Markov chains continued
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BFN/bfn

Section 3.6 Random Walk

$$X_{n+1} = X_n + Z_{n+1}, \quad Z_i \text{ i.i.d}$$

Integer indexed random variable. Can be generalised to a general state space. Frequently

$$\begin{aligned}
X_{n+1} &= X_n + \Delta, \quad |\Delta| \leq 1, \Delta \in \mathbb{N} \\
\mathbb{P}(\Delta = 1) &= p \\
\mathbb{P}(\Delta = -1) &= q \\
\mathbb{P}(\Delta = 0) &= r = 1 - p - q
\end{aligned}$$

Unrestricted/restricted

$$X_{n+1} = \max(\min(X_n + \Delta, N), 0)$$

Random walk is a Markov Chain

$$\boldsymbol{P} = \left\| \begin{array}{cccccccc}
1-p & p & 0 & 0 & 0 & \dots & 0 & 0 \\
q & r & p & 0 & 0 & \dots & 0 & 0 \\
0 & q & r & p & 0 & \dots & 0 & 0 \\
0 & 0 & q & r & p & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots\ddots\ddots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & r & p \\
0 & 0 & 0 & 0 & 0 & \dots & 1-p & p
\end{array} \right\|$$

Define

$$\begin{aligned}
T &= \min\{n \geq 0 : X_n \in \{0, n\}\} \\
u_k &= \mathbb{P}(X_T = 0 | X_0 = k) \\
u_0 &= 1 \\
u_N &= 0 \\
u_k &= q u_{k-1} + r u_k + p u_{k+1}
\end{aligned}$$

Assume $r = 0$

$$\begin{aligned}
(p+q)u_k &= qu_{k-1} + pu_{k+1} \\
p(u_{k+1} - u_k) &= q(u_k - (u_{k-1})) \\
u_{k+1} - u_k &= \frac{q}{p}(u_k - (u_{k-1})) \\
x_{k+1} &= \frac{q}{p}x_k
\end{aligned}$$

With solution

$$u_k = \begin{cases} \frac{N-k}{N} & \text{for } p = q = \frac{1}{2} \\ \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} & \text{for } p \neq q \end{cases}$$

$$\begin{aligned}
v_k &= \mathbb{E}(T|X_0 = k) \\
v_k &= 1 + qv_{k-1} + rv_k + pv_{k+1}
\end{aligned}$$

For $p = q = \frac{1}{2}$

$$v_k = k(N-k) \quad (3.53)$$

The general case is in 3.61

Section 3.7 Another look at first step analysis/first step analysis revisited

The setup $N+1$ (N finite) staes of which (the first) r is transient. The transition matrix $\boldsymbol{P}$ is partitioned accordingly

$$\begin{aligned}
\boldsymbol{P} &= \left\| \begin{matrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{matrix} \right\|, \quad \boldsymbol{Q} \text{ is } r \times r \\
T &= \min\{n \geq 0; X_n \in \{r, r+1, \ldots, N\}
\end{aligned}$$

We have

$$\begin{aligned}
u_{kj} &= \mathbb{P}\{X_T = j|X_0 = k\} \\
v_k &= \mathbb{E}[T|X_0 = k]
\end{aligned}$$

which we can find (last week found) using first step analysis. We now follwow an approach by direct derivation

$W_{ij}^{(n)}$ Expected time spent in/visits to state j starting in state i during first n time steps.

$$
\begin{aligned}
W_{ij}^{n} &= \mathbb{E}\left[\sum_{k=0}^{n} \mathbf{1}\{X_k = j\} \middle| X_0 = i\right] \\
\mathbf{1}\{X_j = j\} &= \begin{cases} 0 & \text{if } X_k \neq j \\ 1 & \text{if } X_k = j \end{cases} \\
W_{ij}^{n} &= \sum_{k=0}^{n} \mathbb{E}\left(\mathbf{1}\{X_k = j | X_0 = i\}\right) = \sum_{k=0}^{n} \mathbb{P}\{X_k = j | X_0 = i\} = \sum_{k=0}^{n} P_{ij}^{(n)} \\
\boldsymbol{P} &= \boldsymbol{P}^n \\
\boldsymbol{P}^2 &= \left\| \begin{matrix} \boldsymbol{Q} & \boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| \left\| \begin{matrix} \boldsymbol{Q} & \boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| = \left\| \begin{matrix} \boldsymbol{Q}^2 & \boldsymbol{R} + \boldsymbol{Q}\boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| \\
\boldsymbol{P}^{(n)} &= \boldsymbol{P}^n = \left\| \begin{matrix} \boldsymbol{Q}^n & \boldsymbol{R} + \boldsymbol{Q}\boldsymbol{R} \ldots \boldsymbol{Q}^{n-1}\boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| = \left\| \begin{matrix} \boldsymbol{Q}^n & \sum_{k=0}^{n-1} \boldsymbol{Q}^k \boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| \\
W_{ij}^{(0)} &= \delta_{ij}, \quad \delta_{ij} = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \\
W_{ij}^{(n)} &= \delta_{ij} + Q_{ij} + Q_{ij}^{(2)} + \ldots Q_{ij}^{(n)} \\
\boldsymbol{W}^{(n)} &= \boldsymbol{I} + \boldsymbol{Q} + \boldsymbol{Q}^2 + \ldots \boldsymbol{Q}^n = \boldsymbol{I} + \boldsymbol{Q}\boldsymbol{W}^{(n-1)} \\
\boldsymbol{W} &= \lim_{n\to\infty} W^{(n)} \\
\boldsymbol{W} &= \boldsymbol{I} + \boldsymbol{Q}\boldsymbol{W} \\
\boldsymbol{W}(\boldsymbol{I} - \boldsymbol{Q}) &= \boldsymbol{I} \\
\boldsymbol{W} &= (\boldsymbol{I} - \boldsymbol{Q})^{-1}, \quad \text{fundamental matrix} \\
(\boldsymbol{I} - \boldsymbol{Q})^{-1} &= \sum_{k=0}^{\infty} \boldsymbol{Q}, \quad \text{All eigenvalues of } \boldsymbol{Q} \text{ are strictly within the unit circle} \\
v_i &= \sum_{j=0}^{r-1} W_{ij} \\
\boldsymbol{v} &= \boldsymbol{e} + \boldsymbol{Q}\boldsymbol{v} \\
\boldsymbol{v} &= (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{e} \\
u_{ij}^{(n)} &= \mathbb{P}\{T \leq n, X_T = j | X_0 = i\} \\
\boldsymbol{P}^{(n)} &= \boldsymbol{P}^n = \left\| \begin{matrix} \boldsymbol{Q}^n & \sum_{k=0}^{n-1} \boldsymbol{Q}^k \boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{matrix} \right\| \\
\boldsymbol{U}^{(n)} &= \sum_{k=0}^{n-1} \boldsymbol{Q}^k \boldsymbol{R} \\
\boldsymbol{U} &= \lim_{n\to\infty} \boldsymbol{U}^{(n)} = \boldsymbol{W}\boldsymbol{R} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}
\end{aligned}
$$

Section 2.1 Conditional Expectation

Conditional distribution

$$\begin{aligned}
& \mathbb{P}\{X_1 = x_1, X_2 = x_2\} \\
\mathbb{P}\{X_2 = x_2 | X_1 = x_1\} &= \frac{\mathbb{P}\{X_1 = x_1, X_2 = x_2\}}{\mathbb{P}\{X_1 = x_1\}} \\
\mathbb{E}(X_2 | X_1 = x_1) &= \sum_{x_2} x_2 \mathbb{P}(X_2 | X_1 = x_1) = h(x_1) \\
h(X_1) & \quad \text{Is a random variable, written as } h(X_1) = \mathbb{E}(X_2|X_1) \\
\mathbb{E}(\mathbb{E}(X_2|X_1)) &= \mathbb{E}(h(X_1)) = \sum_{x_1} h(x_1) \mathbb{P}(X_1 = x_1) \\
&= \sum_{x_1} \left(\sum_{x_2} x_2 \mathbb{P}(X_2 | X_1 = x_1) \right) \mathbb{P}(X_1 = x_1) \\
&= \sum_{x_2} x_2 \sum_{x_1} \mathbb{P}(X_2 | X_1 = x_1) \mathbb{P}(X_1 = x_1) = \sum_{x_2} x_2 \sum_{x_1} \mathbb{P}\{X_1 = x_1, X_2 = x_2\} \\
&= \sum_{x_2} x_2 \mathbb{P}\{X_2 = x_2\} = \mathbb{E}(X_2)
\end{aligned}$$

Similarly

$$\mathbb{E}(\mathbb{E}(g(X_2)|X_1)) = \mathbb{E}(g(X_2))$$

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(X_2) &= \mathbb{E}(X_2^2) - \mathbb{E}(X_2)^2 = \mathbb{E}(\mathbb{E}(X_2^2|X_1)) - \mathbb{E}(\mathbb{E}(X_2|X_1))^2 = \mathbb{E}\left(\mathbb{V}\mathrm{ar}(X_2|X_1) + \mathbb{E}(X_2|X_1)^2\right) - \mathbb{E}(\mathbb{E}(X_2|X_1))^2 \\
&= \mathbb{E}(\mathbb{V}\mathrm{ar}(X_2|X_1)) + \mathbb{E}\left(\mathbb{E}(X_2|X_1)^2\right) - \mathbb{E}(\mathbb{E}(X_2|X_1))^2 = \mathbb{E}\left(\mathbb{V}\mathrm{ar}(X_2|X_1)\right) + \mathbb{V}\mathrm{ar}\left(\mathbb{E}(X_2|X_1)\right)
\end{aligned}$$

Section 2.3 Random Sum

Given $N \in \mathbf{N}$ and Z_i i.i.d.

$$\begin{aligned}
X &= \sum_{i=1}^{N} Z_i \\
\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|N)) = \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{N} Z_i | N\right)\right) = \mathbb{E}\left(\sum_{i=1}^{N} \mathbb{E}(Z_i|N)\right) = \mathbb{E}\left(\sum_{i=1}^{N} \mathbb{E}(Z_i)\right) = \mathbb{E}(N\mathbb{E}(Z_i)) = \mathbb{E}(N)\mathbb{E}(Z_i) \\
\mathbb{V}\mathrm{ar}(X) &= \mathbb{E}(\mathbb{V}\mathrm{ar}(X|N)) + \mathbb{V}\mathrm{ar}(\mathbb{E}(X|N)) = \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{N} Z_i|N\right)\right) + \mathbb{V}\mathrm{ar}(\mathbb{E}(X|N)) \\
&= \mathbb{E}\left(\sum_{i=1}^{N} \mathbb{V}\mathrm{ar}\left(Z_i|N\right)\right) + \mathbb{V}\mathrm{ar}(N\mathbb{E}(Z_i)) = \mathbb{E}\left(\sum_{i=1}^{N} \mathbb{V}\mathrm{ar}\left(Z_i\right)\right) + \mathbb{E}(Z_i)^2 \mathbb{V}\mathrm{ar}(N) \\
&= \mathbb{E}\left(N\mathbb{V}\mathrm{ar}\left(Z_i\right)\right) + \mathbb{E}(Z_i)^2 \mathbb{V}\mathrm{ar}(N) = \mathbb{E}(N)\mathbb{V}\mathrm{ar}\left(Z_i\right) + \mathbb{E}(Z_i)^2 \mathbb{V}\mathrm{ar}(N)
\end{aligned}$$

Section 3.9.2 Probability generating functions

But now we are more ambitous and want to find the distribution of X. We introduce probability generating functions

$$Z \in \mathbf{N}_0 : \mathbb{P}(Z = k) = p_k, \quad k \in \mathbf{N}_0$$

$$\phi(s) = \mathbb{E}\left(s^Z\right) = \sum_{k=0}^{\infty} s^k \mathbb{P}\{Z = k\} = \sum_{k=0}^{\infty} s^k p_k$$

The power series is convergent for $|s| \leq 1$.

$$\begin{aligned}
Z_1 &\sim \text{geo}(p), \quad \mathbb{P}\{Z_1 = k\} = (1-p)^k p \\
\phi_1(s) &= \mathbb{E}\left(s^{Z_1}\right) = \sum_{k=0}^{\infty} s^k (1-p)^k p = p \sum_{k=0}^{\infty} (s(1-p))^k = \frac{p}{1 - s(1-p)} \\
Z_2 &\sim \text{Pois}(\mu), \quad \mathbb{P}(Z_2 = k) = \frac{\mu^k}{k!} e^{-\mu} \\
\phi_2(s) &= \sum_{k=0}^{\infty} s^k \frac{\mu^k}{k!} e^{-\mu} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(s\mu)^k}{k!} = e^{-\mu} e^{s\mu} = e^{-\mu(1-s)}
\end{aligned}$$

Recover probabilities (rarely done though)

$$\begin{aligned}
\phi(0) &= p_0, \quad \phi'(s) = \sum_{k=1}^{\infty} k s^{k-1}, \quad \phi'(0) = p_1, \quad \text{etc.} \\
\mathbb{E}(Z) &= \sum_{k=0}^{\infty} k p_k = \sum_{k=1}^{\infty} k p_k \\
\phi''(s) &= \sum_{k=2}^{\infty} k(k-1) s^{k-2} \\
\mathbb{E}(Z) &= \phi'(s)|_{s=1} \\
\mathbb{E}(Z(Z-1)) &= \phi''(s)|_{s=1} \\
\mathbb{V}\text{ar}(Z) &= \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = \mathbb{E}(Z(Z-1)) + \mathbb{E}(Z) - \mathbb{E}(Z)^2 = \phi''(1) + \phi'(1) - \phi'(1)^2
\end{aligned}$$

$$\begin{aligned}
Z_1 : \quad & \mathbb{E}\left(s_1^Z\right) = \phi_1(s), \quad Z_2 : \mathbb{E}\left(s_2^Z\right) = \phi_2(s), \quad \text{independent} \\
Z &= Z_1 + Z_2 \\
\phi(s) &= \mathbb{E}\left(s^Z\right) = \mathbb{E}\left(s^{Z_1+Z_2}\right) = \mathbb{E}\left(s_1^Z s_2^Z\right) = \mathbb{E}\left(s_1^Z\right) \mathbb{E}\left(s_2^Z\right) = \phi_1(s)\phi_2(s) \\
Z_1 &\sim \text{Pois}(\mu_1), \quad Z_2 \sim \text{Pois}(\mu_2) \\
\phi(s) &= \phi_1(s)\phi_2(s) = e^{-\mu_1(1-s)} e^{-\mu_2(1-s)} = e^{-(\mu_1+\mu_2)(1-s)} \\
Z &\sim \text{Pois}(\mu_1 + \mu_2)
\end{aligned}$$

Back to random sum

$$
\begin{aligned}
X &= \sum_{i=1}^{N} Z_i, \quad \text{with } \phi_N(s), \phi_{Z_i}(s) \\
\phi(s) &= \mathbb{E}\left(s^X\right) = \mathbb{E}\left(s^{\sum_{i=1}^N Z_i}\right) = \mathbb{E}\left(\mathbb{E}\left(s^{\sum_{i=1}^N Z_i}\middle| N\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\prod_{i=1}^{N} s^{Z_i}\middle| N\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\prod_{i=1}^{N} s^{Z_i}\middle| N\right)\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{N} \mathbb{E}\left(s^{Z_i}\middle| N\right)\right) = \mathbb{E}\left(\prod_{i=1}^{N} \mathbb{E}\left(s^{Z_i}\right)\right) = \mathbb{E}\left(\prod_{i=1}^{N} \phi_{Z_i}(s)\right) = \mathbb{E}\left(\phi_{Z_i}(s)^N\right) = \phi_N(\phi_{Z_i}(s)) \\
Z_i &\sim \text{Be}(p), \quad \phi_{Z_i}(s) = 1 - p + ps \\
N &\sim \text{Pois}(\mu), \quad \phi_N(s) = e^{-\mu(1-s)} \\
\phi_X(s) &= \phi_N(\phi_{Z_i}(s)) = e^{-\mu(1-(1-p+ps))} = e^{-p\mu(1-s)} \\
X &\sim \text{Pois}(p\mu)
\end{aligned}
$$

Section Branching Processes

$$
\begin{aligned}
X_{n+1} &= \sum_{i=1}^{X_n} Z_{ni}, \quad \mathbb{E}(Z_{ni}) = \mu, \mathbb{V}\text{ar}(Z_{ni}) = \sigma^2, \mathbb{E}\left(Z_{ni}\right) = \phi(s) \\
\mathbb{E}(X_{n+1}) &= \mathbb{E}(X_n)\mathbb{E}(Z_{ni}) = \mathbb{E}(X_n)\mu = \mu^{n+1} \\
\mathbb{V}\text{ar}(X_{n+1}) &= \mathbb{E}((X_n)\mathbb{V}\text{ar}(Z_{ni}) + \mathbb{E}(Z_{ni})^2\mathbb{V}\text{ar}(X_n) = \sigma^2\mathbb{E}((X_n) + \mu^2\mathbb{V}\text{ar}(X_n) \\
\phi_n(s) &= \mathbb{E}\left(s^{X_n}\right) = \phi_{n-1}(\phi(s))
\end{aligned}
$$