Chance-constrained optimisation

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42586 – Decisions under Uncertainty – 30 April 2022

Learning Objectives

At the end of this lecture, a student should be able to:

- Formulate **Chance-constrained** linear programs
- Explain and discuss how such problems are **fundamentally different** from other approaches studied so far
- Use analytical techniques to **reformulate** chance-constrained linear problems
- Appraise **limitations** of chance-constrained linear problems

Outline

1. We need more certainty...
2. Chance-constrained linear programming
3. Reformulation of chance constraints

1. **We need more certainty**

Starting from a basic (stochastic) linear program

Consider the following linear program:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \leq b \\ & \mathbf{x} \geq 0 \end{aligned}$$

Definition (Stochastic linear program (with uncertainty in constraints))

Considering a cost vector $\mathbf{c}$, decision vector $\mathbf{x}$, as well as matrix A and right-hand side $\mathbf{b}$ to define constraints, a *stochastic linear program* (with uncertainty in constraints) writes

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A(\omega)\,\mathbf{x} \leq b(\omega) \\ & \mathbf{x} \geq 0 \end{aligned}$$

where ω is an uncertain parameter.

* In the most general case, one can uncertainty in both cost function and constraints

An example (Uncertainty in b only)

Cargo stowage planning

- you have 2 types of containers with 2 heights, and revenue you can get from those

Type	quantity	height [m]	revenue [DKK]
1	x_1	$h_1 = 6$	$c_1 = 100.000$
2	x_2	$h_2 = 2$	$c_2 = 20.000$

- since you pile them up (layers of 10 of the same type), you have a maximum height ω they may collapse en-route, as a function of wind and sea conditions. It is uncertain though...

- The problem writes

$$\begin{aligned} \min_{x_1, x_2} \quad & -(c_1 x_1 + c_2 x_2) \\ \text{s.t.} \quad & x_1 h_1 + x_2 h_2 \leq \omega \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

(Note that x_1 and x_2 are integers, and not continuous variables)

An example (Uncertainty in A only)

Energy dispatch

- To meet a given energy demand D, you can use two energy supply types, gas and wind, with different characteristics

Type	quantity	max.	unit cost [DKK]	delivery
gas	x_1	100	$c_1 = 20$	1
wind	x_2	100	$c_2 = 3$	ω

  i.e., delivery from the gas supplier is certain (so, $1 \times x_1$), and that from the wind supplier is uncertain (so, $\omega \times x_2$)...

- You want to meet the demand at a minimum cost

- The problem writes

$$
\begin{aligned}
\min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\
\text{s.t.} \quad & x_1 + x_2 \omega \geq D \\
& 0 \leq x_1 \leq 100 \\
& 0 \leq x_2 \leq 100
\end{aligned}
$$

Deterministic reformulation

As for other stochastic linear programs, we want to find a *deterministic reformulation*, which we can then manage...

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1. **expectation constraints** - the problem then becomes

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   by replacing ω by its expectation... *(Easy, but no certainty!)*

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2. **worst case constraints** - the problem is then

$$\begin{aligned} \min_{x} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A(\omega)\,\mathbf{x} \leq b(\omega), \quad \forall \omega \\ & \mathbf{x} \geq 0 \end{aligned}$$

   by making sure constraints are respected for all values of ω... *(maximum certainty, but getting complicated!)*

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3. **chance constraints** - we relax the worst case, and want to satisfy constraints with a certain probability...

Chance-constrained linear programs

In the more general case, we want to satisfy the constraints, but with **a given level of certainty**

Definition (chance-constrained linear program)

Considering a cost vector $\mathbf{c}$, decision vector $\mathbf{x}$, as well as matrix A and right-hand side $\mathbf{b}$ to define constraints, a *chance-constrained linear program* writes

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathrm{P}[A(\omega)\,\mathbf{x} \leq b(\omega)] \geq 1-\epsilon \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $1-\epsilon$ is the *certainty level* (i.e., minimum probability that the constraints are satisfied).
We also say that the solution is then ϵ-*reliable*.

Pros and cons:

Pros: This gives us a great trade off between the lack of certainty of the **expected constraints** and the overly conservative view of **worst-case constraints**

Cons: Only for simple cases are these problems easy to solve... In the general case, chance-constrained optimization problems are **very difficult** to solve!

Revisiting the cargo stowage example

Given a *certainty level* $1-\epsilon$, for the containers not to collapse, we can write the resulting chance-constrained linear program:

$$\begin{aligned}
\min_{x_1, x_2} \quad & -(c_1 x_1 + c_2 x_2) \\
\text{s.t.} \quad & \mathrm{P}\left[\, x_1 h_1 + x_2 h_2 \leq \omega \,\right] \geq 1-\epsilon \\
& x_1,\ x_2 \geq 0
\end{aligned}$$

Revisiting the energy dispatch problem

Given the energy demand D, and a *certainty level* $1-\epsilon$ (called reliability) that demand will be met, we can write the resulting chance-constrained linear program:

$$\begin{aligned}
\min_{x_1,x_2} \quad & c_1x_1 + c_2x_2 \\
\text{s.t.} \quad & \mathrm{P}\left[\, x_1 + x_2\omega \geq D \,\right] \geq 1-\epsilon \\
& 0 \leq x_1 \leq 100 \\
& 0 \leq x_2 \leq 100
\end{aligned}$$

3. **Reformulation of chance constraints**

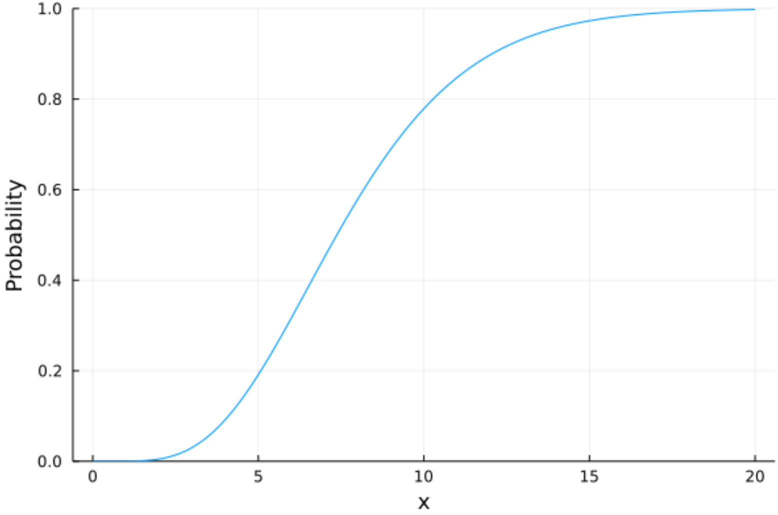
(Keep in mind that we focus on simple cases only... it is not always that straightforward!!)

The key and central idea

In all cases (uncertainty in A and $\mathbf{b}$), we end up with a constraint like $\mathrm{P}\,[\,\text{event which is a function of } \omega\,] \geq 1 - \epsilon$

We can use F_ω the cumulative distribution function (c.d.f.) for ω to reformulate that probability... using

$$F_\omega(x) = \mathrm{P}[\omega \leq x]$$



Looking at the container stowage example

The chance constraint originally is

$$\mathrm{P}\left[\, x_1 h_1 + x_2 h_2 \leq \omega \,\right] \geq 1 - \epsilon$$

while we have a c.d.f. F_ω for the uncertain parameter ω.

Looking at the container stowage example

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while we have a c.d.f. F_ω for the uncertain parameter ω.

We have

$$\begin{aligned}
\mathrm{P}\left[\, x_1 h_1 + x_2 h_2 \leq \omega \,\right] &= 1 - \mathrm{P}\left[\, \omega \leq x_1 h_1 + x_2 h_2 \,\right] \\
&= 1 - F_\omega(x_1 h_1 + x_2 h_2)
\end{aligned}$$

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It then means that

$$\mathrm{P}\left[x_1 h_1 + x_2 h_2 \leq \omega\right] \geq 1 - \epsilon \quad \Leftrightarrow \quad 1 - F_\omega(x_1 h_1 + x_2 h_2) \geq 1 - \epsilon$$

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leading to

$$F_\omega(x_1 h_1 + x_2 h_2) \le \epsilon$$

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leading to

$$F_\omega(x_1 h_1 + x_2 h_2) \leq \epsilon$$

So that we finally obtain the useful final form for that constraint

$$x_1 h_1 + x_2 h_2 \leq F_\omega^{-1}(\epsilon)$$

The problem is then different depending upon the certainty level

- Your client gives you a certainty level $1-\epsilon$
- You use the reformulation and the c.d.f. to obtain the final form of the chance constraint

If $1-\epsilon = 0.5$:

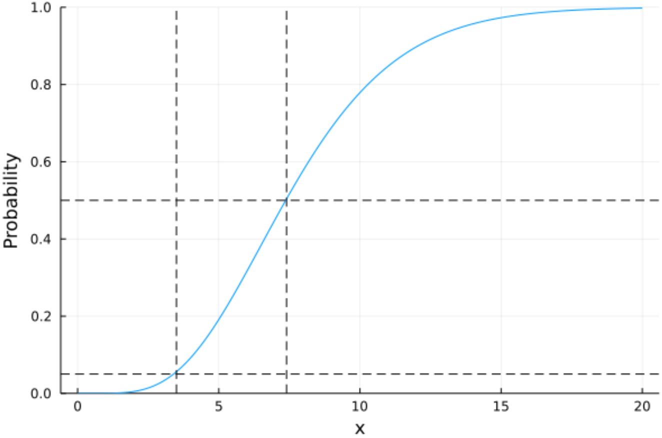
$$\begin{aligned} \min_{x_1, x_2} \quad & -(c_1 x_1 + c_2 x_2) \\ \text{s.t.} \quad & x_1 h_1 + x_2 h_2 \leq 7.4 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Solution: $x_1 = 1$ and $x_2 = 0$

If $1-\epsilon = 0.95$:

$$\begin{aligned} \min_{x_1, x_2} \quad & -(c_1 x_1 + c_2 x_2) \\ \text{s.t.} \quad & x_1 h_1 + x_2 h_2 \leq 3.5 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Solution: $x_1 = 0$ and $x_2 = 1$



For the energy dispatch problem...

The chance constraint originally is

$$\mathrm{P}\left[x_1 + \omega x_2 \geq D\right] \geq 1 - \epsilon$$

while we have a c.d.f. F_ω for the uncertain parameter ω.

With the same process as before we get to the final reformulation of the chance constraint:

$$\frac{D - x_1}{x_2} \leq F_\omega^{-1}(\epsilon), \qquad \text{or similarly,} \qquad x_1 + x_2 F_\omega^{-1}(\epsilon) \geq D$$

Two examples (for D=100):

Imagine that for $1 - \epsilon = 0.5$, $F_\omega^{-1}(\epsilon) = 0.9$, then

$$\begin{aligned} \min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & x_1 + 0.9 x_2 \geq D \\ & 0 \leq x_1 \leq 100 \\ & 0 \leq x_2 \leq 100 \end{aligned}$$

Solution: $x_1 = 10$ and $x_2 = 100$ (for a cost of 500)

Imagine that for $1 - \epsilon = 0.9$, $F_\omega^{-1}(\epsilon) = 0.5$, then

$$\begin{aligned} \min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & x_1 + 0.5 x_2 \geq D \\ & 0 \leq x_1 \leq 100 \\ & 0 \leq x_2 \leq 100 \end{aligned}$$

Solution: $x_1 = 50$ and $x_2 = 100$ (for a cost of 1300)

Wrapping up

- Expected utility maximization problem **do not give any certainty** on potential outcomes!
- We then use **chance-constrained programming** instead
- These can be **reformulated in simple cases** only
- In the general case, these may be **very difficult problems** to solve

Thanks for your attention!