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A characterization of quintic helices

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Abstract

A polynomial curve of degree 5, α, is a helix if and only if both $\|\alpha'\|$ and $\|\alpha' \wedge \alpha''\|$ are polynomial functions.
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1. Introduction

In [4] the authors study the notion of polynomial curves which made a constant angle with a fixed line in space. Such curves are called there helices and we will use here the same terminology. In fact, a curve α is a helix if given a unitary vector $\overrightarrow{u}$ along the fixed line (the axis of the helix) then $\langle \overrightarrow{u}, \overrightarrow{t} \rangle = c$, where $c \in \mathbb{R}$ and $\overrightarrow{t} = \alpha'/\|\alpha'\|$ is fthe tangent vector. However, in other contexts those curves are also named generalized helices and the term “helices” is reserved to curves in a cylinder with the same property.

We refer to the introduction of the cited paper and some other papers like [2,3] for the relationship between such curves and some problems in the realm of computer-aided design of curves and surfaces. The only fact we want to recall here is that, for real applications, it seems clear that the suitable curves are the quintic helices.

As it is said in [4], any polynomial helix, α, must be a Pythagorean hodograph (PH) curve, i.e., $\|\alpha'\|^2$ is a perfect square of a polynomial. Moreover, this condition is sufficient in the cubical case: all PH cubics are helices.

As it is also said in the cited paper (along the lines after formula (11)), another necessary condition to be a helix is the fact that $\|\alpha' \wedge \alpha''\|^2/\|\alpha'\|^2$ (denoted there by ρ) must also be a perfect square of a polynomial. However, here we will focus on the expression $\|\alpha' \wedge \alpha''\|^2$ instead of ρ. Note that, as $\|\alpha'\|^2$ is a perfect square the expression $\|\alpha' \wedge \alpha''\|^2 = \|\alpha'\|^2 \rho$ must also be a perfect square of a polynomial. The easiest way to see that a polynomial helix verifies that $\|\alpha' \wedge \alpha''\|$ is a polynomial is the following: the argument that shows that a polynomial helix must be PH, there applied to the tangent vector, $\overrightarrow{t} = \alpha'/\|\alpha'\|$, can also be applied to the binormal vector, $\overrightarrow{b} = \alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|$.(See the proof of Proposition 1 for details.) The consequence now is that $\|\alpha' \wedge \alpha''\|$ is polynomial.

In this short paper we will go a little bit forward and show that both conditions are sufficient in the quintic case. Moreover, we will show an example of polynomial curve of degree 7 verifying both conditions but not being a helix.

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2. Spatial PH curves

We will use the quaternion representation of spatial PH curves. Given a quaternion polynomial

$$\mathscr{A}(t) = u(t) + \mathbf{i}v(t) + \mathbf{j}p(t) + \mathbf{k}q(t),$$

the product

$$\alpha'(t) = \mathscr{A}(t)\mathbf{i}\mathscr{A}^*(t)$$

defines a spatial PH, α', whose components are

$$\begin{aligned} x' &= u^2 + v^2 - p^2 - q^2, \\ y' &= 2(uq + vp), \\ z' &= 2(vq - up), \end{aligned} \tag{2.1}$$

and such that $\|\alpha'\|^2 = (x')^2 + (y')^2 + (z')^2 = (u^2 + v^2 + p^2 + q^2)^2$.

In terms of the Hopf map (see [2, Theorem 4.2])

$$H : \mathbb{C}^2 \to \mathbb{R}^3$$

defined by $H(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2)$, and taking

$$z_1(t) = u(t) + \mathbf{i}v(t), \quad z_2(t) = q(t) + \mathbf{i}p(t), \tag{2.2}$$

the derivative of the curve can be written as

$$\alpha'(t) = H(z_1(t), z_2(t)).$$

Definition 1. A curve α is said a PH curve of second class (2-PH curve) if both $\|\alpha'\|$ and $\|\alpha' \wedge \alpha''\|$ are polynomial functions.

It is easy to check that 2-PH curves are examples of curves with a rational Frenet–Serret frame (see [6]).

Lemma 1. *The Frenet–Serret frame of any 2-PH curve is made of rational vectorial functions.*

Proof. Simply recall that

$$\overrightarrow{t} = \frac{\alpha'}{\|\alpha'\|}, \quad \overrightarrow{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|} \quad \text{and} \quad \overrightarrow{n} = \overrightarrow{b} \wedge \overrightarrow{t}. \qquad \square$$

Proposition 1. *Polynomial helices are 2-PH curves.*

Proof. Let us recall that if a curve is a helix, then, not only the tangent vector makes a constant angle with the axis, but also the binormal vector, see the classical references [1,5]. In fact, if $\overrightarrow{u}$ is a unitary vector that determines the axis of the helix then

$$\langle \overrightarrow{u}, \overrightarrow{t} \rangle = c, \quad \langle \overrightarrow{u}, \overrightarrow{b} \rangle = \sqrt{1 - c^2},$$

where $c \in \mathbb{R}$ is a constant, $\overrightarrow{t} = \alpha'/\|\alpha'\|$ is the tangent vector and $\overrightarrow{b} = \alpha' \wedge \alpha''/\|\alpha' \wedge \alpha''\|$ is the binormal vector of the curve.

The previous expressions are equivalent to

$$\langle \overrightarrow{u}, \alpha' \rangle = c\ \|\alpha'\|, \quad \langle \overrightarrow{u}, \alpha' \wedge \alpha'' \rangle = \sqrt{1 - c^2}\|\alpha' \wedge \alpha''\|.$$

If our curve α is polynomial then it is a PH and also the norm of $\|\alpha' \wedge \alpha''\|$ is polynomial, indeed

$$\|\alpha'\| = \frac{1}{c}\langle \overrightarrow{u}, \alpha' \rangle, \quad \|\alpha' \wedge \alpha''\| = \frac{1}{\sqrt{1 - c^2}}\langle \overrightarrow{u}, \alpha' \wedge \alpha'' \rangle. \qquad \square$$

3. Algebraic 2-PH curve characterization

Proposition 2. *Let α be a spatial PH curve whose tangent vector is defined by the functions u, v, p, q as in* (2.1) *and let z_1, z_2 be the associated complex functions as in* (2.2). *Then, $\|\alpha' \wedge \alpha''\|$ is a polynomial function if and only if there is a complex polynomial function $z(t)$ and a real polynomial function $\omega(t)$ such that*

$$z_2^2\left(\frac{z_1}{z_2}\right)' = \omega z^2. \tag{3.1}$$

Proof. A straightforward computation shows that

$$\|\alpha' \wedge \alpha''\|^2 = 4\|\alpha'\|^2((u'q - uq' - v'p + vp')^2 + (u'p - up' + v'q - vq')^2).$$

Therefore, $\|\alpha' \wedge \alpha''\|$ is a polynomial function if and only if $(u'q - uq' - v'p + vp')^2 + (u'p - up' + v'q - vq')^2$ is a perfect square of a polynomial. Since both terms, $u'q - uq' - v'p + vp'$ and $u'p - up' + v'q - vq'$, are polynomials, we can apply the well-known result about Pythagorean curves, see [3, Section 17.2]: there is a polynomial function, $\omega(t)$, and a complex polynomial function, $z(t)$, such that

$$(u'q - uq' - v'p + vp') + \mathbf{i}(u'p - up' + v'q - vq') = \omega z^2.$$

An algebraic manipulation using the functions z_1 and z_2 defined in (2.2) allows to write the left-hand member as

$$(u' + \mathbf{i}v')(q + \mathbf{i}p) - (u + \mathbf{i}v)(q' + \mathbf{i}p') = z_1'z_2 - z_1z_2' = z_2^2\left(\frac{z_1}{z_2}\right)',$$

and the statement follows. □

Example 1. Let us check this result in the two examples shown in [4].

The first example is defined by the four quadratic polynomials

$$u(t) = t^2 - 3t, \quad v(t) = t^2 - 5t + 10, \quad p(t) = -2t^2 + 3t + 5, \quad q(t) = t^2 - 9t + 10.$$

Therefore, the complex functions $z_1(t) = (t^2 - 3t) + \mathbf{i}(t^2 - 5t + 10)$ and $z_2(t) = (t^2 - 9t + 10) + \mathbf{i}(-2t^2 + 3t + 5)$ verify expression (3.1) for

$$\omega(t) = 1, \quad z(t) = \sqrt{1 - 7\mathbf{i}}\,(t - (1 + 2\mathbf{i})).$$

The second example is defined by

$$u(t) = -19t^2 + 12t + 5, \quad v(t) = -22t^2 + 18t + 1,$$
$$p(t) = 15t^2 - 12t - 1, \quad q(t) = -31t^2 + 24t + 3.$$

Now, expression (3.1) holds for

$$\omega(t) = 26(3 - 7t + 3t^2), \quad z(t) = \sqrt{-1 + \mathbf{i}}.$$

If we consider 2-PH curves of degree 5, then the functions $z_1(t)$ and $z_2(t)$ are quadratic polynomials and the term on the left of the expression

$$z_1'z_2 - z_1z_2' = \omega z^2$$

is a polynomial of degree 2. Therefore, there are two possibilities for the pair $\omega(t), z(t)$. The first is $\omega(t)$ be a quadratic function and $z(t)$ a constant function. The second, $\omega(t)$ be a constant and $z(t)$ a linear polynomial. As we will see, each possibility corresponds to one of the two classes of quintic helices studied in [4]: the general helices and the monotone helices.

4. Characterization of quintic helices

We study first the case when $\omega(t)$ is constant, and without loss of generality we can suppose that $\omega(t)=1$.

Lemma 2. *Monotone quintic helices are characterized by a constant $\omega(t)$.*

Proof. The complex polynomials $z_1(t)$ and $z_2(t)$ are of degree less or equal than 2. The first possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$z_1(t)=a(t-r_1)(t-r_2),$$

$$z_2(t)=b(t-r_3)(t-r_4),$$

where $a, b, r_i \in \mathbb{C}$. An easy computation gives us that

$$z_1'z_2 - z_1z_2' = ab((r_1+r_2-r_3-r_4)t^2 + 2(r_3r_4-r_1r_2)t + (r_3+r_4)r_1r_2 - (r_1+r_2)r_3r_4).$$

If $\omega(t)=1$ this expression is the square of a complex polynomial function of degree 1, $z(t)=mt+n$, if and only if

$$ab(r_1+r_2-r_3-r_4)=m^2,$$

$$ab(r_3r_4-r_1r_2)=mn,$$

$$ab((r_3+r_4)r_1r_2-(r_1+r_2)r_3r_4)=n^2.$$

From the first two equations we get

$$m=\pm\sqrt{ab}\sqrt{r_1+r_2-r_3-r_4}, \quad n=\frac{ab(r_3r_4-r_1r_2)}{m}.$$

Substituting in the last equation

$$r_1r_2r_3+r_1r_2r_4-r_1r_3r_4-r_2r_3r_4=\frac{r_1^2r_2^2-2r_1r_2r_3r_4+r_3^2r_4^2}{r_1+r_2-r_3-r_4}.$$

After some algebraic manipulation we can rewrite this equation as

$$(r_1-r_3)(r_2-r_3)(r_1-r_4)(r_2-r_4)=0.$$

Therefore, $z_1'z_2-z_1z_2'=z^2$ if and only if $z_1(t)$ and $z_2(t)$ share a linear factor. In this case $\gcd(z_1(t), z_2(t)) \neq$ constant which is the characterization of monotone helices (see [4, Section 3.1]). Indeed,

$$\gcd(x', y', z') = |\gcd(u+\mathbf{i}v, p-\mathbf{i}q)|^2 = |\gcd(z_1, -\mathbf{i}z_2)|^2 = |\gcd(z_1, z_2)|^2.$$

The second possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$z_1(t)=a(t-r_1)(t-r_2),$$

$$z_2(t)=b(t-r_3),$$

where $a, b, r_i \in \mathbb{C}$. A similar analysis shows that $r_3=r_1$ or $r_3=r_2$, and the same conclusion holds.

The last possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$z_1(t)=a(t-r_1)(t-r_2),$$

$$z_2(t)=b,$$

where $a, b, r_i \in \mathbb{C}$. It is easy to check that this case leads to a contradiction. □

For the case when $z(t)$ is constant we will use a description of PH quintics based on quaternions, see [4]. A spatial quintic helix is defined by a quadratic polynomial

$$\mathscr{A}(t) = \mathscr{A}_0 + \mathscr{A}_1 t + \mathscr{A}_2 t^2,$$

with quaternion coefficients

$$\mathscr{A}_0 = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k},$$

$$\mathscr{A}_1 = b + b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$

$$\mathscr{A}_2 = c + c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}.$$

In terms of the functions u, v, p, q: $u(t) = a + bt + ct^2$, $v(t) = a_x + b_x t + c_x t^2$, $p(t) = a_y + b_y t + c_y t^2$ and $q(t) = a_z + b_z t + c_z t^2$.

Lemma 3. *Let $z_1(t) = u(t) + \mathbf{i}v(t)$ and $z_2(t) = q(t) + \mathbf{i}p(t)$ be the quadratic polynomials of a quintic 2-PH curve defined by three quaternions $\{\mathscr{A}_i\}$, $i = 0, 1, 2$. If $z(t)$ in (3.1) is constant then $\mathscr{A}_1 = c_0\mathscr{A}_1 + c_2\mathscr{A}_2$, for suitable real scalars c_0, c_2.*

Remark 1. In [4] the authors use a Bézier quadratic polynomial

$$\mathscr{A}(t) = \mathscr{A}_0(1-t)^2 + 2\mathscr{A}_1 t(1-t) + \mathscr{A}_2 t^2.$$

We have used here the usual basis of polynomials instead of the Bernstein basis because computations are easier. The statement of the previous lemma remains true for Bézier quaternion coefficients due to just a change of basis.

Proof. Let us suppose that

$$\omega(t) = m_0 + m_1 t + m_2 t^2, \quad z(t) = \mathrm{e}^{\mathbf{i}\theta},$$

where, $m_0, m_1, m_2, \theta \in \mathbb{R}$ and without loss of generality, we assume that $|z| = 1$.

We use now Proposition 2 and compute the expression

$$z_1' z_2 - z_1 z_2' = \omega z^2.$$

The real part of the left-hand term can be written as

$$u'q - uq' - v'p + vp' = (a_x b_y - ab_z + a_z b - a_y b_x) + 2(a_x c_y - ac_z - a_y c_x + a_z c)t + (b_z c - b_y c_x + b_x c_y - bc_z)t^2,$$

and the imaginary part as

$$u'p - up' + v'q - vq' = (a_y b + a_z b_x - ab_y - a_x b_z) + 2(a_y c + a_z c_x - ac_y - a_x c_z)t + (b_y c + b_z c_x - bc_y - b_x c_z)t^2.$$

Analogously, the real part of the right-hand term can be written as

$$(m_0 + m_1 t + m_2 t^2)\cos(2\theta),$$

and the imaginary part as

$$(m_0 + m_1 t + m_2 t^2)\sin(2\theta).$$

Therefore, the condition $z_1' z_2 - z_1 z_2' = \omega z^2$ can be translated into a set of six equations. By equating the coefficients of t we can deduce that

$$\theta = \frac{1}{2}\arctan\left(\frac{a_y c + a_z c_x - ac_y - a_x c_z}{a_z c - a_y c_x + a_x c_y - ac_z}\right)$$

and

$$m_1 = 2\sqrt{(a_y c + a_z c_x - a c_y - a_x c_z)^2 + (a_z c - a_y c_x + a_x c_y - a c_z)^2}.$$

Substituting these values into the other four equations and solving the resulting linear system we obtain

$$\mathscr{A}_1 = \frac{2m_0}{m_1}\mathscr{A}_0 + \frac{2m_2}{m_1}\mathscr{A}_2. \qquad \square$$

Theorem 1. *A quintic polynomial curve is a helix if and only if it is a 2-PH curve.*

Proof. We have proved in Proposition 1 that any polynomial helix is a 2-PH curve.

Reciprocally, if $\|\alpha' \wedge \alpha''\|$ is polynomial then by Proposition 2 we know that $z_2^2(z_1/z_2)' = \omega z^2$. In the quintic case, the only possibilities are ω constant or z constant. If ω is constant then by Lemma 2 the curve is a monotone helix.

If z is constant, then by Lemma 3 the quaternions defining the curve are linear dependents and by Proposition 1 in [4] we know that the curve is a helix. $\square$

Remark 2. In higher dimensions it is possible to find 2-PH curves being not helices. For example,

$$\alpha(t) = \left(-3t + t^3 + \frac{t^5}{5} + \frac{t^7}{21}, 3t^2 - \frac{t^4}{2}, -2t^3\right)$$

is a polynomial curve of degree 7 verifying

$$\|\alpha'\| = \frac{1}{3}(9 + 9t^2 + 3t^4 + t^6), \quad \|\alpha' \wedge \alpha''\| = 2(1 + t^2)(9 + 9t^2 + 3t^4 + t^6)$$

but

$$\frac{\tau}{\kappa} = \frac{-9 + 9t^4 + 2t^6}{9(1 + t^2)^2},$$

so, it does not satisfy Lancret's theorem (see, for example [5]) and the curve is not a generalized helix.

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