Growth Curve Model

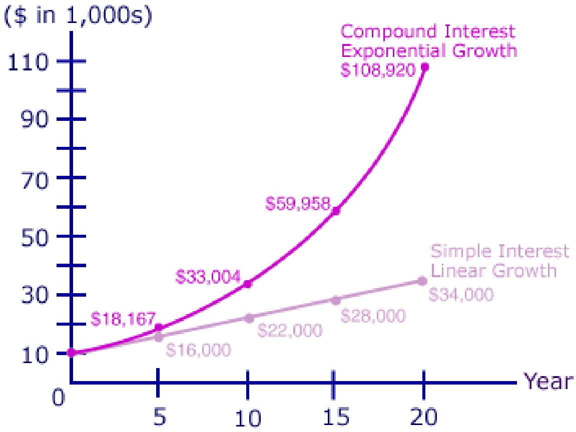
Dr. Natalia Khorunzhina

CBS

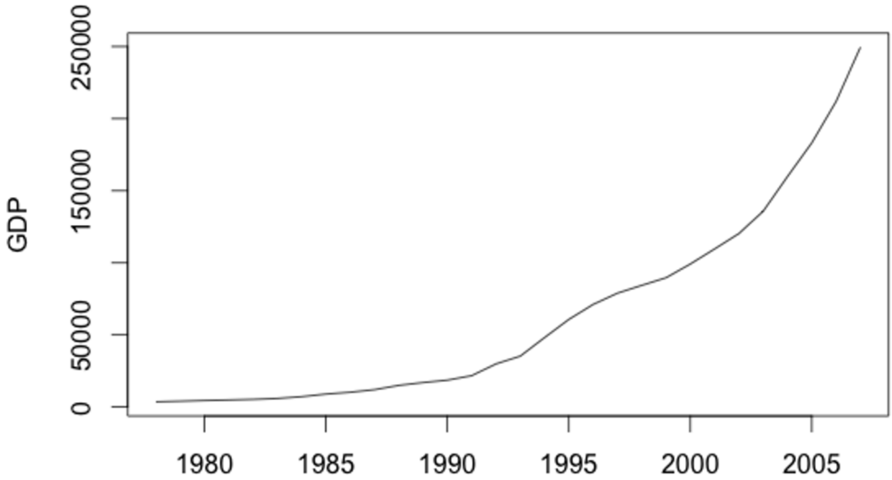
Objective and Format of the Class

- Continue with models of trend, consider models of exponential trend;
- Discuss structural interpretation of models of exponential trend;
- Testing for whether the regression errors can be forecasted;
- Application: modeling and forecasting fast-food chain growth;
- In-class lab monetary exercise: modeling and forecasting volume of currency in circulation.

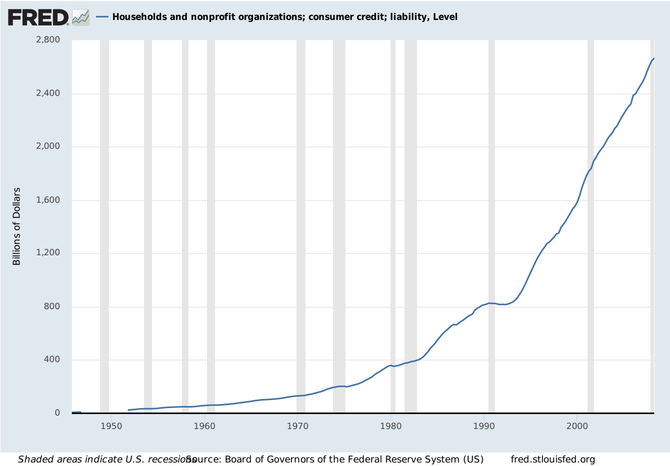
Exponential Growth vs Linear Growth



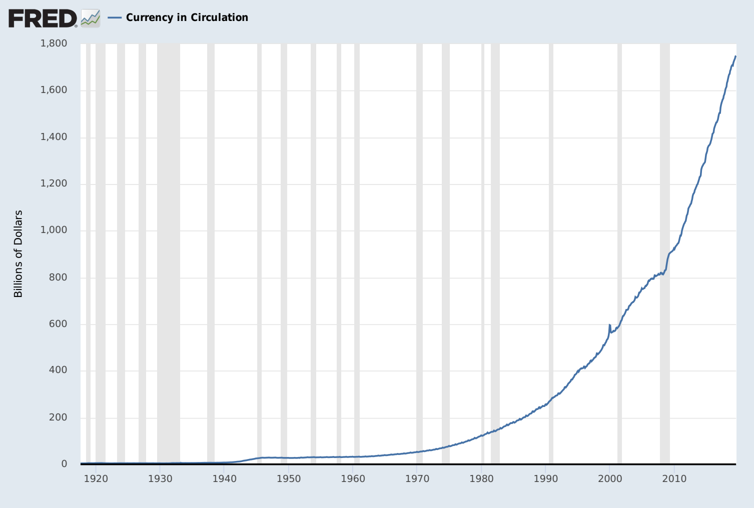
GDP in China



Debt for households and nonprofit organizations in the USA

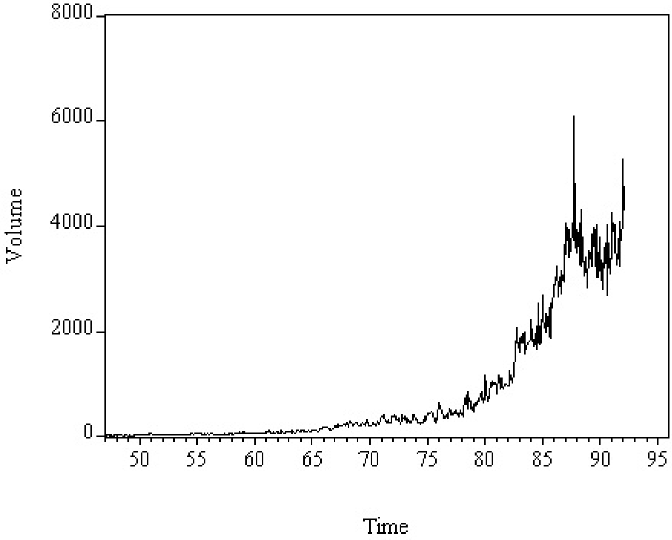


Monetary example: Currency in Circulation

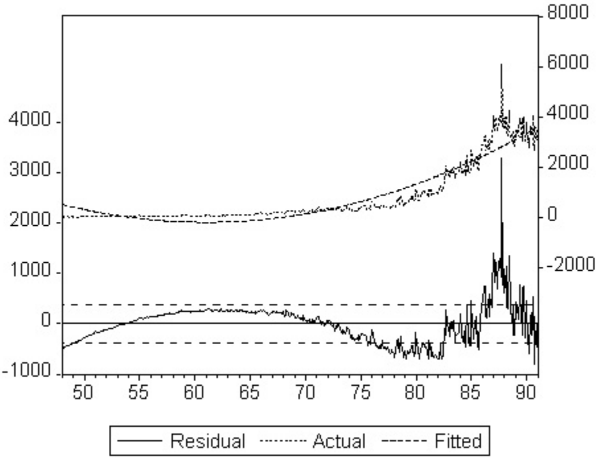


Shaded areas indicate U.S. recessions Source: Board of Governors of the Federal Reserve System (US) fred.stlouisfed.org

Volume on the New York Stock Exchange



Quadratic Trend
Volume on the New York Stock Exchange



Exponential Trends

- Most economic series which are growing (aggregate output, such as GDP, investment, consumption) are exponentially increasing
  - Percentage changes are stable in the long run
- These series cannot be fit by a linear trend
- We can fit a linear trend to their (natural) logarithm

Exponential Trend
Volume on the New York Stock Exchange

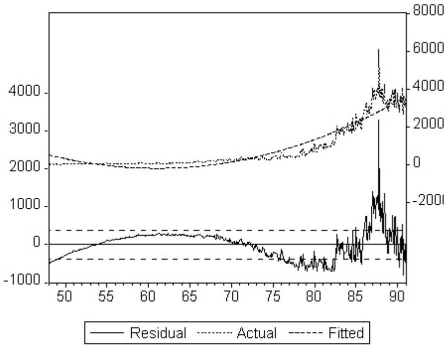


Figure: Quadratic Trend

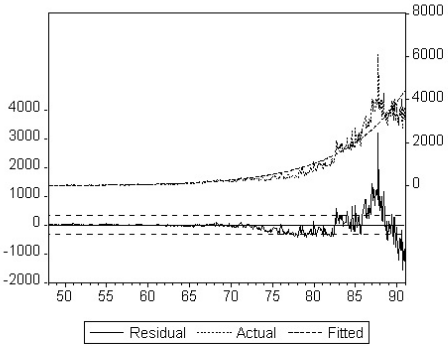


Figure: Exponential Trend

Logarithmic Rules

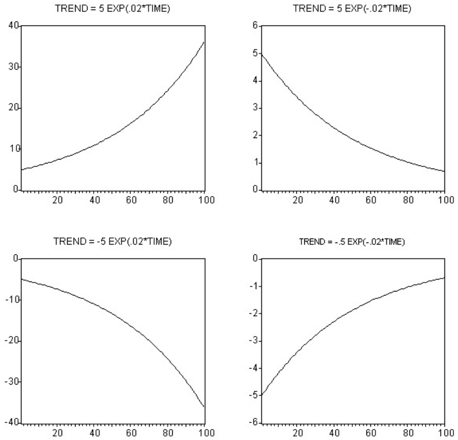
A logarithm is the power to which a number must be raised in order to get some other number. Common logarithm with base equal to 10, is denoted *log*; natural logarithm with base $e = 2.718$ is denoted ln.

Rules for both "log" and "ln"	Separate for "log" and "ln"
1. $\ln x \rightarrow -\infty \; when \; x \rightarrow 0$	
2. $\ln x \rightarrow \infty \; when \; x \rightarrow \infty$	6. $\mathrm{y} = \ln \mathrm{x} <=> \mathrm{x} = \mathrm{e}^{\mathrm{y}}$
3. $\ln(a \times b) = \ln(a) + \ln(b)$	7. $\ln e = 1$
4. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$	8. $y = \log x <=> x = 10^{y}$
5. $\ln(a^{r}) = r \times \ln(a)$	9. $\log 10 = 1$

Models of Exponential Trend

Exponential trend is modelled using power (exponential) function:

$$Trend_t = B_0 e^{(\beta_1 Time_t)}$$



Exponential Trend

We can formulate the model as:

$$\begin{aligned} y_t &= T_t \epsilon_t \\ &= B_0 e^{\beta_1 t} \epsilon_t \\ &= e^{\beta_0} e^{\beta_1 t} \epsilon_t \\ &= e^{\beta_0 + \beta_1 t} \epsilon_t \end{aligned}$$

where the error term ϵ_t is multiplicative rather than additive.

Exponential Trend

The model

$$y_t = T_t \epsilon_t$$

can be transformed by taking logarithms on both sides:

$$\ln(y_t) = \ln(T_t) + \ln(\epsilon_t)$$

Redefine $u_t = \ln(\epsilon_t)$, and note that we formulated $T_t = e^{\beta_0 + \beta_1 t}$, therefore $\ln(T_t) = \beta_0 + \beta_1 t$. Then our model can be written as:

$$\ln(y_t) = \beta_0 + \beta_1 t + u_t$$

The exponential trend model is linear after taking (natural) logarithms. This model is typically estimated by a linear model after taking logs of the variable to forecast.

Rate of growth

Some of you may be familiar with the compound interest problems:

$$y = B(1 + r)^t$$

- B is the initial amount
- r is the growth rate, percent increase each period
- t is the time period

If we observe y over time, can we recover the initial amount B and the percent increase each period r?
In logs, this problem can be written as

$$\ln y = \ln B + t \ln(1 + r)$$

Then $\ln B$ and $\ln(1 + r)$ can be estimated, and B and r recovered.

Growth Curve Model for Estimation of Rate of Growth

If we want to learn the growth rate of the forecasted variable, a convenient way to formulate the model of exponential trend is:

$$y_t = \beta_0(\beta_1^t)\epsilon_t$$

where the error term ϵ_t is multiplicative rather than additive.

We must transform such a nonlinear model to one that is linear in the parameters. The model can be transformed by taking logarithms on both sides:

$$\ln(y_t) = \ln(\beta_0) + \ln(\beta_1)t + \ln(\epsilon_t)$$

If we let $\alpha_0 = \ln(\beta_0)$, $\alpha_1 = \ln(\beta_1)$ and $u_t = \ln(\epsilon_t)$, the transformed model becomes:

$$\ln(y_t) = \alpha_0 + \alpha_1 t + u_t$$

Growth Curve Model

Since $\alpha_1 = \ln(\beta_1)$, it follows that $\beta_1 = e^{\alpha_1}$.

The model

$$y_t = \beta_0(\beta_1^t)\epsilon_t = [\beta_0(\beta_1^{t-1})]\beta_1\epsilon_t \approx (y_{t-1})\beta_1\epsilon_t$$

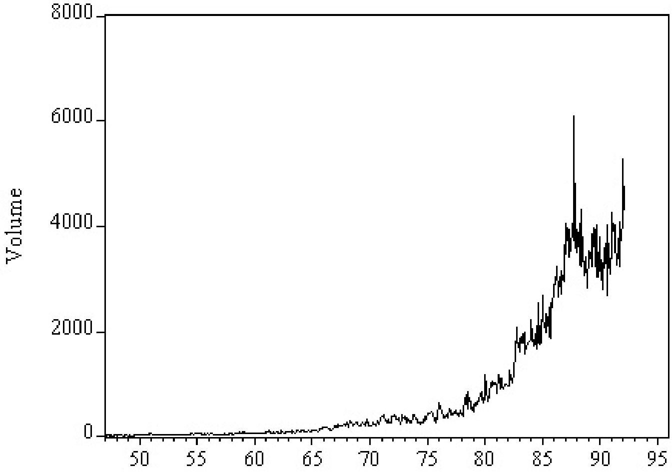
implies that we expect y_t to be approximately β_1 times y_{t-1}. In our application the point estimate of β_1 is 1.293, therefore we estimate y_t to be approximately 1.293 times y_{t-1}, and thus we estimate y_t to be

$$100(\hat{\beta}_1 - 1)\% = 100(1.293 - 1)\% = 29.3\%$$

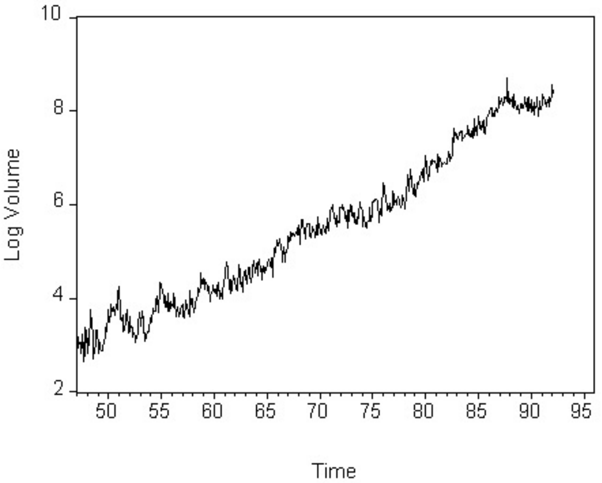
greater than y_{t-1}. Here, 100(1.293-1)% = 29.3% is the point estimate of the **growth rate** $100(\hat{\beta}_1 - 1)\%$.

Also, $\hat{\beta}_1$ is the estimate of $(1 + r)$ in the compound interest problem. Therefore, $r = \hat{\beta}_1 - 1$.

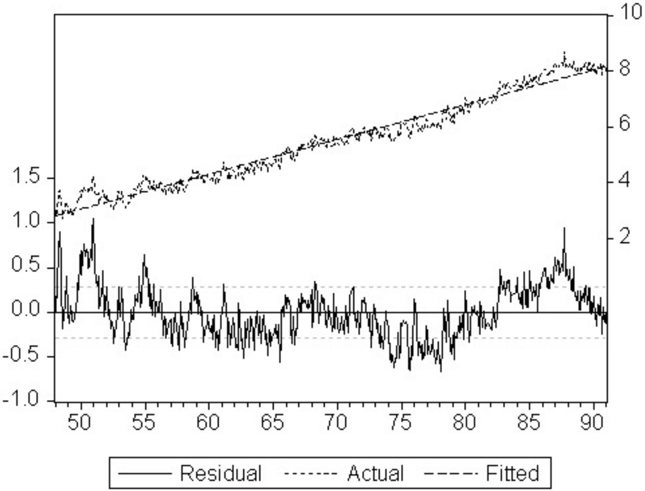
Implementation - start with plotting the original series: Volume on the New York Stock Exchange



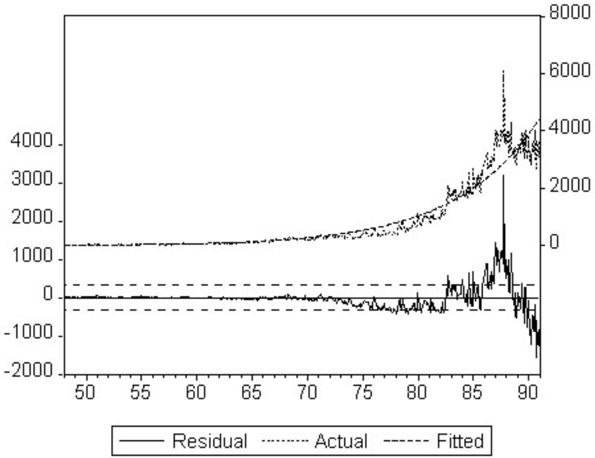
Take logs and see if you get a linear trend
Log Volume on the New York Stock Exchange



Linear Trend
Log Volume on the New York Stock Exchange



Exponential Trend
Volume on the New York Stock Exchange



Regression Errors

- Time-series models are constructed as linear functions of fundamental forecasting errors e_t, also called **innovations** or **shocks**
- These basic building blocks satisfy
  - $Ee_t = 0$
  - $var(e_t) = Ee_t^2 = \sigma^2$
  - Serially uncorrelated
  - These errors e_t are called **white noise**
- In general, if you see an error e_t, it should be interpreted as white noise. We will write
  - e_t is WN$(0, \sigma^2)$

Regression Errors

- White noise processes are linearly unforecastable
- A stronger condition is unforecastable.
- The innovations e_t are **unforecastable** if
  - $E(e_t | \Omega_{t-1}) = 0$
  - This means the best forecast is zero
- For some purposes, we will assume the errors are unforecastable

Durbin-Watson Statistic

- The errors from a good forecasting model should be unforecastable.
- Therefore, it is important to examine whether there are patterns in our forecast errors.
- The **Durbin-Watson statistic** tests for correlation over time, called **serial correlation**.
- If the errors made by a forecasting model are serially correlated, then they are forecastable.

The Durbin-Watson test works within the context of the model:

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$$

$$\varepsilon_t = \varphi \varepsilon_{t-1} + v_t$$

$$v_t \overset{iid}{\sim} N(0, \sigma^2)$$

$$DW = \frac{\sum_{t=2}^{T} (e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2}$$

Durbin-Watson Statistic

- The regression disturbance is serially correlated when $\varphi \neq 0$
- With help of Durbin-Watson statistic, we test whether $\varphi = 0$
- When $\varphi \neq 0$, we say that ε_t follows an autoregressive process of order 1, or AR(1).
- Durbin-Watson statistic takes values between 0 and 4. If all is well, DW should be around 2.
- A rough rule of thumb is, if DW is less than 1.5 (case of positive serial correlation), there may be cause for alarm.

Durbin-Watson Statistic Interpretation

- Recall that we test that there is no serial correlation (main hypothesis) in the errors against the alternative hypothesis that there is a positive autocorrelation.
- Suppose, the estimated DW statistic is equal to 0.80 with p-value 0.001. How do we interpret this?
- If, in reality, $\varphi = 0$ (or negative), then the chance of getting DW statistic of this small would be **at most** 0.1% (=0.001 $\times$ 100%).
- This is **strong evidence** for the presence of **positive autocorrelation.**

Durbin-Watson Statistic Interpretation

- Now suppose, the estimated DW statistic is equal to 1.54 with p-value 0.116. How do we interpret this?
- If, in reality, $\varphi = 0$ (or negative), then the chance of getting DW statistic of this small would be 11.6% (=0.116 $\times$ 100%), which means "chances are large" in statistics.
- This is evidence for the presence of **no significant autocorrelation.**

Another useful feature of log-transforming the data

... is to get the **growth rates** of the data. Often it is preferable to work with growth rates when

- your objective is to forecast growth rate, or
- your model requires data with the properties of growth rates.

Some useful notation:

- The first lag of y_t is y_{t-1}; its jth lag is y_{t-j}
- The first difference of a series, Δy, is its change between periods $t-1$ and t. That is, $\Delta y_t = y_t - y_{t-1}$
- The first difference of the logarithm of y_t is $\Delta ln(y_t) = ln(y_t) - ln(y_{t-1})$
- The percentage change of a time series y_t between periods $t-1$ and t is approximately $100\Delta ln(y_t)$, where the approximation is most accurate when the percentage change is small.