Stochastic Optimization

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42586 – Decisions under uncertainty - April 16, 2024

Learning objectives

After end of this lecture, a student should be able to:

- Describe a **stochastic linear program** (with two stages)
- Recognise **here-and-now** and **recourse** decision variables
- Formulate a **stochastic linear program** based on a problem description
- Reformulate a stochastic linear program to obtain the **deterministic equivalent**
- **Implement and solve** stochastic linear programs

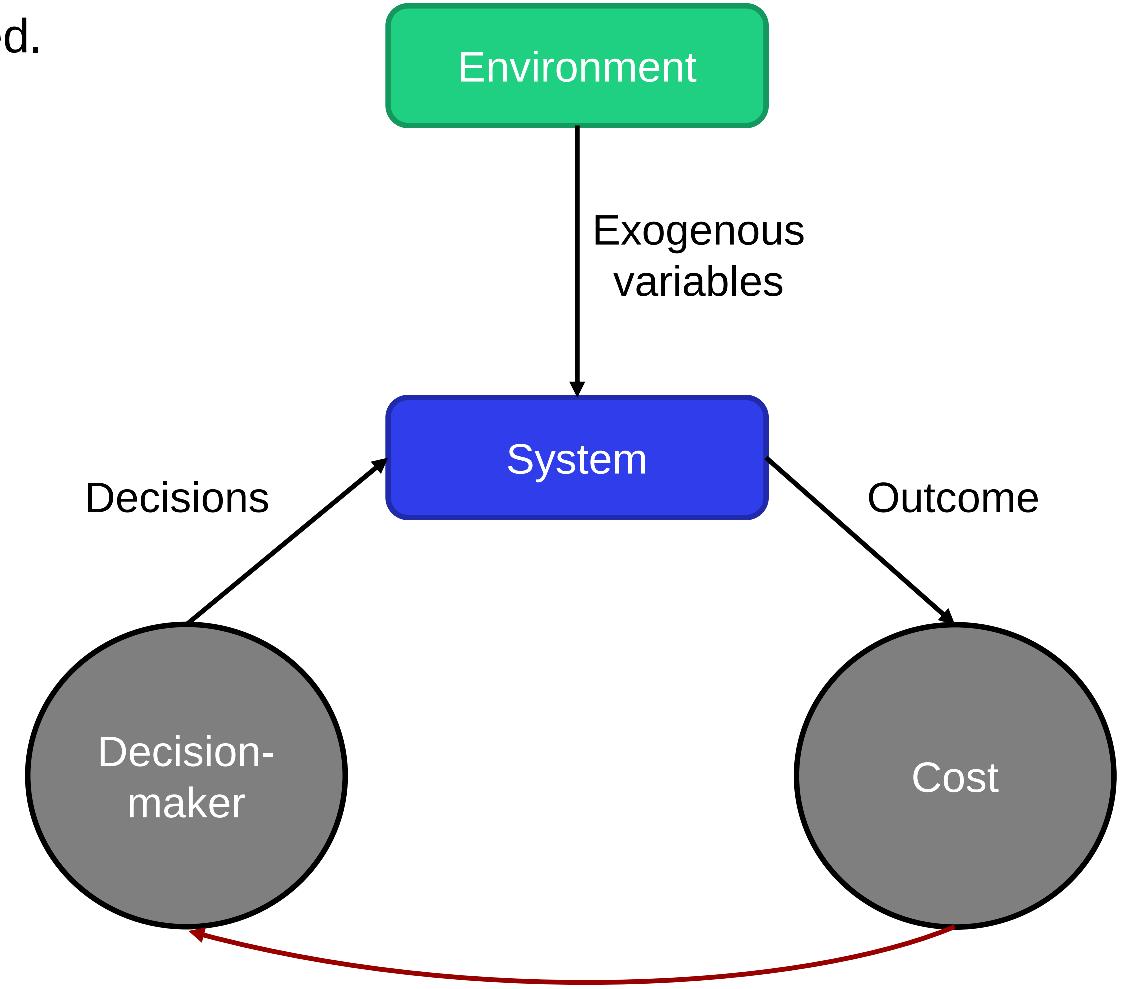
Here-and-now decisions

We make decisions well-knowing that the future can change

Can you give some examples ?

How can we handle uncertainty?

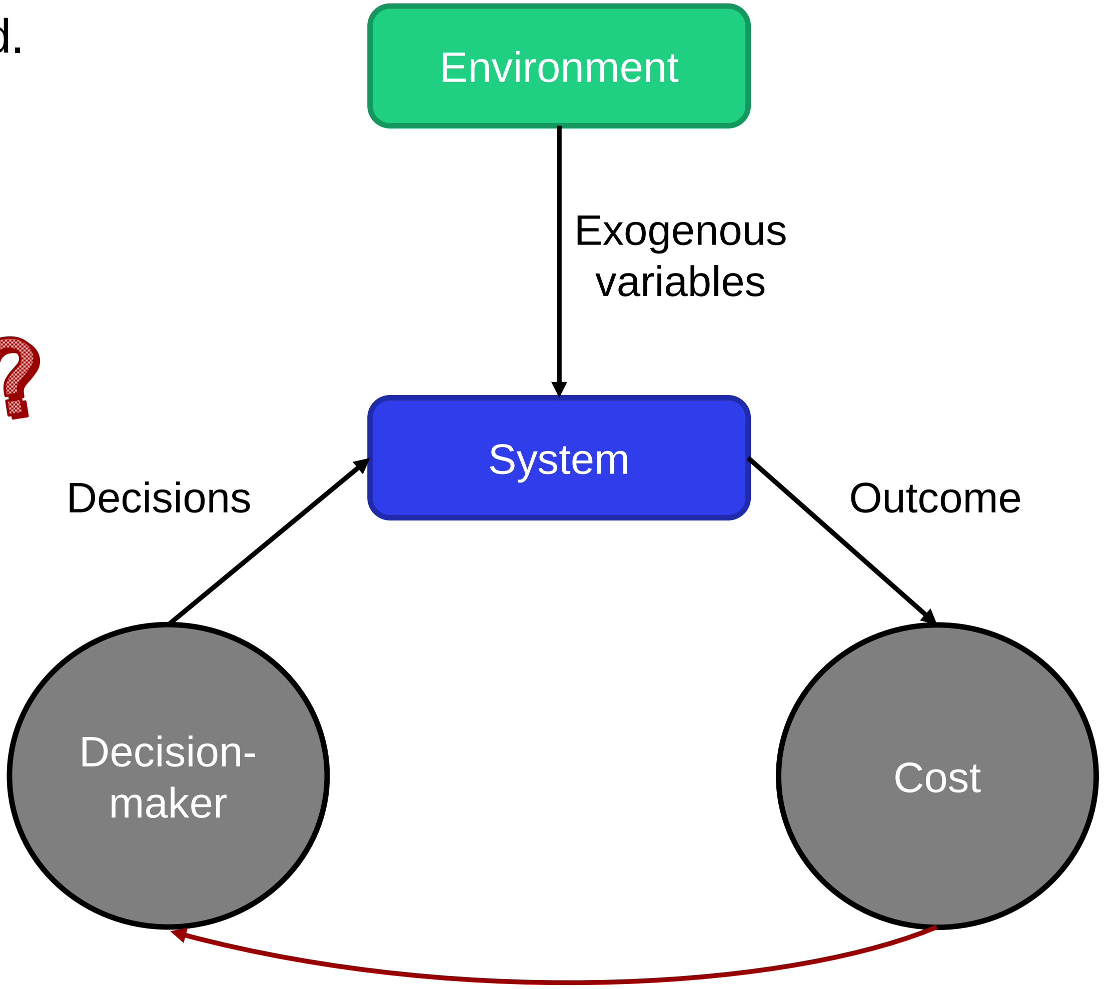
Consider the tools you already learned.



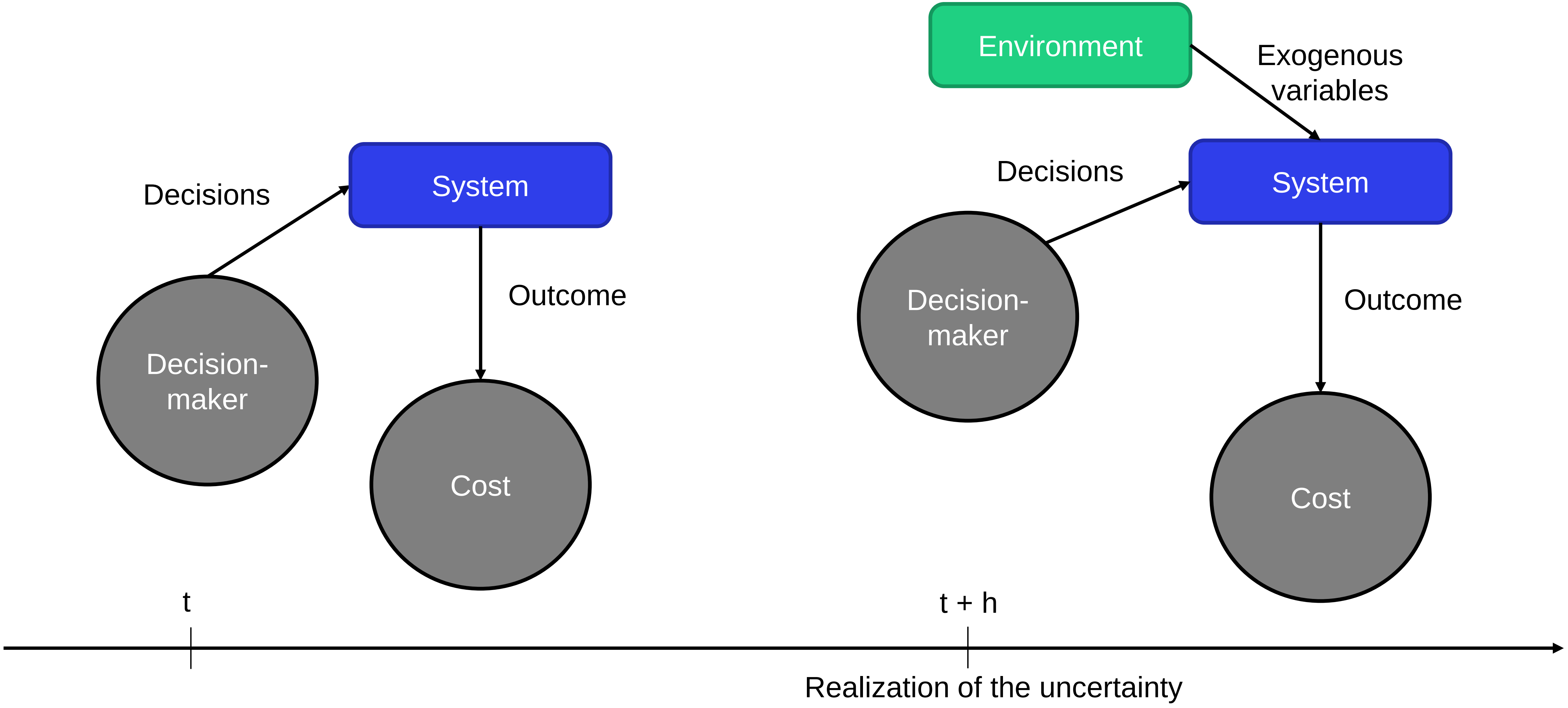
How can we handle uncertainty?

Consider the tools you already learned.

What if we could plan for the future?



What if we could plan for the future?



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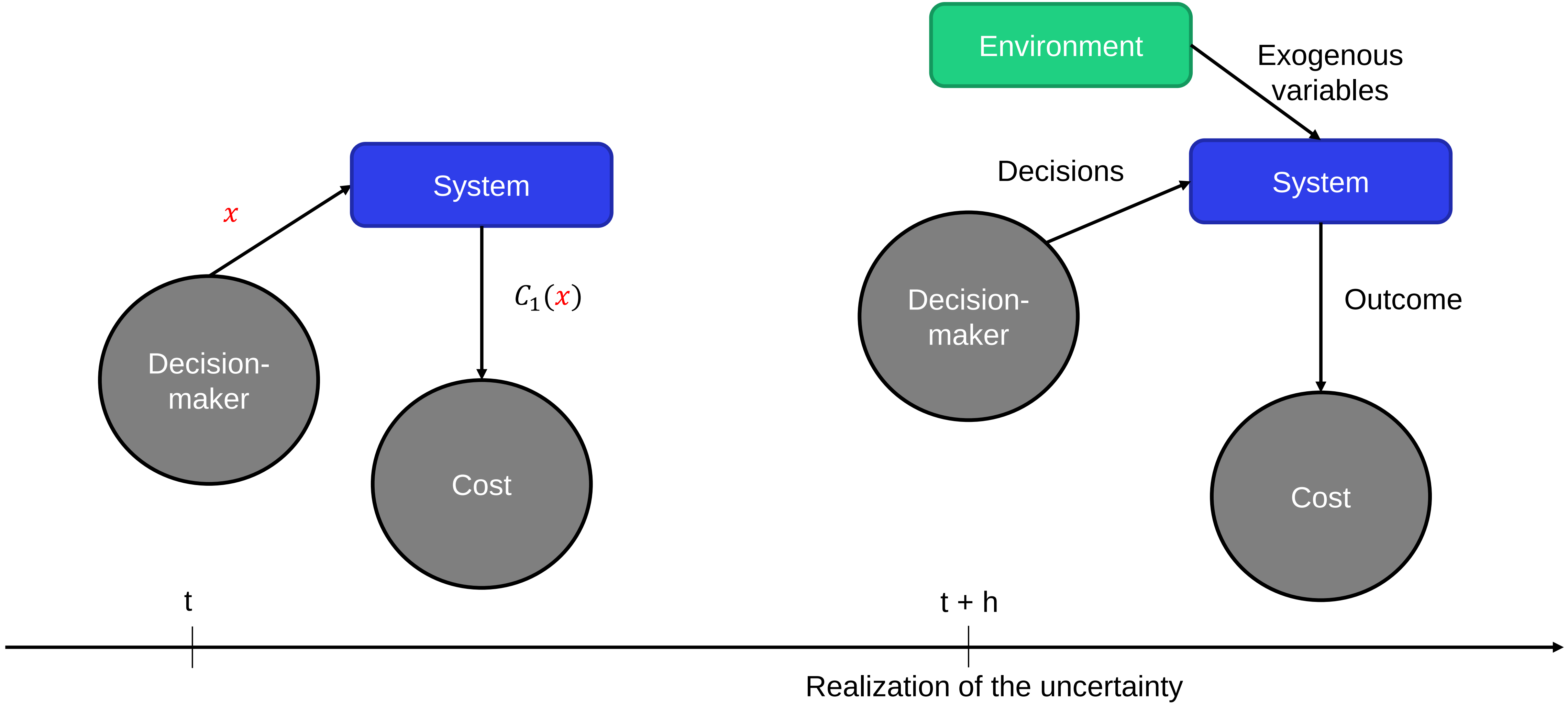

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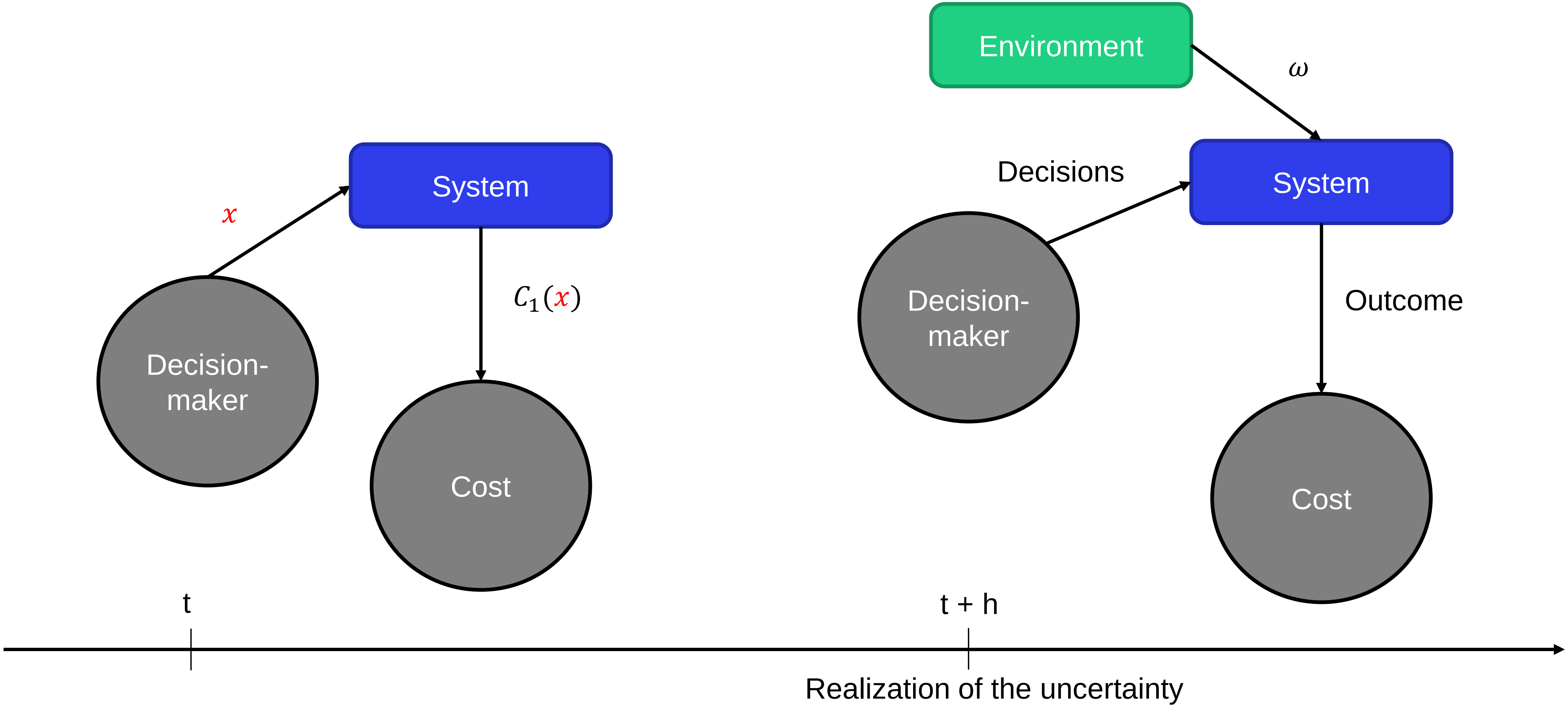
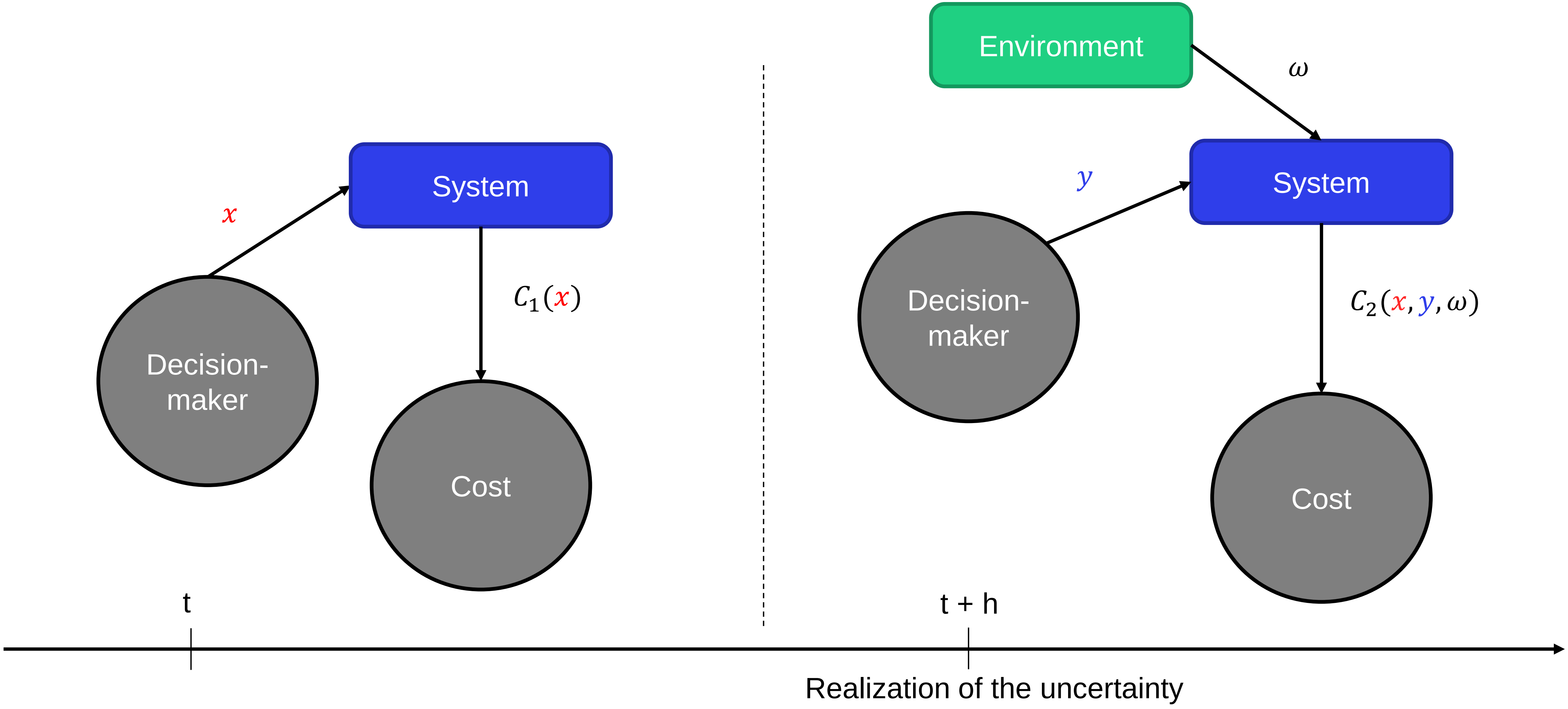


What if we could plan for the future?



The farmer’s problem

It is winter and a farmer needs to decide on how much land needs to be dedicated to each farmed crop: wheat, corn, and sweet beets.

Some of the production goes to feed cattle, and sugar beets are subject to the EU quote system.

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

As an optimization problem

Decisions:

- How much of the land needs to be devoted to wheat, corn, and sweet beets?
- How much of the crops do we sell?
- How much of the crops do we buy?

Objective:

- Minimize cost/maximize profit

Constraints:

- Cannot plant more acres then available.
- Must have enough wheat.
- Must have enough corn.
- Differentiate sweet beets over EU quota.

As an optimization problem

Decisions:

- How much of the land needs to be devoted to wheat, corn, and sweet beets?

  x_1, x_2, x_3 where $\{1,2,3\}$ is the set of all crops $\{wheat, corn, sweet\ beets\}$

- How much of the crops do we sell?

  $w_1, w_2, w_3 (w_4)$ where $w_4$represents the beets sold over the quota

- How much of the crops do we buy?

  y_1, y_2, y_3

Objective:

- Minimize cost/maximize profit

  $\min 150\, x_1 + 230\, x_2 + 260\, x_3 + 238\, y_1 + 210\, y_2 - 170\, w_1 - 150\, w_2 - 36\, w_3 - 10\, w_4$

As an optimization problem

Constraints:

- Cannot plant more acres then available.

$$x_1 + x_2 + x_3 \leq 500$$

- Must have enough wheat.

$$2.5x_1 + y_1 - w_1 \geq 200$$

- Must have enough corn.

$$3x_2 + y_2 - w_2 \geq 240$$

- Differentiate sweet beets over EU quota.

$$w_3 + w_4 \leq 20x_3$$
$$w_3 \leq 6000$$

$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0$$

As an optimization problem

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3 + 238\, y_1 + 210\, y_2 - 170\, w_1 - 150\, w_2 - 36\, w_3 - 10\, w_4$$

$s.t.$

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 500 \\ 2.5x_1 + y_1 - w_1 &\geq 200 \\ 3x_2 + y_2 - w_2 &\geq 240 \\ w_3 + w_4 &\leq 20x_3 \\ w_3 &\leq 6000 \\ x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 &\geq 0 \end{aligned}$$

Optimal solution

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: $118,600			

The farmer’s problem

What if the summer is too dry?

The yield is not certain

Lets test with +/- 20% yield

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

Optimal solutions (assuming correlation)

- Good season 20% more yield

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	–	6000
Purchase (T)	–	–	–
Overall profit: $167,667			

- Average season

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: $118,600			

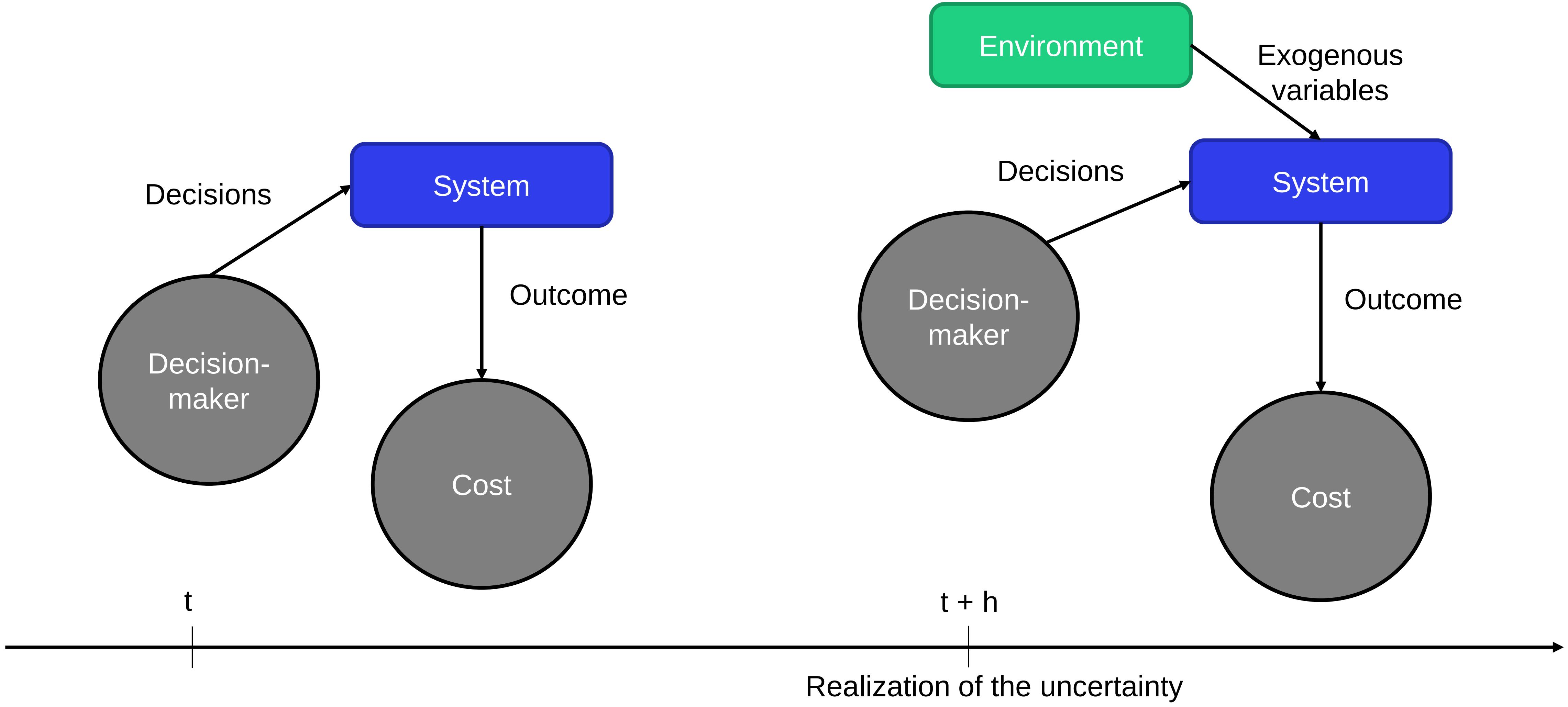
- Bad season 20% less yield

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	100	25	375
Yield (T)	200	60	6000
Sales (T)	–	–	6000
Purchase (T)	–	180	–
Overall profit: $59,950			

How to decide?

If I decide now how much to plant, I can decide later how much to sell and buy

What if we could plan for the future?



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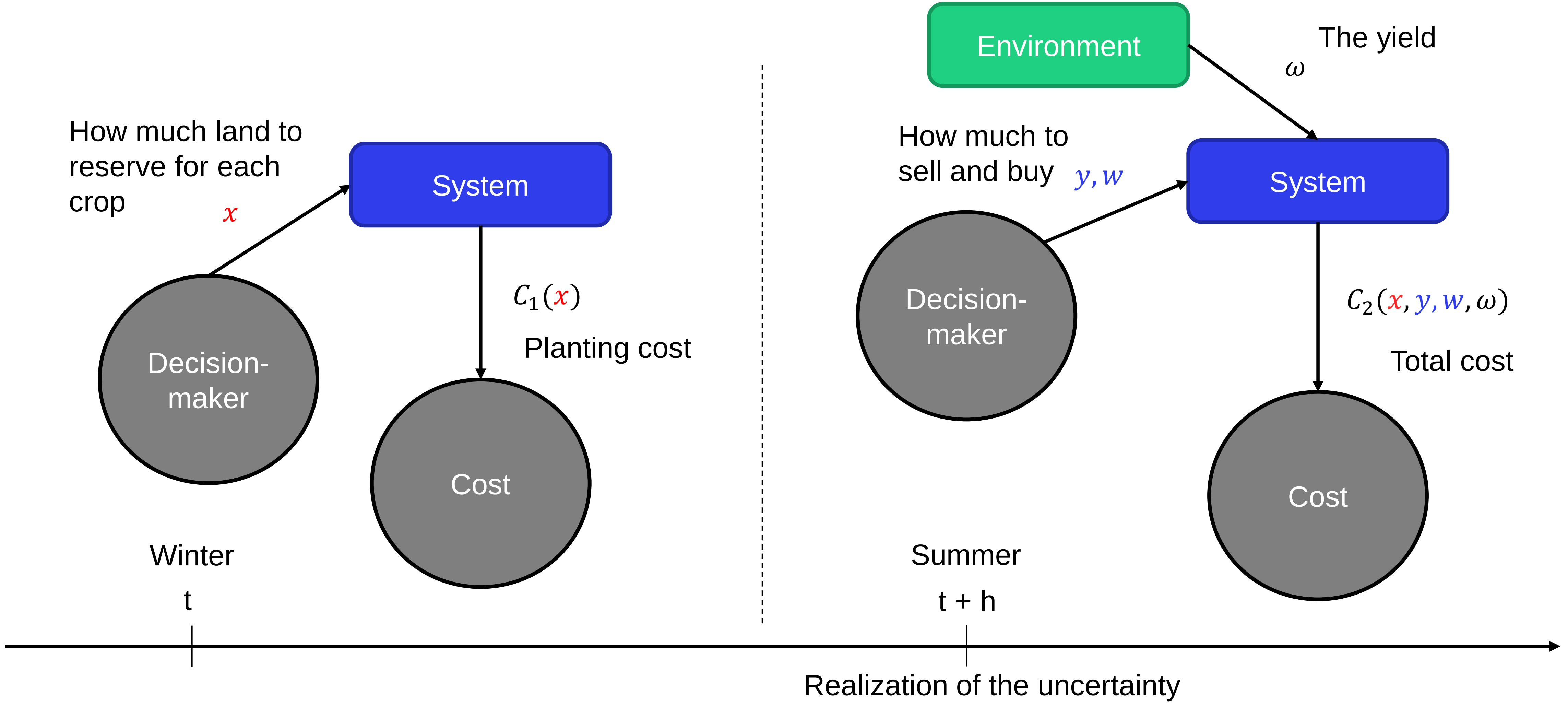


A scenario representation

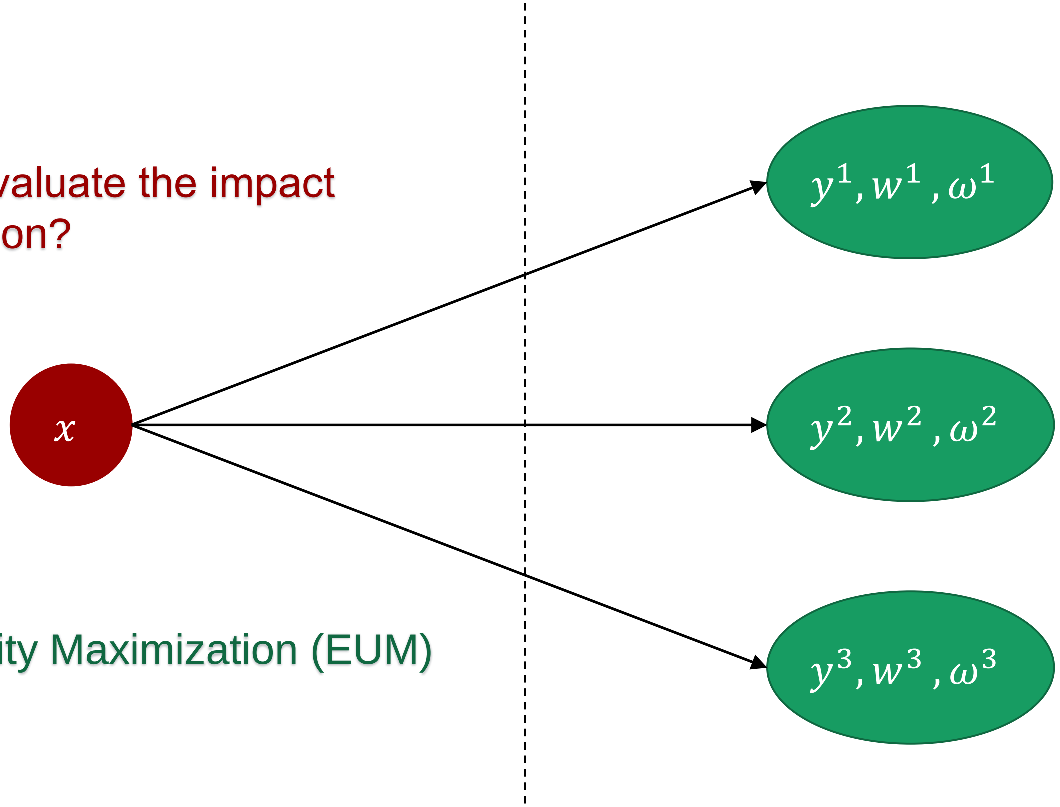
1st – Stage decision

How much land to reserve for each crop

2nd – Stage decision / Recourse action

How much to sell and buy

OK…but how do we evaluate the impact of the 1st –stage decision?



We can use the Expected Utility Maximization (EUM)

The index represents the scenario:

1. *Good season*
2. *Average season*
3. *Bad season*

The vector ω represents the uncertain variable. In this case the yield at each season.

A scenario representation

1st – Stage decision

How much land to reserve for each crop

We can use the Expected Utility Maximization (EUM)

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3 + E_\omega\left[C(x, y, w, \omega)\right]$$

$$s.t.$$

$$x_1 + x_2 + x_3 \leq 500$$

$$x_1, x_2, x_3 \geq 0$$

2nd – Stage decision / Recourse action

How much to sell and buy

y^1, w^1, ω^1

y^2, w^2, ω^2

y^3, w^3, ω^3

The index represents the scenario:

1. *Good season*
2. *Average season*
3. *Bad season*

The vector ω represents the uncertain variable. In this case the yield at each season.

A scenario representation

1st – Stage decision

How much land to reserve for each crop

We can use the Expected Utility Maximization (EUM)

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3 + E_\omega \left[C(x, y, w, \omega)\right]$$

$s.t.$

$$x_1 + x_2 + x_3 \le 500$$
$$x_1, x_2, x_3 \ge 0$$

2nd – Stage decision Recourse action

How much to sell and buy

$$C\left(x, y^i, w^i, \omega^i\right) = 238\, y_1^i + 210\, y_2^i$$
$$-170\, w_1^i - 150\, w_2^i - 36\, w_3^i - 10\, w_4^i$$

$s.t.$

$$\omega_1^i x_1 + y_1^i - w_1^i \ge 200$$
$$\omega_2^i x_2 + y_2^i - w_2^i \ge 240$$
$$w_3^i + w_4^i \le \omega_3^i x_3$$
$$w_3^i \le 6000$$
$$y_1^i, y_2^i, w_1^i, w_2^i, w_3^i, w_4^i \ge 0$$

A scenario representation

1st – Stage decision

How much land to reserve for each crop

We can use the Expected Utility Maximization (EUM)

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3 + \min \sum_{i=1}^{3} p_i\, C\left(x, y^i, w^i, \omega^i\right)$$

$s.t.$

$$x_1 + x_2 + x_3 \le 500$$
$$x_1, x_2, x_3 \ge 0$$

Yield	Good	Avg.	Bad
Wheat	3	2.5	2
Corn	3.6	3	2.4
Beet	24	20	16

2nd – Stage decision / Recourse action

How much to sell and buy
(Good season)

$$C\left(x, y^2, w^2, \omega^2\right) = 238\, y_1^2 + 210\, y_2^2 - 170\, w_1^2 - 150\, w_2^2 - 36\, w_3^2 - 10\, w_4^2$$

$s.t.$

$$3x_1 + y_1^2 - w_1^2 \ge 200$$
$$3.6x_2 + y_2^2 - w_2^2 \ge 240$$
$$w_3^2 + w_4^2 \le 24x_3$$
$$w_3^i \le 6000$$
$$y_1^2, y_2^2, w_1^2, w_2^2, w_3^2, w_4^2 \ge 0$$

Yield	Good	Avg.	Bad
Wheat	3	2.5	2
Corn	3.6	3	2.4
Beet	24	20	16

Extensive form

$$\min 150\,x_1 + 230\,x_2 + 260\,x_3$$
$$+\frac{1}{3}\left(238\,y_1^1 + 210\,y_2^1 - 170\,w_1^1 - 150\,w_2^1 - 36\,w_3^1 - 10\,w_4^1\right)$$
$$+\frac{1}{3}\left(238\,y_1^2 + 210\,y_2^2 - 170\,w_1^2 - 150\,w_2^2 - 36\,w_3^2 - 10\,w_4^2\right)$$
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$s.t.$

$$x_1 + x_2 + x_3 \le 500$$

$$3x_1 + y_1^1 - w_1^1 \ge 200 \qquad w_3^1 + w_4^1 \le 24x_3$$
$$2.5x_1 + y_1^2 - w_1^2 \ge 200 \qquad w_3^2 + w_4^2 \le 20x_3$$
$$2x_1 + y_1^3 - w_1^3 \ge 200 \qquad w_3^3 + w_4^3 \le 16x_3$$

$$3.6x_2 + y_2^1 - w_2^1 \ge 240 \qquad w_3^1 \le 6000$$
$$3x_2 + y_2^2 - w_2^2 \ge 240 \qquad w_3^2 \le 6000$$
$$2.4x_2 + y_2^3 - w_2^3 \ge 240 \qquad w_3^3 \le 6000$$

$$x_1, x_2, x_3, y_1^i, y_2^i, w_1^i, w_2^i, w_3^i, w_4^i \ge 0 \text{ for all } i$$

Optimal solution

		Wheat	Corn	Sugar Beets
First Stage	Area (acres)	170	80	250
$s=1$ Above	Yield (T)	510	288	6000
	Sales (T)	310	48	6000 (favor. price)
	Purchase (T)	–	–	–
$s=2$ Average	Yield (T)	425	240	5000
	Sales (T)	225	–	5000 (favor. price)
	Purchase (T)	–	–	–
$s=3$ Below	Yield (T)	340	192	4000
	Sales (T)	140	–	4000 (favor. price)
	Purchase (T)	–	48	–
Overall profit: $108,390				

Stochastic Linear Program (General Formulation)

- $x = [x_1, x_2, \dots, x_n]^\top, x \in \mathbb{R}_+^n$, a vector of here-and-now decision variables
- $c = [c_1, c_2, \dots, c_n]^\top, c \in \mathbb{R}^n$, the unit cost vector associated to these variables
- $A \in \mathbb{R}^{m_h \times n}$, the matrix gathering the coefficients of the m_h linear equality constraints for the n here-and-now decision variables
- $b \in \mathbb{R}^{m_h}$, the corresponding right-hand side.

A Stochastic Linear Program (SLP) is an optimization problem, for here-and-now decisions x and an uncertainty ω, of the form

$$\begin{aligned} \min \quad & c^\top x + \mathbb{E}_\omega[Q(x,\omega)] \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $\mathbb{E}_\omega[Q(x,\omega)]$ is the expected recourse cost over ω.

Second-stage value function

The second-stage value function $Q(x, \omega)$ is actually not a function of y, since y can be deduced from x and ω.

- $y = [y_1, y_2, \dots, y_l]^\top, y \in \mathbb{R}_+^l$, a vector of recourse variables
- $q(\omega) = [q_1(\omega), q_2(\omega), \dots, q_l(\omega)]^\top, q(\omega) \in \mathbb{R}^l$, the unit cost vector of these variables
- $T(\omega) \in \mathbb{R}^{m_w \times n}, W(\omega) \in \mathbb{R}^{m_w \times l}$, the matrices gathering the coefficients of the m_w linear equality constraints linking all the variables
- $h(\omega) \in \mathbb{R}^{m_w}$, the corresponding right-hand side.

The second-stage value function is defined as another minimization problem

$$Q(x, \omega) = \begin{array}{ll} min & q(\omega)^\top y \\ s.t. & T(\omega)x + W(\omega)y = h(\omega) \\ & y \geq 0 \end{array}$$

Complete formulation

The SLP in its complete formulation, here-and-now decisions x, recourse decisions y and uncertainty ω.

$$\min_{x} c^\top x + \mathbb{E}_\omega[\min_{y} q(\omega)^\top y]$$

$$\begin{aligned} s.t. \quad & Ax = b \\ & T(\omega)x + W(\omega)y = h(\omega) \\ & x, y \geq 0 \end{aligned}$$

Complete formulation

The SLP in its complete formulation, here-and-now decisions x, recourse decisions y and uncertainty ω.

$$
\begin{aligned}
&\min_{x} c^\top x + \mathbb{E}_\omega[\min_{y} q(\omega)^\top y] \\
&s.t. \quad Ax = b \\
&\qquad\quad T(\omega)x + Wy = h(\omega) \\
&\qquad\quad x, y \geq 0
\end{aligned}
$$

In this course we will only focus on the special case where W is not a function of ω.
This case is called **Stochastic Linear Program with fixed recourse**.

Deterministic equivalent

Let p_k be the probability that scenario k is realised. Sample Average Approximation (SAA) can be used for continuous distributions.

$$\begin{aligned} \min_{x}\ & c^\top x + \mathbb{E}_\omega[\min_{y} q(\omega)^\top y] \\ s.t.\ & Ax = b \\ & T(\omega)x + Wy = h(\omega) \\ & x, y \geq 0 \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} \min_{x}\ & c^\top x + \min_{y} \sum_{k=1}^{K} p_k q_k^\top y_k \\ s.t.\ & \\ & Ax = b \\ & T_k x + W_k y_k = h_k && \forall\ k = 1, \dots, K \\ & x \geq 0 \\ & y_k \geq 0 && \forall\ k = 1, \dots, K \end{aligned}$$

Deterministic equivalent

Let p_k be the probability that scenario k is realised. Sample Average Approximation (SAA) can be used for continuous distributions.

$$
\begin{aligned}
\min_{x}\ & c^\top x + \mathbb{E}_\omega[\min_{y} q(\omega)^\top y] \\
s.t.\ \ & Ax = b \\
& T(\omega)x + Wy = h(\omega) \\
& x, y \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{x,y}\ & c^\top x + \sum_{k=1}^{K} p_k q_k^\top y_k \\
s.t.\ \ & \\
& Ax = b \\
& T_k x + W_k y_k = h_k && \forall\ k = 1, \dots, K \\
& x \geq 0 \\
& y_k \geq 0 && \forall\ k = 1, \dots, K
\end{aligned}
$$

Deterministic equivalent

$$
\begin{aligned}
\min_{x,y}\ & c^{\top}x + \sum_{k=1}^{K} p_k q_k^{\top} y_k \\
s.t.\ & \\
& Ax = b \\
& T_k x + W_k y_k = h_k && \forall\, k = 1, \dots, K \\
& x \ \geq 0 \\
& y_k \geq 0 && \forall\, k = 1, \dots, K
\end{aligned}
$$

$$
\begin{aligned}
\min\ & 150\, x_1 + 230\, x_2 + 260\, x_3 \\
& + \frac{1}{3}\left(238\, y_1^1 + 210\, y_2^1 - 170\, w_1^1 - 150\, w_2^1 - 36\, w_3^1 - 10\, w_4^1\right) \\
& + \frac{1}{3}\left(238\, y_1^2 + 210\, y_2^2 - 170\, w_1^2 - 150\, w_2^2 - 36\, w_3^2 - 10\, w_4^2\right) \\
& + \frac{1}{3}\left(238\, y_1^3 + 210\, y_2^3 - 170\, w_1^3 - 150\, w_2^3 - 36\, w_3^3 - 10\, w_4^3\right)
\end{aligned}
$$

$s.t.$

$$x_1 + x_2 + x_3 \leq 500$$

$$
\begin{aligned}
3x_1 + y_1^1 - w_1^1 &\geq 200 \qquad & w_3^1 + w_4^1 &\leq 24x_3 \\
2.5x_1 + y_1^2 - w_1^2 &\geq 200 & w_3^2 + w_4^2 &\leq 20x_3 \\
2x_1 + y_1^3 - w_1^3 &\geq 200 & w_3^3 + w_4^3 &\leq 16x_3 \\
3.6x_2 + y_2^1 - w_2^1 &\geq 240 & w_3^1 &\leq 6000 \\
3x_2 + y_2^2 - w_2^2 &\geq 240 & w_3^2 &\leq 6000 \\
2.4x_2 + y_2^3 - w_2^3 &\geq 240 & w_3^3 &\leq 6000
\end{aligned}
$$

$$x_1, x_2, x_3, y_1^i, y_2^i, w_1^i, w_2^i, w_3^i, w_4^i \geq 0 \text{ for all } i$$

Deterministic equivalent

$$\min_{x,y} c^\top x + \sum_{k=1}^{K} p_k q_k^\top y_k$$

$s.t.$

$$Ax = b$$
$$T_k x + W_k y_k = h_k \qquad \forall\, k = 1, \dots, K$$
$$x \ge 0$$
$$y_k \ge 0 \qquad \forall\, k = 1, \dots, K$$

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3$$
$$+\frac{1}{3}\left(238\, y_1^1 + 210\, y_2^1 - 170\, w_1^1 - 150\, w_2^1 - 36\, w_3^1 - 10\, w_4^1\right)$$
$$+\frac{1}{3}\left(238\, y_1^2 + 210\, y_2^2 - 170\, w_1^2 - 150\, w_2^2 - 36\, w_3^2 - 10\, w_4^2\right)$$
$$+\frac{1}{3}\left(238\, y_1^3 + 210\, y_2^3 - 170\, w_1^3 - 150\, w_2^3 - 36\, w_3^3 - 10\, w_4^3\right)$$

$s.t.$

$$x_1 + x_2 + x_3 \le 500$$

$$3x_1 + y_1^1 - w_1^1 \ge 200$$
$$2.5x_1 + y_1^2 - w_1^2 \ge 200$$
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$$3.6x_2 + y_2^1 - w_2^1 \ge 240$$
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$$w_3^3 + w_4^3 \le 16x_3$$

$$w_3^1 \le 6000$$
$$w_3^2 \le 6000$$
$$w_3^3 \le 6000$$

$$x_1, x_2, x_3, y_1^i, y_2^i, w_1^i, w_2^i, w_3^i, w_4^i \ge 0 \quad \text{for all } i$$

$$\min_{x,y} \; c^\top x + \sum_{k=1}^{K} p_k q_k^\top y_k$$

$$s.t.$$

$$Ax = b$$

$$T_k x + W_k y_k = h_k \qquad \forall\, k = 1,\dots,K$$

$$x \geq 0$$

$$y_k \geq 0 \qquad \forall\, k = 1,\dots,K$$

$$c = \begin{bmatrix} 150 \\ 230 \\ 260 \end{bmatrix} \qquad p = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \qquad q_k = \begin{bmatrix} 238 \\ 210 \\ -170 \\ -150 \\ -36 \\ -10 \end{bmatrix} \; \forall\, k \in 1..3$$

$$\min 150\, x_1 + 230\, x_2 + 260\, x_3$$
$$+\frac{1}{3}\left(238\, y_1^1 + 210\, y_2^1 - 170\, w_1^1 - 150\, w_2^1 - 36\, w_3^1 - 10\, w_4^1\right)$$
$$+\frac{1}{3}\left(238\, y_1^2 + 210\, y_2^2 - 170\, w_1^2 - 150\, w_2^2 - 36\, w_3^2 - 10\, w_4^2\right)$$
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$$s.t.$$

$$x_1 + x_2 + x_3 \leq 500$$

$$3x_1 + y_1^1 - w_1^1 \geq 200$$
$$2.5x_1 + y_1^2 - w_1^2 \geq 200$$
$$2x_1 + y_1^3 - w_1^3 \geq 200$$

$$3.6x_2 + y_2^1 - w_2^1 \geq 240$$
$$3x_2 + y_2^2 - w_2^2 \geq 240$$
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$$w_3^1 + w_4^1 \leq 24x_3$$
$$w_3^2 + w_4^2 \leq 20x_3$$
$$w_3^3 + w_4^3 \leq 16x_3$$

$$w_3^1 \leq 6000$$
$$w_3^2 \leq 6000$$
$$w_3^3 \leq 6000$$

$x_1, x_2, x_3, y_1^i, y_2^i, w_1^i, w_2^i, w_3^i, w_4^i \geq 0$ for all i

Wrapping up