Uncertainty characterization and propagation

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42586 – Decisions under Uncertainty – 6 February 2024

Learning Objectives

At the end of this lecture (and related hands-on session), a student should be able to:

- Describe various types of **uncertainties**
- Understand **how uncertainty propagates** and its consequences
- Formulate and discuss **sample average approximation**
- Formulate and discuss the **fallacy of the averages**

Outline

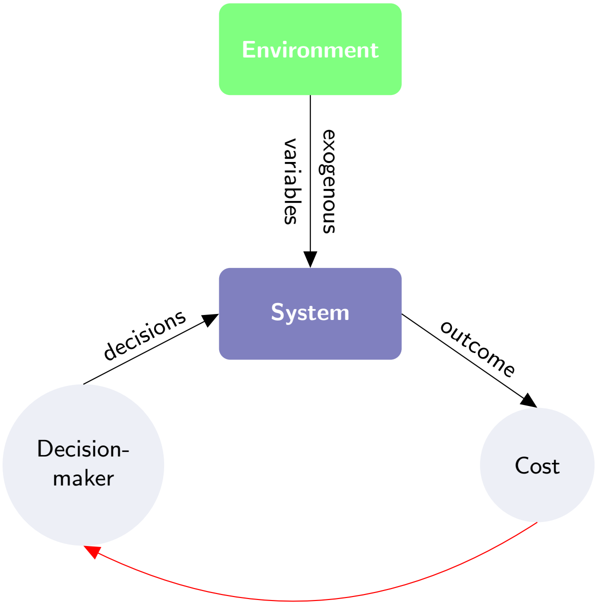
1. Uncertainty in our decision framework
2. Random variables
3. Uncertainty propagation and loss evaluation
4. Scenarios and the sample average approximation
5. The fallacy of the averages

1. **Uncertainty in our decision framework**

Remember our framework for decision-making

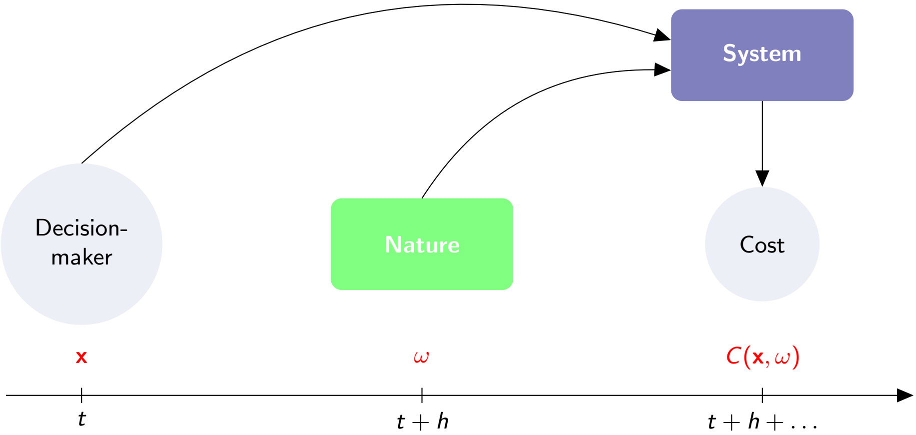
In any decision-making problem, we expect to have

- an **environment**, that imposes conditions seen as exogenous variables
- a **system** we can act upon
- a **decision-maker** who can act on the system
- potential **costs**, **restrictions** and **constraints** on decisions and the system
- an **outcome** that may yield a cost (or benefit) to the decision-maker



There is a temporal component too...

It may be seen as a game between the decision-maker and Nature.



Therefore, that also means that the decision-maker is aiming to forecast ω (or, at least, describe what *may* happen).

Uncertainties from Nature

There are so many different forms and sources of uncertainty to consider!

- weather and climate uncertainties, in e.g. construction, insurance, energy, etc.

- demand uncertainties, in e.g. logistics, supply chain, hospitals, energy, etc.

- price uncertainties, in e.g. real estate, commodity markets, energy, etc.

Uncertainties and deep uncertainties

There are known knowns; there are things we know we know.
We also know there are known unknowns; that is to say we know there are some things we do not know.
But there are also unknown unknowns - the ones we don't know we don't know.

Donald Rumsfeld

- Some uncertainties can be quantified and modelled, some cannot!
- Advanced decision-making under deep uncertainty is out of the scope of the course
- We will make the assumption that the uncertainties we deal with can be modelled

If interested in decision-making under deep uncertainty, first start by reading "The black Swan" by Nassim N. Taleb...

2. **Random variables**

Discrete and binary outcomes

One can readily model uncertainty ω with discrete and binary outcomes based on a **Bernoulli** random variable with probability p, $\omega \sim \mathrm{B}(p)$

- Two possible outcomes 0 and 1
- Their probability of occurrence are p for $\omega = 1$ and $(1 - p)$ for $\omega_{=}0$
- Consequently,

$$\mathbb{E}[\omega] = p$$
$$\mathrm{Var}[\omega] = p(1 - p)$$

- It can be generalized to more than 2 outcomes...
- **Potential use:** failure of machine, weather events, success in gamble, etc.

Discrete outcomes more generally

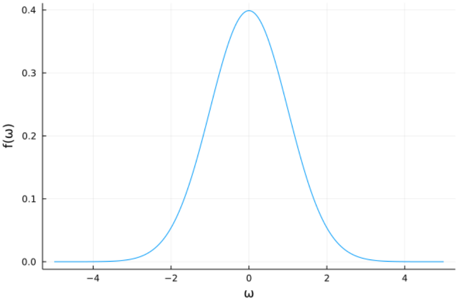
For discrete events with more than 2 potential outcomes, alternative approaches may be considered, i.e.

- **Binominal**, $\omega \sim \mathrm{B}(n, p)$ (outcome of n Bernoulli trials with probability p)
  - Probability mass function: $\mathrm{P}[\omega = k] = \binom{n}{k} p^k (1-p)^{n-k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  - Expectation: $\mathbb{E}[\omega] = np$
  - Variance: $\mathrm{Var}[\omega] = np(1-p)$
- **Poisson**, $\omega \sim \mathrm{Pois}(\lambda)$ (λ is referred to as intensity)
  - Probability mass function: $\mathrm{P}[\omega = k] = \frac{\lambda^k e^{-k}}{k!}$
  - Expectation: $\mathbb{E}[\omega] = \lambda$
  - Variance: $\mathrm{Var}[\omega] = \lambda$
- **Potential use:** arrival processes e.g. at hospitals, shops, etc.

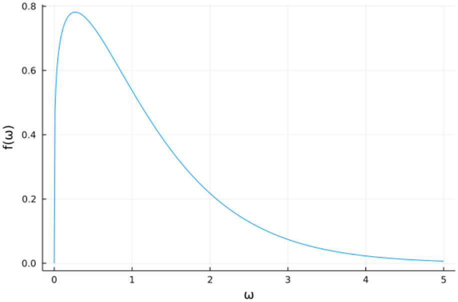
Continuous outcomes

There are so many possibilities to model continuous outcomes...

Gaussian, $\omega \sim \mathcal{N}(\mu, \sigma^2)$ – $\omega \in \mathbb{R}$



Weibull, $\omega \sim \text{Weibull}(\lambda, k)$ – $\omega \in \mathbb{R}^+$



etc.

Potential use:

- weather variables (e.g., wind speed, precipitation and temperature),
- market prices
- demand for a product or service, etc.

Continuous distributions - Uniform

Uniform random variables represent processes where X is equally likely to be "near" any point in that interval

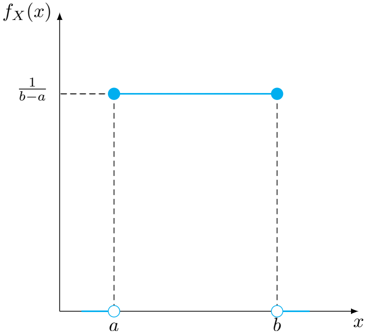
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} a < x < b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In general:

$$E[X] = \frac{b+a}{2}$$

$$Var[X] = \frac{1}{12}(b-a)^2$$

$$F(x) = P\{X \leq x\} = \int_a^b (b-a)^{-1} dx = \frac{x-a}{b-a}$$



pdf of uniform distribution

3. **Uncertainty propagation and loss evaluation**

- In most cases (and probably yours, before to take this course), the **decision-maker wants to work in a deterministic framework**.
- Instead of accepting the uncertainty in ω, a single value $\hat{\omega}$ is used (e.g., a forecast of the potential realization of ω)

Definition (Deterministic decision-making)

Based on an estimate $\hat{\omega}$ for ω, the deterministic decision-maker solves the following minimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & C(\mathbf{x}, \hat{\omega}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_{\mathbf{x}} \end{aligned}$$

with $\mathbf{x}$ the vector of decisions to be made, and $C(\mathbf{x}, \omega)$ the cost function.

What do you think happens in practice?

Deterministic decision-making

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What do you think happens in practice?

The solution $\mathbf{x}^*$ is only optimal if $\hat{\omega}$ realizes, which is quite unlikely...!
(Unless ω truly is deterministic)

Example: When to leave for the airport?

13:18 — 190 — E — M2 — 14:30

1 t 12 min, 2 skift, 45,00 kr. med Rejsekort

⚠ **Stoppestedet DTU, Akademivej for linje 190 mod Holte er flyttet.**

Pris & køb | **Detaljer**

13:19 — 150S — M2 — 14:10

51 min, 1 skift, 38,50 kr. med Rejsekort

Pris & køb | **Detaljer**

13:29 — 150S — M2 — 14:22

53 min, 1 skift, 38,50 kr. med Rejsekort

Pris & køb | **Detaljer**

13:39 — 150S — M2 — 14:30

51 min, 1 skift, 38,50 kr. med Rejsekort

Pris & køb | **Detaljer**

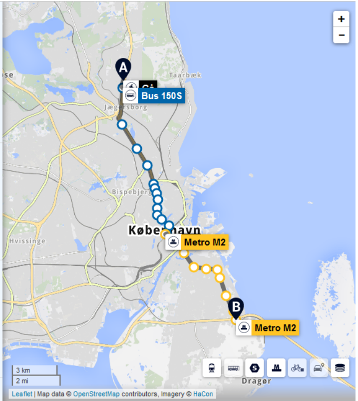
13:47 — 190 — A — M2 — 14:42

55 min, 2 skift, 45,00 kr. med Rejsekort

⚠ **Linje 190 mod Lyngby St. betjener ikke stoppestedet Sti til Bauneporten**

⚠ **Stoppestedet DTU, Akademivej for linje 190 mod Holte er flyttet.**

Pris & køb | **Detaljer**



Example: When to buy Christmas presents (as a father)?

Setup:

- A (overly rational) father is tasked to buy Christmas presents for his kids
- He wants to buy them "just on time"!
- The website says that expected delivery time is $\hat{\omega} = 8$ days...
- We are on December 14th

He then solves the following **optimization problem**:

$$\begin{aligned} \min_x \quad & C(x, \hat{\omega}) \\ \text{s.t.} \quad & x \in 14, \cdots, 23 \end{aligned}$$

where

- The cost function is such that

$$C(x, \omega) = \begin{cases} \varepsilon(d - (x + \omega)), & d - (x + \omega) \geq 0 \quad (\varepsilon \text{ small, e.g., } \varepsilon = 1) \\ \gamma((x + \omega) - d), & d - (x + \omega) < 0 \quad (\gamma \text{ large, e.g., } \gamma = 4) \end{cases}$$

- d is the 23rd, so as to get the presents just in time

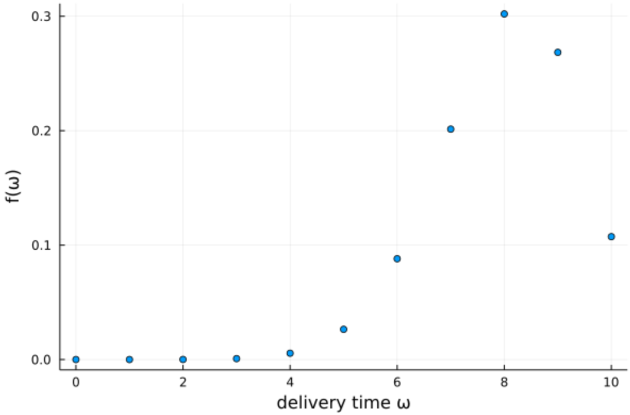
The **solution** is easy to find: the father buys presents on $x^* = 15$ December (whatever the values of ε and γ).

Consequences of neglected uncertainty

When the website says that the expected delivery time is $\hat{\omega} = 8$ days, it is really an *expected value* in the mathematical sense of the term!

The delivery time actually follows a Binomial distribution

$\omega \sim B(10, 0.8)$

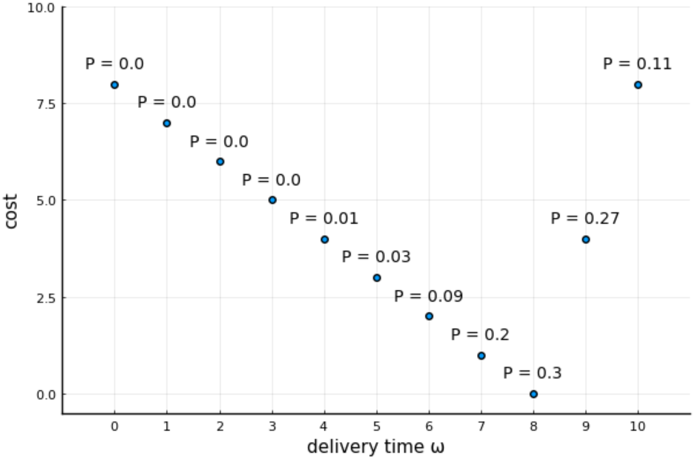


Bad luck: On average, the delivery time is 8 days, but

- there is only a 30% chance the delivery time *actually* is 8 days, and
- there is 39% chance it is more than 8 days... and the presents will be late!

Cost of neglected uncertainty

Since we know the distribution for ω and the cost parameters ($\varepsilon = 1$ and $\gamma = 4$), we can assess the cost of the delivery time **is not** 8 days (assuming we order on December 15th)



This plot shows the costs if different delivery times materialize, as well as the related probability of occurrence

How to interpret it?

4. **Scenarios and sample average approximation**

The sample average approximation

In the general case (EUM framework), we want to solve a **stochastic optimization problem** of the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbb{E}_\omega[C(\mathbf{x}, \omega)] \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_{\mathbf{x}} \end{aligned}$$

with $\mathbf{x}$ the vector of decisions to be made, and ω the uncertain parameter.

However, it is most often not possible to have an *analytical formulation* for that expectation, which could be "workable" in an optimization framework. It can be approximated though...

Definition (Sample Average Approximation (SAA))

In an EUM framework, the sample average approximation of the stochastic optimization problem at hand is

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{k=1}^{K} p_k C(\mathbf{x}, \omega_k) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_{\mathbf{x}} \end{aligned}$$

where K is the number of samples considered, and p_k their associated probability of occurence, $0 \leq p_k \leq 1$, $\sum_k p_k = 1$.

And it is a **deterministic optimization problem**!

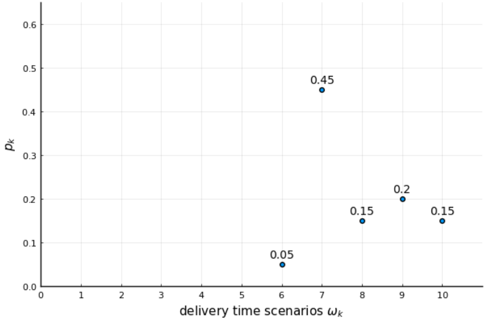
Scenarios

Thanks to sample average approximation (SAA), our focus is now to have scenarios ω_k $(k = 1, \ldots, K)$ for our uncertainty, and their associated probabilities p_k.

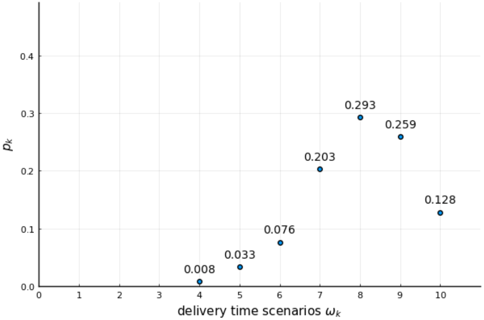
- they may be provided to you by a forecaster
- they may be readily obtained by sampling from a proposal distribution (i.e., your own model)

Sampling for the Christmas gift example (assuming we make a correct choice for the distribution):

$K = 20$



$K = 1000$



Solving the Christmas gift problem based on SAA

- Let us use the scenarios sampled before, and the SAA, to solve the actual stochastic optimization problem, i.e.

$$\begin{aligned} \min_x \quad & \mathbb{E}_\omega[C(x,\omega)] \\ \text{s.t.} \quad & x \in \mathcal{A}_x \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \min_x \quad & \sum_{k=1}^{K} p_k C(x,\omega_k) \\ \text{s.t.} \quad & x \in \mathcal{A}_x \end{aligned}$$

  with the details of the problem (cost function and constraints) defined previously.

- For the sake of example, we use $\varepsilon = 1$ and $\gamma = 4$.

Depending on the sampling, the SAA version of the problem is sightly different:

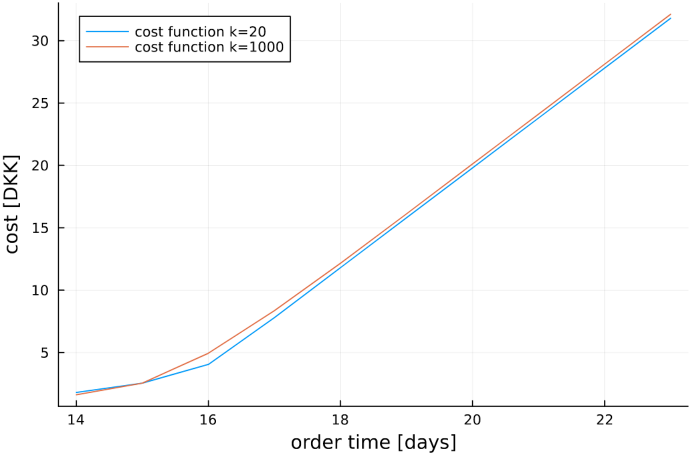
- for $K = 20$ one has

$$0.05C(x,6) + 0.45C(x,7) + 0.15C(x,8) + 0.2C(x,9) + 0.15C(x,10)$$

- for $K = 1000$, the cost function is

$$0.008C(x,4) + 0.033C(x,5) + 0.076C(x,6) + 0.203C(x,7) + 0.293C(x,8) + 0.259C(x,9) + 0.128C(x,10)$$

Scenarios



In both cases, the optimal solution is $x^* = 14$ December (so, a bit earlier...)

5. **The fallacy of the averages**

What is the fallacy of averages?

This is the whole reason why we need to use stochastic optimization for decision-making under uncertainty!

Definition (Fallacy of averages)

It is **not** equivalent to consider

(i) minimization of the expected cost over ω, and

(ii) minimization of the cost for the expected ω.

More formally,

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbb{E}_\omega[C(\mathbf{x},\omega)] \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_\mathbf{x} \end{aligned} \quad \neq \quad \begin{aligned} \min_{\mathbf{x}} \quad & C(\mathbf{x},\mathbb{E}[\omega]) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_\mathbf{x} \end{aligned}$$

For our Christmas gift problem:

- the problem solved by the father is that on the right (minimization of the cost for the expected ω)
- the problem we solved with SAA is that on the left (minimization of the expected cost over ω)
- as expected the two solutions are different ($x^* = 15$ days and $x^* = 14$ days, respectively)

Note that there are a few special cases, where these problems are equivalent...

Wrapping up

- There are many **alternative approaches to uncertainty modelling**
- In general, stochastic optimization problems can be **recast as an equivalent deterministic problem** (with SAA for instance)
- The key motivation for decision-making under uncertainty is the **fallacy of the averages**
- Now that we are done with basics, we will explore **more advanced topics!**

Thanks for your attention!