```julia
#using Pkg
#Pkg.add("Plots")
#Pkg.add("Distributions")
#Pkg.add("LaTeXStrings")
using Plots, LaTeXStrings, Distributions
```

1 Lecture note - The newsvendor problem

In this lecture, we will focus on one of the fundamental problems in stochastic optimization, the **newsvendor problem**.

Before jumping into the newsvendor problem, it is important to understand the concept of **hedging**. If you search online information about hedging, you will most likely find articles relating to the financial sector. However, hedging can be applied to many domains. We can think of hedging in the process of making a decision with the aim of minimizing potential losses stemming from uncertainty.

> ***Definition*** Hedging consists of making decisions in order to decrease (and hopefully control) potential losses.

Usually, we consider two types of hedging. The first is the one where you position yourself in terms of your decision to minimize potential losses due to uncertainty. The other is where you test an instrument or a setup to reduce your exposure to the potential costs due to uncertainty.

To clarify this concept, we will use an example that will illustrate these two types of hedging.

1.1 Hedging in a game of dice

You are playing a gambling game based on a single 6-sided dice. Before the game begins, you have to make a choice. You have to guess which side the dice will show, hence deciding on a number between 1 and 6. Each time you play, you earn 5 DKK plus 3 DKK times the number you have selected. The dice is then thrown. You will now have to pay back 10 DKK times the difference between the number you have chosen and the outcome of the dice.

If you want to maximize your earning expectations (hence minimizing your potential losses), **what number would you choose?**

Let's try to find a solution to this problem. The first thing to do is to choose a strategy (maximax, minimax, regret, expected utility maximization, etc.). This lecture will focus on Expected Utility Maximization (EUM).

Let us start by writing the **loss function**, which we are trying to minimize. Let x be the decision variable, and let ω be the random variable representing the realization of the outcome of the dice.

Since we are using a EUM strategy, the loss function is an expectation.

$$L(x,\omega) = \mathbb{E}\left[-5 - 3x + 10|x-\omega|\right]$$

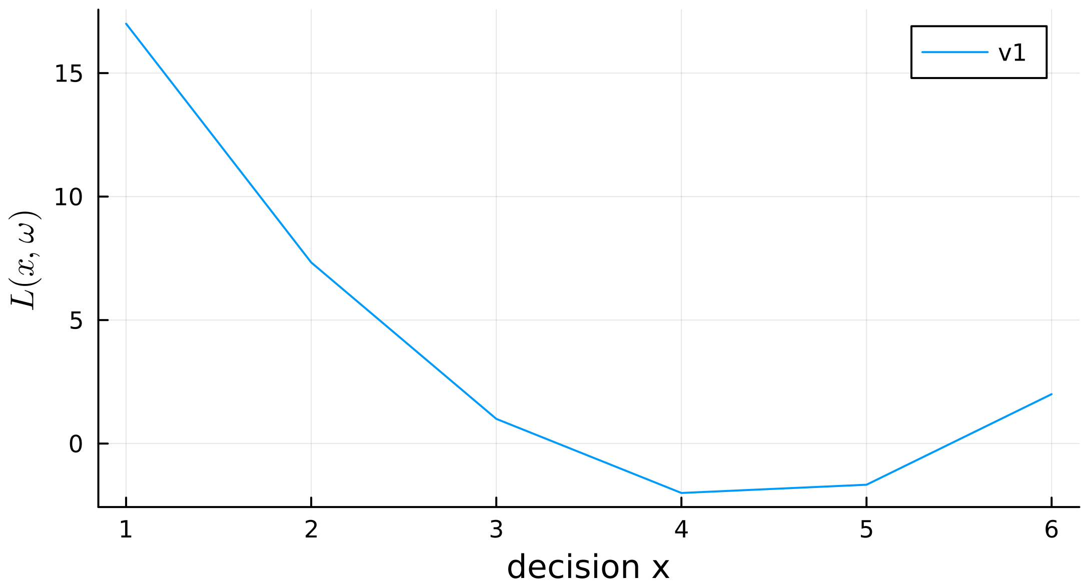
Remember, we are minimizing; hence the incomes will be negative and the costs positive. The first part of the expectation is the fixed income per game (5) and the income for the number we chose ($3x$). The second part of the function is the cost, meaning what we need to pay back if the dice does not match the number we chose ($10|x-\omega|$).

Since the first part of the function does not include any random variables, we can move it out of the expectation (since it is a constant). We also split the expectation into the two possible options: $\omega < x$ and $\omega > x$ since they have different probabilities:

$$L(x,\omega) = \mathbb{E}\left[-5 - 3x + 10|x-\omega|\right] = -5 - 3x + 10\left(P[\omega < x]\mathbb{E}_{\omega<x}[x-\omega] + P[\omega > x]\mathbb{E}_{\omega>x}[\omega - x]\right)$$

Let us now implement the loss function and plot its value for each choice of x.

```
function L(x,f,s,c)
    p1 = (x-1)/6 # The probability of the dice roll to be < x
    p2 = (6-x)/6 # The probability of the dice roll to be > x
    ω1 = collect(1:x-1) # The set of numbers less than x
    ω2 = collect(x+1:6) # The set of numbers greater than x
    if isempty(ω1) #special case in which x is 1
        return -f -s*x +c*(p2*mean(ω2.-x))
    elseif isempty(ω2)#special case in which x is 6
        return -f -s*x +c*(p1*mean(x.-ω1))
    else
        return -f -s*x +c*(p1*mean(x.-ω1) + p2*mean(ω2.-x))
    end
end
x = collect(1:6) # The decision space
y = [L(i,5,3,10) for i in x] # The value of the loss function
plot(x,y,xlabel="decision x",ylabel=L"L(x,\omega)",label="v1",
  ↪size=(550,300))
```



The graph shows us the expected loss of each value of the decision. We are interested in negative losses as those represent positive expected winnings. The graph shows that this can be achieved only by selecting 4 or 5, with 4 being the optimal choice. Notice that the cost associated with the risk of selecting the wrong number is quite high compared to the potential winnings. This is a game where you could potentially lose a lot of money.

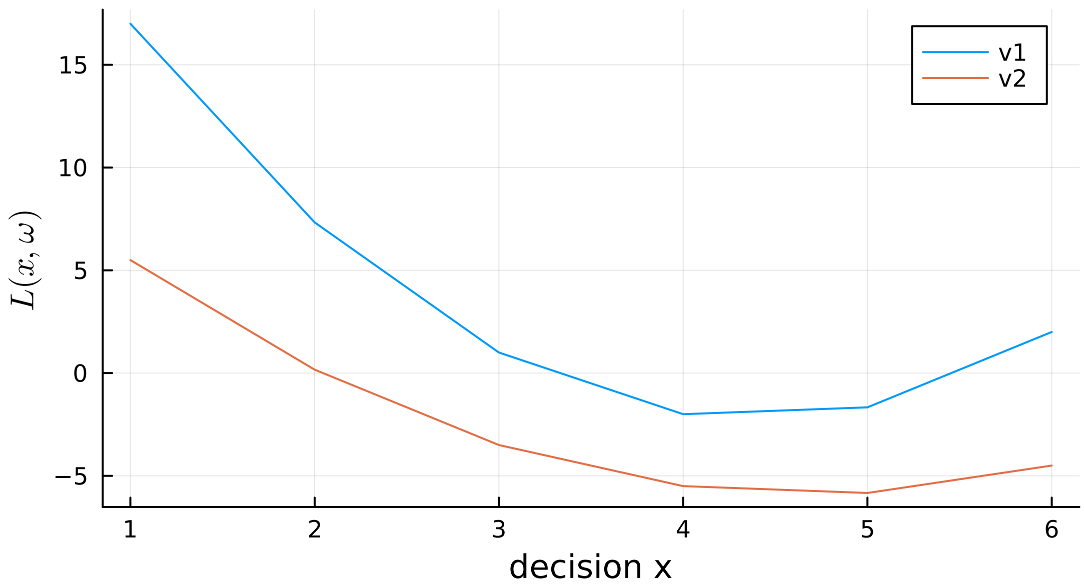
Let us now consider the same hedging game in a setting where we want to test an instrument to manage the risk. For our dice example, consider the possibility of buying an insurance policy. The policy reduces your potential losses from 10 DKK times the difference between your decision and the outcome of the dice to 5 DKK. To access this policy, you will have to pay by reducing your gain from 3 DKK times the number you choose to 2 DKK.

With the new insurance policy, our loss function will look as follows.

$$L(x,\omega) = \mathbb{E}\left[-5-2x+5|x-\omega|\right] = -5-2x+5\left(P[\omega<x]\mathbb{E}_{\omega<x}[x-\omega]+P[\omega>x]\mathbb{E}_{\omega>x}[\omega-x]\right)$$

Let's now plot the loss function and evaluate its effect.

```
plot(x,y,xlabel="decision x",ylabel=L"L(x,\omega)",label="v1", size=(550,300))
# The result of the loss function for the new reward values
y2 = [L(i,5,2,5) for i in x]
plot!(x,y2,label="v2")
```



The graph shows how the insurance policy manages the potential loss. We can now select more numbers for which we can expect a positive gain (with five being the optimal decision). Also, notice that the curve defined by the loss function is flatter than the original one, meaning that the cost coming from uncertainty is less. In conclusion, the insurance policy is a good investment.

1.2 The newsvendor problem

Let us move toward a more generalised version of hedging by discussing the newsvendor problem, one of the fundamental problems studied in stochastic optimisation. It takes its name from the early 19th newsvendors in the USA (as the one in the picture). Consider the decision problem those newsvendors had to make every day. Each morning they would have to buy several newspapers to be sold on the street. If the vendor does not buy enough newspapers, there will be a loss in terms of potential revenue. On the other hand, if too many newspapers were bought, the unsold newspaper would represent an economic loss as they were worth nothing the day after.

Now, if the newsvendor knew the exact demand, the optimal choice would be to buy as many newspapers as the demand. Unfortunately, demand is volatile; hence the newsvendor is faced with a difficult decision.

In this lecture, we look at the classical version of the newsvendor problem where:

- We focus on one-shot decisions. The decision we make now does not have an impact on the next decision.
- The outcome is uncertain.
- We know the marginal profit and loss
- The aim is to maximise expected utility

Before going into a specific example, let us formalise the definition of the problem.

Let ω be the uncertain parameter, and x be the decision we need to make before the realisation of ω. Let then π_u be the penalty for underestimating the uncertain parameter $x < \omega$ (called *underage* penalty), and π_o the penalty for over estimating $x > \omega$ (called *overage* penalty).

The expected utility maximisation (under cost minimisation form) would be:

$$\mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+]$$

> ***Definition*** In its generic form, the newsvendor problem is a one-period expected utility maximisation problem for an uncertain quantity ω, with known underage and overage penalties π_u and π_o, respectively. The optimal decision x^* to be made is the solution of
>
> $$\min_x \ \mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+]$$

Example 1: At the farmers market. Sarah owns a food truck selling a vegetarian dish of the day. She needs to prepare the food for sale at the farmers' market the next day. As her dish is mainly based on raw vegetables, anything she cannot sell will go to waste. Sarah needs to decide how much food to prepare. She knows that each food unit has a production cost of 30 DKK, and each portion of food is sold for 75 DKK.

Sarah keeps track of her sales, and based on what she has sold in previous markets, she estimates to sell an average of 150 portions of food with a standard deviation of 15.3 (the demand is normally distributed). *How much food should Sarah prepare?*

It is easy to see that the above problem can be modelled as a newsvendor problem, so let's see if we can help Sarah make the right decision. The decision variable x represents the food units that need to be prepared, and the stochastic parameter ω is the demand. Let $c = 30$ be the cost of a unit of food, and $p = 75$ be its sales price. What is missing now is to identify the overage and underage penalties.

Let's start by considering the underage case. Remember that the underage penalty is paid when $x < \omega$, hence when we prepare less food than the demand. In this case, Sarah is losing potential revenue. For each unit of food not sold, the underage penalty would be $\pi_u = p - c = 45$ (the marginal revenue loss). In the opposite case, where $\omega < x$, for each unsold unit of food, the production cost is lost and the overage penalty is $\pi_o = c = 30$.

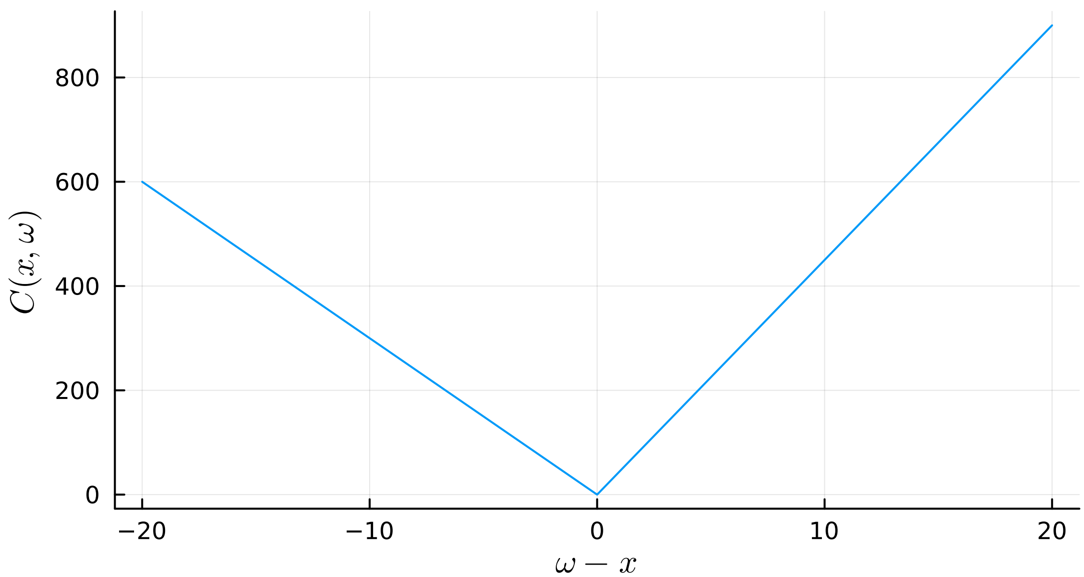
Given that we know (or have an estimation) of the demand distribution, all that is left is to solve

$$\min_x \ \mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+] = \min_x \ \mathbb{E}[45(\omega - x)_+ + 30(x - \omega)_+]$$

Before we look at how to solve it, let's look at the loss function.

```
# loss function where y represent the result of ω - x
loss_e1(y) = (y>=0) ? 45*y : 30*-y
ω_x = collect(-20:20)
y = [loss_e1(i) for i in ω_x]

plot(ω_x,y,xlabel=L"\omega -x",ylabel=L"C(x,\omega)",legend = false,␣
  ↪size=(550,300))
annotate!(10, 550, text(L"\pi_u"))
annotate!(-10, 400, text(L"\pi_o"))
```



The plot above is a classic example of the loss function in newsvendor problems. It is a piece-wise convex function with two segments, each with its onw slope. One represented by the underage penalty, and one by the overage penalty. It turns out that there exists an analytical function that can be used to solve the newsvendor problem.

Theorem: Given an uncertain parameter ω with distribution $f(\omega)$ (and associated CDF $F(\omega)$), as well as overage amd underage penalties π_o and π_u, respectively, the optimal solution x^* of the newsvendor problem is

$$x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right)$$

The term $\frac{\pi_u}{\pi_u+\pi_o}$ is called *critial ratio* or *critical fractile*.

Let's now try to build some intuition on this solution approach. Consider the following cases:

1. $\pi_u \simeq 0$ and π_o is very large
2. $\pi_o \simeq 0$ and π_u is very large
3. $\pi_u = \pi_o$

In case 1, the cost of having underage is nearly zero, meaning that it is almost free to make a decision lower than the realization of the random parameter. In our example, it would mean that we would produce the minimum amount of food by looking over to the left of the normally distributed demand. Case 2 is the opposite. Here it is almost free to overshoot the demand; hence we want to produce as much as possible (we are all the way to the right of the demand distribution). Finally, in case 3, since the penalties are the same, we want to choose the mean value.

Let's build up more intuition. Looking at the graph above (and the values of the penalties), we can see that the underage penalty is higher than the overage ($\pi_u > \pi_o$). When this is the case, it is reasonable to assume that the optimal value of the decision will be higher than the mean of the random distribution. This is the case since it is cheaper to overshoot the demand. The opposite is also true when $\pi_o > \pi_u$.

Proof: Let us now prove the theorem we discussed. We start by remembering that to solve the newsvendor problem we have to solve the following minimization problem.

$$\min_x \ \mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+]$$

Since we are working with a continuous convex function, we know (basic principle of convex optimization) that the optimal solution can be found where the function's derivative is equal to 0. This means that the solution x^* should be such that $\frac{d}{dx}\mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+] = 0$. We can now rewrite the formula of the expectation in integral form (since we are using a continuous distribution) where $f(\omega)$ is the PDF of the random variable

$$\int_{-\infty}^{x} \pi_o(x - \omega) f(\omega) d\omega + \int_{x}^{\infty} \pi_u(\omega - x) f(\omega) d\omega$$

Before we go ahead, we need to remember the formula for *differentiation under the integral sign*.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,t)dt = \left[g(x,b(x))\frac{d}{dx}b(x)\right] - \left[g(x,a(x))\frac{d}{dx}a(x)\right] + \left[\int_{a(x)}^{b(x)} \frac{d}{dx}g(x,t)dt\right]$$

Let's apply this formula to each of the two integrals in our equation. For the overage, we have that

$$\begin{aligned}
\frac{d}{dx}\int_{-\infty}^{x} \pi_o(x-\omega)f(\omega)d\omega &= \left[\pi_o(x-x)\frac{d}{dx}x\right] - \left[\pi_o(x+\infty)\frac{d}{dx}-\infty\right] + \left[\int_{-\infty}^{x}\frac{d}{dx}\pi_o(x-\omega)f(\omega)d\omega\right] \\
&= [0] - [0] + \left[\int_{-\infty}^{x}\frac{d}{dx}\pi_o x f(\omega) - \frac{d}{dx}\pi_o \omega f(\omega)d\omega\right] \\
&= \int_{-\infty}^{x} \pi_o f(\omega) d\omega
\end{aligned}$$

The first two parts of the equation become zero. The first is because the function itself is 0 at x, and the second is because the integral of a constant is zero (∞ is considered a constant as it is not a function of x). We then solve the last differencial.

We can then solve the integral, and since the CDF ($F(x)$) is the area under the curve of the PDF ($f(x)$) the solution of the integral is

$$\pi_o \left(F(x) - F(-\infty)\right) = \pi_o F(x)$$

We can follow the steps for the underage part of the expectations and get

$$\begin{aligned}\frac{d}{dx}\int_x^\infty \pi_u(\omega - x)f(\omega)d\omega &= \int_x^\infty -\pi_u(\omega)f(\omega)d\omega \\ &= -\pi_u\left(F(\infty) - F(x)\right) \\ &= -\pi_u(1 - F(x))\end{aligned}$$

If we now combine these two results with the optimality condition, we get

$$\pi_o F(x^*) - \pi_u(1 - F(x^*)) = 0$$

We can then reshuffle the terms to obtain

$$F(x^*) = \frac{\pi_u}{\pi_u + \pi_o}$$

This means that the optimal value x^* can be found when the probability that $\omega - x \leq 0$ is $\frac{\pi_u}{\pi_u+\pi_o}$. Essentially we can find x^* by calculating the quantile at that point in the CDF, hence the final result is

$$x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right) \qquad \blacksquare$$

Now that we have proved the analytical solution is correct let's also check it empirically on the example. Remember that we are aiming to solve the following problem

$$\min_x \; \mathbb{E}[\pi_u(\omega - x)_+ + \pi_o(x - \omega)_+] = \min_x \; \mathbb{E}[45(\omega - x)_+ + 30(x - \omega)_+]$$

According to the analytical formula, we have that

$$x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right) = F^{-1}\left(\frac{45}{75}\right) = F^{-1}(0.6) = 153.87$$

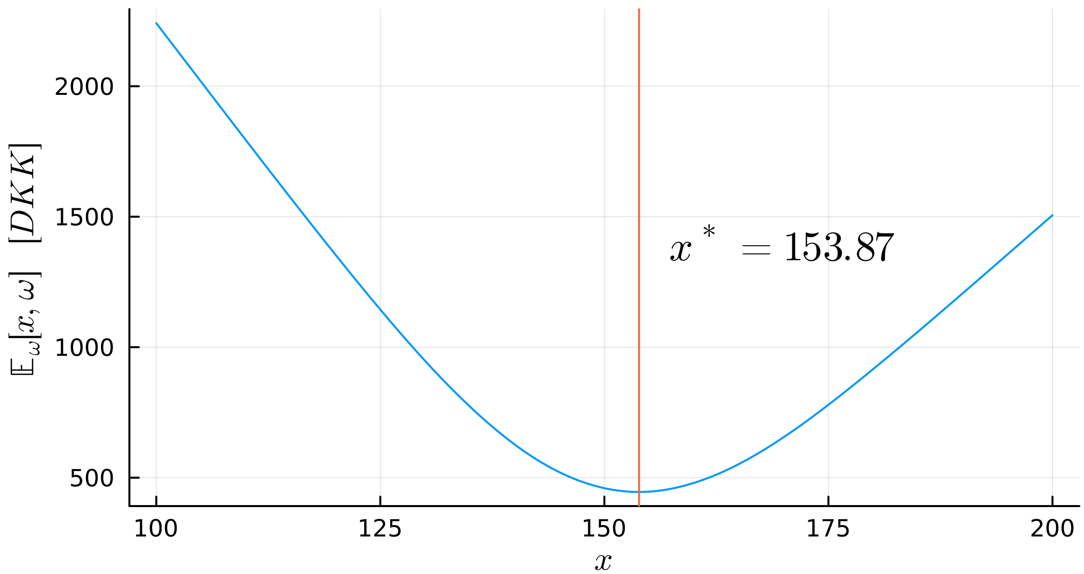
You can find the quantile by graphically inspecting the CDF. by looking at a distribution table. Or using Julia, as shown below.

```
# The demand distribution with μ=150 and σ=15.3
Demand = Normal(150,15.3)
#The solution given by the quantile
println("Solution: ",quantile(Demand,0.6))
```

```
Solution: 153.87621067797772
```

Let's now try to verify the result empirically using simulation.

```
πo = 30 # overage
πu = 45 # underage
x = collect(100:200)
ω = rand(Demand,10000) # Demand sampling
cv = zeros(length(x)) # Mean cost
for i in 1:length(x) # Repetitions
    # Simulation
    cv[i] = mean(πo .* (x[i].-ω) .* ((x[i].-ω).>=0)
                 + πu .* (ω.-x[i]) .* ((ω.-x[i]).>0))
end
plot(x,cv,xlabel=L"x",ylabel=L"\mathbb{E}_\omega[x,\omega] \quad␣
  ↪[DKK]",legend=false, size=(550,300))
plot!([153.87], seriestype="vline")
annotate!(170, 1400, text(L"x^* = 153.87"))
```



As we can see, the plotted results of the simulation show the optimal solution to also be at 153.87.

Example 2: - DTU Friday bar. The bar manager has to order (draft) beer on Monday to be ready when Friday comes. The demand for beer ω is an uncertain parameter that follows the distribution F_ω. You will need to decide how many litres of beer to order. Each litre of beer has a cost $c = 10$ DKK/L and a sales price of $s = 15$ DKK/L. If the bar runs out of beer, there is the possibility of sending a rush order. The cost of beer in rush orders is $p = 30$ DKK/L. If the bar has a surplus of beer, the manager can sell it at a discounted price of $z = 7$ DKK/L to various weekend parties. It is a requirement that the demand is fulfilled every friday. *What is the optimal amount of beer the manager should order?*

We can solve this problem by modelling it as a newsvendor problem. To do so, we need to find the overage and underage costs, and we need to know the demand distribution F_ω. Looking at historical data, the manager estimates that the demand is normally distributed with mean $\mu = 160$ and standard deviation $\sigma = 4$. Let's now compute the overage and underage costs.

We start by making some an observation. The way the problem is posed, the manager will always fulfil the demand (by ordering extra beer as needed). This means that the revenue from the actual demand is always constant, which does not affect the decision. Let's now start with calculating the underage cost. Remember that the underage cost is the cost we pay for underestimating the demand, meaning the cost of buying less beer than needed ($\omega - x > 0$). Since the revenue is constant, we are interested in the changes in cost that appear from making rush orders. The difference in cost is the price of the rush order minus the normal order price ($\pi_u = p - c$). You can see this as the penalty for ordering too little beer.

The overage cost is the cost of ordering more beer than needed. If the beer was thrown away, this penalty would be equal to the cost of the beer (c). However, we know that the manager can always sell the beer at a discount, reducing the losses. The overage cost is then represented by the beer cost minus the discounted sales price ($\pi_o = c - z$).

Before we solve the example, let me show you that we could have mathematically obtained the same result. Let us write the cost function that needs to be minimized.

$$C(x,\omega) = \underbrace{\color{red}{cx}}_{\color{red}{\text{buy beer}}} \quad \underbrace{\color{blue}{-s\omega}}_{\color{blue}{\text{sell beer}}} \quad \underbrace{\color{red}{+p(\omega - x)_+}}_{\color{red}{\text{buy missing beer}}} \quad \underbrace{\color{blue}{-z(x-\omega)_+}}_{\color{blue}{\text{recýlcle beer surplus}}}$$

The red in the equation represents costs, and the blue represents income. The goal is to go from this to the expectation formula of the newsvendor problem. To do so, we can use a simple trick that is often used. We are going to include a null-sum element in the equation given by $c\omega - c\omega$. As the null-sum gives zero, it will have no impact on the equation, which becomes

$$C(x,\omega) = \underbrace{\color{red}{cx}}_{\color{red}{\text{buy beer}}} \quad \underbrace{+c\omega - c\omega}_{=0} \quad \underbrace{\color{blue}{-s\omega}}_{\color{blue}{\text{sell beer}}} \quad \underbrace{\color{red}{+p(\omega - x)_+}}_{\color{red}{\text{buy missing beer}}} \quad \underbrace{\color{blue}{-z(x-\omega)_+}}_{\color{blue}{\text{recýlcle beer surplus}}}$$

Now we go ahead and start regrouping terms. We can combine the beer sales $-s\omega$ with the cost of those sales $c\omega$ to obtain the benefit of selling beer $-(s-c)\omega$. I will also group the cost of buying the beer cx with the other part of the null-sum element $-c\omega$, though I am still unsure what is represents. The equation now looks as follows.

$$C(x,\omega) = \underbrace{\color{blue}{-(s-c)\omega}}_{\color{blue}{\text{benefit from selling beer}}} \quad \underbrace{+c(x-\omega)}_{\text{what to do with that?}} \quad \underbrace{\color{red}{+p(\omega - x)_+}}_{\color{red}{\text{buy missing beer}}} \quad \underbrace{\color{blue}{-z(x-\omega)_+}}_{\color{blue}{\text{recýlcle beer surplus}}}$$

Notice that since the cost of buying missing beer ($+p(\omega - x)_+$) and the surplus coming from recycled beer ($-z(x-\omega)_+$) are dependent on the positive or negative outcomes of $(x - \omega)$, we can combine them with $+c(x-\omega)$ resulting in the following.

$$C(x,\omega) = \underbrace{-(s-c)\omega}_{\text{benefit from selling beer}} \underbrace{+(p-c)(\omega-x)_+}_{\text{penalty for missing beer}} \underbrace{+(c-z)(x-\omega)_+}_{\text{penalty for beer surplus}}$$

Taking the expectation of this cost, we have

$$\mathbb{E}_\omega[C(x,\omega)] = \mathbb{E}_\omega\left[-(s-c)\omega + (p-c)(\omega-x)_+ + (c-z)(x-\omega)_+\right].$$

Since $-(s-c)\omega$ is a constant (it is not affected by the decision variable), we can remove it for the purpose of the optimisation (the decision will not change). Let's also consider the underage and overage penalties, respectively: $\pi_u = (p-c)$ and $\pi_o = (c-z)$. The resulting expectation is now the same as that of the newsvendor problem.

$$\mathbb{E}_\omega[C(x,\omega)] = \mathbb{E}_\omega\left[\pi_u(\omega-x)_+ + \pi_o(x-\omega)_+\right]$$

Now that we have shown that we can also, mathematically, reach the same conclusion, let's go ahead and solve the example.

```
c,p,z = 10, 30, 7 # Cost components
F= Normal(160,4)  # Demand distribution μ=160, σ=4
πu = p-c # the overage penalty
πo = c-z # the underage penalty
cp = πu/(πu+πo) # The critical ratio
xstar = quantile(F,cp) #The quintile at the critical ratio

println("The optimal beer order size is $xstar litres.")
```

```
The optimal beer order size is 164.49735292627454 litres.
```

1.3 The newsvendor problem with discrete uncertain parameters

So far, we have looked at the newsvendor problem, assuming that the uncertain parameters follow a continuous distribution. Sometimes, however, this is not the case, and we need to be able to handle discrete distributions as well.

Fortunately, the analytical solution to the newsvendor problem can be easily adapted to the discrete distribution case.

Theorem: Consider a newsvendor problem with an uncertain parameter ω taking decrete values D_i for $i \in \{1, 2, \ldots, n\}$, with corresponding probabilities p_i, as well as overage and underage penalties π_o and π_u, respectively. the optimal solution x^* is equal to the minimum ω quantity for which

$$Prob(\omega \le D_i) = \sum_{k=1}^{i} p_k \ge \frac{\pi_u}{\pi_u + \pi_o}$$

The proof of the theorem by Spyros Reveliotis can be found on the course homepage.

Let's now look at the practical implication of this theorem with an example.

Example 3: Course TAs. A lecturer needs to hire a number of TAs for a course starting next semester. The number of students taking the course varies from year to year, and consequently, the number of TAs needed. Based on historical data, the probability distribution of the TAs required for a course is given by the following mass probability function.

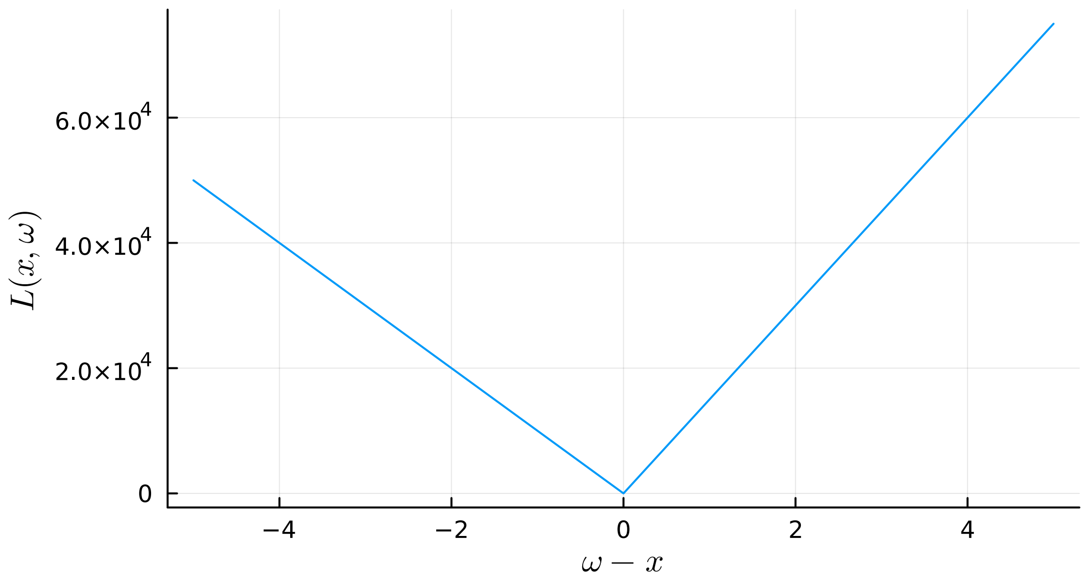
p_i	0.2	0.3	0.25	0.15	0.1
TAs	1	2	3	4	5

A specific student-to-TA ratio is required for each course; hence if the lecturer hires too few TAs, the missing ones would have to be found among the department's PhD students. Each TA costs $c = 10,000$ DKK, while a PhD would cost $z = 25,000$ DKK. What is the optimal number of TAs the lecturer should hire?

We start by calculating the overage and underage penalties. The overage penalty, in this case, is the cost of hiring one extra TA; hence we have that $\pi_o = c = 10,000$. The underage penalty is the marginal cost we have to pay extra for hiring a PhD; hence we have that $\pi_u = z - c = 15,000$.

Before solving the problem analytically, let's draw some intuitive conclusions by looking at the loss function.

```
# loss function where y represent the result of ω - x
loss_e3(y) = (y>=0) ? 15000*y : 10000*-y
ω_x = collect(-5:5)
y = [loss_e3(i) for i in ω_x]
plot(ω_x,y,xlabel=L"\omega -x",ylabel=L"L(x,\omega)",legend = false,
  size=(550,300))
annotate!(2, 40000, text(L"\pi_u"))
annotate!(-2, 30000, text(L"\pi_o"))
```



The graph, and the penalty values, show that the loss is more significant if we underestimate the demand for TAs. We expect the optimal number of TAs to be higher than the mean value.

Let's solve the problem using the analytical formula and check if this is true. We start by calculating the critical ratio.

$$\frac{\pi_u}{\pi_u + \pi_o} = \frac{15}{25} = 0.6$$

Now, we can calculate the optimal value by following the theorem and looking at the minimum value for which $\sum_{k=1}^{i} p_k \geq \frac{\pi_u}{\pi_u + \pi_o}$. Let's do this manually first.

- $Prob(\omega < 1) = 0 \rightarrow$ is this strictly less than $\frac{\pi_u}{\pi_u + \pi_o} = 0.6$? Yes.
- $Prob(\omega < 2) = 0.2 \rightarrow$ is this strictly less than 0.6? Yes.
- $Prob(\omega < 3) = 0.5 \rightarrow$ is this strictly less than 0.6? Yes.
- $Prob(\omega < 4) = 0.75 \rightarrow$ is this strictly less than 0.6? No.

This means that the minimum value, and hence the optimal solution, is 3.

Finally, let's see how we can accomplish this with Julia.

```
# The demand distribution
C = Categorical([0.2, 0.3, 0.25, 0.15, 0.1])
# the penalties
πu, πo = 15, 10
# we can use the quantile function also in a discrete distribution
solution = quantile(C,πu/(πu+πo))
println("The optimal number of TAs is $solution.")
```

```
The optimal number of TAs is 3.
```