Random Walk

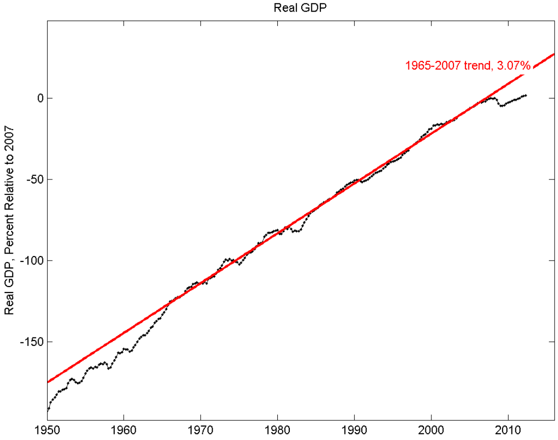
Dr. Natalia Khorunzhina

CBS

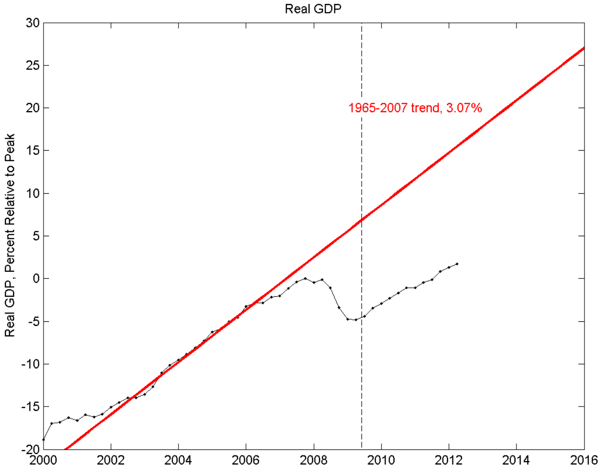
Deterministic or Stochastic Trend?

- Recall, that AR, MA and ARMA model require stationarity.
- Data with a trend violate stationarity, because the mean of the series is not constant over time, but develops (grows or declines) along the trend
- Thus far, we solved it by assuming *deterministic* trend.
- But what if the trend is *stochastic*?

Real GDP and its trend with a solid historical pattern OR The trend is your friend until the bend at the end



GDP declined 5% in the recession, and started growing 2.4% since then



AR(1) Process

- The first-order autoregressive process, AR(1) is

$$y_t = \beta y_{t-1} + e_t$$

where e_t is WN(0, σ^2)

Variance of AR(1)

- Take variance of both sides of

$$y_t = \beta y_{t-1} + e_t$$

- Thus

$$\begin{aligned} \operatorname{var}(y_t) &= \operatorname{var}(\beta y_{t-1} + e_t) \\ &= \operatorname{var}(\beta y_{t-1}) + \operatorname{var}(e_t) \\ &= \beta^2 \operatorname{var}(y_{t-1}) + \sigma^2 \end{aligned}$$

- If y is variance stationary, we solve and find

$$\operatorname{var}(y_t) = \operatorname{var}(y_{t-1}) = \frac{\sigma^2}{1-\beta^2}$$

$|\beta| < 1$

- We calculated that

$$\mathrm{var}(y_t) = \beta^2 \,\mathrm{var}(y_{t-1}) + \sigma^2$$

- When $|\beta|=1$, then

$$\mathrm{var}(y_t) = \mathrm{var}(y_{t-1}) + \sigma^2 > \mathrm{var}(y_{t-1})$$

  so the variance is increasing with t

- $|\beta|=1$ is inconsistent with variance stationarity.
- $|\beta|<1$ is necessary for stationarity.

Random Walk

- An AR(1) with β=1 is known as a random walk or unit root process

$$y_t = y_{t-1} + e_t$$

- By back-substitution

$$y_t = y_0 + \sum_{i=0}^{t} e_{t-i}$$

- The past never disappears. Shocks have permanent effects

Random Walk

The autoregressive process with a unit coefficient:

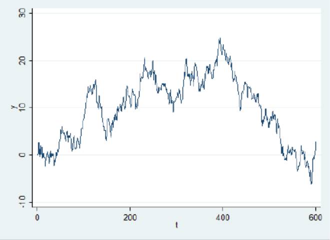
$$y_t = y_{t-1} + \varepsilon_t$$

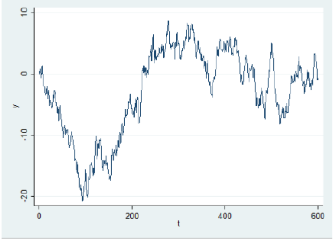
where $\varepsilon_t \sim WN(0, \sigma^2)$.

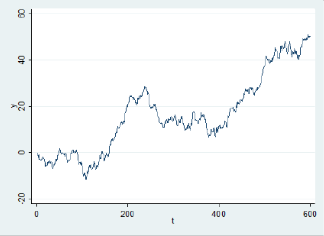
Imagine the random-walk model to be a drunk man, who walks back and forth randomly either left or right.

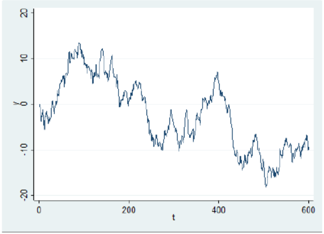
Imagine a particle that can move only in one dimension. It moves one step in a randomly determined direction per unit of time with the moves symmetrically distributed around 0 and with a variance of σ^2. The direction of these moves is independent of other moves, but the move starts from the previous location of the particle.

More Example on Random Walk









Random Walk

Unlike the stationary processes with deterministic trend, random walk does not have a particular level to which it returns.

If a shock lowers the value of a random walk, we expect it to stay permanently lower.

The trend simply begins anew from the series' new location.

Thus shocks to random walks have completely permanent effect: a unit shock forever moves the expected future path of the series by one unit.

Random Walk

Random Walk Models and Finance:
The random walk hypothesis is a financial theory stating that stock market prices evolve according to a random walk and thus cannot be predicted. It is consistent with the *efficient-market hypothesis*.

Random walk models are used in economic and business analysis:

- modeling GDP
- Personal Consumption expenditures
- Labor supply

Random Walk with Drift

Consider random walk with a trend:

$$y_t = \delta + y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

In unit root models, trend is called a **drift.**

Every period y_t grows by the drift δ.

Random walk is a model of **stochastic trend.**

Random Walk with Drift

Examples

$$y_t = 0.1 + y_{t-1} + e_t$$
$$e_t \sim N(0,1)$$

Random Walk Properties

The random walk is:

$$y_t = y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

By substitution:

$$\begin{aligned} y_t &= y_{t-1} + \varepsilon_t \\ &= y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= \ldots \\ &= y_0 + \sum_{i=1}^{t} \varepsilon_i \end{aligned}$$

Random Walk Properties

Assuming the process started at time 0 from some constant value y_0, we have:

$$\begin{aligned} E(y_t) &= E(y_0) + E(\sum_{i=1}^{t} \varepsilon_i) \\ &= E(y_0) + E(\varepsilon_1 + ... + \varepsilon_{t-1} + \varepsilon_t) \\ &= E(y_0) + tE(\varepsilon) \\ &= E(y_0) \end{aligned}$$

and

$$\begin{aligned} var(y_t) &= var(y_0) + var(\sum_{i=1}^{t} \varepsilon_i) \\ &= 0 + var(\varepsilon_1 + ... + \varepsilon_{t-1} + \varepsilon_t) \\ &= 0 + tvar(\varepsilon) \\ &= t\sigma^2 \end{aligned}$$

Random Walk with Drift

The random walk with a drift is:

$$y_t = \delta + y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

By substitution:

$$\begin{aligned} y_t &= \delta + y_{t-1} + \varepsilon_t \\ &= 2\delta + y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= 3\delta + y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ &= \ldots \\ &= t\delta + y_0 + \sum_{i=1}^{t} \varepsilon_i \end{aligned}$$

Random Walk Properties

Assuming the process started at time 0 from some constant value y_0, we have:

$$\begin{aligned} E(y_t) &= E(t\delta) + E(y_0) + E(\sum_{i=1}^{t} \varepsilon_i) \\ &= t\delta + E(y_0) + E(\varepsilon_1 + ... + \varepsilon_{t-1} + \varepsilon_t) \\ &= t\delta + E(y_0) + tE(\varepsilon) \\ &= t\delta + E(y_0) \end{aligned}$$

and

$$\begin{aligned} var(y_t) &= var(t\delta) + var(y_0) + var(\sum_{i=1}^{t} \varepsilon_i) \\ &= 0 + 0 + var(\varepsilon_1 + ... + \varepsilon_{t-1} + \varepsilon_t) \\ &= 0 + 0 + tvar(\varepsilon) \\ &= t\sigma^2 \end{aligned}$$

Recognizing Random Walk: ACF and PACF

If a series is random walk, its autocorrelation function is not well defined because its variance is infinite.

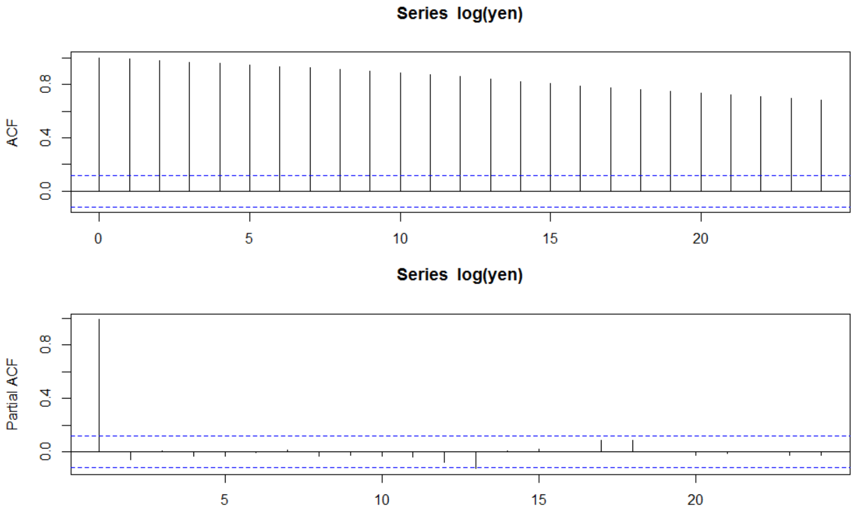
But the sample autocorrelation function can still be mechanically computed in the usual way.

One evidence on the random walk is that the sample ACF will tend to damp extremely slow, or fail to damp at all.

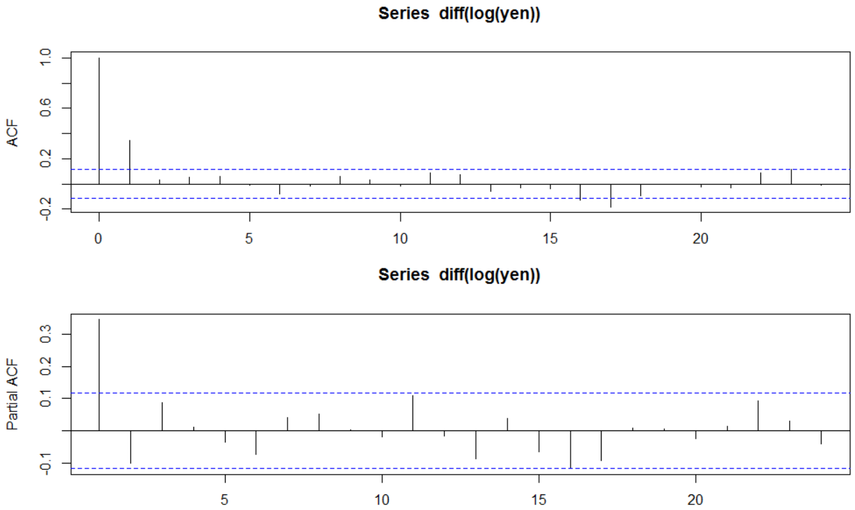
In contrast, the sample PACF on random walk process will damp quickly: it will be close to 1 at lag 1, but will be small quickly thereafter.

However, first differences of random walk produce a white noise ($y_t - y_{t-1} = \Delta y_t = \varepsilon_t$), because, by assumption $\varepsilon_t \sim WN(0, \sigma^2)$. Therefore we expect to see no significant ACF and PACF on first differences.

Recognizing Random Walk: ACF and PACF



Random Walk and ACF and PACF, first differences



Formal Test for Unit Root

Start with the simple AR(1) process:

$$y_t = \beta y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

We can regress y_t on y_{t-1} and then use the standard t-test for testing $\beta = 1$.

But remember that t-statistic is for the null hypothesis that $\beta = 0$, while we want to test that $\beta = 1$.

Formal Test for Unit Root

We can trick the system by rewriting the regression as

$$\begin{aligned} y_t &= \beta y_{t-1} + \varepsilon_t \\ y_t - y_{t-1} &= \beta y_{t-1} - y_{t-1} + \varepsilon_t \\ y_t - y_{t-1} &= (\beta - 1) y_{t-1} + \varepsilon_t \\ \Delta y_t &= \varphi y_{t-1} + \varepsilon_t \end{aligned}$$

where $\varphi = \beta - 1$, and testing the null hypothesis that $\varphi = 0$.

The test statistic follows the Dickey-Fuller Distribution (not the normal t-distribution).

Testing for Unit Roots

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################
Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.1853873  0.0681252   2.721  0.00692 **
z.lag.1     -0.0320276  0.0116882  -2.740  0.00655 **
tt          -0.0001432  0.0000533  -2.686  0.00768 **
z.diff.lag1  0.3885315  0.0594140   6.539 3.03e-10 ***
z.diff.lag2 -0.1072465  0.0632457  -1.696  0.09108 .
z.diff.lag3  0.1066671  0.0590731   1.806  0.07207 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.02596 on 273 degrees of freedom
Multiple R-squared: 0.1555, Adjusted R-squared:  0.14
F-statistic: 10.05 on 5 and 273 DF,  p-value: 7.566e-09
Value of test-statistic is: -2.7402 3.1295 3.8205

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.98 -3.42 -3.13
phi2  6.15  4.71  4.05
phi3  8.34  6.30  5.36
```

We test the null hypothesis that there is unit root in z. We do NOT reject that there is unit root if |test statistic|<|critical value|. Here, we do not reject that there is unit root process because test statistic in abs.values is smaller than abs. critical values at all levels of significance

Estimation of Random Walk Process

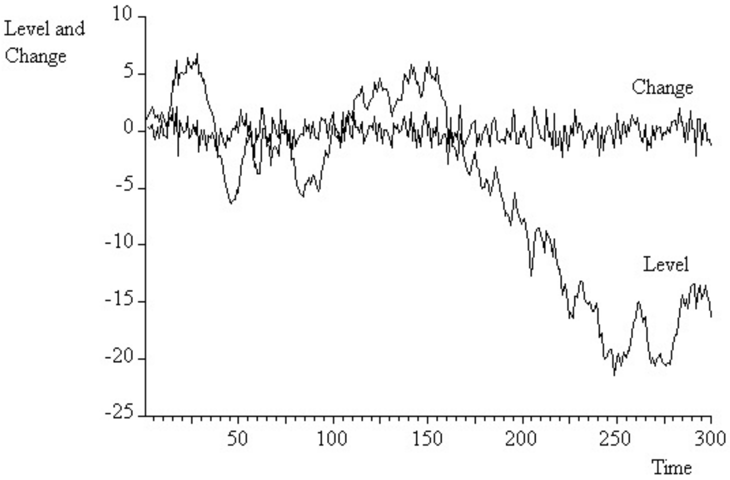
Random walk is somewhat ill behaved

But its difference is the ultimate well-behaved series: zero-mean white noise:

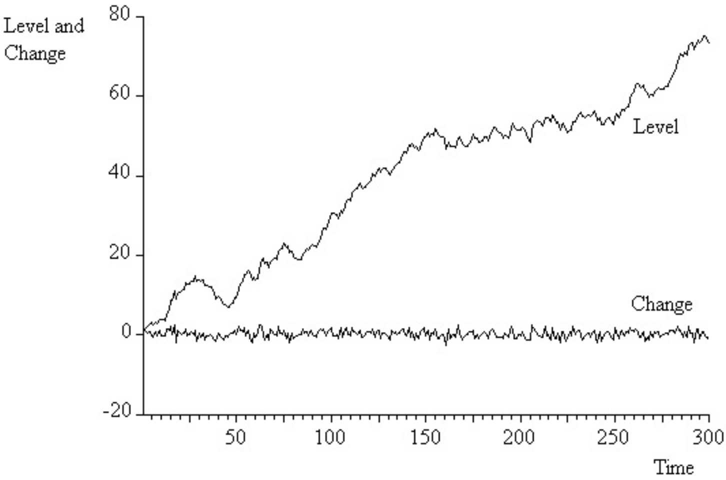
$$y_t - y_{t-1} = \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

Random Walk: Level and Change



Random Walk With Drift: Level and Change



Estimation of Random Walk

$$y_t = y_{t-1} + \varepsilon_t$$

Under unit root process the series y_t are not covariance stationary.

However, the differences in y_t and y_{t-1}:

$$y_t - y_{t-1} = \Delta y_t = \varepsilon_t$$

are covariance stationary, because, by assumption $\varepsilon_t \sim WN(0, \sigma^2)$.

We say that a nonstationary series is integrated if its nonstationarity is appropriately ”undone” by differencing.

Estimation of Random Walk

If only one difference is required, we say the series is integrated of order 1, or $I(1)$ for short.

More generally, if d differences are required, the series is $I(d)$.

Once the series is differenced, it is more likely to become stationary, but we still have to check that. We can apply as many differences as needed to reach stationarity in the differenced data

We can apply our standard tools to analyze the cycles with the help of AR, MA or ARMA processes.

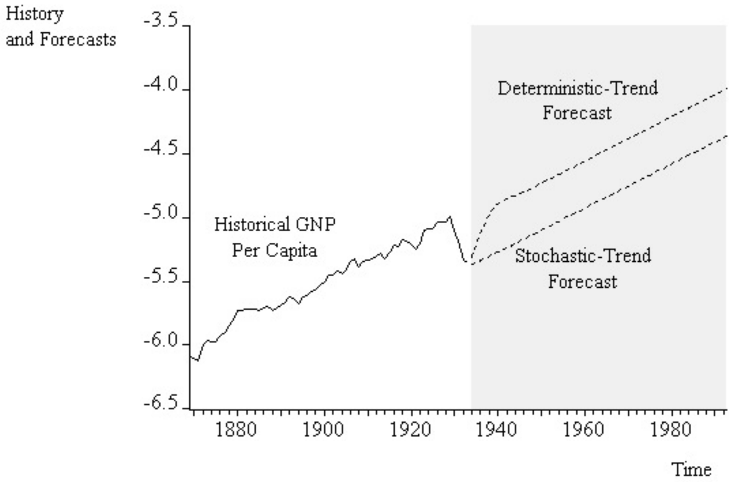
ARIMA(p,d,q)

We use **ARIMA(p,d,q)** models for estimation of random walk processes.

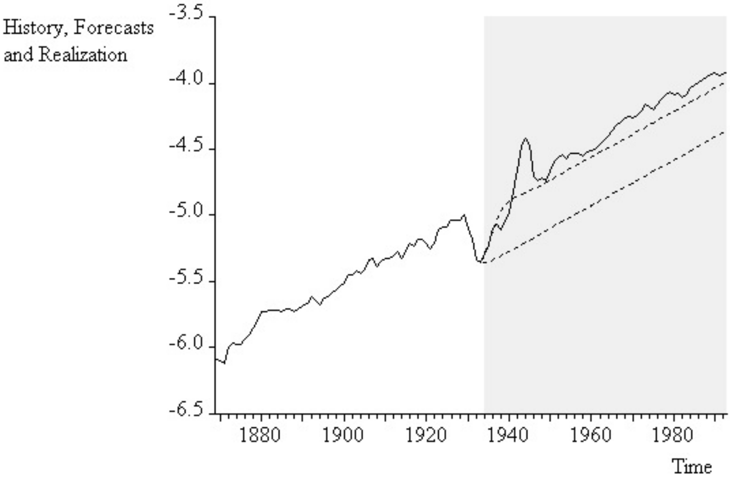
ARIMA stands for AutoRegressive *Integrated* Moving Average model.

The ARIMA(p,1,q) process is just a stationary ARMA(p,q) process in first differences.

Forecast Example: U.S. Per Capita GNP, History and Two Forecasts



U.S. Per Capita GNP, History and Two Forecasts



Forecasting: Important Implications of ARIMA(p,1,q) models

Shocks to ARIMA(p,1,q) process have permanent effects.

The implications for forecasting is that due to shock persistence the optimal forecast does not revert to a trend (or a mean).

The variance of an ARIMA(p,1,q) process grows without bound as time progresses.

Due to increasing variance the uncertainty associated with our forecasts increases without a bound as the forecast horizon grows.

We can make ARIMA(p,1,q) process stationary by taking first differences.

Forecasting: Important Implications of ARIMA(p,1,q) models

Consider a simple random walk model, which is AR(1) process with a unit coefficient:

$$y_t = \beta y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$. The optimal 1-period-ahead forecast is:

$$\hat{y}_{T+1} = \beta y_T$$

2-period-ahead forecast:

$$\begin{aligned} \hat{y}_{T+2} &= \beta \hat{y}_{T+1} \\ &= \beta(\beta y_T) \\ &= \beta^2 y_T \end{aligned}$$

h-period-ahead forecast:

$$\hat{y}_{T+h} = \beta^h y_T$$

Important Implications of ARIMA(p,1,q) models

h-period-ahead forecast:

$$\hat{y}_{T+h} = \beta^h y_T$$

If $\beta = 1$, which is the case for random walk case, the optimal forecast is simply the current value:

$$\hat{y}_{T+h} = y_T$$

regardless the horizon.

Any random shock that moves the series up or down today, also moves the optimal forecast up or down, at all horizons.

Forecast of ARIMA(p,1,q) process

The optimal point forecast is the last available value of the series:

$$\hat{y}_{T+h} = y_T$$

regardless the horizon.

The forecast intervals are constructed based on the variance of the forecast errors.

The forecast error variance is the sum of h error variances, $h\sigma^2$.

An h-step-ahead 95% interval forecast for any future horizon is:

$$y_T \pm 1.96\sqrt{h\sigma^2} = y_T \pm 1.96\sigma\sqrt{h}$$

Forecast Example