Lag Operator for Time-Series Models

Dr. Natalia Khorunzhina

CBS

The lag operator

Learning lag operators will help us formulate time-series models

Let L be the lag operator and y_t be a time series

$Ly_t = y_{t-1}$

$L^2 y_t = L(y_{t-1}) = y_{t-2}$

In general, $L^j y_t = y_{t-j}$

The lag operator and multiplication are commutative:
$L(ay_t) = aLy_t = ay_{t-1}$ where a is a constant

The lag operator is distributive over addition:
$L(y_t + x_t) = Ly_t + Lx_t = y_{t-1} + x_{t-1}$

The lag operator on differences

Lag operator is convenient for describing the process of *differencing*.

A first difference can be written as
$\Delta y_t = y_t - y_{t-1} = y_t - Ly_t = (1-L)y_t$

Therefore, first difference is represented by $(1-L)$

Second-order differences are represented as follows:

$(y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) = y_t - 2y_{t-1} + y_{t-2} = y_t - 2Ly_t + L^2 y_t =$
$(1 - 2L + L^2)y_t = (1-L)^2 y_t$

In general, a dth-order difference can be written as $(1-L)^d y_t$

However, if you take *seasonal* differencing, then
$y_t - y_{t-s} = y_t - L^s y_t = (1 - L^s)y_t$, where s is a constant for seasonality in the data

The lag operator and AR models

- The AR(1) model: $Y_t = \beta_0 + \beta_1 Y_{t-1} + \varepsilon_t$
- $(1 - \beta_1 L) Y_t = \beta_0 + \varepsilon_t$
- The AR(2) model: $Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \varepsilon_t$
- $(1 - \beta_1 L - \beta_2 L^2) Y_t = \beta_0 + \varepsilon_t$
- The AR(p) model: $Y_t = \beta_0 + \beta_1 Y_{t-1} + \cdots + \beta_p Y_{t-p} + \varepsilon_t$
- $(1 - \beta_1 L - \cdots - \beta_p L^p) Y_t = \beta_0 + \varepsilon_t$
- Thus, the AR(p) model is Y_t multiplied by a polynomial of order p.

The lag operator and MA models

- MA(1): $Y_t = c_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1}$
- MA(1) using the Lag operator: $Y_t = c_0 + (1 + \theta_1 L)\varepsilon_t$
- MA(2): $Y_t = c_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$
- MA(2): $Y_t = c_0 + (1 + \theta_1 L + \theta_2 L^2)\varepsilon_t$
- MA(q): $Y_t = c_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$
- MA(q): $Y_t = c_0 + (1 + \theta_1 L + \cdots + \theta_q L^q)\varepsilon_t$

The lag operator and non-seasonal ARIMA models

Once we start combining components to form more complicated models, it is much easier to work with the lag operator.

If we combine differencing with AR and a MA model, we obtain a non-seasonal ARIMA model:
$\Delta Y_t = \beta_0 + \beta_1 \Delta Y_{t-1} + \cdots + \beta_p \Delta Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$

This equation can be written in Lag operator notation as:
$(1 - \beta_1 L - \cdots - \beta_p L^p)(1 - L) Y_t = \beta_0 + (1 + \theta_1 L + \cdots + \theta_q L^q)\varepsilon_t$

In general, ARIMA(p,d,q) can be compactly written as:
$(1 - \beta_1 L - \cdots - \beta_p L^p)(1 - L)^d Y_t = \beta_0 + (1 + \theta_1 L + \cdots + \theta_q L^q)\varepsilon_t$

The lag operator and seasonal ARIMA models

A seasonal ARIMA model is formed by including additional seasonal terms

$$ARIMA(p, d, q)(P, D, Q)[s]$$

where s is the number of observations per year, (p, d, q) is the order of non-seasonal part of the model, and (P, D, Q) is the order of the seasonal part of the model.

The seasonal part of the model consists of terms that are similar to the non-seasonal components of the model, but involve Lag of the seasonal period. For example, an ARIMA(2,1,2)(1,1,1)[4] without a constant can be written as

$$(1-\beta_1 L-\beta_2 L^2)(1-B_1 L^s)(1-L)(1-L^s)Y_t = (1+\theta_1 L+\theta_2 L^2)(1+\Theta_1 L^s)\varepsilon_t$$

where B_1 is an autoregressive coefficient on seasonal AR part and Θ_1 is a moving average coefficient on seasonal MA part.

The additional seasonal terms are simply multiplied by the non-seasonal terms