

SGD Ex

July 12, 2022

[P4]

E1

Def. 17:

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle = \langle \nabla f(y) - \nabla f(x), y - x \rangle \quad (1)$$

$\forall x, y \in \mathbb{R}^d$:

$$\begin{aligned} \mu \|x - y\|^2 &\leq 2D_f(x, y), \\ \frac{\mu}{2} \|x - y\|^2 &\leq D_f(x, y), \\ \frac{\mu}{2} \|x - y\|^2 &\leq D_f(y, x), \\ D_f(x, y) + \frac{\mu}{2} \|x - y\|^2 &\leq D_f(x, y) + D_f(y, x), \\ D_f(x, y) + \frac{\mu}{2} \|x - y\|^2 &\stackrel{(1)}{\leq} \langle \nabla f(x) - \nabla f(y), x - y \rangle. \end{aligned} \quad (2)$$

E2

$$\begin{aligned} D_f(x, y) + \frac{\mu}{2} \|x - y\|^2 &\leq \langle \nabla f(x) - \nabla f(y), x - y \rangle, \\ \langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \underbrace{D_f(x, y)}_{\geq \frac{\mu}{2} \|x - y\|^2} + \frac{\mu}{2} \|x - y\|^2, \\ \langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \frac{\mu}{2} \|x - y\|^2 + \frac{\mu}{2} \|x - y\|^2, \\ \langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \mu \|x - y\|^2. \end{aligned} \quad (3)$$

[P6]

E17

(Equation 34):

$$\begin{aligned}\langle a, b \rangle &\leq \frac{\|a\|^2}{2t} + \frac{t\|b\|^2}{2}, \\ \langle a, b \rangle &\leq \frac{\langle a, a \rangle}{2t} + \frac{t\langle b, b \rangle}{2}, \\ 2t\langle a, b \rangle &\leq \langle a, a \rangle + t^2\langle b, b \rangle, \\ 0 &\leq \langle a, a \rangle + \langle tb, tb \rangle - \langle a, tb \rangle - \langle tb, a \rangle, \\ 0 &\leq \|a - tb\|^2.\end{aligned}\tag{4}$$

(Equation 35):

$$\begin{aligned}\|a + b\|^2 &\leq 2\|a\|^2 + 2\|b\|^2, \\ \langle a, a \rangle + \langle b, b \rangle + 2\langle a, b \rangle &\leq 2\langle a, a \rangle + 2\langle b, b \rangle, \\ 0 &\leq \langle a, a \rangle + \langle b, b \rangle - 2\langle a, b \rangle, \\ 0 &\leq \|a - b\|^2.\end{aligned}\tag{5}$$

(Equation 36):

$$\begin{aligned}\frac{1}{2}\|a\|^2 - \|b\|^2 &\leq \|a + b\|^2, \\ \frac{1}{2}\langle a, a \rangle - \langle a, a \rangle &\leq \langle a, a \rangle + \langle b, b \rangle + 2\langle a, b \rangle, \\ \langle a, a \rangle - 2\langle b, b \rangle &\leq 2\langle a, a \rangle + 2\langle b, b \rangle + 4\langle a, b \rangle, \\ 0 &\leq \langle a, a \rangle + \langle 2b, 2b \rangle + \langle a, 2b \rangle + \langle 2b, a \rangle, \\ 0 &\leq \|a + 2b\|^2.\end{aligned}\tag{6}$$

E19

For random vector $X \in \mathbb{R}^d$:

$$\mathbf{Var}[X] := \mathbf{E} [\|X - \mathbf{E}[X]\|^2]. \tag{7}$$

Markov's inequality:

$$\text{Prob}(X \geq t) \leq \frac{\mathbf{E}[X]}{t}. \tag{8}$$

Proof of Chebyshev's inequality using Markov's inequality:

$$\text{Prob}(\|X - \mathbf{E}[X]\|^2 \geq t^2) \leq \frac{\mathbf{E}[\|X - \mathbf{E}[X]\|^2]}{t^2}.$$

Since

$$\text{Prob}(\|X - \mathbf{E}[X]\|^2 \geq t^2) = \text{Prob}(\|X - \mathbf{E}[X]\| \geq t), \tag{9}$$

then

$$\text{Prob}(\|X - \mathbf{E}[X]\| \geq t) \leq \frac{\mathbf{Var}[X]}{t^2}. \tag{10}$$

[P7]

E24

If

$$f = \frac{1}{n} \sum_{i=1}^n f_i,$$

then

$$D_f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x) - \frac{1}{n} \sum_{i=1}^n f_i(y) - \frac{1}{n} \sum_{i=1}^n \langle \nabla f_i(y), x - y \rangle,$$

$$D_f(x, y) = \frac{1}{n} \sum_{i=1}^n (f_i(x) - f_i(y) - \langle \nabla f_i(y), x - y \rangle),$$

$$D_f(x, y) = \frac{1}{n} \sum_{i=1}^n D_{f_i}(x, y).$$

E26

If $\sigma_\star^2 = 0$, then

$$\begin{aligned} \sigma_\star^2 &= \left(\frac{1}{n^2} \sum_{i=1}^n \frac{\|\nabla f_i(x^\star)\|^2}{p_i} \right) - \|\nabla f(x^\star)\|^2 = 0 \\ &= \left(\frac{1}{n^2} \sum_{i=1}^n \frac{\|np_i \nabla f(x^\star)\|^2}{p_i} \right) - \|\nabla f(x^\star)\|^2 = 0 \\ &= p_i \sum_{i=1}^n (\|\nabla f(x^\star)\|^2) - \|\nabla f(x^\star)\|^2 = 0 \\ &= np_i \|\nabla f(x^\star)\|^2 - \|\nabla f(x^\star)\|^2 = 0, \\ \sigma_\star^2 = 0 &\implies np_i \nabla f(x^\star) = \nabla f(x^\star). \end{aligned}$$

[P8]

E33

Let

$$\chi_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}.$$

Since

$$p_i = \frac{1}{n},$$

and

$$|S| = \tau,$$

then

$$\mathbf{E}[\chi_i] = \text{Prob}(i \in S) = \sum_{i=1}^n p_i \chi_i = \frac{1}{n} \sum_{i=1}^n \chi_i = \frac{\tau}{n}.$$

E35

For any vectors, $b_1, \dots, b_n \in \mathbb{R}^d$:

$$\begin{aligned} \left\| \sum_{i=1}^n b_i \right\|^2 - \sum_{i=1}^n \|b_i\|^2 &= \underbrace{\sum_{i=1}^n \langle b_i, b_i \rangle + \sum_{i \neq j} \langle b_i, b_j \rangle}_{\left\| \sum_{i=1}^n b_i \right\|^2} - \sum_{i=1}^n \langle b_i, b_i \rangle, \\ \left\| \sum_{i=1}^n b_i \right\|^2 - \sum_{i=1}^n \|b_i\|^2 &= \sum_{i \neq j} \langle b_i, b_j \rangle. \end{aligned}$$

[P9]

E37

Assumptions of $\mathcal{C} : \mathbb{R}^d \rightarrow \mathbb{R}^d$:

1. $\mathbf{E}[\mathcal{C}(x)] = x, \quad \forall x \in \mathbb{R}^d$
2. $\mathbf{E}[\|\mathcal{C}(x) - x\|^2] \leq \omega \|x\|^2 + \delta, \quad \forall x \in \mathbb{R}^d, \quad \exists \omega, \delta \geq 0$

Proof of convergence for CGD with $n = 1$:

Since $\mathcal{C} \in \mathbb{B}^d(\omega)$,

$$\mathbf{E}[\|g(x)\|^2] = \mathbf{E}[\|\mathcal{C}(\nabla f(x))\|^2] \leq (\omega + 1)\|\nabla f(x)\|^2. \quad (11)$$

In case of $\nabla f(y) = 0$,

$$\begin{aligned} G(x, y) &:= \mathbf{E}[\|g(x) - \nabla f(y)\|^2] \\ &= \mathbf{E}[\|g(x)\|^2] \\ &\stackrel{(11)}{\leq} (\omega + 1)\|\nabla f(x) - \nabla f(y)\|^2, \\ &\leq 2(\omega + 1)L D_f(x, y). \end{aligned}$$

In case of $\nabla f(y) \neq 0$,

$$\begin{aligned} G(x, y) &:= \mathbf{E}[\|g(x) - \nabla f(y)\|^2] \\ &= \mathbf{E}[\|g(x) - \nabla f(x)\|^2] + \|\nabla f(x) - \nabla f(y)\|^2 \\ &= \mathbf{E}[\|\mathcal{C}(\nabla f(x)) - \nabla f(x)\|^2] + \|\nabla f(x) - \nabla f(y)\|^2 \\ &\leq \omega \|\nabla f(x)\|^2 + \delta + \|\nabla f(x) - \nabla f(y)\|^2 \\ &= \omega \|\nabla f(x) - \nabla f(y) + \nabla f(y)\|^2 + \|\nabla f(x) - \nabla f(y)\|^2 + \delta \\ &\leq 2\omega \|\nabla f(x) - \nabla f(y)\|^2 + 2\omega \|\nabla f(y)\|^2 + \|\nabla f(x) - \nabla f(y)\|^2 + \delta \\ &= (2\omega + 1)\|\nabla f(x) - \nabla f(y)\|^2 + 2\omega \|\nabla f(y)\|^2 + \delta \\ &\leq 2 \underbrace{(2\omega + 1)L D_f(x, y)}_A + \underbrace{2\omega \|\nabla f(y)\|^2 + \delta}_C. \end{aligned}$$

If $0 < \gamma < \frac{1}{A}$, then

$$\mathbf{E}[\|x^k - x^*\|^2] \leq (1 - \gamma\mu)^k \|x^0 - x^*\|^2 + \frac{2\gamma\omega \|\nabla f(x^*)\|^2 + \gamma\delta}{\mu}.$$

E39

Lemma 51:

if $\mathcal{C}(x) = x, \forall x$ (no master compression) and $\omega_i = \omega, \forall i$, then

$$G(x, y) \leq 2 \underbrace{\left(L + 2L_{\max} \frac{\omega}{n} \right)}_A D_f(x, y) + 2 \underbrace{\frac{\omega}{n} \sigma^2(y)}_{C(y)},$$

where

$$\sigma^2(y) := \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(y)\|^2.$$

If $\sigma^2(y) = 0$, then

$$G(x, y) \leq 2 \underbrace{\left(L + L_{\max} \frac{\omega}{n} \right)}_A D_f(x, y).$$

Proof:

If $\nabla f(y) \neq 0$, then

$$\begin{aligned} G(x, y) &:= \mathbf{E} [\|g(x) - \nabla f(y)\|^2] \\ &= \mathbf{E} [\|g(x) - \nabla f(x)\|^2] + \|\nabla f(x) - \nabla f(y)\|^2 \\ &\leq \mathbf{E} [\|g(x) - \nabla f(x)\|^2] + 2LD_f(x, y), \end{aligned} \tag{12}$$

and

$$g(x) = \mathcal{C}(\hat{g}(x)) = \hat{g}(x) = \frac{1}{n} \sum_{i=1}^n g_i(x). \tag{13}$$

where

$$g_i(x) = \mathcal{C}_i(\nabla f_i(x)).$$

Estimate

$$\begin{aligned} \mathbf{E} [\|g(x) - \nabla f(x)\|^2] &\stackrel{(13)}{=} \mathbf{E} [\|\mathcal{C}(\hat{g}(x)) - \nabla f(x)\|^2] \\ &= \mathbf{E} [\|\hat{g}(x) - \nabla f(x)\|^2] \\ &= \mathbf{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \underbrace{(g_i(x) - \nabla f_i(x))}_{a_i} \right\|^2 \right] \\ &= \frac{1}{n^2} \mathbf{E} \left[\sum_{i=1}^n \|a_i\|^2 + \sum_{i \neq j} \langle a_i, a_j \rangle \right] \\ &= \frac{1}{n^2} \sum_{i=1}^n \mathbf{E} [\|a_i\|^2] + \sum_{i \neq j} \mathbf{E} [\langle a_i, a_j \rangle] \\ &= \frac{1}{n^2} \sum_{i=1}^n \mathbf{E} [\|a_i\|^2] + \sum_{i \neq j} \underbrace{\langle \mathbf{E}[a_i], \mathbf{E}[a_j] \rangle}_0 \\ &\leq \frac{1}{n^2} \sum_{i=1}^n \omega_i \|\nabla f_i(x)\|^2 \\ &= \frac{\omega}{n^2} \sum_{i=1}^n \|\nabla f_i(x)\|^2. \end{aligned}$$

Next, bound

$$\begin{aligned}
\|\nabla f_i(x)\|^2 &= \|\nabla f_i(x) - \nabla f_i(y) + \nabla f_i(y)\|^2 \\
&\leq 2\|\nabla f_i(x) - \nabla f_i(y)\|^2 + 2\|\nabla f_i(y)\|^2 \\
&\leq 4L_i D_{f_i}(x, y) + 2\|\nabla f_i(y)\|^2.
\end{aligned}$$

Combine everything:

$$\begin{aligned}
G(x, y) &\leq \mathbf{E} [\|g(x) - \nabla f(x)\|^2] + 2LD_f(x, y) \\
&\leq \frac{\omega}{n^2} \sum_{i=1}^n \|\nabla f_i(x)\|^2 + 2LD_f(x, y) \\
&\leq \frac{\omega}{n^2} \sum_{i=1}^n (4L_i D_{f_i}(x, y) + 2\|\nabla f_i(y)\|^2) + 2LD_f(x, y) \\
&= 2\frac{\omega}{n} \left(2 \sum_{i=1}^n \frac{1}{n} L_i D_{f_i}(x, y) + \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(y)\|^2 \right) + 2LD_f(x, y) \\
&\leq 2\frac{\omega}{n} (2L_{\max} D_f(x, y) + \sigma^2(y)) + 2LD_f(x, y) \\
&= 2(L + 2L_{\max}) D_f(x, y) + 2\frac{\omega}{n} \sigma^2(y).
\end{aligned} \tag{14}$$

Else, if $\nabla f(y) = 0$, then

$$\begin{aligned}
G(x, y) &= \mathbf{E} [\|\hat{g}(x)\|^2] \\
&= \mathbf{E} [\|\hat{g}(x) - \mathbf{E}[\hat{g}(x)]\|^2] + \|\mathbf{E}[\hat{g}(x)]\|^2 \\
&= \mathbf{E} [\|\hat{g}(x) - \nabla f(x)\|^2] + \|\nabla f(x)\|^2 \\
&\leq \left(\frac{\omega}{n^2} \sum_{i=1}^n \|\nabla f_i(x)\|^2 \right) + \|\nabla f(x) - \nabla f(y) + \nabla f(y)\|^2 \\
&\leq \left(\frac{\omega}{n^2} \sum_{i=1}^n \|\nabla f_i(x)\|^2 \right) + 2\|\nabla f(x) - \nabla f(y)\|^2 + 2\underbrace{\|\nabla f(y)\|^2}_0 \\
&\leq \left(\frac{\omega}{n^2} \sum_{i=1}^n \|\nabla f_i(x)\|^2 \right) + 2LD_f(x, y) \\
&\leq \dots \text{ same as (14), from the third line} \\
&= 2(L + 2L_{\max}) D_f(x, y) + 2\frac{\omega}{n} \sigma^2(y).
\end{aligned} \tag{15}$$

[P10]

E41

Let

$$p_i = \text{Prob}(i \in S),$$

where

$$S \subseteq \{1, 2, \dots, d\},$$

then

$$\mathbf{E}[|S|] = \mathbf{E} \left[\sum_{i=1}^d |S_i| \right] = \sum_{i=1}^d \mathbf{E}[|S_i|] = \sum_{i=1}^d 1p_i + 0(1 - p_i) = \sum_{i=1}^d p_i.$$

E42

If $\mathbf{E}[\mathbf{C}^\top \mathbf{C}]$ is finite, then $\forall x \neq 0$:

$$\begin{aligned} x^\top \mathbf{E}[\mathbf{C}^\top \mathbf{C}]x &\geq 0 \\ \mathbf{E}[x^\top \mathbf{C}^\top \mathbf{C}x] &\geq 0 \\ x^\top \mathbf{C}^\top \mathbf{C}x &\geq 0 \\ (\mathbf{C}x)^\top (\mathbf{C}x) &\geq 0. \end{aligned}$$

[P11]

E47

Define base case:

$$\begin{aligned} C_{1,2} &:= C_1 \circ C_2 \in \mathbb{B}^d(\underbrace{(\omega_1 + 1)(\omega_2 + 1) - 1}_{\omega_{1,2}}), \\ C_{1,3} &:= C_1 \circ C_2 \circ C_3 = C_{1,2} \circ C_3, \\ \omega_{1,3} &:= (\omega_{1,2} + 1)(\omega_3 + 1) - 1 \\ &= ((\omega_1 + 1)(\omega_2 + 1) - 1 + 1)(\omega_3 + 1) - 1 \\ &= (\omega_1 + 1)(\omega_2 + 1)(\omega_3 + 1) - 1, \\ C_{1,n} &:= C_1 \circ C_2 \circ \dots \circ C_n = C_{1,n-1} \circ C_n, \\ \omega_{1,n} &:= (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_n + 1) - 1 = (\omega_{1,n-1} + 1)(\omega_n + 1) - 1. \end{aligned}$$

By induction, the base case is clear. Next, if $n = k$, assume

$$\begin{aligned} C_{1,k} &:= C_1 \circ C_2 \circ \dots \circ C_k = C_{1,k-1} \circ C_k, \\ \omega_{1,k} &:= (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1) - 1 = (\omega_{1,k-1} + 1)(\omega_k + 1) - 1 \end{aligned}$$

is true. Then for $n = k + 1$:

$$\begin{aligned} C_{1,k+1} &:= C_1 \circ C_2 \circ \dots \circ C_k \circ C_{k+1} = C_{1,k-1} \circ C_k \circ C_{k+1}, \\ \omega_{1,k+1} &:= (\omega_{1,k} + 1)(\omega_{k+1} + 1) - 1 = (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1)(\omega_{k+1} + 1) - 1 \\ &:= ((\omega_{1,k-1} + 1)(\omega_k + 1) - 1 + 1)(\omega_{k+1} + 1) - 1 = (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1)(\omega_{k+1} + 1) - 1 \\ &:= ((\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1))(\omega_{k+1} + 1) - 1 = (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1)(\omega_{k+1} + 1) - 1, \\ \omega_{1,k+1} &= (\omega_1 + 1)(\omega_2 + 1)\dots(\omega_k + 1)(\omega_{k+1} + 1) - 1. \end{aligned}$$

E48

Define

$$\min\{a_i, b_i\} = \begin{cases} a_i, & \text{if } a_i < b_i \\ b_i, & \text{if } a_i > b_i \end{cases}.$$

Thus

$$\sum_i \min\{a_i, b_i\} = \begin{cases} \sum_i a_i, & \text{if } a_i < b_i, \forall i \\ \sum_i b_i, & \text{if } a_i > b_i, \forall i \end{cases}.$$

In case of inequality, define

$$\begin{aligned} I &:= \{i | a_i < b_i\}, \\ J &:= \{i | a_i > b_i\}. \end{aligned}$$

Thus

$$\sum_i \min\{a_i, b_i\} < \begin{cases} \sum_i a_i, & \text{if } |I| > |J| \\ \sum_i b_i, & \text{if } |J| > |I| \end{cases} .$$