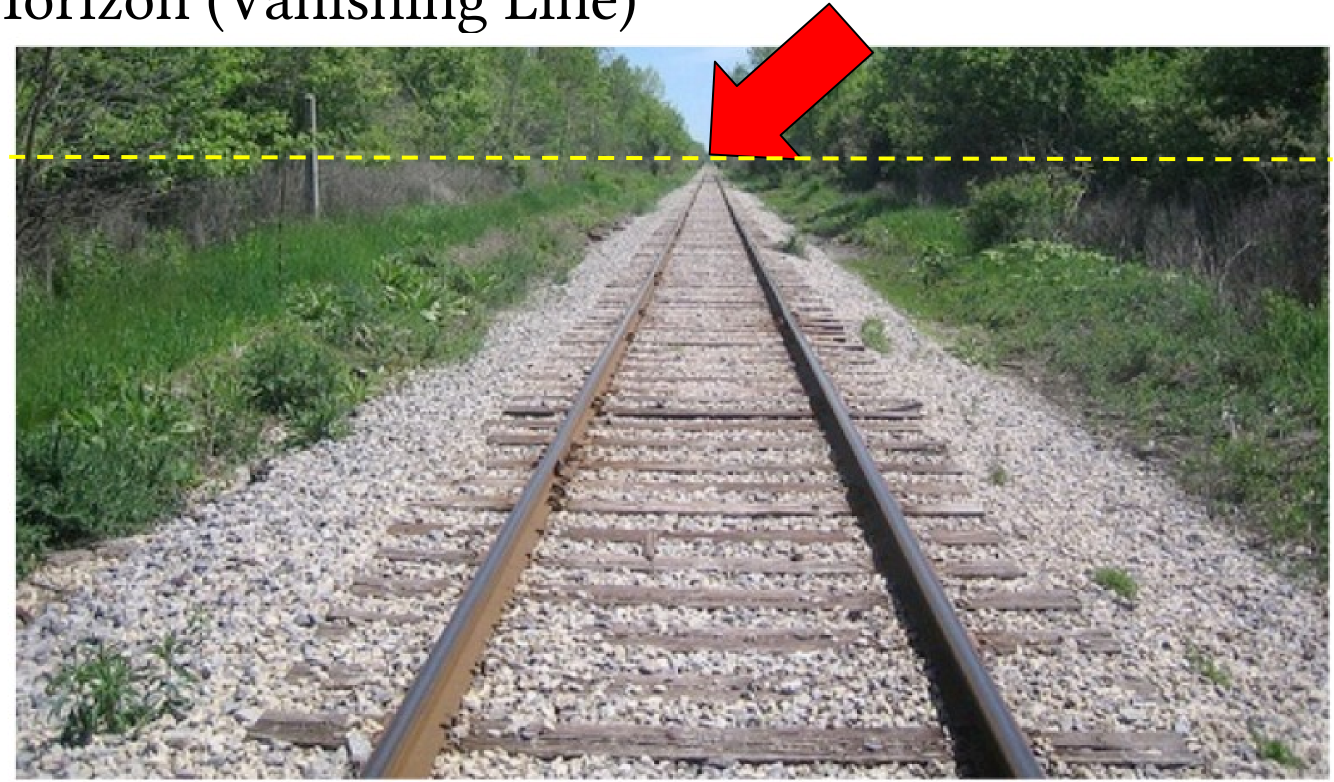
Perspective transformation properties

- Angles are not preserved
- Parallel lines intersect!
  - In the so-called **vanishing point**



Perspective transformation properties

- Parallel lines that lies in the same plane have vanishing points on a line
- The Horizon (Vanishing Line)



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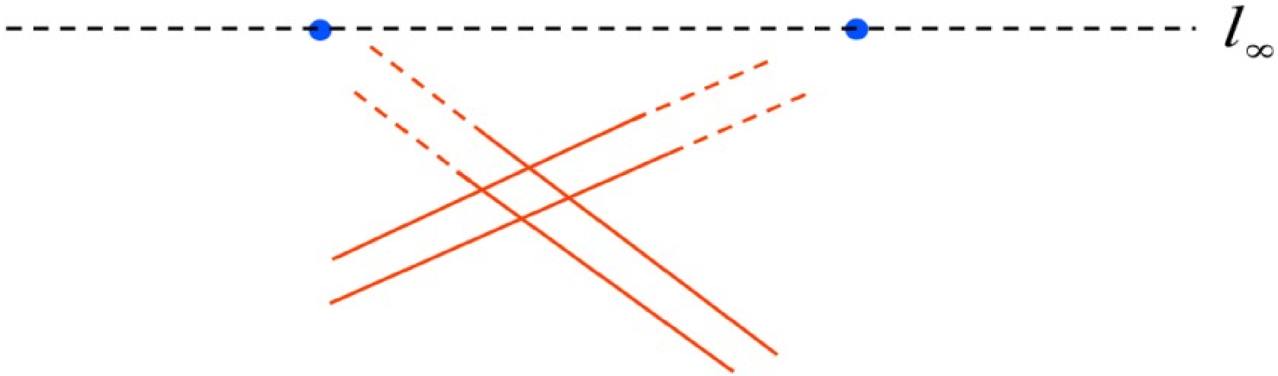
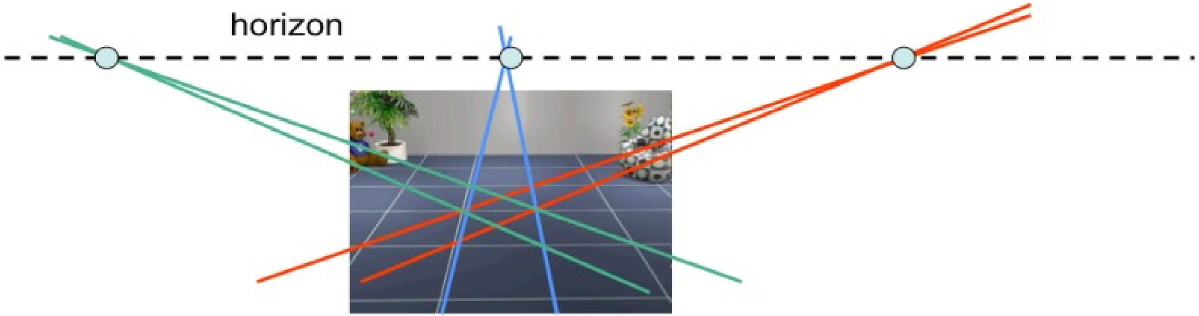
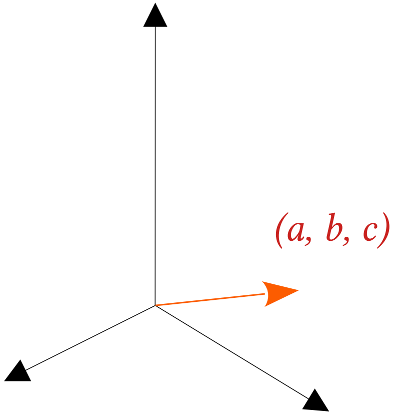




Lines become Lines

- A 3D straight line can be expressed as:

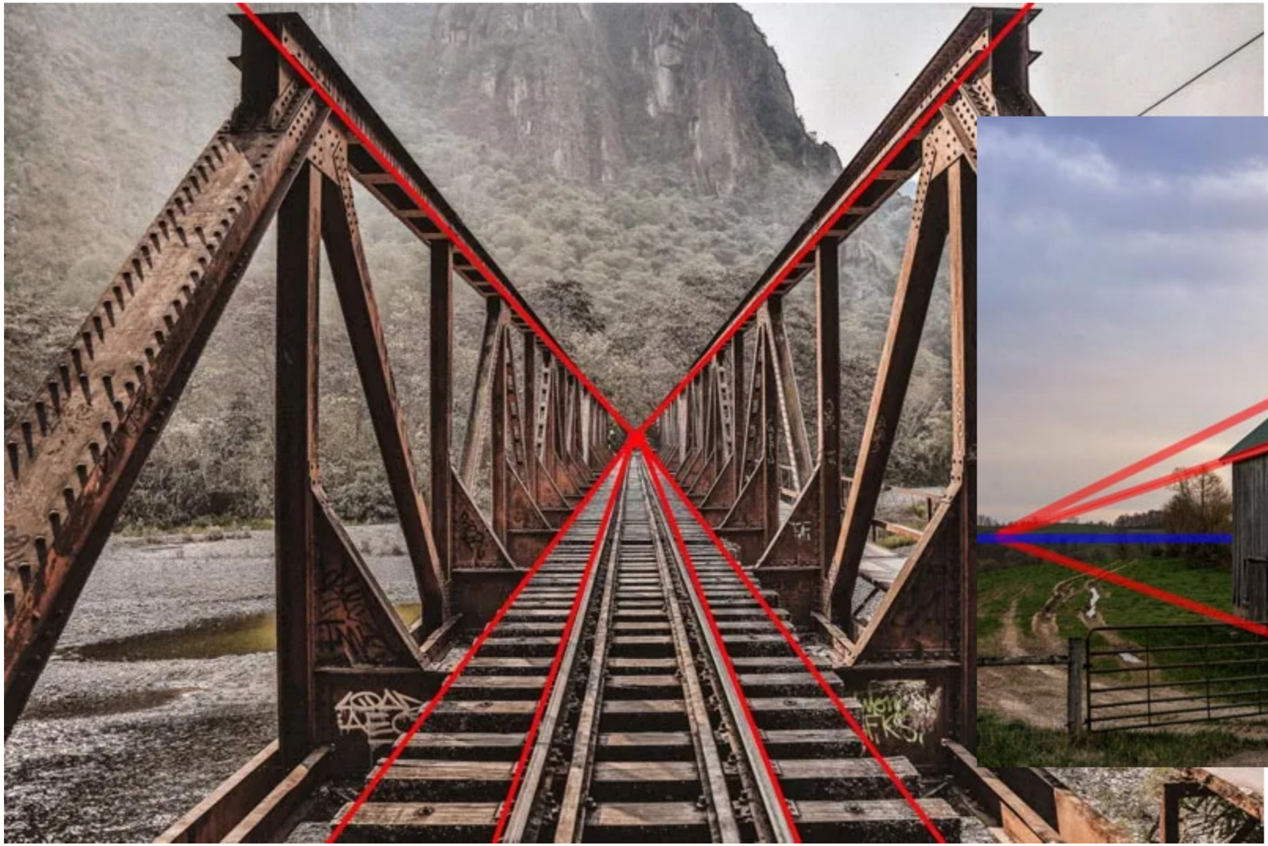
$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases} \longrightarrow \begin{cases} x'(t) = f\dfrac{(x_0+at)}{(z_0+ct)} \\ y'(t) = f\dfrac{(y_0+bt)}{(z_0+ct)} \end{cases}$$



$$\begin{cases} \lim_{t\to\pm\infty} x'(t) = f\dfrac{a}{c} \\ \lim_{t\to\pm\infty} y'(t) = f\dfrac{b}{c} \end{cases}$$

- x_0, y_0, z_0 represent a point of the line, where t is the distance from that point.
- a, b, c then encode the direction
- Parallel lines have the same a, b, and c params!
- Points at infinity are not at infinity in the image

Vanishing lines



Pin Hole Geometry Recap

UNIVERSITÀ DI PARMA

Intrinsic

$$M = K \cdot \begin{bmatrix} I & 0 \end{bmatrix} \cdot E = \boxed{K} \cdot \begin{bmatrix} \boxed{R \quad T} \end{bmatrix} \in R^{3 \times 4}$$

Extrinsic

$$P' = M \cdot P_w = K \cdot \begin{bmatrix} R & T \end{bmatrix} \cdot P_w$$

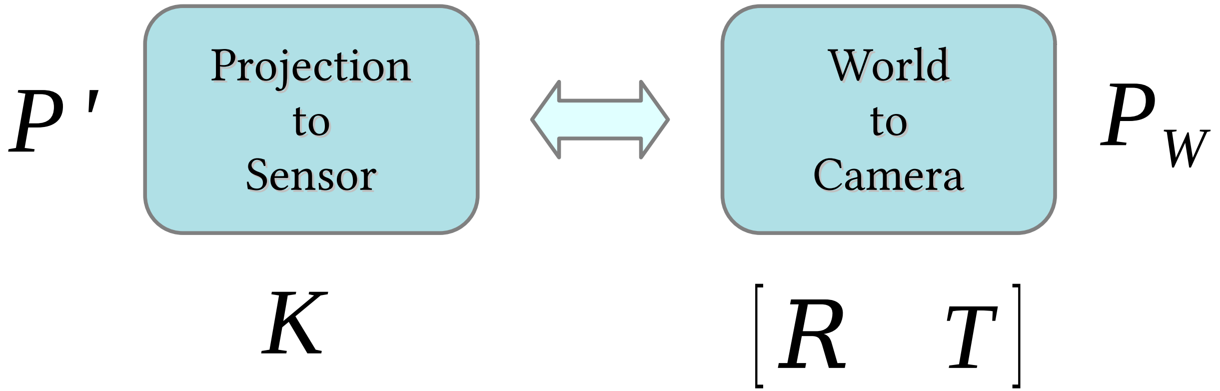
3D Homogeneous

2D Homogeneous

Model the perspective transformation from 3D to 2D

Ideal Camera

- We can see this transformation as the combination of
  - World to camera
  - Projection to sensor

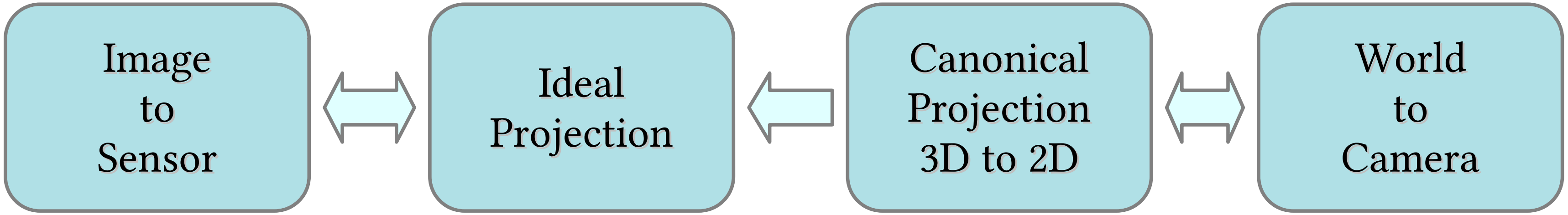


Ideal Camera

- The process can be further decomposed
- Also image to sensor can be further decomposed
  - Euclidean, affine, and general

Arrows have a meaning!

P'



P_W

k, l, c_x, c_y, θ

$$\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} R & T \end{bmatrix}$$