SGD Ex

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[P4]

E1

Def. 17:

$$D_f(x,y) + D_f(y,x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle = \langle \nabla f(y) - \nabla f(x), y - x \rangle \tag{1}$$

$\forall x, y \in \mathbb{R}^d$:

$$\begin{aligned}
\mu||x-y||^2 &\leq 2D_f(x,y), \\
\frac{\mu}{2}||x-y||^2 &\leq D_f(x,y), \\
\frac{\mu}{2}||x-y||^2 &\leq D_f(y,x), \\
D_f(x,y) + \frac{\mu}{2}||x-y||^2 &\leq D_f(x,y) + D_f(y,x), \\
D_f(x,y) + \frac{\mu}{2}||x-y||^2 &\overset{(1)}{\leq} \langle \nabla f(x) - \nabla f(y), x - y \rangle.
\end{aligned} \tag{2}$$

E2

$$\begin{aligned}
D_f(x,y) + \frac{\mu}{2}||x-y||^2 &\leq \langle \nabla f(x) - \nabla f(y), x - y \rangle, \\
\langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \underbrace{D_f(x,y)}_{\geq \frac{\mu}{2}||x-y||^2} + \frac{\mu}{2}||x-y||^2, \\
\langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \frac{\mu}{2}||x-y||^2 + \frac{\mu}{2}||x-y||^2, \\
\langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \mu||x-y||^2.
\end{aligned} \tag{3}$$

[P6]

E17

(Equation 34):

$$\begin{aligned}
\langle a, b\rangle &\leq \frac{||a||^2}{2t} + \frac{t||b||^2}{2},\\
\langle a, b\rangle &\leq \frac{\langle a, a\rangle}{2t} + \frac{t\langle b, b\rangle}{2},\\
2t\langle a, b\rangle &\leq \langle a, a\rangle + t^2\langle b, b\rangle,\\
0 &\leq \langle a, a\rangle + \langle tb, tb\rangle - \langle a, tb\rangle - \langle tb, a\rangle,\\
0 &\leq ||a - tb||^2.
\end{aligned} \tag{4}$$

(Equation 35):

$$\begin{aligned}
||a + b||^2 \leq 2||a||^2 + 2||b||^2,\\
\langle a, a\rangle + \langle b, b\rangle + 2\langle a, b\rangle \leq 2\langle a, a\rangle + 2\langle b, b\rangle,\\
0 \leq \langle a, a\rangle + \langle b, b\rangle - 2\langle a, b\rangle,\\
0 \leq ||a - b||^2.
\end{aligned} \tag{5}$$

(Equation 36):

$$\begin{aligned}
\frac{1}{2}||a||^2 - ||b||^2 \leq ||a + b||^2,\\
\frac{1}{2}\langle a, a\rangle - \langle a, a\rangle \leq \langle a, a\rangle + \langle b, b\rangle + 2\langle a, b\rangle,\\
\langle a, a\rangle - 2\langle b, b\rangle \leq 2\langle a, a\rangle + 2\langle b, b\rangle + 4\langle a, b\rangle,\\
0 \leq \langle a, a\rangle + \langle 2b, 2b\rangle + \langle a, 2b\rangle + \langle 2b, a\rangle,\\
0 \leq ||a + 2b||^2.
\end{aligned} \tag{6}$$

E19

For random vector $X \in \mathbb{R}^d$:

$$\mathbf{Var}[X] := \mathbf{E}\left[||X - \mathbf{E}[X]||^2\right]. \tag{7}$$

Markov's inequality:

$$\text{Prob}(X \geq t) \leq \frac{\mathbf{E}[X]}{t}. \tag{8}$$

Proof of Chebyshev's inequality using Markov's inequality:

$$\text{Prob}(||X - \mathbf{E}[X]||^2 \geq t^2) \leq \frac{\mathbf{E}\left[||X - \mathbf{E}[X]||^2\right]}{t^2}.$$

Since

$$\text{Prob}(||X - \mathbf{E}[X]||^2 \geq t^2) = \text{Prob}(||X - \mathbf{E}[X]|| \geq t), \tag{9}$$

then

$$\text{Prob}(||X - \mathbf{E}[X]|| \geq t) \leq \frac{\mathbf{Var}[X]}{t^2}. \tag{10}$$

[P7]

E24

If

$$f = \frac{1}{n}\sum_{i=1}^{n} f_i,$$

then

$$\begin{aligned}
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n} f_i(x) - \frac{1}{n}\sum_{i=1}^{n} f_i(y) - \frac{1}{n}\sum_{i=1}^{n}\langle \nabla f_i(y), x-y\rangle,\\
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n}(f_i(x) - f_i(y) - \langle \nabla f_i(y), x-y\rangle),\\
D_f(x,y) &= \frac{1}{n}\sum_{i=1}^{n} D_{f_i}(x,y).
\end{aligned}$$

E26

If $\sigma_\star^2 = 0$, then

$$\begin{aligned}
\sigma_\star^2 &= \left(\frac{1}{n^2}\sum_{i=1}^{n}\frac{||\nabla f_i(x^\star)||^2}{p_i}\right) - ||\nabla f(x^\star)||^2 = 0\\
&= \left(\frac{1}{n^2}\sum_{i=1}^{n}\frac{||np_i\nabla f(x^\star)||^2}{p_i}\right) - ||\nabla f(x^\star)||^2 = 0\\
&= p_i\sum_{i=1}^{n}(||\nabla f(x^\star)||^2) - ||\nabla f(x^\star)||^2 = 0\\
&= np_i||\nabla f(x^\star)||^2 - ||\nabla f(x^\star)||^2 = 0,\\
\sigma_\star^2 = 0 &\implies np_i\nabla f(x^\star) = \nabla f(x^\star).
\end{aligned}$$

[P8]

E33

Let

$$\chi_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}.$$

Since

$$p_i = \frac{1}{n},$$

and

$$|S| = \tau,$$

then

$$\mathbf{E}[\chi_i] = \mathrm{Prob}(i \in S) = \sum_{i=1}^{n} p_i\chi_i = \frac{1}{n}\sum_{i=1}^{n}\chi_i = \frac{\tau}{n}.$$

E35

For any vectors, $b_1, ..., b_n \in \mathbb{R}^d$:

$$\left\|\sum_{i=1}^n b_i\right\|^2 - \sum_{i=1}^n \|b_i\|^2 = \underbrace{\sum_{i=1}^n \langle b_i, b_i\rangle + \sum_{i\neq j}\langle b_i, b_j\rangle}_{\left\|\sum_{i=1}^n b_i\right\|^2} - \sum_{i=1}^n \langle b_i, b_i\rangle,$$

$$\left\|\sum_{i=1}^n b_i\right\|^2 - \sum_{i=1}^n \|b_i\|^2 = \sum_{i\neq j}\langle b_i, b_j\rangle.$$

[P9]

E37

Assumptions of $\mathcal{C} : \mathbb{R}^d \to \mathbb{R}^d$:

1. $\mathbf{E}[\mathcal{C}(x)] = x, \quad \forall x \in \mathbb{R}^d$
2. $\mathbf{E}\left[||\mathcal{C}(x) - x||^2\right] \leq \omega||x||^2 + \delta, \quad \forall x \in \mathbb{R}^d, \quad \exists \omega, \delta \geq 0$

Proof of convergence for CGD with $n = 1$:
Since $\mathcal{C} \in \mathbb{B}^d(\omega)$,

$$\mathbf{E}\left[||g(x)||^2\right] = \mathbf{E}\left[||\mathcal{C}(\nabla f(x))||^2\right] \leq (\omega + 1)||\nabla f(x)||^2. \tag{11}$$

In case of $\nabla f(y) = 0$,

$$\begin{aligned} G(x, y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\ &= \mathbf{E}\left[||g(x)||^2\right] \\ &\overset{(11)}{\leq} (\omega + 1)||\nabla f(x) - \nabla f(y)||^2, \\ &\leq 2(\omega + 1)LD_f(x, y). \end{aligned}$$

In case of $\nabla f(y) \neq 0$,

$$\begin{aligned} G(x, y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\ &= \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \\ &= \mathbf{E}\left[||\mathcal{C}(\nabla f(x)) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \\ &\leq \omega||\nabla f(x)||^2 + \delta + ||\nabla f(x) - \nabla f(y)||^2 \\ &= \omega||\nabla f(x) - \nabla f(y) + \nabla f(y)||^2 + ||\nabla f(x) - \nabla f(y)||^2 + \delta \\ &\leq 2\omega||\nabla f(x) - \nabla f(y)||^2 + 2\omega||\nabla f(y)||^2 + ||\nabla f(x) - \nabla f(y)||^2 + \delta \\ &= (2\omega + 1)||\nabla f(x) - \nabla f(y)||^2 + 2\omega||\nabla f(y)||^2 + \delta \\ &\leq 2\underbrace{(2\omega + 1)L}_{A} D_f(x, y) + \underbrace{2\omega||\nabla f(y)||^2 + \delta}_{C}. \end{aligned}$$

If $0 < \gamma < \frac{1}{A}$, then

$$\mathbf{E}\left[||x^k - x^*||^2\right] \leq (1 - \gamma\mu)^k||x^0 - x^*|| + \frac{2\gamma\omega||\nabla f(x^*)||^2 + \gamma\delta}{\mu}.$$

E39

Lemma 51:
if $\mathcal{C}(x) = x, \forall x$ (no master compression) and $\omega_i = \omega, \forall i$, then

$$G(x,y) \leq 2\underbrace{\left(L + 2L_{\max}\frac{\omega}{n}\right)}_{A} D_f(x,y) + \underbrace{2\frac{\omega}{n}\sigma^2(y)}_{C(y)},$$

where

$$\sigma^2(y) := \frac{1}{n}\sum_{i=1}^{n} ||\nabla f_i(y)||^2.$$

If $\sigma^2(y) = 0$, then

$$G(x,y) \leq 2\underbrace{\left(L + L_{\max}\frac{\omega}{n}\right)}_{A} D_f(x,y).$$

Proof:
If $\nabla f(y) \neq 0$, then

$$\begin{aligned} G(x,y) &:= \mathbf{E}\left[||g(x) - \nabla f(y)||^2\right] \\ &= \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + ||\nabla f(x) - \nabla f(y)||^2 \\ &\leq \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + 2LD_f(x,y), \end{aligned} \tag{12}$$

and

$$g(x) = \mathcal{C}(\hat{g}(x)) = \hat{g}(x) = \frac{1}{n}\sum_{i=1}^{n} g_i(x). \tag{13}$$

where

$$g_i(x) = \mathcal{C}_i(\nabla f_i(x)).$$

Estimate

$$\begin{aligned} \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] &\overset{(13)}{=} \mathbf{E}\left[||\mathcal{C}(\hat{g}(x)) - \nabla f(x)||^2\right] \\ &= \mathbf{E}\left[||\hat{g}(x) - \nabla f(x)||^2\right] \\ &= \mathbf{E}\left[\left|\left|\frac{1}{n}\sum_{i=1}^{n}\underbrace{(g_i(x) - \nabla f_i(x))}_{a_i}\right|\right|^2\right] \\ &= \frac{1}{n^2}\mathbf{E}\left[\sum_{i=1}^{n}||a_i||^2 + \sum_{i\neq j}\langle a_i, a_j\rangle\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}\left[||a_i||^2\right] + \sum_{i\neq j}\mathbf{E}\left[\langle a_i, a_j\rangle\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}\left[||a_i||^2\right] + \sum_{i\neq j}\langle\underbrace{\mathbf{E}\left[a_i\right]}_{0}, \underbrace{\mathbf{E}\left[a_j\right]}_{0}\rangle \\ &\leq \frac{1}{n^2}\sum_{i=1}^{n}\omega_i||\nabla f_i(x)||^2 \\ &= \frac{\omega}{n^2}\sum_{i=1}^{n}||\nabla f_i(x)||^2. \end{aligned}$$

Next, bound

$$\begin{aligned}||\nabla f_i(x)||^2 &= ||\nabla f_i(x) - \nabla f_i(y) + \nabla f_i(y)||^2 \\ &\le 2||\nabla f_i(x) - \nabla f_i(y)||^2 + 2||\nabla f_i(y)||^2 \\ &\le 4L_i D_{f_i}(x,y) + 2||\nabla f_i(y)||^2.\end{aligned}$$

Combine everything:

$$\begin{aligned}G(x,y) &\le \mathbf{E}\left[||g(x) - \nabla f(x)||^2\right] + 2LD_f(x,y) \\ &\le \frac{\omega}{n^2}\sum_{i=1}^n ||\nabla f_i(x)||^2 + 2LD_f(x,y) \\ &\le \frac{\omega}{n^2}\sum_{i=1}^n \left(4L_iD_{f_i}(x,y) + 2||\nabla f_i(y)||^2\right) + 2LD_f(x,y) \\ &= 2\frac{\omega}{n}\left(2\sum_{i=1}^n \frac{1}{n}L_iD_{f_i}(x,y) + \frac{1}{n}\sum_{i=1}^n ||\nabla f_i(y)||^2\right) + 2LD_f(x,y) \\ &\le 2\frac{\omega}{n}\left(2L_{\max}D_f(x,y) + \sigma^2(y)\right) + 2LD_f(x,y) \\ &= 2(L + 2L_{\max})D_f(x,y) + 2\frac{\omega}{n}\sigma^2(y).\end{aligned} \tag{14}$$

Else, if $\nabla f(y) = 0$, then

$$\begin{aligned}G(x,y) &= \mathbf{E}\left[||\hat{g}(x)||^2\right] \\ &= \mathbf{E}\left[||\hat{g}(x) - \mathbf{E}\left[\hat{g}(x)\right]||^2\right] + ||\mathbf{E}\left[\hat{g}(x)\right]||^2 \\ &= \mathbf{E}\left[||\hat{g}(x) - \nabla f(x)||^2\right] + ||\nabla f(x)||^2 \\ &\le \left(\frac{\omega}{n^2}\sum_{i=1}^n ||\nabla f_i(x)||^2\right) + ||\nabla f(x) - \nabla f(y) + \nabla f(y)||^2 \\ &\le \left(\frac{\omega}{n^2}\sum_{i=1}^n ||\nabla f_i(x)||^2\right) + 2||\nabla f(x) - \nabla f(y)||^2 + 2||\underbrace{\nabla f(y)}_{0}||^2 \\ &\le \left(\frac{\omega}{n^2}\sum_{i=1}^n ||\nabla f_i(x)||^2\right) + 2LD_f(x,y) \\ &\le \text{... same as (14), from the third line} \\ &= 2(L + 2L_{\max})D_f(x,y) + 2\frac{\omega}{n}\sigma^2(y).\end{aligned} \tag{15}$$

[P10]

E41

Let

$$p_i = \mathrm{Prob}(i \in S),$$

where

$$S \subseteq \{1, 2, ..., d\},$$

then

$$\mathbf{E}[|S|] = \mathbf{E}\left[\sum_{i=1}^d |S_i|\right] = \sum_{i=1}^d \mathbf{E}[|S_i|] = \sum_{i=1}^d 1p_i + 0(1-p_i) = \sum_{i=1}^d p_i.$$

E42

If $\mathbf{E}[\mathbf{C}^\top\mathbf{C}]$ is finite, then $\forall x \neq 0$:

$$\begin{aligned} x^T\mathbf{E}[\mathbf{C}^\top\mathbf{C}]x &\geq 0 \\ \mathbf{E}[x^T\mathbf{C}^\top\mathbf{C}x] &\geq 0 \\ x^T\mathbf{C}^\top\mathbf{C}x &\geq 0 \\ (\mathbf{C}x)^\top(\mathbf{C}x) &\geq 0. \end{aligned}$$

[P11]

E47

Define base case:

$$\begin{aligned} C_{1,2} &:= C_1 \circ C_2 \in \mathbb{B}^d(\underbrace{(\omega_1+1)(\omega_2+1)-1}_{\omega_{1,2}}), \\ C_{1,3} &:= C_1 \circ C_2 \circ C_3 = C_{1,2} \circ C_3, \\ \omega_{1,3} &:= (\omega_{1,2}+1)(\omega_3+1)-1 \\ &= ((\omega_1+1)(\omega_2+1)-1+1)(\omega_3+1)-1 \\ &= (\omega_1+1)(\omega_2+1)(\omega_3+1)-1, \\ C_{1,n} &:= C_1 \circ C_2 \circ ... \circ C_n = C_{1,n-1} \circ C_n, \\ \omega_{1,n} &:= (\omega_1+1)(\omega_2+1)...(\omega_n+1)-1 = (\omega_{1,n-1}+1)(\omega_n+1)-1. \end{aligned}$$

By induction, the base case is clear. Next, if $n = k$, assume

$$\begin{aligned} C_{1,k} &:= C_1 \circ C_2 \circ ... \circ C_k = C_{1,k-1} \circ C_k, \\ \omega_{1,k} &:= (\omega_1+1)(\omega_2+1)...(\omega_k+1)-1 = (\omega_{1,k-1}+1)(\omega_k+1)-1 \end{aligned}$$

is true. Then for $n = k+1$:

$$\begin{aligned} C_{1,k+1} &:= C_1 \circ C_2 \circ ... \circ C_k \circ C_{k+1} = C_{1,k-1} \circ C_k \circ C_{k+1}, \\ \omega_{1,k+1} &:= (\omega_{1,k}+1)(\omega_{k+1}+1)-1 = (\omega_1+1)(\omega_2+1)...(\omega_k+1)(\omega_{k+1}+1)-1 \\ &:= ((\omega_{1,k-1}+1)(\omega_k+1)-1+1)(\omega_{k+1}+1)-1 = (\omega_1+1)(\omega_2+1)...(\omega_k+1)(\omega_{k+1}+1)-1 \\ &:= ((\omega_1+1)(\omega_2+1)...(\omega_k+1))(\omega_{k+1}+1)-1 = (\omega_1+1)(\omega_2+1)...(\omega_k+1)(\omega_{k+1}+1)-1, \\ \omega_{1,k+1} &= (\omega_1+1)(\omega_2+1)...(\omega_k+1)(\omega_{k+1}+1)-1. \end{aligned}$$

E48

Define

$$\min\{a_i, b_i\} = \begin{cases} a_i, & \text{if } a_i < b_i \\ b_i, & \text{if } a_i > b_i \end{cases}.$$

Thus

$$\sum_i \min\{a_i, b_i\} = \begin{cases} \sum_i a_i, & \text{if } a_i < b_i, \forall i \\ \sum_i b_i, & \text{if } a_i > b_i, \forall i \end{cases}.$$

In case of inequality, define

$$\begin{aligned} I &:= \{i | a_i < b_i\}, \\ J &:= \{i | a_i > b_i\}. \end{aligned}$$

Thus

$$\sum_i \min\{a_i, b_i\} < \begin{cases} \sum_i a_i, & \text{if } |I| > |J| \\ \sum_i b_i, & \text{if } |J| > |I| \end{cases}.$$

[P12]

E55

The DCGD-SHIFT has the same exact steps in the algorithm as DCGD ($n \geq 1$ case), the difference is the gradient estimator:

$$g_h(x) := \frac{1}{n}\sum_{i=1}^{n} g_{h_i}(x) = \frac{1}{n}\sum_{i=1}^{n} h_i + \mathcal{C}_i(\nabla f_i(x) - h_i). \tag{16}$$

which means the gradients on the workers are shifted and then compressed. In order to prove the convergence theorem, first decompose

$$\mathbf{E}\left[||g_h(x^k) - \nabla f(x^\star)||^2\right] = \mathbf{E}\left[||g_h(x^k) - \nabla f(x^k)||^2\right] + ||\nabla f(x^k) - \nabla f(x^\star)||^2.$$

Then, bound

$$\begin{aligned}
\mathbf{E}\left[||g_h(x^k) - \nabla f(x^k)||^2\right] &= \mathbf{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathcal{C}_i(\nabla f_i(x^k) - h_i) + h_i - \nabla f_i(x^k)}_{b_i^k}\right\|^2\right] \\
&= \frac{1}{n^2}\mathbf{E}\left[\sum_i ||b_i^k||^2 \sum_{i\neq j}\langle b_i^k, b_j^k\rangle\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}\left[||b_i^k||^2\right] + \frac{1}{n^2}\sum_{i\neq j}\underbrace{\langle \mathbf{E}[b_i^k], \mathbf{E}[b_j^k]\rangle}_{0} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{E}\left[\left\|\mathcal{C}_i(\nabla f_i(x^k) - h_i) + h_i - \nabla f_i(x^k)\right\|^2\right] \\
&\leq \frac{1}{n^2}\sum_{i=1}^{n}\omega_i||\nabla f_i(x^k) - h_i||^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\omega_i||\nabla f_i(x^k) - \nabla f_i(x^\star) - (h_i - \nabla f_i(x^\star))||^2 \\
&\leq \frac{2}{n^2}\sum_{i=1}^{n}\omega_i||\nabla f_i(x^k) - \nabla f_i(x^\star)||^2 + \omega_i||h_i - \nabla f_i(x^\star)||^2 \\
&\leq \frac{2}{n^2}\sum_{i=1}^{n}2\omega_i L_i D_{f_i}(x^k, x^\star) + \frac{2}{n^2}\sum_{i=1}^{n}\omega_i||h_i - \nabla f_i(x^\star)||^2 \\
&\leq \frac{4}{n}\max(L_i\omega_i)\frac{1}{n}\sum_{i=1}^{n}D_{f_i}(x^k, x^\star) + \frac{2}{n^2}\sum_{i=1}^{n}\omega_i||h_i - \nabla f_i(x^\star)||^2 \\
&\leq \frac{4}{n}\max(L_i\omega_i)D_{f_i}(x^k, x^\star) + \frac{2}{n^2}\sum_{i=1}^{n}\omega_i||h_i - \nabla f_i(x^\star)||^2.
\end{aligned}$$

Thus

$$\mathbf{E}\left[||g_h(x^k) - \nabla f(x^\star)||^2\right] \leq 2\underbrace{\left(L + \frac{2}{n}\max(\omega_i L_i)\right)}_{A} D_f(x^k, x^\star) + \underbrace{\frac{2}{n^2}\sum_{i=1}^{n}\omega_i||h_i - \nabla f_i(x^\star)||^2}_{C}.$$

[P13]

E56

Thm 94: whenever $B = 0$ and $M = 0$, then $\frac{B+M\tilde{B}}{M} = 0$, proof:
First, the stepsize γ satisfies

$$0 < \gamma < \frac{1}{\mu}. \tag{17}$$

Then the iterates $\{x^k, \sigma^k\}$ satisfy

$$\mathbf{E}[d^k] \leq (1-\gamma\mu)^k d^0 + \frac{C\gamma}{\mu}. \tag{18}$$

where

$$d^k := \left\|x^k - x^\star\right\|^2. \tag{19}$$

From Lemma 95, it is clear

$$\mathbf{E}[d^{k+1}] \leq (1-\gamma\mu)\mathbf{E}[d^k] + C\gamma^2.$$

By recurrence, we obtain

$$\mathbf{E}[d^k] \leq (1-\gamma\mu)^k d^0 + \frac{C\gamma}{\mu}.$$

[P14]

E57

In case of arbitrary p, the gradient estimator of L-SVRG is

$$g^k := g(x^k) - g(y^k) + \nabla f(y^k).$$

Hence, the unbiasedness:

$$\begin{aligned}
\mathbf{E}[g^k|x^k, y^k] &= \mathbf{E}[g(x^k) - g(y^k) + \nabla f(y^k)|x^k, y^k] \\
&= \mathbf{E}[g(x^k)|x^k, y^k] - \mathbf{E}[g(y^k)|x^k, y^k] + \mathbf{E}[\nabla f(y^k)|x^k, y^k] \\
&= \nabla f(x^k) - \nabla f(y^k) + \nabla f(y^k) \\
&= \nabla f(x^k).
\end{aligned}$$

E58

If

$$g(x) = \nabla f(x) + \xi,$$

then

$$\begin{aligned} g^k &= (\nabla f(x^k) + \xi) - (\nabla f(y^k) + \xi) + \nabla f(y^k) \\ &= \nabla f(x^k), \end{aligned}$$

which is exactly GD's gradient estimator, where in this case p does not have any role, since the gradient estimator does not depend on y^k anymore. The convergence rate in this case, with stepsize $\gamma = \frac{1}{6A''}$ is

$$\mathbf{E}[d^k] \le \left(1 - \frac{\mu}{6A''}\right)^k d^0,$$

where

$$d^k := \left\|x^k - x^\star\right\|^2.$$

Thus

$$k \ge \frac{6A''}{\mu} \log \frac{1}{\epsilon},$$

which is equal to GD's rate of $\mathcal{O}\left(\frac{L}{\mu} \log \frac{1}{\epsilon}\right)$.

[P15]

E62

The algorithm with (200) as the update rule is equivalent to CGD if we set

$$\begin{aligned} x^k &:= h_i^k, \\ g^k &:= \mathcal{C}_i^k(\nabla f_i(x^k) - h_i^k). \end{aligned}$$

Corollary 49 says that, if $0 < \gamma \le \frac{1}{(\omega+1)L}$ and $\nabla f(x^\star) = 0$, then

$$\mathbf{E}\left[\left\|x^k - x^\star\right\|^2\right] \le (1 - \gamma\mu)^k \left\|x^0 - x^\star\right\|^2.$$

In case of one step iteration, then

$$\mathbf{E}\left[\left\|x^{k+1} - x^\star\right\|^2\right] \le (1 - \gamma\mu) \left\|x^k - x^\star\right\|^2. \tag{20}$$

Since the optimization problem is in the form of

$$\max_{h_i} \phi_i^k(h_i) := -\frac{1}{2}\left\|h_i - \nabla f_i(x^k)\right\|^2, \tag{21}$$

then the solution is

$$\nabla \phi_i^k(h_i) = \nabla f_i(x^k) - h_i = 0 \implies \nabla f_i(x^k) = h_i. \tag{22}$$

which corresponds to $\nabla f(x^\star) = 0$ in CGD's case. Also, it can be noticed that ϕ_i^k is 1-smooth and 1-strongly convex, i.e., $\mu = L = 1$. Thus, with $0 < \alpha \le \frac{1}{w_i+1}$, (200) is equivalent to (20).

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The update rule

$$h^{k+1} = h^k - \alpha \mathcal{C}(h^k - \nabla f(x^k))$$

can be interpreted as a *descent* rule instead of *ascent*, and minimize rather than maximize. The solution for the minimization of ϕ_i^k is (22), the only difference is the compressed shifted gradient is

$$\tilde{g}^k := h^k - \nabla f(x^k) = -(\nabla f(x^k) - h^k),$$

where this does not change the convergence properties since $\left\|\tilde{g}^k\right\|^2$ is considered. Finally, we have

$$h_i^{k+1} = h_i^k - \alpha \mathcal{C}_i(\tilde{g}_i^k)$$

which is the interpretation of descent. Hence (200) still holds.