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Single View Reconstruction

Summary

- Calibration math recap
- Vanishing points and lines
- Calibration using a single view
- Estimation of geometry using a single view

Credits

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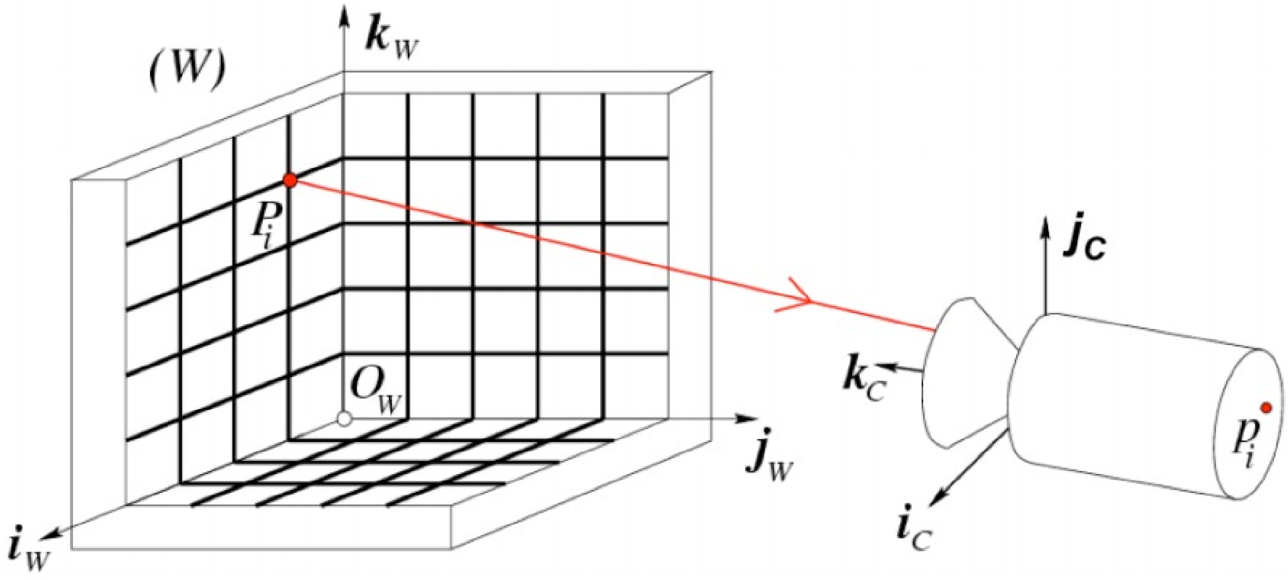
- [FP] D. A. Forsyth and J. Ponce. **Computer Vision: A Modern Approach (2nd Edition)**. Prentice Hall, 2011.
- [HZ] R. Hartley and A. Zisserman. **Multiple View Geometry in Computer Vision. Cambridge University Press**, 2003.
- CS231A · **Computer Vision: from 3D reconstruction to recognition, Prof. Silvio Savarese** – Stanford University

Camera model



$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = M\ P_i$$

In pixels

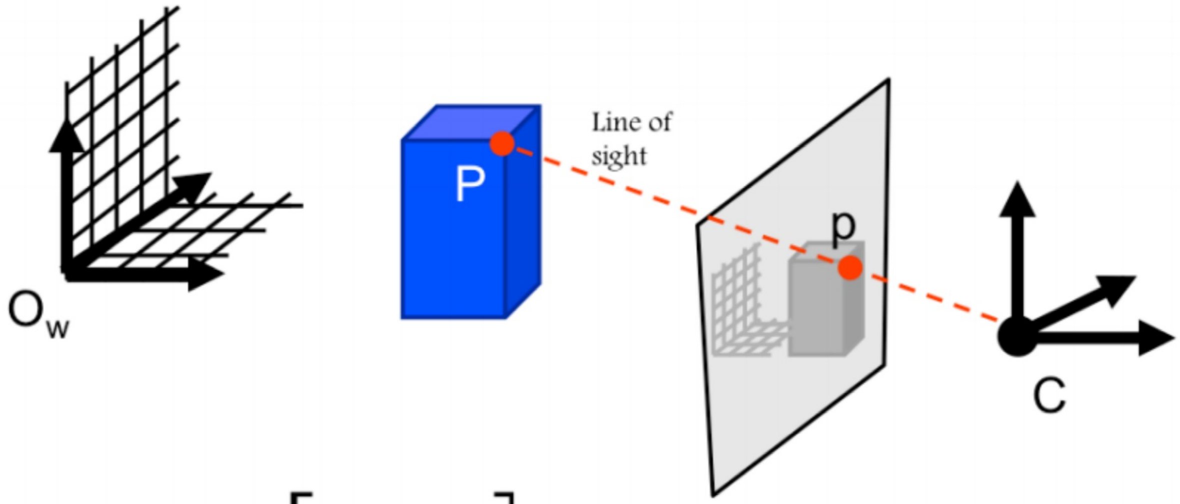
World ref. system

$$M = K\begin{bmatrix} R & T \end{bmatrix}$$

11 unknowns

Need at least 6 correspondences

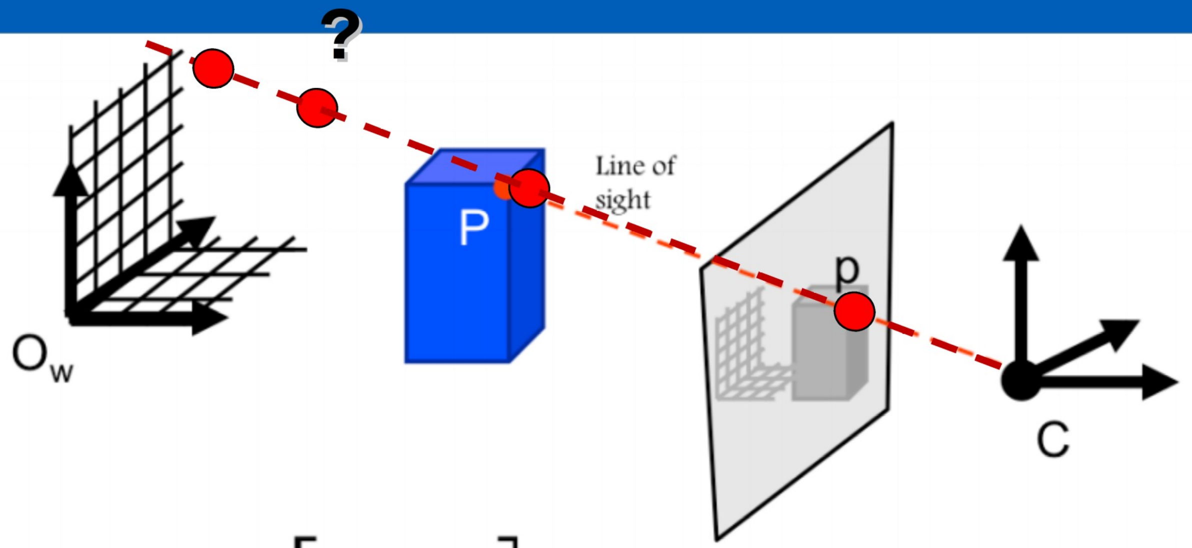
Camera model



$$M = K\begin{bmatrix} R & T \end{bmatrix}$$

- After calibration we know the transformation world $\rightarrow$ camera
  - We can discover p if we know P

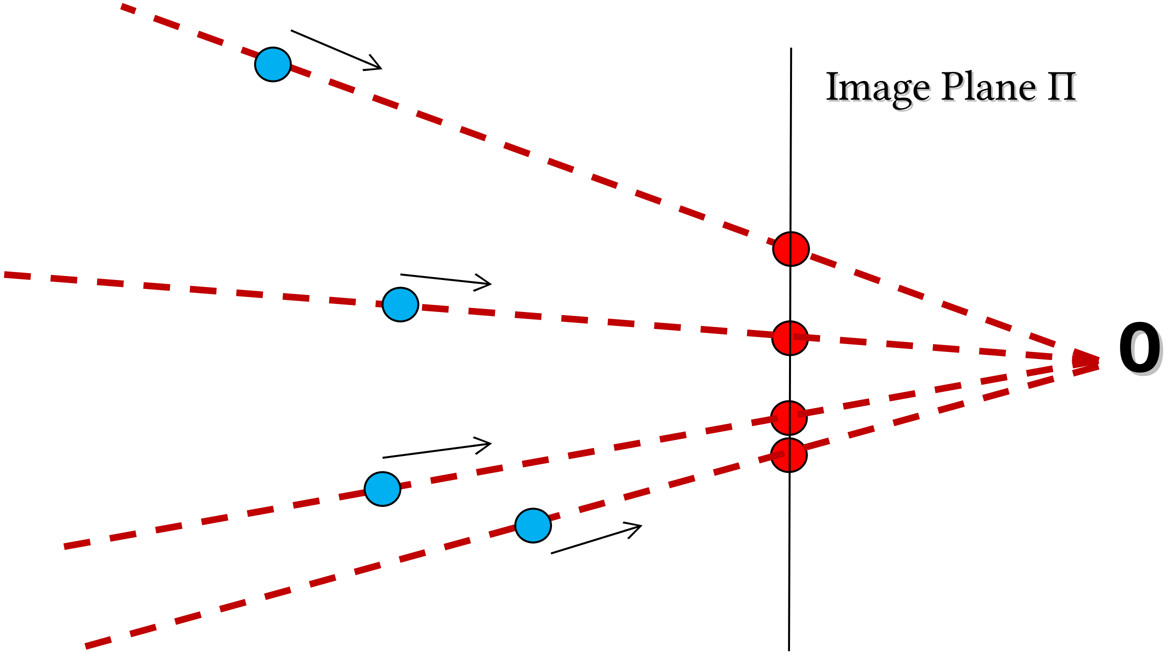
Camera model



$$M = K\begin{bmatrix} R & T \end{bmatrix}$$

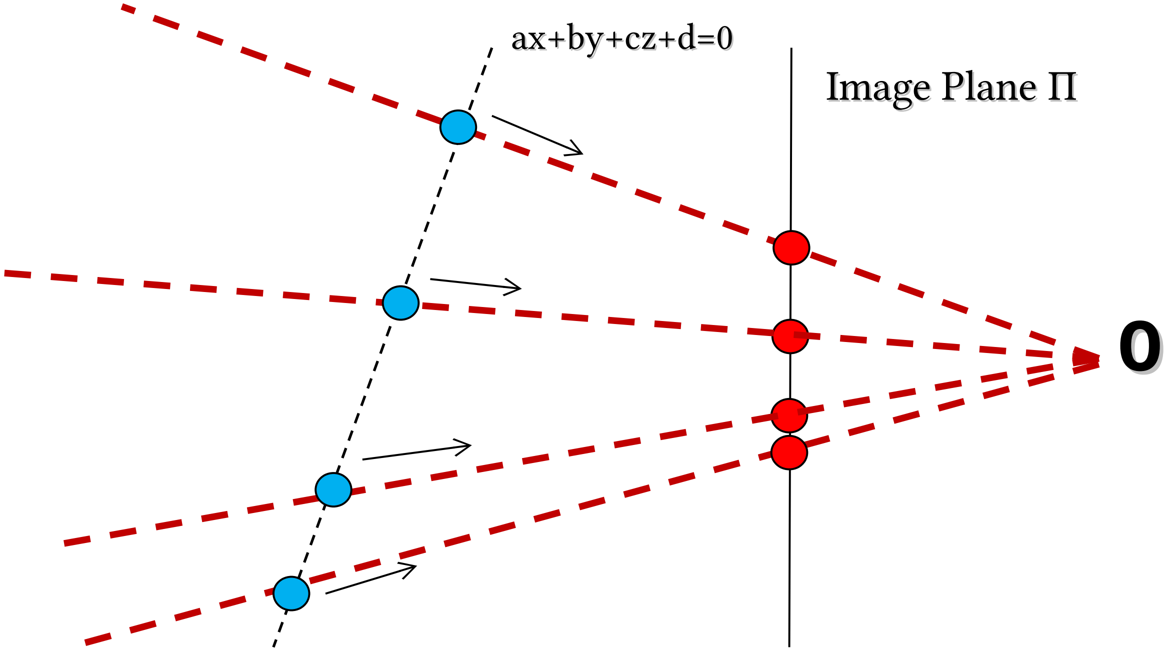
- Can we say where is P when we know only p?
  - NO: any P on the line of sight can generate p...

Ill posed problem



- Projective space → we lost one dimension
- 3D [x,y,z] are projected in [x/z,y/z,1]

Add a constraint



- Just image that the (red) image points are projection of points that lie on the same plane
- In such a case for each image point there is only one world point (cyan)

Add a constraint

- Given a plane aX + bY + cZ + d = 0

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = M \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} u \\ v \\ w \\ 0 \end{bmatrix} = \overset{\boxed{\text{4x4}}}{\begin{bmatrix} M \\ a\,b\,c\,d \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} M \\ a\,b\,c\,d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \\ 1 \\ 0 \end{bmatrix} \quad \xrightarrow[\textit{euclidean}]{} \quad \begin{bmatrix} X/W \\ Y/W \\ Z/W \end{bmatrix}$$

- We can obtain euclidean world coordinates of pixel (u,v)

Specific case, plane Z=0

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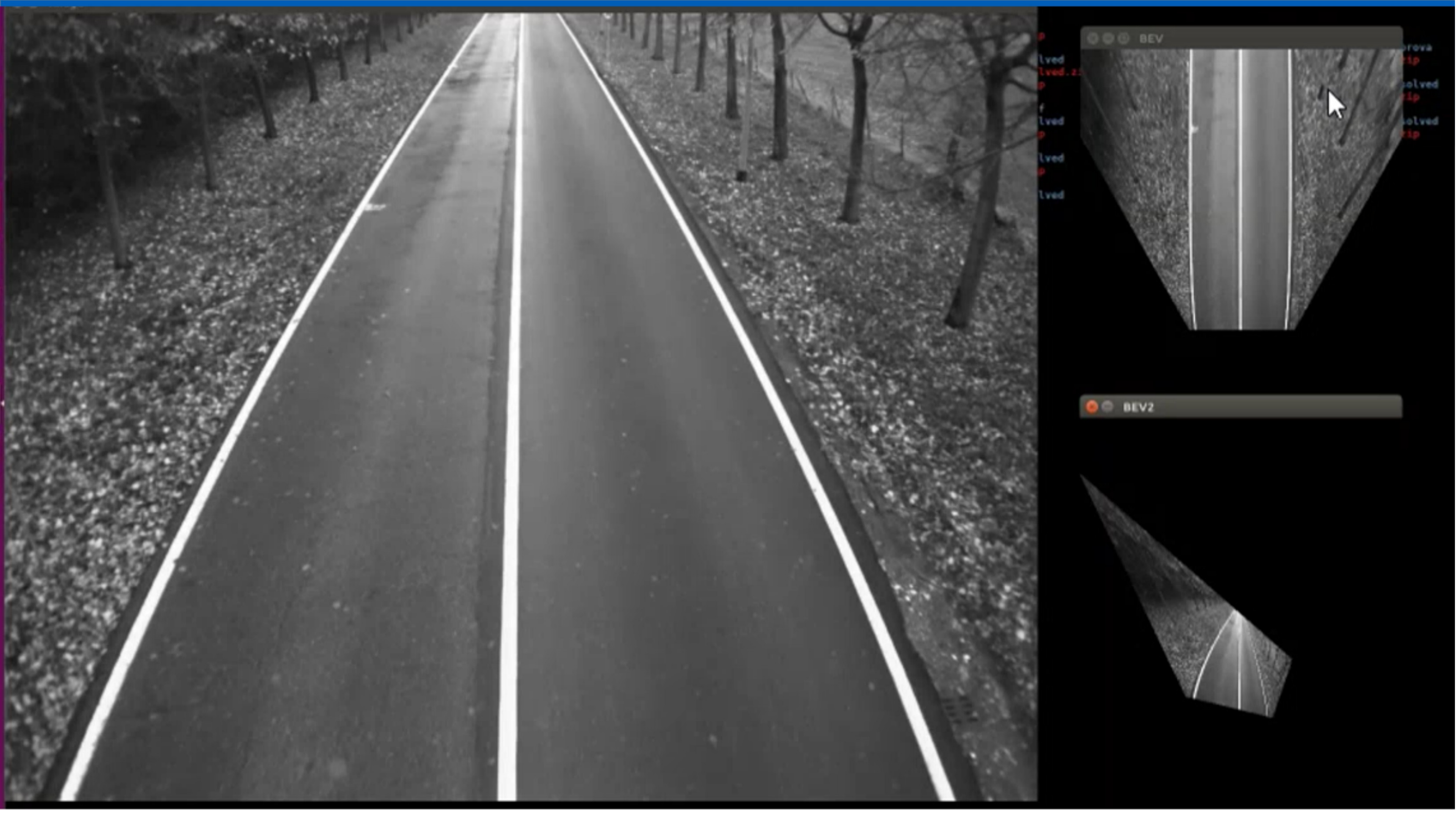
- Consider a very specific plane ⟶ Z=0

$$\begin{bmatrix} X \\ Y \\ 0 \\ W \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} = H \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

- H is 3x3
- This actually works for every possible plane!


BEV
BEV2

Single View Reconstruction

http://www.robots.ox.ac.uk/~vgg/projects/SingleView/models/hut/hutme.wrl

Single View Reconstruction

http://www.robots.ox.ac.uk/~vgg/projects/SingleView/models/hut/hutme.wrl

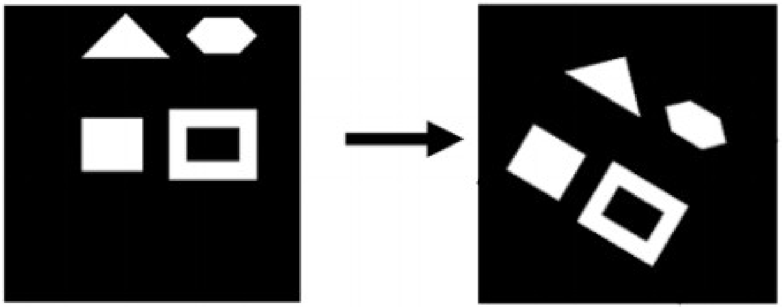
Transformations in 2D

- Isometric transformations
- Similarity transformations
- Affine transformations
- Projective transformations or **homographies**

Isometric Transformation

- Concatenation of rotation and translation
- 3 degrees of freedom
  - 2 for translation and 1 for rotation
- **It preserves distance**
- Motion of rigid object

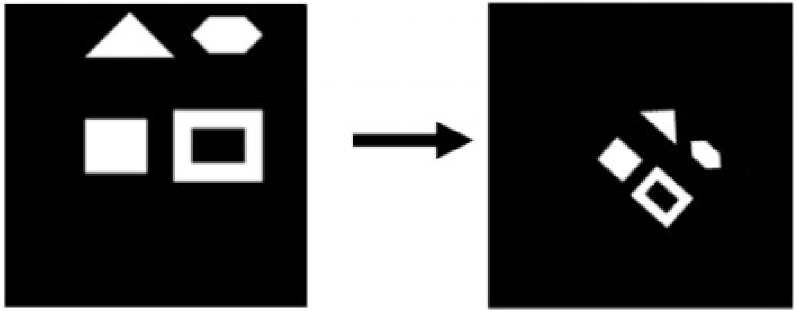
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_i \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



R and t 2×2 matrices

Similarity Transformation

- Concatenation of rotation, translation, and scale
- 4 degrees of freedom
  - 2 for translation, 1 for rotation, 1 for scale
- **It preserves shape**
- Angles among lines & Lenghts ratio

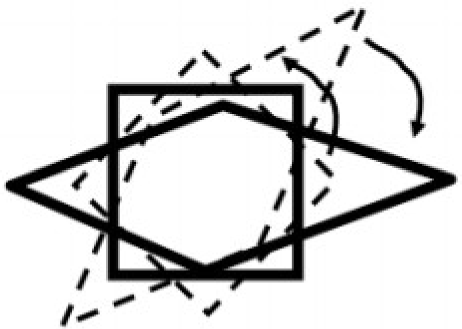


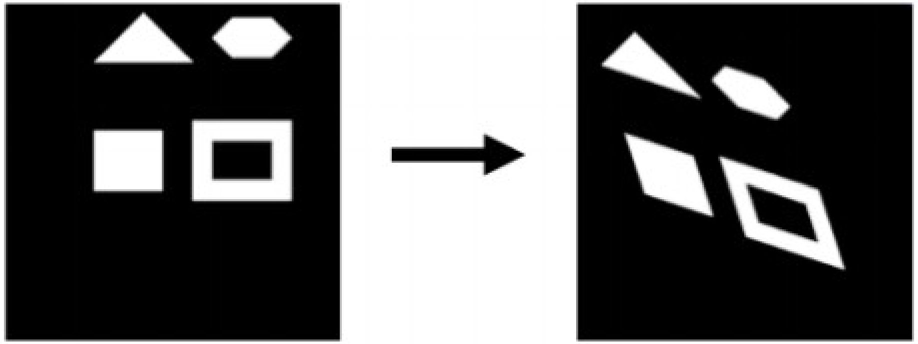
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} SR & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad S = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

SR is still a 2×2 matrix

Affine Transformation

- **It preserves ??**





$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \qquad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Affine Transformation

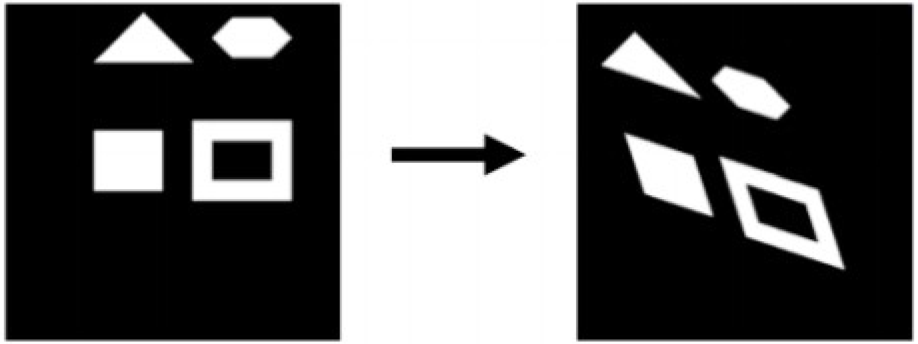
- Matrix A can be seen as a Single Value Decomposition of 2 ortogonal matrices and a diagonal one

$$A = UDV^T = (UV^T)(VDV^T) = R(\theta)\cdot R(-\phi)\cdot D\cdot R(\phi)$$

$$UV^T = R(\theta), \quad V^T = R(\phi), \quad D = D$$

Affine Transformation

- **It preserves**
  - **Parallel lines**
  - **Areas ratio**
  - **Collinear segment ratio**
- 6 degrees of freedom



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

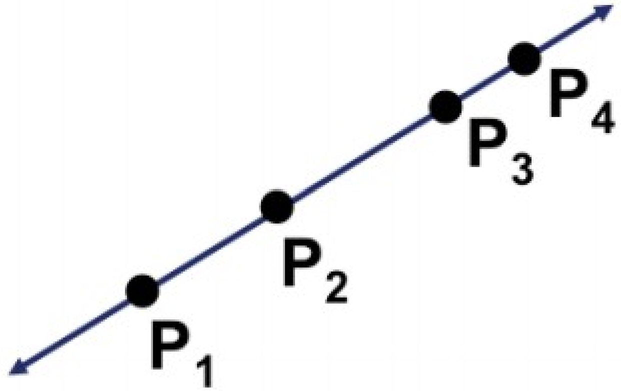
$$A \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \qquad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Projective Transformation

- **It preserves**
  - **Collinearity**
  - **Cross-ratio for 4 collinear points**
- 8 degrees of freedom

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ v & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Cross-ratio for 4 collinear points



[Eq. 9]

$$\frac{\|\mathbf{P}_3 - \mathbf{P}_1\| \, \|\mathbf{P}_4 - \mathbf{P}_2\|}{\|\mathbf{P}_3 - \mathbf{P}_2\| \, \|\mathbf{P}_4 - \mathbf{P}_1\|}$$

$$\mathbf{P}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

Affine transformation vs Homograpy

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- It can be noticed that “up” to the affine transformation z is left unchanged

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Omographies also affect z!

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ v & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine transformation vs Homograpy

- Affine transformations:
  - move the (image) point
  - originally the projection of a (world) point
  - in another point
  - that it is the projection of another (world) point at the same distance as the previous one
  - $(x,y,z) \longrightarrow (x',y',z)$
- Omographies do not have that constraint!
  - Let's consider the last H_P line as [a b c]
    - $(x,y,z) \longrightarrow (x',y',ax+by+cz)$

Planar Homograpy

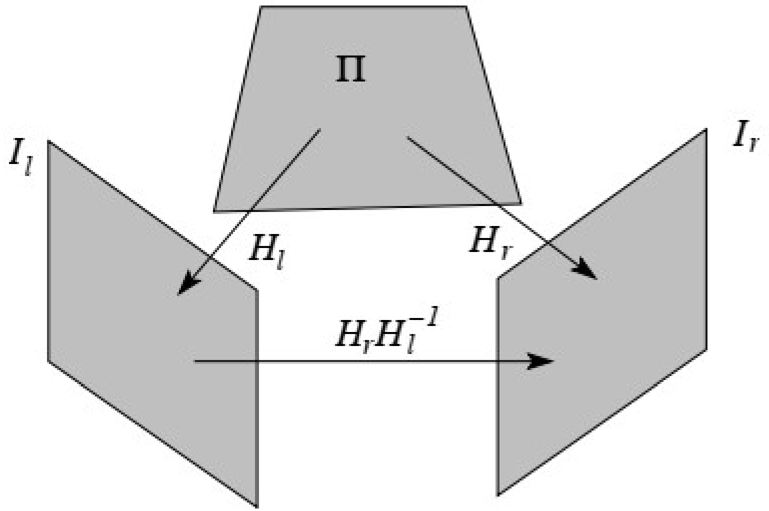
- A plane in the $\mathbb{R}^3$ space is
  - $ax+by+cz+d=0 \longrightarrow ax+by+cz=-d$
- Projection of points that lies on such plane are (x’, y’, -d)
- Homographies transform points that lie on a plane on another plane: plane ⟶ another plane with a different point of view

Planar Homograpy

- Consider 2 planes
  - $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$
- It can be easily demonstrated that we can compute an homograpy that bring from one plane to the other
- We can compute H and H' homographies
- We can then combine them to obtain another projection
  - $H_{\Pi} = (H)\cdot(H')^{-1}(d/d')$

Planar Homograpy

- We can see this in a different fashion
- The projection of a plane on another 2 planes can be computed using a homographic transformation



Planar Homography

- Assuming the 2 planes are the sensor planes of 2 cameras
- The induced homography can be computed using intrinsic camera parameters and their relative position

$$H_{\Pi} = K'\left(R + \frac{tn^T}{d}\right)K^{-1}$$

- K and K' are intrinsic camera matrices
- R and t are relative rototranslation matrices
- n·p=d is plane equation

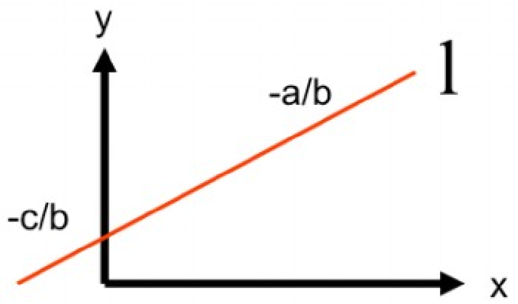
Planar Homography

- Homographies transform planes in another planes
  - ax+by+cz+d → (x’, y’, d)
- What happens when the plane is ax+by+cz=0?
  - (x’,y’,0) is a point to infinity…

2D Lines

$$ax + by + c = 0$$

$$l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



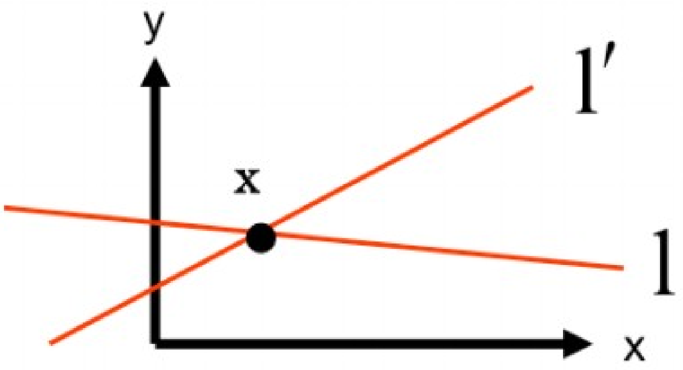
If $x = [x_1, x_2]^T \in l$

$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

[Eq. 10]

2D Lines Intersection

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \quad \text{[Eq. 11]}$$

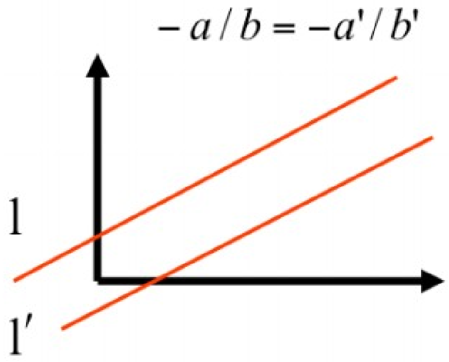


$$\mathbf{l} \times \mathbf{l}' \perp \mathbf{l} \quad \rightarrow (\mathbf{l} \times \mathbf{l}') \cdot \mathbf{l} = 0 \quad \rightarrow x \in l \quad \text{[Eq. 12]}$$

$$\mathbf{l} \times \mathbf{l}' \perp \mathbf{l}' \quad \rightarrow \underbrace{(\mathbf{l} \times \mathbf{l}')}_{\mathbf{x}} \cdot \mathbf{l}' = 0 \quad \rightarrow x \in l' \quad \text{[Eq. 13]}$$

2D Parallel Lines Intersection

$$\mathrm{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix}, \mathrm{x}_3 \neq 0$$



$$\mathrm{l} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix}$$

$$x_\infty = \begin{bmatrix} x'_1 \\ x'_2 \\ 0 \end{bmatrix}$$

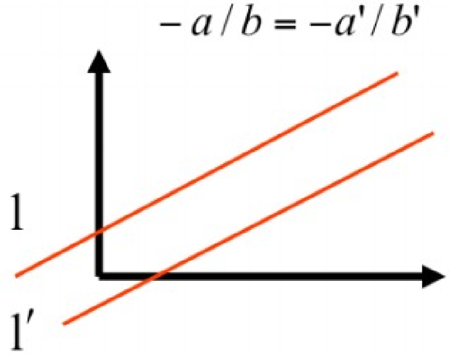
$$l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

$$\rightarrow l \times l' \propto \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} = x_\infty \quad \text{Eq.13]}$$

- The result is a point to infinity
  - **Ideal point**

2D Ideal Points

$$\mathrm{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix}, \mathrm{x}_3 \neq 0$$



$$\mathrm{l} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} \qquad l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

Note: the line l = $[a\ b\ c]^T$ pass trough the ideal point x_∞

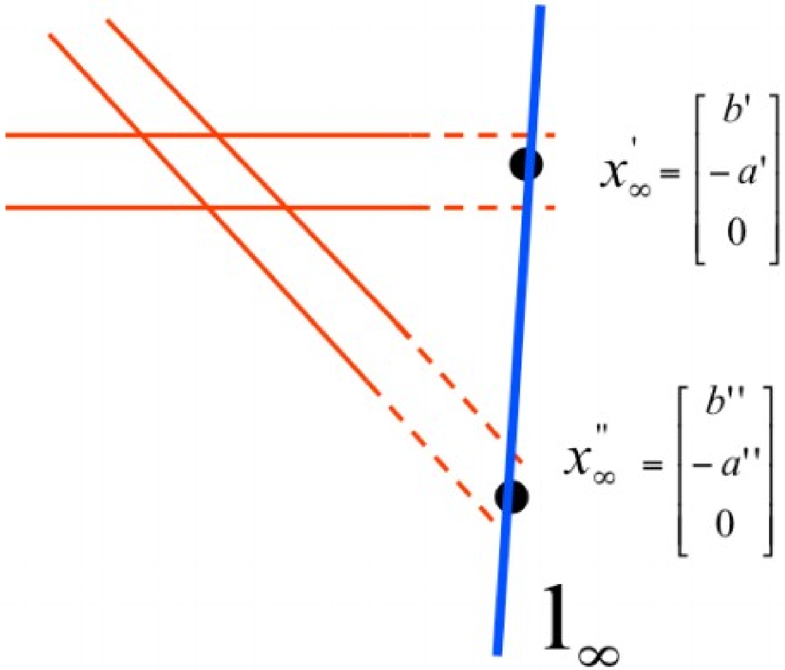
$$\mathrm{l}^T x_\infty = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} = 0 \quad \text{[Eq. 15]}$$

- This is true for every straight lines with the same slope

Ideal line

$$\mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Indeed: $$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$



- All ideal points lie on the same line $\longrightarrow$ ideal line or vanishing line
  - set of “directions” of lines in the plane

Ideal points transformation

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$

$$p' = H\ p$$

is it a point at infinity?

$$H\ p_\infty = ? = \begin{bmatrix} A & t \\ v & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

…no!

[Eq. 17]

$$H_A\ p_\infty = ? = \begin{bmatrix} A & t \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p'_x \\ p'_y \\ 0 \end{bmatrix}$$

[Eq. 18]

An affine transformation of a point at infinity is still a point at infinity

Ideal line transformation

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$

$$l' = H^{-T}\, l$$

[Eq. 19]

is it a line at infinity?

$$H^{-T}\, l_{\infty} = ? = \begin{bmatrix} A & t \\ v & b \end{bmatrix}^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ b \end{bmatrix}$$

…no!

[Eq. 20]

$$H_A^{-T}\, l_{\infty} = ? \quad = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix}^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[Eq. 21]

3D planes e lines

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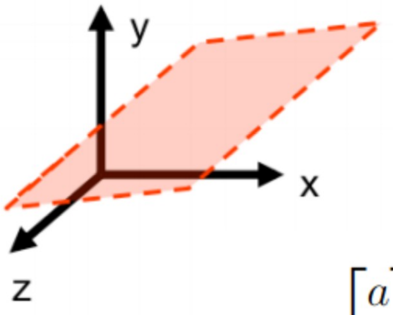
[HZ] Ch 3.2.2

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \qquad \Pi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



$$x^T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = ax_1 + bx_2 + cx_3 + d = 0$$

$$x \in \Pi \leftrightarrow x^T \Pi = 0$$

[Eq. 22]

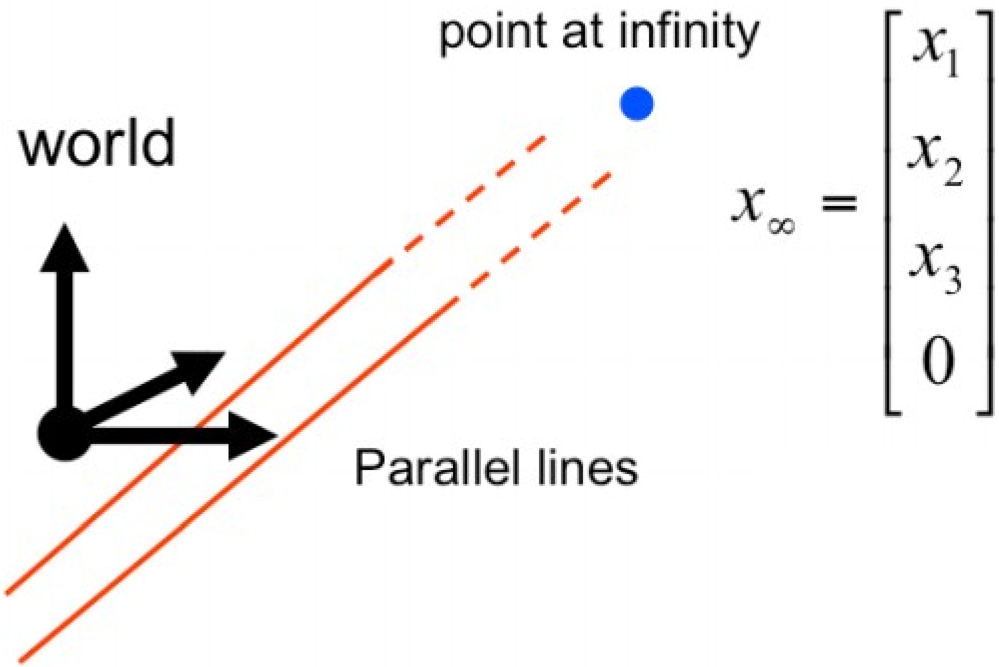
$$ax + by + cz + d = 0$$

[Eq. 23]

- Lines have 4 degree of freedom
  - They can be seen as intersections of 2 planes

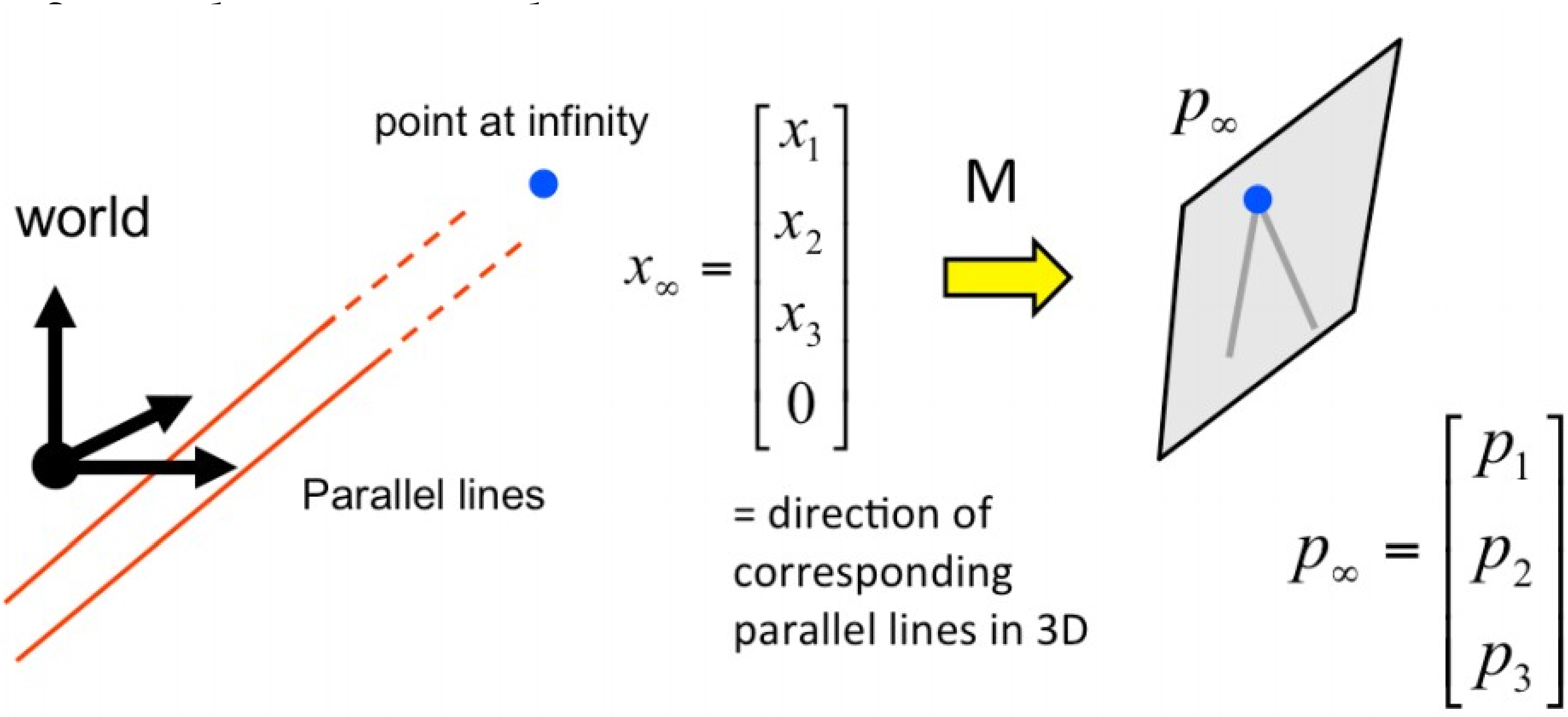
3D vanishing points

- Similar to 2D case

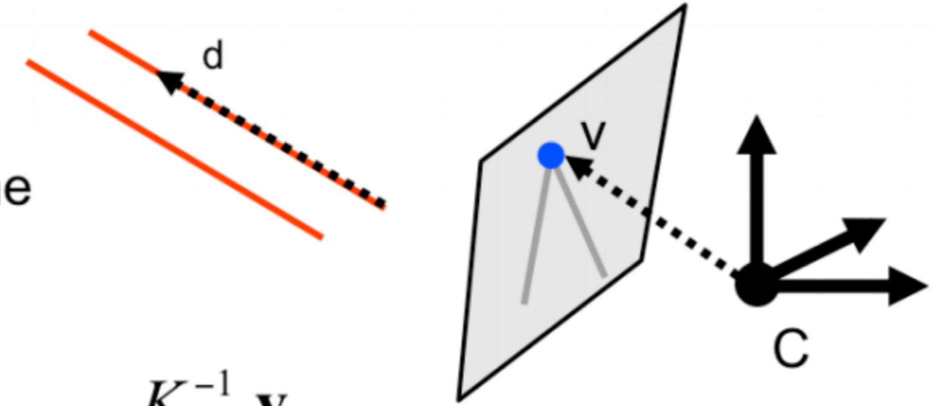


3D vanishing points

- A projective transformation M can bring point to infinity in a



Vanishing points vs Camera parameters



d = direction of the line
= [a, b, c]^T

$$\mathbf{v} = K\,\mathbf{d}$$

[Eq. 24]

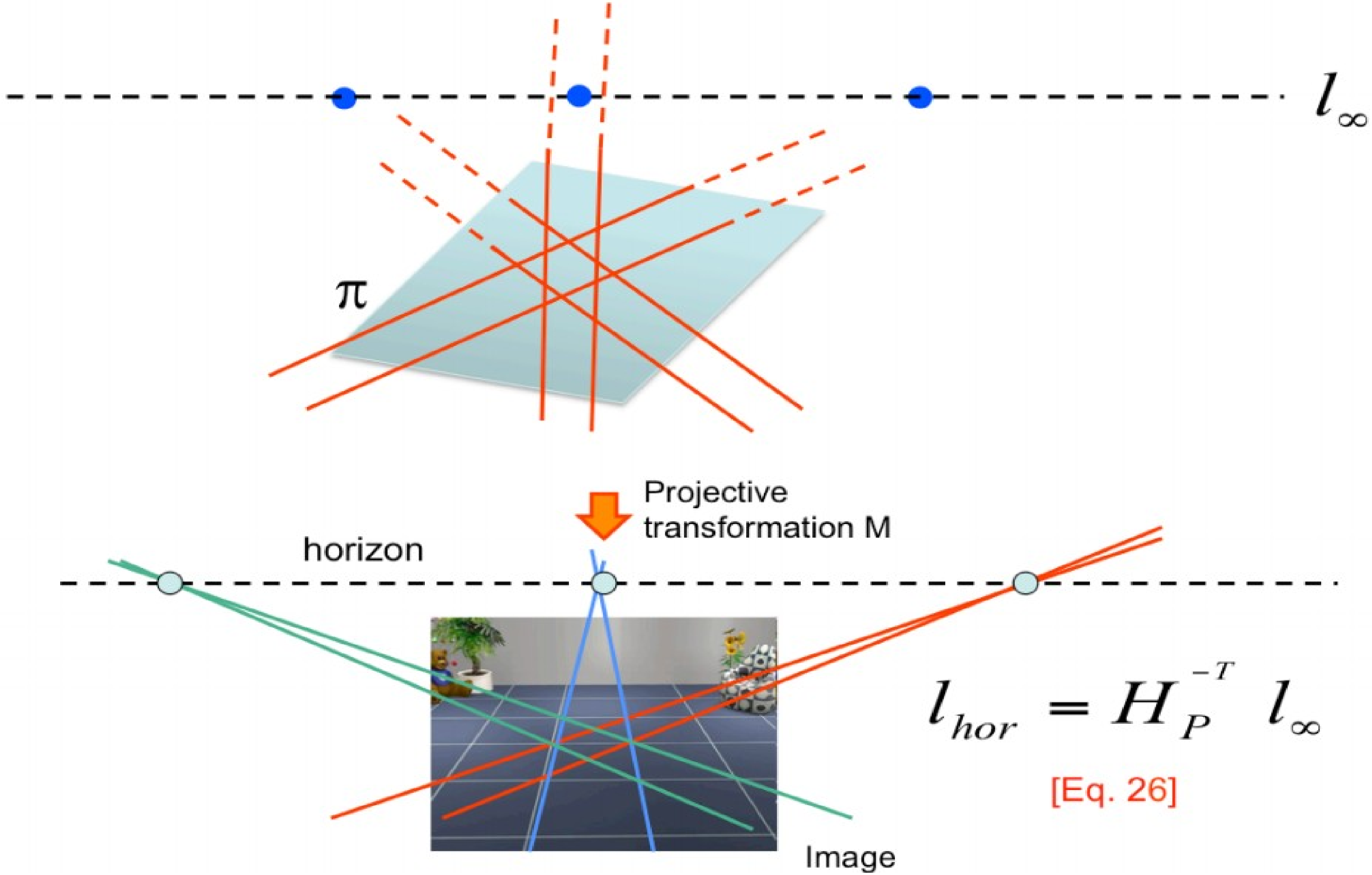
$$\mathbf{d} = \frac{K^{-1}\,\mathbf{v}}{\left\|K^{-1}\,\mathbf{v}\right\|}$$

[Eq. 25]

Proof:

$$X_\infty = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \xrightarrow{M} \mathbf{v} = \mathbf{M}\,\mathbf{X}_\infty = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} = \mathbf{K}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

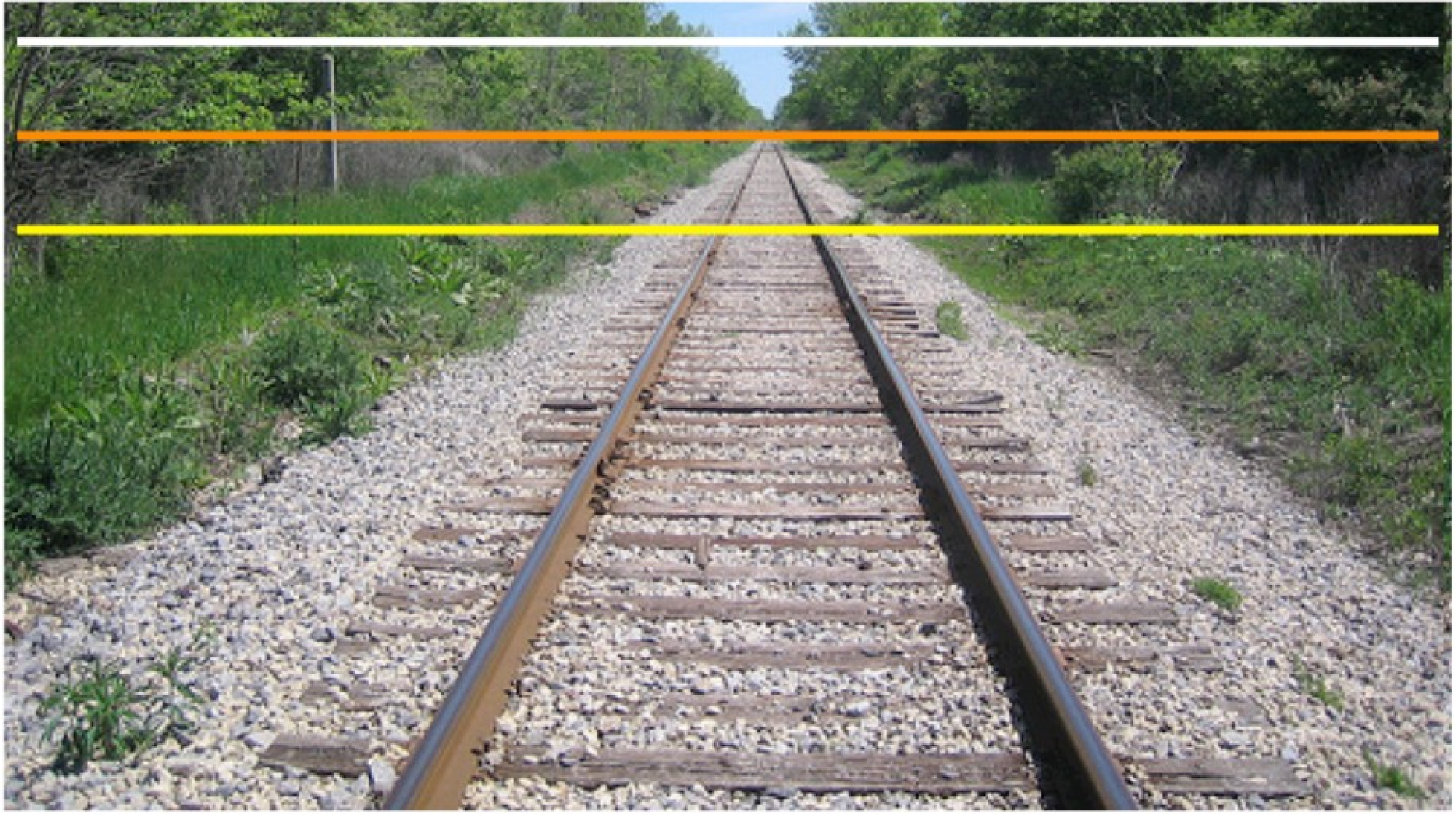
Horizon line



$$l_{hor} = H_P^{-T} \; l_\infty$$

[Eq. 26]

Horizon line



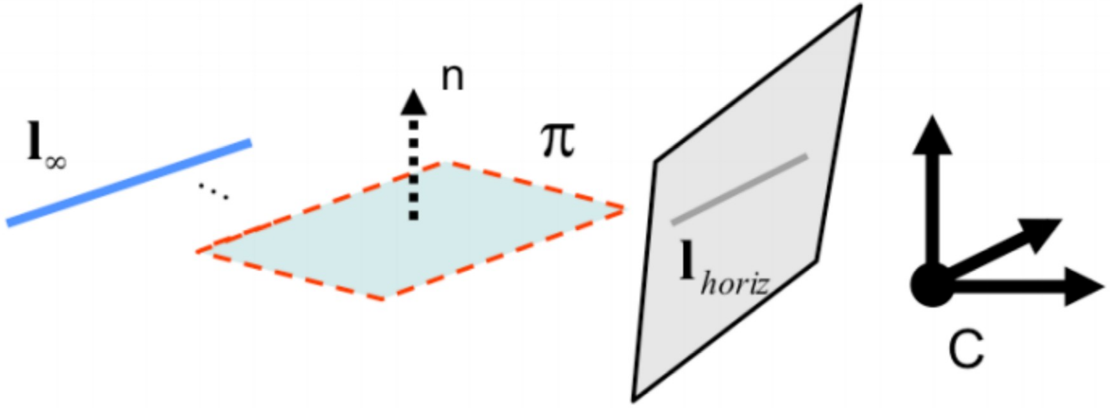
Humans intuitively deduce properties about the image

Horizon line

If lines intersect on horizon line they are parallel

Horizon line vs Camera parameters

[HZ] Ch 8.6.2



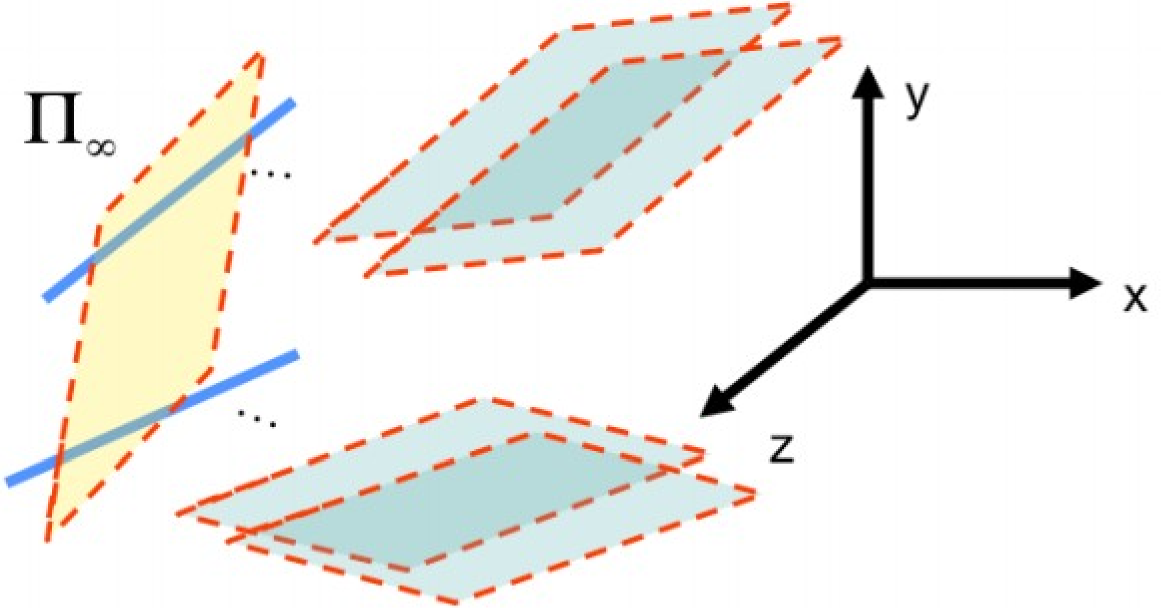
we can estimate the orientation of the ground plane!

$$\mathbf{n} = K^T \mathbf{l}_{horiz}$$

[Eq. 27]

Planes at infinity

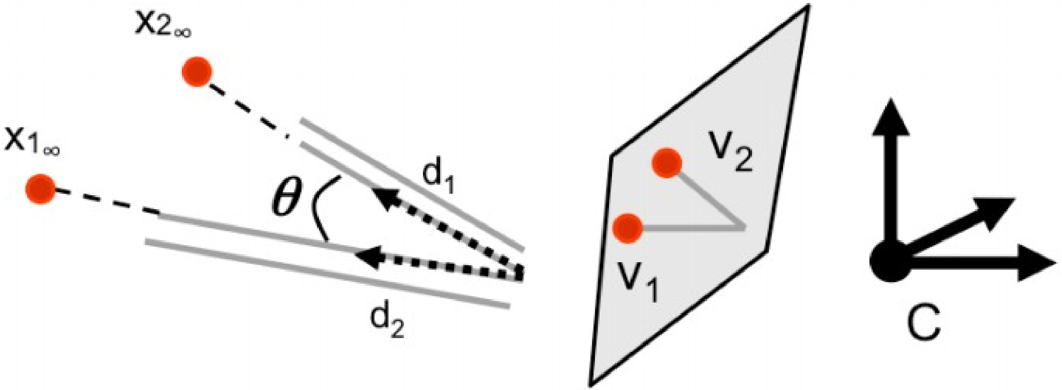
- Parallel planes intersect on a line at infinity
- 2 or more lines at infinity define a plane at infinity



$$\Pi_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

plane at infinity

Angle between 2 pair of parallel lines



$$\cos\theta = \frac{v_1^T \omega\, v_2}{\sqrt{v_1^T \omega\, v_1}\sqrt{v_2^T \omega\, v_2}}$$

[Eq. 28]

$$\omega = (K\,K^T)^{-1}$$

If $\theta = 90 \quad \rightarrow \quad \boxed{v_1^T \omega\, v_2 = 0}$ [Eq. 29]

Scalar equation

Small recap

$$\boxed{\mathbf{v} = K\,\mathbf{d}}$$

[Eq. 24]

$$\boxed{\mathbf{n} = \mathrm{K}^{\mathrm{T}}\mathbf{l}_{\mathrm{horiz}}}$$

[Eq. 27]

$$\cos\theta = \frac{\mathrm{v}_1^{\mathrm{T}}\boldsymbol{\omega}\,\mathrm{v}_2}{\sqrt{\mathrm{v}_1^{\mathrm{T}}\boldsymbol{\omega}\,\mathrm{v}_1}\sqrt{\mathrm{v}_2^{\mathrm{T}}\boldsymbol{\omega}\,\mathrm{v}_2}}$$

[Eq. 28]

$$\xrightarrow{\theta = 90}\quad \boxed{\mathrm{v}_1^{\mathrm{T}}\boldsymbol{\omega}\,\mathrm{v}_2 = 0}$$

[Eq. 29]

$$\boxed{\omega = (K\,K^T)^{-1}}$$

Estimating geometry from a single image

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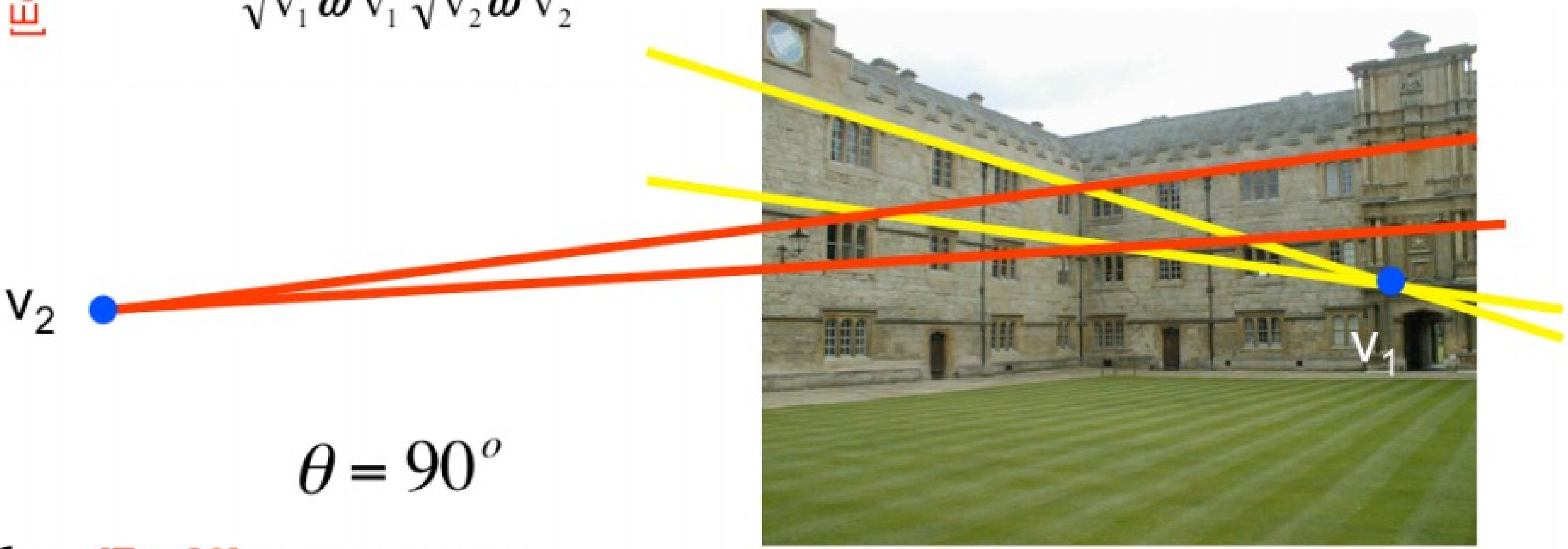
[Eq. 28]

$$\cos\theta = \frac{v_1^T \omega\, v_2}{\sqrt{v_1^T \omega\, v_1}\sqrt{v_2^T \omega\, v_2}}$$



$$\theta = 90^o$$

[Eq. 29]

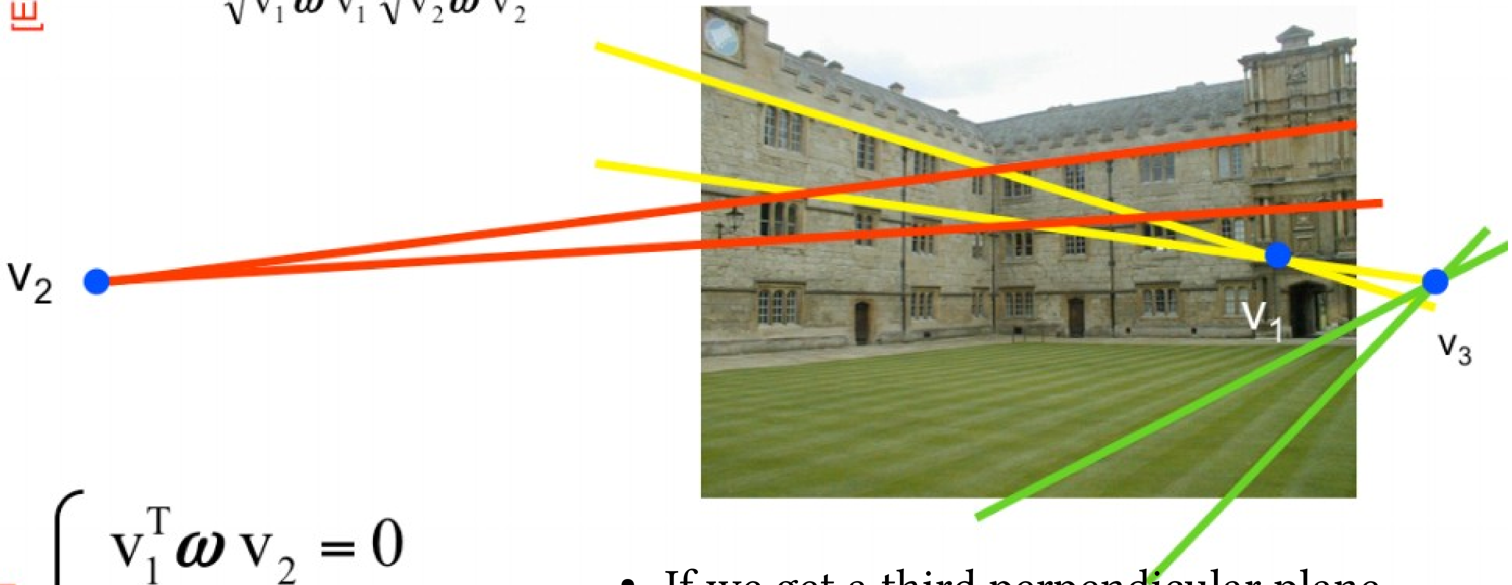
$$\begin{cases} v_1^T \omega\, v_2 = 0 \\ \omega = (K\, K^T)^{-1} \end{cases} \longrightarrow$$

- Identify two 3D planes and a pair of parallel lines for each plane → v_1, v_2 and eq. 28
- 3D perpendicular → eq. 29
- Scalar eq. 29 is still not enough (K has 5 DOF)

Estimating geometry from a single image

[Eq. 28]

$$\cos\theta = \frac{v_1^T \omega\, v_2}{\sqrt{v_1^T \omega\, v_1}\,\sqrt{v_2^T \omega\, v_2}}$$



[Eqs. 31]

$$\begin{cases} v_1^T \omega\, v_2 = 0 \\ v_1^T \omega\, v_3 = 0 \\ v_2^T \omega\, v_3 = 0 \end{cases}$$

- If we get a third perpendicular plane…
- Scalar eq. 29 is still not enough (3 vs 5 DOF)

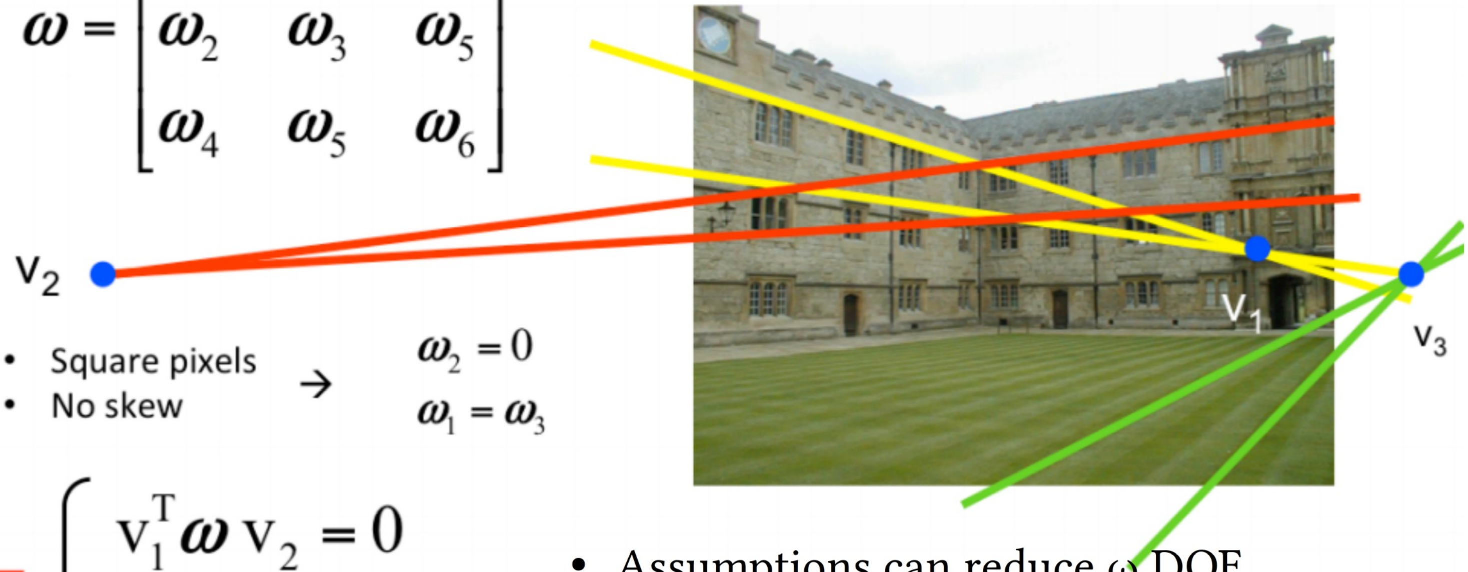
Small recap about K

$$K = \begin{bmatrix} \alpha & -\alpha\cot(\theta) & c_x \\ 0 & \dfrac{\beta}{\sin(\theta)} & c_y \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

Square pixels
No Skew

Estimating geometry from a single image

$$\omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_4 \\ \omega_2 & \omega_3 & \omega_5 \\ \omega_4 & \omega_5 & \omega_6 \end{bmatrix}$$



- Square pixels
- No skew

$\rightarrow$ $\omega_2 = 0$, $\omega_1 = \omega_3$

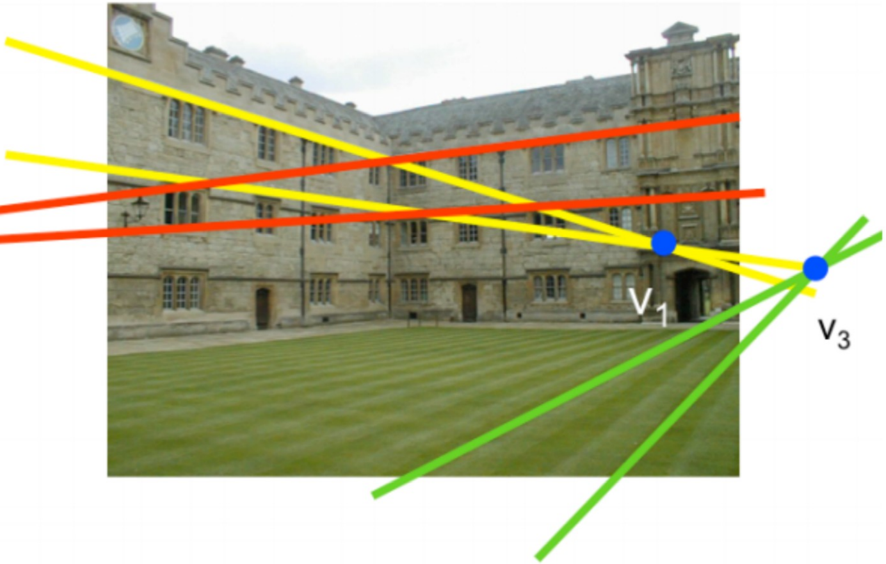
[Eqs. 31]
$$\begin{cases} v_1^T \omega\, v_2 = 0 \\ v_1^T \omega\, v_3 = 0 \\ v_2^T \omega\, v_3 = 0 \end{cases}$$

- Assumptions can reduce ω DOF

Estimating geometry from a single image

$$\omega = \begin{bmatrix} \omega_1 & 0 & \omega_4 \\ 0 & \omega_1 & \omega_5 \\ \omega_4 & \omega_5 & \omega_6 \end{bmatrix}$$

known up to scale



- Square pixels
- No skew

→ $\omega_2 = 0$, $\omega_1 = \omega_3$

[Eqs. 31]

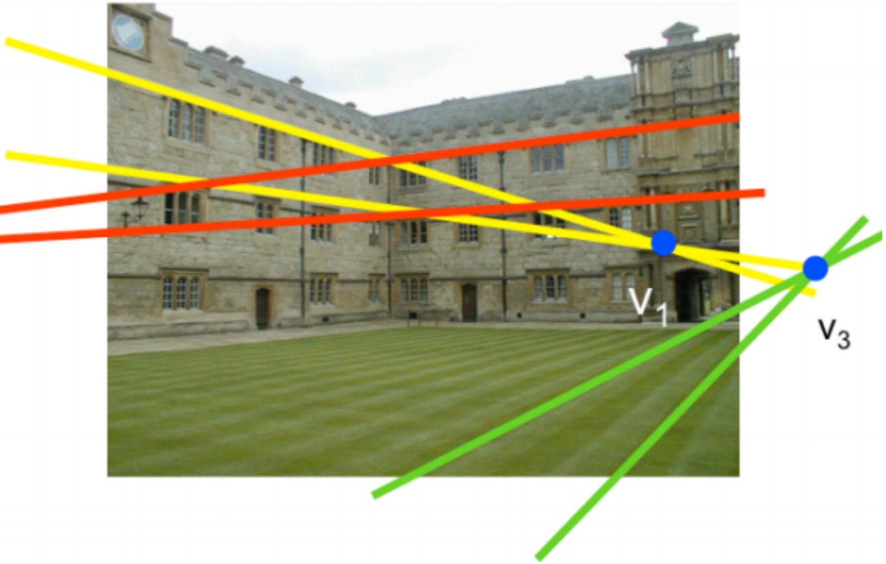
$$\begin{cases} v_1^T \boldsymbol{\omega}\, v_2 = 0 \\ v_1^T \boldsymbol{\omega}\, v_3 = 0 \\ v_2^T \boldsymbol{\omega}\, v_3 = 0 \end{cases}$$

→ Compute ω !

Estimating geometry from a single image

Fact. Cholesky [HZ] Ch pg. 582

$$\omega = \begin{bmatrix} \omega_1 & 0 & \omega_4 \\ 0 & \omega_1 & \omega_5 \\ \omega_4 & \omega_5 & \omega_6 \end{bmatrix}$$ known up to scale



- Square pixels
- No skew

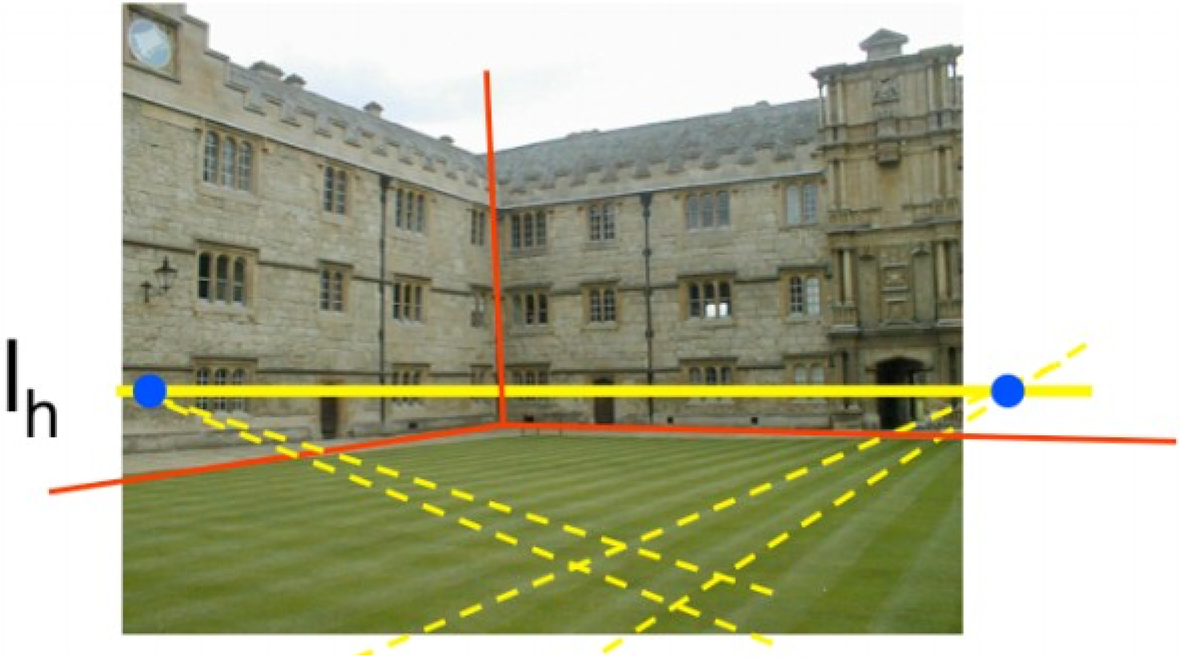
$\rightarrow$ $\omega_2 = 0$, $\omega_1 = \omega_3$

[Eqs. 31]

$$\begin{cases} v_1^T \omega\, v_2 = 0 \\ v_1^T \omega\, v_3 = 0 \\ v_2^T \omega\, v_3 = 0 \end{cases}$$

$$\omega = (K\,K^T)^{-1} \longrightarrow K$$

Estimating geometry from a single image



[Eq. 27]

$$K \text{ known} \rightarrow \mathbf{n} = K^T \mathbf{l}_{\text{horiz}}$$

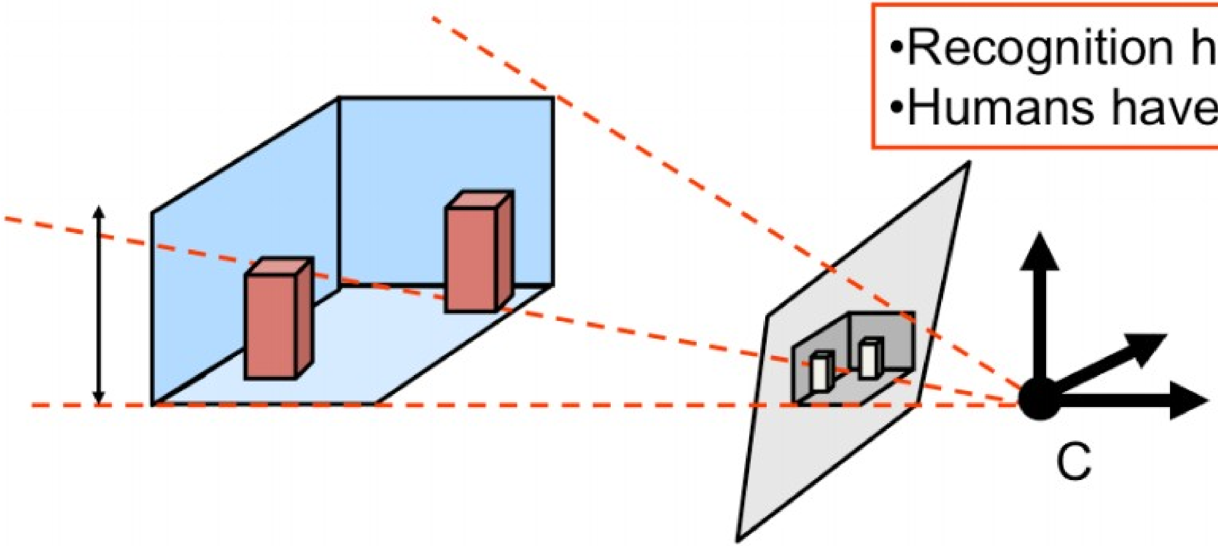
Eq. 27 allows to compute corresponding lines at infinity
We can then select orientation discontinuities (red lines)
And now we can compute the orientation of all planes

Example

http://www.robots.ox.ac.uk/~vgg/projects/SingleView/models/merton/merton.wrl

Example

Small recap



- Recognition helps reconstruction!
- Humans have learnt this

- Now we are able to compute K and some 3D world info using a single camera!
- Scale still unknown
  - We can recover having other info about what we are looking at
- Please this is not a unmanned procedure!

Not an automatic procedure...