Recursive Lexicographical Search: Finding all Markov Perfect Equilibria in Directional Dynamic Games

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Next: Formal definition of DDG

1. Strategy specific partial order over D
2. Anti-cycling condition to rule out loops
3. Consistency of strategy specific partial orders

$\Rightarrow$ Definition of DDG

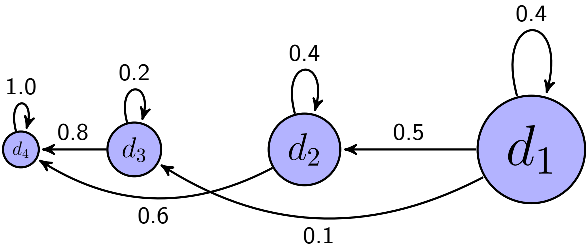
Strategy-specific partial order over D

- Let σ be a feasible Markovian strategy in the game, and let $\rho(d'|(d,x),\sigma(d,x))$ be the *conditional hitting probability* of the state $d' \in D$ starting from the state $s = (d,x)$.
- **Definition** $d' \succ_\sigma d$ *iff*
  $\exists x \in X \quad \rho(d'|(d,x),\sigma(d,x)) > 0$ and
  $\forall x' \in X \quad \rho(d|(d',x'),\sigma(d',x')) = 0.$
- **Lemma** $\succ_\sigma$ is a *strict partial order of D,* i.e. it is irreflexive, asymmetric, and transitive.

The No-Loop (Anti-Cycling) Condition

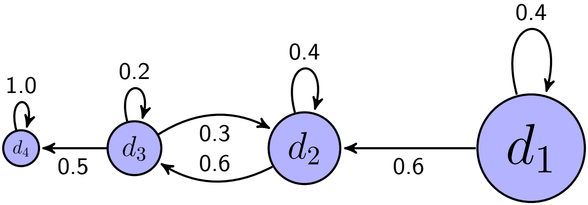
- The partial order $\succ_\sigma$ defines a clear notion of *directionality* in the strategy-induced law of motion of the game $\mathcal{G}$. (cycles) in any subset of *comparable* elements of D
- There are two ways (d', d) can be non-comparable w.r.t $\succ_\sigma$: a) there may be no communication between d' and d, b) there may be a *loop* (cycle) between d' and d.
- **Definition: No Loop Condition** $d' \not\succ_\sigma d$ and $d' \neq d$ *iff and* $\forall x' \in X$ $\rho(d|(d', x'), \sigma(d', x')) = 0$.

Bargaining over a stochastically shrinking pie



Notice that d_2 and d_3 are not comparable under the induced partial order $\succ_\sigma$. However the no-loop condition is satisified.

Bargaining over a shrinking/growing pie



This game induces the same partial order on D, $\succ_\sigma$, but the no-loop condition fails due to the loop (cycle) between d_2 and d_3.

Consistency in Induced Partial Orders and DDGs

- **Definition** If σ and σ' are two feasible, Markovian strategies of $\mathcal{G}$, we say the induced partial orders of D are *consistent* if

$$d' \succ_{\sigma} d \Longrightarrow d \not\succ_{\sigma'} d'$$

- Essentially, the two strategies should not induce contradicting partial orders

Definition of the Dynamic Directional Games

Definition (DDG)

A *Dynamic Directional Game* (DDG) is a finite state Markovian game $\mathcal{G}$ that satisfies the following two conditions:

1. Every feasible, Markovian strategy σ satisfies the No-Loop Condition
2. Every pair of feasible, Markovian strategies σ and σ' induce consistent partial orderings, $\succ_\sigma$ and $\succ_{\sigma'}$, respectively.

Next: Stages and stage games

1. Strategy independent partial order over D
2. DAG to represent the directionality of the game
3. Recursion on the game DAG to form partition of totally ordered subsets of D
4. Stages on the state space and induced subgames of DDG

$\Rightarrow$ Definition of the stage games of DDG

Strategy-Independent Partial Order for a DDG

- **Definition** Let $\succ_\sigma$ and $\succ_{\sigma'}$ be two strategy-induced partial orders of D. We say that $\succ_{\sigma'}$ is a *refinement* of $\succ_\sigma$ *iff* $\forall d, d' \in D$ we have $d' \succ_\sigma d \Longrightarrow d' \succ_{\sigma'} d$
- **Definition** Let $\{\succ_\sigma |\sigma \in \Sigma(\mathcal{G})\}$ be a set of partial orders of D induced by the set of all feasible Markovian strategies in the game $\mathcal{G}$, $\Sigma(\mathcal{G})$. Then let $\succ_{\mathcal{G}}$ be the *join* (or coarsest common refinement) of the the set of partial orders $\{\succ_\sigma |\sigma \in \Sigma(\mathcal{G})\}$.
- **Theorem** The join partial order $\succ_{\mathcal{G}}$ exists, is strategy-independent, and equals the transitive closure of the union of the partial orders in the set $\{\succ_\sigma |\sigma \in \Sigma(\mathcal{G})\}$.