1 Note on Chance Constraints

All the optimization problems under uncertainty that we have studied until now have required all the contraints to be fulfilled. In this note we will learn about chance constraints, which will allow us to solve stochastic optimization problems where some constraints only need to hold with a cetain level of probability.

Chance contrained optimization problems, in the general case, can be very difficult to solve. In this course, we will focus on some special cases that are easy to solve. The general assumption is that the uncertain parameter follows a statistical distribution with know Cummulative Distribution Function (CDF), and that the contraints are linear.

Consider the following stochastic optimization problem, where ω is a random vector.

$$\begin{aligned} \min_{\boldsymbol{x}} \ & c^\top x \\ s.t. \ & A(\omega)x \le b(\omega) \\ & x \ge 0 \end{aligned}$$

As we have done before, we aim to find a deterministic formulation. In order to do so, we have the following options.

Option 1) Expectation constraints

Here we replace the uncertain parameter ω by its expectation. This is easy but gives us no certainty about the feasibility of the contraints.

$$\begin{aligned} \min_{\boldsymbol{x}} \ & c^\top x \\ s.t. \ & A(\mathbb{E}[\omega])x \le b(\mathbb{E}[\omega]) \\ & x \ge 0 \end{aligned}$$

Option 2) Worst case constraints

We optimize for all possible scenarios and make sure the constraints will hold for each of them. The model is now more complex, and certainty is maximized.

$$\begin{aligned} \min_{\boldsymbol{x}} \ & c^\top x \\ s.t. \ & A(\omega_i)x \le b(\omega_i) \quad \forall i \in \{1, ..., |\omega|\} \\ & x \ge 0 \end{aligned}$$

Option 3) Chance constraints

We relax the worst case scenario and only aim at satisfying constraints with a certain probability 1- ϵ.

$$\begin{aligned} &\min_{x} \; c^{\top}x \\ &s.t. \; P[A(\mathbb{E}[\omega])x \leq b(\mathbb{E}[\omega])] \geq 1-\epsilon \\ &\quad\; x \geq 0 \end{aligned}$$

1.1 Simple chance constraints

In this course we will consider only two simple cases: a) the case where the uncertain parameter is part of the coefficients A of the decision variables, and b) the case where the unceratain parameter is on the righ-hand-side b of the constraint.

1.1.1 Case of the uncertainty on the right-hand-side

Let's start with the case there the uncerainty is in the right-hand side b. We will illustrate the concept using an example. Consider a container ship. For simplicity, assume the ship carries 2 types of containers $i \in \{1,2\}$, with specific height h_i and revenue c_i. The containers are stacked on top of each other, but due to wheater conditions, the maximum height a container stack can reach is ω. Stacks that are over this height are at risk of collaps on route. As wheather conditions are uncertain, the parameter ω is random.

Focusing our attention at a single stack, we want to calculate the number of containers of each type to load on the vessel. The problem can be formulated as follows.

$$\begin{aligned} &\min_{x} \; -c_1x_1 - c_2x_2 \\ &s.t. \; h_1x_1 + h_2x_2 \leq \omega \\ &\quad\; x \in \mathbb{Z}_+ \end{aligned}$$

The objective of the model is to maximize revenue, while the constraint makes sure that we do not exceed the height limit. Since the weather can change drastically, we know that solving for the worst-case scenario would be too conservative. We are interested in solving the problem so the probability that the constraints holds is at least $1-\epsilon$ e.g. 95. With this in mind, we can now rewrite the model as follows.

$$\begin{aligned} &\min_{x} \; -c_1x_1 - c_2x_2 \\ &s.t. \; P[h_1x_1 + h_2x_2 \leq \omega] \geq 1-\epsilon \\ &\quad\; x \geq 0 \\ &\quad\; x \in \mathbb{Z}_+ \end{aligned}$$

No matter which case we are trying to solve (whether the uncertainty is in A or in b), we always end up in a constraint of the form

$$P[\text{event which is a function of } \omega] \geq 1 - \epsilon.$$

The key to the deterministic reformulation resides in the definition of the comulative distribution function (CDF)

$$F_\omega(x) = P[\omega \leq x].$$

Let's now use the result above to try and reformulate the contraint of our problem into deterministic form. The original constraint is

$$P[h_1 x_1 + h_2 x_2 \leq \omega] \geq 1 - \epsilon$$

Since we want to use the CDF fomula we need to play around with the equation so that it is in the form $P[\omega \leq x]$. Let's apply some transformations

$$\begin{aligned} P[h_1 x_1 + h_2 x_2 \leq \omega] &= 1 - P[\omega \geq h_1 x_1 + h_2 x_2] \\ &= 1 - F_\omega(h_1 x_1 + h_2 x_2) \end{aligned}$$

We have taken the complement of the probability in order to reverse the inequality. The resulting probablity represents the CDF of the random vector ω. With this we can now transform the original constraint.

$$\begin{aligned} P[h_1 x_1 + h_2 x_2 \leq \omega] \geq 1 - \epsilon &= 1 - F_\omega(h_1 x_1 + h_2 x_2) \geq 1 - \epsilon \\ &= F_\epsilon(h_1 x_1 + h_2 x_2) \leq \epsilon \\ &= h_1 x_1 + h_2 x_2 \leq F_\omega^{-1}(\epsilon) \end{aligned}$$

We have now transformed the constraints in determenistic form and can solve the model the usual way. The resulting model is

$$\begin{aligned} \min_x \; & -c_1 x_1 - c_2 x_2 \\ s.t. \; & h_1 x_1 + h_2 x_2 \leq F_\omega^{-1}(\epsilon) \\ & x \geq 0 \\ & x \in \mathbb{Z}_+ \end{aligned}$$

Let's implement the model and see what happens when we solve it for different values of $1 - \epsilon$.

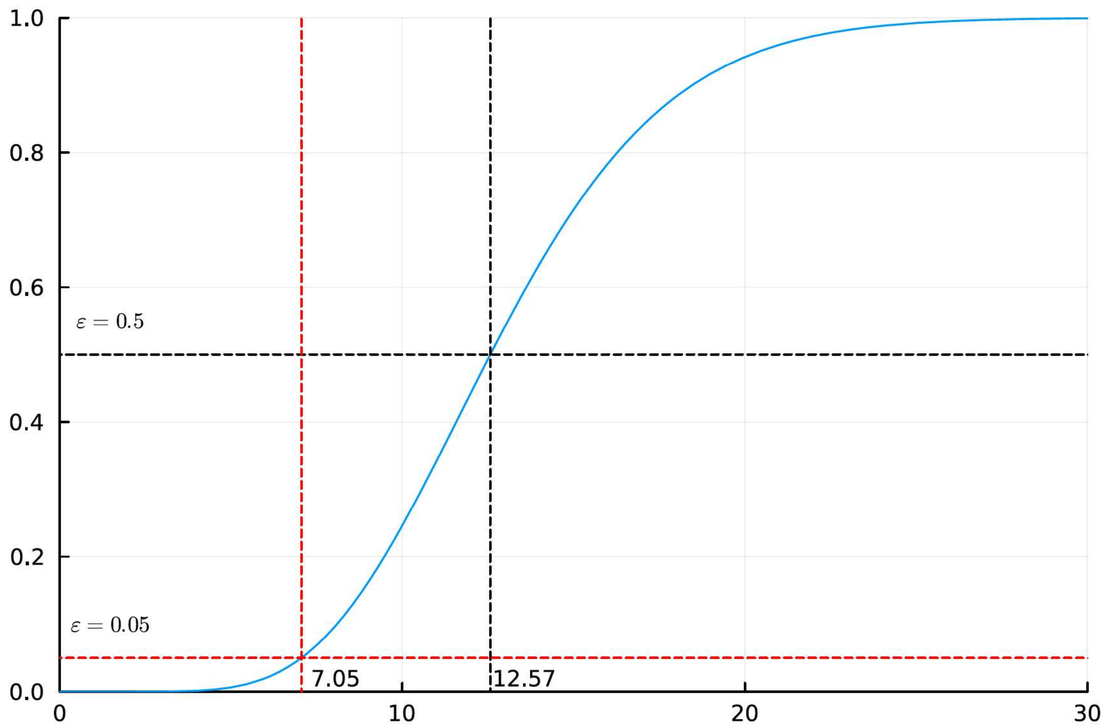
```julia
using JuMP, HiGHS, Distributions, Plots, LaTeXStrings
```

```julia
# Revenues
c = [20000 30000]

# Heights
h = [2.6 2.9]

```

```
# Assume the uncertain parameter follows a Gamma distribution with k = 10 and θ = 1.3
ω = Gamma(10,1.3)

# Let's now implement the model into a function so that we can more easilty solve it for
# different certainty levels.

function solve_problem(c,h,ω,ϵ)
    limit = quantile(ω,ϵ)
    m = Model(HiGHS.Optimizer)
    @variable(m,x[1:2],Int)
    @objective(m,Min,-c[1]x[1]-c[2]x[2])
    @constraint(m,h[1]x[1]+h[2]x[2]<=limit)
    @constraint(m,[i in 1:2],x[i]>=0)

    # This function makes sure that the output of the solver is not printed
    set_optimizer_attribute(m,"output_flag",false)
    optimize!(m)

    return objective_value(m), value.(x)
end;
```

Before we solve the model, let's have a look at how the CDF of the uncertain parameter looks like and at how the fractile values change.

```
# I prepare a function to plot
f(x) = cdf(ω,x)
# I plot the CDF function, passing the limits of the plot
plot(f,ylim=(0,1),xlim=(0,30),legend=false)
# Let's now plot the quantile values for ϵ=0.5 (50%) and ϵ=0.05 (95%)
ϵ=0.5
plot!([ϵ], seriestype="hline", linestyle=:dash, color="black")
plot!([quantile(ω,ϵ)], seriestype="vline", linestyle=:dash, color="black")
annotate!([quantile(ω,ϵ)+1],0.02,text(round(quantile(ω,ϵ),digits=2),8))
annotate!(1.5,ϵ+0.05,text(L"\epsilon = 0.5",8))

ϵ=0.05
plot!([ϵ], seriestype="hline", linestyle=:dash, color="red")
plot!([quantile(ω,ϵ)], seriestype="vline", linestyle=:dash, color="red")
annotate!([quantile(ω,ϵ)+1],0.02,text(round(quantile(ω,ϵ),digits=2),8))
annotate!(1.5,ϵ+0.05,text(L"\epsilon = 0.05",8))
```



Notice how the fratile value for $\epsilon = 0.05$ ($1 - \epsilon =$ 95%) is much smaller than the fractile at $\epsilon = 0.5 (50\%)$, indicating that the solution is more restictive.

Let's now solve the problem for these two values of ϵ and compare the solutions.

```
obj, x = solve_problem(c,h,ω,0.05)
println("For 95% confidence we load $(x[1]) $(h[1])-high containers␣
  ↪and $(x[2]) $(h[2])-high containers.")
obj, x = solve_problem(c,h,ω,0.50)
println("For 50% confidence we load $(x[1]) $(h[1])-high containers␣
  ↪and $(x[2]) $(h[2])-high containers.")
```

```
For 95% confidence we load 0.0 2.6-high containers and 2.0 2.9-high␣
  ↪containers.
For 50% confidence we load 0.0 2.6-high containers and 4.0 2.9-high␣
  ↪containers.
```

As expected the model favous containers with higher revenue, and allows for more containers when the level of confidence that the constraint holds is lower.

For sake of clarity, let's write up the resuting model for e.g. $\epsilon = 0.5$.

$$
\begin{aligned}
\min_{x} \; & -c_1x_1 - c_2x_2 \\
s.t. \; & h_1x_1 + h_2x_2 \leq 12.57 \\
& x \geq 0 \\
& x \in \mathbb{Z}_+ .
\end{aligned}
$$

1.1.2 Case of the uncertainty in the A coefficients

Let's again use an example to illustrate the concept. Consider an energy dispatch problem. An energy company needs to deliver an energy demand D. The company can produce energy either with gas or wind. For both types the maximum energy unit that can be produced is 100. Producing energy with gas is always feasible at a cost $c_1 = 20$, while wind is uncertain but has a lower production cost $c_2 = 3$. We want to satisfy the demand at minimum cost.

Let $x_1 \in \mathbb{R}_+$ be the amount of energy based on gas and $x_2 \in \mathbb{R}_+$ the one based on wind. Also, let ω be a random vector representing the fraction of wind based energy actually produced, hence for a given value of ω, the wind energy actually produced is ωx_2.

Let us now formulate the problem as a stocahstic linear program.

$$
\begin{aligned}
\min_{x} \; & c_1x_1 + c_2x_2 \\
s.t. \; & x_1 + \omega x_2 \geq D \\
& 0 \leq x_1 \leq 100 \\
& 0 \leq x_2 \leq 100
\end{aligned}
$$

We want to solve this program using chance constraints, hence we reformulate it as follows.

$$
\begin{aligned}
\min_{x} \; & c_1x_1 + c_2x_2 \\
s.t. \; & P[x_1 + \omega x_2 \geq D] \geq 1 - \epsilon \\
& 0 \leq x_1 \leq 100 \\
& 0 \leq x_2 \leq 100
\end{aligned}
$$

We now go ahead and try to find a deterministic reformulation of the probabilistic constraint. We start by looking at the probability of the constraint.

$$
\begin{aligned}
P[x_1 + \omega x_2 \geq D] &= 1 - P\left[x_1 + \omega x_2 \leq D\right] \\
&= 1 - P\left[\omega \leq \frac{D - x_1}{x_2}\right] \\
&= 1 - F_\omega\left(\frac{D - x_1}{x_2}\right)
\end{aligned}
$$

Now we apply the result to the constraint of the model.

$$
\begin{aligned}
P[x_1 + \omega x_2 \geq D] \leq 1 - \epsilon = 1 - F_\omega\left(\frac{D - x_1}{x_2}\right) &\geq 1 - \epsilon \\
= F_\omega\left(\frac{D - x_1}{x_2}\right) &\leq \epsilon \\
= \frac{D - x_1}{x_2} &\leq F_\omega^{-1}(\epsilon) \\
= x_1 + F_\omega^{-1}(\epsilon) x_2 &\geq D
\end{aligned}
$$

We can now substitute this result to the original model an obtain the deterministic reformulation.

$$
\begin{aligned}
\min_{x}\ & c_1 x_1 + c_2 x_2 \\
s.t.\ & x_1 + F_\omega^{-1}(\epsilon) x_2 \geq D \\
& 0 \leq x_1 \leq 100 \\
& 0 \leq x_2 \leq 100
\end{aligned}
$$

1.1.3 Important note

What we have seen in this lecture is the simplest version of chance constraints where we assume only one uncertain parameter with a known distribution and CDF. The general case is often much more complex.