Stochastic optimization and the newsvendor problem

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42586 – Decisions under Uncertainty – 9 April 2024

Learning Objectives

At the end of this lecture (and related hands-on session, next week), a student should be able to:

- Describe a **newsvendor problem**
- Recognize a number of **hedging problems** as newsvendor problems
- Explain (and prove) the **analytical solution** of newsvendor problems
- **Implement and solve** newsvendor problems
- Discuss the **limitations** of such problems

Outline

1. Hedging
2. The newsvendor problem
3. Solution of the newsvendor problem
4. Limitations

1. **Hedging**

What is hedging?

Definition (Hedging)

Hedging consists in making a decision in order to decrease (and hopefully control) potential losses

Illustrative example, based on a dice game:

- You say a number, between 1 and 6 (that is your *decision*)
- I give you 5 DKK, whatever happens, plus 3 DKK times the number you chose
- We throw the dice
- You pay me back 10 DKK times the difference between the number you chose and the outcome of the dice.

Which number do you choose?

Solving that problem

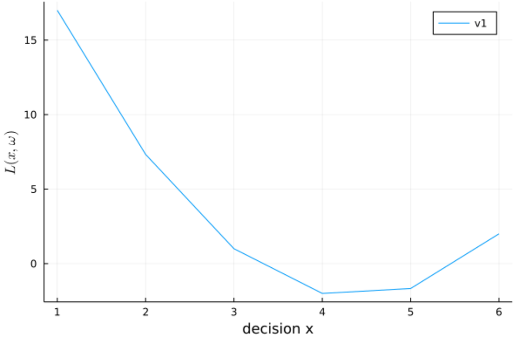
Steps:

1. Choose a **strategy** (maximax, minimax regret, expected utility maximization, etc.): we focus on EUM here
2. Write the **loss function** (to minimize), i.e.,

$$\begin{aligned} L(x,\omega) &= \mathbb{E}[-5-3x+10|x-\omega|] \\ &= -5-3x+10\left(\mathrm{P}[\omega<x]\,\mathbb{E}_{\omega<x}[x-\omega]+\mathrm{P}[\omega>x]\,\mathbb{E}_{\omega>x}[\omega-x]\right) \end{aligned}$$

3. **Find its minimum** (analytically and visually)

- the EUM decision is 4
- only for 4 and 5 you get a revenue (in expectation)
- you could actually loose quite a bit of money!



A hedge as an insurance policy

More than simply positioning yourself, a **hedge** may more generally consist in a taking an **insurance policy**.

Illustrative example, based on a dice game:

- The set up and rules are the same as before...
- But, I give you the choice to reduce your potential loss: 5 times the difference, instead of 10 times previously...
- in exchange, as a way to pay for that insurance policy, I also reduce your gain: 2 times the number instead of 3 times

Are you willing to decrease your upfront revenue, and do you still choose the same number?

Comparing the outcomes

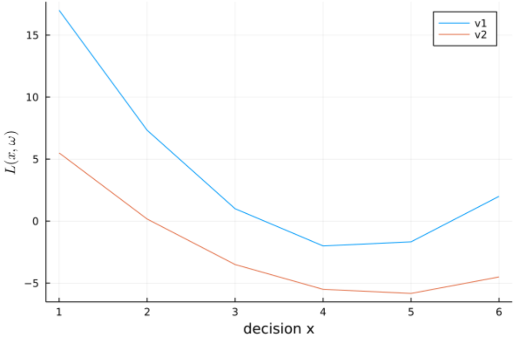
Steps:

1. We stick to our EUM strategy
2. The loss function (to minimize) for the second problem is,

$$
\begin{aligned}
L(x,\omega) &= \mathbb{E}[-5 - 2x + 5|x-\omega|] \\
&= -5 - 2x + 5\left(\mathrm{P}[\omega < x]\,\mathbb{E}_{\omega<x}[x-\omega] + \mathrm{P}[\omega > x]\,\mathbb{E}_{\omega>x}[\omega - x]\right)
\end{aligned}
$$

3. **Find its minimum** (analytically and visually)

- the EUM decision is 5
- the potential loss in less
- we are overall less exposed to uncertainty-related costs



2. **The newsvendor problem**

General description

- The newsvendor problem is one of the most classical problems in **stochastic optimization** (or **statistical decision theory**)

- It can be traced back to:

FY Edgeworth (1888). The mathematical theory of banking. *Journal of the Royal Statistical Society* 51(1): 113–127

even though in this paper the problem is about how much a bank should keep in its reserves to satisfy request for withdrawal (i.e., the *bank-cash-flow problem*)

- It applies to varied problems as long as:
    - one-shot possibility to decide on the quantity of interest
    - outcome is uncertain
    - known marginal profit and loss
    - the aim is to maximize expected utility!

More formally

Let us reinterpret the conditions of the problem:

1. we have an **uncertain parameter** ω
2. we need to make a **decision variable** x before we know the realization of ω
3. there will be **penalties** as a function of the difference between x and ω:
   - if $x > \omega$ the penalty is π_o (so-called *overage* penalty)
   - if $x < \omega$ the penalty is π_u (so-called *underage* penalty)
4. we want to solve an **expected utility maximization** problem (here, form cost minimization form), i.e. ,for

$$\mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+]$$

Definition (Newsvendor problem)

In its most generic form, a newsvendor problem is a one-period expected utilization maximization problem, for an uncertain quantity ω, and with known underage and overage penalties π_u and π_o, respectively.
The optimal decision x^* to be made is the solution of

$$\min_x \quad \mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+]$$

Newsvendor Example

Let us reinterpret the conditions of the problem:

- A newsvendor is getting ready for his daily sales, and needs to decide how many newspaper to buy.
- Each newspaper costs 0.50 cents, and the newsvendor can re-sell them at a price of 1 dollar.
- The experienced newsvendor has a fair idea of how the demand is distributed.
- If there are any newpapers left, the newsvendor can salvage some of the cost by selling them back to the published for 0.25 cents a piece.

Identify:

- The decision variable
- The uncertain parameter
- The overage and underage penalties
- The EUM function

Newsvendor Example

Let us reinterpret the conditions of the problem:
Identify:

- The decision variable: The number of newspapers to buy
- The uncertain parameter: The number of newspapers sold
- The overage and underage penalties:
  - Overage $= 0.5 - 0.25 = 0.25$
  - Underage $= 1 - 0.5 = 0.5$
- The EUM function:

$$\mathbb{E}[0.5(\omega - x)_+ + 0.25(x - \omega)_+]$$

Newsvendor Example

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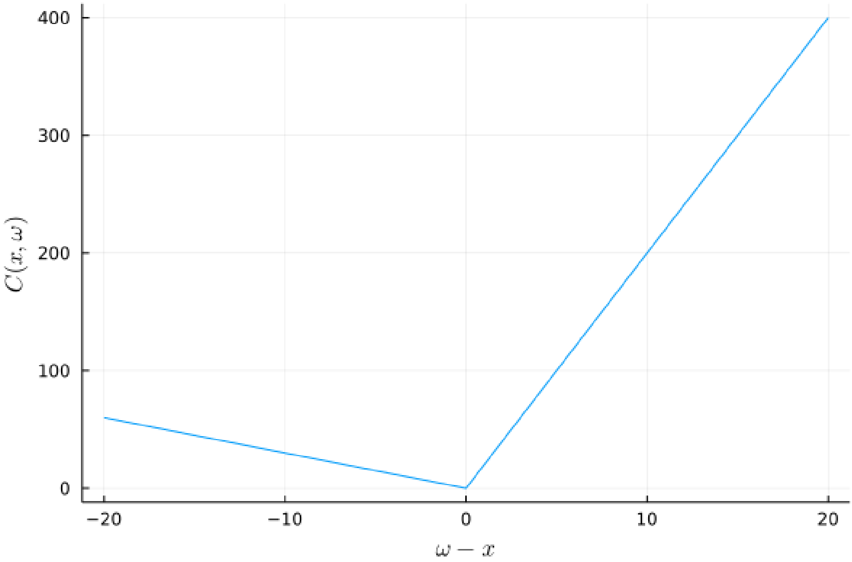
Assume the demand is normally distributed with μ=50 and σ=10.

Newsvendor Example



Theorem (Newsvendor solution)

Given an uncertain parameter ω with distribution $f(\omega)$ (and associated c.d.f. $F(\omega)$), as well as overage and underage penalties π_o and π_u, respectively, the optimal solution x^* of the newsvendor problem is

$$x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right)$$

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Associated intuition... Consider the following cases:

- $\pi_u \simeq 0$ and π_o very large $\longrightarrow$ virtually free to make decisions lower than the realization
- $\pi_o \simeq 0$ and π_u very large $\longrightarrow$ virtually free to make decisions higher than the realization
- $\pi_u = \pi_o \longrightarrow$ Equally expensive to go above or below the realization

Proof

Reminder about the expected value:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

What we are trying to optimize

$$\min_x \mathbb{E}[\pi_u(x-\omega)_+ \ + \pi_o(\omega - x)_+]$$

To obtain the optimal solution we solve

$$\frac{d}{dx}\mathbb{E}[\pi_u(x-\omega)_+ \ + \pi_o(\omega - x)_+] = 0$$

Rewriting the expectation, we have

$$\frac{d}{dx}\int_{-\infty}^{x} \pi_o(x-\omega) f(\omega) d\omega + \int_{x}^{\infty} \pi_u(\omega - x) f(\omega) d\omega$$

Proof

Rewriting the expectation, we have

$$\frac{d}{dx}\int_{-\infty}^{x}\pi_o(x-\omega)f(\omega)d\omega+\int_{x}^{\infty}\pi_u(\omega-x)f(\omega)d\omega$$

Using differentiation under the integral sign, for the overage part

$$\frac{d}{dx}\int_{-\infty}^{x}\pi_o(x-\omega)f(\omega)d\omega=\int_{-\infty}^{x}\pi_o f(\omega)d\omega=\pi_o F(x)$$

Using differentiation under the integral sign, for the underage part

$$\frac{d}{dx}\int_{x}^{\infty}\pi_u(\omega-x)f(\omega)d\omega=\int_{x}^{\infty}-\pi_u f(\omega)d\omega=-\pi_u(1-F(x))$$

Proof

We substitute with the optimal solution x^*

$$\frac{d}{dx}\mathbb{E}[\pi_u(x-\omega)_+ + \pi_o(\omega - x)_+] = \pi_o F(x^*) - \pi_u(1 - F(x^*) = 0$$

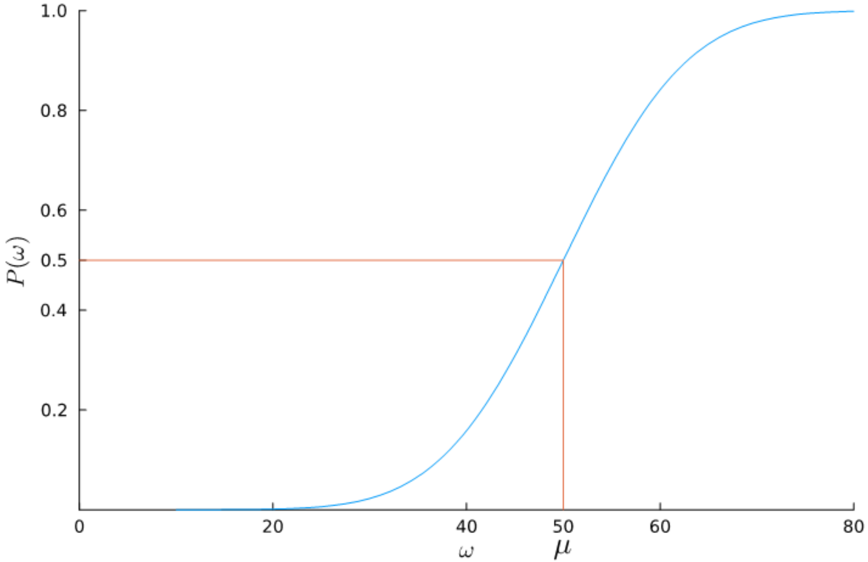
We then find the critical ratio

$$F(x^*) = \frac{\pi_u}{\pi_o + \pi_u}$$

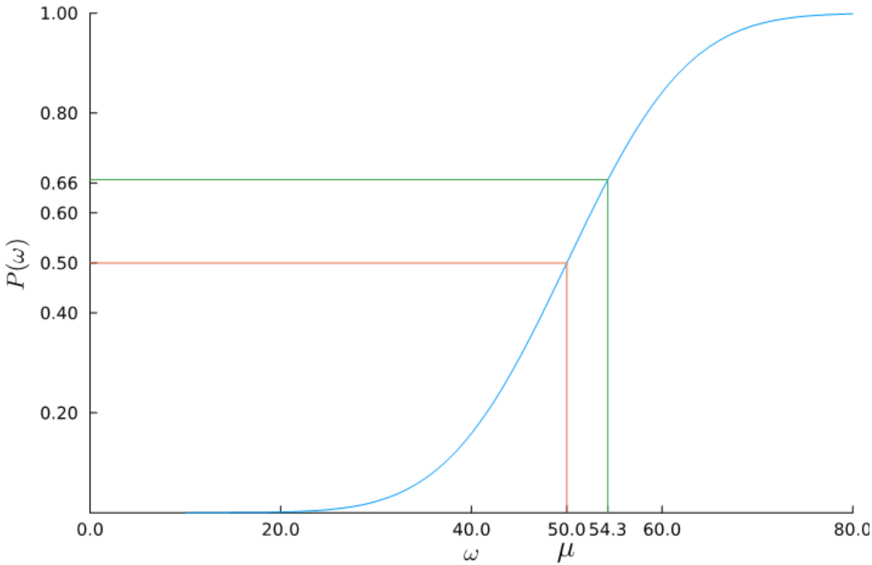
The optimal value is then

$$x^* = F^{-1}\left(\frac{\pi_u}{\pi_o + \pi_u}\right)$$

CDF of the Newsvendor Example



CDF of the Newsvendor Example



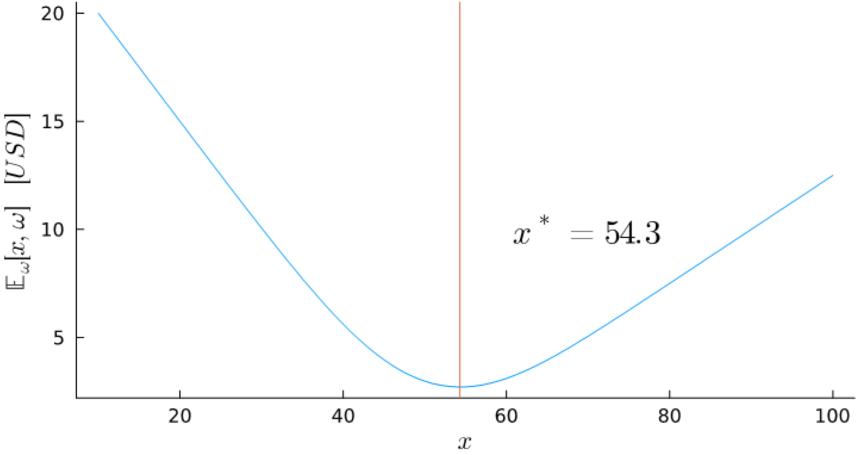
Empirical verification through simulation

Let's simulate the earnings of a newsvendor

```
x = collect(10:100)

ω = rand(N,10000) # Demand sampling

cv = zeros(length(x)) # Mean cost

for i in 1:length(x) # Factorial experimentation

   # Simulation
   cv[i] = mean(πo .* (x[i].-ω) .* ((x[i].-ω).>=0)

            + πu .* (ω.-x[i]) .* ((ω.-x[i]).>0))

end

plot(x,cv,xlabel=L"x",ylabel=L"\mathbb{E}_\omega[x,\omega] \quad [USD]",
       legend=false, size=(550,300))

plot!([opt], seriestype="vline")
```

Empirical verification through simulation

Let's simulate the earnings of a newsvendor



A working example

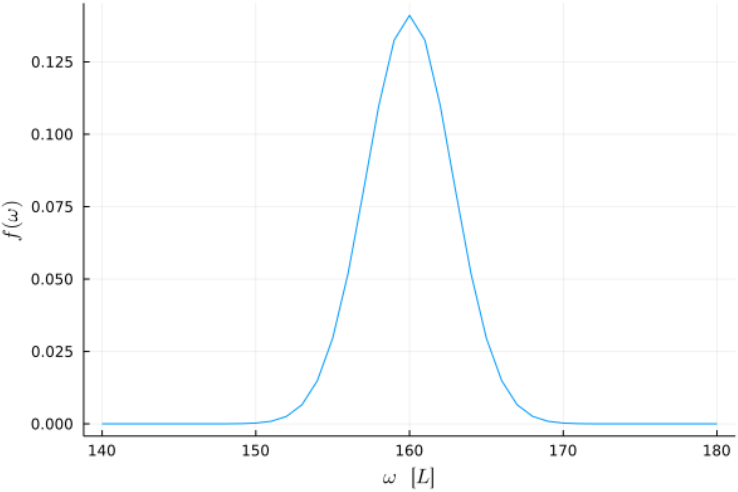
Let us focus on a very simple example (DTU bar manager):

- you have been appointed "DTU bar manager" for DTU friday bar
- you need to order (draft) beer on Monday to be sure to be ready when Friday comes!
- the uncertain parameter is how much demand ω for beer they will be (ω follows a given distribution)
- the decision variable x is how much you order
- prices are:
  - c when you order (ex: $c = 10$ DKK/L)
  - s when you sell (ex: $c = 15$ DKK/L)
  - p for any "live" order, if not enough (ex: $p = 30$ DKK/L)
  - z if too much, to be recycled at another party (ex: $z = 7$ DKK/L)

Do you see the newsvendor problem?

And, we are missing a description of the uncertainty

A description of the uncertainty (so, a distribution for ω) is a necessary input, generally provided by a forecaster



For our example, let us say that $\omega \sim \mathcal{N}(\mu = 160, \sigma^2 = 16)$...

Solution

Observation:

- Since we need to always fulfill the demand, the sales income is constant, hence it has no impact on the optimization.

Overage penalty:

- Each liter of extra beer costs c but since we can sell it at a price of z the overage penalty is

$$\pi_o = c - z$$

Underage penalty:

- If we do not have enough beer, we need to buy it at a price p. However, we are only interested in how much more it would have costed, so we need to subtract the original cost c, giving

$$\pi_u = p - c$$

Solution

And, for our example:

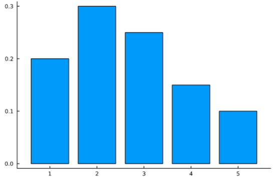
$$\pi_o = c - z = 10 - 7 = 3$$

$$\pi_u = p - c = 30 - 10 = 20$$

- The Critical Ratio: $\frac{\pi_u}{\pi_u + \pi_o} = \frac{20}{23} \simeq 0.87$
- The optimal solution is: $x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right) = F^{-1}(0.87) = 164.5\text{L}$

The newsvendor problem with discrete uncertain parameters

- The DTU Management department needs to assign a number of teaching assistances (TAs) to a course.
- Each TA has costs 10.000 DKK. If not enough TA are assigned to the course, the department will need to use PhD students to cover the course needs. Each PhD student is associated with a cost of 25.000 DKK.
- The number of required TAs is given by the following probability mass function



TAs (ω)	1	2	3	4	5
$P(\omega)$	0.2	0.3	0.25	0.15	0.1

Consider a newsvendor problem with an uncertain parameter ω taking discrete values D_i for $i \in \{1, 2, \dots, n\}$, with corresponding probabilities p_i, as well as overage and underage penalties π_o and π_u, respectively.

The optimal solution x^* is equal to the minimum ω quantity for which

$$P(\omega \le D_i) = \sum_{k=1}^{i} p_k \ge \frac{\pi_u}{\pi_u + \pi_o}$$

The newsvendor problem with discrete uncertain parameters

- Overage penalty: If we have too many TAs, we pay a cost equal to the salary of the extra TA

$$\pi_o = 10.000$$

- Underage penalty: If we do not have enough TAs we need to pay the extra cost imposed by a PhD student

$$\pi_u = 25.000 - 10.000 = 15.000$$

- Critical Ratio:

$$\frac{\pi_u}{\pi_u + \pi_o} = 15/25 = 0.6$$

- Optimal solution:

$$\begin{array}{llll} P(\omega < 1) = 0 & \longrightarrow & \text{is strictly less than } 0.6 & \text{Yes} \\ P(\omega < 2) = 0.2 & \longrightarrow & \text{is strictly less than } 0.6 & \text{Yes} \\ P(\omega < 3) = 0.5 & \longrightarrow & \text{is strictly less than } 0.6 & \text{Yes} \\ P(\omega < 4) = 0.75 & \longrightarrow & \text{is strictly less than } 0.6 & \text{No} \end{array}$$

The optimal number of TAs is 3

4. **Limitations**

There are a few in practice

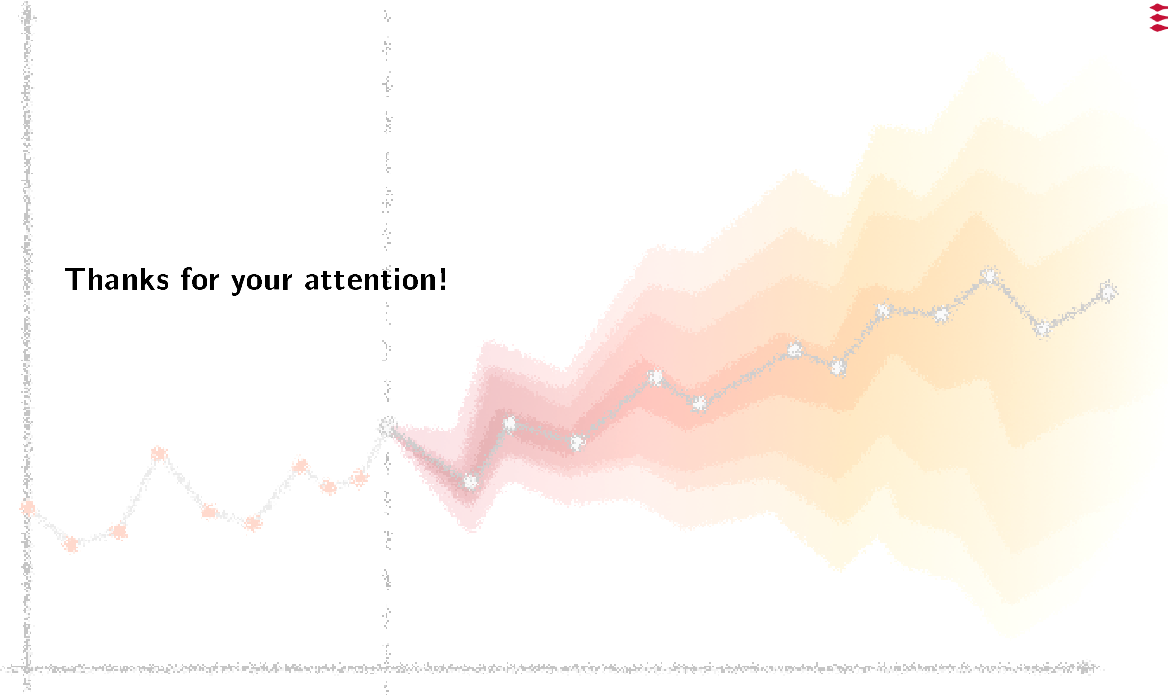
Please remember some of the things I mentioned on the way, e.g.

- It is an **expected utility maximization** problem
- The penalties π_o and π_u are **assumed to be known**...
  - quite often these are not known beforehand (for instance in trading problems)
  - and even more annoying, they might be a function of your decision!
- The distribution for the uncertain parameter ω is also **assumed to be known**
  - in practice, it is most often a forecast
  - and forecasts are always wrong!
  - possibly also, one is provided with a number of scenarios, not a distribution

Wrapping up

- Newsvendor problems can be found **everywhere** (trading, logistics and supply chain, etc.)
- We looked at the basic version, but there are also many **general extensions**
- They provide a simple **analytical solution** to a broad range of practical problems
- There are also many **limitations** though...

DTU



Thanks for your attention!