SGD Ex

July 11, 2022

[P4]

E1

Def. 17:

$$D_f(x,y) + D_f(y,x) = \langle \nabla f(x) - \nabla f(y), x-y \rangle = \langle \nabla f(y) - \nabla f(x), y-x \rangle \tag{1}$$

$\forall x, y \in \mathbb{R}^d$:

$$\begin{aligned}
\mu ||x-y||^2 &\leq 2D_f(x,y), \\
\frac{\mu}{2} ||x-y||^2 &\leq D_f(x,y), \\
\frac{\mu}{2} ||x-y||^2 &\leq D_f(y,x), \\
D_f(x,y) + \frac{\mu}{2} ||x-y||^2 &\leq D_f(x,y) + D_f(y,x), \\
D_f(x,y) + \frac{\mu}{2} ||x-y||^2 &\overset{(1)}{\leq} \langle \nabla f(x) - \nabla f(y), x-y \rangle.
\end{aligned} \tag{2}$$

E2

$$\begin{aligned}
D_f(x,y) + \frac{\mu}{2} ||x-y||^2 &\leq \langle \nabla f(x) - \nabla f(y), x-y \rangle, \\
\langle \nabla f(x) - \nabla f(y), x-y \rangle &\geq \underbrace{D_f(x,y)}_{\geq \frac{\mu}{2}||x-y||^2} + \frac{\mu}{2} ||x-y||^2, \\
\langle \nabla f(x) - \nabla f(y), x-y \rangle &\geq \frac{\mu}{2} ||x-y||^2 + \frac{\mu}{2} ||x-y||^2, \\
\langle \nabla f(x) - \nabla f(y), x-y \rangle &\geq \mu ||x-y||^2.
\end{aligned} \tag{3}$$

[P6]

E17

(Equation 34):

$$\begin{aligned}
\langle a, b\rangle &\leq \frac{||a||^2}{2t} + \frac{t||b||^2}{2}, \\
\langle a, b\rangle &\leq \frac{\langle a, a\rangle}{2t} + \frac{t\langle b, b\rangle}{2}, \\
2t\langle a, b\rangle &\leq \langle a, a\rangle + t^2\langle b, b\rangle, \\
0 &\leq \langle a, a\rangle + \langle tb, tb\rangle - \langle a, tb\rangle - \langle tb, a\rangle, \\
0 &\leq ||a - tb||^2.
\end{aligned} \tag{4}$$

(Equation 35):

$$\begin{aligned}
||a + b||^2 \leq 2||a||^2 + 2||b||^2, \\
\langle a, a\rangle + \langle b, b\rangle + 2\langle a, b\rangle \leq 2\langle a, a\rangle + 2\langle b, b\rangle, \\
0 \leq \langle a, a\rangle + \langle b, b\rangle - 2\langle a, b\rangle, \\
0 \leq ||a - b||^2.
\end{aligned} \tag{5}$$

(Equation 36):

$$\begin{aligned}
\frac{1}{2}||a||^2 - ||b||^2 \leq ||a + b||^2, \\
\frac{1}{2}\langle a, a\rangle - \langle a, a\rangle \leq \langle a, a\rangle + \langle b, b\rangle + 2\langle a, b\rangle, \\
\langle a, a\rangle - 2\langle b, b\rangle \leq 2\langle a, a\rangle + 2\langle b, b\rangle + 4\langle a, b\rangle, \\
0 \leq \langle a, a\rangle + \langle 2b, 2b\rangle + \langle a, 2b\rangle + \langle 2b, a\rangle, \\
0 \leq ||a + 2b||^2.
\end{aligned} \tag{6}$$

E19

For random vector $X \in \mathbb{R}^d$:

$$\mathbf{Var}[X] := \mathbf{E}\left[||X - \mathbf{E}[X]||^2\right]. \tag{7}$$

Markov's inequality:

$$\text{Prob}(X \geq t) \leq \frac{\mathbf{E}[X]}{t}. \tag{8}$$

Proof of Chebyshev's inequality using Markov's inequality:

$$\text{Prob}(||X - \mathbf{E}[X]||^2 \geq t^2) \leq \frac{\mathbf{E}\left[||X - \mathbf{E}[X]||^2\right]}{t^2}.$$

Since

$$\text{Prob}(||X - \mathbf{E}[X]||^2 \geq t^2) = \text{Prob}(||X - \mathbf{E}[X]|| \geq t), \tag{9}$$

then

$$\text{Prob}(||X - \mathbf{E}[X]|| \geq t) \leq \frac{\mathbf{Var}[X]}{t^2}. \tag{10}$$

[P7]

E24

If

$$f = \frac{1}{n}\sum_{i=1}^{n} f_i,$$

then

$$\begin{aligned}
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n} f_i(x) - \frac{1}{n}\sum_{i=1}^{n} f_i(y) - \frac{1}{n}\sum_{i=1}^{n}\langle \nabla f_i(y), x-y\rangle,\\
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n}(f_i(x) - f_i(y) - \langle \nabla f_i(y), x-y\rangle),\\
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n} D_{f_i}(x,y).
\end{aligned}$$

E26

If $\sigma_\star^2 = 0$, then

$$\begin{aligned}
\sigma_\star^2 &= \left(\frac{1}{n^2}\sum_{i=1}^{n}\frac{||\nabla f_i(x^\star)||^2}{p_i}\right) - ||\nabla f(x^\star)||^2 = 0\\
&= \left(\frac{1}{n^2}\sum_{i=1}^{n}\frac{||np_i\nabla f(x^\star)||^2}{p_i}\right) - ||\nabla f(x^\star)||^2 = 0\\
&= p_i\sum_{i=1}^{n}(||\nabla f(x^\star)||^2) - ||\nabla f(x^\star)||^2 = 0\\
&= np_i||\nabla f(x^\star)||^2 - ||\nabla f(x^\star)||^2 = 0,\\
\sigma_\star^2 = 0 &\implies np_i\nabla f(x^\star) = \nabla f(x^\star).
\end{aligned}$$

[P8]

E33

Let

$$\chi_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}.$$

Since

$$p_i = \frac{1}{n},$$

and

$$|S| = \tau,$$

then

$$\mathbf{E}[\chi_i] = \mathrm{Prob}(i \in S) = \sum_{i=1}^{n} p_i\chi_i = \frac{1}{n}\sum_{i=1}^{n}\chi_i = \frac{\tau}{n}.$$

E35

For any vectors, $b_1, ..., b_n \in \mathbb{R}^d$:

$$\left\|\sum_{i=1}^n b_i\right\|^2 - \sum_{i=1}^n \|b_i\|^2 = \underbrace{\sum_{i=1}^n \langle b_i, b_i\rangle + \sum_{i\neq j}\langle b_i, b_j\rangle}_{\left\|\sum_{i=1}^n b_i\right\|^2} - \sum_{i=1}^n \langle b_i, b_i\rangle,$$

$$\left\|\sum_{i=1}^n b_i\right\|^2 - \sum_{i=1}^n \|b_i\|^2 = \sum_{i\neq j}\langle b_i, b_j\rangle.$$

[P9]

E37

Assumptions of $\mathcal{C} : \mathbb{R}^d \to \mathbb{R}^d$:

1. $\mathbf{E}[\mathcal{C}(x)] = x, \quad \forall x \in \mathbb{R}^d$
2. $\mathbf{E}\left[||\mathcal{C}(x) - x||^2\right] \leq \omega||x||^2 + \delta, \quad \forall x \in \mathbb{R}^d, \quad \exists \omega, \delta \geq 0$

Proof of convergence for CGD with $n = 1$:
Since $\mathcal{C} \in \mathbb{B}^d(\omega)$,

$$\mathbf{E}\left[||g(x)||^2\right] = \mathbf{E}\left[||\mathcal{C}(\nabla f(x))||^2\right] \leq (\omega+1)||\nabla f(x)||^2. \tag{11}$$

In case of $\nabla f(y) = 0$,

$$\begin{aligned}
G(x,y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\
&= \mathbf{E}\left[||g(x)||^2\right] \\
&\overset{(11)}{\leq} (\omega+1)||\nabla f(x) - \nabla f(y)||^2, \\
&\leq 2(\omega+1)LD_f(x,y).
\end{aligned}$$

In case of $\nabla f(y) \neq 0$,

$$\begin{aligned}
G(x,y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\
&= \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \\
&= \mathbf{E}\left[||\mathcal{C}(\nabla f(x)) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \\
&\leq \omega||\nabla f(x)||^2 + \delta + ||\nabla f(x) - \nabla f(y)||^2 \\
&= \omega||\nabla f(x) - \nabla f(y) + \nabla f(y)||^2 + ||\nabla f(x) - \nabla f(y)||^2 + \delta \\
&\leq 2\omega||\nabla f(x) - \nabla f(y)||^2 + 2\omega||\nabla f(y)||^2 + ||\nabla f(x) - \nabla f(y)||^2 + \delta \\
&= (2\omega+1)||\nabla f(x) - \nabla f(y)||^2 + 2\omega||\nabla f(y)||^2 + \delta \\
&\leq 2\underbrace{(2\omega+1)L}_{A} D_f(x,y) + \underbrace{2\omega||\nabla f(y)||^2 + \delta}_{C}.
\end{aligned}$$

If $0 < \gamma < \frac{1}{A}$, then

$$\mathbf{E}\left[||x^k - x^*||^2\right] \leq (1-\gamma\mu)^k||x^0 - x^*|| + \frac{2\gamma\omega||\nabla f(x^*)||^2 + \gamma\delta}{\mu}.$$

E39

Lemma 51:
if $\mathcal{C}(x) = x, \forall x$ (no master compression) and $\omega_i = \omega, \forall i$, then

$$G(x,y) \leq 2 \underbrace{\left(L + 2L_{\max}\frac{\omega}{n}\right)}_{A} D_f(x,y) + \underbrace{2\frac{\omega}{n}\sigma^2(y)}_{C(y)},$$

where

$$\sigma^2(y) := \frac{1}{n}\sum_{i=1}^n ||\nabla f_i(y)||^2.$$

If $\sigma^2(y) = 0$, then

$$G(x,y) \leq 2 \underbrace{\left(L + L_{\max}\frac{\omega}{n}\right)}_{A} D_f(x,y).$$

Proof:
If $\nabla f(y) \neq 0$, then

$$\begin{aligned} G(x,y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\ &= \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \qquad (12) \\ &\leq \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + 2LD_f(x,y), \end{aligned}$$

and

$$g(x) = \mathcal{C}(\hat{g}(x)) = \hat{g}(x) = \frac{1}{n}\sum_{i=1}^n g_i(x). \qquad (13)$$

where

$$g_i(x) = \mathcal{C}_i(\nabla f_i(x)).$$

Estimate

$$\begin{aligned} \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] &\overset{(13)}{=} \mathbf{E}\left[||\mathcal{C}(\hat{g}(x)) - \nabla f(x)||^2\right] \\ &= \mathbf{E}\left[||\hat{g}(x) - \nabla f(x)||^2\right] \\ &= \mathbf{E}\left[\left|\left|\frac{1}{n}\sum_{i=1}^n \underbrace{(g_i(x) - \nabla f_i(x))}_{a_i}\right|\right|^2\right] \\ &= \frac{1}{n^2}\mathbf{E}\left[\sum_{i=1}^n ||a_i||^2 + \sum_{i\neq j}\langle a_i, a_j\rangle\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n \mathbf{E}\left[||a_i||^2\right] + \sum_{i\neq j}\mathbf{E}\left[\langle a_i, a_j\rangle\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n \mathbf{E}\left[||a_i||^2\right] + \sum_{i\neq j}\langle \underbrace{\mathbf{E}\left[a_i\right]}_{0}, \underbrace{\mathbf{E}\left[a_j\right]}_{0}\rangle \\ &\leq \frac{1}{n^2}\sum_{i=1}^n \omega_i ||\nabla f_i(x)||^2 \\ &= \frac{\omega}{n^2}\sum_{i=1}^n ||\nabla f_i(x)||^2. \end{aligned}$$

Next, bound

$$\begin{aligned}
||\nabla f_i(x)||^2 &= ||\nabla f_i(x) - \nabla f_i(y) + \nabla f_i(y)||^2 \\
&\le 2||\nabla f_i(x) - \nabla f_i(y)||^2 + 2||\nabla f_i(y)||^2 \\
&\le 4L_i D_{f_i}(x,y) + 2||\nabla f_i(y)||^2.
\end{aligned}$$

Combine everything:

$$\begin{aligned}
G(x,y) &\le \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + 2LD_f(x,y) \\
&\le \frac{\omega}{n^2}\sum_{i=1}^{n} ||\nabla f_i(x)||^2 + 2LD_f(x,y) \\
&\le \frac{\omega}{n^2}\sum_{i=1}^{n} \left(4L_i D_{f_i}(x,y) + 2||\nabla f_i(y)||^2\right) + 2LD_f(x,y) \\
&= 2\frac{\omega}{n}\left(2\sum_{i=1}^{n}\frac{1}{n}L_i D_{f_i}(x,y) + \frac{1}{n}\sum_{i=1}^{n}||\nabla f_i(y)||^2\right) + 2LD_f(x,y) \\
&\le 2\frac{\omega}{n}\left(2L_{\max}D_f(x,y) + \sigma^2(y)\right) + 2LD_f(x,y) \\
&= 2(L + 2L_{\max})D_f(x,y) + 2\frac{\omega}{n}\sigma^2(y).
\end{aligned} \tag{14}$$

Else, if $\nabla f(y) = 0$, then

$$\begin{aligned}
G(x,y) &= \mathbf{E}\left[||\hat{g}(x)||^2\right] \\
&= \mathbf{E}\left[||\hat{g}(x) - \mathbf{E}\left[\hat{g}(x)\right]||^2\right] + ||\mathbf{E}\left[\hat{g}(x)\right]||^2 \\
&= \mathbf{E}\left[||\hat{g}(x) - \nabla f(x)||^2\right] + ||\nabla f(x)||^2 \\
&\le \left(\frac{\omega}{n^2}\sum_{i=1}^{n}||\nabla f_i(x)||^2\right) + ||\nabla f(x) - \nabla f(y) + \nabla f(y)||^2 \\
&\le \left(\frac{\omega}{n^2}\sum_{i=1}^{n}||\nabla f_i(x)||^2\right) + 2||\nabla f(x) - \nabla f(y)||^2 + 2||\underbrace{\nabla f(y)}_{0}||^2 \\
&\le \left(\frac{\omega}{n^2}\sum_{i=1}^{n}||\nabla f_i(x)||^2\right) + 2LD_f(x,y) \\
&\le \text{... same as (14), from the third line} \\
&= 2(L + 2L_{\max})D_f(x,y) + 2\frac{\omega}{n}\sigma^2(y).
\end{aligned} \tag{15}$$

[P10]

E41

Let

$$p_i = \text{Prob}(i \in S),$$

where

$$S \subseteq \{1, 2, ..., d\},$$

then

$$\mathbf{E}[|S|] = \mathbf{E}\left[\sum_{i=1}^{d}|S_i|\right] = \sum_{i=1}^{d}\mathbf{E}[|S_i|] = \sum_{i=1}^{d} 1p_i + 0(1-p_i) = \sum_{i=1}^{d} p_i.$$

E42

If $\mathbf{E}[\mathbf{C}^\top \mathbf{C}]$ is finite, then $\forall x \neq 0$:

$$\begin{aligned} x^T \mathbf{E}[\mathbf{C}^\top \mathbf{C}] x &\geq 0 \\ \mathbf{E}[x^T \mathbf{C}^\top \mathbf{C} x] &\geq 0 \\ x^T \mathbf{C}^\top \mathbf{C} x &\geq 0 \\ (\mathbf{C}x)^\top (\mathbf{C}x) &\geq 0. \end{aligned}$$

[P11]