# Exponential Smoothing

Dr. Natalia Khorunzhina

CBS

# Exponential Smoothing

Smoothing methods give a more flexible “formal” representation of the data.

Smoothing methods do not require “best fitting models”, but they not always produce “optimal forecasts”.

Rather it is simply a way to tell a computer to draw a smooth line through the data, and to extrapolate the smooth line in a reasonable and replicable way.

Smoothing may not be an “academic” way of modeling, but smoothing techniques are useful in many situations when model-based methods can’t or shouldn’t be used:

- when the data series is very short
- when the forecasting task is voluminous
- sometime they deliver a reasonable forecast

# Exponential Smoothing

Exponential smoothing provides a forecasting method most effective when the components (trend and seasonal factors) of the series change over time.

This method weights the observed series values unequally:

- most recent observations are weighted more heavily
- remote observation provide a smaller weight

Unequal weighting is accompanied by using **smoothing constant**, which determine how much weight is given to each observation.

# The objective for this class is to cover:

- simple exponential smoothing
- trend corrected exponential smoothing
- Holt-Winters methods

# Simple Exponential Smoothing

Simple Exponential Smoothing is used if the mean (or the level) of a series remains constant:

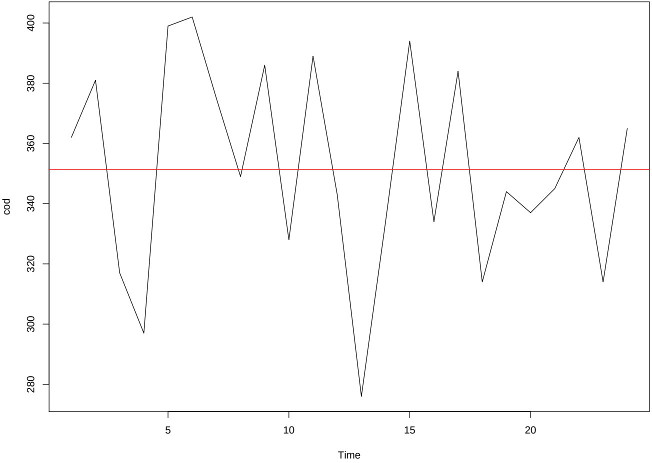
$$y_t = \delta_0 + \varepsilon_t$$

In this case the least squares estimate of the mean $\delta_0$ is

$$\delta_0 = \bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t$$

When we compute the point estimate of $\delta_0$, we are giving equal weight to each of the observed values of the series $y_1, y_2, ..., y_T$.

# Example: Monthly Cod Catch (measured in tons) and its Mean



# Simple Exponential Smoothing

When the mean of the series is changing slowly over time, the equal weight may not be appropriate.

Instead, we may give recent observations more weight than remote observations.

The simple exponential smoothing method is used for forecasting a series when there is no trend or seasonal pattern, but when the mean of the series $y_t$ is slowly changing over time.

# Simple Exponential Smoothing

We begin the simple exponential smoothing procedure by calculating an initial estimate $\ell_0$ of the level of the series at time 0, by averaging over half the data series.

If the series is over 24 months, then

$$\ell_0 = \frac{1}{12}\sum_{t=1}^{12} y_t$$

Next, assume that at the end of time period $T-1$ we have an estimate for the level of the series. Then assuming in period $T$ we obtain a new observation $y_T$, we can update $\ell_{T-1}$ to $\ell_T$, which is the new estimate of the level in period $T$.

We compute the updated estimate by using the **smoothing equation**

$$\ell_T = \alpha y_T + (1-\alpha)\ell_{T-1}$$

# Simple Exponential Smoothing

We compute the updated estimate by using the **smoothing equation**

$$\ell_T = \alpha y_T + (1 - \alpha)\ell_{T-1}$$

where $\alpha$ is a **smoothing constant**, between 0 and 1.

The smoothing equation says that $\ell_T$, the estimate of the level made in period $T$, equals to the fraction $\alpha$ (for example 0.1) of the newly observed series observation $y_T$ plus a fraction $(1 - \alpha)$ (for example 0.9) of $\ell_{T-1}$, the estimate of the level made in period $T - 1$.

The more the level of the process is changing, the more a newly observed series value should influence our estimate, and the larger the smoothing constant $\alpha$ should be.

# Simple Exponential Smoothing

Let's return to our estimate of $\ell_0$. Using $y_1$ and trying an arbitrary value for $\alpha$ (say 0.1) we can update $\ell_0$ to $\ell_1$, an estimate in period 1 of the level of the series:

$$\ell_1 = \alpha y_1 + (1 - \alpha)\ell_0$$

Once we compute $\ell_1$ as the forecast made in period 1 for $y_2$, and since we see $y_2$, we can compute a forecast error $y_2 - \ell_1$.

Next, using $y_2$, we further update $\ell_1$ to $\ell_2$:

$$\ell_2 = \alpha y_2 + (1 - \alpha)\ell_1$$

and compute the forecast error $y_3 - \ell_2$.

The procedure is continued through the entire number of periods of historical data.

# Simple Exponential Smoothing

The procedure is continued through the entire number of periods of historical data.

Having collected all forecasting errors, we can find a “good” value for $\alpha$ (instead of some initial value that we started with) by minimizing the value of sum of squared forecast errors.

Fortunately, we do not have to do it manually, everything is taken care of by the “HoltWinters( )” function in R.

# Forecast with Simple Exponential Smoothing

In the simple exponential smoothing method, a point forecast at time $T$ of any value $y_{T+h}$ is the last estimate $\ell_T$:

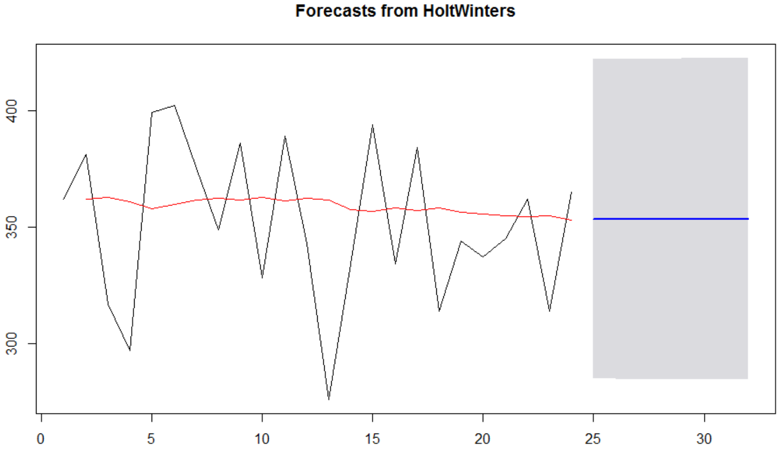
$$\hat{y}_{T+h} = \ell_T$$

The formula for prediction intervals is skipped, but may be computed by the computer program.

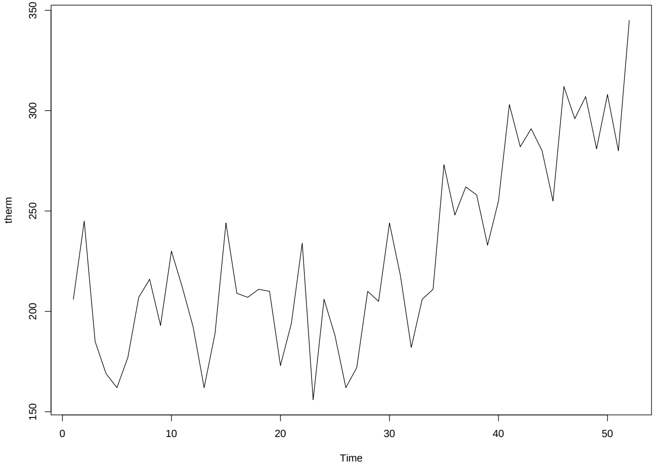
We do expect to be less accurate when we forecast further in the future.

The prediction intervals will get larger.

# Example: Forecast of Monthly Cod Catch (measured in tons)



# Next Example: Weekly Thermostat Sales



# Holt's Trend Corrected Exponential Smoothing

Suppose that the series display a linear trend. If the series is increasing (or decreasing) at a fixed rate, then the series may be described by the linear trend model:

$$y_t = \delta_0 + \delta_1 t + \varepsilon_t$$

The level at time $T$ is $\delta_0 + \delta_1 T$, while the level at time $T - 1$ is $\delta_0 + \delta_1(T - 1)$. Thus the increase (or decrease) in the level of the series from period $T - 1$ to period $T$ is

$$[\delta_0 + \delta_1 T] - [\delta_0 + \delta_1(T - 1)] = \delta_1$$

The fixed rate of increase (or decrease) $\delta_1$ is called the **growth rate.**

# Holt's Trend Corrected Exponential Smoothing

Holt's Trend Corrected Exponential Smoothing is appropriate when both the level and the growth rate are changing.

To implement Holt's trend corrected exponential smoothing, we let:

- $\ell_{T-1}$ denote the estimate of the level of the series in time $T-1$
- $b_{T-1}$ denote the estimate of the growth rate of the series in time $T-1$

If we observe a new series value $y_T$ in period $T$, we use two smoothing equations to update the estimates of $\ell_{T-1}$ and $b_{T-1}$.

# Holt's Trend Corrected Exponential Smoothing

If we observe a new series value $y_T$ in period $T$, we use two smoothing equations to update the estimates of $\ell_{T-1}$ and $b_{T-1}$.

The estimate of the level in period $T$ uses the *smoothing constant* $\alpha$:

$$\ell_T = \alpha y_T + (1 - \alpha)[\ell_{T-1} + b_{T-1}]$$

$\ell_T$ equals to a fraction $\alpha$ of the newly observed series value $y_T$ plus a fraction $(1 - \alpha)$ of $[\ell_{T-1} + b_{T-1}]$, which is the forecast of the level of the series in period $T$.

# Holt's Trend Corrected Exponential Smoothing

The estimate of the growth rate of the series in period $T$ uses the *smoothing constant* $\beta$:

$$b_T = \beta[\ell_T - \ell_{T-1}] + (1 - \beta)b_{T-1}$$

$b_T$ equals to a fraction $\beta$ of $[\ell_T - \ell_{T-1}]$, which is an estimate of the difference between the levels in period $T$ and $T-1$, plus a fraction $(1-\beta)$ of $b_{T-1}$, the estimate of the growth rate made in period $T$.

# Forecast with Holt's Trend Corrected Exponential Smoothing

The point forecast made in period $T$ for $y_{T+h}$ is

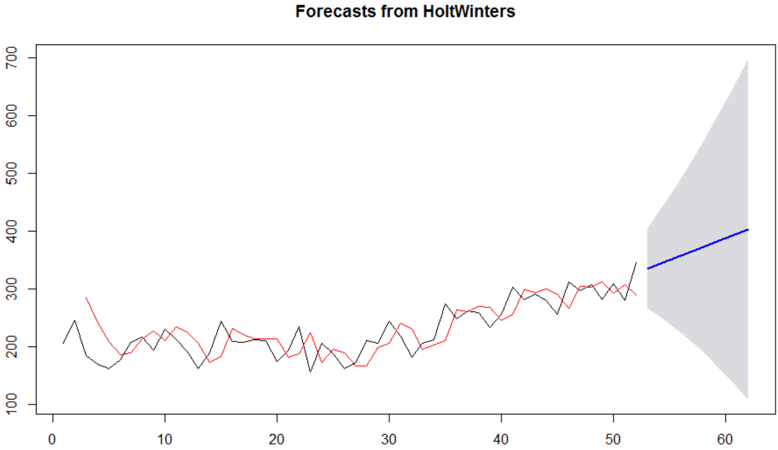
$$\hat{y}_{T+h} = \ell_T + hb_T,$$

where $h$ is the forecast horizon. The formula for prediction intervals is skipped, but may be computed by the computer program.

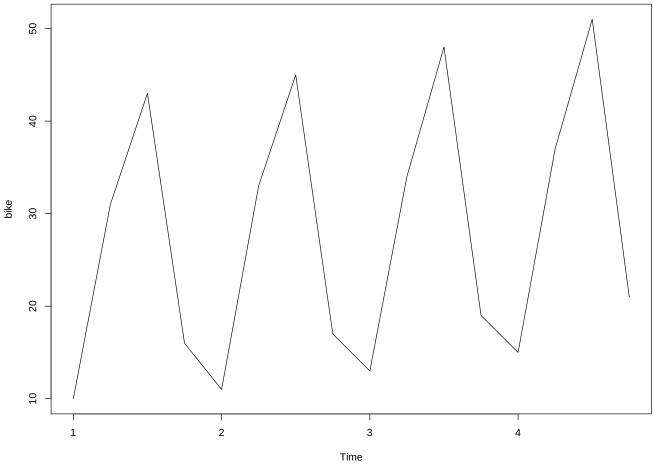
We do expect to be less accurate when we forecast further in the future.

The prediction intervals will get larger.

# Example: Forecast of the Weekly Thermostat Sales



# Final Example: Mountain Bike Sales



# Holt-Winters Exponential Smoothing

If the series display a linear trend and a fixed seasonal pattern $S_t$, then the series may be described by the model:

$$y_t = (\delta_0 + \delta_1 t) + S_t + \varepsilon_t$$

In the time series regression, we use dummy variables to model $S_t$.

For this model, the level of the series at time $T - 1$ is $\delta_0 + \delta_1(T - 1)$ and at time $T$ is $\delta_0 + \delta_1 T$.

Hence, the growth rate in the level from one time period to the next is $\delta_1$.

# Holt-Winters Exponential Smoothing

Holt-Winters method is appropriate when a series has a linear trend with an additive seasonal pattern for which the level, the growth rate, and the seasonal pattern *may be changing*.

To implement the Holt-Winters method, let

- $\ell_{T-1}$ denote the estimate of the **level** of the series in time $T-1$
- $b_{T-1}$ denote the estimate of the **growth rate** of the series in time $T-1$
- $s_{T-L}$ denote the “most recent” estimate of the **seasonal factor** for the season corresponding to period $T$.

Here $L$ denotes the number of seasons in a year. $T-L$ denotes the period occurring one year prior to period $T$. Furthermore, the subscript $T-L$ of $s_{T-L}$ denotes the fact that the series value in time $T-L$ is the most recent value observed in the season being analyzed, and thus is the most recent value used to help find $s_{T-L}$.

# Holt-Winters Exponential Smoothing

Then suppose we observe a new series value $y_T$ in period $T$. The estimate of the level in period $T$ uses the *smoothing constant* $\alpha$:

$$\ell_T = \alpha(y_T - s_{T-L}) + (1-\alpha)[\ell_{T-1} + b_{T-1}]$$

where $(y_T - s_{T-L})$ is the deseasonalized observation in time $T$.

# Holt-Winters Exponential Smoothing

The estimate of the growth rate of the series in period $T$ uses the *smoothing constant* $\beta$:

$$b_T = \beta[\ell_T - \ell_{T-1}] + (1-\beta)b_{T-1}$$

$b_T$ equals to a fraction $\beta$ of $[\ell_T - \ell_{T-1}]$, which is an estimate of the difference between the levels in period $T$ and $T-1$, plus a fraction $(1-\beta)$ of $b_{T-1}$, the estimate of the growth rate made in period $T$.

# Holt-Winters Exponential Smoothing

The new estimate of the seasonal factor $s_T$ in period $T$ uses the *smoothing constant* $\gamma$:

$$s_T = \gamma[y_T - \ell_T] + (1-\gamma)s_{T-L}$$

where $y_T - \ell_T$ is an estimate of the newly observed seasonal component.

# Forecast with Holt-Winters Exponential Smoothing

The point forecast made in period $T$ for $y_{T+h}$ is

$$\hat{y}_{T+h} = \ell_T + hb_T + s_{T-L+h},$$

where $h$ is the forecast horizon. The formula for prediction intervals is skipped, but may be computed by the computer program.

We do expect to be less accurate when we forecast further in the future.

The prediction intervals will get larger.

# Example: Forecast of the Mountain Bike Sales