Analysis of stochastic linear programs

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Learning Objectives

At the end of this lecture (and related hands-on session), a student should be able to:

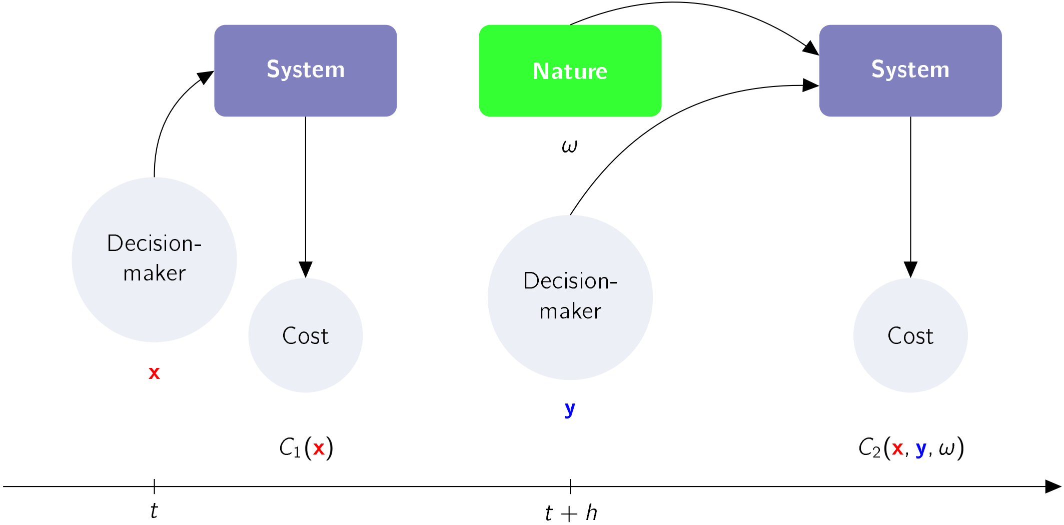
- Appraise **benefits and caveats** of stochastic linear programs
- Define and discuss the **value of the stochastic solution** (VSS)
- Explain **various concepts** related to problems solutions (e.g. value of perfect information, expected value of expected solution)
- Appraise **key relationships and bounds** that illustrate the fallacy of averages

Outline

1. The newsvendor as a stochastic linear program
2. From expected value problem to the value of the stochastic solution
3. Illustration based on the newsvendor problem

Making a decision now... knowing you will have another chance

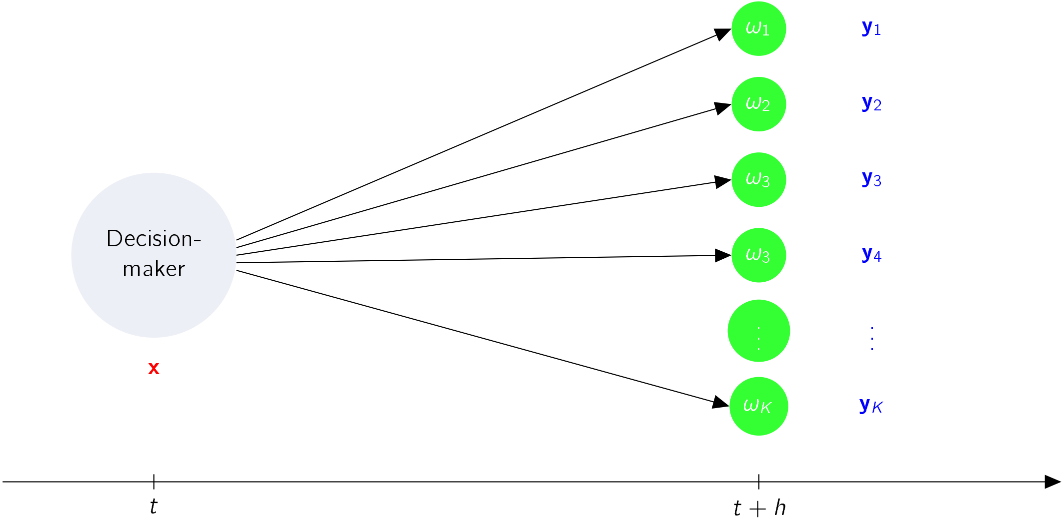
In many problems of decision-making under uncertainty, one has to make a decision now, but with a chance to perform some form of "correction" later on...



$\mathbf{x}$: here-and-now decisions $\quad$ $\mathbf{y}$: recourse decisions

Visualization based on scenarios

Obviously at time t, the decision-maker want to account for what *may* happen at time $t + h$



This may possibly rely on K scenarios for the uncertain parameter ω

2. **The newsvendor as a stochastic linear program**

The energy trading newsvendor problem

It is a newsvendor problem, but on the production side!

- Renewable energy producers have to sell their production in advance in electricity markets (quantity x)
- They are not sure of what they will actually produce though (uncertain parameter ω)

In practice:

- probabilistic forecast F for the uncertain production ω
- unit price c for what they sell in the market (in advance)
- unit price $c + \delta$ for any missing energy ($\delta \geq 0$)
- unit price $c - \eta$ for any surplus energy ($\eta \geq 0$)

We know that the optimal quantity to sell in advance is $x^* = F^{-1}\left(\frac{\pi_u}{\pi_u + \pi_o}\right)$

Formulation as a stochastic linear program

To write it as a stochastic linear program we need:

- an **uncertain parameter** ω (the actual energy production)
- a **here-and-now** decision x,i.e., the sold production
- **recourse** variables
  - the quantity of energy produced, or sold, y_1
  - the quantity of energy missing y_2 (if production is less than the contract x)
  - and the surplus of energy y_3 (in case the production is greater than the contract x)

Eventually this yields

$$\min_{x} \quad (-c)x + \mathbb{E}_\omega\left[Q(x,\omega)\right]$$

$$\begin{aligned} Q(x,\omega) = \min_{y_1,y_2,y_3} \; & (c+\delta)y_2 - (c-\eta)y_3 \\ \text{s.t.} \quad & y_1 + y_2 = x \\ & y_1 + y_3 = \omega \\ & x, y_1, y_2, y_3 \geq 0 \end{aligned}$$

And in an extensive form (or, deterministic equivalent)

- Instead of distributional information for ω,
- one is provided with K scenarios ω_k ($k = 1, \dots, K$) and associated probabilities p_k ($k = 1, \dots, K$)
- we hence need to have recourse variables $y_{1,k}$, $y_{2,k}$, $y_{3,k}$ also for each scenario

The **extensive form** of the problem then writes

$$
\begin{aligned}
\min_{x, y_{1,k}, y_{2,k}, y_{3,k}} \quad & (-c)x + \sum_{k=1}^{K} p_k \big((c+\delta) y_{2,k} - (c-\eta) y_{3,k} \big) \\
\text{s.t.} \quad & y_{1,k} + y_{2,k} = x, \quad k = 1, \dots, K \\
& y_{1,k} + y_{3,k} = \omega_k, \quad k = 1, \dots, K \\
& x, y_{1,k}, y_{2,k}, y_{3,k} \geq 0
\end{aligned}
$$

The final problem is a linear program with

- $1 + 3 \times K$ variables
- $2 \times K$ constraints

An example

The decision-maker is provided with **5 equiprobable scenarios** ω_k:

- $\omega_1 = 46.38$, and with $p_1 = 1/5$
- $\omega_2 = 53.95$, and with $p_2 = 1/5$
- $\omega_3 = 43.93$, and with $p_3 = 1/5$
- $\omega_4 = 30.08$, and with $p_4 = 1/5$
- $\omega_5 = 47.24$, and with $p_5 = 1/5$

Cost parameters:

- $c = 100$ DKK/MWh (for here-and-now decision)
- $\delta = 20$ DKK/MWh (penalty for missing energy)
- $\eta = 10$ DKK/MWh (penalty for surplus energy)

After solving the SLP with these scenarios, we obtain the **here-and-now** decision $x^* = 43.93$

The **objective function value** is -4344.87 DKK, hence including:

- first-stage revenue (negative since we look at a minimization problem): $-43.93 \times 100 = -4393$ DKK
- expected recourse revenue (also negative): $-4344.87 - (-4393) = 48.13$ DKK

3. **From expected value problem to the value of the stochastic solution**

Remember the fallacy of averages?

I told you at the time **it is the whole reason why we need to use stochastic optimization** for decision-making under uncertainty!

Definition (Fallacy of averages)

It is **not** equivalent to consider

(i) minimization of the expected cost over ω, and

(ii) minimization of the cost for the expected ω.

More formally,

$$\begin{aligned} \min_{x} \quad & \mathbb{E}_{\omega}[C(\mathbf{x}, \omega)] \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_{x} \end{aligned} \quad \neq \quad \begin{aligned} \min_{x} \quad & C(\mathbf{x}, \mathbb{E}[\omega]) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{A}_{x} \end{aligned}$$

How does that translate to the case of stochastic linear programs (with fixed recourse)?

The more general setup of the stochastic linear program

In the general case, we have

- $\mathbf{x} \in \mathbb{R}^n$ the here-and-now decision variables
- $\mathbf{y} \in \mathbb{R}^l$ the recourse decision variables
- ω is the uncertain parameter

The "basic" program we are interested in is

$$\begin{aligned} \min_{\mathrm{x}} \quad & \phi(\mathbf{x}, \omega) = \mathbf{c}^\top \mathbf{x} \quad + \quad \min_{\mathrm{y}} \left[\mathbf{d}^\top \mathbf{y} \mid T(\omega)\mathbf{x} + W\mathbf{y} = \mathbf{h}(\omega),\ \ \mathbf{y} \geq 0 \right] \\ \text{s.t} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

And, what could be the different ways to solve it?

There may be 3 different ways to look at how to solve such a program...

WS

WS: wait-and-see solution

What you solve is:

$$\mathrm{WS} = \mathbb{E}_\omega \left[\min_{\mathbf{x}} \phi(\mathbf{x}, \omega) \right]$$

EV

EV: expected-value solution

What you solve is:

$$\mathrm{EV} = \min_{\mathbf{x}} \phi \left(\mathbf{x}, \mathbb{E}_\omega[\omega] \right)$$

RP

RP: recourse program solution

What you solve is:

$$\mathrm{RP} = \min_{\mathbf{x}} \mathbb{E}_\omega \left[\phi(\mathbf{x}, \omega) \right]$$

Again, in the general case, **these 3 approaches are not equivalent!**

(in practice, expectations are replaced by weighted averages over scenarios ω_k for ω)

Expected result of the expected-value solution

Imagine that you implement the expected value solution...

What would be the outcome over the potential scenarios that may realize?

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Definition (EEV)

The *expected result of the expected-value solution* (EEV) is defined as

$$\mathrm{EEV} = \mathbb{E}_{\omega}\left[\phi\left(\mathbf{x}(\bar{\omega}), \omega\right)\right],$$

where $\mathbf{x}(\bar{\omega})$ is the solution of the EV problem

Expected value of perfect information

Imagine that you you could be an oracle, and know in advance what scenario is the one that will really happen...

How better off would you be than if using a stochastic linear program (for which you think there are different scenarios ω_k and associated probabilities p_k)?

Expected value of perfect information

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How better off would you be than if using a stochastic linear program (for which you think there are different scenarios ω_k and associated probabilities p_k)?

Definition (EVPI)

The *expected value of perfect information* (EVPI) is defined as

$$\mathrm{EVPI} = \mathrm{RP} - \mathrm{WS},$$

which represents how much you would gain, in expectation, if you could know the actual outcome for ω, instead of having to consider these scenarios in a stochastic program.

Value of the stochastic solution

You have been told about the fallacy of averages...

What is the difference between the outcome of the stochastic linear program and of the expected-value solution?

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Definition (VSS)

The *value of the stochastic solution* (VSS) is defined as

$$\mathrm{VSS} = \mathrm{EEV} - \mathrm{RP},$$

which represents your benefits from using a stochastic linear program instead of solving the problem based on the expected value for ω

Key relationships and bounds

There are 3 important results that one needs to remember:

1. **Fundamental ordering**:

$$\text{EEV} \geq \text{RP} \geq \text{WS} \geq \text{EV}$$

(Remember that these objective function values mean different things, so it is not to be seen as an ordering of the quality of the various solutions...)

2. **Bounds** on the **expected value of perfect information** (EVPI):

$$0 \leq \text{EVPI} \leq \text{EEV} - \text{EV}$$

(non-negative, and at most the expected recourse costs for the expected-value solution)

3. **Bounds** on the **value of the stochastic solution** (VSS):

$$0 \leq \text{VSS} \leq \text{EEV} - \text{EV}$$

(non-negative, and at most the expected recourse costs for the expected-value solution)

Wrapping up

- Always remember the **fallacy of averages**!
- The **value of the stochastic solution** is always greater than (or equal to) 0 (hence, stochastic linear programs provide superior solutions)
- We can also quantify the **value of perfect information** in decision-making under uncertainty
- There are a number of **limitations** we cannot cover in that course, e.g., related to
  - the size of the scenario set,
  - how good these represent the uncertainty parameter ω...

Thanks for your attention!