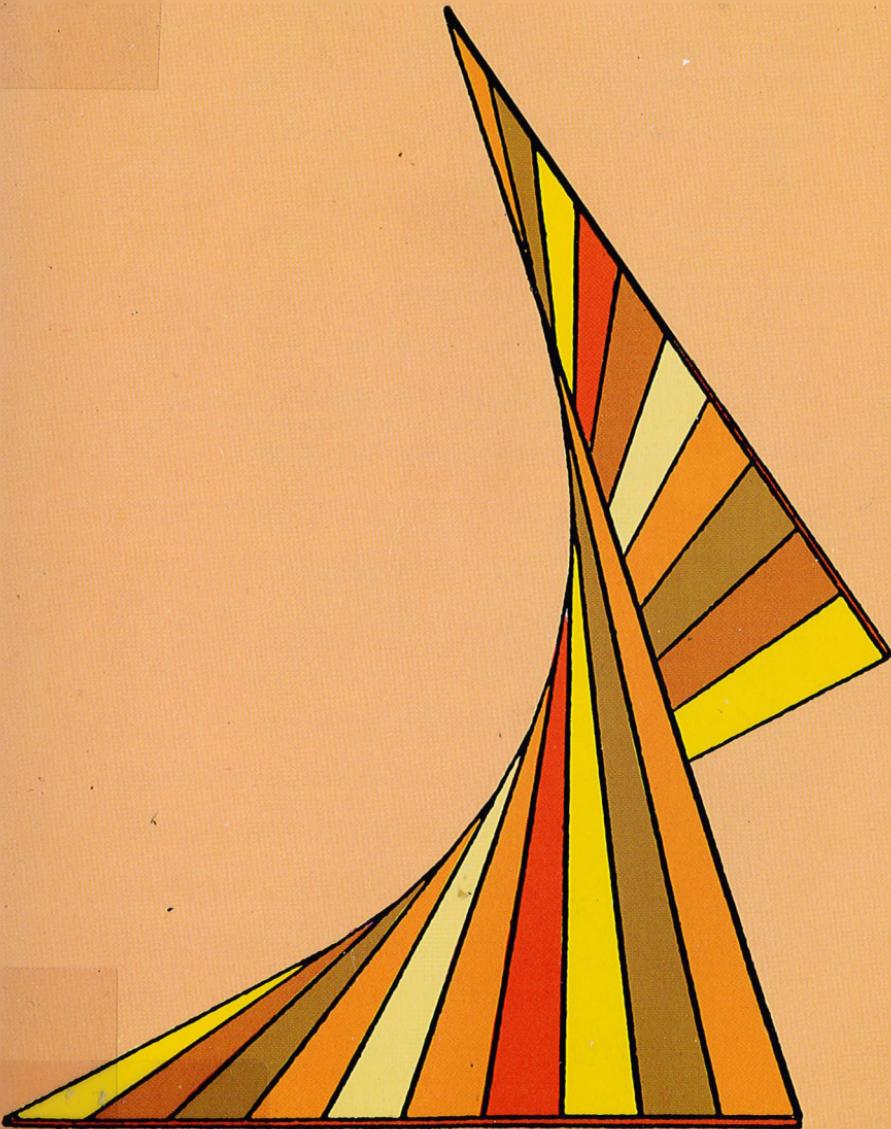


DIFFERENTIAL GEOMETRY



Erwin Kreyszig

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BY
ERWIN KREYSZIG

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PREFACE

THIS book provides an introduction to the differential geometry of curves and surfaces in three-dimensional Euclidean space. We first consider some basic concepts and facts of analytic geometry which will be useful for later investigations. The theory of space curves is presented in the second chapter. We then proceed to the foundations of the theory of surfaces. Problems closely related to the first and second fundamental forms are considered in the third and fourth chapter. Chapter V is devoted to geodesics. Several types of mappings of surfaces which are of theoretical or practical importance figure in Chapter VI, including some mappings of the sphere into the plane which are frequently used when constructing maps of the globe. In this connexion different types of special surfaces occur necessarily. This chapter is therefore related to Chapter VIII on special surfaces. The absolute differential calculus and the displacement of Levi-Civit  , which is of interest especially in connexion with the theory of relativity, are investigated in Chapter VII. As is natural the results obtained in Chapters III and IV yield the foundations of the Chapters V-VIII.

In the theory of surfaces we make full use of the tensor calculus, which is developed as needed, cf. Sections 27-33. The student will quickly find that this calculus becomes a simple tool as soon as he is accustomed to the few basic concepts and rules, especially to the 'summation convention', cf. Section 27. He will perceive that the tensor method is helpful in achieving a simplification of the analytic formalism of many investigations. Hence tensors are important tools in modern differential geometry.

The presentation in this book may also be considered as a preparation for the Riemannian geometry of n dimensions.

As is well known, tensors are of increasing importance not only in mathematics, but also in the application of mathematics to physics and engineering. Since the problems treated in differential geometry by means of tensor calculus are relatively perspicuous, they enable us to understand not only the formalism but also the nature and essential background of this calculus. The student will thus gain by being able to apply his knowledge of tensors to fields other than that of differential geometry. In using tensor calculus one should never forget that the purpose of this calculus lies in its applications to certain problems; it is a tool only, albeit a very powerful one.

We should mention that many of the topics considered in this book can also be investigated by means of outer differential forms. Cf. E. Cartan, *Les Systèmes différentiels extérieurs et leurs applications géométriques* (Paris, 1945), W. Blaschke, *Einführung in die Differentialgeometrie* (Berlin, 1950).

In writing this book, the experiences which I gained during the period of personal co-operation with Professor H. Behnke (University of Münster in Westphalia), as well as his printed lecture notes on differential geometry, were of help to me. I have tried to present the whole subject-matter in the simplest possible form consistent with the needs of mathematical rigour, and to convey a clear idea of the geometric significance of the different concepts, methods, and results. For this reason also, numerous figures and examples are included in the text.

In order to lessen the reader's difficulties, especially for those who are approaching differential geometry for the first time, the discussion is relatively detailed. The selection of topics included in this book has been made with great care, consideration being given to the didactic point of view as well as the theoretical and practical importance of the different aspects of the subject.

Problems are to be found at the end of almost every section, and the solutions are listed at the end of the book. These exercises should help the reader to become familiar with the material presented in the text and, what is more important, to get acquainted with the manner of reasoning in differential geometry.

Differential geometry has various relations to other fields of mathematics. Besides the calculus other branches, such as function theory, the calculus of variations, and the theory of differential equations, are also basically important in differential geometry. On the other hand, differential geometry is an essential part of the foundations of some applied sciences, for instance physics, geodesy, and geography. Differential geometry has therefore what we may call a 'general character'; I have tried to stress this point of view in connexion with several topics.

This book is a free translation of my *Differentialgeometrie* which appeared in the series *Mathematik und ihre Anwendungen in Physik und Technik* ('Mathematics and its applications to physics and technical science') (Series A, vol. 25) of the Akademische Verlagsgesellschaft, Geest und Portig, Leipzig, Germany. Some minor changes have been made in the course of translation.

Professor H. Behnke (University of Münster) and Professor H. Graf (Technical University of Darmstadt) have made valuable suggestions to

me. Professor S. Bergman (Stanford University) and Professor E. Ullrich (University of Giessen) have checked Sections 64 and 84, respectively. Professor C. Loewner (Stanford University), Professor M. Riesz (University of Lund), Professor H. S. M. Coxeter (University of Toronto), Professor P. Scherk (University of Saskatchewan), Professor M. Barnet (Technical University of Karlsruhe), and Professor O. Biberstein (University of Ottawa) have read the manuscript carefully, and I have obtained valuable suggestions from all of them in the course of numerous personal discussions. The translated manuscript has been checked by Professor H. S. M. Coxeter, Professor G. F. D. Duff (University of Toronto), Professor J. T. Duprat (University of Ottawa), Professor R. C. Fisher (Ohio State University, Columbus), and Professor L. Sauvè (St. Patrick's College, Ottawa). I wish to express my gratitude to all of them and also to the University of Toronto Press, for their efficient co-operation.

E. K.

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PRELIMINARIES

1. Notation. In this section the meaning of some symbols occurring very frequently in our text will be explained and the page on which these symbols are introduced for the first time indicated. The reader will find another index of definitions, formulae, and theorems at the end of this book.

- Page 4 x_1, x_2, x_3 : Cartesian coordinates in three-dimensional Euclidean space R_3 .
- 9 Bold-face letters \mathbf{a}, \mathbf{y} , etc.: Vectors in space R_3 ; the components of these vectors will be denoted by $a_1, a_2, a_3; y_1, y_2, y_3$, etc.
- 28 s arc length of a curve. Derivatives with respect to s will be denoted by dots, e.g.

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{ds} \equiv (\dot{x}_1, \dot{x}_2, \dot{x}_3) \equiv \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds} \right).$$

An arbitrary parameter figuring in the representation of a curve will usually be denoted by t . Derivatives with respect to t will be characterized by primes, e.g.

$$\mathbf{x}' \equiv \frac{d\mathbf{x}}{dt}, \quad \mathbf{x}'' \equiv \frac{d^2\mathbf{x}}{dt^2}, \quad \text{etc.}$$

- 29 $\mathbf{t} = \dot{\mathbf{x}}$ unit tangent vector of a curve C : $\mathbf{x}(s)$.
- 34 $\mathbf{p} = \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|}$ unit principal normal vector of that curve.
- 36 $\mathbf{b} = \mathbf{t} \times \mathbf{p}$ unit binormal vector of that curve.
- 34 $\kappa = \frac{1}{\rho}$ curvature, ρ radius of curvature of a curve.
- 38 τ torsion of a curve.
- 75 u^1, u^2 coordinates on a surface.

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2},$$

$$\mathbf{x}_\alpha \equiv \frac{\partial \mathbf{x}}{\partial u^\alpha}, \quad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \quad \text{etc.}$$

Page 84 **Summation convention.** If in a product an index figures twice, once as a subscript and once as a superscript, summation must be carried out with respect to that index from 1 to 2; the summation sign will be omitted.

Example: $a^\alpha b_\alpha \equiv \sum_{\alpha=1}^2 a^\alpha b_\alpha = a^1 b_1 + a^2 b_2$. For further details and examples see Section 27.

- 82 $d\mathbf{x} \cdot d\mathbf{x} = g_{\alpha\beta} du^\alpha du^\beta$: first fundamental form.
- 86 $g = g_{11}g_{22} - g_{12}^2$: discriminant of the first fundamental form.
- 90 Superscript: contravariant index.
- 93 Subscript: covariant index.
- 104 $g^{\alpha\beta}$ contravariant components of the metric tensor.
- 107 \mathbf{n} unit normal vector to a surface, $\mathbf{n}_\alpha \equiv \frac{\partial \mathbf{n}}{\partial u^\alpha}$, etc.
- 119 $-d\mathbf{x} \cdot d\mathbf{n} = b_{\alpha\beta} du^\alpha du^\beta$: second fundamental form.
- 122 $\kappa_n = \frac{1}{R}$ normal curvature of a surface.
- 125 $b = b_{11}b_{22} - b_{12}^2$: discriminant of the second fundamental form.
- 129 κ_1, κ_2 principal curvatures of a surface.
- 131 K Gaussian curvature of a surface.
- 131 H mean curvature of a surface.
- 140 $\Gamma_{\alpha\beta\gamma}$ Christoffel symbols of the first kind.
- 141 $\Gamma_{\alpha\beta}{}^\gamma$ Christoffel symbols of the second kind.
- 144 $R_{\alpha\beta\gamma\epsilon}, R^\alpha{}_{\beta\gamma\epsilon}$: components of the curvature tensors.
- 154 κ_g geodesic curvature.
- 186 $d\mathbf{n} \cdot d\mathbf{n} = c_{\alpha\beta} du^\alpha du^\beta$: third fundamental form.

2. Nature and purpose of differential geometry. In differential geometry properties of geometric configurations (curves, surfaces) are investigated by means of differential and integral calculus. All our considerations will take place in three-dimensional Euclidean space and will, in general, be restricted to real geometric configurations. We will, however, occasionally extend our methods to the complex domain.

A geometric property is called *local*, if it does not pertain to the geometric configuration as a whole but depends only on the form of the configuration in an (arbitrary small) neighbourhood of a point under consideration. For instance, the curvature of a curve is a local property. Since differential

geometry is concerned mainly with local properties, it is primarily a *geometry in the small* or a *local geometry*.

This fact does not exclude the possibility of considering geometric configurations as a whole. This kind of investigation belongs to what we call *global differential geometry* or *differential geometry in the large*. In this book we will consider only a small number of global problems, for example, in connexion with the theorem of Gauss-Bonnet. We may say that global problems are problems in which ‘macroscopic’ properties are related to ‘microscopic’ ones. For further study in this field see, for example, W. Blaschke, *Vorlesungen über Differentialgeometrie* (3 vols., Berlin, 1945, 1923, and 1929).

As is natural, concepts, methods, and results of analytic geometry will be constantly used in differential geometry. The following sections are consequently devoted to a brief review of some of the topics from analytic geometry which we will need for our further investigations. We may restrict ourselves to the analytic geometry of three-dimensional Euclidean space in which all our considerations will take place.

3. Concept of mapping. Coordinates in Euclidean space.

The concept of mapping is of basic importance in differential geometry.

Let M and M' be two sets of points in three-dimensional Euclidean space R_3 . (M or M' may contain all points of R_3 or only a subset of these points.) If a rule T is stated which associates a point P' of M' to every point P of M we say that a *mapping* or *transformation* (more exactly: *point transformation*) of the set M into the set M' is given. P' is called the *image point* of P , and P is called an *inverse image point* of P' . The set of the image points of all points of M is called the *image* of M . If every point of M' is an image point of at least one point of M the mapping is called a *mapping of M onto M'* .

A mapping T of M onto M' is called *one-to-one* if the image points of any pair of different points of M are different points of M' . Then there exists the *inverse mapping* of T , denoted by T^{-1} , which maps M' onto M such that every point P' of M' is mapped onto that point P of M which corresponds to P' with respect to the mapping T .

The set of all points whose distance from a point P is smaller than a positive number η is called a *neighbourhood* of P . Consequently this neighbourhood consists of all points in the interior of a sphere of radius η with centre at P . There are arbitrarily many different neighbourhoods of P each of which corresponds to a certain value of η . A mapping of a set M into a set M'

is said to be *continuous at a point P* of M if, for every neighbourhood U' of the image P' of P there exists a neighbourhood U of P whose image is contained in U' . The mapping is said to be *continuous* if it is continuous at every point of M . A one-to-one continuous mapping whose inverse mapping is also continuous is called a *topological mapping*. Point sets which can be topologically mapped onto each other are said to be *homeomorphic*.

A mapping is called a *rigid motion* if any pair of image points has the same distance as the corresponding pair of inverse image points.

We will now discuss some basic facts of the analytic geometry of the three-dimensional Euclidean space R_3 which we will need in our later investigations.

We first introduce a right-handed system of orthogonal parallel coordinates x_1, x_2, x_3 whose unit points on the axes, that is, the points with coordinates $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively, have the distance 1 from the origin, cf. Fig. 1. Such a special right-handed system will be called a *Cartesian coordinate system*.

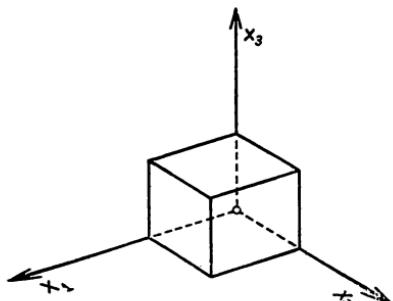


FIG. 1. Right-handed system of orthogonal parallel coordinates

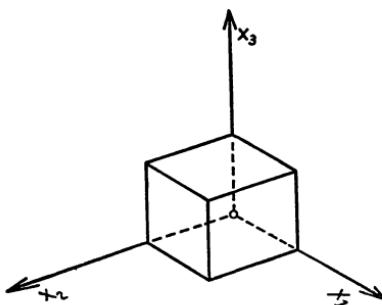


FIG. 2. Left-handed system of orthogonal parallel coordinates

In general, a coordinate system is called *right-handed* if the axes, in their natural order, assume the same sort of orientation as the thumb, index finger, and middle finger of the right hand. A system is said to be *left-handed* if the axes, in their natural order, assume the same sort of orientation as the thumb, index finger, and middle finger of the left hand, cf. Fig. 2.

The notation x_1, x_2, x_3 for the coordinates is more convenient than the familiar x, y, z , for it enables us to use the abbreviated form (x_i) for the coordinates x_1, x_2, x_3 of a point.

Any other Cartesian coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ is related to the given one by a special linear transformation of the form

$$(3.1 \text{ a}) \quad \tilde{x}_i = \sum_{k=1}^3 a_{ik} x_k + b_i, \quad (i = 1, 2, 3)$$

whose coefficients satisfy the conditions

$$(3.1 \text{ b}) \quad \sum_{i=1}^3 a_{ik} a_{il} = \delta_{kl} = \begin{cases} 0 & (k \neq l) \\ 1 & (k = l), \end{cases} \quad (k, l = 1, 2, 3),$$

and

$$(3.1 \text{ c}) \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1.$$

The quantity δ_{kl} is called the *Kronecker symbol*.

The transition from one Cartesian coordinate system to another can be effected by a certain rigid motion of the axes of the original system. Such a motion is composed of a suitable translation and a suitable rotation. A rigid motion which carries a Cartesian coordinate system into another Cartesian coordinate system is called a *direct congruent transformation* (or *displacement*). We will now investigate (3.1) in somewhat greater detail.

Let m and n be natural numbers. A system of $m \cdot n$ quantities arranged in a rectangular array of m horizontal rows and n vertical columns is called a *matrix*. The quantities are called *elements* of that matrix. If m equals n the matrix is said to be *square*, and the number n is called the *order* of the matrix.

The coefficients a_{ik} figuring in (3.1) form a quadratic matrix

$$A = (a_{ik}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The corresponding determinant (3.1 c) will be denoted by

$$\det A = \det(a_{ik}).$$

If in particular A equals the '*unit matrix*'

$$(\delta_{ik}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then (3.1 a) is of the form

$$(3.2) \quad \tilde{x}_i = x_i + b_i, \quad (i = 1, 2, 3).$$

This transformation corresponds to a translation of the coordinate system. If, moreover, $b_i = 0$ ($i = 1, 2, 3$), we obtain from (3.2)

$$(3.3) \quad \bar{x}_i = x_i, \quad (i = 1, 2, 3),$$

i.e. the transformed coordinates are the same as the original ones. Such a special transformation is called *the identical transformation*.

If $b_i = 0$ ($i = 1, 2, 3$), and the coefficients a_{ik} are arbitrary but such that the conditions (3.1 b) and (3.1 c) are satisfied, then (3.1 a) corresponds to a rotation of the coordinate system with the origin as centre. A rotation is also called a *direct orthogonal transformation*.

We note that a transformation of the form

$$\bar{x}_i = \sum_{k=1}^3 a_{ik} x_k, \quad \sum_{i=1}^3 a_{ik} a_{il} = \delta_{kl}, \quad \det(a_{ik}) = -1,$$

can be geometrically interpreted as a motion composed of a rotation about the origin and a reflection in a plane. A transformation of this type is called an *opposite orthogonal transformation*. It transforms a right-handed coordinate system into a left-handed one and vice versa. An example of a reflection in a plane (in this particular case, in the $x_2 x_3$ -coordinate plane) is given by the transformation $\bar{x}_1 = -x_1$, $\bar{x}_2 = x_2$, $\bar{x}_3 = x_3$. Direct and opposite orthogonal transformations are called *orthogonal transformations*, and the corresponding matrices are referred to as *orthogonal matrices*. A transformation which is composed of translations, rotations, and an odd number of reflections is called an *opposite congruent transformation*. We should note that every translation or rotation can be composed of two suitable reflections.

We will now point out that (3.1) can be interpreted in two different ways:

(*Alias.*) Formerly we interpreted (3.1) as a coordinate transformation; (x_i) and (\bar{x}_i) are then the coordinates of one and the same point with respect to two different Cartesian coordinate systems.

(*Alibi.*) The relation (3.1) can also be interpreted as a mapping or point transformation. Then (x_i) and (\bar{x}_i) represent the coordinates of two different points with respect to one and the same Cartesian coordinate system; that is, the coordinate system remains fixed and the location of the points is changed.

Both interpretations are closely related to each other. For, in consequence of the above remarks, the transition from one interpretation to the other can be effected in the following manner. Instead of imposing a direct congruent transformation on the given Cartesian coordinate system, one can just as well move the geometric configuration, that is, change its

location in space, while the coordinate system is kept fixed. Geometric configurations which differ only by their location in space are said to be *congruent* to each other. Two congruent geometric configurations can always be made to coincide by means of a displacement. (*Remark:* Since we do not include reflections the concept of congruence is used here in a narrower sense than is usual.)

If we wish to investigate geometric facts analytically we have to introduce a coordinate system. Consequently, a coordinate system is a useful tool but no more. A property of a geometric configuration consisting of points of R_3 will be called *geometric* if it is independent of the special choice of the Cartesian coordinates. In other words, a property of a configuration is called a geometric property if it is an invariant with respect to direct congruent transformations of the coordinate system, or, what is the same, with respect to direct congruent transformations of the configuration under consideration.

In order to consider this fact more precisely and from a general point of view it is advantageous to use the concept of a group and to mention an important connexion between geometry and group theory.

We assume the mappings

$$(3.4) \quad T: \quad \bar{x}_i = F_i(x_1, x_2, x_3), \quad (i = 1, 2, 3),$$

used in the following to be one-to-one mappings which are defined in the whole space R_3 , i.e. they are one-to-one mappings of the space R_3 onto itself. If several mappings of this kind are given, for example,

$$\bar{x}_i = F_i(x_1, x_2, x_3), \quad \bar{x}_j = \bar{F}_j(\bar{x}_1, \bar{x}_2, \bar{x}_3), \quad (i, j = 1, 2, 3),$$

we can conceive these mappings to be effected successively; in this manner we obtain what is called a *composite mapping* or *product* of these mappings. In the case of the foregoing example we have

$$\bar{x}_j = \bar{F}_j(F_1(x_1, x_2, x_3), F_2(x_1, x_2, x_3), F_3(x_1, x_2, x_3)).$$

A set G of mappings (3.4) is called a *group of mappings* or a *transformation group* if G has the following properties:

1. The identical mapping $\bar{x}_i = x_i$ ($i = 1, 2, 3$), is contained in G .
2. The inverse mapping T^{-1} of any mapping T contained in G is also an element of G .
3. For any arbitrary pair of (not necessarily distinct) mappings T_1, T_2 contained in G the product mapping of these mappings is also an element of G .

The direct congruent transformations form a group.

By means of every one of these groups we may classify the geometric configurations in R_3 into *equivalence classes*: Two configurations are called equivalent with respect to a certain group (that is, they are contained in one and the same equivalence class) if that group contains a mapping which maps one of the configurations onto the other.

In the case of the group of direct congruent transformations this concept of equivalence is identical with the concept of congruence as defined above.

The three conditions involved in the definition of a group, if considered from the standpoint of classification, are nothing but the axioms of equivalence:

1. *Every geometric configuration is equivalent to itself.*
2. *If a configuration A is equivalent to a configuration B, then B is also equivalent to A.*
3. *If a configuration A is equivalent to a configuration B and B is equivalent to a configuration C, then A is also equivalent to C.*

Let now a certain group G of mappings be given and consider geometric configurations each of which, for the sake of simplicity, is an ordered system of a certain finite number of points of R_3 . Every property of each of these objects can be analytically described by a certain expression which is a function of the coordinates of the points of the configuration under consideration. We denote by I such an expression, corresponding to a certain property of an object. This property will be called geometric with respect to a group G if and only if the numerical value of I remains unaffected by any mapping contained in G . Then I is said to be an *invariant* of the group G .

For example, the distance

$$d = \sqrt{\sum_{i=1}^3 (y_i - x_i)^2}$$

of two points (x_i) and (y_i) of R_3 is an invariant with respect to the group of the displacements; such a motion will, in general, cause a change of the values of the coordinates, but d will retain its value.

In consequence of the foregoing reasoning one may consider a geometry to be identical with the theory of invariants of a certain transformation group. The investigation of geometry from that 'group theoretical' point of view has been initiated by F. Klein [1], [2, vol. i, p. 460]† in his so-called *Erlangen Program* (1872). Klein's idea created a certain order among the

† Cf. the bibliography at the end of this book.

different types of geometry and also had a considerable influence on the development of geometry.

For example, affine geometry corresponds to the group of affine transformations,

$$\bar{x}_i = \sum_{k=1}^3 a_{ik} x_k + b_i, \quad \det(a_{ik}) \neq 0,$$

projective geometry to the group of projective transformations,

$$\bar{x}_i = \frac{\sum_{k=1}^3 b_{ik} x_k + b_{i4}}{\sum_{k=1}^3 b_{4k} x_k + b_{44}}, \quad \det(b_{ik}) \neq 0$$

(where the determinant $\det(b_{ik})$ of the coefficients is of order 4), etc.

Problem 3.1. Do the following mappings form a group or not?

- (a) Translations, (b) rotations about a fixed axis, the angle of every rotation being always an integral multiple of 120° , (c) direct congruent transformations, (d) direct orthogonal transformations, (e) opposite orthogonal transformations.

4. Vectors in Euclidean space. Vectors are essential tools which we will constantly need in our investigations. The concept of a vector was first used by W. Snellius (1581–1626) and L. Euler (1707–83).

A line segment S in Euclidean space R_3 is uniquely determined by its end points P and Q . The segment is said to be *directed* if we designate one of the two points as the initial point and the other as the terminal point. Obviously, there are two possibilities in ‘directing’ a segment or giving it a *sense*.

A directed segment is called a *vector*. The length of the segment is called the *length* or *magnitude* of the vector.

We define: *Two vectors are equal if and only if they are parallel, have the same length, and have the same sense.*

Hence a vector may be arbitrarily displaced parallel to itself, or, what amounts to the same, its initial point may be chosen arbitrarily.

A vector of length 1 is called a *unit vector*.

Vectors in Euclidean space R_3 will be denoted by bold-face letters, e.g. $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$, etc. The length of a vector \mathbf{a} will be denoted by $|\mathbf{a}|$.

Let us denote by \mathbf{z} the vector which we obtain by directing the segment S with end points P and Q such that P becomes the initial point and Q the terminal point, cf. Fig. 3. We assume that in R_3 a certain Cartesian coordinate system has been introduced with respect to which P has the

coordinates (x_i) and Q has the coordinates (y_i) . As follows from (3.2), the values of the coordinate differences

$$(4.1) \quad z_i = y_i - x_i, \quad (i = 1, 2, 3),$$

remain unchanged if \mathbf{z} undergoes a translation. These quantities z_1, z_2, z_3

are called the *components* of the vector \mathbf{z} with respect to that coordinate system. We write $\mathbf{z} = (z_1, z_2, z_3)$, or $\mathbf{z} = (z_i)$. Similarly, the components of a vector \mathbf{a} will be denoted by a_1, a_2, a_3 , etc.

If we introduce the *null vector* $\mathbf{0}$, that is, a vector with components $0, 0, 0$, then every ordered triple of numbers z_1, z_2, z_3 , also the triple $0, 0, 0$ can be chosen as the components of a certain vector. The correspondence between the ordered triples of numbers and the vectors is one-to-one.

Moreover, if the null vector is introduced, addition of vectors (see below) can be done without exceptional cases.

Since vector components are invariant with respect to translations, the transformation (3.1) corresponds to a transformation of the components of a vector which is of the form

$$(4.2) \quad \bar{z}_i = \sum_{k=1}^3 a_{ik} z_k, \quad \sum_{i=1}^3 a_{ik} a_{il} = \delta_{kl}, \quad \det(a_{ik}) = 1.$$

The transformation property (4.2) is basic for vectors in Euclidean space: A vector in that space can also be defined as an ordered system of three numbers, called components, which behave according to (4.2) if a transformation (3.1) is imposed. From this definition it follows that the differences of the corresponding coordinates of two points of R_3 are the components of a vector. Hence the present definition of a vector is equivalent to the one given previously. While the first definition may be easier to grasp, the second has the advantage that it can be extended to more general spaces, cf. Section 29.

Vectors are useful for the following two reasons: Many geometrical (and physical) concepts are vectors. Rules of vector calculation can be defined which in several respects are similar to the rules governing the system of real numbers and which enable us to conduct many investigations without

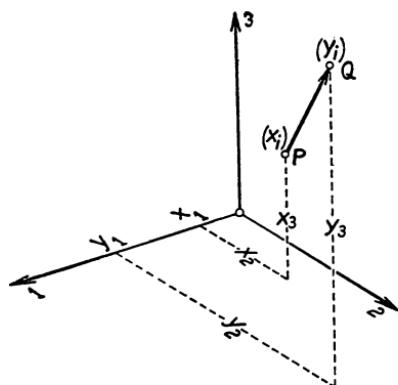


FIG. 3

regard to the special choice of the coordinate system. We will discuss the basic rules of vector calculus in the following section.

As was shown at the beginning of this section, a vector may be arbitrarily translated; its initial point may be chosen arbitrarily. It is sometimes advantageous, however, to choose a certain fixed point P as the initial point of a vector. Then the vector is said to be *bound* at P . A vector whose initial point is left undetermined is sometimes called a *free vector*.

A vector having the origin of the chosen coordinate system as initial point and a point Q as terminal point is called the *position vector* of Q with respect to that particular coordinate system,

cf. Fig. 4. The components of the position vector have the same numerical value as the coordinates of the point Q . Given a fixed coordinate system, any point in space can be uniquely determined by a certain position vector. It should be stressed that the origin of the coordinate system will in general change if the coordinates are transformed according to (3.1); hence the components of the position

vector with respect to the transformed coordinate system will be obtained by using (3.1) and not (4.2).

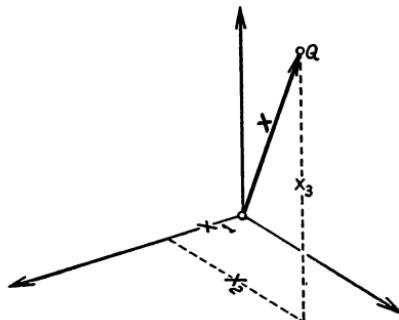


FIG. 4. Position vector

5. Basic rules of vector calculus in Euclidean space. The *product* ka of a vector a by a real number k is, by definition, a vector having the components ka_i . Obviously, the vectors ka and a are parallel to each other; they have the same sense if k is positive, and opposite sense if k is negative. Instead of $(-1)a$ we write simply $-a$. The vectors $-a$ and a are parallel to each other and are of the same length but of opposite sense.

The sum

$$(5.1) \quad \mathbf{c} = \mathbf{a} + \mathbf{b}$$

of two vectors \mathbf{a} and \mathbf{b} is, by definition, a vector \mathbf{c} with components

$$(5.2) \quad c_i = a_i + b_i, \quad (i = 1, 2, 3).$$

Instead of $\mathbf{a} + (-\mathbf{b})$ we write simply $\mathbf{a} - \mathbf{b}$. Vector addition is associative,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

and commutative,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

Furthermore, we have

$$\begin{aligned} \mathbf{0} + \mathbf{a} &= \mathbf{a}, \\ \mathbf{a} + (-\mathbf{a}) &= \mathbf{0}, \\ k(\mathbf{a} + \mathbf{b}) &= k\mathbf{a} + k\mathbf{b}, \\ (k_1 + k_2)\mathbf{a} &= k_1\mathbf{a} + k_2\mathbf{a}, \\ k_1(k_2\mathbf{a}) &= (k_1 k_2)\mathbf{a} = k_1 k_2 \mathbf{a}, \\ 1\mathbf{a} &= \mathbf{a}, \end{aligned}$$

where k , k_1 , and k_2 are real numbers.

The expression

$$(5.3) \quad \mathbf{c} \cdot \mathbf{d} = \sum_{i=1}^3 c_i d_i = c_1 d_1 + c_2 d_2 + c_3 d_3$$

is called the *scalar product* of the vectors $\mathbf{c} = (c_i)$ and $\mathbf{d} = (d_i)$, where c_i and d_i are the components of these vectors with respect to a Cartesian coordinate system. Scalar products were first used by H. Grassmann (1809–77). As can be seen from (5.3), the scalar product is a number. Instead of $\mathbf{c} \cdot \mathbf{d}$ the notation \mathbf{cd} , without dot, can also be found in the literature.

Scalar multiplication is commutative,

$$\mathbf{c} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{c},$$

and distributive with respect to addition,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

The scalar product is invariant with respect to coordinate transformations of the form (3.1). Using (4.2) we obtain from (5.3)

$$\bar{\mathbf{c}} \cdot \bar{\mathbf{d}} = \sum_{i=1}^3 \left(\left(\sum_{k=1}^3 a_{ik} c_k \right) \left(\sum_{l=1}^3 a_{il} d_l \right) \right) = \sum_{k=1}^3 \sum_{l=1}^3 \delta_{kl} c_k d_l = \sum_{k=1}^3 c_k d_k = \mathbf{c} \cdot \mathbf{d}.$$

The scalar product can be geometrically interpreted in the following manner: We choose a point with coordinates (x_i) to be the initial point of a vector $\mathbf{z} = (z_i)$ and denote by (y_i) the coordinates of the corresponding terminal point of \mathbf{z} . Then, by the Theorem of Pythagoras, \mathbf{z} has the length

$$(5.4) \quad |\mathbf{z}| = \sqrt{\left(\sum_{i=1}^3 (y_i - x_i)^2 \right)} = \sqrt{\left(\sum_{i=1}^3 z_i^2 \right)} = \sqrt{(\mathbf{z} \cdot \mathbf{z})}.$$

Hence the scalar product of a vector by itself is equal to the square of its length. In general, if \mathbf{c} and \mathbf{d} are two arbitrary vectors,

$$(5.5) \quad \mathbf{c} \cdot \mathbf{d} = |\mathbf{c}| |\mathbf{d}| \cos \alpha,$$

where α ($0 \leq \alpha \leq \pi$), denotes the angle between \mathbf{c} and \mathbf{d} . In order to prove (5.5) we transform the Cartesian coordinates in such a way that \mathbf{c} lies in the x_1 -axis and \mathbf{d} lies in the $x_1 x_2$ -plane. Then $\mathbf{c} = (c_1, 0, 0)$ and $\mathbf{d} = (d_1, d_2, 0)$.

The value of the scalar product $\mathbf{c} \cdot \mathbf{d}$ is invariant under the coordinate transformation imposed. As follows from (5.3), we have in the transformed coordinate system $\mathbf{c} \cdot \mathbf{d} = c_1 d_1 = |\mathbf{c}| |\mathbf{d}| \cos \alpha$, cf. Fig. 5.

Relation (5.5) enables us to represent angles between vectors in terms of scalar products,

$$(5.6) \quad \cos \alpha = \frac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}| |\mathbf{d}|} = \frac{\mathbf{c} \cdot \mathbf{d}}{\sqrt{(\mathbf{c} \cdot \mathbf{c}) (\mathbf{d} \cdot \mathbf{d})}}.$$

As can be seen from (5.3) or (5.5) the scalar product of a vector by the null vector is zero; moreover we have from (5.5)

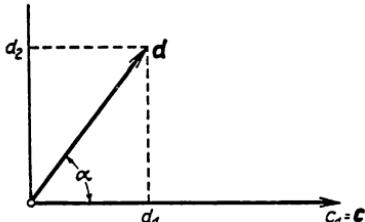


FIG. 5

Theorem 5.1. Two non-null vectors are orthogonal to each other if and only if their scalar product vanishes.

The vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$

of two vectors \mathbf{a} and \mathbf{b} is defined by the following expression:

$$(5.7) \quad \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & a_1 & b_1 \\ \mathbf{e}_2 & a_2 & b_2 \\ \mathbf{e}_3 & a_3 & b_3 \end{vmatrix}$$

where \mathbf{e}_i denotes a unit vector having the (positive) direction of the i th coordinate axis of the Cartesian coordinate system in space R_3 .

The vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ can be interpreted as a vector orthogonal to \mathbf{a} and \mathbf{b} in such a way that the vectors \mathbf{a} , \mathbf{b} , \mathbf{v} , in this order, have a right-handed sense, i.e. the same sense as the positive rays of the coordinate axes, taken in their natural order; the absolute value $|\mathbf{v}|$ of \mathbf{v} equals the area of a parallelogram with sides \mathbf{a} and \mathbf{b} . We have

$$(5.8) \quad |\mathbf{v}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha,$$

where α ($0 \leq \alpha \leq \pi$), denotes the angle between \mathbf{a} and \mathbf{b} , cf. Fig. 6. Developing the determinant (5.7) by the first column we obtain the following explicit expressions for the components of \mathbf{v} :

$$(5.7') \quad v_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2, \quad v_2 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = a_3 b_1 - a_1 b_3, \\ v_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

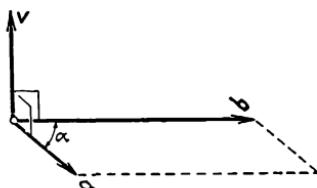


FIG. 6

Since the determinant in (5.7) changes sign when the last two columns are interchanged, we have

$$(5.9) \quad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = -\mathbf{v}.$$

Such a multiplication is consequently called anticommutative. From the definition of vector multiplication we easily derive the following rules:

$$(5.10) \quad \begin{aligned} (k\mathbf{a}) \times \mathbf{b} &= k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b}) \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \end{aligned}$$

where k is a real number. The relation

$$(5.11) \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

is called the *Identity of Lagrange*. From (5.11) follows

$$(5.12) \quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Applications of vector products are numerous in differential geometry. Other applications can be found in physics as is well known.

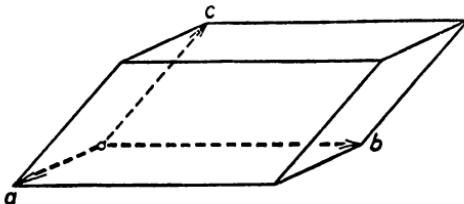


FIG. 7

In our later investigations we shall often meet scalar products of a vector by a vector product. If we form the scalar product of a vector \mathbf{a} and a vector product $\mathbf{b} \times \mathbf{c}$ then, in consequence of the definitions (5.3) and (5.7) we obtain

$$(5.13) \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We write

$$(5.14) \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{abc}|$$

and call this expression the *mixed product*, *scalar triple product*, or *determinant* of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

As can be found by a simple computation, the mixed product is invariant under direct congruent transformations of the coordinate system and changes its sign under a reflection. The absolute value of the mixed product $|\mathbf{abc}|$ can be geometrically interpreted as the volume of a parallelepiped

having the edge vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} , cf. Fig. 7. In consequence of the rules governing determinants we find

$$|\mathbf{abc}| = |\mathbf{bca}| = |\mathbf{cab}| \quad (\text{cyclic permutation}),$$

$$|\mathbf{abc}| = -|\mathbf{bac}| \quad \text{etc.,}$$

$$|(\mathbf{a} + \mathbf{d}) \mathbf{b} \mathbf{c}| = |\mathbf{abc}| + |\mathbf{dbc}|,$$

$$|k\mathbf{a} \mathbf{b} \mathbf{c}| = k|\mathbf{abc}|,$$

where k denotes a scalar. From (5.12) we obtain

$$(5.15) \quad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = |\mathbf{abd}| \mathbf{c} - |\mathbf{abc}| \mathbf{d}.$$

We assume that \mathbf{a} is a vector whose components a_i are differentiable functions of a variable t . Then the derivative $\mathbf{a}'(t)$ of the vector $\mathbf{a} = \mathbf{a}(t)$ with respect to t is defined as the vector with components $a'_i(t)$, where the prime denotes the derivative with respect to t . By applying the usual rules of calculus we immediately find

$$(5.16) \quad \begin{aligned} (\mathbf{a} \cdot \mathbf{b})' &= \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}', \\ (\mathbf{a} \times \mathbf{b})' &= \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}', \\ |\mathbf{abc}|' &= |\mathbf{abc}| + |\mathbf{ab}'\mathbf{c}| + |\mathbf{a'b'}\mathbf{c}| \end{aligned}$$

where all the vectors occurring are assumed to have components which are differentiable functions of a variable t .

We finally consider the concept of linear dependence of vectors. p vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(p)}$ are said to be *linearly dependent* if and only if there are p real numbers k_1, k_2, \dots, k_p , not all zero, such that

$$(5.17) \quad \sum_{i=1}^p k_i \mathbf{a}^{(i)} = \mathbf{0}.$$

If no such p numbers exist the vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(p)}$ are said to be *linearly independent*.

If the vectors $\mathbf{a}^{(i)}$ are linearly independent then (5.17) can hold only if $k_1 = k_2 = \dots = k_p = 0$. Conversely, if (5.17) holds only if every k_i ($i = 1, 2, \dots, p$), vanishes then the vectors $\mathbf{a}^{(i)}$ are linearly independent. If at least one of the vectors $\mathbf{a}^{(i)}$ is a null vector these vectors are always linearly dependent.

The geometric interpretation of the linear dependence of two non-null vectors is that the vectors are parallel to each other. From (5.8) we obtain

Theorem 5.2. *Two vectors are linearly dependent if and only if their vector product is the null vector.*

From (5.13) we obtain

Theorem 5.3. *Three vectors are linearly dependent if and only if the determinant of these vectors vanishes.*

The linear dependence of three vectors means that these vectors, if bound at the same point, lie in one and the same plane. Four or more vectors of the three-dimensional Euclidean space R_3 are always linearly dependent, cf. Problem 5.3.

Problem 5.1. Give a representation of a plane using a scalar product.

Problem 5.2. Prove the invariance of the mixed product under direct congruent transformations of the coordinate system.

Problem 5.3. Prove: Four vectors of the three-dimensional Euclidean space are always linearly dependent.

II

THEORY OF CURVES

6. The concept of a curve in differential geometry. We will now investigate curves in three-dimensional Euclidean space R_3 . We restrict ourselves to real curves. We assume that a Cartesian coordinate system has been introduced in R_3 . Then every point in space can be uniquely determined by its position vector

$$\mathbf{x} = (x_1, x_2, x_3).$$

In order to introduce the concept of a curve we start from a real vector function

$$(6.1) \quad \mathbf{x} = \mathbf{x}(t), \quad (a \leq t \leq b).$$

In (6.1), the components

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = x_3(t), \quad (a \leq t \leq b),$$

of the vector \mathbf{x} are (single-valued) functions of a real variable t defined in the interval $I: a \leq t \leq b$. By means of (6.1), to every value of t a point of R_3 is associated whose position vector is $\mathbf{x}(t)$; the vector function (6.1) determines a point set M in R_3 . (6.1) is called a *parametric representation* of the set M , and the variable t is called the *parameter* of this representation.

Since we will apply calculus to geometric problems we have to require that the vector function (6.1) possess a certain number of derivatives. Moreover, we wish to exclude the trivial case where M consists of one point only. Consequently we make the following

Assumptions :

(1) *The functions $x_i(t)$ ($i = 1, 2, 3$) are r (≥ 1) times continuously differentiable in I (where the value of r will depend on the problem under consideration).*

(2) *For every value of t in I , at least one of the three functions*

$$x'_i(t) \equiv \frac{dx_i(t)}{dt}$$

is different from zero.

A representation of the form (6.1) satisfying these conditions will be called an *allowable parametric representation*.

As follows from Assumption (2) the representation (6.1) can be solved for t , at least in the small,

$$(6.2) \quad t = t(x_i), \quad (i = 1, 2, \text{ or } 3),$$

and the function (6.2) is also r times continuously differentiable in its interval of definition.

If in a certain subinterval \bar{I} of I the function $x'_1(t)$ is different from zero we may introduce x_1 as new independent variable and derive from (6.1) a representation of the form

$$(6.3) \quad x_2 = x_2(x_1), \quad x_3 = x_3(x_1),$$

which is valid in \bar{I} .

With respect to Assumption (1) we may notice that the existence and continuity of a finite number of derivatives of $\mathbf{x}(t)$ will, in general, be sufficient to guarantee the validity of the results to be obtained later. In most cases it is unnecessary to assume that the functions $x_i(t)$ are analytic.

Obviously, we may obtain other vector functions from (6.1) by imposing a transformation

$$(6.4) \quad t = t(t^*).$$

The point set represented by a new vector function thus obtained remains the same. The choice of the function (6.4) is arbitrary; we have only to make sure that the range of values of this function includes the aforementioned interval I and that the new parametric representation $\mathbf{x} = \mathbf{x}(t^*)$ satisfies Assumptions (1) and (2). Hence we make the following

Assumptions :

(0*) *The function (6.4) is (at least) defined in an interval I^* : $a^* \leqslant t^* \leqslant b^*$, $t(a^*) = a$, $t(b^*) = b$ (or $t(a^*) = b$, $t(b^*) = a$), and the range of values corresponding to the interval I^* lies in the interval I .*

(1*) *The function (6.4) is r ($\geqslant 1$) times continuously differentiable in the interval I^* .*

(2*) *The derivative dt/dt^* is different from zero everywhere in I^* .*

A transformation (6.4) satisfying these conditions is said to be an *allowable parametric transformation*.

In consequence of these assumptions there exists the inverse transformation $t^* = t^*(t)$ which is a strictly monotonic and r times continuously differentiable function.

Remark: The point set M represented by (6.1) can, in general, also be represented in another way, namely by means of two implicit functions,

$$(6.5) \quad F(x_1, x_2, x_3) = 0, \quad G(x_1, x_2, x_3) = 0.$$

If, in connexion with certain problems, the relation between the points of M and the different values of t is of minor interest, then a representation of the form (6.5) may be advantageous; but the standpoint of consideration

is then somewhat different from that which corresponds to representations of the form (6.1). In our later investigation, representations of the form (6.1)—including the special case (6.3)—will be by far the most important ones. We should also note that (6.5) is not as general as (6.1), cf. Section 7.

By means of the allowable transformations (6.4) we may classify the allowable representations of the form (6.1) into equivalence classes: Two allowable representations are said to be equivalent, or to belong to the same class, if there is an allowable transformation (6.4) transforming the one representation into the other. The axioms of equivalence, stated in Section 3, are satisfied, as can easily be seen.

Definition 6.1. *A point set in space R_3 which can be represented by the allowable representations of an equivalence class (and is then ordered in a certain manner) is called an arc of a curve.*

A certain point set may correspond to several equivalence classes; the arcs obtained by the choice of allowable representations of different classes are then, by definition, different from each other. For further remarks see Section 7.

The functional correspondence of the points of an arc to the values of t , given by an allowable representation (6.1), is continuous: Given a number $\epsilon > 0$, there exists a number $\delta(\epsilon) > 0$ such that from $|t_1 - t_2| \leq \delta$ it follows that $|\mathbf{x}(t_1) - \mathbf{x}(t_2)| \leq \epsilon$, where t_1 and t_2 are values of t in the interval I . This can be proved as follows. The functions $x_i(t)$ are differentiable in the closed interval I and consequently continuous in I . Hence, given an arbitrary number $\epsilon > 0$, there is a number $\delta(\epsilon) > 0$ such that $|x_i(t_1) - x_i(t_2)| \leq \frac{1}{2}\epsilon$ if $|t_1 - t_2| \leq \delta$. Therefore

$$|\mathbf{x}(t_1) - \mathbf{x}(t_2)| = \sqrt{\left(\sum_{i=1}^3 \{x_i(t_1) - x_i(t_2)\}^2 \right)} \leq \sqrt{3(\frac{1}{2}\epsilon)^2} < \epsilon.$$

It may happen that the vector function (6.1) has the same value for different values of t , that is, a certain point of M corresponds to several values of t . Such a point is said to be a *multiple point* of the arc. An arc of a curve having no multiple point is called *simple*.

If an arc is simple the correspondence between its points and the values of the parameter t is one-to-one. In this case, not only the functional relation of the points to the values of t is continuous but also the inverse relation, i.e. the relation of the values of t to the points: Given a number $\epsilon^* > 0$ there exists a number $\delta^*(\epsilon^*)$ such that, for any two values \tilde{t} and \tilde{t}' in I , from $|\mathbf{x}(\tilde{t}) - \mathbf{x}(\tilde{t}')| \leq \delta^*$ there follows $|\tilde{t} - \tilde{t}'| \leq \epsilon^*$. If this were not true we could find a number $\eta > 0$ and an infinite sequence of pairs $\tilde{t}_n, \tilde{t}'_n$,

$a \leqslant \tilde{t}_n < \tilde{t}_n \leqslant b$, such that $|\mathbf{x}(\tilde{t}_n) - \mathbf{x}(\tilde{t}_n)| \rightarrow 0$ if $n \rightarrow \infty$ and, at the same time, $\tilde{t}_n - \tilde{t}_n > \eta$ for each $n = 1, 2, \dots$. Now, the points $(\tilde{t}_n, \tilde{t}_n)$ belong to a closed domain (a square) of the $\tilde{t}\tilde{t}$ -plane. Hence a suitable sub-sequence of the sequence of these points converges to a point, say (\tilde{t}, \tilde{t}) , $\tilde{t} \geqslant \tilde{t}$, of this domain. When considering only the values of n corresponding to that sub-sequence we consequently have $\tilde{t}_n \rightarrow \tilde{t}$, $\tilde{t}_n \rightarrow \tilde{t}$ if $n \rightarrow \infty$; from $\tilde{t}_n - \tilde{t}_n > \eta$ it follows that $\tilde{t} - \tilde{t} \geqslant \eta$. Now, on the other hand, $\mathbf{x}(\tilde{t}_n) \rightarrow \mathbf{x}(\tilde{t})$ and $\mathbf{x}(\tilde{t}_n) \rightarrow \mathbf{x}(\tilde{t})$ if $n \rightarrow \infty$, and consequently $|\mathbf{x}(\tilde{t}) - \mathbf{x}(\tilde{t})| = 0$ or $\mathbf{x}(\tilde{t}) = \mathbf{x}(\tilde{t})$. Since the arc is assumed to be simple, we have $\tilde{t} = \tilde{t}$, in contradiction to $\tilde{t} - \tilde{t} \geqslant \eta$.

It follows from the foregoing investigation that a simple arc is a topological image of a segment (namely of the interval I of the t -axis). A topological image of a segment is also called a *Jordan arc*. However, in consequence of our assumption of differentiability, not every Jordan arc is an arc of a curve in the sense of Definition 6.1.

Definition 6.2. *A point set is called a curve if it can be represented by an equivalence class of allowable representations of the form (6.1) whose interval I is not assumed to be closed or bounded, but which are such that one always obtains an arc of a curve if the values of the parameter t are restricted to any closed and bounded subinterval of I .*

A curve is said to be *closed* if it possesses at least one allowable representation whose vector function $\mathbf{x}(t)$ is periodic,

$$\mathbf{x}(t+\omega) = \mathbf{x}(t), \quad \omega > 0, \quad (-\infty < t < \infty).$$

The circle $\mathbf{x}(t) = (r \cos t, r \sin t, 0)$ is an example of a simple closed curve. Every simple closed curve is a topological image of the circle. (However, because of Assumptions (1) and (2), not every topological image of a circle is a simple closed curve in the sense of our definition.)

A curve (or an arc) is said to be *plane* if all its points lie in a plane.

A function of one (or several) variables which is r times continuously differentiable, that is, which possesses continuous (partial) derivatives up to the order r , inclusively, is called a *function of class r* . A *vector function* is said to be of class r if one of its component functions is of this class, and if the other ones are at least of this class. A curve is said to be of class r if it can be represented by a vector function $\mathbf{x}(t)$ of class r .

7. Further remarks on the concept of a curve. In order to gain a better understanding of the considerations contained in the previous section we will now add some further remarks on the concept of a curve.

The following example taken from physics will perhaps be helpful in

making clear the idea of the classification of the allowable representations. When considering the path of a moving particle in mechanics, it is natural to choose a parametric representation with time t as parameter. A transformation of the parameter means that the motion is changed in time, without altering the geometric shape of the path in space.

The special choice of the parametric representation of a curve or an arc is of minor interest. But one may not conclude from this fact that the point set is the only thing that matters in the concept of a curve. Consider Fig. 8a. We assume that we obtain an arc in the sense of Definition 6.1 if the point set is traversed in the order $ABCDBE$. But we will certainly not obtain an arc if the order $ABDCBDE$ is chosen, since, in this case, at B the first derivative of the corresponding vector function does not exist. We assume that the point set in Fig. 8b consists of three circular arcs, AC , CD , and DE . The corresponding tangent (cf. Section 10) is continuous for both of the orders $ABCDBE$ and $ABDCBDE$, but the curvature (cf. Section 12) is continuous at B only if the first order is chosen. This example shows that one and the same point set may correspond to several arcs with conflicting differential geometric properties.

We will now compare the different types of representations

$$(6.1) \quad \mathbf{x} = \mathbf{x}(t),$$

$$(6.3) \quad x_2 = x_2(x_1), \quad x_3 = x_3(x_1)$$

and

$$(6.5) \quad F(x_1, x_2, x_3) = 0, \quad G(x_1, x_2, x_3) = 0$$

somewhat more in detail. We have already stressed that (6.3) is a special case of (6.1). We may notice the fact that a curve, represented by (6.1), cannot always be represented as a whole in the form (6.3) (or in a similar form with x_2 or x_3 as independent variable); this difficulty will occur when one value of the independent variable corresponds to several values of the dependent one, since the usual concept of a function requires a *unique* correlation of the function values to the arguments. For instance, in the case of the circle $\mathbf{x}(t) = (r \cos t, r \sin t, 0)$, a representation of the form (6.3) is given by

$$x_2 = \pm\sqrt{(r^2 - x_1^2)}, \quad x_3 = 0.$$

If we choose the positive sign of the root we have a representation of the

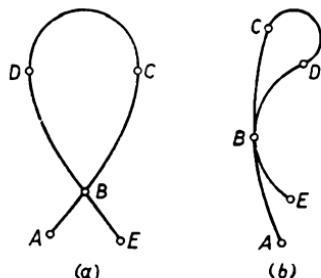


FIG. 8

semicircle in the upper half-plane; if we choose the negative sign the semicircle in the other half-plane is represented by this expression. The representation (6.3) means geometrically that the orthogonal projections of the arc into the x_1x_2 - and x_1x_3 -plane are given, cf. Fig. 9.

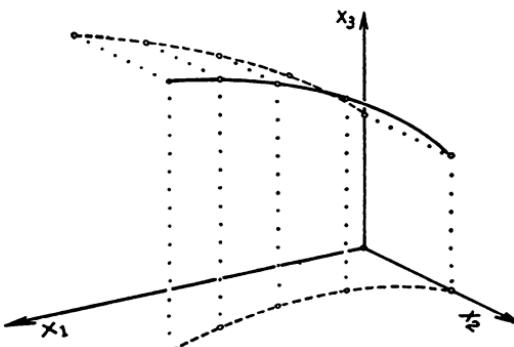


FIG. 9

We will now state a further remark on the representation (6.5). Each of the two equations in (6.5) represents a surface in space R_3 (provided certain conditions are satisfied, cf. Section 24). The point set corresponding to (6.5) is the intersection of these two surfaces. It is not always possible to represent the point set of a curve in the form (6.5). For example, difficulties arise if we attempt to represent the point set of the curve $\mathbf{x}(t) = (t, t^2, t^3)$ as an intersection of two surfaces of second order. It can be proved that these surfaces, besides the point set under consideration, always have a straight line in common. For instance, the intersection of the surfaces $x_1x_3 - x_2^2 = 0$, $x_1^2 + x_2^2 - x_1x_3 - x_2 = 0$ consists of the aforementioned point set and of the x_3 -axis.

The earliest investigations of curves by means of analysis and the first precise definition of the concept of a curve were made by R. Descartes [1] in 1637. The definition given by Descartes corresponds to what we nowadays call an algebraic curve, that is, a curve which can be represented by means of two algebraic equations relating the Cartesian coordinates (or one such equation if curves in a plane are considered). Parametric representations of (some special) curves were already used by Euler [2] in 1748, but the fact that parametric representations are the most suitable ones (since all three coordinates x_i play the same role) was not acknowledged until much later. The subsequent historical development of the concept of a curve is closely related to the development of the concept of a function.

Problem 7.1. A one-to-one and continuous image, that is, a topological image, of a segment is called a Jordan arc, cf. Section 6. One should perceive that the two conditions (one-to-one, continuous) are both necessary in order to exclude point sets which are two-dimensional. Prove: (a) The interval $0 < t \leq 1$ can be mapped in a one-to-one manner onto the square $0 < x_1 \leq 1, 0 < x_2 \leq 1$ of the $x_1 x_2$ -plane. (b) The interval $0 \leq t \leq 1$ can be uniquely and continuously mapped onto the square $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$ of the $x_1 x_2$ -plane.

8. Examples of special curves. We will now consider some special curves and their representations.

Example 1. A straight line in space passing through a point $(x_i) = (a_i)$ can be represented in the following parametric form

$$(8.1) \quad \mathbf{x}(t) = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t).$$

The coefficients b_i are proportional to the three *direction cosines* of the line, that is, the cosines of the angles between the line and the positive rays of the coordinate axes. As is well known the sum of the squares of the direction cosines equals 1. Hence we obtain the direction cosines of the above straight line by dividing the coefficients b_i by $\sqrt{b_1^2 + b_2^2 + b_3^2}$. If $b_1 \neq 0$ we find from (8.1)

$$t = (x_1 - a_1)/b_1$$

and consequently

$$x_2 = a_2 + \frac{b_2}{b_1}(x_1 - a_1), \quad x_3 = a_3 + \frac{b_3}{b_1}(x_1 - a_1),$$

which is a representation of the form (6.3). Hence, the planes spanned by the line and its orthogonal projections in the $x_1 x_2$ - and $x_1 x_3$ - coordinate planes can be represented in the form

$$b_2 x_1 - b_1 x_2 + a_2 b_1 - a_1 b_2 = 0,$$

$$b_3 x_1 - b_1 x_3 + a_3 b_1 - a_1 b_3 = 0;$$

this is, at the same time, a representation of the straight line (8.1) of the form (6.5).

Example 2. We consider next an ellipse with centre at the origin of the coordinate system in space. If the principal axes have the lengths $2a$ and $2b$ and coincide with the x_1 - and x_2 -axis, respectively, the ellipse has the parametric representation

$$(8.2) \quad \mathbf{x}(t) = (a \cos t, b \sin t, 0).$$

If in particular $a = b = r$ we obtain a circle of radius r ,

$$(8.3) \quad \mathbf{x}(t) = (r \cos t, r \sin t, 0).$$

These curves are closed; indeed, in (8.2), the functions $x_i(t)$ are periodic. From (8.2) we can obtain a representation of the form (6.5):

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0, \quad x_3 = 0.$$

Example 3. The folium of Descartes, introduced by R. Descartes, [2, p. 11] in 1638, can be represented in parametric form as follows:

$$\mathbf{x}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}, 0 \right).$$

This curve lies in the first, second, and fourth quadrant of the $x_1 x_2$ -plane, cf. Fig. 10a, and has a double point at $(x_1, x_2) = (0, 0)$; the part of the curve

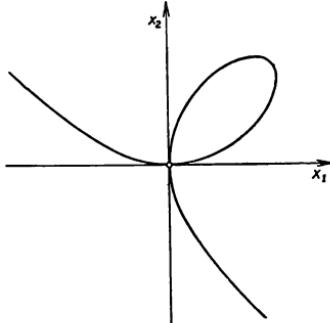


FIG. 10a. Folium of Descartes

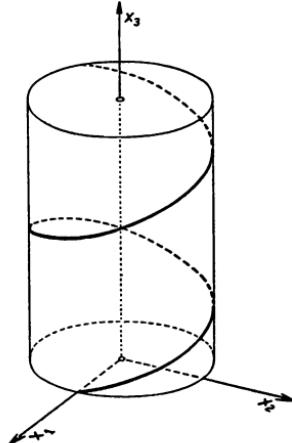


FIG. 10b. Circular helix

in the second quadrant corresponds to values of t from -1 to 0 , the loop in the first quadrant to values between 0 and ∞ , and the part in the fourth quadrant to values from $-\infty$ to -1 .

Example 4. A simple but important example of a twisted curve is given by the circular helix

$$(8.4) \quad \mathbf{x}(t) = (r \cos t, r \sin t, ct) \quad (c \neq 0).$$

The orthogonal projection of the helix into the $x_1 x_2$ -plane is the circle

$$x_1^2 + x_2^2 - r^2 = 0, \quad x_3 = 0,$$

which is the intersection of this plane with the cylinder of revolution on which the helix lies. Projecting the helix orthogonally into the x_2x_3 -plane we obtain the sine curve

$$x_2 - r \sin \frac{x_3}{c} = 0, \quad x_1 = 0.$$

A cosine curve is obtained by projecting the helix orthogonally into the x_1x_3 -plane.

Problem 8.1. Represent the folium of Descartes and the curve

$$\mathbf{x}(t) = \left(\frac{at}{t^2+1}, \frac{b(t^2-1)}{t^2+1}, 0 \right) \quad (a \neq 0, b \neq 0)$$

in the form (6.5).

Problem 8.2. Let r and ϕ be polar coordinates in the x_1x_2 -plane. Represent the following curves in the form (6.5) using Cartesian coordinates:

- (a) Spiral of Archimedes $r = a\phi$ ($a \neq 0$),
- (b) Cissoid of Diocles (200 B.C.) $r = 2c \sin^2\phi / \cos\phi$ ($c \neq 0$),
- (c) Conchoid of Nicomedes (200 B.C.) $r = \frac{a}{\cos\phi} + c$ ($a \neq 0, c \neq 0$),
- (d) Trisectrix of Maclaurin (1720) $r = a/\cos \frac{1}{3}\phi$ ($a \neq 0$).

Further examples of special curves will be considered in Section 21.

9. Arc length. The length of a segment with end points (x_i) and (y_i) is given by the expression

$$l = \sqrt{\left(\sum_{i=1}^3 (y_i - x_i)^2 \right)}.$$

Hence when defining the length of an arc C of a curve we may approximate C by a broken line of chords connecting the end points of C (cf. Fig. 11) since the length of such a broken line can be immediately determined if the end points of every chord are known. This suggests the following

Definition 9.1. Let $\mathbf{x}(t)$ ($a \leq t \leq b$), be an allowable representation of an arc C with initial point A and terminal point B . We denote by $l(Z)$ the length of a broken line Z of chords whose end points lie on C and, taken in the natural order, correspond to the values $a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$ of t . Let n increase arbitrarily and in such a way that $\sigma(Z) = \max_{1 \leq i \leq n} (t_i - t_{i-1})$

tends to zero. If then $l(Z)$ tends to a limit s , C is said to be rectifiable, and s is called the length of the arc C .

We will now prove that, in consequence of Assumption (1), Section 6, C is rectifiable; it is sufficient to choose the value $r = 1$.

Theorem 9.1. Let $\mathbf{x}(t)$ ($a \leq t \leq b$) be an allowable representation of an arc C of a curve of class $r \geq 1$. Then the arc C has the length

$$(9.1) \quad s = \int_a^b \sqrt{\left\{ \sum_{i=1}^3 \left(\frac{dx_i}{dt} \right)^2 \right\}} dt = \int_a^b \sqrt{(\mathbf{x}' \cdot \mathbf{x}')} dt.$$

s is independent of the special choice of the allowable parametric representation.

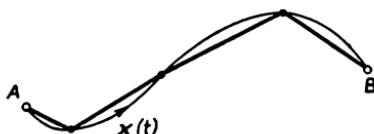


FIG. 11

Proof. We choose $n+1$ points P_ν of the arc corresponding to the values

$$(9.2) \quad a = t_0 < t_1 < \dots < t_n = b.$$

Let Z be the broken line of chords $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$, and denote its length by $l(Z)$. Then

$$(9.3) \quad l(Z) = \sum_{\nu=1}^n l_\nu,$$

where l_ν is the length of the chord with initial point $P_{\nu-1}$ and terminal point P_ν . For each $n = 1, 2, \dots$ we choose any subdivision (9.2) so that

$$\sigma(Z) = \max_{1 \leq \nu \leq n} (t_\nu - t_{\nu-1})$$

tends to zero as $n \rightarrow \infty$. We will prove that (9.3) then tends to a limit which is given by (9.1) and is independent of the special choice of (9.2). We have

$$l_\nu = \sqrt{\left(\sum_{i=1}^3 \{x_i(t_\nu) - x_i(t_{\nu-1})\}^2 \right)}.$$

According to the mean value theorem of differential calculus,

$$(9.4) \quad l_\nu = (t_\nu - t_{\nu-1}) \sqrt{\left(\sum_{i=1}^3 x_i'^2(t_\nu^{(i)}) \right)}$$

where $t_{\nu-1} < t_\nu^{(i)} < t_\nu$. Hence we obtain

$$l(Z) = \sum_{\nu=1}^n l_\nu = \sum_{\nu=1}^n \left((t_\nu - t_{\nu-1}) \sqrt{\left(\sum_{i=1}^3 x_i'^2(t_\nu^{(i)}) \right)} \right)$$

which is equivalent to

$$(9.5) \quad l(Z) = \sum_{\nu=1}^n \left((t_\nu - t_{\nu-1}) \sqrt{\left(\sum_{i=1}^3 x_i'^2(t_\nu) \right)} + \sum_{\nu=1}^n (t_\nu - t_{\nu-1}) \left[\sqrt{\left(\sum_{i=1}^3 x_i'^2(t_\nu^{(i)}) \right)} - \sqrt{\left(\sum_{i=1}^3 x_i'^2(t_\nu) \right)} \right] \right).$$

The expression in brackets [...] is the difference of the distances of the points $(x'_i(t_\nu^{(i)}))$ and $(x'_i(t_\nu))$ from the origin of the Cartesian coordinate system; we denote this distance by d_ν . In consequence of the triangle inequality, $|d_\nu|$ is at most equal to the distance

$$a_\nu = \sqrt{\left(\sum_{i=1}^3 (x'_i(t_\nu^{(i)}) - x'_i(t_\nu))^2 \right)}$$

between those points. We have

$$(9.6) \quad a_\nu \leq \sum_{i=1}^3 |x'_i(t_\nu^{(i)}) - x'_i(t_\nu)|,$$

as can be easily seen by squaring both sides of (9.6) and comparing the expression thus obtained with the preceding one. Since, by assumption, $\mathbf{x}(t)$ is (at least) once continuously differentiable, for every given number $\epsilon > 0$ there exists a number $\delta(\epsilon) > 0$ such that

$$|x'_i(\alpha) - x'_i(\beta)| < \epsilon/3, \quad (i = 1, 2, 3)$$

if $|\alpha - \beta| \leq \delta(\epsilon)$, $a \leq \alpha$, $\beta \leq b$. Consequently, if $\sigma(Z) \leq \delta(\epsilon)$ the right-hand side of (9.6) becomes smaller than ϵ . Hence also $|d_\nu| < \epsilon$. We thus obtain

$$\left| \sum_{\nu=1}^n (t_\nu - t_{\nu-1}) d_\nu \right| < \epsilon \sum_{\nu=1}^n (t_\nu - t_{\nu-1}) = \epsilon(b-a).$$

If $\sigma(Z) \rightarrow 0$ we may assume the right-hand side of this expression to be arbitrarily small. Hence in (9.5) the last sum tends to zero while the first one tends to the integral (9.1) since the function $\sqrt{\left(\sum_{i=1}^3 x_i'^2(t) \right)}$ is continuous in the interval I : $a \leq t \leq b$. Finally we will prove that s is independent of the special choice of the allowable parametric representation of C . Let t^* be another allowable parameter and $I^*: a^* \leq t^* \leq b^*$ the interval corresponding to I . In consequence of the Assumptions (1*) and (2*), cf. Section 6, the function dt/dt^* is continuous in I^* . Introducing t^* in (9.1) we find

$$\begin{aligned} s &= \int_{a^*}^{b^*} \sqrt{\left\{ \sum_{i=1}^3 \left(\frac{dx_i}{dt} \right)^2 \right\}} \left| \frac{dt}{dt^*} \right| dt^* = \int_{a^*}^{b^*} \sqrt{\left\{ \sum_{i=1}^3 \left(\frac{dx_i}{dt} \frac{dt}{dt^*} \right)^2 \right\}} dt^* \\ &= \int_{a^*}^{b^*} \sqrt{\left\{ \sum_{i=1}^3 \left(\frac{dx_i}{dt^*} \right)^2 \right\}} dt^*. \end{aligned}$$

This completes the proof.

If we replace in (9.1) the fixed value b by a variable one, t , then s becomes a function of t . Also a may be replaced by any other fixed value $t = t_0$, $a \leq t_0 \leq b$. In this manner we obtain the integral

$$(9.1') \quad s(t) = \int_{t_0}^t \sqrt{(\mathbf{x}' \cdot \mathbf{x}') dt}.$$

The function $s(t)$ is called the *arc length* of C ; it has the following geometric significance: If $t > t_0$ then $s(t)$ is positive and is equal to the length of the arc $P(t_0)P(t)$ of C . If $t < t_0$ then $s(t)$ is negative, and the length of $P(t_0)P(t)$ is given by $-s(t)$ (> 0).

Instead of

$$s'^2 = \sum_{i=1}^3 x_i'^2 = \mathbf{x}' \cdot \mathbf{x}'$$

we may write symbolically

$$(9.7) \quad ds^2 = \sum_{i=1}^3 dx_i^2 = d\mathbf{x} \cdot d\mathbf{x};$$

ds is called the *element of arc* or *linear element* of C .

The arc length may be used as parameter in the parametric representation of a curve. In consequence of the Assumptions (1) and (2), cf. Section 6, $s(t)$ is a continuous and strictly monotonic function of t . Furthermore, by Assumption (2), everywhere in I

$$s'(t)^2 = \sum_{i=1}^3 x_i'^2(t) \neq 0$$

and also

$$\frac{dx_i}{ds} = \frac{x_i'(t)}{s'(t)} \neq 0$$

for at least one value of $i = 1, 2, 3$. Finally, in consequence of Assumption (1), $s'(t)$ is of class $r-1$. Hence $\mathbf{x}(s)$ is an allowable representation of class r , and s is an allowable parameter. s is called the *natural parameter*. We have already noted that the point of the arc corresponding to $s = 0$ can be chosen arbitrarily, and that points of the arc for which $t < t_0$ correspond to negative values of s .

We shall see that the choice of a parametric representation with s as parameter simplifies many investigations.

Derivatives with respect to the arc length s will always be denoted by dots while derivatives with respect to any other allowable parameter will be denoted by primes, e.g.,

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds}, \quad \ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{ds^2}, \quad \mathbf{x}' = \frac{d\mathbf{x}}{dt}, \quad \mathbf{x}'' = \frac{d^2\mathbf{x}}{dt^2}, \quad \text{etc.}$$

Example. In the case of the circular helix (8.4) we find

$$\mathbf{x}' = (-r \sin t, r \cos t, c), \quad \mathbf{x}' \cdot \mathbf{x}' = r^2 + c^2$$

and therefore

$$(9.8) \quad s(t) = t\sqrt{r^2 + c^2}.$$

Example (cf. Fig. 12). The representation

$$(9.9) \quad x_1 = t, \quad x_2 = \begin{cases} t \sin 1/t & (t \neq 0) \\ 0 & (t = 0), \end{cases} \quad x_3 = 0, \quad 0 \leq t \leq t_1,$$

is continuous, even at $t = 0$, but the corresponding point set (which does not form an arc in the sense of our definition) has no length.

Problem 9.1. Represent the circular helix by a parametric representation with arc length s as parameter.

Problem 9.2. Determine the arc length of the catenary

$$\mathbf{x}(t) = \left(t, a \cosh \frac{t}{a}, 0 \right).$$

Problem 9.3. Determine the arc length of an ellipse. The integral occurring cannot be evaluated in terms of elementary functions. Develop this integral as a series of powers of the numerical eccentricity of the ellipse.

Problem 9.4. Prove that (9.9) does not possess a length.

10. Tangent and normal plane. Let C be an arbitrary curve in the space R_3 and let $\mathbf{x}(s)$ be a parametric representation of C with arc length s as parameter. Two points of C , corresponding to the values s and $s+h$ of the parameter, determine a chord of C whose direction is given by the vector $\mathbf{x}(s+h) - \mathbf{x}(s)$, cf. Fig. 13, hence also by the vector

$$\frac{\mathbf{x}(s+h) - \mathbf{x}(s)}{h}.$$

The vector

$$(10.1) \quad \mathbf{t}(s) = \lim_{h \rightarrow 0} \frac{\mathbf{x}(s+h) - \mathbf{x}(s)}{h} = \frac{d\mathbf{x}}{ds} = \dot{\mathbf{x}}(s)$$

is called the *unit tangent vector* to the curve C at the point $\mathbf{x}(s)$. This vector

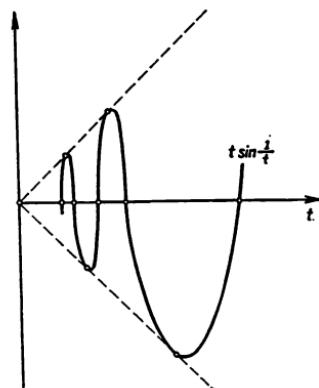


Fig. 12

exists, in consequence of Assumption (1), Section 6, and, indeed, is a unit vector, because

$$|\mathbf{t}|^2 = \mathbf{t} \cdot \mathbf{t} = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} = 1.$$

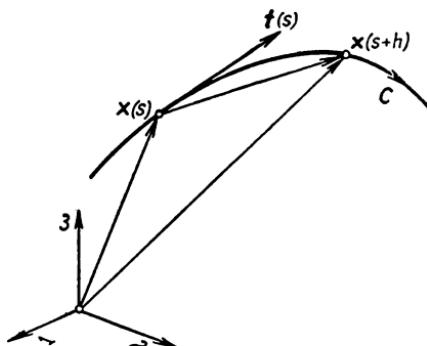


FIG. 13

Its sense corresponds to increasing values of s and thus depends on the choice of the parametric representation. Introducing any other allowable parameter t we have

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} \frac{dt}{ds} = \frac{\mathbf{x}'}{\sqrt{(\mathbf{x}' \cdot \mathbf{x}')}},$$

hence

$$(10.2) \quad \mathbf{t}(t) = \frac{\mathbf{x}'}{|\mathbf{x}'|}.$$

The straight line passing through a point P of C in the direction of the corresponding unit tangent vector is called the *tangent* to the curve C at P . According to (10.1), the position of the tangent to C at P is the limit position of a straight line through P and through another point P_1 of C if P_1 tends to P . Obviously, the tangent can be represented in the form

$$(10.3) \quad \mathbf{y}(u) = \mathbf{x} + u\mathbf{t},$$

where \mathbf{x} and \mathbf{t} depend on the point of C under consideration and u is a real variable. The point $\mathbf{y}(0)$ is the point of contact between C and the tangent. In (10.3) we may replace the unit vector \mathbf{t} by any vector parallel to \mathbf{t} , e.g. by \mathbf{x}' . Then we obtain another parametric representation of the tangent,

$$(10.3') \quad \mathbf{y}(v) = \mathbf{x} + v\mathbf{x}'.$$

The totality of all vectors bound at a point P of C which are orthogonal to the corresponding unit tangent vector lie in a plane. This plane is called the *normal plane* to C at P .

Example. In the case of the circular helix (8.4) we have (9.8):

$$s' = \sqrt{(r^2 + c^2)} \equiv w.$$

Hence the unit tangent vector to this curve is of the form

$$\mathbf{t} = \left(-\frac{r}{w} \sin t, \frac{r}{w} \cos t, \frac{c}{w} \right).$$

From (10.3) we obtain the following representation of the tangent:

$$\mathbf{y}(u) = \left\{ r \left(\cos t - \frac{u}{w} \sin t \right), r \left(\sin t + \frac{u}{w} \cos t \right), c \left(t + \frac{u}{w} \right) \right\}.$$

Using (10.3') we find a simpler one,

$$\mathbf{y}(v) = \{r(\cos t - v \sin t), r(\sin t + v \cos t), c(t + v)\}.$$

As can be seen from the above expression, the angle between the tangent to the circular helix and the x_3 -axis is constant.

We should mention that straight lines are the only curves whose tangent direction is constant; this can be seen by integrating the vector equation $\mathbf{x}' = \text{const.}$

In antiquity only tangents of special curves (conic sections, spiral of Archimedes) were known. The general concept of a tangent was introduced during the seventeenth century, in connexion with the basic concepts of calculus; Fermat, Descartes, and Huyghens made important contributions to the tangent problem; a complete solution was obtained by Leibniz [1] in 1677. The first analytical expression of a tangent was given by Monge [1] in 1785.

Problem 10.1. Determine the point of intersection between the x_1 -axis and the tangent to the curve $\mathbf{x}(t) = (t, t^2, 0)$ at that point which corresponds to $t = 1$.

Problem 10.2. Determine the curve of intersection between the $x_1 x_2$ -plane and the tangents to the curve $\mathbf{x}(t) = (t, Bt^2, Ct^n)$. B and C are constants; n is a natural number.

Problem 10.3. Given two unit circles in the $x_1 x_2$ -plane with centres at $(0, 0, 0)$ and $(1, -1, 0)$, respectively. Determine those straight lines which are tangent to both circles.

11. Osculating plane. Let $\mathbf{x}(t)$ be a parametric representation of a curve C . Two points of C , say P and P_1 , determine a straight line S . If P_1 tends to P then S tends to the tangent to C at P . It is natural to ask for the limit position of a plane E passing through three points P , P_1 , and P_2 of C if P_1 and P_2 both tend to P .

We assume the representation $\mathbf{x}(t)$ to be of class $r \geq 2$. We denote by t , $t+h_1$, and $t+h_2$, respectively, the parametric values of P , P_1 , and P_2 . The chords PP_1 and PP_2 of C are given by the vectors $\mathbf{a}_i = \mathbf{x}(t+h_i) - \mathbf{x}(t)$, ($i = 1, 2$), cf. Fig. 14. That is, these vectors, if linearly independent, span

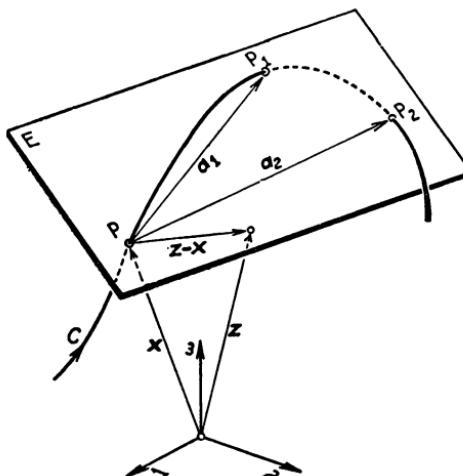


FIG. 14

the plane E . This plane is consequently also spanned by the vectors $\mathbf{v}^{(1)} = \mathbf{a}_i/h_i$ or by the vectors

$$\mathbf{v}^{(1)} \text{ and } \mathbf{w} = \frac{2(\mathbf{v}^{(2)} - \mathbf{v}^{(1)})}{h_2 - h_1}.$$

By means of the Taylor formula we have

$$\mathbf{x}(t+h_i) = \mathbf{x}(t) + h_i \mathbf{x}'(t) + \frac{h_i^2}{2!} \mathbf{x}''(t) + o(h_i^2),$$

where $o(h_i^2)$ is a vector whose components are $o(h_i^2)$. (Definition of this so-called *Landau symbol*: Let $f(z)$ be a function which is different from zero in an interval containing the point $z = 0$. Let $g(z)$ be another function defined in this interval and such that the quotient $g(z)/f(z)$ tends to zero if z tends to zero. Then $g(z)$ is said to be $o(f(z))$, read ‘small o of $f(z)$ ’.) Hence we obtain

$$(11.1) \quad \begin{aligned} \mathbf{v}^{(1)} &= \mathbf{x}'(t) + \frac{1}{2} h_1 \mathbf{x}''(t) + o(h_1), \\ \mathbf{w} &= \mathbf{x}''(t) + o(1). \end{aligned}$$

Consequently, if $h_i \rightarrow 0$, then $\mathbf{v}^{(1)} \rightarrow \mathbf{x}'(t)$ and $\mathbf{w} \rightarrow \mathbf{x}''(t)$. These vectors, if linearly independent, determine the limiting position of the plane E passing through the above three points of the curve. The plane spanned by $\mathbf{x}'(t)$

and $\mathbf{x}''(t)$ is called the *osculating plane* of the curve C at P . The name 'osculating plane' was first used by Tinseau (1780).

We denote by \mathbf{z} the position vector of any point of the osculating plane O of C . The vector $\mathbf{z} - \mathbf{x}$ lies in O , i.e. the vectors $\mathbf{z} - \mathbf{x}$, \mathbf{x}' , and \mathbf{x}'' are linearly dependent. In consequence of Theorem 5.3 we thus obtain the following representation of the osculating plane of C at P :

$$(11.2) \quad |(\mathbf{z} - \mathbf{x}) \mathbf{x}' \mathbf{x}''| = 0, \quad (\mathbf{x}' \times \mathbf{x}'' \neq \mathbf{0}).$$

In (11.2), the vectors \mathbf{x} , \mathbf{x}' , and \mathbf{x}'' depend on P .

The osculating plane passes through the tangent. The intersection of the osculating plane with the corresponding normal plane is called the *principal normal*; it will be considered in the following section.

An exceptional case occurs when the vectors \mathbf{x}' and \mathbf{x}'' are linearly dependent. If this is true for all values of the parameter t , then the curve is a straight line, cf. Problem 11.1. The osculating plane of a plane curve is always the plane of the curve.

At the moment we confine ourselves to the consideration of a simple example; further properties of the osculating plane will be investigated in Section 19.

Example. The osculating plane O of the circular helix (8.4):

$$\mathbf{x}(t) = (r \cos t, r \sin t, ct)$$

can be represented in the form

$$\begin{vmatrix} z_1 - r \cos t & -r \sin t & -r \cos t \\ z_2 - r \sin t & r \cos t & -r \sin t \\ z_3 - ct & c & 0 \end{vmatrix} = 0,$$

which is equivalent to

$$z_1 c \sin t - z_2 c \cos t + (z_3 - ct)r = 0.$$

O passes through the tangent to the helix, which may easily be checked by inserting the representation of the tangent given in Section 10. O passes also through the straight line $\mathbf{z}^*(a) = (a \cos t, a \sin t, ct)$ where t is fixed and a is a real parameter. This line is parallel to the $x_1 x_2$ -plane. It passes through the axis of the cylinder on which the helix lies as well as through a point P of the helix. Moreover, it is orthogonal to the tangent to the helix at P . Hence this line is the principal normal to the helix.

Problem 11.1. Given an allowable representation $\mathbf{x}(t)$ of class $r \geq 2$ of a curve C . Prove: If, for every value of t , the vectors $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$ are linearly dependent then C is a straight line.

12. Principal normal, curvature, osculating circle. We will now introduce the important concept of the curvature of a curve.

Let a curve C be given by an allowable parametric representation $\mathbf{x}(s)$ of class $r \geq 2$ with arc length s as parameter. Differentiating the relation $\mathbf{t} \cdot \mathbf{t} = 1$ we obtain $\mathbf{t} \cdot \dot{\mathbf{t}} = 0$. Hence, if the vector

$$(12.1) \quad \ddot{\mathbf{t}} = \ddot{\mathbf{x}},$$

is not the null vector, it is orthogonal to the unit tangent vector \mathbf{t} and consequently lies in the normal plane to C at the point under consideration; $\ddot{\mathbf{t}}$ also lies in the osculating plane, cf. Section 11. The unit vector

$$(12.2) \quad \mathbf{p}(s) = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|}$$

which has the direction and sense of $\ddot{\mathbf{t}}$ is called the *unit principal normal vector* to the curve C at the point $\mathbf{x}(s)$. The straight line passing through this point and containing $\mathbf{p}(s)$ is called the *principal normal* to the curve C at this point. Since $\ddot{\mathbf{t}}$ lies in the corresponding osculating plane as well as in the normal plane, the principal normal is the intersection of these planes.

The absolute value of the vector $\dot{\mathbf{t}}$,

$$(12.3) \quad \kappa(s) = |\dot{\mathbf{t}}(s)| = \sqrt{\{\ddot{\mathbf{x}}(s) \cdot \ddot{\mathbf{x}}(s)\}}, \quad (\kappa \geq 0),$$

is called the *curvature* of the curve C at the point $\mathbf{x}(s)$. The reciprocal of the curvature,

$$(12.4) \quad \rho(s) = \frac{1}{\kappa(s)}, \quad (\kappa > 0),$$

is called the *radius of curvature* of the curve C at the point $\mathbf{x}(s)$.

Straight lines are the only curves whose curvature vanishes identically, because from $\kappa \equiv 0$ we obtain $\mathbf{x}(s) = \mathbf{a}s + \mathbf{c}$ where \mathbf{a} is a constant unit vector and \mathbf{c} is also a constant vector.

It will be advantageous for our later considerations to use a special letter for the vector $\dot{\mathbf{t}}$. We set

$$(12.5) \quad \mathbf{k}(s) = \dot{\mathbf{t}}(s)$$

and call this vector the *curvature vector* of the curve C .

We now mention the following fact: while the sense of the unit tangent vector to a curve depends on the orientation of the curve resulting from the choice of a certain parametric representation, the unit principal normal vector is independent of the orientation of the curve; its sense does not change if the parameter s is replaced by the parameter $s^* = -s$ or any other allowable parameter.

The point M on the positive ray of the principal normal at distance $\rho(s)$ from the corresponding point P of the curve C is called the *centre of curvature*. The circle in the osculating plane whose radius is ρ and whose centre is M is called *osculating circle* or *circle of curvature* of C at P .

Obviously, the osculating circle passes through P ; the tangents to C and to the osculating circle at P coincide; this fact will be considered in detail in Section 19.

From the definition of M it follows that its position vector \mathbf{z} is of the form

$$(12.6) \quad \mathbf{z} = \mathbf{x} + \rho \mathbf{p} = \mathbf{x} + \rho^2 \mathbf{t}, \quad (\kappa > 0).$$

We will now derive from (12.3) an analytic expression of the curvature which is valid when C is represented by any allowable parametric representation $\mathbf{x}(t)$ of class $r \geq 2$. In consequence of (12.3), (12.2) is equivalent to $\mathbf{t} = \kappa \mathbf{p}$. Taking the vector product of both sides of this equation by \mathbf{t} we find

$$\mathbf{t} \times \dot{\mathbf{t}} = \dot{\mathbf{x}} \times \ddot{\mathbf{x}} = \mathbf{t} \times \kappa \mathbf{p}.$$

Since \mathbf{t} and \mathbf{p} are orthogonal unit vectors, using (5.8) we obtain

$$\kappa = |\dot{\mathbf{x}} \times \ddot{\mathbf{x}}| = |\mathbf{x}' \times \mathbf{x}''| \left(\frac{dt}{ds} \right)^3.$$

Hence, by means of (9.7),

$$(12.7) \quad \kappa = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3},$$

which, in consequence of (5.11), is equivalent to

$$(12.7) \quad \kappa = \frac{\sqrt{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}}{(\mathbf{x}' \cdot \mathbf{x}')^{\frac{3}{2}}}.$$

If in particular $t = s$, then (12.7) reduces to the form (12.3).

Example 1. If C is a circle of radius r then the radius of curvature also has the constant value r . Indeed, when C is represented by

$$\mathbf{x}(t) = (r \cos t, r \sin t, 0),$$

cf. (8.3), then $\mathbf{x}' = (-r \sin t, r \cos t, 0)$, $\mathbf{x}'' = (-r \cos t, -r \sin t, 0)$, and, by means of (12.7), $\rho = r$. Formula (12.6) yields $\mathbf{z} = (0, 0, 0)$; the centre of curvature and the centre of the circle coincide.

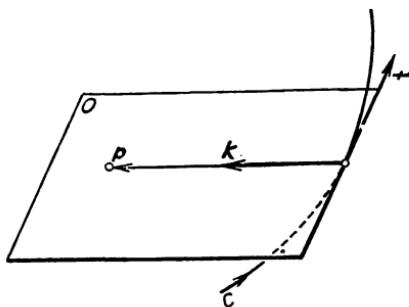


FIG. 15

Example 2. The curvature of the circular helix $\mathbf{x}(t) = (r \cos t, r \sin t, ct)$ is constant. By means of (12.7) we obtain

$$\kappa = \frac{r}{r^2 + c^2}.$$

If c tends to zero $\rho = 1/\kappa$ tends to the radius r of the circle of intersection between the $x_1 x_2$ -plane and the cylinder of revolution on which the helix lies.

Problem 12.1. Find a representation of the position vector of the centre of curvature of a curve $\mathbf{x}(t)$ in terms of $\mathbf{x}(t)$ and its derivatives.

Problem 12.2. Prove: The locus of the centres of curvature of a circular helix C is a circular helix C^* . Determine a circular helix C for which the cylinder of C^* coincides with that of C .

13. Binormal. Moving trihedron of a curve. We again consider an arbitrary curve C , given by an allowable representation $\mathbf{x}(s)$ of class $r \geq 2$ where s denotes the arc length of C . To every point of the curve at which $\kappa > 0$, we have associated two unit vectors, the unit tangent vector $\mathbf{t}(s) = \dot{\mathbf{x}}(s)$ and the unit principal normal vector

$$\mathbf{p}(s) = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|} = \rho(s) \ddot{\mathbf{x}}(s), \quad (\kappa > 0).$$

These two vectors are orthogonal. We now introduce a unit vector

$$(13.1) \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{p}(s) \quad (\kappa > 0)$$

which is orthogonal to \mathbf{t} and \mathbf{p} and is called the *unit binormal vector* of the curve C at the point $\mathbf{x}(s)$. The orthogonality is an immediate consequence of the definition of the vector product, cf. Section 5. From this definition we see also that the vectors \mathbf{t} , \mathbf{p} , \mathbf{b} , in this order, have the right-handed sense. The fact that \mathbf{b} is a unit vector can be seen from (5.8).

To every point of a curve at which $\kappa > 0$ we have now associated three orthogonal unit vectors. The triple \mathbf{t} , \mathbf{p} , \mathbf{b} is called the *moving trihedron* of the curve, cf. Fig. 16.

For several investigations it is convenient to use the Cartesian coordinate system whose axes are the straight lines passing through a point of the curve in the direction of the corresponding vectors \mathbf{t} , \mathbf{p} , and \mathbf{b} , respectively. The motion of the trihedron along the curve will be considered in Section 16.

The straight line passing through a point $\mathbf{x}(s)$ of a curve C in the direction of the corresponding unit binormal vector $\mathbf{b}(s)$ is called the *binormal* to C at the point $\mathbf{x}(s)$.

The vectors \mathbf{t} and \mathbf{p} span the osculating plane S of the curve. The vectors \mathbf{p} and \mathbf{b} span the normal plane N . The plane spanned by the vectors \mathbf{t} and \mathbf{b} is called the *rectifying plane*. Since \mathbf{t} , \mathbf{p} , and \mathbf{b} are the normal vectors to the

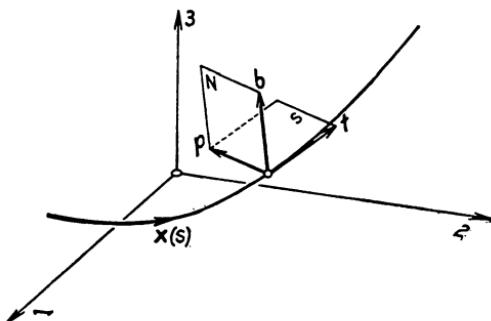


FIG. 16. Moving trihedron

normal, the rectifying, and the osculating plane, respectively, we have the following representations:

- $$(13.2) \quad \begin{aligned} (a) \text{ Normal plane: } & (\mathbf{z} - \mathbf{x}) \cdot \mathbf{t} = 0, \\ (b) \text{ Rectifying plane: } & (\mathbf{z} - \mathbf{x}) \cdot \mathbf{p} = 0, \quad (\kappa > 0), \\ (c) \text{ Osculating plane: } & (\mathbf{z} - \mathbf{x}) \cdot \mathbf{b} = 0, \quad (\kappa > 0). \end{aligned}$$

(11.2) may be obtained from (13.2 (c)) by taking into account (13.1).

Problem 13.1. Given two twisted curves which are in a one-to-one correspondence so that at corresponding points the tangents are parallel. Prove that at corresponding points the principal normals, and the binormals are parallel.

14. Torsion. We consider an arbitrary twisted curve which has a parametric representation $\mathbf{x}(s)$ of class $r \geq 3$, with arc length s as parameter. The curvature $\kappa(s)$, cf. Section 12, measures the rate of change of the tangent when moving along the curve. It measures, so to speak, the deviation of the curve from a straight line in the neighbourhood of any of its points. The torsion to be introduced now should, roughly speaking, measure the magnitude and sense of deviation of a curve from the osculating plane in the neighbourhood of the corresponding point of the curve, or, in other words, the rate of change of the osculating plane; we assume that $\kappa > 0$ at this point.

Since the osculating plane is spanned by the unit tangent vector \mathbf{t} and by the unit principal normal vector \mathbf{p} , the unit binormal vector \mathbf{b} is the normal vector to this plane, cf. Section 13. In the case of a plane curve the

osculating planes coincide with the plane of the curve. Then $\mathbf{b}(s)$ is a constant vector; we have $\dot{\mathbf{b}} = \mathbf{0}$. It is therefore natural to consider the vector \mathbf{b} in our present investigation.

Differentiating the scalar products

$$\mathbf{b} \cdot \mathbf{b} = 1 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{t} = 0$$

with respect to the arc length s we obtain

$$\mathbf{b} \cdot \dot{\mathbf{b}} = 0 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{t} + \mathbf{b} \cdot \dot{\mathbf{t}} = 0.$$

Since $\dot{\mathbf{t}} = \kappa \mathbf{p}$ we find from this that

$$\mathbf{b} \cdot \mathbf{t} = -\mathbf{b} \cdot \dot{\mathbf{t}} = -\kappa \mathbf{b} \cdot \mathbf{p} = 0.$$

Hence, if the vector \mathbf{b} is not the null vector it is orthogonal to the vectors \mathbf{t} and \mathbf{b} , and consequently lies in the principal normal. We may set

$$(14.1) \quad \mathbf{b}(s) = -\tau(s) \mathbf{p}(s), \quad (\kappa > 0).$$

Scalar multiplication of (14.1) by \mathbf{p} yields

$$(14.2) \quad \tau(s) = -\mathbf{p}(s) \cdot \dot{\mathbf{b}}(s), \quad (\kappa > 0).$$

$\tau(s)$ is called the *torsion* of the curve C at the point $\mathbf{x}(s)$.

The sign of the torsion has a geometric significance. It has been chosen so that right-handed curves (for example, right-handed helices, that is, helices for which, in (8.4), $c > 0$) have a positive torsion $\tau(s)$, as will be seen in what follows.

The name ‘torsion’ was first used by L. I. de la Vallée [1] in 1825. Curvature and torsion are also known as *first* and *second curvature*, and a twisted curve is therefore called a *curve of double curvature*; we will not use these names.

Let us now derive a representation of $\tau(s)$ in terms of the function $\mathbf{x}(s)$ and its derivatives. Inserting (13.1) in (14.2) we obtain

$$\tau = -\mathbf{p} \cdot \dot{\mathbf{b}} = -\mathbf{p} \cdot \frac{d}{ds}(\dot{\mathbf{x}} \times \mathbf{p}) = -\mathbf{p} \cdot ((\ddot{\mathbf{x}} \times \mathbf{p}) + (\dot{\mathbf{x}} \times \dot{\mathbf{p}})).$$

Now $\mathbf{p} \cdot (\dot{\mathbf{x}} \times \mathbf{p}) = |\mathbf{p} \dot{\mathbf{x}} \mathbf{p}| = 0$. Inserting $\mathbf{p} = \rho \ddot{\mathbf{x}}$, cf. (12.2)–(12.4) we find

$$\tau = -|\rho \ddot{\mathbf{x}} \dot{\mathbf{x}} (\rho \ddot{\mathbf{x}} + \rho \ddot{\mathbf{x}})|.$$

Interchanging the first two columns and simplifying the last one we have

$$\tau = \rho^2 |\dot{\mathbf{x}} \dot{\mathbf{x}} \ddot{\mathbf{x}}|.$$

In consequence of (12.3) and (12.4) this expression is equivalent to

$$(14.3) \quad \tau = \frac{|\dot{\mathbf{x}} \dot{\mathbf{x}} \ddot{\mathbf{x}}|}{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}.$$

We pointed out in Section 9 that the positive sense of a curve, corresponding to increasing values of s , can be arbitrarily chosen. One orientation can be changed into the other, for instance, by means of the transformation $s = -s^*$. Under this transformation the functions $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are multiplied by a factor -1 while $\ddot{\mathbf{x}}$ remains unchanged. The right-hand side of (14.3) remains therefore unaffected by this transformation or by any other allowable transformation which changes the orientation of the curve. Hence the sign of the torsion must have a geometric significance. Since $\mathbf{x}(s)$ is assumed to be of class $r \geq 3$ we may write

$$\mathbf{x}(s+h) = \mathbf{x}(s) + h\dot{\mathbf{x}}(s) + \frac{h^2}{2!}\ddot{\mathbf{x}}(s) + \frac{h^3}{3!}\ddot{\mathbf{x}}(s) + o(h^3).$$

The vectors $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ lie in the osculating plane. The vector $\frac{h^3}{6}\ddot{\mathbf{x}}(s) + o(h^3)$ will therefore determine the sense in which, in a neighbourhood of the point $\mathbf{x}(s)$, the curve leaves the osculating plane. Because of continuity, the difference between this vector and the vector $h^3\ddot{\mathbf{x}}(s)/6$ is arbitrarily small for sufficiently small values of h . The vector $\ddot{\mathbf{x}}(s)$ will therefore also determine the aforementioned sense. The curve is right-handed if, at the point $\mathbf{x}(s)$, for increasing values of s , it leaves the osculating plane to the positive side of this plane which is determined by the sense of the unit binormal vector $\mathbf{b}(s)$. In this case the angle between the vectors $\ddot{\mathbf{x}}$ and $\mathbf{b} = \dot{\mathbf{x}} \times \mathbf{p} = \dot{\mathbf{x}} \times \rho \ddot{\mathbf{x}}$ must be smaller than $\frac{1}{2}\pi$. Since $\rho > 0$, the same must be true for the angle between the vectors $\ddot{\mathbf{x}}$ and $\dot{\mathbf{x}} \times \ddot{\mathbf{x}}$. Hence $|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}| > 0$, and in consequence of (14.3) we must also have $\tau > 0$. That is, right-handed curves have positive torsion.

The reciprocal of the torsion,

$$\sigma = \frac{1}{\tau}, \quad (\tau \neq 0),$$

is called the *radius of torsion*.

Imposing on $\mathbf{x}(s)$ an allowable transformation of class $r \geq 3$ we obtain from (14.3)

$$(14.4) \quad \tau = \frac{|\mathbf{x}' \mathbf{x}'' \mathbf{x}'''|}{(\mathbf{x}' \times \mathbf{x}'') \cdot (\mathbf{x}' \times \mathbf{x}'')} = \frac{|\mathbf{x}' \mathbf{x}'' \mathbf{x}'''|}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}, \quad (\kappa > 0),$$

cf. Problem 14.1.

We mention the following simple but important fact:

Theorem 14.1. *A curve, with non-vanishing curvature, is plane if and only if its torsion vanishes identically.*

Proof. If the curve is plane, \mathbf{b} is constant, i.e. $\mathbf{b} = \mathbf{0}$. In consequence of (14.2) therefore also $\tau = 0$. Conversely, if $\tau = 0$ then, by means of (14.2), $\mathbf{b} = \text{const}$. Integrating $\mathbf{b} \cdot \mathbf{t} = 0$ where $\mathbf{b} = \text{const}$ we have $\mathbf{b} \cdot \mathbf{x}(s) = \text{const}$,

that is, the curve represented by $\mathbf{x}(s)$ lies in a plane orthogonal to the constant vector \mathbf{b} .

Example. The circular helix (8.4)

$$\mathbf{x}(t) = (r \cos t, r \sin t, ct)$$

is right-handed if $c > 0$ and left-handed if $c < 0$, cf. Fig. 17. The curvature of this curve is given by the expression

$$\kappa = \frac{r}{r^2 + c^2},$$

cf. Section 12. Since $\mathbf{x}' \cdot \mathbf{x}' = r^2 + c^2$, $\mathbf{x}'' \cdot \mathbf{x}'' = r^2$, and $\mathbf{x}' \cdot \mathbf{x}'' = 0$ we obtain from (14.4)

$$\tau = \frac{1}{r^2(r^2 + c^2)} \begin{vmatrix} -r \sin t & r \cos t & c \\ -r \cos t & -r \sin t & 0 \\ r \sin t & -r \cos t & 0 \end{vmatrix} = \frac{c}{r^2 + c^2}.$$

If this curve is right-handed ($c > 0$) then $\tau > 0$, if it is left-handed ($c < 0$) then $\tau < 0$.

Problem 14.1. Prove (14.4).

Problem 14.2. Determine the torsion of the curve $\mathbf{x}(t) = (t, t^2/2, t^3/6)$.

15. Formulae of Frenet. The first derivatives $\dot{\mathbf{t}}$, $\dot{\mathbf{p}}$, and $\dot{\mathbf{b}}$ of the unit tangent vector \mathbf{t} , the unit principal normal vector $\mathbf{p}(s)$, and the unit binormal vector $\mathbf{b}(s)$ can be represented as a linear combination of these vectors \mathbf{t} , \mathbf{p} , and \mathbf{b} ; the corresponding formulae are called the formulae of Frenet [1]; they are of fundamental importance in the theory of curves.

Let us at first explain the reason for the existence of formulae of this type: in three-dimensional space R_3 four (or more) vectors are always linearly dependent, cf. Problem 5.3. This means that any vector in this space can always be represented as a linear combination of three linearly independent vectors. Now, since \mathbf{t} , \mathbf{p} , and \mathbf{b} are linearly independent, any other vector can be represented as a linear combination of these three vectors, in particular also the vectors $\dot{\mathbf{t}}$, $\dot{\mathbf{p}}$, and $\dot{\mathbf{b}}$, provided these vectors exist and are continuous functions of s .

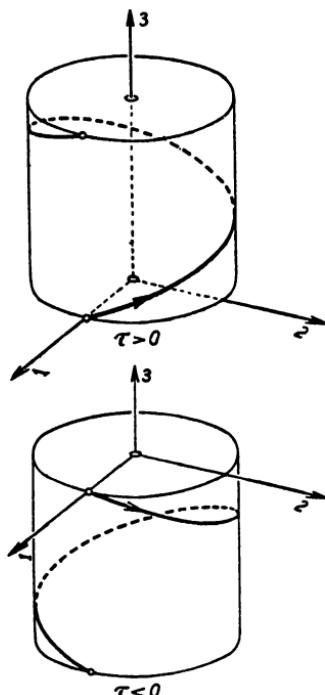


FIG. 17

We already know two of the three formulae of Frenet,
 cf. (12.2) and (12.3), and $\dot{\mathbf{t}} = \kappa \mathbf{p}$,
 $\dot{\mathbf{b}} = -\tau \mathbf{p}$,

cf. (14.1). We will now find an analogous representation of the vector $\dot{\mathbf{p}}$. Differentiating $\mathbf{p} \cdot \mathbf{p} = 1$ we have $\mathbf{p} \cdot \dot{\mathbf{p}} = 0$. Hence, if the vector $\dot{\mathbf{p}}$ is not the null vector it is orthogonal to \mathbf{p} and consequently of the form

$$\dot{\mathbf{p}} = at + cb.$$

Scalar multiplication by \mathbf{t} and \mathbf{b} , respectively, yields

$$a = \dot{\mathbf{p}} \cdot \mathbf{t} \quad \text{and} \quad c = \dot{\mathbf{p}} \cdot \mathbf{b},$$

since $\mathbf{b} \cdot \mathbf{t} = 0$. Differentiating $\mathbf{p} \cdot \mathbf{t} = 0$ we find $\dot{\mathbf{p}} \cdot \mathbf{t} + \mathbf{p} \cdot \dot{\mathbf{t}} = 0$, that is,

$$\dot{\mathbf{p}} \cdot \mathbf{t} = -\mathbf{p} \cdot \dot{\mathbf{t}} = -\kappa \mathbf{p} \cdot \mathbf{p} = -\kappa,$$

since $\dot{\mathbf{t}} = \kappa \mathbf{p}$. Differentiating $\mathbf{p} \cdot \mathbf{b} = 0$ and using (14.2) we obtain

$$\dot{\mathbf{p}} \cdot \mathbf{b} = -\mathbf{p} \cdot \dot{\mathbf{b}} = \tau.$$

We thus obtain the *formulae of Frenet*:

$$(15.1) \quad \begin{aligned} \dot{\mathbf{t}} &= \kappa \mathbf{p}, \\ \dot{\mathbf{p}} &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \dot{\mathbf{b}} &= -\tau \mathbf{p}, \end{aligned} \quad (\kappa > 0),$$

or, in matrix notation,

$$(15.1') \quad \begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{p}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{p} \\ \mathbf{b} \end{pmatrix}.$$

The coefficient matrix is skew-symmetric.

Formulae analogous to those of Frenet also occur in the theory of surfaces and will be considered in Section 45. Applications of the formulae of Frenet are numerous, cf. in the following.

Example. A curve is called a *general or cylindrical helix*, if its tangent makes a constant angle with a fixed line in space. Circular helices have this property, cf. the example in Section 10. From the results obtained in Section 13 we see that the curvature and torsion of a circular helix are constant and consequently the ratio of these quantities is also constant. The latter property is characteristic for all helices, as follows from

Theorem 15.1 (Lancret [1]). *A twisted curve of class $r \geq 3$ with non-vanishing curvature is a general helix if and only if, at all of its points, the ratio of its curvature and torsion is the same, $\tau(s) : \kappa(s) = \text{const}$.*

Proof. (a) Let a curve C be a general helix, that is, by definition, its unit tangent vector \mathbf{t} makes a constant angle α_0 ($0 < |\alpha_0| \leq \frac{1}{2}\pi$) with a fixed direction in space, determined, say, by a unit vector \mathbf{c} ,

$$(15.2) \quad \mathbf{c} \cdot \mathbf{t} = \cos \alpha_0.$$

Differentiating (15.2) we have

$$\mathbf{c} \cdot \dot{\mathbf{t}} = \kappa \mathbf{c} \cdot \mathbf{p} = 0.$$

Since $\kappa > 0$ we find

$$(15.3) \quad \mathbf{c} \cdot \mathbf{p} = 0,$$

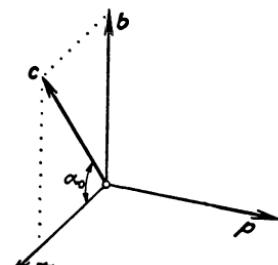


FIG. 18

that is, at any point of C the principal normal to C is orthogonal to the vector \mathbf{c} ; binding \mathbf{c} at a point of C it lies in the corresponding rectifying plane which contains also the unit tangent vector \mathbf{t} and the unit binormal vector \mathbf{b} . Formula (15.2) is therefore equivalent to

$$\mathbf{c} \cdot \mathbf{b} = \sin \alpha_0,$$

cf. Fig. 18. Differentiating (15.3) and using (15.1) we obtain

$$\mathbf{c} \cdot \dot{\mathbf{p}} = \mathbf{c} \cdot (-\kappa \mathbf{t} + \tau \mathbf{b}) = -\kappa \cos \alpha_0 + \tau \sin \alpha_0 = 0.$$

Hence $\tau(s)/\kappa(s) = \cot \alpha_0 = \text{const.}$

(b) Conversely, let us now assume that the ratio of curvature and torsion of the curve C is constant,

$$(15.4) \quad \frac{\tau(s)}{\kappa(s)} = c_0 = \text{const},$$

or, $c_0 \kappa - \tau = 0$. By (15.1), we obtain from this relation

$$c_0 \mathbf{t} + \mathbf{b} = (c_0 \kappa - \tau) \mathbf{p} = \mathbf{0}.$$

We integrate,

$$c_0 \mathbf{t} + \mathbf{b} = \mathbf{c}^*,$$

where \mathbf{c}^* ($\neq \mathbf{0}$) denotes a constant vector. Taking the scalar product of the unit tangent vector \mathbf{t} to C by the unit vector

$$\mathbf{c} = \frac{\mathbf{c}^*}{|\mathbf{c}^*|} = \frac{c_0 \mathbf{t} + \mathbf{b}}{\sqrt{(1+c_0^2)}}$$

we find

$$\mathbf{c} \cdot \mathbf{t} = \frac{c_0}{\sqrt{(1+c_0^2)}} = \text{const} \quad (< 1).$$

Hence the vectors \mathbf{c} and \mathbf{t} make a constant angle, and C is therefore a general helix.

Problem 15.1. Derive the formula of Frenet for $\dot{\mathbf{p}}$ from the other two formulae by proving that the matrix in (15.1') must be skew-symmetric.

16. Motion of the trihedron, vector of Darboux. When a point moves along a curve C the corresponding trihedron makes a motion which we will now investigate. This consideration turns out to be a kinematic interpretation of the formulae of Frenet, first made by G. Darboux [1, vol. 1] in 1887.

Since the vectors \mathbf{t} , \mathbf{p} , and \mathbf{b} have always the same mutual position and the constant length 1 we may imbed these vectors in a rigid body K which performs the same motion as the trihedron. Our problem may consequently be considered as a problem of kinematics of rigid bodies. We start from the following fundamental

Theorem 16.1. (G. Mozzi and A. Cauchy.) *Any motion of a rigid body in space is, at every instant, an (infinitesimal) screw motion.*

By definition, a *screw motion* is composed of a translation T along a straight line L and a rotation R about L such that the angle of rotation is proportional to the translation during corresponding times. (The path of any point P of the moving body K , not on the axis, is therefore a circular helix.)

Theorem 16.1 was first communicated by G. Mozzi [1] in 1763; the earliest correct proof is due to A. Cauchy [1] who rediscovered the theorem.

We exclude the translation of the moving trihedron from our investigation and consider only its rotation. Thus we assume that the trihedron undergoes a translation and is then bound at a fixed point, say at the origin of the Cartesian coordinate system in space.

A rotation of a rigid body can be simply and uniquely described by a 'rotation vector'.

Definition 16.1. *A vector \mathbf{d} is called rotation vector of a rotation if it has the following properties:*

- (1) \mathbf{d} has the direction of the axis of rotation.
- (2) The sense of \mathbf{d} is such that the rotation has the clockwise sense if one looks from the initial point of \mathbf{d} to its terminal point.
- (3) The absolute value $|\mathbf{d}|$ of \mathbf{d} equals the 'angular velocity' ω of the rotation, that is, the velocity of points at distance 1 from the axis of rotation.

Let P be any point of a rotating body K and let z be the distance of P from the axis of rotation. We denote by \mathbf{r} the position vector of P referred

to a coordinate system with origin on the axis of rotation. The derivative

$$\mathbf{v} = \mathbf{r}'$$

of \mathbf{r} with respect to the time t is called the *velocity vector* of P ; this vector has the direction of the motion of P , and its absolute value $|\mathbf{v}|$ is the velocity of P . If α denotes the angle between the position vector \mathbf{r} and the rotation vector \mathbf{d} , cf. Fig. 19, then

$$|\mathbf{v}| = \omega z = \omega |\mathbf{r}| \sin \alpha = |\mathbf{d} \times \mathbf{r}|,$$

and, in consequence of the definition of \mathbf{d} ,

$$(16.1) \quad \mathbf{v} = \mathbf{d} \times \mathbf{r}.$$

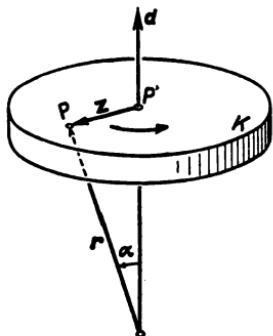


FIG. 19

connected with the trihedron is of the form

$$(16.2) \quad \mathbf{r} = ut + v\mathbf{p} + w\mathbf{b}.$$

Assuming that the point moving along the curve C under consideration has the constant velocity 1 we may equate the arc length s of C with the time t . The velocity vector of P thus is of the form

$$(16.3) \quad \mathbf{v} = \dot{\mathbf{r}} = u\mathbf{t} + v\dot{\mathbf{p}} + w\dot{\mathbf{b}}.$$

As follows from (15.1) the vectors \mathbf{t} , $\dot{\mathbf{p}}$, and $\dot{\mathbf{b}}$ lie in the same plane which, in Fig. 20, is denoted by E . This plane is uniquely determined since $\kappa > 0$. According to (16.1) the vector \mathbf{d} is orthogonal to E , that is, to \mathbf{t} and $\dot{\mathbf{p}}$, and therefore has the direction of the vector

$$\mathbf{t} \times \dot{\mathbf{p}} = \kappa \mathbf{p} \times (-\kappa \mathbf{t} + \tau \mathbf{b}) = \kappa^2 (\mathbf{t} \times \mathbf{p}) + \kappa \tau (\mathbf{p} \times \mathbf{b}) = \kappa(\kappa \mathbf{b} + \tau \mathbf{t});$$

hence it is of the form

$$(16.4) \quad \mathbf{d} = c(\tau \mathbf{t} + \kappa \mathbf{b}).$$

In order to determine the constant c we set $v = w = 0$ in (16.2) and (16.3). Then, in consequence of (16.1),

$$\mathbf{t} = \mathbf{d} \times \mathbf{t}$$

and, by inserting (16.4) in this expression, we find

$$\mathbf{t} = c(\tau \mathbf{t} + \kappa \mathbf{b}) \times \mathbf{t} = c\kappa \mathbf{p}.$$

Comparing this with (15.1) we have $c = 1$ and thus finally the following result:

Theorem 16.2. *The rotation vector of the trihedron of a curve C : $\mathbf{x}(s)$ of class $r \geq 3$ with non-vanishing curvature, when a point moves along C with constant velocity 1, is given by the expression*

$$(16.5) \quad \mathbf{d} = \tau \mathbf{t} + \kappa \mathbf{b}.$$

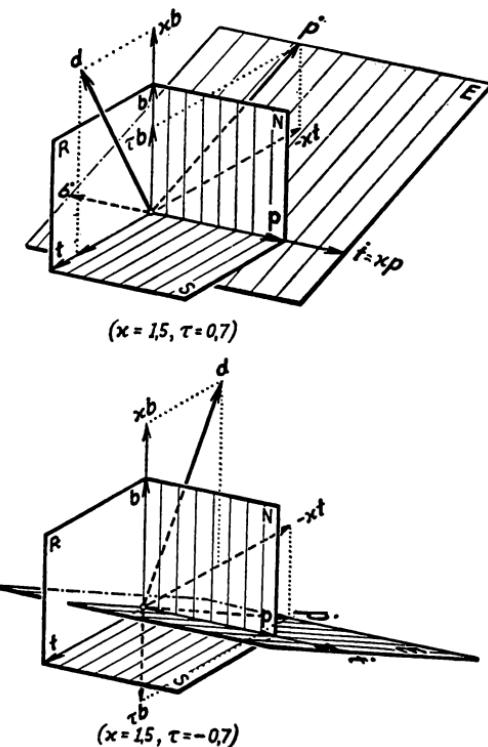


FIG. 20

Here \mathbf{d} is called the *vector of Darboux*. If a curve is plane then \mathbf{d} has the direction of the binormal; in this particular case the binormal is the axis of rotation, and $\omega = \kappa$.

In consequence of (16.1)–(16.3) we can now write the formulae of Frenet in the form

$$(16.6) \quad \dot{\mathbf{t}} = \mathbf{d} \times \mathbf{t}, \quad \dot{\mathbf{p}} = \mathbf{d} \times \mathbf{p}, \quad \dot{\mathbf{b}} = \mathbf{d} \times \mathbf{b}.$$

Problem 16.1. Determine the curves for which the corresponding vector of Darboux has constant direction in space.

17. Spherical images of a curve. We will now continue our investigation of the vectors of the moving trihedron of a curve $C: \mathbf{x}(s)$ of class $r \geq 3$ with non-vanishing curvature. We again assume that these vectors undergo a parallel displacement and become bound at the origin O of the Cartesian coordinate system in space. Then the terminal points of these vectors $\mathbf{t}(s)$, $\mathbf{p}(s)$, and $\mathbf{b}(s)$ lie on the unit sphere S with centre O and generate, in general, three curves on S which are called the *tangent indicatrix*, the *principal normal indicatrix*, and the *binormal indicatrix*, respectively, of the curve C , cf. Fig. 21.

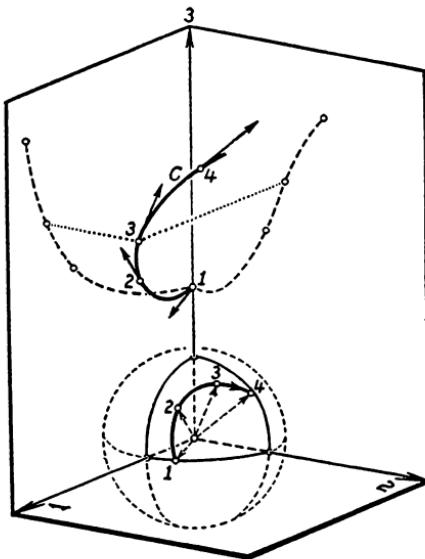


FIG. 21. Curve C and corresponding tangent indicatrix

The linear elements ds_T , ds_P , and ds_B of these indicatrices or *spherical images* can be easily obtained by means of (15.1). Since $\mathbf{t}(s)$, $\mathbf{p}(s)$, and $\mathbf{b}(s)$ are the vector functions representing these curves we find

$$(17.1) \quad \begin{aligned} ds_T^2 &= \dot{\mathbf{t}} \cdot \dot{\mathbf{t}} \, ds^2 = \kappa^2 \mathbf{p} \cdot \mathbf{p} \, ds^2 = \kappa^2 \, ds^2, \\ ds_P^2 &= \dot{\mathbf{p}} \cdot \dot{\mathbf{p}} \, ds^2 = (-\kappa \mathbf{t} + \tau \mathbf{b}) \cdot (-\kappa \mathbf{t} + \tau \mathbf{b}) \, ds^2 = (\kappa^2 + \tau^2) \, ds^2, \\ ds_B^2 &= \dot{\mathbf{b}} \cdot \dot{\mathbf{b}} \, ds^2 = \tau^2 \mathbf{p} \cdot \mathbf{p} \, ds^2 = \tau^2 \, ds^2. \end{aligned}$$

Curvature and torsion appear here as quotients of linear elements; choosing the orientation of the spherical image induced by the orientation of the curve C we have from (17.1)

$$(17.2) \quad \kappa = \frac{ds_T}{ds}, \quad |\tau| = \frac{ds_B}{ds}.$$

Moreover, from (17.1) we obtain the *Equation of Lancret*

$$(17.3) \quad ds_P^2 = ds_T^2 + ds_B^2.$$

Since curvature and torsion are also known as first and second curvatures the expression $\sqrt{(ds_T^2 + ds_B^2)}$ is sometimes called the *third or total curvature* of a curve.

Different curves may have the same spherical images. Simple examples illustrating this fact are circles in the same plane, with arbitrary radius and centre, and also circular helices $\mathbf{x}(t) = (r \cos t, r \sin t, ct)$ on coaxial cylinders for which the ratio $r:c$ is the same.

Problem 17.1. Find the curves for which (a) the tangent indicatrix, (b) the binormal indicatrix degenerates to a point. What does it mean when a spherical image is a closed curve?

Problem 17.2. Investigate the spherical images of the circular helix (8.4).

18. Shape of a curve in the neighbourhood of any of its points (canonical representation). In order to investigate the form of a curve C in a (sufficiently small) neighbourhood of any of its points we expand the vector function $\mathbf{x}(s)$ by which C may be represented according to Taylor's formula. Assuming that $\mathbf{x}(s)$ is of class $r \geq 3$ we have

$$(18.1) \quad \mathbf{x}(s) = \mathbf{x}(0) + \sum_{\nu=1}^3 \frac{s^\nu}{\nu!} \frac{d^\nu \mathbf{x}(0)}{ds^\nu} + \mathbf{o}(s^3)$$

where the vector $\mathbf{o}(s^3)$ has the meaning indicated in Section 11. Since the point of C corresponding to $s = 0$ may be chosen arbitrarily we may consider C in the neighbourhood of this point, without loss of generality. The vectors occurring in (18.1) can be represented in terms of the vectors of the trihedron. We have $\dot{\mathbf{x}} = \mathbf{t}$ and, according to the formulae of Frenet,

$$(18.2) \quad \begin{aligned} \ddot{\mathbf{x}} &= \dot{\mathbf{t}} = \kappa \mathbf{p}, \\ \ddot{\mathbf{x}} &= \ddot{\mathbf{t}} = \dot{\kappa} \mathbf{p} + \kappa \dot{\mathbf{p}} = \dot{\kappa} \mathbf{p} - \kappa^2 \mathbf{t} + \kappa \tau \mathbf{b}. \end{aligned}$$

We assume that the Cartesian coordinate system in space was chosen so that $s = 0$ corresponds to the origin and the vectors $\mathbf{t}(0)$, $\mathbf{p}(0)$, and $\mathbf{b}(0)$ lie in the positive rays of the x_1 -, x_2 -, and x_3 -axes, respectively. Then

$$(18.3) \quad \mathbf{t}(0) = (1, 0, 0), \quad \mathbf{p}(0) = (0, 1, 0), \quad \mathbf{b}(0) = (0, 0, 1).$$

By means of (18.2) and (18.3) we obtain from (18.1) the so-called canonical representation of the curve C ,

$$(18.4) \quad \begin{aligned} x_1(s) &= s - \frac{\kappa_0^2}{3!} s^3 + o(s^3), \\ x_2(s) &= \frac{\kappa_0}{2} s^2 + \frac{\dot{\kappa}_0}{3!} s^3 + o(s^3), \\ x_3(s) &= \frac{\kappa_0 \tau_0}{6} s^3 + o(s^3), \end{aligned}$$

where κ_0 and τ_0 , respectively, denote the value of the curvature and torsion at the point $s = 0$. When we discard all the terms in each series except the leading term we find

$$(18.5) \quad \mathbf{x}(s) \approx \left(s, \frac{\kappa_0}{2} s^2, \frac{\kappa_0 \tau_0}{6} s^3 \right), \quad (\kappa_0 > 0, \tau_0 \neq 0).$$

By eliminating s from (18.5) we obtain the following representations of the orthogonal projections of this approximative curve:

(in the osculating plane O): $x_2 = \frac{\kappa_0}{2} x_1^2$ (quadratic parabola),

(in the rectifying plane R): $x_3 = \frac{\kappa_0 \tau_0}{6} x_1^3$ (cubical parabola),

(in the normal plane N): $x_3 = \frac{\sqrt{2}}{3} \frac{\tau_0}{\sqrt{\kappa_0}} x_1^{\frac{3}{2}}$ (semi-cubical parabola),

cf. Fig. 22. The graph of the approximative curve in space is shown in Fig. 23. This curve has a right-handed screw motion if τ_0 is positive and a left-handed one if τ_0 is negative; that means, if the curve is traversed in the positive direction, it pierces the osculating plane at the point $s = 0$ from the side of the negative or positive binormal according as the torsion at $s = 0$ is positive or negative. While the curve has no singularity, its projection on the normal plane has a cusp.

Problem 18.1. Let s ($\leq \pi r$) be the length of an arc C of a circle of radius r with end points P_0 and P . The corresponding chord $P_0 P$ has the length

$$h = 2r \sin \frac{s}{2r} = s - \frac{s^3}{3!(2r)^2} + \dots$$

If C is sufficiently small the difference $|h - s|$ is approximately given by the term $s^3/3!(2r)^2$, that is, is of order s^3 . Prove that the difference of the lengths of a sufficiently small arc of any curve and the corresponding chord is always of order s^3 .

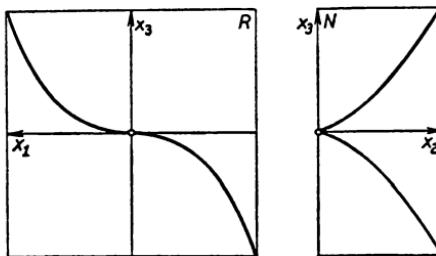


FIG. 22

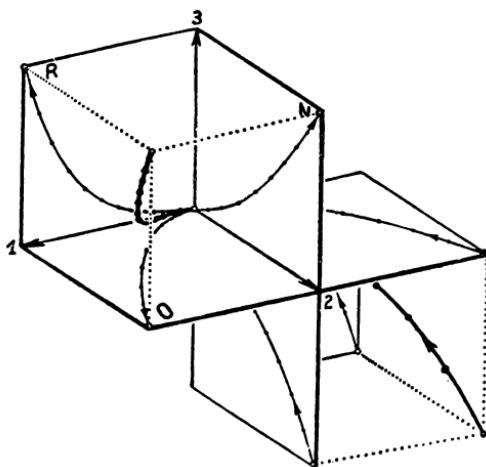


FIG. 23

19. Contact, osculating sphere. Let us consider two curves C and C^* which have a point, say P_0 , in common. In a neighbourhood of P_0 these curves may ‘more or less’ differ from each other. We will investigate this situation in detail and characterize it in a precise manner by introducing the concept of *contact*.

First of all there are two possible cases: Either

- (a) the tangents to C and C^* at P_0 are different from each other,
- or (b) they coincide.

Let C be allowably represented by $\mathbf{x}(t) = (f_1(t), f_2(t), f_3(t))$ and C^* by $\mathbf{x}(t^*) = (g_1(t^*), g_2(t^*), g_3(t^*))$ and let P_0 correspond to the values $t = t_0$ and $t^* = t_0^*$. Then in both cases (a) and (b)

$$\mathbf{x}(t_0) = \mathbf{x}(t_0^*).$$

Moreover, in case (b)—and only in this case—the two vectors

$$\frac{d\mathbf{x}}{dt} \Big|_{t=t_0} \quad \text{and} \quad \frac{d\mathbf{x}}{dt^*} \Big|_{t^*=t_0^*}$$

are parallel since these vectors are tangent to C and C^* at P_0 . We introduce now the arc lengths of C and C^* as parameters in such a manner that both curves at P_0 have the same orientation. Let

$$(19.1) \quad (C): \quad \mathbf{x}(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s)),$$

$$(C^*): \quad \mathbf{x}(s^*) = (\beta_1(s^*), \beta_2(s^*), \beta_3(s^*))$$

be the representations thus obtained; they are of class r if the above representations are of this class, cf. Section 9. We denote by s_0 and s_0^* the values of the parameters corresponding to P_0 . Using (19.1) in case (b) we now have simply

$$\alpha_i(s_0) = \beta_i(s_0^*), \quad \frac{d\alpha_i}{ds} \Big|_{s=s_0} = \frac{d\beta_i}{ds^*} \Big|_{s^*=s_0^*}, \quad (i = 1, 2, 3).$$

While we can visually distinguish only between the two cases (a) and (b) we are able to characterize the case (b) more in detail by analytical methods, that is, by taking into account the higher derivatives of $\mathbf{x}(s)$ and $\mathbf{x}(s^*)$. For this purpose we make the following

Definition 19.1. A curve C has contact of order m (exactly) with a curve C^* at a point P_0 if, when using the representations (19.1), at P_0 :

$$(19.2 \text{ a}) \quad \alpha_i(s) = \beta_i(s^*), \quad \frac{d^\mu \alpha_i}{ds^\mu} = \frac{d^\mu \beta_i}{ds^{*\mu}}, \quad (\mu = 1, 2, \dots, m; i = 1, 2, 3),$$

and, if also the derivatives of order $m+1$ at P_0 exist,

$$(19.2 \text{ b}) \quad \frac{d^{m+1} \alpha_i}{ds^{m+1}} \neq \frac{d^{m+1} \beta_i}{ds^{*\mu+1}}, \quad (i = 1, 2, 3).$$

Of course this definition has a meaning only for those values of m which are at most equal to the class of the functions $\mathbf{x}(s)$ and $\mathbf{x}(s^*)$.

According to this definition, in the above case (a) C has contact of order 0 with C^* . Contact of first order is also known as an *ordinary contact*,

contact of second order is known as *stationary contact* or *osculation*, and contact of third order as *stationary osculation* or *superosculuation*.

Definition 19.1 may be geometrically interpreted as follows. We consider the development of $\mathbf{x}(s)$ in powers of $s - s_0$, according to Taylor's formula. The curve obtained by taking the sum of the first terms of the above development up to the term containing the power $(s - s_0)^k$ (inclusively) is called ' k th approximating curve of C at P_0 '. In this terminology, the tangent at P_0 is the first approximating curve, etc. Then contact of order m is equivalent to the coincidence of the first, second,..., m th approximating curves of the curves under consideration at the point of contact.

In order to continue our present investigation it is necessary to consider also surfaces, but actually only very special simple types. Postponing a detailed investigation to Section 24 it is, at this stage, sufficient to say that surfaces are point sets in Euclidean space which, at least in a sufficiently small neighbourhood of any of their points, can be represented in the form

$$G(x_1, x_2, x_3) = 0,$$

where G is a function of class $m \geq 1$ whose partial derivatives of the first order do not all vanish at any point.

Let us now define the contact of a curve with a surface.

Definition 19.2. *A curve C has contact of order m (exactly) with a surface S at a point P_0 if there exists at least one curve C^* on S which has a contact of order m with C at P_0 and there does not exist a curve on S which has a contact of order greater than m with C at P_0 .*

We will now consider planes and spheres which have a contact with an arbitrary given curve.

First it is clear that a plane which has contact of first order with a curve C at a point P_0 must pass through the tangent to C at P_0 . At any point of C there exists a whole pencil of planes of this type, in which the corresponding osculating plane of C is also contained.

Theorem 19.1. *At any of its points, a curve has contact of second order (at least) with its corresponding osculating plane.*

Proof. Let $C: \mathbf{x}(s)$ be any curve. We develop $\mathbf{x}(s)$ in a series of powers of $s - s_0$, by Taylor's formula, where s_0 may correspond to any point P_0 of C . When we take the terms of this expansion only up to that which contains $\dot{\mathbf{x}}(s_0)$ (inclusively) we obtain a representation of a curve C^* . This curve has a contact of second order with C at P_0 . Since the osculating plane of C at P_0 is spanned by the vectors $\dot{\mathbf{x}}(s_0)$ and $\ddot{\mathbf{x}}(s_0)$, the curve C^* lies in this

plane. By Definition 19.2, C therefore has contact of at least second order with that osculating plane at P_0 . If $\tau \neq 0$ at P_0 , the contact is exactly of second order as can be seen, for example, from (18.1)–(18.4).

As a preparation for the investigation of spheres which have contact with a curve we state the following

Lemma 19.2. *Let C be a curve which can be represented by an allowable representation $\mathbf{x}(s) = (\alpha_i(s))$ of class $r \geq m$, with arc length s as parameter. Furthermore, let C have a point $P_0: s = s_0$ in common with a surface S which, in a neighbourhood of P_0 , has a representation $G(x_1, x_2, x_3) = 0$ of class $r \geq m$. Then C has contact of order m (exactly) with S at P_0 if and only if the function*

$$p(s) = G(\alpha_1(s), \alpha_2(s), \alpha_3(s))$$

and its derivatives with respect to s , up to the m -th inclusively, vanish at P_0 , while the $(m+1)$ -th derivative—if it exists—does not vanish at this point,

$$(19.3) \quad p(s_0) = 0, \quad \left. \frac{d^\mu p}{ds^\mu} \right|_{s=s_0} = 0 \quad (\mu = 1, 2, \dots, m), \quad \left. \frac{d^{m+1}p}{ds^{m+1}} \right|_{s=s_0} \neq 0.$$

Proof. If C has contact of order m with S at P_0 then, according to Definition 19.2, there exists a curve C^* on S which has a contact of order m (exactly) with C at P_0 . We denote by s^* the arc length of C^* . Let

$$\mathbf{x}(s^*) = (\beta_i(s^*))$$

be a representation of C^* so that, at P_0 , C and C^* have the same orientation. The point P_0 may correspond to the value $s^* = s_0^*$. Since C^* lies on S we first have

$$(19.4 \text{ a}) \quad G(\beta_1(s^*), \beta_2(s^*), \beta_3(s^*)) = 0$$

and, by successive differentiation with respect to s^* ,

$$(19.4 \text{ b}) \quad \sum_{i=1}^3 \frac{\partial G}{\partial x_i} \frac{d\beta_i}{ds^*} = 0, \quad \sum_{i=1}^3 \sum_{k=1}^3 \frac{\partial^2 G}{\partial x_i \partial x_k} \frac{d\beta_i}{ds^*} \frac{d\beta_k}{ds^*} + \sum_{i=1}^3 \frac{\partial G}{\partial x_i} \frac{d^2 \beta_i}{ds^{*2}} = 0,$$

etc. All of these relations are identities in s^* . Since C has contact of order m with C^* at P_0 , according to (19.2 a), we obtain

$$(19.5 \text{ a}) \quad \alpha_i(s_0) = \beta_i(s_0^*), \quad \left. \frac{d^\mu \alpha_i}{ds^\mu} \right|_{s=s_0} = \left. \frac{d^\mu \beta_i}{ds^{*\mu}} \right|_{s^*=s_0^*}, \quad \mu = 1, 2, \dots, m, \\ (i = 1, 2, 3),$$

and, if the $(m+1)$ -th derivative also exists, according to (19.2 b)

$$(19.5 \text{ b}) \quad \left. \frac{d^{m+1} \alpha_i}{ds^{m+1}} \right|_{s=s_0} \neq \left. \frac{d^{m+1} \beta_i}{ds^{*m+1}} \right|_{s^*=s_0^*}, \quad (i = 1, 2, 3).$$

Setting $s^* = s_0^*$ in (19.4), by means of (19.5), we can eliminate from (19.4) the derivatives of the functions $\beta_i(s_0^*)$. This procedure yields (19.3). We will now prove the converse. By hypothesis, $\partial G/\partial x_i \neq 0$ at any point for at least one value of $i = 1, 2, 3$. If at P_0 , say, $\partial G/\partial x_3 \neq 0$ then, at least in a certain neighbourhood of P_0 we can represent the surface S in the form $x_3 = h(x_1, x_2)$. In consequence of (19.3) this equation and the equations obtained by differentiating it m times are satisfied at P_0 by the functions $\alpha_i(s)$ and its derivatives. Setting $s^* = s$, $\beta_1(s^*) = \alpha_1(s)$ and $\beta_2(s^*) = \alpha_2(s)$, then $\beta_3 = h(\beta_1, \beta_2)$ is determined so that (19.5) is also satisfied for $i = 3$. This completes the proof.

This lemma permits us a more detailed geometric characterization of the contact of a curve with a surface.

Theorem 19.3. *Let C be a curve of class $r \geq m+1$ which has contact of order m with a surface S of class r at a point P_0 . If m is even then C pierces S at P_0 . If m is odd then, in a sufficiently small neighbourhood U of P_0 , C lies on one side of S .*

Proof. Let C be represented by $\mathbf{x}(s) = (\alpha_i(s))$ so that P_0 corresponds to $s = 0$ and let $G(x_1, x_2, x_3) = 0$ be a representation of S valid in a neighbourhood of P_0 . Then, by means of Taylor's formula, we have, in consequence of (19.3),

$$p(s) = G(\alpha_1(s), \alpha_2(s), \alpha_3(s)) = \frac{s^{m+1}}{(m+1)!} \frac{d^{m+1}p(\theta s)}{ds^{m+1}}, \quad (0 < \theta < 1).$$

Since $\frac{d^{m+1}p(0)}{ds^{m+1}} \neq 0$, cf. (19.3), for sufficiently small values of $|s|$ also $\frac{d^{m+1}p(\theta s)}{ds^{m+1}} \neq 0$ and, in consequence of the continuity of the $(m+1)$ th derivative the sign is the same in both cases. Let m be even. Then $s^{m+1} > 0$ if $s > 0$ and $s^{m+1} < 0$ if $s < 0$. If m is odd then $s^{m+1} > 0$ for all values of s . This completes the proof.

We will now consider spheres which have contact with an arbitrary given curve C : $\mathbf{x}(s)$ where s denotes the arc length of C . Let P be a common point of C and a sphere S of radius R with centre M . Denoting by \mathbf{a} the position vector of M we can represent S in the form

$$(19.6) \quad G(x_1, x_2, x_3) = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) - R^2 = 0.$$

In order that C should have contact of first order with S at P we must have at P

$$(19.7) \quad \frac{dp}{ds} = \frac{d}{ds}[G(x_1(s), x_2(s), x_3(s))] = 2(\mathbf{x} - \mathbf{a}) \cdot \dot{\mathbf{x}} = 2(\mathbf{x} - \mathbf{a}) \cdot \mathbf{t} = 0,$$

that is, the centre M of any sphere which has contact of first order with the curve C at P lies in the normal plane N to C at P , cf. (13.2(a)). The radius corresponding to a certain choice of M can then be obtained from (19.6). The vector \mathbf{a} is now of the form

$$(19.8) \quad \mathbf{a} = \mathbf{x} + \alpha \mathbf{p} + \beta \mathbf{b},$$

where α and β are real numbers. We now assume that the curvature κ of C at P is not zero. In order that C have a contact of second order with S at P , in addition to (19.7) we must have at P

$$(19.9) \quad \frac{d^2 \mathbf{p}}{ds^2} = 2(\mathbf{t} \cdot \mathbf{t} + (\mathbf{x} - \mathbf{a}) \cdot \dot{\mathbf{t}}) = 2(1 + (\mathbf{x} - \mathbf{a}) \cdot \kappa \mathbf{p}) = 0.$$

By inserting (19.8) into (19.9) we find the condition $1 - \alpha\kappa = 0$, i.e. $\alpha = \rho$. This means that the two-parameter family of spheres with centres (19.8) which have contact (at least) of first order with C at P contains a one-parameter family of spheres having contact of second order (at least) with C at P ; the position vectors of the centres of these spheres are of the form

$$(19.10) \quad \mathbf{a} = \mathbf{x} + \rho \mathbf{p} + \beta \mathbf{b}, \quad (\kappa > 0).$$

Since β may have any real value, (19.10) is a representation of a straight line parallel to the binormal and passing through the centre of curvature corresponding to P , cf. (12.6). This line is called the *polar axis* of the curve C at P .

In order that C have contact of third order with a sphere S at P , in addition to (19.7) and (19.9) we must have at P

$$(19.11)$$

$$\frac{d^3 \mathbf{p}}{ds^3} = 2(\kappa \mathbf{t} \cdot \mathbf{p} + (\mathbf{x} - \mathbf{a}) \cdot (\kappa \mathbf{p} + \kappa \dot{\mathbf{p}})) = 2(\mathbf{x} - \mathbf{a}) \cdot (\kappa \mathbf{p} - \kappa^2 \mathbf{t} + \kappa \tau \mathbf{b}) = 0.$$

Inserting (19.10) into (19.11) we obtain

$$\rho \dot{\kappa} + \beta \kappa \tau = 0$$

and therefore, if $\tau \neq 0$ at P ,

$$\beta = -\frac{\dot{\kappa} \rho^2}{\tau} = \frac{\dot{\rho}}{\tau}.$$

This means that the one-parameter family of spheres with centres (19.10), which have contact of second order with C at P , in general contains just one sphere S_0 which has contact of third order (at least) with C at P . S_0 is called the *osculating sphere* of the curve C at the point P . Inserting the value of β just obtained we find the position vector \mathbf{a} of the centre of S_0 :

$$(19.12) \quad \mathbf{a} = \mathbf{x} + \rho \mathbf{p} + \frac{\dot{\rho}}{\tau} \mathbf{b}, \quad (\kappa > 0, \tau \neq 0).$$

If $\kappa = 0$ at P the centre of S_0 lies in the osculating plane of C at P and coincides with the centre of curvature. From (19.12) we obtain the radius R_S of S_0 :

$$(19.13) \quad R_S = |\mathbf{x} - \mathbf{a}| = \sqrt{\{\rho^2 + (\dot{\rho}/\tau)^2\}}.$$

If the curvature of a curve is constant the radius of the osculating spheres is also constant. The circular helix is of this type, cf. Section 12.

We may sum up our results as follows:

Theorem 19.4. *The centre of any sphere which has contact of first order (at least) with a curve C at a point P lies in the normal plane to C at P . The centre of any sphere which has a contact of second order (at least) with C at a point P , where $\kappa > 0$, lies on the polar axis of C corresponding to P ; all these spheres intersect the osculating plane of C at P along the circle of curvature of C at P which has a contact of second order (at least) with C at P . The osculating sphere, with centre and radius determined by (19.12) and (19.13), respectively, has contact of third order (at least) with C .*

Osculating spheres were first considered by Fuss [1] in 1806. Formula (19.12) is due to B. de Saint Venant [1].

The surface generated by the polar axes of a curve will be investigated in Section 87.

Problem 19.1. Investigate the contact of the surface

$$S: x_1^2 + x_2^2 + x_3^2 - r^2 = 0$$

with the curve C : $\mathbf{x}(s) = (s, r, 0)$.

Problem 19.2. Prove that, in general, the osculating plane is the only plane which has a contact of second order with a curve.

Problem 19.3. Determine the conditions which a curve must satisfy in order that the centres of the osculating spheres be the same for all of its points.

20. Natural equations of a curve. So far a curve C was always represented in the form (6.1): $\mathbf{x} = \mathbf{x}(t)$ where the arbitrary allowable parameter t was sometimes specified by the arc length s of C . Clearly, the analytic form of such a representation depends on the choice of the coordinate system in space. Therefore the question arises whether there is a possibility of characterizing a curve in a manner independent of the coordinates, except for the position of the curve in space, that is, to within direct congruent transformations. The earliest investigation of this problem was already made more than 200 years ago, by L. Euler [1].

When trying to find such a representation we have to look for quantities which are independent of the choice of the coordinates and parameter but depend only on the nature of the curve, that is, on its geometric shape. The arc length s , the curvature κ , and the torsion τ are quantities of this kind. Two independent functional relations between s , κ , and τ are called *natural* or *intrinsic equations* of the corresponding curve. We shall use the two natural equations

$$\kappa = \kappa(s), \quad \tau = \tau(s)$$

in our further investigations.

We will prove that, if $\kappa(s)$ and $\tau(s)$ are continuous in an interval, these functions determine an arc of a curve uniquely, except for its position in space. As a consequence of this theorem, which will be proved shortly, we may state the fact that any invariant with respect to direct congruent transformations which can be associated with a curve is completely determined when the corresponding functions $\kappa(s)$ and $\tau(s)$ are given.

The solution of our problem turns out to be very simple if $\kappa(s)$ and $\tau(s)$ are analytic functions. Since, in consequence of (15.1), the coefficients of the Taylor series development of the function $x(s)$ representing the curve involve only the functions $\kappa(s)$ and $\tau(s)$ and their derivatives, $x(s)$ is uniquely determined when analytic functions $\kappa(s)$ and $\tau(s)$ are given.

Returning to the general case we have the important

Theorem 20.1. *Let $\kappa(s) (> 0)$ and $\tau(s)$ be continuous functions of a real variable s , defined in an interval $I: 0 \leq s \leq a$. Then there exists one and only one arc $x(s)$ of a curve, determined up to a direct congruent transformation, with arc length s whose curvature and torsion, respectively, are given by the above functions $\kappa(s)$ and $\tau(s)$.*

Proof. (a) *Existence.* $\kappa(s)$ and $\tau(s)$ being given, the formulae (15.1) of Frenet are three systems of ordinary linear differential equations in the components of the unit vectors t , p , and b of the moving trihedron,

$$(20.1) \quad \begin{cases} t_i(s) = \kappa(s)p_i(s) \\ \dot{p}_i(s) = -\kappa(s)t_i(s) + \tau(s)b_i(s) \\ \dot{b}_i(s) = -\tau(s)p_i(s) \end{cases}, \quad \begin{matrix} (i = 1, 2, 3, \\ \kappa > 0). \end{matrix}$$

For the sake of simplicity we set

$$(20.1^*) \quad t = v_1, \quad p = v_2, \quad b = v_3.$$

Therefore we have, written in vector form, differential equations of the type

$$\dot{v}_j = \sum_{k=1}^3 c_{jk} v_k, \quad (j = 1, 2, 3).$$

We write these equations again in scalar form, omitting the index i by which the different components of the vectors v_j are denoted, that is, we write (20.1) in the form

$$(20.1') \quad \dot{v}_j = \sum_{k=1}^3 c_{jk} v_k, \quad (j = 1, 2, 3).$$

In (20.1') the elements of the coefficient matrix

$$(c_{jk}) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

are continuous functions in the closed interval $I: 0 \leq s \leq a$ and consequently bounded in I ,

$$(20.2) \quad |c_{jk}| \leq \frac{k}{3}, \quad k > 0.$$

We have to prove that there exists a solution of (20.1) or of (20.1') satisfying given initial conditions, say

$$v_j(0) = v_j^0, \quad (j = 1, 2, 3),$$

where we have to assume that $|v_j^0| \leq 1$, since these quantities are components of unit vectors. The proof of existence will be carried out by means of iteration. The basic idea of this method is as follows: if there exist functions $v_j(s)$ continuous in I and satisfying (20.1') then, for all values of s in I ,

$$(20.3) \quad v_j(s) = v_j^0 + \int_0^s \sum_{k=1}^3 c_{jk}(\sigma) v_k(\sigma) d\sigma$$

must hold. Replacing the functions v_k in the integrand of (20.3) by the initial values v_k^0 we obtain three first approximate functions

$$(20.4) \quad v_j^{(1)} = v_j^0 + \int_0^s \sum_{k=1}^3 c_{jk} v_k^0 d\sigma.$$

If we insert these first approximate functions $v_j^{(1)}$ into the integrand of (20.3) we obtain three second approximate functions $v_j^{(2)}$, etc. By inserting the $(n-1)$ th approximate functions $v_k^{(n-1)}$ we obtain three n th approximate functions

$$(20.5) \quad v_j^{(n)} = v_j^0 + \int_0^s \sum_{k=1}^3 c_{jk} v_k^{(n-1)} d\sigma, \quad (j = 1, 2, 3).$$

We have to prove that when n increases arbitrarily $v_j^{(n)}$ tends to a limit

function satisfying the initial condition. Since $|v_j^0| \leq 1$ we obtain from (20.2) and (20.4)

$$|v_j^{(1)} - v_j^0| \leq ks;$$

similarly,

$$|v_j^{(2)} - v_j^{(1)}| \leq k^2 \frac{s^2}{2},$$

and in general

$$(20.6) \quad |v_j^{(n)} - v_j^{(n-1)}| \leq k^n \frac{s^n}{n!}.$$

Now

$$(20.7) \quad v_j^{(n)} = v_j^0 + (v_j^{(1)} - v_j^0) + (v_j^{(2)} - v_j^{(1)}) + \dots + (v_j^{(n)} - v_j^{(n-1)}).$$

In this representation the absolute value of each term is at most equal to the corresponding term of the sum

$$(20.8) \quad v_j^0 + ks + k^2 \frac{s^2}{2!} + \dots + k^n \frac{s^n}{n!}.$$

If n tends to infinity the expression (20.8) tends to

$$v_j^0 - 1 + e^{ks}.$$

Hence, the series

$$(20.8') \quad v_j^0 + (v_j^{(1)} - v_j^0) + \dots + (v_j^{(n)} - v_j^{(n-1)}) + \dots$$

converges uniformly in I and

$$\lim_{n \rightarrow \infty} v_j^{(n)} = v_j.$$

The approximate functions $v_j^{(n)}$ are differentiable with respect to s in I , cf. (20.5), and therefore also continuous. This means that, according to (20.8'), v_j is represented by a uniformly convergent series of continuous functions and is therefore continuous. (20.5) is equivalent to

$$(20.9) \quad v_j + (v_j^{(n)} - v_j) = v_j^0 + \int_0^s \sum_{k=1}^3 c_{jk} v_k d\sigma + \int_0^s \sum_{k=1}^3 c_{jk} (v_k^{(n-1)} - v_k) d\sigma.$$

If n increases arbitrarily, then, for all values of s in I , the functions $|v_j^{(n)} - v_j|$ and $|v_k^{(n-1)} - v_k|$ tend uniformly to zero. Therefore we obtain from (20.9)

$$v_j = v_j^0 + \int_0^s \sum_{k=1}^3 c_{jk} v_k d\sigma$$

and from this

$$\dot{v}_j = \sum_{k=1}^3 c_{jk} v_k;$$

furthermore,

$$v_j(0) = v_j^0,$$

that is, the three continuous limit functions v_j are differentiable and satisfy the formulae of Frenet and also the given initial conditions.

(b) (*Uniqueness of the solutions.*) We will now prove that there exists only one arc C of a curve, except for its position in space, corresponding to the above natural equations. We use the notation (20.1*). We consider the above solution $\mathbf{v}_j, \mathbf{v}_j(0) = \mathbf{v}_j^0 (j = 1, 2, 3)$, of the given differential equations and a second solution $\mathbf{v}_j^* (j = 1, 2, 3)$, which satisfies the initial conditions $\mathbf{v}_j^*(0) = \mathbf{v}_j^{*0}$. There exists one and only one displacement which carries these initial conditions into the others. We now let the curve defined by \mathbf{v}_j^* undergo this displacement. Then $\mathbf{v}_j^*(0) = \mathbf{v}_j(0)$. We have to prove that

$$\mathbf{v}_j^*(s) = \mathbf{v}_j(s)$$

for all values of s in I . We will first give the proof for all values of s in the sub-interval \tilde{I} of I , defined by

$$s \leq \frac{3}{2k}.$$

Let v_1, v_2, v_3 denote corresponding components of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let s_j be that value of s in \tilde{I} for which $|v_j^* - v_j|$ assumes its maximum value, say M_j . In consequence of (20.1) and (20.2) we find

$$M_1 = |v_1(s_1) - v_1^*(s_1)| = \left| \int_0^{s_1} (\dot{v}_1 - \dot{v}_1^*) ds \right| = \left| \int_0^{s_1} \kappa(v_2 - v_2^*) ds \right| \leq k \frac{M_2}{2k},$$

hence

$$M_1 \leq \frac{1}{2} M_2.$$

Using the second and third formula of Frenet we similarly find

$$M_2 \leq \frac{1}{2}(M_1 + M_3), \quad M_3 \leq \frac{1}{2}M_2.$$

Adding these three inequalities we have

$$M_1 + M_3 \leq \frac{1}{2}(M_1 + M_3).$$

Since, by definition, $M_j \geq 0$, we thus have $M_j = 0 (j = 1, 2, 3)$. This means that for all values of s in \tilde{I} the relations $v_j^*(s) = v_j(s)$ hold. Repeating this argument we obtain $v_j^*(s) = v_j(s)$ for all values of s in I . This completes the proof of the uniqueness.

(c) We will now prove that, if the vectors $\mathbf{v}_1^0, \mathbf{v}_2^0, \mathbf{v}_3^0$ form an orthonormal system, for any value of $s (0 \leq s \leq a)$ the vectors

$$\mathbf{v}_j(s) = (v_{j1}(s), v_{j2}(s), v_{j3}(s)), \quad (j = 1, 2, 3),$$

also form such a system. That is, we have to prove that the matrix

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} \equiv \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

is orthogonal for any value of $s (0 \leq s \leq a)$, if it is orthogonal for $s = 0$. It

suffices to prove that the column vectors of this matrix satisfy the relations

$$(20.10) \quad \sum_{i=1}^3 v_{i\alpha}(s)v_{i\beta}(s) = \delta_{\alpha\beta}, \quad (\alpha, \beta = 1, 2, 3)$$

since the row vectors then satisfy the analogous relations as is well known. If the derivative with respect to s of the sum in (20.10) vanishes then this sum is constant and, for reasons of continuity, equal to its value $\delta_{\alpha\beta}$ at $s = 0$. In consequence of the formulae of Frenet,

$$\dot{v}_{ji} = \sum_{k=1}^3 c_{jk} v_{ki},$$

we have

$$\frac{d}{ds} \left[\sum_{i=1}^3 v_{i\alpha} v_{i\beta} \right] = \sum_{i=1}^3 (\dot{v}_{i\alpha} v_{i\beta} + v_{i\alpha} \dot{v}_{i\beta}) = \sum_{i=1}^3 \sum_{k=1}^3 (c_{ik} v_{k\alpha} v_{i\beta} + v_{i\alpha} c_{ik} v_{k\beta})$$

or, when interchanging i and k in the second term of each summand,

$$\frac{d}{ds} \left[\sum_{i=1}^3 v_{i\alpha} v_{i\beta} \right] = \sum_{i=1}^3 \sum_{k=1}^3 (c_{ik} + c_{ki}) v_{k\alpha} v_{i\beta}.$$

Since $c_{ik} = -c_{ki}$, this expression vanishes. The determinant of an orthogonal matrix has the constant values ± 1 , depending on the orientation of its row (or column) vectors. For reasons of continuity the orientation of the vectors $\mathbf{v}_1(s)$, $\mathbf{v}_2(s)$, $\mathbf{v}_3(s)$ is the same as that of the given vectors \mathbf{v}_1^0 , \mathbf{v}_2^0 , \mathbf{v}_3^0 . Hence we conclude that, if the given vectors \mathbf{v}_1^0 , \mathbf{v}_2^0 , \mathbf{v}_3^0 form a right-handed orthonormal system then the vectors $\mathbf{v}_1(s)$, $\mathbf{v}_2(s)$, $\mathbf{v}_3(s)$ ($0 \leq s \leq a$), are uniquely determined and also form such a system. The representation of the corresponding arc can be obtained by integration,

$$\mathbf{x}(s) = \mathbf{x}_0 + \int_0^s \mathbf{t}(\sigma) d\sigma, \quad (0 \leq s \leq a),$$

where \mathbf{x}_0 denotes a constant vector. The initial conditions can differ in \mathbf{x}_0 and in the direction of the trihedron \mathbf{t}^0 , \mathbf{p}^0 , \mathbf{b}^0 ; there is exactly one direct congruent transformation which transforms a certain initial condition into another given one. The given functions $\kappa(s)$ and $\tau(s)$ remain unaffected by this transformation. Hence, in consequence of the uniqueness of the solution, any two solutions corresponding to different initial conditions become identical under a suitable direct congruent transformation.

21. Examples of curves and their natural equations. In addition to the preceding section we will now consider some examples of curves and their natural equations.

We will at first point out the following fact concerning plane curves. If $\alpha(s)$ denotes the directed angle from the positive ray of the x_1 -axis to the unit tangent vector $t(s)$ to a plane curve C then $\alpha(s)$ is continuous and $t = (\cos \alpha, \sin \alpha)$, $dt/d\alpha = (-\sin \alpha, \cos \alpha)$, and $(dt/d\alpha) \cdot t = 0$. The unit vector $dt/d\alpha$ is therefore orthogonal to t , that is, $dt/d\alpha = \pm p$. Now, from (15.1), $\dot{t} = (dt/d\alpha)(d\alpha/ds) = \kappa p$. Therefore, since $\kappa \geq 0$, by comparison, $\kappa = |d\alpha/ds|$. On an arc of C , on which $\kappa \neq 0$, when choosing a suitable orientation, we finally obtain

$$(21.1) \quad \kappa = \frac{d\alpha}{ds}.$$

Example 1. The natural equations

$$\rho = r = \text{const}, \quad \tau = 0$$

yield a circle of radius r . Indeed, using (21.1), we have

$$dx_1 = ds \cos \alpha = r \cos \alpha d\alpha, \quad dx_2 = ds \sin \alpha = r \sin \alpha d\alpha$$

and therefore

$$x_1 = r \int_0^\alpha \cos \phi d\phi = r \sin \alpha, \quad x_2 = r \int_0^\alpha \sin \phi d\phi = r(1 - \cos \alpha),$$

or

$$x_1^2 + (x_2 - r)^2 = r^2.$$

Example 2. From the natural equations

$$s\rho = c^2, \quad \tau = 0 \quad (c \neq 0)$$

we obtain a plane curve which is known as *spiral of Cornu*, cf. Fig. 24. By means of (21.1) we find

$$\alpha = \int \frac{d\sigma}{\rho} = \frac{1}{c^2} \int_0^s \sigma d\sigma = \frac{s^2}{2c^2},$$

and, since $dx_1 = ds \cos \alpha$, $dx_2 = ds \sin \alpha$, therefore

$$x_1 = \int_0^s \cos \alpha(\sigma) d\sigma = \int_0^s \cos \frac{\sigma^2}{2c^2} d\sigma = \frac{c}{\sqrt{2}} \int_0^{\Phi} \frac{\cos \alpha}{\sqrt{\alpha}} d\alpha,$$

$$x_2 = \int_0^s \sin \alpha(\sigma) d\sigma = \int_0^s \sin \frac{\sigma^2}{2c^2} d\sigma = \frac{c}{\sqrt{2}} \int_0^{\Phi} \frac{\sin \alpha}{\sqrt{\alpha}} d\alpha.$$

These integrals are called the *Fresnel integrals*; they were first considered by A. J. Fresnel (1788–1827) in connexion with refraction problems. They cannot be evaluated in terms of elementary functions; for tables see

E. Jahnke and F. Emde, *Tables of Functions*, New York, 1945. Since

$$\int_0^\infty \frac{\cos \alpha}{\sqrt{\alpha}} d\alpha = \int_0^\infty \frac{\sin \alpha}{\sqrt{\alpha}} d\alpha = \sqrt{\frac{\pi}{2}},$$

the coordinates of the 'limit points' of the spiral are

$$x_1 = x_2 = \pm \frac{c}{2} \sqrt{\pi},$$

cf. Fig. 24. For further details cf. Kreyszig [2], [3].

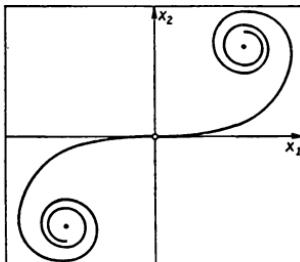


FIG. 24

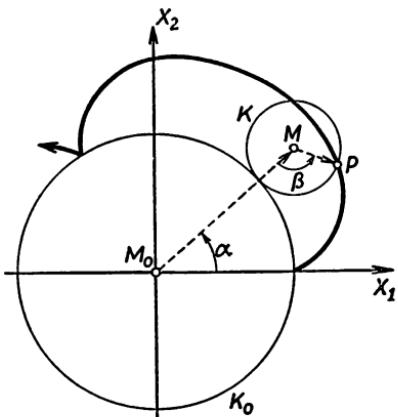


FIG. 25

Example 3. The natural equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ of spherical curves with $\tau \neq 0$ must be so that the relation

$$\rho\tau + \frac{d(\rho)}{ds} \left(\frac{\dot{\rho}}{\tau} \right) = 0$$

is satisfied, cf. Problem 19.3.

Example 4. An important class of plane curves are the cycloids. We first consider the *ordinary epicycloid*. Let K_0 be a fixed circle of radius r_0 and let K be a circle of radius r , which lies in the same plane and has a contact of first order with K_0 at a point; K is assumed to lie exterior to K_0 and vice versa. Now, when K rolls on K_0 without sliding a fixed point P of K generates a plane curve which is known as an ordinary epicycloid, cf. Fig. 25. We denote by M_0 and M the respective centres of K_0 and K . If the segment M_0M , starting from the positive ray of the x_1 -axis, has rotated through an angle α (cf. Fig. 25) then the angle between M_0M and MP is $\beta = \alpha r_0/r$; MP

makes therefore the angle

$$\alpha + \beta = \left(\frac{r_0}{r} + 1 \right) \alpha$$

with the x_1 -axis. We can thus represent the ordinary epicycloid in the form

$$x_1 = (r_0 + r) \cos \alpha - r \cos \left(\frac{r_0 + r}{r} \alpha \right),$$

$$x_2 = (r_0 + r) \sin \alpha - r \sin \left(\frac{r_0 + r}{r} \alpha \right).$$

This curve has the arc length

$$s = a \cos \frac{r_0}{2r} \alpha, \quad a = 4 \frac{r}{r_0} (r_0 + r),$$

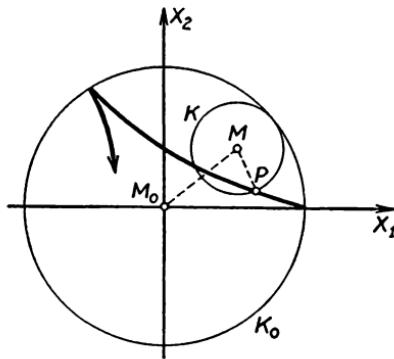


FIG. 26

where the point $s = 0$ is chosen so that it corresponds to $\alpha = (r/r_0)\pi$. The radius of curvature is

$$\rho = b \sin \frac{r_0}{2r} \alpha, \quad b = \frac{4r}{r_0 + 2r} (r_0 + r).$$

Hence the natural equations of the epicycloid are

$$(21.2) \quad \frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1, \quad \tau = 0, \quad (a > b).$$

The *ordinary hypocycloid*, cf. Fig. 26, is generated by the above point P when K is rolling on K_0 in the same manner as before but lies inside K_0 . This curve can therefore be represented in the form

$$x_1 = (r_0 - r) \cos \alpha + r \cos \left(\frac{r_0 - r}{r} \alpha \right),$$

$$x_2 = (r_0 - r) \sin \alpha - r \sin \left(\frac{r_0 - r}{r} \alpha \right).$$

If r_0 tends to infinity, then a and b tend to $4r$, and the natural equation (21.2) is then of the form

$$s^2 + \rho^2 = 16r^2.$$

The corresponding curve is called an *ordinary cycloid*, cf. Fig. 27; it is generated by a point P of K if K rolls along a straight line.

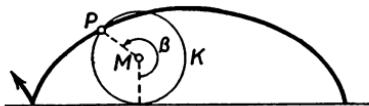


FIG. 27

Cycloids are important in connexion with technical problems, for example, in the theory of cog-wheels. The earliest investigation of a special cycloid is due to the famous painter Albrecht Dürer [1].

Problem 21.1. Prove that all plane curves having the natural equation $\rho = as + b$ ($a \neq 0$) are logarithmic spirals.

Problem 21.2. Determine the natural equations of the catenary

$$\frac{x_2}{c} - \cosh \frac{x_1}{c} = 0, \quad x_3 = 0,$$

cf. Problem 9.2.

22. Involutes and evolutes. Involutes and evolutes are curves associated with a curve in the following manner:

In general, the tangents to a curve $x(s)$ generate a surface, called the *tangent surface* of this curve (for detailed investigation of this surface see Section 58). In consequence of the representation (10.3) of the tangent we obtain immediately the following representation of the corresponding tangent surface:

$$(22.1) \quad \mathbf{y}(s, u) = \mathbf{x}(s) + u\mathbf{t}(s),$$

where s denotes the arc length of the curve and u is a real parameter whose absolute value equals the distance of a point P of the surface from the point P_0 of contact of the corresponding tangent with the curve; u is positive if the directed segment P_0P has the same sense as \mathbf{t} and is negative otherwise.

Involutes of a curve are curves on the corresponding tangent surface which are orthogonal to the generating tangents.

The angle γ ($0 \leq \gamma \leq \pi$) between two intersecting curves is defined as the angle between the unit tangent vectors to these curves at the intersection point.

We consider a curve $C: \mathbf{x}(s)$ of class $r \geq 2$ with non-vanishing curvature $\kappa(s)$. Since any involute of the curve C lies on the corresponding tangent surface we can represent it in the form

$$\mathbf{z}(s) = \mathbf{x}(s) + u(s)\mathbf{t}(s),$$

where $u(s)$ denotes an $(r-1)$ times continuously differentiable function to be determined so that the above orthogonality condition is satisfied. According to this representation the unit tangent vector to the involute is

$$\dot{\mathbf{z}} = \mathbf{t} + \frac{d}{ds}(u\mathbf{t}) = \mathbf{t} + u\kappa\mathbf{p} + \dot{u}\mathbf{t}.$$

In consequence of the orthogonality

$$\dot{\mathbf{z}} \cdot \mathbf{t} = 0$$

must hold. Therefore $1 + \dot{u} = 0$ or

$$u(s) = c - s,$$

where c is a constant. From this we obtain the following representation of the involutes of the curve $\mathbf{x}(s)$:

$$(22.2) \quad \mathbf{z}(s) = \mathbf{x}(s) + (c-s)\mathbf{t}(s).$$

Every real value of c corresponds to exactly one of these infinitely many involutes of the given curve C . Since $\kappa > 0$ we have $\dot{\mathbf{z}} \neq 0$ for all values of $s \neq c$.

If C is a plane curve its tangent surface is a plane which coincides with the plane of the curve. In this case the involutes are therefore plane curves.

We may interpret (22.2) in the following manner: If a thread, lying on a curve, is wound off so that the unwound portion of it is always held taut in the direction of the tangent to the curve, while the rest of it lies on the curve, then every point of the thread generates an involute of the curve during this motion.

Previously we started from a given curve and determined its involutes (Fig. 28). Let us now treat the converse problem: Let a curve C be given and determine a curve C^* so that the given curve is an involute of C^* . The curve C^* is then called an *evolute* of C .

The names ‘involute’ and ‘evolute’ were introduced by Chr. Huyghens (1629–95) in 1665, cf. [1].

We will now find a representation $\mathbf{y}(s)$ of the evolutes C^* of the curve $C: \mathbf{x}(s)$, assuming that C is of class $r \geq 3$ and has non-vanishing curvature

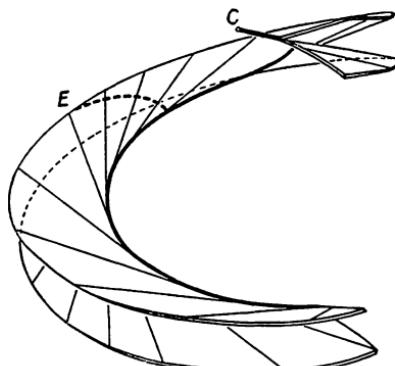


FIG. 28. Circular helix C ; tangent surface of C , and an involute E

$\kappa(s)$. By the definition of an involute, the tangents to C^* must intersect the given curve C orthogonally. An evolute may consequently be represented in the form

$$\mathbf{y}(s) = \mathbf{x}(s) + q(s)\mathbf{a}(s),$$

where $\mathbf{a}(s)$ is a unit vector in the normal plane corresponding to a point $\mathbf{x}(s)$ of the curve and, obviously, $|q(s)|$ is then the distance between that point $\mathbf{x}(s)$ and the corresponding point $\mathbf{y}(s)$ of the evolute. In consequence of the definition of the involute the vector \mathbf{a} must be tangent to the curve $\mathbf{y}(s)$; therefore

$$\dot{\mathbf{y}} = \frac{dy}{ds} = \beta \mathbf{a},$$

where β is a scalar. Using the preceding representation we obtain from this the condition

$$\mathbf{t} + q\mathbf{a} + q\dot{\mathbf{a}} = \beta \mathbf{a}.$$

Since $\mathbf{a} \cdot \mathbf{a} = 1$ and therefore $\mathbf{a} \cdot \dot{\mathbf{a}} = 0$ and also $\mathbf{t} \cdot \mathbf{a} = 0$ we find $q = \beta$; consequently

$$\mathbf{t} + q\dot{\mathbf{a}} = 0.$$

The vector \mathbf{a} lies in the normal plane and is therefore of the form

$$\mathbf{a} = \mathbf{p} \sin \alpha + \mathbf{b} \cos \alpha,$$

where α denotes the angle between \mathbf{a} and the unit binormal vector $\mathbf{b}(s)$ to the curve C , cf. Fig. 29. Differentiating this representation with respect to s , inserting it into the foregoing equation, and using (15.1), we find

$$\mathbf{t} + q[(-\kappa \mathbf{t} + \tau \mathbf{b}) \sin \alpha + \dot{\alpha} \mathbf{p} \cos \alpha - \tau \mathbf{p} \cos \alpha - \dot{\alpha} \mathbf{b} \sin \alpha] = 0.$$

Since \mathbf{t} , \mathbf{p} , and \mathbf{b} are linearly independent vectors, the coefficient of each of these vectors must vanish. We thus obtain $1 - \kappa q \sin \alpha = 0$, or

$$q \sin \alpha = \rho,$$

and furthermore

$$(\dot{\alpha} - \tau) \cos \alpha = 0, \quad (\tau - \dot{\alpha}) \sin \alpha = 0.$$

Now, since for any value of α at least one of the functions $\cos \alpha$ and $\sin \alpha$ is different from zero, $\dot{\alpha} = \tau$ must hold. Integrating this relation we have

$$\alpha(s) = \int_0^s \tau(\sigma) d\sigma + k^*$$

where k^* is a constant of integration. By means of these conditions we

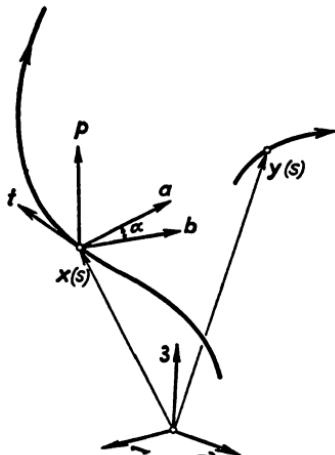


FIG. 29

obtain the desired representation of an evolute of the curve C : $\mathbf{x}(s)$ in the form

$$(22.3) \quad \mathbf{y}(s) = \mathbf{x}(s) + \rho(s)[\mathbf{p}(s) + \mathbf{b}(s) \cot \alpha(s)], \quad \alpha(s) = \int_0^s \tau(\sigma) d\sigma + k^*,$$

where $\mathbf{p}(s)$ and $\mathbf{b}(s)$ are the unit principal and unit binormal vector to the curve C . Each value of the constant k^* corresponds to exactly one of the infinitely many evolutes of the curve C .

Comparing (22.3) and (19.10) we see that any point $\mathbf{y}(s)$ of any evolute lies on the polar axis of the curve at the corresponding point. Furthermore, we stress the following simple property of the evolutes which is an immediate consequence of (22.3):

Theorem 22.1. *Let $\mathbf{y}^{(1)}(s)$ and $\mathbf{y}^{(2)}(s)$ be two different evolutes of the same curve $\mathbf{x}(s)$. Then the angle of intersection of the tangents to $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$, corresponding to any value of s , is the same for all values of s .*

If a curve is plane ($\tau = 0$) then, in (22.3) the angle α is constant. Plane curves have only one plane evolute; this evolute corresponds to the value $k^* = \frac{1}{2}\pi$ and lies in the plane of the curve; it is the locus of the centres of curvature of the curve, cf. (12.6). The other evolutes are general helices on the cylinder which is orthogonal to the plane of the curve and intersects it along the aforementioned plane evolute.

Problem 22.1. Prove Theorem 22.1.

23. Bertrand curves. Two curves which, at any of their points, have a common principal normal are called *Bertrand curves*, cf. J. Bertrand [1].

Example 1. If C is a *plane* curve we can always find a curve C^* such that C and C^* are Bertrand curves. Indeed, if E is the plane evolute of C then all involutes C^* of E have the same principal normal as the given curve C , because, by definition of the involute, these curves intersect orthogonally any tangent to E , cf. Fig. 30. C and any of the curves C^* are therefore Bertrand curves. Moreover, it follows from (22.2) that the distance, measured along the common principal normal, between corresponding points of any two of these involutes is constant. Two plane curves are therefore called *plane parallel curves* if they have common principal normals, this name was introduced by G. W. Leibniz (1646–1716), cf. [2].

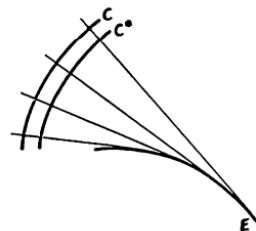


FIG. 30

Example 2. The principal normals to a circular helix C (cf. Sections 8 and 12) intersect the axis of rotation of the cylinder Z on which C lies at right angles. The points of intersection of these normals with any cylinder Z^* , coaxial with Z , determine a circular helix C^* , cf. Fig. 31.

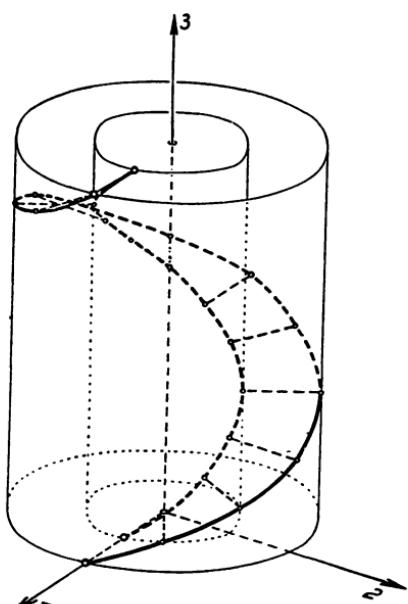


FIG. 31

Obviously, C and C^* are Bertrand curves, and there exist infinitely many helices C^* of this type which, together with C , are Bertrand curves.

If two curves $C: \mathbf{x}(t)$ ($\kappa > 0$) and $C^*: \mathbf{x}^*(t)$ are Bertrand curves then, by definition, C^* can be represented in the form

$$(23.1) \quad \mathbf{x}^*(t) = \mathbf{x}(t) + a(t)\mathbf{p}(t),$$

where $\mathbf{p}(t)$ is the unit principal normal vector to C and the scalar $a(t)$ is the (directed) distance of a point P^* of C^* from the corresponding point P of C ; $a(t)$ has a positive sign if the sense from P to P^* is that of $\mathbf{p}(t)$ and a negative sign otherwise. We will prove shortly that a is a constant, that is, is independent of t .

While for every plane curve C there exists a curve C^* so that C and C^*

are Bertrand curves, this property does not hold in general, for twisted curves.

Theorem 23.1. For a given curve C there exists a curve C^* so that C and C^* are Bertrand curves if and only if the curvature κ and the torsion τ of C satisfy a linear relation with constant coefficients,

$$(23.2) \quad c_1\kappa(t) + c_2\tau(t) = 1.$$

Proof. (a) We assume that $C: \mathbf{x}(t)$ and $C^*: \mathbf{x}^*(t)$ are Bertrand curves. We denote by s and s^* the arc length of C and C^* , respectively. Starting from (23.1) we will first prove that a is a constant. Since a is the (directed) distance of corresponding points of C and C^* , a is a constant if and only if

$$(23.3) \quad \frac{d}{dt}(a^2) = \frac{d}{dt}[(\mathbf{x}^* - \mathbf{x}) \cdot (\mathbf{x}^* - \mathbf{x})] = 2(\mathbf{x}^* - \mathbf{x}) \cdot (\mathbf{x}'^* - \mathbf{x}')$$

vanishes. But this is true since the vector $\mathbf{x}^* - \mathbf{x}$ lies in the common prin-

cipal normal while the vectors \mathbf{x}'' and \mathbf{x}' are tangent to C^* and C , respectively, that is, they are orthogonal to that principal normal. We denote by \mathbf{t} and \mathbf{t}^* , respectively, the unit tangent vectors to C and C^* . We have

$$(23.4) \quad \frac{d}{dt}(\mathbf{t}^* \cdot \mathbf{t}) = \mathbf{t}'' \cdot \mathbf{t} + \mathbf{t}^* \cdot \mathbf{t}',$$

this expression vanishes since the vectors \mathbf{t}' and \mathbf{t}'' lie in the common principal normal. Denoting by α the angle between the tangents at corresponding points we thus have

$$(23.5) \quad \mathbf{t}^* \cdot \mathbf{t} = \cos \alpha = \text{const},$$

cf. (5.5), that is, at all points of C and C^* the angle between the tangents is the same. Instead of the arbitrary parameter t we introduce now the arc length s of C as parameter, writing (23.1) in the form

$$\mathbf{x}^*(s) = \mathbf{x}(s) + a\mathbf{p}(s).$$

Then

$$(23.6) \quad \begin{aligned} \cos \alpha &= \mathbf{t}^* \cdot \mathbf{t} = \frac{d\mathbf{x}^*}{ds} \frac{ds}{ds^*} \cdot \mathbf{t} = \frac{ds}{ds^*} (\mathbf{t} + a\dot{\mathbf{p}}) \cdot \mathbf{t} \\ &= \frac{ds}{ds^*} (1 - a\kappa) = \text{const} \end{aligned}$$

and $|\mathbf{t}^* \times \mathbf{t}| = \left| \frac{ds}{ds^*} ([\mathbf{t} + a(-\kappa\mathbf{t} + \tau\mathbf{b})] \times \mathbf{t}) \right| = \left| \frac{ds}{ds^*} a\tau\mathbf{p} \right|.$

From this, in consequence of (5.8), (23.5), and $|\mathbf{p}| = 1$, it follows that

$$(23.7) \quad a\tau \frac{ds}{ds^*} = \pm \sin \alpha = \text{const}.$$

From (23.6) and (23.7) we obtain

$$(23.8) \quad \frac{1 - a\kappa}{a\tau} = c = \text{const}.$$

Setting $a = c_1$ and $c = c_2/a$, (23.8) becomes identical with (23.2).

(b) The condition (23.2) is also sufficient. We start from a curve C : $\mathbf{x}(s)$, with arc length s , introduce a curve C^* :

$$(23.9) \quad \mathbf{x}^*(s) = \mathbf{x}(s) + c_1 \mathbf{p}(s)$$

and prove that, if (23.2) is satisfied, C and C^* are Bertrand curves. Differentiating (23.9) with respect to s and making use of (15.1) we find

$$(23.10) \quad \frac{d\mathbf{x}^*}{ds} = \mathbf{t} + c_1 \dot{\mathbf{p}} = (1 - c_1 \kappa)\mathbf{t} + c_1 \tau \mathbf{b}.$$

This vector is tangent to C^* . If we write (23.2) in the form

$$1 - c_1 \kappa = cc_1 \tau, \quad c = \frac{c_2}{c_1}$$

and insert this in (23.10) we obtain

$$\frac{d\mathbf{x}^*}{ds} = c_1 \tau (ct + \mathbf{b}).$$

Hence, if the orientation of C^* has been suitably chosen, the unit tangent vector to C^* is of the form

$$\mathbf{t}^* = \frac{ct + \mathbf{b}}{\sqrt{(1+c^2)}}.$$

From this, by (15.1),

$$\frac{d\mathbf{t}^*}{ds} = \frac{1}{\sqrt{(1+c^2)}}(c\kappa - \tau)\mathbf{p}.$$

Since this vector is the derivative of the tangent vector to C^* it lies in the principal normal to C^* , and this normal coincides therefore with that of C . Hence C and C^* are Bertrand curves. This completes the proof.

It follows from this proof that we can write (23.2) in the form

$$(23.11) \quad \kappa + \tau \cot \alpha = 1/a, \quad (\alpha \neq 0)$$

where a has the above meaning and α is the angle between the tangents at corresponding points. (We have to exclude the trivial case $\alpha = 0$ which corresponds to plane or coinciding curves.)

From Theorem 23.1 we obtain the following interesting fact:

Theorem 23.2. *A circular helix C is the only twisted curve for which more than one curve C^* exists such that C and C^* are Bertrand curves.*

Proof. If, for a given twisted curve C there are two curves, say C^* , represented by (23.1) and C^{**} , represented by

$$\mathbf{x}^{**}(t) = \mathbf{x}(t) + a^{**}\mathbf{p}(t), \quad (a^{**} \neq a),$$

so that C and C^* as well as C and C^{**} are Bertrand curves then C must satisfy, in addition to (23.11), another relation of the same type,

$$\kappa + \tau \cot \alpha^{**} = \frac{1}{a^{**}}, \quad (\alpha^{**} \neq 0).$$

From both relations we see that both the curvature κ and the torsion τ of C must be constant. Since the natural equations determine a curve up to its position in space and since circular helices have constant curvature and torsion, the curve C is a circular helix. Furthermore, we see that there are infinitely many curves (which must also be circular helices) so that C and any of these curves are Bertrand curves.

In the case of curves of constant curvature the following fact holds:

Theorem 23.3. *For a given twisted curve C of constant curvature $\kappa = \kappa_0$ (> 0) there always exists a curve C^* such that C and C^* are Bertrand curves. If, in particular, C is not a circular helix, C^* is a twisted curve of the same constant curvature $\kappa^* = \kappa_0$. Then each of these two curves is the locus of the centres of curvature of the other (cf. Section 12). At corresponding points P and P^* the unit tangent vectors to these curves are orthogonal; the normal plane of C at P coincides with the osculating plane of C^* at P^* .*

Proof. If not only the curvature but also the torsion is constant, C is a circular helix; this particular case has already been considered. We may therefore assume that τ is not constant. Then (23.11) takes the form

$$\kappa_0 + \tau \cot \alpha = \frac{1}{a}, \quad \tau \neq \text{const}, \quad \alpha \neq 0.$$

The only solution of this equation is $\kappa_0 = 1/a$, $\cot \alpha = 0$, and so at corresponding points the tangents to both curves are orthogonal. Since the curves have in addition a common principal normal the statement about the osculating and normal planes follows. Since $a = 1/\kappa_0 = \rho_0$ in consequence of (23.1) C^* is the locus of the centres of curvature of C , cf. (12.6). If one of two Bertrand curves is plane, the other must also be plane. Therefore, in our case, C^* must be a twisted curve, that is, $\tau^* \not\equiv 0$. C^* must thus satisfy the relation

$$\kappa^* + \tau^* \cot \alpha^* = \frac{1}{a^*}, \quad (\alpha^* \neq 0),$$

cf. (23.11). From this, $x = x^* + a^* p^*$, $p^* = \pm p$, and (23.1) we obtain the relations

$$|a^*| = a = \rho_0$$

and $\alpha^* = \frac{1}{2}\pi$. Therefore $\kappa^* = \kappa_0$; C is the locus of the centres of curvature of C^* .

We mention finally the following interesting property:

Theorem 23.4. *The product of the torsions of Bertrand curves is constant.*

Problem 23.1. Prove Theorem 23.4.

III

CONCEPT OF A SURFACE. FIRST FUNDAMENTAL FORM. FOUNDATIONS OF TENSOR-CALCULUS

24. Concept of a surface in differential geometry. We will now investigate geometric properties of surfaces in three-dimensional Euclidean space R_3 by means of differential geometry. The reasoning which will lead us to a definition of a surface is similar to that which led us to the concept of a curve.

We assume that in R_3 a system of Cartesian coordinates x_1, x_2, x_3 has been introduced. Then any point in space can be uniquely determined by its position vector $\mathbf{x} = (x_1, x_2, x_3)$. We start from a real single-valued vector function

$$(24.1) \quad \mathbf{x}(u^1, u^2) = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2))$$

of two real variables u^1, u^2 which is defined in a simply-connected and bounded domain B of the u^1u^2 -plane. By (24.1), to any point (u^1, u^2) of B there is associated a point of R_3 with position vector $\mathbf{x}(u^1, u^2)$. We denote by M the point set in R_3 which is obtained when the variables u^1, u^2 vary in B . (24.1) is said to be a *parametric representation* of this set M . The variables u^1, u^2 are called the *parameters* of this representation.

Remark: The reasons for writing the variables in the form u^1, u^2 , with superscripts, instead of u_1, u_2 , will be explained in Section 29.

In order to be able to apply differential calculus to geometric problems we must require the existence of a certain number of partial derivatives of $\mathbf{x}(u^1, u^2)$ with respect to u^1 and u^2 . Moreover, we have to exclude the possibility that (24.1) represents merely a curve. We therefore make the following Assumptions :

(1) *The function $\mathbf{x}(u^1, u^2)$ is of class $r \geq 1$ in B (cf. Section 6). Each point of the set M , represented by $\mathbf{x}(u^1, u^2)$, corresponds to just one ordered pair (u^1, u^2) in B .*

(2) *The Jacobian matrix*

$$(24.2) \quad J = \begin{pmatrix} \frac{\partial x_1}{\partial u^1} & \frac{\partial x_1}{\partial u^2} \\ \frac{\partial x_2}{\partial u^1} & \frac{\partial x_2}{\partial u^2} \\ \frac{\partial x_3}{\partial u^1} & \frac{\partial x_3}{\partial u^2} \end{pmatrix}$$

is of rank 2 in B (cf. the remark below).

(By definition, a matrix is said to be of *rank R* if it contains at least one non-vanishing determinant of order *R* while any determinant of higher order—if any are contained in the matrix—vanishes.)

A representation of the form (24.1) satisfying the Assumptions (1) and (2) is said to be an *allowable representation*.

Notation. Partial derivatives of $\mathbf{x}(u^1, u^2)$ will be denoted as follows:

$$\mathbf{x}_\alpha \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{u}^\alpha}, \quad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \quad \text{etc.}$$

The three determinants of second order contained in *J* are the components of the vector product of the vectors

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1} \quad \text{and} \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2}.$$

This product is different from the null vector if and only if \mathbf{x}_1 and \mathbf{x}_2 are linearly independent vectors, cf. Theorem 5.2. Assumption (2) is therefore the necessary and sufficient condition for the linear independence of those vectors.

Assumption (2) thus excludes the possibility that the point set *M* represented by (24.1) is that of a curve. This could happen if the functions $x_i(u^1, u^2)$ were of the form $x_i(t(u^1, u^2))$. For example,

$$\mathbf{x}(u^1, u^2) = (u^1 + u^2, (u^1 + u^2)^2, (u^1 + u^2)^3)$$

is a representation of the curve $\mathbf{x}(t) = (t, t^2, t^3)$; the corresponding matrix *J* is of rank 1 ($t \neq 0$), that is, does not satisfy Assumption (2).

Remark. The case where the matrix (24.2) is of rank $R < 2$ everywhere in *B* or in a subdomain of *B* has to be excluded. However, it will occasionally be necessary to admit the case where for certain pairs (u^1, u^2) the matrix (24.2) is of rank $R < 2$ (for instance, in the case of tangent surfaces and envelopes). The corresponding points of *M* are called *singular points with respect to the representation*; they are called *singular points of M* if they are singular with respect to every allowable representation of *M*. Simple examples will be given in the following section.

Assumption (2) enables us to solve locally for u^1, u^2 a suitable pair of the three functions $x_i(u^1, u^2)$. Then we can represent one of the quantities x_i as a function of the two others. If the determinant of the first two rows of *J* is different from zero in a certain subdomain *B'* of *B* then u^1 and u^2 can be represented as functions of x_1 and x_2 ,

$$u^1 = \phi(x_1, x_2), \quad u^2 = \psi(x_1, x_2).$$

By inserting this in (24.1) we obtain a representation of the form

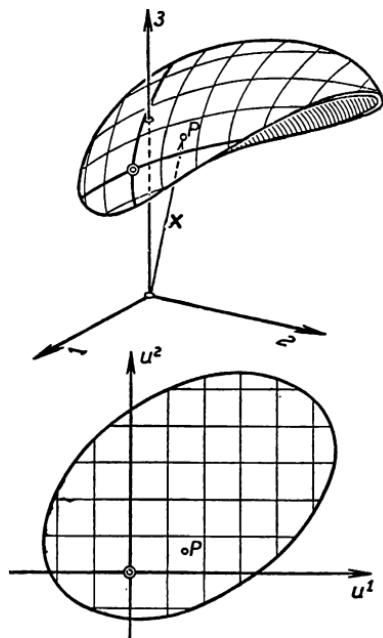


FIG. 32

$$(24.3) \quad x_3 = F(x_1, x_2)$$

valid in B' .

(24.1) is not the only possibility of a parametric representation of the point set M . By imposing a transformation

$$(24.4) \quad u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2), \quad (\alpha = 1, 2),$$

we obtain a new parametric representation $\mathbf{x}(\bar{u}^1, \bar{u}^2)$ of M . Of course we will require that this representation satisfies the Assumptions (1) and (2). We therefore make the following

Assumptions :

(0*) The functions (24.4) are defined in a domain \bar{B} such that the corresponding range of values includes the domain B , cf. (24.1).

(1*) The functions (24.4) are of class $r (\geq 1)$ everywhere in \bar{B} , and (24.4) is a one-to-one transformation.

(2*) The Jacobian

$$D = \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix}$$

of the transformation (24.4) is different from zero everywhere in \bar{B} .

A transformation of the form (24.4) satisfying these conditions is said to be an *allowable coordinate transformation*.

Remark: As is well known the conditions that the transformation (24.4) be one-to-one and that its Jacobian be different from zero are independent of each other. For example,

$$u^1 = e^{\bar{u}^1} \cos \bar{u}^2, \quad u^2 = e^{\bar{u}^1} \sin \bar{u}^2$$

is a transformation which, when considered for all values of \bar{u}^1, \bar{u}^2 , is not one-to-one but whose Jacobian

$$D = \begin{vmatrix} e^{\bar{u}^1} \cos \bar{u}^2 & -e^{\bar{u}^1} \sin \bar{u}^2 \\ e^{\bar{u}^1} \sin \bar{u}^2 & e^{\bar{u}^1} \cos \bar{u}^2 \end{vmatrix} = e^{2\bar{u}^1}$$

is different from zero everywhere in the $\bar{u}^1\bar{u}^2$ -plane. The transformation

$$u^1 = (\bar{u}^1)^3, \quad u^2 = \bar{u}^2$$

is one-to-one everywhere in the (real) $\bar{u}^1\bar{u}^2$ -plane but its Jacobian

$$D = \begin{vmatrix} 3(\bar{u}^1)^2 & 0 \\ 0 & 1 \end{vmatrix} = 3(\bar{u}^1)^2$$

vanishes at every point of the \bar{u}^2 -axis ($\bar{u}^1 = 0$).

Remark: The point set M determined by (24.1) can, in general, also be represented by an implicit function,

$$(24.5) \quad G(x_1, x_2, x_3) = 0.$$

If (24.5) is of class $r \geq 1$ and if, in a neighbourhood U of a certain point of M , at least one of the three first partial derivatives of G is different from zero then (24.5) can be solved for one of the three variables in U , and this variable can be introduced as independent variable. In this manner we obtain a representation of the form (24.3) which is a special case of (24.1). In so far as local properties are concerned the representations (24.1) and (24.5), satisfying the above conditions, are equivalent.

By means of the allowable transformations (24.4) we may classify the allowable representations (24.1) into equivalence classes: two allowable representations are said to be equivalent, that is, belong to the same class if there is an allowable transformation (24.4) transforming the one representation into the other. The axioms of equivalence, stated in Section 3, are satisfied.

Definition 24.1. *A point set in space R_3 which can be represented by the allowable representations of an equivalence class is called a portion of a surface.*

A certain point set may correspond to several equivalence classes; the portions of a surface obtained by the choice of allowable representations of different classes are then, by definition, different from each other.

We shall always assume that all points of the portions of surfaces under consideration are regular points, unless otherwise stated.

In consequence of Assumption 1 the correspondence between the points of a portion of a surface S and the ordered pairs u^1, u^2 is one-to-one. u^1 and u^2 are called *coordinates on S*. The curves $u^1 = \text{const}$ and $u^2 = \text{const}$ are called the *coordinate curves* of the u^1u^2 -coordinate system.

A portion of a surface is a topological image of the domain B ; however, because of the assumptions on $x(u^1, u^2)$, not every topological image of a simply connected domain is a portion of a surface in the sense of our definition.

Definition 24.2. A union U of portions of surfaces is called a surface if every two portions S and S' of U can be joined by finitely many portions $S = S_1, S_2, \dots, S_{n-1}, S_n = S'$ of U in such a manner that the intersection of two subsequent portions S_i ($i = 1, 2, \dots, n-1$), is a portion of a surface.

If we choose any arbitrary allowable representations (of class $r \geq 1$) of S_i, S_{i+1} , then S_i is mapped into the parametric plane; this mapping is one-to-one (except at singular points). The same is true for S_{i+1} . Hence there exists a mapping of class r between the two plane images of the intersection D of S_i and S_{i+1} , that is, an allowable transformation of class r between the coordinates corresponding to S_i and S_{i+1} . Furthermore, the above definition includes the possibility that the intersection of certain portions of U may consist of curves or isolated points.

A portion of a surface is said to be of class r if it can be represented by an allowable representation of class r . A surface is said to be of class r if each of its portions is of class r .

A surface S is said to be simple if each point P of S has a three-dimensional neighbourhood $U(P)$ such that the part of S in $U(P)$ is a topological image of a plane circular disc.

25. Further remarks on the representation of surfaces, examples. Representations of surfaces of the form (24.1) were first used by Euler [4]. Gauss [3] was the first to make general use of these representations.

If the matrix (24.2) has the rank $R = 1$ everywhere in B then $\mathbf{x}(u^1, u^2)$ represents a curve. The fact that $R = 1$ at certain points of B may be due either to the special choice of the representation or to the geometric shape of the point set M .

Example 1. The x_1x_2 -plane can be represented in the form
(25.1) (a) $\mathbf{x}(u^1, u^2) = (u^1, u^2, 0)$ or (b) $\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, 0)$.
In (25.1 a), u^1 and u^2 are Cartesian coordinates; in (25.1 b), u^1 and u^2 are polar coordinates. The corresponding matrix (24.2) has the form

$$(a) J_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (b) J_b = \begin{pmatrix} \cos u^2 & -u^1 \sin u^2 \\ \sin u^2 & u^1 \cos u^2 \\ 0 & 0 \end{pmatrix}.$$

J_a has the rank 2. J_b has the rank 2 except for $u^1 = 0$; that is, the point $\mathbf{x} = \mathbf{0}$ is singular with respect to the representation (25.1 b).

Example 2. The sphere of radius r with centre at $\mathbf{x} = (0, 0, 0)$ can be represented in the form (24.5), e.g. by

$$\sum_{i=1}^3 x_i^2 - r^2 = 0.$$

From this we can obtain a representation of the form (24.3),

$$x_3 = \pm\sqrt{(r^2 - x_1^2 - x_2^2)};$$

depending on the choice of the sign this is a representation of one of the two hemispheres $x_3 \geq 0$ and $x_3 \leq 0$. A parametric representation of the sphere under consideration is

$$\mathbf{x}(u^1, u^2) = (r \cos u^2 \cos u^1, r \cos u^2 \sin u^1, r \sin u^2)$$

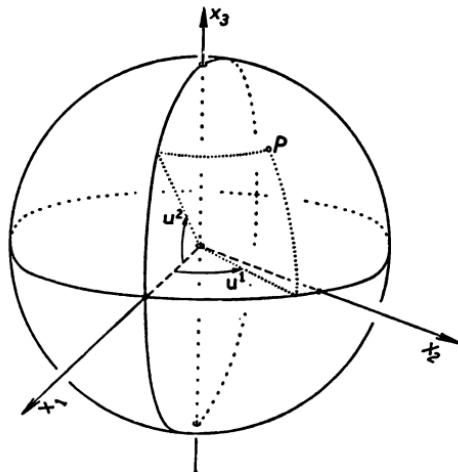


FIG. 33

or, at length,

$$(25.2) \quad \left. \begin{aligned} x_1 &= r \cos u^2 \cos u^1 \\ x_2 &= r \cos u^2 \sin u^1 \\ x_3 &= r \sin u^2 \end{aligned} \right\}, \quad 0 \leq u^1 < 2\pi, \quad -\frac{1}{2}\pi \leq u^2 \leq \frac{1}{2}\pi.$$

This coordinate system is used especially in geography for determining the latitude and longitude of points on the globe. Indeed, the coordinate curves $u^1 = \text{const}$ and $u^2 = \text{const}$ are the 'meridians' and 'parallels', respectively; the 'equator' is given by $u^2 = 0$ and the 'poles' by $u^2 = \pm\frac{1}{2}\pi$. At the poles the corresponding matrix

$$J = \begin{pmatrix} -r \cos u^2 \sin u^1 & -r \sin u^2 \cos u^1 \\ r \cos u^2 \cos u^1 & -r \sin u^2 \sin u^1 \\ 0 & r \cos u^2 \end{pmatrix}$$

is of rank 1, i.e. these points are singular with respect to the representation (25.2); every coordinate curve $u^1 = \text{const}$ passes through these points,

and the curves $u^2 = \pm \frac{1}{2}\pi$ degenerate into points. (*Remark:* The parametric representation of the sphere normally used in mathematics can be obtained from (25.2) by interchanging $\sin u^2$ and $\cos u^2$; in later investigations of mappings of the globe, (25.2) will be convenient.)

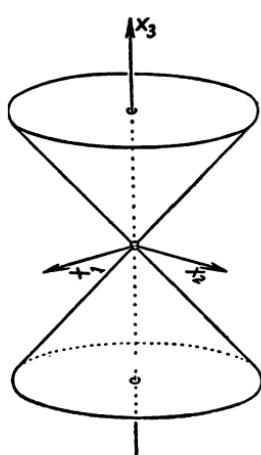


FIG. 34

Example 3. A cone of revolution with apex at $\mathbf{x} = (0, 0, 0)$ and with x_3 -axis as axis of revolution can be represented in the form

$$(25.3) \quad a^2(x_1^2 + x_2^2) - x_3^2 = 0.$$

The resulting representation of the form (24.3),

$$x_3 = \pm a\sqrt{(x_1^2 + x_2^2)},$$

represents one of the two portions $x_3 \geq 0$ and $x_3 \leq 0$ of this cone, depending on the choice of the sign of the square root. Now

$$(25.4) \quad \mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, au^1)$$

is a parametric representation of the cone of the form (24.1). The curves $u^1 = \text{const}$ are circles parallel to the $x_1 x_2$ -plane while the curves

$u^2 = \text{const}$ are the generating straight lines of the cone. The corresponding matrix (24.2),

$$J = \begin{pmatrix} \cos u^2 & -u^1 \sin u^2 \\ \sin u^2 & u^1 \cos u^2 \\ a & 0 \end{pmatrix}$$

is of rank 1 at $u^1 = 0$; the apex is a singular point of the cone.

Problem 25.1. What types of surface are determined by the following representations:

$$(a) \quad \mathbf{x}(u^1, u^2) = (0, u^1, u^2), \quad (b) \quad \mathbf{x}(u^1, u^2) = (u^1 + u^2, u^1 + u^2, u^1),$$

$$(c) \quad \mathbf{x}(u^1, u^2) = (a \cos u^1, a \sin u^1, u^2).$$

Investigate the behaviour of the corresponding matrices (24.2) and find representations of the form (24.5).

Problem 25.2. Find a parametric representation of the cylinder generated by a straight line G which moves along a curve

$$C: \quad \mathbf{x}(s) = (h_1(s), h_2(s), h_3(s))$$

and is always parallel to the x_3 -axis; consider the corresponding matrix (24.2).

Problem 25.3. Find a representation of the form (24.5) of the following surfaces:

- (a) Ellipsoid: $\mathbf{x}(u^1, u^2) = (a \cos u^2 \cos u^1, b \cos u^2 \sin u^1, c \sin u^2)$.
- (b) Elliptic paraboloid: $\mathbf{x}(u^1, u^2) = (au^1 \cos u^2, bu^1 \sin u^2, (u^1)^2)$.
- (c) Hyperbolic paraboloid: $\mathbf{x}(u^1, u^2) = (au^1 \cosh u^2, bu^1 \sinh u^2, (u^1)^2)$.
- (d) Hyperboloid of two sheets:

$$\mathbf{x}(u^1, u^2) = (a \sinh u^1 \cos u^2, b \sinh u^1 \sin u^2, c \cosh u^1).$$

What kind of coordinate curves do you have in each case?

26. Curves on a surface, tangent plane to a surface. In many investigations of geometric properties of a surface suitable curves on that surface are involved. We will now consider representations of curves on surfaces.

A curve on a surface S : $\mathbf{x}(u^1, u^2)$ can be determined by a parametric representation

$$(26.1) \quad u^1 = u^1(t), \quad u^2 = u^2(t)$$

of class $r \geq 1$, where the parameter t is a real variable. Indeed, when inserting (26.1) in the representation $\mathbf{x}(u^1, u^2)$ of S we obtain a representation of the form (6.1); we assume that this representation satisfies the Assumptions (1) and (2), cf. Section 6. The element of arc of a curve on a surface will be considered in the next section.

We will sometimes also represent curves on surfaces in the form

$$(26.1') \quad u^2 = u^2(u^1)$$

or in the form

$$(26.1'') \quad h(u^1, u^2) = 0.$$

Example. A cylinder of revolution S , whose axis of revolution is the x_3 -axis of the coordinate system in space, can be represented in the form

$$\mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2).$$

The generating straight lines G of S , considered as curves on S , can be represented in the form $u^1 = \text{const}$; the variable u^2 plays the role of a parameter on each of those lines. The circles on S parallel to the $x_1 x_2$ -plane are determined by $u^2 = \text{const}$. Any circular helix (8.4):

$$\mathbf{x}(t) = (r \cos t, r \sin t, ct)$$

is a curve on S and can be represented in the form (26.1) by

$$u^1 = t, \quad u^2 = ct$$

or, in the form (26.1'), by $u^2 = cu^1$.

Of particular importance are the coordinate curves $u^1 = \text{const}$ and $u^2 = \text{const}$ on a surface S : $\mathbf{x}(u^1, u^2)$, cf. Section 24. These curves correspond to parallels to the coordinate axes in the $u^1 u^2$ -plane.

A set of curves on a surface S which depend continuously on a (real) parameter is said to be a *one-parameter family of curves* on S . Two one-parameter families of curves on a surface S are called a *net of curves* on S if through every point P of S there passes one and only one curve of each of these families and if the two curves have distinct direction at P .

The coordinate curves on a simple surface form a net which consists of the families of curves $u^1 = \text{const}$ and $u^2 = \text{const}$.

A family of curves can also be determined by a suitable linear differential equation of first order,

$$\frac{du^2}{du^1} = F(u^1, u^2),$$

since the general solution of such an equation involves an arbitrary constant. In Sections 39 and 41 we will also determine nets of curves by suitable differential equations.

The direction of the tangent to a curve C : $u^1(t), u^2(t)$ on a surface S : $\mathbf{x}(u^1, u^2)$ is determined by the vector

$$(26.2) \quad \mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial u^1} \frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2} \frac{du^2}{dt} = \mathbf{x}_1 u^1' + \mathbf{x}_2 u^2',$$

which, in general, depends on t . The vector \mathbf{x}' is a linear combination of the vectors \mathbf{x}_1 and \mathbf{x}_2 which are tangential to the coordinate curves passing through the point P on S under consideration. We assume that P is a regular point of S , that is, the vectors \mathbf{x}_1 and \mathbf{x}_2 are linearly independent, cf. Section 24. These vectors span a plane $E(P)$ called the *tangent plane* at P to the surface S . Obviously $E(P)$ contains the tangent to any curve on S at P passing through this point. Cf. Fig. 35.

The tangent plane $E(P)$ can be represented in the form

$$(26.3) \quad \mathbf{y}(q^1, q^2) = \mathbf{x} + q^1 \mathbf{x}_1 + q^2 \mathbf{x}_2, \quad (\mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0}).$$

q^1 and q^2 are the coordinates of the points of $E(P)$; obviously, the position vector \mathbf{x} of P depends on P only while the vectors \mathbf{x}_1 and \mathbf{x}_2 depend on P as well as on the choice of the coordinates u^1, u^2 on S ; that is, if P is a fixed point on S and the coordinates u^1, u^2 have been chosen in a certain manner the vectors involved in (26.3) are uniquely determined. P is the origin of the $q^1 q^2$ -coordinate system. Cf. Fig. 35.

Since the vector $\mathbf{y} - \mathbf{x}$ lies in the plane $E(P)$ the vectors $\mathbf{y} - \mathbf{x}$, \mathbf{x}_1 , and \mathbf{x}_2

are linearly dependent; we can therefore represent $E(P)$ also in the form

$$(26.4) \quad |(y - x) \ x_1 \ x_2| = 0,$$

cf. Theorem 5.3.

Let now a surface be represented in the form (24.5): $G(x_1, x_2, x_3) = 0$. When inserting in this representation the components x_i as functions of

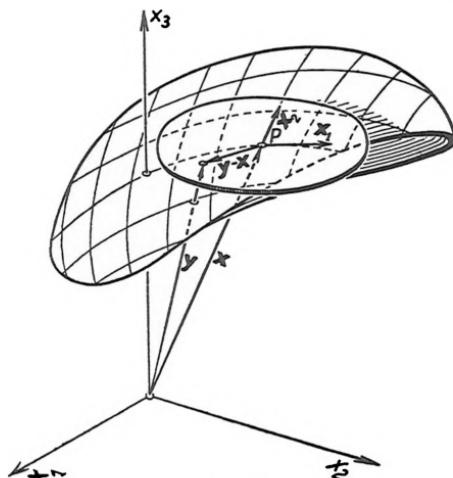


FIG. 35

u^1, u^2 the resulting representation becomes an identity in u^1, u^2 ; therefore the partial derivatives with respect to u^1 and u^2 vanish,

$$(26.5) \quad \frac{\partial G}{\partial u^\alpha} = \frac{\partial G}{\partial x_1} \frac{\partial x_1}{\partial u^\alpha} + \frac{\partial G}{\partial x_2} \frac{\partial x_2}{\partial u^\alpha} + \frac{\partial G}{\partial x_3} \frac{\partial x_3}{\partial u^\alpha} = 0, \quad (\alpha = 1, 2).$$

Multiplying the first of these equations by $\partial x_3 / \partial u^2$, the second by $\partial x_3 / \partial u^1$ and subtracting the relations thus obtained we find

$$\frac{\partial G}{\partial x_1} \left(\frac{\partial x_1}{\partial u^2} \frac{\partial x_3}{\partial u^1} - \frac{\partial x_1}{\partial u^1} \frac{\partial x_3}{\partial u^2} \right) + \frac{\partial G}{\partial x_2} \left(\frac{\partial x_2}{\partial u^2} \frac{\partial x_3}{\partial u^1} - \frac{\partial x_2}{\partial u^1} \frac{\partial x_3}{\partial u^2} \right) = 0.$$

The analogous relation between $\partial G / \partial x_1$ and $\partial G / \partial x_3$ can be similarly obtained. The two relations together show that the quantities $\partial G / \partial x_i$ are proportional to the subdeterminants (taken with a suitable sign) corresponding to the elements $y_i - x_i$ of (26.4) when the determinant (26.4) is developed by the first column. We therefore have the following result: If a surface S is represented in the form (24.5) the tangent plane $E(P)$ to S at a point P

can be represented in the form

$$(26.6) \quad \sum_{i=1}^3 (y_i - x_i) \frac{\partial G}{\partial x_i} = 0,$$

where \mathbf{y} is the position vector of the points of $E(P)$ and \mathbf{x} is the position vector of P .

Furthermore, we see from (26.6) that the vector $\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_3} \right)$ is orthogonal to the tangent plane $E(P)$.

27. First fundamental form. Concept of Riemannian geometry. **Summation convention.** As was pointed out in the preceding section, any curve on a surface S : $\mathbf{x}(u^1, u^2)$ can be represented in the form (26.1):

$$u^1 = u^1(t), \quad u^2 = u^2(t).$$

We will now determine the element of arc of such a curve. This problem will lead us to the introduction of the so-called first fundamental form which is of basic importance.

From (9.7) and (26.2) we immediately find

$$ds^2 = (\mathbf{x}_1 du^1 + \mathbf{x}_2 du^2) \cdot (\mathbf{x}_1 du^1 + \mathbf{x}_2 du^2)$$

$$\text{or} \quad ds^2 = \mathbf{x}_1 \cdot \mathbf{x}_1 (du^1)^2 + 2\mathbf{x}_1 \cdot \mathbf{x}_2 du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 (du^2)^2.$$

We set

$$(27.1) \quad \mathbf{x}_\alpha \cdot \mathbf{x}_\beta = g_{\alpha\beta},$$

that is,

$$\mathbf{x}_1 \cdot \mathbf{x}_1 = g_{11}, \quad \mathbf{x}_1 \cdot \mathbf{x}_2 = g_{12}, \quad \mathbf{x}_2 \cdot \mathbf{x}_1 = g_{21}, \quad \mathbf{x}_2 \cdot \mathbf{x}_2 = g_{22}.$$

$$\text{Obviously} \quad g_{21} = g_{12}.$$

Using this notation we have

$$(27.2) \quad ds^2 = \sum_{\alpha=1}^2 \sum_{\beta=1}^2 g_{\alpha\beta} du^\alpha du^\beta$$

or,

$$(27.2') \quad ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1 du^2 + g_{22}(du^2)^2.$$

This quadratic form is called the *first fundamental form*.

Besides (27.1) the following notations, introduced by Gauss, can be found in the literature:

$$(27.1') \quad g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G.$$

Then

$$(27.2'') \quad ds^2 = E(du^1)^2 + 2F du^1 du^2 + G(du^2)^2.$$

We will use the notation (27.1) exclusively since it seems to be more convenient than (27.1') as will be explained below.

Example. A sphere can be represented in the form (25.2):

$$\mathbf{x}(u^1, u^2) = (r \cos u^2 \cos u^1, r \cos u^2 \sin u^1, r \sin u^2).$$

The first partial derivatives are

$$\mathbf{x}_1 = (-r \cos u^2 \sin u^1, r \cos u^2 \cos u^1, 0)$$

and

$$\mathbf{x}_2 = (-r \sin u^2 \cos u^1, -r \sin u^2 \sin u^1, r \cos u^2).$$

Therefore

$$g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = r^2 \cos^2 u^2, \quad g_{12} = \mathbf{x}_1 \cdot \mathbf{x}_2 = 0, \quad g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = r^2.$$

The first fundamental form corresponding to (25.2) is therefore given by the expression

$$(27.3) \quad ds^2 = r^2 \cos^2 u^2 (du^1)^2 + r^2 (du^2)^2.$$

In Sections 35 and 36 we shall see that the first fundamental form enables us to measure arc lengths, angles, and areas on a surface; the first fundamental form therefore defines a '*metric*' on a surface.

We mention that the coefficients $g_{\alpha\beta}$ are the components of a tensor which is called *metric tensor* or *fundamental tensor*. (The concept of a tensor will be defined in Section 30.)

A metric which is defined by a quadratic differential form is called a *Riemannian metric*. The corresponding geometry is called a *Riemannian geometry*, and the space in which such a metric has been introduced is called a *Riemannian space*. In consequence of the metric defined by (27.2) surfaces are two-dimensional Riemannian spaces, and the geometry in these surfaces is a Riemannian geometry.

We will investigate the differential geometry of surfaces by means of tensor calculus. As we have already stressed in the introduction to this book, this method will cause no essential difficulties as soon as the reader becomes accustomed to the few rules which govern tensor calculus. A great advantage of this method lies in the fact that it can immediately be generalized to Riemannian spaces of higher dimension which have assumed increasing importance during the last few decades. In addition, many aspects of the theory of surfaces are simplified when treated with the aid of tensor calculus which thus leads to a better and deeper insight into several problems of differential geometry.

Although we restrict our consideration to surfaces, that is, to two-dimensional Riemannian spaces, we will now make a brief remark on the

Riemannian geometry of spaces of higher dimension in order to show that the above generalization is immediate.

The student will know that in analytic geometry the restriction to a certain dimension n , for instance, $n = 3$, can be overcome without difficulty; the resulting theory becomes more unified and includes results valid in the plane or in three-dimensional space as particular cases. Proceeding from two-dimensional Riemannian spaces, that is, from surfaces, to Riemannian spaces of higher dimension involves a very similar process, in principle. When dealing with spaces of any (finite) dimension, many problems, methods, and results remain the same as in two dimensions, although new and interesting 'problems enter into consideration.

In the general case of n dimensions we have to introduce n real coordinates u^1, u^2, \dots, u^n and, in order to obtain a Riemannian space, we must introduce a metric defined by a quadratic differential form

$$(27.2'') \quad ds^2 = \sum_{\alpha=1}^n \sum_{\beta=1}^n g_{\alpha\beta} du^\alpha du^\beta.$$

In the particular case $n = 2$ (27.2'') is identical with (27.2). Without loss of generality we may assume the coefficients $g_{\alpha\beta}$ to be symmetric as is the case on surfaces, that is, $g_{\alpha\beta} = g_{\beta\alpha}; \alpha, \beta = 1, 2, \dots, n$. The assumption that the form (27.2'') is positive definite (cf. next section) may be omitted; for example, this is done in the theory of relativity, for physical reasons.

In many investigations tensor calculus will enable us to simplify the formalism to a great extent. No previous knowledge is assumed, since the whole calculus will be developed here, at least in so far as it will be required for our investigations. We will now introduce an essential simplification in the notation of formulae. In the course of our investigations, multiple summations will occur very often. In order to make formulae of this type more manageable we introduce the following convention which is generally used in Riemannian geometry.

SUMMATION CONVENTION. If in a product a letter figures twice, once as superscript and once as subscript, summation must be carried out from 1 to 2 with respect to this letter. The summation sign Σ will be omitted.

Example 1. $a^\alpha b_\alpha \equiv \sum_{\alpha=1}^2 a^\alpha b_\alpha \equiv a^1 b_1 + a^2 b_2.$

Example 2. Instead of (27.2):

$$ds^2 = \sum_{\alpha=1}^2 \sum_{\beta=1}^2 g_{\alpha\beta} du^\alpha du^\beta$$

we now simply write

$$(27.2*) \quad ds^2 = g_{\alpha\beta} du^\alpha du^\beta.$$

$$\text{Example 3.} \quad c^{\gamma\omega} d_{\alpha\omega} \equiv \sum_{\omega=1}^2 c^{\gamma\omega} d_{\alpha\omega} \equiv c^{\gamma 1} d_{\alpha 1} + c^{\gamma 2} d_{\alpha 2}.$$

In this case we have only to sum with respect to ω since γ and α occur only once.

We call an index with respect to which summation must be carried out a *summation index* or *dummy index*; the other indices are said to be *free indices*. Summation indices may be changed during a computation while a change of free indices is not allowed. Of course, letters which already denote other indices may not be used as dummy indices; different dummy indices must be used to indicate different independent summations.

Example 4.

$$\begin{aligned} a^\alpha b_\alpha &= a^i b_i = a^1 b_1 + a^2 b_2, \\ ds^2 &= g_{\alpha\beta} du^\alpha du^\beta = g_{\mu\nu} dw^\mu dw^\nu, \\ a^\alpha b_\alpha + a^\gamma c_\gamma &= a^\alpha (b_\alpha + c_\alpha). \end{aligned}$$

In expressions of the form $\partial u^\beta / \partial u^\alpha$, $x_\alpha = \partial \mathbf{x} / \partial u^\alpha$, etc., α will be considered as subscript.

Example 5. $\frac{\partial \mathbf{x}}{\partial u^\alpha} \frac{du^\alpha}{ds} = \frac{\partial \mathbf{x}}{\partial u^1} \frac{du^1}{ds} + \frac{\partial \mathbf{x}}{\partial u^2} \frac{du^2}{ds}.$

Problem 27.1. Determine the first fundamental form of a plane with respect to polar coordinates.

Problem 27.2. Determine the first fundamental form of the cylinder

$$\mathbf{x}(u^1, u^2) = (h_1(u^1), h_2(u^1), u^2)$$

and of a cylinder of revolution.

Problem 27.3. Determine the first fundamental form of a surface represented in the form (24.3).

Problem 27.4. Determine the first fundamental form corresponding to Cartesian and spherical coordinates in space.

28. Properties of the first fundamental form. A quadratic form

$$F(x, y) = a_{11} x^2 + 2a_{12} xy + a_{22} y^2$$

of two real variables x and y is called *positive definite* if it has a positive value for any pair $(x, y) \neq (0, 0)$. $F(x, y)$ is positive definite if and only if

$$a_{11} > 0 \quad \text{and} \quad a = a_{11} a_{22} - a_{12}^2 > 0;$$

a is called the *discriminant* of the form.

Since our investigations are restricted to real curves and surfaces we have

Theorem 28.1. *At regular points of a surface the first fundamental form (27.2) is positive definite.*

Proof. This is an immediate consequence of the geometric meaning of the first fundamental form. Indeed, the above necessary and sufficient condition is satisfied. From (27.1) we have $g_{11} > 0$. In order that the matrix (24.2) be of rank 2 the sum of the squares of the second-order determinants contained in (24.2) must be positive,

$$(28.1) \quad \left(\frac{\partial x_1}{\partial u^1} \frac{\partial x_2}{\partial u^2} - \frac{\partial x_2}{\partial u^1} \frac{\partial x_1}{\partial u^2} \right)^2 + \left(\frac{\partial x_2}{\partial u^1} \frac{\partial x_3}{\partial u^2} - \frac{\partial x_3}{\partial u^1} \frac{\partial x_2}{\partial u^2} \right)^2 + \left(\frac{\partial x_3}{\partial u^1} \frac{\partial x_1}{\partial u^2} - \frac{\partial x_1}{\partial u^1} \frac{\partial x_3}{\partial u^2} \right)^2 > 0.$$

As can be easily verified this sum is the *discriminant g of the first fundamental form*,

$$(28.2) \quad g = \det(g_{\alpha\beta}) = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - (g_{12})^2 > 0, \quad (\mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0}).$$

We consider now the behaviour of the first fundamental form under a transformation of the coordinates on a surface.

Theorem 28.2. *If we introduce new coordinates \bar{u}^1, \bar{u}^2 on a surface $\mathbf{x}(u^1, u^2)$ by means of an allowable coordinate transformation*

$$u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2), \quad (\alpha = 1, 2),$$

the coefficients $g_{\alpha\beta}$ of the first fundamental form with respect to u^1, u^2 and the coefficients $\bar{g}_{\mu\nu}$ of this form with respect to \bar{u}^1, \bar{u}^2 are related as follows:

$$(28.3 \text{ a}) \quad \bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial u^\beta}{\partial \bar{u}^\nu},$$

and conversely

$$(28.3 \text{ b}) \quad g_{\alpha\beta} = \bar{g}_{\mu\nu} \frac{\partial \bar{u}^\mu}{\partial u^\alpha} \frac{\partial \bar{u}^\nu}{\partial u^\beta}.$$

Proof. The element of arc has a geometric meaning which is independent of the choice of the coordinates on the surface and is therefore an invariant with respect to transformations of these coordinates. We have

$$du^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\mu} d\bar{u}^\mu, \quad (\alpha = 1, 2).$$

If we insert this in

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta$$

we obtain

$$ds^2 = \bar{g}_{\mu\nu} d\bar{u}^\mu d\bar{u}^\nu$$

where $\bar{g}_{\mu\nu}$ is given by (28.3 a). The inverse $\bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2)$ ($\alpha = 1, 2$) of the

above transformation exists and is of the same class, cf. Section 24. Therefore the partial derivatives $\partial \bar{u}^\alpha / \partial u^\beta$ exist. This fact enables us to prove (28.3 b) in the same way as (28.3 a).

We consider finally the transformation behaviour of the discriminant g of the first fundamental form.

Theorem 28.3. *If the coordinates u^1, u^2 undergo an allowable transformation $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$ ($\alpha = 1, 2$), then*

$$(28.4) \quad \bar{g} = D^2 g, \quad g = \bar{D}^2 \bar{g}$$

where \bar{g} is the discriminant of the first fundamental form with respect to the coordinates \bar{u}^1, \bar{u}^2 and

$$D = \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix}, \quad \bar{D} = \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)} = \begin{vmatrix} \frac{\partial \bar{u}^1}{\partial u^1} & \frac{\partial \bar{u}^1}{\partial u^2} \\ \frac{\partial \bar{u}^2}{\partial u^1} & \frac{\partial \bar{u}^2}{\partial u^2} \end{vmatrix}$$

are the Jacobians of the coordinate transformation and of its inverse

Proof. In consequence of (28.3) we have

$$\bar{g} = \bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2 = g_{\kappa\lambda} g_{\mu\nu} \frac{\partial u^\kappa}{\partial \bar{u}^1} \frac{\partial u^\nu}{\partial \bar{u}^2} \left(\frac{\partial u^\lambda}{\partial \bar{u}^1} \frac{\partial u^\mu}{\partial \bar{u}^2} - \frac{\partial u^\lambda}{\partial \bar{u}^2} \frac{\partial u^\mu}{\partial \bar{u}^1} \right).$$

Since, if $\lambda = \mu$ the expression in brackets is zero, we have

$$\bar{g} = (g_{\kappa 1} g_{2\nu} - g_{\kappa 2} g_{1\nu}) \frac{\partial u^\kappa}{\partial \bar{u}^1} \frac{\partial u^\nu}{\partial \bar{u}^2} \left(\frac{\partial u^1}{\partial \bar{u}^1} \frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2} \frac{\partial u^2}{\partial \bar{u}^1} \right).$$

If $\kappa = \nu$ the expression in the first brackets is zero; therefore

$$\bar{g} = \{g_{11} g_{22} - (g_{12})^2\} \left(\frac{\partial u^1}{\partial \bar{u}^1} \frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2} \frac{\partial u^2}{\partial \bar{u}^1} \right)^2,$$

that is, $\bar{g} = D^2 g$. The other relation in (28.4) can be similarly proved.

Problem 28.1. A surface S generated by a curve rotating about a fixed straight line A is called a *surface of revolution*. A is called the *axis* of S . Find a parametric representation of S and of the corresponding first fundamental form.

Problem 28.2. A surface S is called a *right conoid* if it can be generated by a moving straight line G intersecting a fixed straight line G_0 so that G and G_0 are always orthogonal. Find a parametric representation of a right conoid and determine the corresponding first fundamental form.

Problem 28.3. Prove that if two coordinate transformations, say

$\bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2)$ ($\alpha = 1, 2$), $\bar{w}^\beta = \bar{w}^\beta(\bar{u}^1, \bar{u}^2)$ ($\beta = 1, 2$), are imposed successively, the discriminants \bar{g} and g , corresponding to the respective coordinates \bar{w}^β and u^α are connected by a relation of the form (28.4).

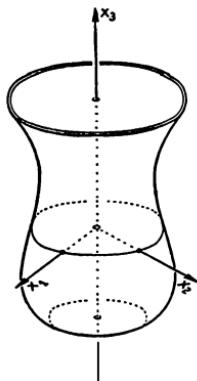


FIG. 36. Surface of revolution

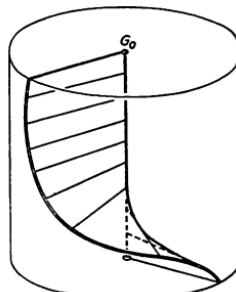


FIG. 37. Right conoid

Problem 28.4. Prove that $D\bar{D} = 1$, cf. (28.4).

Problem 28.5. If a twisted curve C rotates about a fixed axis A and, at the same time, is displaced parallel to A so that the velocity of displacement is always proportional to the angular velocity of rotation, then C generates a surface S which is called a *general helicoid*. Find a parametric representation of S .

29. Contravariant and covariant vectors. In this and the following sections an introduction to tensor calculus will be given. The basic ideas are contained in this section while the material in the following ones is only a simple generalization of those ideas. We will just introduce all concepts formally and later, in Section 32, we will treat geometric interpretations and relations to the theory of surfaces more in detail.

In the preceding section we investigated the behaviour of the coefficients $g_{\alpha\beta}$ of the first fundamental form with respect to an allowable coordinate transformation

$$(29.1) \quad u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2), \quad (\alpha = 1, 2);$$

we obtained the important formulae (28.3). We will immediately see that this problem is a part of a more general one. The general problem will be investigated in the course of this and the following sections; in doing so we will gain knowledge of the transformation behaviour of all quantities which will interest us in the sequel.

We first repeat the fact that the inverse

$$(29.1') \quad \bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2), \quad (\alpha = 1, 2),$$

of an allowable transformation (29.1) exists and is of the same class as (29.1).

In this and the following two sections we drop the restriction of the dimension ($n = 2$) and consider Riemannian spaces of any dimension n (cf. Section 27); there will be no difficulty at all. Let u^1, u^2, \dots, u^n be the corresponding coordinates. Instead of (29.1) we have now to consider transformations of the form

$$(29.2) \quad u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n), \quad (\alpha = 1, 2, \dots, n).$$

We assume that (29.2) is of class $r \geq 1$ and that the inverse transformation

$$(29.2') \quad \bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2, \dots, u^n), \quad (\alpha = 1, 2, \dots, n),$$

exists and is of the same class. A transformation of the form (29.2) satisfying these conditions will be called an *allowable transformation*. We assume that the allowable transformations to be considered form a group (cf. Section 3).

The summation convention must be slightly modified in this and the next two sections: summations must be carried out from 1 to n (where, in the particular case of surfaces, $n = 2$).

If we write (29.2) and (29.2') in the form

$$u^\alpha = f^\alpha(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n), \quad \bar{u}^\alpha = h^\alpha(u^1, u^2, \dots, u^n), \quad (\alpha = 1, 2, \dots, n),$$

we obtain the identities

$$u^\alpha = f^\alpha(h^1, h^2, \dots, h^n), \quad \bar{u}^\alpha = h^\alpha(f^1, f^2, \dots, f^n).$$

If we differentiate with respect to u^β the first of these two systems of formulae we find

$$\frac{\partial u^\alpha}{\partial u^\beta} = \frac{\partial f^\alpha}{\partial h^\gamma} \frac{\partial h^\gamma}{\partial u^\beta}, \quad (\alpha, \beta = 1, 2, \dots, n).$$

Since u^α and u^β are independent if $\alpha \neq \beta$, the value of the left-hand side of these relations is 0 or 1, according as $\alpha \neq \beta$ or $\alpha = \beta$. We therefore obtain the relations

$$(29.3 \text{ a}) \quad \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\gamma}{\partial u^\beta} = \delta_\beta^\alpha, \quad (\alpha, \beta = 1, 2, \dots, n),$$

and similarly

$$(29.3 \text{ b}) \quad \frac{\partial \bar{u}^\alpha}{\partial u^\gamma} \frac{\partial u^\gamma}{\partial \bar{u}^\beta} = \delta_\beta^\alpha, \quad (\alpha, \beta = 1, 2, \dots, n),$$

where

$$\delta_\beta^\alpha = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta) \end{cases}$$

is the Kronecker symbol.

In consequence of the elementary rules of partial differentiation we have the following relations between the differentials du^1, du^2, \dots, du^n of the coordinates u^α and the differentials $d\bar{u}^1, d\bar{u}^2, \dots, d\bar{u}^n$ of the coordinates \bar{u}^β :

$$d\bar{u}^\beta = \frac{\partial \bar{u}^\beta}{\partial u^\alpha} du^\alpha, \quad (\beta = 1, 2, \dots, n).$$

Hence the differential $d\bar{u}^\beta$ is obtained by multiplying the differentials du^α by $\partial \bar{u}^\beta / \partial u^\alpha$ and summing with respect to α from 1 to n . Conversely

$$du^\gamma = \frac{\partial u^\gamma}{\partial \bar{u}^\beta} d\bar{u}^\beta, \quad (\gamma = 1, 2, \dots, n).$$

We may say that (29.2) *induces* a homogeneous linear transformation on the differentials; the coefficients of this transformation are functions of the coordinates.

An ordered system of n quantities which have the same transformation behaviour as the differentials of the coordinates is called a contravariant tensor of first order or contravariant vector. More precisely:

Definition 29.1. Let an n -tuple of real numbers a^1, a^2, \dots, a^n be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, u^2, \dots, u^n . Furthermore, let there be associated with P an n -tuple of real numbers $\bar{a}^1, \bar{a}^2, \dots, \bar{a}^n$ with respect to any coordinate system $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n$ which can be obtained from the coordinates u^α by an allowable transformation. If these numbers satisfy the relations

$$(29.4 \text{ a}) \quad \bar{a}^\beta = a^\alpha \frac{\partial \bar{u}^\beta}{\partial u^\alpha}, \quad (\beta = 1, 2, \dots, n)$$

(conversely

$$(29.4 \text{ b}) \quad a^\gamma = \bar{a}^\beta \frac{\partial u^\gamma}{\partial \bar{u}^\beta}, \quad (\gamma = 1, 2, \dots, n),$$

we say that a contravariant tensor of first order or contravariant vector at P is given. The quantities a^1, a^2, \dots, a^n and $\bar{a}^1, \bar{a}^2, \dots, \bar{a}^n$ are called the components of this vector in their respective coordinate systems. This vector will be denoted by a^α or \bar{a}^α in the respective coordinate system u^α or \bar{u}^α . The contravariant transformation behaviour (29.4) is indicated by a superscript.

The relations (29.4 b) can be obtained from (29.4 a) by multiplication by $\partial u^\gamma / \partial \bar{u}^\beta$ and summation with respect to β from 1 to n ; in consequence of (29.3 a) we find

$$\frac{\partial u^\gamma}{\partial \bar{u}^\beta} \bar{a}^\beta = a^\alpha \frac{\partial \bar{u}^\beta}{\partial u^\alpha} \frac{\partial u^\gamma}{\partial \bar{u}^\beta} = a^\alpha \delta_\alpha^\gamma = a^\gamma.$$

Since the notation a^α occurring in Definition 29.1 is nothing but an abbreviated notation of the system a^1, a^2, \dots, a^n we may also write a^λ or a^τ or choose any other letter as index.

Any set of n real numbers may be taken as components of a contravariant vector at P with respect to the coordinates u^1, u^2, \dots, u^n ; the components of this vector with respect to the coordinates $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n$ are then determined by (29.4).

The relation of this concept of a vector with that introduced in Section 3 will be considered in Section 32.

In general, the quantities a^1, \dots, a^n (and the corresponding quantities $\bar{a}^1, \dots, \bar{a}^n$) will be given not only at a point but in a certain domain D of the space, that is, they will be functions of the coordinates defined in D . Then we say that in D a *contravariant tensor field of first order* or *contravariant vector field* is given. For the sake of simplicity we will use the name ‘tensor’ (instead of ‘tensor field’) in this case also.

While in the special case of Euclidean spaces the free vector is the appropriate concept, in the general case of a Riemannian space the concept of bound vectors is appropriate. The reason for this difference lies in the fact that, in general, the allowable coordinate transformations (29.2)—in contrast to (3.1)—are non-linear and the derivatives occurring in (29.4) therefore depend on the coordinates while the coefficients a_{ik} in (4.2) are constants.

We will say that a *geometric object* is given at a point P of a Riemannian space if the following conditions hold:

- (1) With respect to every allowable coordinate system one and only one ordered system of N real numbers is given. (These numbers are called the *components* of the geometric object in the respective coordinate system.)
- (2) A law is given which permits the representation of the components of the object with respect to a certain coordinate system in terms
 - (a) of the components of this object with respect to any other coordinate system,
 - and (b) of the values at P of the functions involved in the corresponding coordinate transformation and their derivatives.

Geometric objects can be divided into classes; each class consists of objects having the same transformation law. We already know two important classes of geometric objects:

- (0) Scalars or invariants are geometric objects with one component which is invariant with respect to any coordinate transformation.

(1 a) Contravariant vectors are geometric objects with n components which transform according to (29.4).

Further classes will be introduced below.

Another transformation, the so-called ‘covariant transformation’, is as important as the contravariant transformation (29.4); its importance can be understood by the following consideration. We start from a simple example. The first fundamental form

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta$$

is an invariant with respect to any allowable coordinate transformation, cf. Section 28. This form contains contravariant quantities, namely, the coordinate differentials whose transformation behaviour is known (see above). We now have to ask how the quantities $g_{\alpha\beta}$ must behave under a coordinate transformation in order that ds^2 be an invariant. In the special case of the first fundamental form the answer to this problem has already been given in Section 28. But we may consider this problem as a part of a more general one which we will now attack, starting from the simplest possible case:

Let

$$I = b_\alpha a^\alpha$$

be a linear form of the components of an arbitrary contravariant vector a^α and assume that I is invariant with respect to any allowable coordinate transformation of the form (29.2). Denoting by \bar{a}^β the components of the vector with respect to the coordinates $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n$, we have

$$\bar{b}_\beta \bar{a}^\beta = b_\alpha a^\alpha.$$

We can easily find relations between the coefficients \bar{b}_β and b_α of the form in the two different coordinate systems. In consequence of (29.4 b):

$$a^\alpha = \bar{a}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \quad (\alpha = 1, 2, \dots, n),$$

we have

$$\bar{b}_\beta \bar{a}^\beta = b_\alpha a^\alpha = b_\alpha \bar{a}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\beta}.$$

This relation must hold for any arbitrary vector a^α . Hence, by comparing the coefficients of corresponding components \bar{a}^β on both sides of this formula we find the following relations between the coefficients b_α and \bar{b}_β of the form I in the respective coordinate systems:

$$(29.5 \text{ a}) \quad \bar{b}_\beta = b_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \quad (\beta = 1, 2, \dots, n).$$

Conversely, by (29.4 a),

$$(29.5 \text{ b}) \quad b_\gamma = \bar{b}_\beta \frac{\partial \bar{u}^\beta}{\partial u^\gamma}, \quad (\gamma = 1, 2, \dots, n).$$

(29.5 b) can also be obtained from (29.5 a) by taking into account (29.3).

Definition 29.2. Let an n -tuple of real numbers b_1, b_2, \dots, b_n be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, \dots, u^n . Furthermore, let there be associated with P an n -tuple of real numbers $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n$ with respect to any coordinate system $\bar{u}^1, \dots, \bar{u}^n$ which can be obtained from the coordinates u^α by an allowable transformation. If these numbers satisfy the relations (29.5) we say that a covariant tensor of first order or covariant vector at P is given. The numbers are called the components of this vector in their respective coordinate systems. The vector (in the coordinates u^α) will be denoted by b_α . The covariant transformation behaviour (29.5) is indicated by a subscript.

This definition is again the definition of a vector bound at a point.

We have already mentioned the fact that the tensor calculus is a useful tool in physics. In order to illustrate this we consider the following

Example. If u^1, u^2, u^3 are any coordinates in space R_3 the work done by a force as a result of an infinitesimal displacement du^α is given by the expression

$$dW = p_\alpha du^\alpha$$

where p_α are the components of the force with respect to the coordinates u^α . The work dW is independent of the choice of the coordinates; since du^α is a contravariant vector the force vector must be covariant for dW to be an invariant.

A real function $\phi(u^1, u^2, \dots, u^n)$ which is defined in a certain region and whose value is invariant with respect to all allowable coordinate transformations (29.2) is called a *scalar function*. The gradient $\partial\phi/\partial u^\alpha$ of a scalar function ϕ of class $r \geq 1$ is a covariant vector since

$$\frac{\partial \phi}{\partial \bar{u}^\beta} = \frac{\partial \phi}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \quad (\beta = 1, 2, \dots, n);$$

indeed, this transformation behaviour is of the form (29.5).

30. Contravariant, covariant, and mixed tensors. In this section we will define tensors of arbitrary order. In the sense of this definition a contravariant (or covariant) vector will be a contravariant (or covariant) tensor of first order.

In order to introduce the concept of a tensor of arbitrary order it is necessary that a certain space with a group of coordinate transformations be given, as was done in the preceding section. We again consider an n -dimensional Riemannian space and in it coordinate transformations of the form

$$(30.1) \quad u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n), \quad (\alpha = 1, 2, \dots, n).$$

cf. (29.2) which are of class $r \geq 1$ and such that the inverse transformation

$$(30.1') \quad \bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2, \dots, u^n), \quad (\alpha = 1, 2, \dots, n),$$

exists and is of class r . Transformations satisfying these conditions will again be called *allowable*.

In the preceding section we considered an invariant linear form $I = b_\alpha a^\alpha$. Similarly, we will now consider invariant multilinear forms of the components of contravariant and covariant vectors.

Before we give the definition of a tensor of arbitrary order we will first define tensors of second order which will be important for later use.

Let

$$J = a^\alpha \beta b_\alpha c_\beta$$

be a bilinear form of the components of two arbitrary covariant vectors b_α and c_β which is invariant with respect to any allowable coordinate transformation (30.1). Denoting by a bar the quantities corresponding to the coordinates \bar{u}^α we have $\bar{a}^\gamma \bar{b}_\gamma \bar{c}_\kappa = a^\alpha \beta b_\alpha c_\beta$.

From (29.5) we obtain

$$\bar{a}^\gamma \bar{b}_\gamma \bar{c}_\kappa = a^\alpha \beta b_\alpha c_\beta = a^\alpha \beta \bar{b}_\gamma \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} \bar{c}_\kappa \frac{\partial \bar{u}^\kappa}{\partial u^\beta}.$$

This relation must hold for arbitrary vectors b_α and c_β . Hence comparing the coefficients of every product $\bar{b}_\gamma \bar{c}_\kappa$ on both sides of this relation we obtain the transformation law of the coefficients of the form J ,

$$(30.2 \text{ a}) \quad \bar{a}^\gamma \kappa = a^\alpha \beta \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} \frac{\partial \bar{u}^\kappa}{\partial u^\beta}, \quad (\gamma = 1, 2, \dots, n, \kappa = 1, 2, \dots, n).$$

Multiplying (30.2 a) by $\frac{\partial u^\sigma}{\partial \bar{u}^\gamma} \frac{\partial u^\tau}{\partial \bar{u}^\kappa}$ and summing with respect to γ and κ we find

$$\bar{a}^\gamma \kappa \frac{\partial u^\sigma}{\partial \bar{u}^\gamma} \frac{\partial u^\tau}{\partial \bar{u}^\kappa} = a^\alpha \beta \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\kappa}{\partial u^\beta} \frac{\partial u^\tau}{\partial \bar{u}^\kappa} = a^\alpha \beta \delta_\alpha^\sigma \delta_\beta^\tau,$$

cf. (29.3 a), and therefore

$$(30.2 \text{ b}) \quad a^{\sigma\tau} = \bar{a}^\gamma \kappa \frac{\partial u^\sigma}{\partial \bar{u}^\gamma} \frac{\partial u^\tau}{\partial \bar{u}^\kappa} \quad (\sigma = 1, 2, \dots, n, \tau = 1, 2, \dots, n).$$

Definition 30.1. Let a system of n^2 real numbers $a^{\alpha\beta}$, $\alpha = 1, 2, \dots, n$, $\beta = 1, 2, \dots, n$, be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, u^2, \dots, u^n . Furthermore, let there be associated with P a system of n^2 real numbers $\bar{a}^{\gamma\kappa}$, $\gamma = 1, 2, \dots, n$, $\kappa = 1, 2, \dots, n$, with respect to any coordinate system $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n$ which can be obtained from the coordinates u^α by an allowable coordinate transformation (30.1). If these numbers are related by (30.2) we say that a contravariant tensor of second order is given at P . The numbers are called the components of this tensor in their respective coordinate systems. The tensor (in the coordinates u^α) will be denoted by $a^{\alpha\beta}$.

Again, the contravariant transformation behaviour is indicated by superscripts.

We can arrange the components of a tensor of second order in the form of a square matrix,

$$\begin{pmatrix} a^{11} & a^{12} & \cdot & \cdot & \cdot & a^{1n} \\ a^{21} & a^{22} & \cdot & \cdot & \cdot & a^{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a^{n1} & a^{n2} & \cdot & \cdot & \cdot & a^{nn} \end{pmatrix}.$$

Example. Let p^α and q^β be two contravariant vectors bound at the same point in space. The system of the n^2 quantities

$$r^{\alpha\beta} = p^\alpha q^\beta, \quad (\alpha = 1, 2, \dots, n, \beta = 1, 2, \dots, n),$$

is called the (outer) product of the vectors p^α and q^β . From (29.4) we obtain

$$\bar{r}^{\gamma\kappa} \equiv \bar{p}^\gamma \bar{q}^\kappa = p^\alpha \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} q^\beta \frac{\partial \bar{u}^\kappa}{\partial u^\beta},$$

therefore

$$\bar{r}^{\gamma\kappa} = r^{\alpha\beta} \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} \frac{\partial \bar{u}^\kappa}{\partial u^\beta}.$$

Consequently this product is a contravariant tensor of second order.

We now consider a bilinear form

$$N = a_{\alpha\beta} b^\alpha c^\beta$$

of the components of two contravariant vectors b^α and c^β which is invariant with respect to any coordinate transformation (30.1). Characterizing by a bar the quantities corresponding to the coordinates \bar{u}^α we have

$$\bar{a}_{\gamma\kappa} \bar{b}^\gamma \bar{c}^\kappa = a_{\alpha\beta} b^\alpha c^\beta.$$

From (29.4) it follows that

$$\bar{a}_{\gamma\kappa} \bar{b}^\gamma \bar{c}^\kappa = a_{\alpha\beta} b^\alpha c^\beta = a_{\alpha\beta} \bar{b}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \bar{c}^\kappa \frac{\partial u^\beta}{\partial \bar{u}^\kappa}.$$

This relation must hold for arbitrary vectors b^α and c^β . Hence by comparing the coefficients of the products $\bar{b}^\gamma \bar{c}^\kappa$ on both sides of this relation we find

$$(30.3 \text{ a}) \quad \bar{a}_{\gamma\kappa} = a_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\kappa}, \quad (\gamma = 1, 2, \dots, n, \kappa = 1, 2, \dots, n).$$

Multiplying (30.3 a) by $\frac{\partial \bar{u}^\sigma}{\partial u^\alpha} \frac{\partial \bar{u}^\tau}{\partial u^\sigma}$ and summing with respect to γ and κ we obtain conversely

$$(30.3 \text{ b}) \quad a_{\sigma\tau} = \bar{a}_{\gamma\kappa} \frac{\partial \bar{u}^\gamma}{\partial u^\sigma} \frac{\partial \bar{u}^\kappa}{\partial u^\tau}, \quad (\sigma = 1, 2, \dots, n, \tau = 1, 2, \dots, n).$$

Definition 30.2. Let a system of n^2 real numbers $a_{\alpha\beta}$, $\alpha, \beta = 1, 2, \dots, n$, be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, u^2, \dots, u^n . Furthermore, let there be associated with P a system of n^2 real numbers $\bar{a}_{\gamma\kappa}$, $\gamma, \kappa = 1, 2, \dots, n$, with respect to any coordinate system $\bar{u}^1, \dots, \bar{u}^n$ which can be obtained from the coordinates u^α by an allowable transformation (30.1). If these numbers are related by (30.3) we say that a covariant tensor of second order is given at P . The numbers are called the components of this tensor in their respective coordinate systems.

Again, the covariant transformation behaviour is indicated by subscripts.

Example. The coefficients $g_{\alpha\beta}$ of the first fundamental form

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta$$

are the components of a covariant tensor of second order since this form is of the same type as the above invariant form N . Indeed, the transformation law (28.3) of the coefficients $g_{\alpha\beta}$ is of the form (30.3).

There is a third kind of tensor of second order which we will introduce now. For this purpose we consider an invariant bilinear form

$$L = a_\alpha^\beta b^\alpha c_\beta$$

of the components of a contravariant vector b^α and of a covariant vector c_β . Indicating by a bar the transformed quantities we have

$$\bar{a}_\gamma^\kappa \bar{b}^\gamma \bar{c}_\kappa = a_\alpha^\beta b^\alpha c_\beta$$

and, by (29.4) and (29.5),

$$\bar{a}_\gamma^\kappa \bar{b}^\gamma \bar{c}_\kappa = a_\alpha^\beta b^\alpha c_\beta = a_\alpha^\beta \bar{b}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \bar{c}_\kappa \frac{\partial \bar{u}^\kappa}{\partial u^\beta};$$

therefore

$$(30.4 \text{ a}) \quad \bar{a}_\gamma^\kappa = a_\alpha^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\kappa}{\partial u^\beta}, \quad (\gamma = 1, 2, \dots, n, \kappa = 1, 2, \dots, n),$$

and conversely

$$(30.4 \text{ b}) \quad a_{\sigma}^{\tau} = \bar{a}_{\gamma}^{\kappa} \frac{\partial \bar{u}^{\gamma}}{\partial u^{\sigma}} \frac{\partial u^{\tau}}{\partial \bar{u}^{\kappa}}, \quad (\sigma = 1, 2, \dots, n, \tau = 1, 2, \dots, n).$$

Definition 30.3. Let a system of n^2 real numbers a_{α}^{β} , $\alpha, \beta = 1, 2, \dots, n$, be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, u^2, \dots, u^n , and let there be associated with P a system of n^2 real numbers $\bar{a}_{\gamma}^{\kappa}$, $\gamma, \kappa = 1, 2, \dots, n$, with respect to any coordinate system $\bar{u}^1, \dots, \bar{u}^n$ which can be obtained from the coordinates u^{α} by an allowable transformation (30.1). If these numbers are related by (30.4) we say that a mixed tensor of second order is given at P . The numbers are called the components of this tensor in their respective coordinate system.

In order to introduce the definition of a tensor of arbitrary order (which definition will include all the preceding ones as special cases) let us consider an invariant multilinear form

$$L = h_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s} a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_r} b_{\beta_1} b_{\beta_2} \dots b_{\beta_s}$$

of the components of r contravariant vectors $a^{\alpha_1}, \dots, a^{\alpha_r}$ and s covariant vectors $b_{\beta_1}, \dots, b_{\beta_s}$. By applying (29.4) and (29.5) to these vectors we find the transformation law of the coefficients of the form,

$$(30.5 \text{ a}) \quad h_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s} = h_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s} \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\gamma_1}} \dots \frac{\partial u^{\alpha_r}}{\partial \bar{u}^{\gamma_r}} \frac{\partial \bar{u}^{\kappa_1}}{\partial u^{\beta_1}} \dots \frac{\partial \bar{u}^{\kappa_s}}{\partial u^{\beta_s}}.$$

Multiplying (30.5 a) by

$$\frac{\partial \bar{u}^{\gamma_1}}{\partial u^{\sigma_1}} \dots \frac{\partial \bar{u}^{\gamma_r}}{\partial u^{\sigma_r}} \frac{\partial u^{\tau_1}}{\partial \bar{u}^{\kappa_1}} \dots \frac{\partial u^{\tau_s}}{\partial \bar{u}^{\kappa_s}}$$

and summing with respect to $\gamma_1, \dots, \gamma_r, \kappa_1, \dots, \kappa_s$ we obtain conversely

$$(30.5 \text{ b}) \quad h_{\alpha_1 \dots \alpha_r}^{\tau_1 \dots \tau_s} = \bar{h}_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s} \frac{\partial \bar{u}^{\gamma_1}}{\partial u^{\sigma_1}} \dots \frac{\partial \bar{u}^{\gamma_r}}{\partial u^{\sigma_r}} \frac{\partial u^{\tau_1}}{\partial \bar{u}^{\kappa_1}} \dots \frac{\partial u^{\tau_s}}{\partial \bar{u}^{\kappa_s}}.$$

Definition 30.4. Let a system of n^{r+s} real numbers $h_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$, $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s = 1, 2, \dots, n$, be associated with a point P of an n -dimensional Riemannian space with coordinates u^1, u^2, \dots, u^n and let there be associated with P a system of n^{r+s} real numbers $\bar{h}_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s}$ with respect to any coordinate system $\bar{u}^1, \dots, \bar{u}^n$ which can be obtained from the coordinates u^{α} by an allowable transformation (30.1). If these numbers are related by (30.5) we say that a mixed tensor of order $r+s$, covariant of order r and contravariant of order s , is given at P . The numbers are called the components of this tensor in their respective coordinate systems.

A tensor is called *covariant* or *contravariant* if it has only covariant or only contravariant indices, respectively. Otherwise the tensor is called *mixed*.

A scalar is called a *tensor of order zero*. A tensor of first order is also known as a *vector*. Definition 30.4 includes the Definitions 29.1, 29.2, 30.1–30.3 as particular cases.

We say that in a domain D of the space a tensor field of order $r+s$ is given if the quantities $h_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$ (and the corresponding ones in any allowable coordinate system) are defined in D as functions of the coordinates and satisfy (30.5) at every point of D .

Physical examples of tensor fields of order 0, 1, and 2 are the temperature field, the field of a force, and the field of stress of an elastic body.

One should comprehend that the formula (30.5) is really very simple. To each index of the tensor there corresponds exactly one first partial derivative in each term of the $(r+s)$ -fold sum on the right-hand side of (30.5 a); this derivative involves the above index and the *corresponding* one of the transformed components of the tensor. If the index under consideration is contravariant or covariant, the corresponding derivative is of the form of that in (29.4) or (29.5), respectively.

The position of an index (subscript or superscript) has to be noted carefully. At the beginning of an investigation the order of the indices can be arbitrarily chosen, but it is then fixed in the course of this particular investigation.

When the relative order of two indices, either contravariant or covariant, of a tensor is immaterial the tensor is called *symmetric with respect to these indices*. A tensor, whether covariant or contravariant, is said to be *symmetric* if it is symmetric with respect to all pairs of indices.

Example. The fundamental tensor is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$.

A tensor of second order, $h_{\alpha\beta}$ or $h^{\alpha\beta}$, has n^2 components; if it is symmetric only $n(n+1)/2$ components are different.

When for a tensor two components obtained from one another by the interchange of two particular indices, either contravariant or covariant, differ only in sign, the tensor is said to be *skew-symmetric* (or *alternating*) *with respect to these indices*. A tensor, whether covariant or contravariant, is said to be *skew-symmetric* if it is skew-symmetric with respect to all pairs of indices. The transition from a certain order of the indices to another one can be done step by step by successive permutations of pairs of indices in different manners; it is known that the final result is independent of the choice of the

intermediate steps. If a tensor is skew-symmetric with respect to a pair of indices, say α and β , all its components for which α and β have the same value are zero. Examples of skew-symmetric tensors are contained in Section 33.

The property that a tensor is symmetric or skew-symmetric with respect to certain indices is independent of the choice of the coordinate system, as can be seen from (30.5).

31. Basic rules of tensor calculus. Two tensors are said to be of the same type if they have the same number of contravariant indices and the same number of covariant ones.

By addition of corresponding components of two tensors of the same type, $a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$ and $b_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$, we obtain a tensor with components

$$s_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s} = a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s} + b_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$$

which is called the *sum* of those tensors.

If we multiply every component of a tensor by a scalar we obtain a tensor of the same type. In consequence of this multiplication and of the above addition the totality of tensors of the same type form a vector space.

Any tensor of second order can be represented as a sum of a symmetric tensor and a skew-symmetric one; we have

$$a_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha}) + \frac{1}{2}(a_{\alpha\beta} - a_{\beta\alpha})$$

and the tensor in the first brackets is symmetric while the tensor in the second ones is skew-symmetric.

If we multiply every component of a tensor $a_{\alpha_1 \dots \alpha_p}^{\beta_1 \dots \beta_q}$ by every component of another (arbitrary) tensor $b_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s}$ we obtain a system of products

$$h_{\alpha_1 \dots \alpha_p \gamma_1 \dots \gamma_r}^{\beta_1 \dots \beta_q \kappa_1 \dots \kappa_s} = a_{\alpha_1 \dots \alpha_p}^{\beta_1 \dots \beta_q} b_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s}$$

which is called the *product* or *outer product* of these tensors. As can be seen from (30.5) the product of any two tensors is a tensor; the order of this tensor is the sum of the orders of the two given tensors; the number of covariant, or contravariant, indices of the product is equal to the sum of the numbers of covariant, or contravariant, indices of the given tensors.

Example 1. By means of the transformation law of the tensors $a_{\alpha\beta}$ and $b_{\gamma}{}^{\tau\epsilon}$ we find that the product $a_{\alpha\beta} b_{\gamma}{}^{\tau\epsilon}$ has the transformation law

$$\tilde{a}_{\kappa\lambda} \bar{b}_{\mu}{}^{\nu\sigma} = a_{\alpha\beta} b_{\gamma}{}^{\tau\epsilon} \frac{\partial u^\alpha}{\partial \tilde{u}^\kappa} \frac{\partial u^\beta}{\partial \tilde{u}^\lambda} \frac{\partial u^\gamma}{\partial \tilde{u}^\mu} \frac{\partial u^\tau}{\partial \tilde{u}^\nu} \frac{\partial u^\epsilon}{\partial \tilde{u}^\sigma},$$

that is, this product is a tensor of fifth order, covariant of third order, and contravariant of second order.

Another example of a product of tensors is the product of vectors which was considered in the preceding section.

If we equate two indices of a mixed tensor, a contravariant index and a covariant one, and sum with respect to this pair of indices we obtain a tensor of order one less contravariant and one less covariant, as follows from (30.5); this process is called *contraction* of the given tensor.

Example 2. By contraction of a mixed tensor of second order, for instance, a_α^β , we obtain a scalar,

$$a_\alpha^\alpha = a_1^1 + a_2^2 + \dots + a_n^n,$$

since, in consequence of (30.4) and (29.3),

$$\bar{a}_{\mu\nu}^\gamma = a_\alpha^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\nu}{\partial u^\beta} = a_\alpha^\beta \delta_\alpha^\beta = a_\alpha^\alpha.$$

Example 3. By contraction of the tensor $a_{\alpha\beta}^{\gamma\kappa\epsilon}$ with respect to β and γ we obtain the tensor $a_{\alpha\beta}^{\beta\kappa\epsilon}$ which is of third order, covariant of first order and contravariant of second order. Indeed, in consequence of (30.5) we have

$$\bar{a}_{\mu\nu}^{\nu\sigma\tau} = a_{\alpha\beta}^{\gamma\kappa\epsilon} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial u^\beta}{\partial \bar{u}^\nu} \frac{\partial \bar{u}^\nu}{\partial u^\kappa} \frac{\partial u^\sigma}{\partial \bar{u}^\epsilon} = a_{\alpha\beta}^{\gamma\kappa\epsilon} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial \bar{u}^\sigma}{\partial u^\kappa} \frac{\partial \bar{u}^\tau}{\partial u^\epsilon} \delta_\gamma^\beta,$$

therefore $\bar{a}_{\mu\nu}^{\nu\sigma\tau} = a_{\alpha\beta}^{\beta\kappa\epsilon} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial \bar{u}^\sigma}{\partial u^\kappa} \frac{\partial \bar{u}^\tau}{\partial u^\epsilon}.$

When contraction is applied to the product of two tensors with respect to indices of different factors of the product the resulting tensor is called an *inner product* of those tensors.

Example 4. The inner product of the vectors a^α and b_β is the scalar $a^\alpha b_\alpha$. Contraction may be applied in more than one way. From the tensors $a_{\alpha\beta}$ and $b^{\beta\kappa\epsilon}$ we can obtain different inner products, $a_{\alpha\beta} b^{\beta\kappa\epsilon}$, $a_{\alpha\beta} b^{\gamma\alpha\epsilon}$, etc. Contraction may also be applied more than once. When we apply contraction twice to the above product tensor $a_{\alpha\beta} b^{\beta\kappa\epsilon}$ we obtain covariant vectors such as $a_{\alpha\beta} b^{\alpha\beta\epsilon}$ or $a_{\alpha\beta} b^{\gamma\alpha\beta}$, etc.

Further examples will be included in the following sections.

Problem 31.1. Prove: If $a^{\alpha\beta}$ is a symmetric tensor and $b_{\gamma\kappa}$ is a skew-symmetric one, $a^{\alpha\beta} b_{\alpha\beta} = 0$.

Problem 31.2. Verify that if two coordinate transformations $\bar{u}^\beta = \bar{u}^\beta(u^1, u^2, \dots, u^n)$ ($\beta = 1, 2, \dots, n$), $\bar{u}^\gamma = \bar{u}^\gamma(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n)$ ($\gamma = 1, 2, \dots, n$), are carried out successively the components of a tensor with respect to the coordinates \bar{u}^γ are related to those with respect to the coordinates u^α by formulae of the type (30.5).

32. Vectors in a surface. The contravariant metric tensor. In the preceding three sections we have introduced the important concepts of contravariant and covariant transformation. We have also investigated the basic facts of tensor calculus. We will now apply the knowledge gained to problems of the theory of surfaces. We therefore *restrict the dimension of the Riemannian space again to the particular value $n = 2$, that is, we consider only surfaces. Hence, any summation to be carried out in consequence of the summation convention is to be understood as summation from 1 to 2.*

Only in spaces in which no metric tensor has been introduced, are contravariant and covariant vectors entirely different and independent things. In Riemannian spaces, however, we may correlate these two types of vectors and, moreover, interpret them geometrically in a simple manner.

We consider a surface S embedded in Euclidean space R_3 . We assume that x_1, x_2, x_3 are Cartesian coordinates in R_3 and u^1, u^2 are allowable coordinates on S .

A vector v bound at a regular point P of S which lies in the tangent plane $E(P)$ to S at P is called a *vector in the surface S at P* .

The vectors in S at P are therefore a subset of the set of all vectors bound at P . Of course, any such vector v is uniquely determined by its three components p_1, p_2, p_3 with respect to the Cartesian coordinate system in space R_3 and by the coordinates of P . However, it is natural to determine v with respect to a simple coordinate system in the tangent plane $E(P)$.

$E(P)$ is spanned by the vectors

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1} \quad \text{and} \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2},$$

cf. Section 26. These two vectors are tangent to the coordinate curves through P ; the lengths of these vectors are

$$\sqrt{(\mathbf{x}_1 \cdot \mathbf{x}_1)} = \sqrt{g_{11}} \quad \text{and} \quad \sqrt{(\mathbf{x}_2 \cdot \mathbf{x}_2)} = \sqrt{g_{22}},$$

respectively. We now introduce in $E(P)$ a system of parallel coordinates with origin at P so that the terminal points of the vectors \mathbf{x}_1 and \mathbf{x}_2 are the unit points on the axes of this system.

By definition, any vector v in the surface S at P lies in the tangent plane $E(P)$ and is therefore a linear combination of the vectors \mathbf{x}_1 and \mathbf{x}_2 , that is, is of the form

$$(32.1) \quad \mathbf{v} = a^\alpha \mathbf{x}_\alpha = a^1 \mathbf{x}_1 + a^2 \mathbf{x}_2, \quad (\mathbf{x}_1 \times \mathbf{x}_2 \neq 0).$$

If the coordinates u^1, u^2 on S are kept fixed, v is uniquely determined by the coordinates of the point P and the two numbers a^1 and a^2 . To every

ordered pair a^1, a^2 there corresponds a vector in S at P and vice versa; this correspondence is one-to-one. The numbers a^1, a^2 are the lengths (taken with a suitable sign) of the respective parallel projections of v upon the x_1 - and x_2 -axis of the coordinate system in $E(P)$, measured in units of $\sqrt{g_{11}}$ and $\sqrt{g_{22}}$, respectively. In other words, a^1 and a^2 are the components of v with respect to the coordinate system in $E(P)$. Cf. Fig. 38.

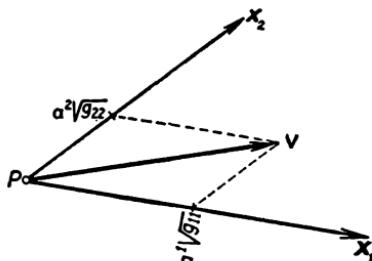


FIG. 38

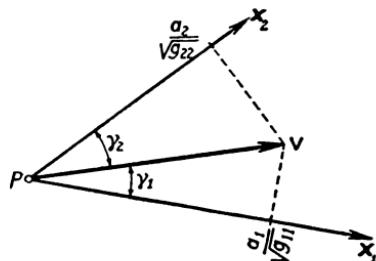


FIG. 39

Let now $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$, $\alpha = 1, 2$, be an allowable coordinate transformation and $\bar{u}^\beta = \bar{u}^\beta(u^1, u^2)$ its inverse. The vector v can be represented with respect to the new coordinates \bar{u}^1, \bar{u}^2 in a manner analogous to (32.1); we write

$$(32.1') \quad v = \bar{a}^\beta x_\beta,$$

$$\text{where} \quad x_\beta \equiv \frac{\partial \mathbf{x}}{\partial \bar{u}^\beta}.$$

Since

$$(32.2) \quad \mathbf{x}_\alpha = x_\beta \frac{\partial \bar{u}^\beta}{\partial u^\alpha},$$

we find

$$v = a^\alpha x_\alpha = a^\alpha x_\beta \frac{\partial \bar{u}^\beta}{\partial u^\alpha}.$$

By comparing this with (32.1') we obtain

$$\bar{a}^\beta = a^\alpha \frac{\partial \bar{u}^\beta}{\partial u^\alpha}, \quad (\beta = 1, 2).$$

These relations are of the form (29.4), that is, if the coordinates on S are transformed the quantities a^1, a^2 undergo a contravariant transformation. Thus a^1, a^2 are called the *contravariant components* of the vector v .

We sum up the results obtained:

Theorem 32.1. *The contravariant components a^α of a vector v in a surface S at a regular point P are the lengths (taken with a suitable sign) of the parallel*

projections of v on the axes of a parallel coordinate system in the tangent plane E(P) to S at P with origin at P and with the terminal points of the vectors $\mathbf{x}_1, \mathbf{x}_2$ as unit points on the axes; those lengths are measured in units of $\sqrt{g_{11}}$ and $\sqrt{g_{22}}$, respectively.

Instead of taking the parallel projections of v on the axes of the above coordinate system we can just as well take its orthogonal projections on these axes. Let γ_α be the angle between v and the \mathbf{x}_α -axis, $\alpha = 1, 2$. Then the length of the orthogonal projection of v upon the \mathbf{x}_α -axis is given by the expression

$$L_\alpha = |\mathbf{v}| \cos \gamma_\alpha = \frac{|\mathbf{x}_\alpha|}{|\mathbf{x}_\alpha|} |\mathbf{v}| \cos \gamma_\alpha = \frac{\mathbf{x}_\alpha \cdot \mathbf{v}}{\sqrt{(\mathbf{x}_\alpha \cdot \mathbf{x}_\alpha)}} = \frac{\mathbf{x}_\alpha \cdot \mathbf{v}}{\sqrt{g_{\alpha\alpha}}}, \quad (\alpha = 1, 2),$$

(do not sum with respect to α).

Hence the number

$$(32.3) \quad a_\alpha = \mathbf{x}_\alpha \cdot \mathbf{v}, \quad (\alpha = 1, 2),$$

is the length (taken with a suitable sign and measured in units of $1/\sqrt{g_{\alpha\alpha}}$) of the orthogonal projection of v on the \mathbf{x}_α -coordinate axis in $E(P)$, $\alpha = 1, 2$. Cf. Fig. 39, p. 102.

Of course these numbers depend on the choice of the coordinates on S . If we introduce new coordinates \bar{u}^1, \bar{u}^2 , then

$$\bar{a}_\beta = \mathbf{x}_\beta \cdot \mathbf{v} = \mathbf{x}_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^\beta} \cdot \mathbf{v} = a_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \quad (\beta = 1, 2).$$

These relations are of the form (29.5). The quantities a_1, a_2 are therefore called the *covariant components* of the vector v.

The correspondence between the ordered pairs (a_1, a_2) and the vectors in the surface S at the point P is one-to-one.

We have thus obtained the following result:

Theorem 32.2. *The covariant components a_α of a vector v in a surface S at a regular point P are the lengths (taken with a suitable sign and measured in units of $1/\sqrt{g_{\alpha\alpha}}$) of the orthogonal projections of v upon the axes of the coordinate system appearing in Theorem 32.1.*

The covariant components of a vector v in S at P can be obtained from its contravariant ones and vice versa. By inserting (32.1) in (32.3) we have

$$a_\alpha = \mathbf{x}_\alpha \cdot \mathbf{v} = \mathbf{x}_\alpha \cdot a^\beta \mathbf{x}_\beta;$$

hence, in consequence of (27.1),

$$(32.4) \quad a_\alpha = g_{\alpha\beta} a^\beta, \quad (\alpha = 1, 2).$$

Since $g = \det(g_{\alpha\beta}) \neq 0$ we can solve this system of linear equations for a^β ; we obtain

$$(32.5) \quad a^\beta = g^{\alpha\beta} a_\alpha, \quad (\beta = 1, 2),$$

where

$$(32.6) \quad g^{11} = \frac{g_{22}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g}.$$

Taking the inner product of both sides of (32.5) and an arbitrary covariant vector b_β we have

$$b_\beta a^\beta = g^{\alpha\beta} a_\alpha b_\beta.$$

The left-hand side of this expression is an invariant and consequently so is the right-hand side. Hence the quantities $g^{\alpha\beta}$ are the components of a contravariant tensor of second order which is called the *contravariant metric tensor*. Since $g_{\alpha\beta}$ is symmetric, $g^{\alpha\beta}$ is also symmetric.

Two tensors $a_{\alpha\beta}$ and $b^{\gamma\kappa}$ with non-vanishing determinant, $\det(a_{\alpha\beta}) \neq 0$, $\det(b^{\gamma\kappa}) \neq 0$, are said to be *conjugate* if the relations

$$a_{\alpha\beta} b^{\beta\gamma} = \delta_\alpha^\gamma = \begin{cases} 0 & (\alpha \neq \gamma) \\ 1 & (\alpha = \gamma) \end{cases}$$

hold, that is, if the corresponding matrices $(a_{\alpha\beta})$ and $(b^{\gamma\kappa})$ are inverse to each other. The covariant and the contravariant metric tensor are conjugate, for

$$(32.7) \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta.$$

In consequence of (32.6) we obtain

$$(32.8) \quad g^* \equiv g^{11} g^{22} - (g^{12})^2 = 1/g.$$

We mention the following fact. The tangent plane is spanned by the covariant basis vectors x_α (see above) or equally well by the contravariant basis vectors

$$x^\alpha = g^{\alpha\beta} x_\beta, \quad (\alpha = 1, 2),$$

in terms of which

$$x_\alpha = g_{\alpha\beta} x^\beta, \quad x_\alpha \cdot x^\beta = \delta_\alpha^\beta.$$

Corresponding to (32.3) we thus have

$$(32.3') \quad a^\alpha = x^\alpha \cdot v, \quad (\alpha = 1, 2).$$

If the coordinate system on the surface S is Cartesian at the point P , that is, if at P

$$g_{\alpha\beta} = \delta_{\alpha\beta} = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta), \end{cases}$$

then at this point $g = 1$ and therefore

$$g^{\alpha\beta} = \delta^{\alpha\beta} = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta). \end{cases}$$

Hence, in this case,

$$a^\alpha = a_\alpha, \quad (\alpha = 1, 2).$$

Indeed, then $\sqrt{g_{\alpha\alpha}} = 1/\sqrt{g_{\alpha\alpha}} = 1$ and the orthogonal projections of the above vector v coincide with its parallel projections. Consequently, in Euclidean spaces, when Cartesian coordinates have been introduced, there is no difference between covariant and contravariant components of a vector.

As we have seen, the transition from covariant to contravariant components (and vice versa) can be accomplished by taking the inner product of the components and the metric tensors. More generally, by means of the metric tensors, from a given tensor we can obtain tensors of the same order but of different type.

Examples.

$$a^{\alpha\beta} = g^{\alpha\gamma}a_\gamma^\beta, \quad a^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\tau}a_{\gamma\tau},$$

$$a_\alpha^\beta = g^{\beta\gamma}a_{\alpha\gamma},$$

$$a_{\alpha\beta} = g_{\alpha\gamma}a^\gamma_\beta, \quad a_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\tau}a^{\gamma\tau},$$

$$a_\kappa^{\alpha\beta} = g_{\kappa\gamma}a^{\gamma\alpha\beta}, \quad a_{\kappa\alpha}^\beta = g_{\kappa\gamma}g_{\alpha\sigma}a^{\gamma\sigma\beta}, \quad a_{\kappa\alpha\beta} = g_{\kappa\gamma}g_{\alpha\sigma}g_{\beta\tau}a^{\gamma\sigma\tau}.$$

We say that these tensors are associate to the given tensors by means of the metric tensors. We speak of this process as *lowering superscripts* by means of $g_{\alpha\beta}$ and *raising subscripts* by means of $g^{\alpha\beta}$.

This process is reversible. For example, if we take the inner product of a mixed tensor a_γ^β and $g^{\alpha\gamma}$,

$$g^{\alpha\gamma}a_\gamma^\beta = a^{\alpha\beta}$$

and then the inner product of $a^{\alpha\beta}$ and $g_{\alpha\kappa}$, we obtain the original mixed tensor,

$$g_{\alpha\kappa}a^{\alpha\beta} = g_{\alpha\kappa}g^{\alpha\gamma}a_\gamma^\beta = \delta_{\kappa}^{\gamma}a_\gamma^\beta = a_\kappa^\beta.$$

In consequence of this possibility of lowering and raising indices it is advantageous to write, for instance, $a_{\alpha_1\dots\alpha_r}{}^{\beta_1\dots\beta_s}$ and not $a_{\alpha_1\dots\alpha_r}^{\beta_1\dots\beta_s}$. At the beginning of an investigation the order of the covariant and contravariant indices of a tensor may be arbitrarily chosen; this order is then fixed in the course of this investigation.

33. Special tensors. Apart from the metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$, some other special tensors are of interest in the theory of surfaces.

As was pointed out in the preceding sections, a sufficiently large number of any functions can be taken as components of a tensor with respect to a certain coordinate system on a surface, and the components of this tensor with respect to any other allowable coordinate system are then determined by means of (30.5).

We take the Kronecker symbols δ_α^β as components of a mixed tensor of second order which is called *mixed ϵ -tensor*. This tensor has the same components with respect to any other coordinate system. Indeed, let \tilde{a}_ρ^σ denote these components; then

$$\tilde{a}_\rho^\sigma = \delta_\alpha^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\sigma}{\partial u^\beta} = \delta_\rho^\sigma.$$

The *covariant ϵ -tensor of second order* is defined by

$$(33.1) \quad \epsilon_{11} = 0, \quad \epsilon_{12} = \sqrt{g}, \quad \epsilon_{21} = -\sqrt{g}, \quad \epsilon_{22} = 0.$$

This tensor is skew-symmetric. In consequence of (30.3) we have

$$\tilde{\epsilon}_{\mu\nu} = \epsilon_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial u^\beta}{\partial \bar{u}^\nu}$$

or, since $\epsilon_{11} = \epsilon_{22} = 0$,

$$\tilde{\epsilon}_{\mu\nu} = \epsilon_{12} \frac{\partial u^1}{\partial \bar{u}^\mu} \frac{\partial u^2}{\partial \bar{u}^\nu} + \epsilon_{21} \frac{\partial u^2}{\partial \bar{u}^\mu} \frac{\partial u^1}{\partial \bar{u}^\nu},$$

that is,

$$\tilde{\epsilon}_{\mu\nu} = \sqrt{g} \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^\mu} & \frac{\partial u^2}{\partial \bar{u}^\mu} \\ \frac{\partial u^1}{\partial \bar{u}^\nu} & \frac{\partial u^2}{\partial \bar{u}^\nu} \end{vmatrix}.$$

Therefore $\tilde{\epsilon}_{11} = \tilde{\epsilon}_{22} = 0$ and

$$\tilde{\epsilon}_{12} = \sqrt{g} \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = \sqrt{g}, \quad \tilde{\epsilon}_{21} = -\sqrt{g} \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = -\sqrt{g},$$

cf. (28.4). The tensor

$$(33.2) \quad \epsilon^{\mu\nu} = \epsilon_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta}$$

is called the *contravariant ϵ -tensor of second order*. Since $\epsilon_{11} = \epsilon_{22} = 0$ we find

$$\epsilon^{\mu\nu} = \sqrt{g}(g^{\mu 1} g^{\nu 2} - g^{\mu 2} g^{\nu 1})$$

and therefore $\epsilon^{11} = \epsilon^{22} = 0$,

$$\epsilon^{12} = \sqrt{g}(g^{11} g^{22} - (g^{12})^2) = \frac{1}{\sqrt{g}},$$

cf. (32.8) and, similarly, $\epsilon^{21} = -\frac{1}{\sqrt{g}}$.

The contravariant ϵ -tensor has therefore the components

$$(33.3) \quad \epsilon^{11} = 0, \quad \epsilon^{12} = \frac{1}{\sqrt{g}}, \quad \epsilon^{21} = -\frac{1}{\sqrt{g}}, \quad \epsilon^{22} = 0.$$

Furthermore $\epsilon^{\alpha\gamma} \epsilon_{\beta\gamma} = \epsilon^{\alpha 1} \epsilon_{\beta 1} + \epsilon^{\alpha 2} \epsilon_{\beta 2} = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta), \end{cases}$

that is,

$$(33.4) \quad \epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta_\beta^\alpha.$$

$$\text{Similarly} \quad \epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = -\delta_\beta^\alpha.$$

A tensor whose components are all zero is called a *null tensor*. If for a tensor all components with respect to a certain coordinate system vanish, the components of this tensor with respect to any allowable coordinate system vanish; this is an immediate consequence of (30.5).

Problem 33.1. Determine the components of the covariant ϵ -tensor, starting from (33.3) and using (33.2).

Problem 33.2. Find a representation of the contravariant metric tensor in terms of the covariant one and ϵ -tensors.

Problem 33.3. Find the associate mixed tensor of the covariant metric tensor.

Problem 33.4. Evaluate the following expressions :

$$\delta_\alpha^\alpha, \quad \delta_\alpha^\beta\delta_\beta^\alpha, \quad \delta_\alpha^\beta\delta_\beta^\gamma\delta_\gamma^\alpha.$$

34. Normal to a surface. As was pointed out in Section 26, to any point of a surface $S: \mathbf{x}(u^1, u^2)$ we can associate the two vectors

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1} \quad \text{and} \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2}.$$

At any regular point P of S these vectors are linearly independent and are tangent to the coordinate curves through P ; they span the tangent plane $E(P)$ to S at P .

The unit vector \mathbf{n} which is orthogonal to those vectors and whose sense is such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{n}$, in this order, form a right-handed triple of vectors, is called the *unit normal vector* to S at P . Cf. Fig. 40, p. 108.

Obviously

$$(34.1) \quad \mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{+\sqrt{g}}$$

where the denominator $+\sqrt{g}$ follows from

$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = |\mathbf{x}_1|^2 |\mathbf{x}_2|^2 \sin^2 \alpha = g_{11} g_{22} (1 - \cos^2 \alpha) = g_{11} g_{22} - (\mathbf{x}_1 \cdot \mathbf{x}_2)^2 = g$;

in this expression, α is the angle between the vectors \mathbf{x}_1 and \mathbf{x}_2 .

The straight line through P in the direction of \mathbf{n} is called the *normal* to the surface S at the point P .

To any regular point of S there now corresponds a trihedron consisting of the vectors \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{n} . In contrast to the trihedron of a curve which

consists of three orthogonal unit vectors, the trihedron of a surface contains, in general, only one unit vector, namely \mathbf{n} , since

$$|\mathbf{x}_1| = \sqrt{g_{11}} \quad \text{and} \quad |\mathbf{x}_2| = \sqrt{g_{22}},$$

as follows from (27.1), and all three of these vectors are mutually orthogonal if and only if $\mathbf{x}_1 \cdot \mathbf{x}_2 = g_{12} = 0$.

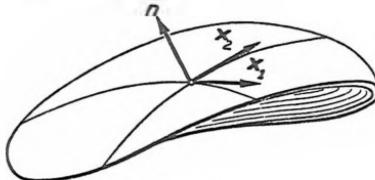


FIG. 40

The sense of the unit normal vector \mathbf{n} depends on the choice of the coordinates on S . After the coordinate transformation

$$u^1 = \bar{u}^2, \quad u^2 = \bar{u}^1$$

the sense of \mathbf{n} will change, since the Jacobian of this transformation has the value -1 . A transformation whose Jacobian is positive preserves the sense of \mathbf{n} while \mathbf{n} changes its sense under a transformation whose Jacobian is negative.

Example. The unit normal vector to the sphere (25.2) is directed away from the interior of the sphere. When, for example, the transformation $u^1 = -\bar{u}^1, u^2 = \bar{u}^2$ is imposed the sense of the normal vector changes.

A surface is said to be *orientable*, if the positive normal direction, when given at an arbitrary point P , can be continued to the whole surface in a unique way, that is, if there does not exist a closed curve C through P which is of such a type that, when displacing continuously the given positive normal direction from P along C and back to P , one obtains in the end a positive normal direction at P opposite to the given one.

A sufficiently small portion of a surface is always orientable. We notice that this may not hold in the large: There are non-orientable surfaces. A well-known example of a surface of this type is the *Möbius strip*, cf. Fig. 41. When a given positive normal direction at P_0 is continuously displaced along C the direction obtained upon returning to P_0 is opposite to the originally given one.

The allowable representations of a portion of a surface may be divided into two classes corresponding to the different possible orientations.

Obviously, any two representations belong to the same class if one of them can be transformed into the other by means of a transformation whose Jacobian is positive.

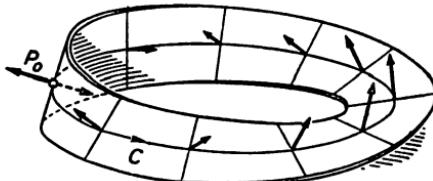


FIG. 41. Möbius strip

35. Measurement of lengths and angles in a surface. We have mentioned already in Section 27 that the first fundamental form enables us to measure lengths, angles, and areas in a surface.

Let $C: u^1 = u^1(t)$, $u^2 = u^2(t)$ be a curve on a surface S : $\mathbf{x}(u^1, u^2)$, cf. Section 26. Then, according to (9.1), the length of an arc of C bounded by the points corresponding to the values $t = t_0$ and $t = t_1$ is given by the integral

$$s = \int_{t_0}^{t_1} \sqrt{(\mathbf{x}' \cdot \mathbf{x}') dt} = \int_{t_0}^{t_1} \sqrt{(\mathbf{x}_\alpha u^\alpha' \cdot \mathbf{x}_\beta u^\beta') dt},$$

that is, in consequence of (27.1)

$$(35.1) \quad s = \int_{t_0}^{t_1} \sqrt{(g_{\alpha\beta} u^\alpha' u^\beta')} dt$$

where primes denote derivatives with respect to t .

We consider now the measurement of angles in a surface.

The scalar product of two vectors in the surface S ,

$$\mathbf{a} = a^\alpha \mathbf{x}_\alpha \quad \text{and} \quad \mathbf{b} = b^\beta \mathbf{x}_\beta,$$

which are bound at the same point P , is given by the expression

$$\mathbf{a} \cdot \mathbf{b} = (a^1 \mathbf{x}_1 + a^2 \mathbf{x}_2) \cdot (b^1 \mathbf{x}_1 + b^2 \mathbf{x}_2)$$

or,

$$(35.2) \quad \mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta,$$

cf. (27.1). In consequence of (32.5) and (32.7) we find

$$g_{\alpha\beta} a^\alpha b^\beta = g_{\alpha\beta} g^{\alpha\gamma} a_\gamma b^\beta = \delta_\beta^\gamma a_\gamma b^\beta = a_\beta b^\beta = g^{\alpha\beta} a_\alpha b_\beta.$$

Hence we may write

$$(35.2') \quad \mathbf{a} \cdot \mathbf{b} = a_\beta b^\beta$$

or

$$(35.2'') \quad \mathbf{a} \cdot \mathbf{b} = g^{\alpha\beta} a_\alpha b_\beta.$$

The vector \mathbf{a} therefore has the length

$$(35.3) \quad |\mathbf{a}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})} = \sqrt{(g_{\alpha\beta} a^\alpha a^\beta)} = \sqrt{(a_\alpha a^\alpha)} = \sqrt{(g^{\alpha\beta} a_\alpha a_\beta)}.$$

We denote by γ the angle between the vectors \mathbf{a} and \mathbf{b} . Then, according to (5.6),

$$(35.4) \quad \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{g_{\alpha\beta} a^\alpha b^\beta}{\sqrt{(g_{\mu\nu} a^\mu a^\nu)} \sqrt{(g_{\sigma\tau} b^\sigma b^\tau)}} \\ = \frac{a_\alpha b^\alpha}{\sqrt{(a_\beta a^\beta)} \sqrt{(b_\gamma b^\gamma)}} = \frac{g^{\alpha\beta} a_\alpha b_\beta}{\sqrt{(g^{\mu\nu} a_\mu a_\nu)} \sqrt{(g^{\sigma\tau} b_\sigma b_\tau)}}.$$

We now consider two curves

$$C: u^\alpha = h^\alpha(t) \quad \text{and} \quad C^*: u^\alpha = h^{\alpha*}(t), \quad (\alpha = 1, 2),$$

on a surface $S: \mathbf{x}(u^1, u^2)$ which intersect at a point P of S . By definition, the angle γ of intersection is the angle between the vector

$$\mathbf{v} = \frac{d}{dt} \{ \mathbf{x}(h^1(t), h^2(t)) \} = \mathbf{x}_\alpha h^\alpha,$$

tangent to C at P , and the vector

$$\mathbf{v}^* = \frac{d}{dt} \{ \mathbf{x}(h^{1*}(t), h^{2*}(t)) \} = \mathbf{x}_\beta h^{\beta*},$$

tangent to C^* at P . According to (35.4) we therefore obtain

$$(35.5) \quad \cos \gamma = \frac{\mathbf{v} \cdot \mathbf{v}^*}{|\mathbf{v}| |\mathbf{v}^*|} = \frac{g_{\alpha\beta} h^\alpha h^{\beta*}}{\sqrt{(g_{\mu\nu} h^\mu h^\nu)} \sqrt{(g_{\sigma\tau} h^{\sigma*} h^{\tau*})}},$$

etc. Hence, C and C^* intersect orthogonally if and only if at P

$$g_{\alpha\beta} h^\alpha h^{\beta*} = 0.$$

The vectors \mathbf{x}_1 and \mathbf{x}_2 are tangent to the respective coordinate curves $u^2 = \text{const.}$ and $u^1 = \text{const.}$ Denoting by ω the angle of intersection of these curves we have

$$(35.6) \quad \cos \omega = \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_1| |\mathbf{x}_2|} = \frac{g_{12}}{\sqrt{(g_{11} g_{22})}}.$$

An allowable coordinate system on a surface S is said to be *orthogonal* if, at any point of S , the coordinate curves intersect orthogonally. From (35.6) we obtain the following important

Theorem 35.1. *A coordinate system on a surface S is orthogonal if and only if at any point of S*

$$(35.7) \quad g_{12} = 0.$$

In many connexions the choice of orthogonal coordinates is of great advantage since the formulae become simpler if such a coordinate system

has been chosen. Examples of orthogonal coordinates have occurred already in Section 27; applications of these coordinates will follow.

For further purposes we shall need the formula

$$(35.8) \quad \sin \gamma = \sqrt{g} \frac{|a^1 b^2 - a^2 b^1|}{\sqrt{(g_{\alpha\beta} a^\alpha a^\beta)} \sqrt{(g_{\mu\nu} b^\mu b^\nu)}} \quad (0 \leq \gamma \leq \pi),$$

which can immediately be obtained from (35.4) by using the relation $\sin^2 \gamma = 1 - \cos^2 \gamma$. Since

$$a^1 b^2 - a^2 b^1 = g^{1\alpha} a_\alpha g^{2\beta} b_\beta - g^{2\alpha} a_\alpha g^{1\beta} b_\beta = \frac{1}{g} (a_1 b_2 - a_2 b_1)$$

we obtain from (35.8)

$$(35.9) \quad \sin \gamma = \frac{|a_1 b_2 - a_2 b_1|}{\sqrt{g} \sqrt{(g^{\alpha\beta} a_\alpha a_\beta)} \sqrt{(g^{\mu\nu} b_\mu b_\nu)}} \quad (0 \leq \gamma \leq \pi).$$

The angle γ ($0 \leq \gamma \leq \pi$) is uniquely determined by its cosine, but not by its sine. If $0 \leq \gamma \leq \pi$ then $\sin \gamma \geq 0$, that is, the right-hand sides of (35.8) and (35.9) must be positive.

Problem 35.1. Introduce orthogonal coordinates on a general helicoid S such that the helices and their orthogonal trajectories on S are coordinate curves. Cf. Problem 28.5.

36. Area. The definition of surface area resembles that of the length of an arc. When introducing the concept of the length we approximated the given arc by a broken line of chords. Similarly, we have now to approximate the given portion of a surface by small portions of planes. The method used will not be exactly analogous to that of Section 9; the reasons for the difference in the two investigations will be explained in Section 37.

Let $\mathbf{x}(u^1, u^2)$ be an allowable representation of a curved surface S . We consider a portion H of S whose orthogonal projection T in the $x_1 x_2$ -plane is one-to-one and whose normal, at any point of H , does not make a right angle with the x_3 -axis. We denote by \bar{H} the image of H under the projection T . The boundary C of H is assumed to be composed of finitely many arcs of curves; then the image \bar{C} of C in the $x_1 x_2$ -plane has the same property and, consequently, the area of \bar{H} exists. We subdivide \bar{H} into finitely many portions \bar{h}_n so that the area of any \bar{h}_n exists. Let h_n be the inverse image of \bar{h}_n . The subdivision of \bar{H} corresponds to a subdivision of H into finitely many portions h_n . Let P_n be an arbitrary point of h_n . We denote by h_n^* the image of h_n with respect to a parallel projection of h_n into the tangent

plane to H at P_n . The projecting lines are assumed to be parallel to the x_3 -axis, cf. Fig. 42. The 'smaller' h_n (that is, the smaller the maximum distance of any two points of h_n) the less h_n^* will differ from h_n . Hence we may approximate the area $A(h_n)$ of h_n by the area $A(h_n^*)$ of the plane portion

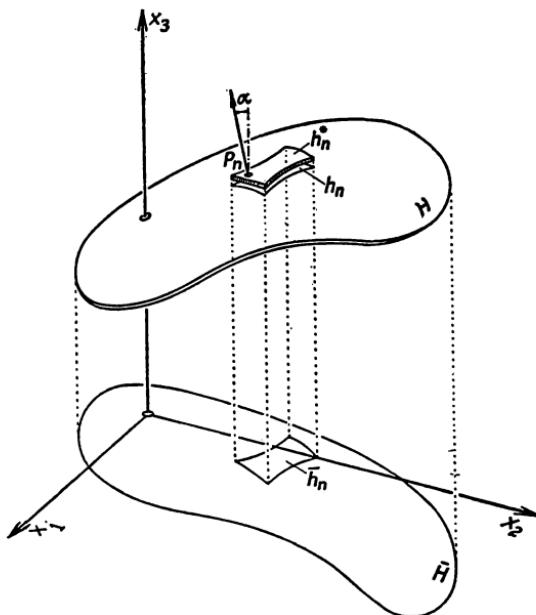


FIG. 42

h_n^* , which area is known. We denote by $\alpha(P_n)$ the angle between the normal to H at P_n and the x_3 -axis. Let $A(\bar{h}_n)$ be the area of \bar{h}_n . Then

$$(36.1) \quad A(h_n^*) = \frac{A(\bar{h}_n)}{\cos \alpha(P_n)}, \quad (\alpha(P_n) \neq \frac{1}{2}\pi).$$

When developing the determinant occurring in (26.4) by its first column we find

$$\sum_{i=1}^3 D_i(y_i - x_i) = 0$$

where

$$D_1 = \frac{\partial(x_2, x_3)}{\partial(u^1, u^2)}, \quad D_2 = \frac{\partial(x_3, x_1)}{\partial(u^1, u^2)}, \quad D_3 = \frac{\partial(x_1, x_2)}{\partial(u^1, u^2)},$$

$$\frac{\partial(x_i, x_k)}{\partial(u^1, u^2)} \equiv \begin{vmatrix} \frac{\partial x_i}{\partial u^1} & \frac{\partial x_i}{\partial u^2} \\ \frac{\partial x_k}{\partial u^1} & \frac{\partial x_k}{\partial u^2} \end{vmatrix}.$$

Therefore

$$\cos \alpha(P_n) = \frac{D_3}{\sqrt{\left(\sum_{i=1}^3 D_i^2\right)}}.$$

This expression holds for one of the two possible directions of the normal to H . If we direct this normal so that $\alpha(P_n) < \frac{1}{2}\pi$ then $\cos \alpha(P_n) > 0$, and

$$(36.2) \quad \cos \alpha(P_n) = \frac{|D_3|}{\sqrt{\left(\sum_{i=1}^3 D_i^2\right)}} = \frac{|D_3|}{\sqrt{g}},$$

cf. (28.1) and (28.2). Let N be the number of portions h_n . We define the area of H by the expression

$$(36.3) \quad A(H) = \lim_{N \rightarrow \infty} \sum_{n=1}^N A(h_n^*),$$

where the limit process has to be carried out in such a manner that the maximum distance between any two points of any h_n^* ultimately becomes arbitrarily small. Using (36.1) and (36.2) we obtain from (36.3)

$$(36.4) \quad A(H) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{\sqrt{g(\bar{P}_n)}}{|D_3(\bar{P}_n)|} A(h_n),$$

where \bar{P}_n is the image of P_n with respect to the above projection T . We will prove that the limit (36.4) exists and is given by the integral

$$(36.5) \quad A(H) = \iint_H \frac{\sqrt{g}}{|D_3|} dx_1 dx_2.$$

Clearly, when introducing the variables $u^1 u^2$ then, by the general law of transformation of double integrals, (36.5) becomes

$$(36.6) \quad A(H) = \iint_U \frac{\sqrt{g}}{|D_3|} |D_3| du^1 du^2 = \iint_U \sqrt{g} du^1 du^2,$$

where U is the domain in the $u^1 u^2$ -plane corresponding to H . Furthermore, (36.6) is invariant with respect to any allowable coordinate transformation. Indeed, when imposing such a transformation, the integrand of (36.6) is multiplied by the absolute value of the corresponding Jacobian while \sqrt{g} is multiplied by the absolute value of the Jacobian corresponding to the inverse transformation, cf. (28.4); the product of these two Jacobians has the value 1, cf. Problem 28.4. Moreover, when applying (36.6) to a portion of a plane, this formula becomes identical with the elementary formula by which plane areas are defined.

We will now prove the above statement. Since the representation of H is allowable, the function $Q \equiv \sqrt{g}/|D_3|$ is continuous and bounded in the domain under consideration. Hence, by applying to (36.5) the mean value theorem on surface integrals we obtain

$$A(H) = \sum_{n=1}^N \iint_{\tilde{h}_n} Q dx_1 dx_2 = \sum_{n=1}^N Q(\bar{P}'_n) A(\tilde{h}_n),$$

where \bar{P}'_n is a point of \tilde{h}_n . Q is uniformly continuous in the domain under consideration. Therefore, a number $\epsilon > 0$ being given we can make all \tilde{h}_n so small that the variation of Q on any \tilde{h}_n becomes smaller than or equal to ϵ . Consequently, we can estimate the difference

$$d \equiv \sum_{n=1}^N [Q(\bar{P}'_n) - Q(\bar{P}_n)] A(\tilde{h}_n)$$

of (36.5) and the (finite) sum occurring in (36.4) in the following manner:

$$|d_n| \leq \sum_{n=1}^N |[Q(\bar{P}'_n) - Q(\bar{P}_n)]| A(\tilde{h}_n) \leq \epsilon \sum_{n=1}^N A(\tilde{h}_n) = \epsilon A(\bar{H}).$$

Hence, the limit occurring in (36.4) exists and is given by (36.5).

We sum up our results:

Definition 36.1. *The area $A(H)$ of a portion H of a surface $S: \mathbf{x}(u^1, u^2)$ is defined by the double integral*

$$(36.7) \quad A(H) = \iint_U \sqrt{g} du^1 du^2,$$

where U denotes the domain in the $u^1 u^2$ -plane corresponding to H . The expression

$$(36.8) \quad dA = \sqrt{g} du^1 du^2$$

is called the element of area of H .

(36.8) can be geometrically interpreted as the area of an ‘infinitesimal parallelogram’ whose sides are the vectors $\mathbf{x}_1 du^1$ and $\mathbf{x}_2 du^2$. Indeed, from the definition of a vector product it follows that the parallelogram has the area

$$|\mathbf{x}_1 du^1 \times \mathbf{x}_2 du^2| = |\mathbf{x}_1 \times \mathbf{x}_2| du^1 du^2.$$

Using Lagrange’s identity (5.11) we find

$$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = (\mathbf{x}_1 \times \mathbf{x}_2) \cdot (\mathbf{x}_1 \times \mathbf{x}_2) = (\mathbf{x}_1 \cdot \mathbf{x}_1)(\mathbf{x}_2 \cdot \mathbf{x}_2) - (\mathbf{x}_1 \cdot \mathbf{x}_2)^2 = g.$$

From this, the result follows.

In Definition 36.1 the square root \sqrt{g} has a positive sign, therefore $A(H) > 0$. In addition to the area as defined above one may consider also

signed areas of portions of surfaces; in this case the area corresponding to one of the two possible orientations is defined by (36.7) while the area corresponding to the other orientation is defined by $-A(H)$.

In consequence of the investigations contained in this and the preceding section we have the following result:

The coefficients $g_{\alpha\beta}$ of the first fundamental form enable us to measure lengths, angles, and areas in a surface; by the tensor $g_{\alpha\beta}$ the metric in a surface is thus completely determined. This is the justification of the name *metric tensor*.

Example. In the case of the sphere (25.2) we have

$$g_{11} = r^2 \cos^2 u^2, \quad g_{12} = 0, \quad g_{22} = r^2, \quad dA = r^2 \cos u^2 du^1 du^2.$$

The total area of the sphere is therefore

$$A = 4\pi r^2.$$

A further simple example is included in Section 43.

37. Remarks on the definition of area. It is natural to compare the investigations which led to the integrals (9.1) and (36.7) by which the arc length and the area, respectively, are defined.

The introduction of the concept of area in a manner completely analogous to that of Section 9 would require the consideration of polyhedra whose vertices lie on the portion of the surface under consideration. But, unless certain restrictions are imposed, this method would not lead to a uniquely determined limit, as can be seen by the following simple example, cf. H. A. Schwarz [1, pp. 309–11].

We consider the cylinder Z : $x_1^2 + x_2^2 = 1$ and undertake to find the area of the portion H of Z defined by $0 \leq x_3 \leq 1$. We construct a portion P of a polyhedron consisting of congruent triangles D in the following manner. We take on H equidistant circles parallel to the $x_1 x_2$ -plane, defined by

$$(37.1) \quad x_3 = 0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, \frac{k}{k}.$$

Obviously, the circles $x_3 = 0$ and $x_3 = 1$ form the boundary of H . We divide each of these circles into n congruent arcs so that the endpoints of the arcs on one circle lie over the midpoints of the arcs of the preceding circle, cf. Fig. 43. When joining the endpoints of the arcs on any circle we obtain a regular n -sided polygon. We now join every side S of these polygons with those two vertices of the adjacent polygons which are at a minimum distance from S . In this manner we obtain the portion P of

a polyhedron consisting of $2kn$ congruent isosceles triangles D , cf. Fig. 43. The sides of the above polygons have the length $2\sin(\pi/n)$, and, according to the theorem of Pythagoras, the altitude of each triangle D is

$$a = \sqrt{\left(\frac{1}{k^2} + \left(1 - \cos \frac{\pi}{n}\right)^2\right)} = \sqrt{\left(\frac{1}{k^2} + 4\sin^4 \frac{\pi}{2n}\right)}.$$

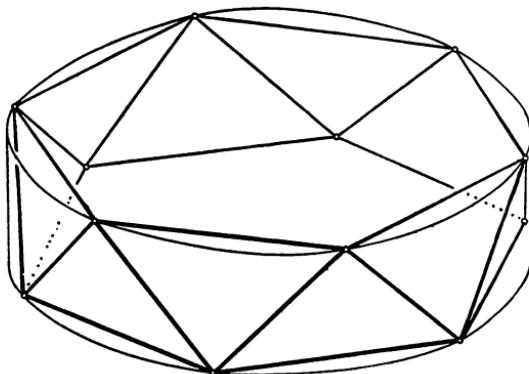


FIG. 43. Two neighbouring circles (37.1) and the corresponding triangles

The area of P therefore becomes

$$A(k, n) = 2n \sin \frac{\pi}{n} \sqrt{\left(1 + 4k^2 \sin^4 \frac{\pi}{2n}\right)}.$$

The limit of this expression depends on the manner in which k and n tend to infinity: First of all, if n is kept fixed, then $\lim_{k \rightarrow \infty} A(k, n) = \infty$. The reason for this fact is simple: The angle between each triangle D and the tangent plane to the cylinder tends to $\frac{1}{2}\pi$, the area of each triangle tends to a finite value different from zero, and the number of triangles increases arbitrarily. In this case P is, so to speak, pleated. Let now both k and n increase arbitrarily so that always $k = n^s$. Then, if $s = 1$, $A(k, n)$ tends to 2π ; if $s = 2$, $A(k, n)$ tends to $2\pi\sqrt{(1 + \pi^4/4)}$; if $s = 3$, $A(k, n)$ tends to infinity as can be seen from

$$A(n^s, n) = 2n \sin \frac{\pi}{n} \sqrt{\left(1 + 4n^{2s} \sin^4 \frac{\pi}{2n}\right)}.$$

We have thus obtained the following result: When a sequence of polyhedra converges to a portion S of a surface the area of these polyhedra may not converge to the area of S . Analogous difficulties do not arise in connexion with the definition of the arc length, since, according to the mean value

theorem of differential calculus, the direction of a chord AB of a curve C tends to the direction of the tangent to C at A if B tends to A .

We finally mention—without proof—the fact that the area of a portion S of a curved surface can be defined by using sequences of polyhedra with vertices on S . In the case of the above example we have

$$A(k, n) \geq 2n \sin \frac{\pi}{n}.$$

In the case of any surface the limit of the areas of polyhedra of the type under consideration (if it exists) is always greater than, or at least equal to, the area of the curved portion S of the surface. If for every sequence of such polyhedra we find the lower limit of the areas, these numbers form a set of numbers associated with S . The area of S can be defined as the lower limit (lower point of accumulation) of this set of numbers.

Problem 37.1. Compute the area of the portion of the cone (25.4) which is determined by $0 \leq x_3 \leq B$.

Problem 37.2. Determine the element of area of a surface represented in the form (24.3): $x_3 = F(x_1, x_2)$.

Problem 37.3. Determine the element of area of an arbitrary surface of revolution, and use the expression obtained to compute the area of the sphere.

SECOND FUNDAMENTAL FORM. GAUSSIAN AND MEAN CURVATURE OF A SURFACE

38. Second fundamental form. In this and the following three sections we will consider the geometric shape of a surface in the neighbourhood of any of its points. Of course, this problem is of fundamental importance. In this connexion it is natural to start from the consideration of the curvature of curves on a surface.

We first state the following simple fact: At any point of a curve C on a surface S the corresponding unit normal vector \mathbf{n} to S lies in the normal

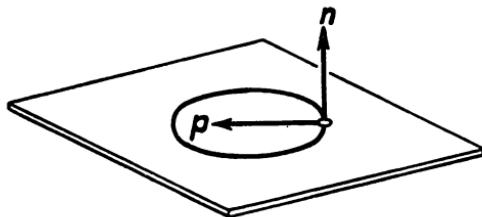


FIG. 44

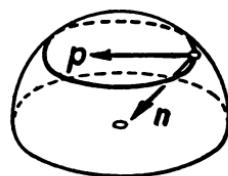


FIG. 45

plane to C which also contains the unit principal normal vector \mathbf{p} to C ; the angle between these two vectors will depend on the geometric shape of C and S in a neighbourhood of the point under consideration. Thus for any plane curve the vectors \mathbf{p} and \mathbf{n} are orthogonal, cf. Fig. 44. In the case of a circle C through a fixed point P of a sphere S , \mathbf{p} and \mathbf{n} lie in the same line if and only if C is a great circle. Assuming that \mathbf{n} is directed towards the interior of the sphere the angle between \mathbf{p} and \mathbf{n} increases with decreasing radius r of C . Cf. Fig. 45.

We now consider an arbitrary surface $S: \mathbf{x}(u^1, u^2)$ of class $r \geq 2$ and on S an arbitrary curve C : $u^1 = u^1(s), \quad u^2 = u^2(s)$

of class $r \geq 2$, where s is the arc length of C . We denote by γ the angle between the unit principal vector \mathbf{p} to C and the unit normal vector \mathbf{n} to S . Since \mathbf{p} and \mathbf{n} are unit vectors we have

$$\cos \gamma = \mathbf{p} \cdot \mathbf{n}, \quad (\kappa > 0).$$

The value of this scalar product will, in general, vary along the curve C .

Since $\mathbf{p} = \dot{\mathbf{x}}/\kappa$, cf. (15.1), we find

$$(38.1) \quad \kappa \cos \gamma = \dot{\mathbf{x}} \cdot \mathbf{n}.$$

We again use the notation

$$\mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial u^\alpha}, \quad \mathbf{x}_{\alpha\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \quad \dots$$

$$\text{Now } \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial u^1} \frac{du^1}{ds} + \frac{\partial \mathbf{x}}{\partial u^2} \frac{du^2}{ds} = \mathbf{x}_\alpha \dot{u}^\alpha$$

and, by differentiation,

$$\ddot{\mathbf{x}} = \mathbf{x}_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta + \mathbf{x}_\alpha \ddot{u}^\alpha.$$

Now $\mathbf{x}_\alpha \cdot \mathbf{n} = 0$. Therefore, from the preceding expression,

$$\ddot{\mathbf{x}} \cdot \mathbf{n} = (\mathbf{x}_{\alpha\beta} \cdot \mathbf{n}) \dot{u}^\alpha \dot{u}^\beta.$$

We now introduce the important abbreviated notation

$$(38.2) \quad b_{\alpha\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{n}, \quad (\alpha = 1, 2; \beta = 1, 2).$$

The values of these scalar products depend on the surface S , but are independent of the curve C on S under consideration. Since $\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha}$ the quantities $b_{\alpha\beta}$ are symmetric,

$$b_{\alpha\beta} = b_{\beta\alpha}.$$

The quadratic form

$$(38.3) \quad b_{\alpha\beta} du^\alpha du^\beta,$$

that is,

$$b_{11}(du^1)^2 + 2b_{12}du^1 du^2 + b_{22}(du^2)^2,$$

is called the *second fundamental form*.

In addition to (38.2) the notation

$$b_{11} = L, \quad b_{12} = b_{21} = M, \quad b_{22} = N$$

can also be found in literature; then the second fundamental form is given by the expression

$$L(du^1)^2 + 2Md u^1 du^2 + N(du^2)^2.$$

For reasons similar to those mentioned in Section 27 we will use the notation (38.2) exclusively.

Differentiating the orthogonality relation $\mathbf{x}_\alpha \cdot \mathbf{n} = 0$ we have

$$\mathbf{x}_{\alpha\beta} \cdot \mathbf{n} + \mathbf{x}_\alpha \cdot \mathbf{n}_\beta = 0,$$

where

$$\mathbf{n}_\beta \equiv \frac{\partial \mathbf{n}}{\partial u^\beta}.$$

By comparing this with (38.2) we obtain the important relations

$$(38.4) \quad b_{\alpha\beta} = -\mathbf{x}_\alpha \cdot \mathbf{n}_\beta.$$

From (9.1) and (27.2) we find

$$(38.5) \quad ds^2 = g_{\alpha\beta} du^\alpha du^\beta = d\mathbf{x} \cdot d\mathbf{x}.$$

From (38.4) we obtain analogously

$$(38.6) \quad b_{\alpha\beta} du^\alpha du^\beta = -d\mathbf{x} \cdot d\mathbf{n}.$$

The second fundamental form is therefore invariant with respect to any allowable coordinate transformation which preserves the sense of \mathbf{n} . Since the coordinate differentials du^α have a contravariant transformation behaviour, cf. Section 29, and the second fundamental form is invariant, the coefficients $b_{\alpha\beta}$ of this form are the components of a covariant tensor of second order with respect to those coordinate transformations.

Now let t be any allowable parameter of the above curve C . Then

$$\dot{u}^\alpha = \frac{du^\alpha}{dt} \frac{dt}{ds} = \frac{u'^\alpha}{s'}.$$

Hence, in consequence of (38.3), (38.1) takes the form

$$\kappa \cos \gamma = \frac{b_{\alpha\beta} u^\alpha u^\beta}{(s')^2} = \frac{b_{\alpha\beta} u^\alpha u^\beta}{g_{\alpha\beta} u^\alpha u^\beta},$$

or

$$(38.7) \quad \kappa \cos \gamma = \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}.$$

Because of (34.1) we can represent the coefficients $b_{\alpha\beta}$ of the second fundamental form in the following manner:

$$b_{\alpha\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{n} = \mathbf{x}_{\alpha\beta} \cdot \left(\frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g}} \right),$$

hence

$$(38.8) \quad b_{\alpha\beta} = \frac{1}{\sqrt{g}} |\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_{\alpha\beta}|.$$

While the first fundamental form is positive definite, cf. Section 28, the second fundamental form may vanish, cf. the following section. The discriminant $b = b_{11} b_{22} - b_{12}^2$ of the second fundamental form will be considered in Section 40.

Problem 38.1. Determine the second fundamental form of a surface represented in the form $x_3 = F(x_1, x_2)$, cf. (24.3).

Problem 38.2. Prove by direct calculation that the coefficients of the second fundamental form are the components of a covariant tensor of second order with respect to coordinate transformations which preserve the sense of the unit normal vector \mathbf{n} .

39. Arbitrary and normal sections of a surface. Meusnier's theorem. Asymptotic lines. We are going to study the geometric shape of a surface S in the neighbourhood of any of its points. For this purpose we have considered curves on S passing through a point of S ; we have obtained the important relation (38.7)

$$\kappa \cos \gamma = \frac{b_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta}{g_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta} = \frac{b_{\alpha\beta} du^\alpha dw^\beta}{g_{\alpha\beta} du^\alpha dw^\beta},$$

where $\kappa = \kappa(s)$ is the curvature of a curve C : $u^1 = u^1(s)$, $u^2 = u^2(s)$ on a surface S : $x(u^1, u^2)$ and $\gamma = \gamma(s)$ is the angle between the unit principal normal vector $\mathbf{p}(s)$ to C and the corresponding unit normal vector $\mathbf{n}(u^1(s), u^2(s))$ to S . We will now find a geometrical interpretation of (38.7).

Obviously, the direction of the unit tangent vector $\mathbf{t} = \dot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha$ to C is determined by the ratio $\dot{u}^1 : \dot{u}^2$. If we consider any fixed point P of S the values of the coefficients $g_{\alpha\beta}$ and $b_{\alpha\beta}$ of the fundamental forms are fixed, that is, independent of the special choice of the curve C on S passing through P . This means that the curvature κ of C at P depends only on the direction of the unit tangent vector and of the unit principal normal vector to C at P . Since these vectors span the osculating plane we have

Theorem 39.1. *All curves of class $r \geq 2$ on a surface S which pass through any fixed point P and have at P the same osculating plane, which does not coincide with the tangent plane of S at P , also have the same curvature at P .*

The set of these curves includes a plane curve, namely the curve of intersection of S and the common osculating plane of the curves at P . We may therefore restrict our investigation to plane curves on S .

We can draw further conclusions from (38.7). For this purpose we consider all curves whose tangents at P have the same direction; if this direction is fixed the right-hand side of (38.7) is constant, that is, the curvature of these curves depends only on the angle γ between their unit principal normal vector \mathbf{p} at P and the corresponding unit normal vector \mathbf{n} to S . Hence, for those curves which have at P the same direction the following relation holds:

$$(39.1) \quad \kappa \cos \gamma = \kappa_n$$

where κ_n , when a fixed tangent direction has been chosen, is a constant. κ_n can be geometrically interpreted as follows: If $\gamma = 0$ then $\kappa = \kappa_n$. If $\gamma = \pi$ then $\kappa = -\kappa_n$. $|\kappa_n|$ is therefore the curvature of the curve of intersection of the surface S and a plane passing through both the tangent to

the curve at P and the normal to S at P . Curves of this type are called *normal sections* of S .

The value of κ_n depends only on the direction of the tangent to the normal section at P . κ_n is called the *normal curvature* of S at P corresponding to the direction of the tangent under consideration. We introduce the *normal curvature vector*

$$\mathbf{k}_n = \kappa_n \mathbf{n}.$$

From (38.7) and (39.1) we obtain

$$(39.2) \quad \kappa_n = \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta},$$

that is, the normal curvature is the quotient of the second and first fundamental forms.

It should be noticed that the sign of the normal curvature is immaterial: We remember that the sense of \mathbf{n} depends on the choice of the coordinates on the surface S , cf. Section 34. If we introduce coordinates on S so that at P the vector \mathbf{n} and the unit principal normal vector \mathbf{p} to a normal section coincide then $\gamma = 0$, and the curvature κ of C at P is equal to the corresponding normal curvature κ_n . If we change the sense of \mathbf{n} then $\gamma = \pi$ and $\kappa = -\kappa_n$. Since we will only be interested in comparing the values of the normal curvature corresponding to different directions at P we could have defined the normal curvature by $-\kappa_n$ as well as by κ_n .

We have already stressed the fact that the normal curvature depends on the direction of the tangent to a normal section at P . Directions at P for which $\kappa_n = 0$ are called *asymptotic directions*. A curve, for which, at any of its points, the tangent has an asymptotic direction is called an *asymptotic curve*. We will consider these curves later.

Excluding the case $\kappa_n = 0$ we may set

$$\kappa_n = \frac{1}{R}, \quad (\kappa_n \neq 0).$$

Obviously, $|R|$ is the radius of curvature of the corresponding normal section at the point under consideration. Since $\kappa = 1/\rho$ we may now write (39.1) in the form

$$(39.3) \quad \rho = R \cos \gamma.$$

In this manner we obtain the following result (cf. Fig. 46):

Theorem 39.2 (J. B. M. Meusnier [1]). *The centre of curvature of all curves on a surface S which pass through an arbitrary (fixed) point P and whose tangents at P have the same direction, different from an asymptotic direction, lie on a circle K of radius $\frac{1}{2}|R|$ which lies in the normal plane and has a contact of first order (at least) with S at P .*

The circles of curvature of all plane sections of S with common tangent at P whose direction is different from an asymptotic direction lie therefore on a sphere.

In consequence of the theorem of Meusnier we may restrict our considerations to the normal sections of S at P .

We will now study asymptotic directions. In consequence of (39.2) κ_n vanishes exactly for those directions for which the second fundamental form vanishes. The differential equation of the asymptotic curves on a surface is therefore

$$(39.4) \quad b_{\alpha\beta} du^\alpha du^\beta = 0.$$

The existence of (real) asymptotic curves, that is, the existence of real solutions of (39.4), depends on the shape of a surface; this will be seen in the following section. The name 'asymptotic direction' will be justified by the consideration of the indicatrix of Dupin, cf. Section 42.

From (39.1) we see that the curvature of a normal section in an asymptotic direction vanishes.

By means of (15.1), (38.1), and (38.7) we obtain from (39.4)

$$(39.5) \quad \dot{x} \cdot n = \kappa p \cdot n = 0.$$

We thus have the following simple

Theorem 39.3. *Any straight line on a surface of class $r \geq 2$ is an asymptotic curve.*

Thus the generating straight lines of a cylinder or a cone are asymptotic curves on these surfaces.

From (39.5) it follows that at each point of an asymptotic curve C , at which $\kappa > 0$, the principal normal of C lies in the tangent plane of the surface. From this we obtain the following important property of asymptotic curves:

Theorem 39.4. *At any point of an asymptotic curve C : $x(t)$, for which $\kappa > 0$, the binormal of C and the normal to the surface coincide; consequently, at any point of C , the osculating plane of C and the tangent plane to the surface then coincide.*

Furthermore, in consequence of (39.4) we obtain

Theorem 39.5. *The coordinate curves of coordinates on a surface are asymptotic curves if and only if with respect to these coordinates*

$$(39.6) \quad b_{11} = 0 \quad \text{and} \quad b_{22} = 0.$$

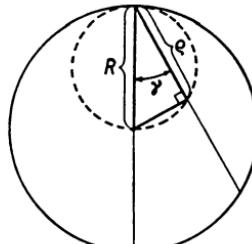


FIG. 46

Of course, coordinates of this type can be introduced on a surface S if and only if at any point of S there are two (different real) asymptotic directions; we will point out in the following section that this condition holds if S consists of 'hyperbolic points' only, that is, points at which the discriminant b of the second fundamental form is negative.

We finally consider two examples (cf. Theorem 39.2):

Example 1. The normal sections of a sphere S are great circles, any other plane section is a small circle. At any point the circles of curvature are identical with the normal sections and therefore lie on S .

Example 2. We consider the cylinder Z : $x(u^1, u^2) = (r \cos u^2, r \sin u^2, u^1)$ and plane sections of Z which pass through any (fixed) point P of Z and whose tangents at P are parallel to the $x_1 x_2$ -plane. The normal section among those sections is a circle of radius r while the other sections are ellipses whose principal axes have the lengths $2a = 2r/\cos\gamma$ and $2b = 2r$, respectively. Here γ is the angle between the plane of the section and the $x_1 x_2$ -plane. According to the theorem of Meusnier the radius of curvature of those ellipses at P is $\rho = r \cos\gamma$. From this and the preceding formulae we obtain the well-known expression $\rho = b^2/a$ for the radius of curvature of an ellipse at the vertex which lies on the principal axis of length $2a$.

Problem 39.1. Prove that the tangent planes to a cone of revolution at the points of any of its generating straight lines coincide.

Problem 39.2. Given the paraboloid of revolution S : $x_3 = x_1^2 + x_2^2$. Determine the radius and centre of the circle of curvature of the normal section of S at a point P : $x_3 = x_3^{(0)}$ whose tangent at P is parallel to the $x_1 x_2$ -plane.

Problem 39.3. Determine the asymptotic curves of a cylinder of revolution by means of (39.4).

40. Elliptic, parabolic, and hyperbolic points of a surface. In order to determine the shape of a surface S in a neighbourhood of any of its points we first considered arbitrary curves on S . We saw that, in consequence of Theorem 39.1, it suffices to consider plane curves on S . Finally, Theorem 39.2 enabled us to restrict our investigation to normal sections of S , that is, curves of intersection of S and planes which are orthogonal to the tangent plane to S at a point P under consideration. That investigation led us to the introduction of the normal curvature (39.2):

$$\kappa_n = \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta};$$

$|\kappa_n|$ is the curvature of the normal section of S at P whose tangent direction is $du^2 : du^1$.

We will now see that there are three different possible forms of S in a neighbourhood of a point at which the second fundamental form does not vanish identically. For this purpose we consider the behaviour of κ_n as a function of the direction $du^2 : du^1$ of the tangent to the normal sections at P .

As was proved in Section 28 the first fundamental form $ds^2 = g_{\alpha\beta} du^\alpha du^\beta$ is positive definite, that is, its value is always positive. Hence the sign of κ_n depends on the second fundamental form only. This form is (positive or negative) definite if and only if at P the discriminant

$$(40.1) \quad b = \det(b_{\alpha\beta}) = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - (b_{12})^2$$

of this form is positive. In this case κ_n has the same sign for all possible directions of the normal sections at P , that is, the centres of curvature of all normal sections lie on the same side of the surface S . Then P is called an *elliptic point* of the surface.

Example. Any point of an ellipsoid

$$\mathbf{x}(u^1, u^2) = (a \cos u^2 \cos u^1, b \cos u^2 \sin u^1, c \sin u^2),$$

cf. Problem 25.3, is elliptic.

If $b = 0$ at a point P of a surface, κ_n does not change sign, but there is exactly one direction where $\kappa_n = 0$, that is, exactly one (real) asymptotic direction. P is then called a *parabolic point* of the surface. Cf. Fig. 48.

Example. Any point of a cylinder or of a cone (with the exception of the apex) is parabolic.

If $b < 0$ at a point P of a surface κ_n does not maintain the same sign for all directions $du^2 : du^1$. More precisely: There exist two (real) asymptotic directions for which $\kappa_n = 0$. These directions separate the directions for which κ_n is positive from those for which κ_n is negative, cf. Fig. 49. P is then called a *hyperbolic* or *saddle point* of the surface.

Since the conditions $b \geq 0$ correspond to a geometric property of the surface they must be invariant with respect to any allowable coordinate transformation $u^\alpha = u^\alpha(\tilde{u}^1, \tilde{u}^2)$ ($\alpha = 1, 2$). Indeed, the coefficients $b_{\alpha\beta}$ of the second fundamental form are the components of a covariant tensor

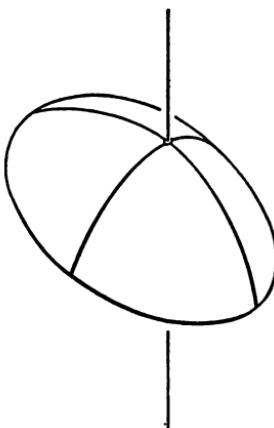


FIG. 47. Shape of a surface in the neighbourhood of an elliptic point

(cf. Section 38), so they have the same transformation behaviour as the components $g_{\alpha\beta}$ of the metric tensor. The discriminant b of the second

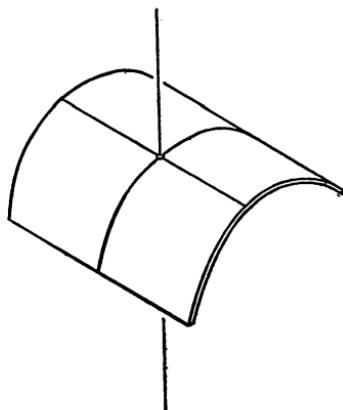


FIG. 48. Shape of a surface in the neighbourhood of a parabolic point

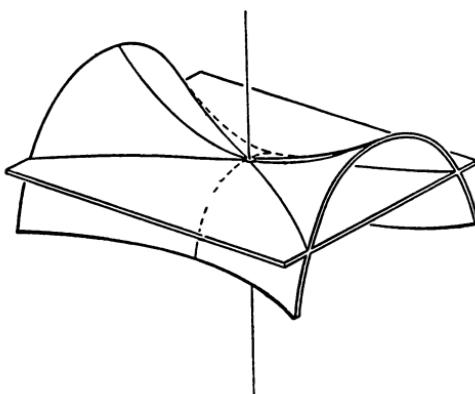


FIG. 49. Shape of a surface in the neighbourhood of a hyperbolic point.
(The figure shows also the tangent plane at that point.)

fundamental form thus has the same transformation law as the discriminant g of the first fundamental form, that is,

$$(40.2) \quad \bar{b} = \left[\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 b,$$

cf. (28.4). Since in (40.2) the square of the Jacobian appears, this holds for all allowable transformations of class $r \geq 2$, and also for those which reverse the sense of the normal vector \mathbf{n} . From this we see that if $b > 0$, $\bar{b} = 0$, or $b < 0$ we have also $\bar{b} > 0$, $\bar{b} = 0$, or $\bar{b} < 0$, respectively.

We sum up our results:

Discriminant b of the second fundamental form	Name	Number of real asymptotic directions	Cf. Fig.
$b > 0$	Elliptic point	0	47
$b = 0$	Parabolic point	1	48
$b < 0$	Hyperbolic point	2	49

Depending on the type of a point P on a surface S , S passes through the tangent plane $E(P)$ to S at P or does not pass through $E(P)$. In order to obtain a clear picture of this important fact we first prove the following

Theorem 40.1. *Let a surface S be given by a representation $\mathbf{x}(u^1, u^2)$ of class $r \geq 2$. If at a point $P: \mathbf{x}(u^1, u^2)$ of S the second fundamental form does not vanish identically then it is equal to twice the directed distance of a point $Q: \mathbf{x}(u^1 + du^1, u^2 + du^2)$ of S from the tangent plane $E(P)$ to S at P , neglecting terms of third and higher order in $|du^1| + |du^2|$. Cf. Fig. 50.*

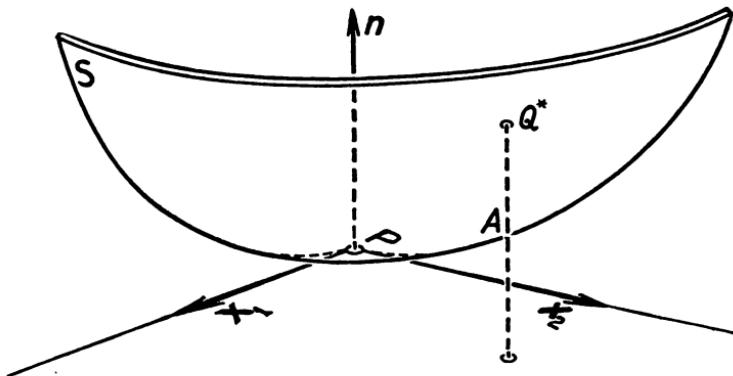


FIG. 50

Proof. We consider a point $Q^*: \mathbf{x}(u^1 + h^1, u^2 + h^2)$ of S . According to Taylor's formula we have

$$(40.3) \quad \mathbf{x}(u^1 + h^1, u^2 + h^2) = \mathbf{x}(u^1, u^2) + h^\alpha \mathbf{x}_\alpha + \frac{1}{2!} h^\alpha h^\beta \mathbf{x}_{\alpha\beta} + \mathbf{o}(|h^1| + |h^2|)^2$$

where the vector \mathbf{o} is defined as in Section 11 and the values of the derivatives at the point $P: \mathbf{x}(u^1, u^2)$ must be taken. The distance of Q^* from the tangent plane $E(P)$ to S at P is therefore

$$(40.4) \quad \begin{aligned} A(Q^*) &= [\mathbf{x}(u^1 + h^1, u^2 + h^2) - \mathbf{x}(u^1, u^2)] \cdot \mathbf{n} \\ &= \frac{1}{2} h^\alpha h^\beta \mathbf{x}_{\alpha\beta} \cdot \mathbf{n} + \mathbf{o}(|h^1| + |h^2|)^2, \end{aligned}$$

where \mathbf{n} is the unit normal vector to S at P . Setting $h^\alpha = du^\alpha$, in consequence of (38.2) we obtain

$$A(Q) = \frac{1}{2} b_{\alpha\beta} du^\alpha du^\beta + o(|du^1| + |du^2|)^2.$$

From this theorem we obtain immediately

Theorem 40.2. *At any elliptic or parabolic point P the surface S , considered in a sufficiently small neighbourhood of P , lies entirely on one side of the corresponding tangent plane $E(P)$. At a hyperbolic point it passes through the tangent plane $E(P)$.*

Problem 40.1. Find the differential equation of the asymptotic curves of a surface of revolution $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2))$ and determine the location of elliptic, parabolic, and hyperbolic points on this surface.

41. Principal curvature. Lines of curvature. Gaussian and mean curvature. In the preceding section we considered the normal curvature κ_n at an arbitrary (fixed) point P of a surface as a function of the direction of the tangent to the normal sections at P . We have obtained the following result: If P is an elliptic or parabolic point the sign of κ_n is the same for all directions; if P is a hyperbolic point there are directions for which $\kappa_n > 0$ as well as directions for which $\kappa_n < 0$; these directions are separated by the two real asymptotic directions (for which $\kappa_n = 0$).

It is natural to ask for directions for which κ_n has an extreme value.

This question does not make sense at points for which κ_n is constant, that is, is independent of the direction of the normal sections. Obviously, this property holds if and only if $b_{\alpha\beta}$ is proportional to $g_{\alpha\beta}$,

$$b_{\alpha\beta} = \lambda(u^1, u^2)g_{\alpha\beta}, \quad (\alpha = 1, 2; \beta = 1, 2).$$

A point of this type is called a *navel point* or *umbilic*. In this case $b = \lambda^2 g$. Now $g > 0$. Hence if $\lambda \neq 0$ we have $b > 0$, and the point is called an *elliptic umbilic*. If $\lambda = 0$, also $b = 0$; such a point is called a *parabolic umbilic*, *flat point*, or *planar point*. For details see Sections 44 and 48.

Let us assume that the point P under consideration is not an umbilic. We first determine those directions (that is, those values of $du^2: du^1$) for which κ_n has an extreme value (which value we shall obtain later). Obviously (39.2) can be written in the form

$$(41.1) \quad (b_{\alpha\beta} - \kappa_n g_{\alpha\beta})l^\alpha l^\beta = 0,$$

where $l^\gamma = du^\gamma$. We set

$$a_{\alpha\beta} = b_{\alpha\beta} - \kappa_n g_{\alpha\beta}.$$

If we differentiate (41.1) with respect to l^γ , treating κ_n as a constant (because $d\kappa_n = 0$ is the necessary condition for κ_n to be a minimum) we obtain

$$\begin{aligned}\frac{\partial}{\partial l^\gamma} (a_{\alpha\beta} l^\alpha l^\beta) &= a_{\alpha\beta} \left(\frac{\partial l^\alpha}{\partial l^\gamma} l^\beta + l^\alpha \frac{\partial l^\beta}{\partial l^\gamma} \right) \\ &= a_{\alpha\beta} (\delta_\gamma^\alpha l^\beta - l^\alpha \delta_\gamma^\beta) = a_{\gamma\beta} l^\beta + a_{\alpha\gamma} l^\alpha = (a_{\gamma\alpha} + a_{\alpha\gamma}) l^\alpha.\end{aligned}$$

Since $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are symmetric we have $a_{\gamma\alpha} = a_{\alpha\gamma}$, and therefore

$$(41.2) \quad (b_{\alpha\gamma} - \kappa_n g_{\alpha\gamma}) l^\alpha = 0, \quad (\gamma = 1, 2).$$

If we eliminate κ_n we find

$$(41.2') \quad \begin{vmatrix} g_{1\alpha} du^\alpha & b_{1\beta} dv^\beta \\ g_{2\alpha} du^\alpha & b_{2\beta} dv^\beta \end{vmatrix} = 0$$

or

$$(41.2'') \quad \begin{vmatrix} (du^2)^2 & -du^1 du^2 & (du^1)^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix} = 0.$$

The roots of the quadratic equation (41.2') determine those directions for which the normal curvature κ_n becomes extreme, as will be seen in the following section. Those directions are called the *principal directions of normal curvature* (or *curvature directions*) at the point P under consideration; the corresponding values of κ_n are called the *principal normal curvatures* of the surface S at P ; we denote them by κ_1 and κ_2 . The centres of curvature of the corresponding normal sections are called the *centres of principal curvature* of the surface S at P .

We will now consider the principal directions in more detail. At any regular point we have

Theorem 41.1. *The roots of equation (41.2') are real. At any point which is not an umbilic the principal directions are orthogonal.*

Proof. The reality of the principal directions is obvious geometrically. We prove the orthogonality. Let $l_{(1)}^\alpha$ and $l_{(2)}^\alpha$ denote those directions and let κ_1 and κ_2 be the corresponding values of κ_n . If we insert this in (41.2) we have

$$(41.3) \quad (b_{\alpha\nu} - \kappa_\nu g_{\alpha\nu}) l_{(\nu)}^\alpha = 0, \quad (\nu = 1, 2).$$

We multiply these respective equations by $l_{(2)}^\nu$ and $l_{(1)}^\nu$, sum with respect to γ and subtract the equations thus obtained. This yields

$$(41.4) \quad (\kappa_2 - \kappa_1) g_{\alpha\nu} l_{(1)}^\alpha l_{(2)}^\nu = 0.$$

Since $\kappa_1 \neq \kappa_2$ the principal directions are orthogonal, cf. (35.5).

A curve on a surface S whose direction at every point is a principal direction is known as a *line of curvature* of S .

Any line of curvature is a solution of the differential equation (41.2'') which is quadratic in du^α . The solutions of (41.2'') are such that through any point of a surface of class $r \geq 3$, which is not an umbilic there pass exactly two curves which, according to Theorem 41.1, intersect orthogonally. In consequence of Theorem 41.1 we thus obtain

Theorem 41.2. *The lines of curvature on any (real) surface S of class $r \geq 3$ are real curves. If S has no umbilics the lines of curvature form an orthogonal net everywhere on S .*

While the existence of (real) asymptotic curves on a surface S depends on the geometric shape of S (cf. the preceding section) the lines of curvature are always real.

Consequently we may always choose coordinates u^1, u^2 on S so that the lines of curvature are the coordinate curves of this system which is allowable at any point of S which is not an umbilic. If we introduce coordinates of that type then (41.2'') must be satisfied for $du^1 = 0$ as well as for $du^2 = 0$, i.e. in (41.2'') the coefficients of $(du^1)^2$ and $(du^2)^2$ must vanish. This leads to the following system of linear equations:

$$(41.5) \quad g_{12} b_{22} - g_{22} b_{12} = 0, \quad g_{11} b_{12} - g_{12} b_{11} = 0.$$

Excluding umbilics the determinant $b_{22}g_{11} - b_{11}g_{22}$ of the coefficients of the system (41.5) is different from zero, and (41.5) has therefore only the trivial solutions $g_{12} = b_{12} = 0$. We thus obtain the important

Theorem 41.3. *The coordinate curves of any allowable coordinate system on a surface S coincide with the lines of curvature if and only if for this system*

$$(41.6) \quad g_{12} = 0 \quad \text{and} \quad b_{12} = 0$$

at any point where those coordinates are allowable.

We will now derive an analytic expression for the principal curvatures κ_1 and κ_2 . If we multiply (41.2) by $g^{\beta\gamma}$ and sum with respect to γ we find

$$(41.7) \quad b_\alpha^\beta du^\alpha - \kappa_n du^\beta = 0, \quad (\beta = 1, 2),$$

$$\text{whence} \quad \kappa_n^2 - b_\alpha^\alpha \kappa_n - \det(b_\alpha^\beta) = 0$$

$$\text{or, since} \quad \det(b_\alpha^\beta) = \frac{b}{g},$$

$$(41.8) \quad \kappa_n^2 - b_{\alpha\beta} g^{\alpha\beta} \kappa_n + \frac{b}{g} = 0.$$

Hence the principal curvatures are the roots of this equation.

In (41.1) to every value of $du^2:du^1$ there corresponds exactly one value of κ_n while, conversely, to every value of κ_n between κ_1 and κ_2 there correspond two values of $du^2:du^1$ as can be seen, for instance, from (41.2'). Now (41.8) is the discriminant of (41.2'), that is, the principal curvatures κ_1 and κ_2 correspond to a double root of (41.2') and therefore to just one direction.

We are now able to introduce the following two fundamental concepts:

Definition 41.1. *The product*

$$(41.9) \quad K = \kappa_1 \kappa_2 = \frac{b}{g}$$

of the two principal curvatures is called the Gaussian curvature (sometimes total curvature) of the surface at the point under consideration.

Definition 41.2. *The arithmetic mean of the principal curvatures,*

$$(41.10) \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}b_{\alpha\beta}g^{\alpha\beta} = \frac{1}{2}b_\alpha^\alpha$$

is called the mean curvature of the surface at the point under consideration.

The quantities K and $|H|$ are invariant with respect to allowable coordinate transformations; the sign of κ_n , and therefore also the sign of H , depends on the orientation of the surface. This is an immediate consequence of the geometric meaning of K and H , but can also easily be verified in a formal manner: According to (28.4) and (40.2) the discriminants g and b are multiplied by the same factor if a coordinate transformation is imposed; hence K is invariant. The inner product $b_{\alpha\beta}g^{\alpha\beta}$ is an invariant, but the sign of $b_{\alpha\beta}$ and therefore also the sign of H depends on the orientation of the surface, that is, on the choice of the coordinates, cf. Section 34.

As was shown in Section 28 we always have $g > 0$. In consequence of the considerations in Section 40 we thus obtain from (41.9)

Theorem 41.4. *At elliptic points the Gaussian curvature is positive, at parabolic ones it vanishes, and at hyperbolic ones it is negative.*

In Section 46 we shall see that the Gaussian curvature K depends only on the coefficients of the first fundamental form (and their derivatives) but not on the second fundamental form.

When the coordinate curves on a surface S are lines of curvature, Equation (41.7) holds with $\kappa = \kappa_1$, $du^2 = 0$ and again with $\kappa = \kappa_2$, $du^1 = 0$. Therefore

$$\kappa_1 = b_1^1, \quad b_1^2 = 0, \quad \kappa_2 = b_2^2, \quad b_2^1 = 0.$$

We thus obtain in this case

$$(41.11) \quad \kappa_1 = \frac{b_{11}}{g_{11}}, \quad \kappa_2 = \frac{b_{22}}{g_{22}}, \quad K = \frac{b_{11}}{g_{11}} \frac{b_{22}}{g_{22}}, \quad H = \frac{1}{2} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right).$$

We finally mention

Theorem 41.5. *The meridians and parallels of any surface of revolution of class $r \geq 2$ are lines of curvature.*

Further theorems on lines of curvature will occur in Sections 68 and 88.

Problem 41.1. Prove Theorem 41.5.

Problem 41.2. Determine the discriminant b of the second fundamental form for a sphere.

42. Euler's theorem. Dupin's indicatrix. The normal curvature κ_n corresponding to any direction can be simply represented in terms of the principal curvatures κ_1 and κ_2 :

Theorem 42.1 (L. Euler [3]). *Let α be the angle between a direction at a point P and the principal direction at P corresponding to κ_1 . Then*

$$(42.1) \quad \kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha.$$

Proof. We assume that P is not an umbilic. We choose coordinates on a surface S so that the coordinate curves are lines of curvature on S . Then, in consequence of (41.6), (39.2) takes the form

$$(42.2) \quad \kappa_n = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2}{ds^2}$$

and $\kappa_1 = \frac{b_{11}}{g_{11}}, \quad \kappa_2 = \frac{b_{22}}{g_{22}},$

cf. (41.11). We can therefore write (42.2) in the form

$$(42.3) \quad \kappa_n = \kappa_1 g_{11}(\dot{u}^1)^2 + \kappa_2 g_{22}(\dot{u}^2)^2.$$

Now the direction du^2/du^1 under consideration is determined by the vector $\dot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha$, and the direction corresponding to κ_1 is given by the vector \mathbf{x}_1 . Since the lines of curvature are orthogonal (cf. Theorem 41.2) and $|\dot{\mathbf{x}}| = 1$, the angle α between the vectors \mathbf{x}_1 and $\dot{\mathbf{x}}$ and the angle $\beta = \frac{1}{2}\pi - \alpha$ between the vectors \mathbf{x}_2 and $\dot{\mathbf{x}}$ are determined by the expressions

$$\cos \alpha = \frac{\mathbf{x}_1 \cdot \dot{\mathbf{x}}}{|\mathbf{x}_1| |\dot{\mathbf{x}}|} = \frac{g_{11} \dot{u}^1}{\sqrt{g_{11}}} = \sqrt{g_{11}} \dot{u}^1,$$

$$\sin \alpha = \cos \beta = \frac{\mathbf{x}_2 \cdot \dot{\mathbf{x}}}{|\mathbf{x}_2| |\dot{\mathbf{x}}|} = \frac{g_{22} \dot{u}^2}{\sqrt{g_{22}}} = \sqrt{g_{22}} \dot{u}^2.$$

If we insert these expressions into (42.3) we obtain (42.1).

The theorem of Euler and the theorem of Meusnier (Theorem 39.2) give complete information on the curvature of any curve on a surface.

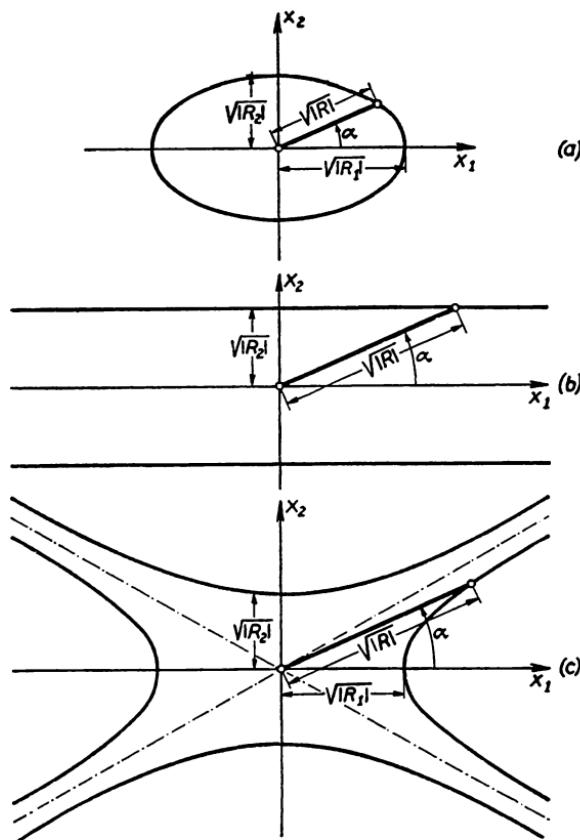


FIG. 51. Dupin indicatrix (a) at an elliptic point,
(b) at a parabolic point, (c) at a hyperbolic point

We may illustrate Euler's theorem in the following simple manner: Let P be any point on a surface S such that $\kappa_n = 1/R \neq 0$ at P . For all directions for which $\kappa_n \neq 0$ we set

$$(42.4) \quad \sqrt{|R|} \cos \alpha = x_1, \quad \sqrt{|R|} \sin \alpha = x_2,$$

that is, we consider a plane with Cartesian coordinates x_1, x_2 and lay off from the origin of those coordinates segments of the lengths $\sqrt{|R(\alpha)|}$ in the direction whose angle with the x_1 -direction is α ; the directions of the coordinate axes thus correspond to the principal directions on the surface at

the point P under consideration. In this manner we obtain the so-called *Dupin indicatrix*, cf. C. Dupin [1]. An analytic representation of this indicatrix can be simply derived as follows. If we multiply both sides of (42.1) by $1/|\kappa_n|$, $\kappa_n \neq 0$, then in consequence of (42.4) we find

$$(42.5) \quad \kappa_1 x_1^2 + \kappa_2 x_2^2 = \pm 1.$$

If the point P under consideration is elliptic, κ_1 and κ_2 have the same sign and the indicatrix is therefore an ellipse. If P is hyperbolic, κ_1 and κ_2 have different signs; then the indicatrix consists of two hyperbolas which have the same asymptotes. This is the reason for the names ‘elliptic’ and ‘hyperbolic’ point and it also justifies the name ‘asymptotic directions’, since these directions are the directions of the asymptotes of the indicatrix.

As an immediate consequence of the above consideration we have

Theorem 42.2. *The principal directions bisect the angles between directions corresponding to the same normal curvature, in particular the angles between the asymptotic directions.*

If $\kappa_1 = 0$ and $\kappa_2 \neq 0$ the point P is parabolic. In this case we obtain from (42.5)

$$x_2^2 = \pm R_2, \quad (R_2 = 1/\kappa_2);$$

that is, the indicatrix consists of a pair of straight lines parallel to the x_1 -axis through the points $\pm\sqrt{|R_2|}$ of the x_2 -axis.

The Dupin indicatrix at a point P of a surface S is closely related to the intersection of S with planes parallel to the tangent plane $E(P)$ to S at P . Indeed, when we choose two such planes which lie on both sides of $E(P)$ at a (small) distance ϵ then, according to Theorem 40.1, the intersection of S and those planes is given by

$$(42.6) \quad \tfrac{1}{2} b_{\alpha\beta} du^\alpha du^\beta = \pm \epsilon,$$

neglecting terms of third and higher order in $|du^1| + |du^2|$. If in particular the coordinates are chosen so that the coordinate curves are lines of curvature on S (42.6) takes the form

$$(42.7) \quad b_{11}(du^1)^2 + b_{22}(du^2)^2 = \pm 2\epsilon.$$

Because of (41.11), equation (42.7) is equivalent to

$$\kappa_1 g_{11}(du^1)^2 + \kappa_2 g_{22}(du^2)^2 = \pm 2\epsilon.$$

Hence we have

Theorem 42.3. *The intersection of a surface S of class $r \geq 2$ and a plane which is parallel to the tangent plane $E(P)$ to S at a point P and near*

to $E(P)$, is approximately a conic section which is similar and similarly placed to the Dupin indicatrix at P , or, if P is a hyperbolic point, to one of the two hyperbolas constituting the Dupin indicatrix.

In the case of a hyperbolic point S intersects both planes while in the case of an elliptic or parabolic point S intersects only one of the two planes.

We see that at an elliptic, hyperbolic, or parabolic point a surface looks approximately the same as a portion of an elliptic or hyperbolic paraboloid or a cylinder, respectively.

43. Torus. A surface obtained by rotating a circle about a coplanar line outside of the circle is called a *torus*. We may illustrate all types of points, elliptic, parabolic, as well as hyperbolic ones, on this surface.

We rotate a circle C of radius r , lying in the x_1x_3 -plane with centre at $(x_1, x_3) = (r_0, 0)$, about the x_3 -axis. We denote by u^2 the directed angle from the positive direction of the x_1 -axis to a point P on C . We may then represent the torus S in the form

$$(43.1) \quad \mathbf{x}(u^1, u^2) = [(r_0 + r \cos u^2) \cos u^1, (r_0 + r \cos u^2) \sin u^1, r \sin u^2],$$

cf. Fig. 52, p. 136. The coefficients of the first fundamental form are

$$g_{11} = (r_0 + r \cos u^2)^2, \quad g_{12} = 0, \quad g_{22} = r^2.$$

Therefore

$$g = r^2(r_0 + r \cos u^2)^2.$$

The coefficients of the second fundamental form are

$$b_{11} = -(r_0 + r \cos u^2) \cos u^2, \quad b_{12} = 0, \quad b_{22} = -r.$$

Since $g_{12} = b_{12} = 0$ the coordinate curves are lines of curvature on S , cf. (41.6). Hence, by (41.11),

$$\kappa_1 = \frac{b_{11}}{g_{11}} = -\frac{\cos u^2}{r_0 + r \cos u^2}, \quad \kappa_2 = \frac{b_{22}}{g_{22}} = -\frac{1}{r}.$$

Consequently

$$(43.2) \quad K = \frac{b}{g} = \kappa_1 \kappa_2 = \frac{\cos u^2}{r(r_0 + r \cos u^2)}.$$

If $u^2 = \pm \frac{1}{2}\pi$ then $K = 0$, that is, the circles of radius r_0 on the torus S consist of parabolic points only. Points on S whose distance from the x_3 -axis is greater than r_0 are elliptic while points whose distance from the x_3 -axis is smaller than r_0 are hyperbolic.

We finally mention that the torus has the total area

$$A = \int_0^{2\pi} \int_0^{2\pi} (r_0 + r \cos u^2) r \, du^1 du^2 = 4\pi^2 r_0 r.$$

This result can be obtained more simply by the *theorem of Pappus*: the area of a surface of revolution is equal to the product of the arc length s of a meridian M (in our case $s = 2\pi r$) and the length L of the path of the centre

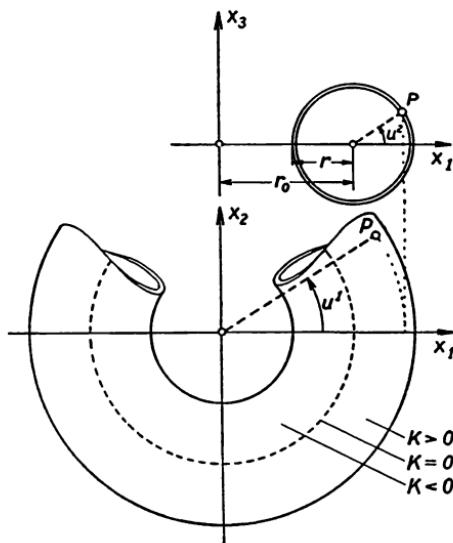


FIG. 52. Torus

of gravity of M when M is rotated through the angle 2π (in our case $L = 2\pi r_0$).

44. Flat points. Saddle points of higher type. A point P of a surface S is called a *flat point*, *planar point*, or *parabolic umbilic* if at P all coefficients $b_{\alpha\beta}$ of the second fundamental form vanish. Clearly, this property of P is independent of the choice of the coordinates on S as follows immediately from the fact that the quantities $b_{\alpha\beta}$ are the components of a tensor.

When classifying the points of a surface (cf. Sections 40 and 41) we excluded planar points. We will now consider two simple characteristic examples of these points.

Example 1. If we rotate the curve $x_3 = x_1^4$ (cf. Fig. 53) in the x_1x_3 -plane about the x_3 -axis of the Cartesian coordinates in space we obtain a surface of revolution which has a planar point at $\mathbf{x} = (0, 0, 0)$. We may represent this surface in the form $(x_1^2 + x_2^2)^2 - x_3^2 = 0$ or in parametric form

$$(44.1) \quad \mathbf{x}(u^1, u^2) = (u^1, u^2, A^2)$$

where

$$A = (u^1)^2 + (u^2)^2.$$

We find

$$g = 1 + 16A^3$$

and, according to (38.8),

$$b_{\alpha\alpha} = \frac{1}{\sqrt{g}} [4A + 8(u^\alpha)^2], \quad (\alpha = 1, 2), \quad b_{12} = \frac{8u^1 u^2}{\sqrt{g}}.$$

At the point P : $(u^1, u^2) = (0, 0)$, which corresponds to $\mathbf{x} = (0, 0, 0)$, all coefficients $b_{\alpha\beta}$ vanish, that is, P is a planar point. At P also $K = 0$ while at any other point $b > 0$ and therefore also $K > 0$.

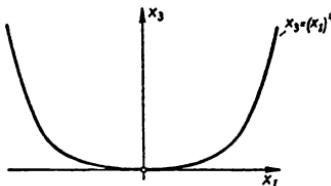


FIG. 53

In that case the surface lies entirely on one side of the tangent plane at P . Let us now consider a planar point at which the tangent plane intersects the surface; we may call such a point a *saddle point of higher type*.

Example 2. While a surface in the neighbourhood of any ordinary saddle point (which is not a planar point, cf. Section 40) has two 'upward' and two 'downward' slopes, a so-called *monkey-saddle* has three upward and three downward slopes, two of the latter for the legs of the monkey and one for his tail, cf. Fig. 54. Obviously, at such a saddle point P the corresponding tangent plane intersects the surface along three curves passing through P . The surface

$$x_3 = x_2^3 - 3x_1^2 x_2 = x_2(x_2 - \sqrt{3}x_1)(x_2 + \sqrt{3}x_1)$$

or, in parametric form,

$$(44.2) \quad \mathbf{x}(u^1, u^2) = (u^1, u^2, (u^2)^3 - 3(u^1)^2 u^2)$$

has a monkey saddle in the neighbourhood of the saddle point $\mathbf{x} = (0, 0, 0)$. We find

$$g_{11} = 1 + 36(u^1)^2(u^2)^2, \quad g_{12} = 18u^1 u^2[(u^1)^2 - (u^2)^2],$$

$$g_{22} = 1 + 9[(u^1)^2 - (u^2)^2]^2,$$

and therefore

$$g(u^1, u^2) = 1 + 9[(u^1)^2 + (u^2)^2]^2;$$

hence $g(0, 0) = 1$. Furthermore, according to (38.8),

$$b_{11} = \frac{-6u^2}{\sqrt{g}}, \quad b_{12} = \frac{-6u^1}{\sqrt{g}}, \quad b_{22} = \frac{6u^2}{\sqrt{g}}, \quad b = -\frac{36}{g}[(u^1)^2 + (u^2)^2].$$

At $P: (u^1, u^2) = (0, 0)$ these coefficients vanish; at this point therefore also $K = 0$ and the normal sections have a point of inflection.

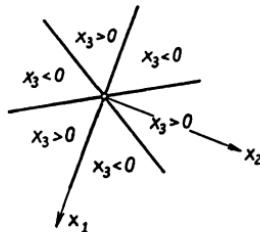
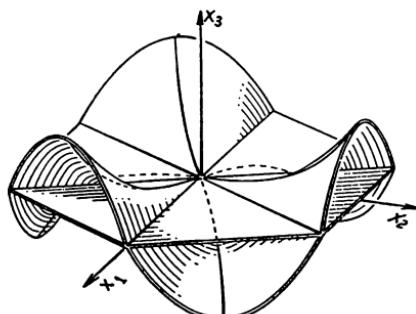


FIG. 54. Monkey saddle

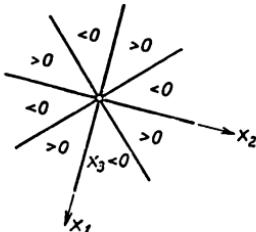
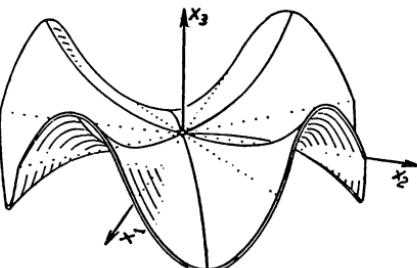


FIG. 55

Similarly we may obtain saddle points of higher kind for which the corresponding tangent plane intersects the surface along four or more curves. For instance, the surface

$$x_3 = x_1 x_2 (x_1^2 - x_2^2)$$

has at $\mathbf{x} = (0, 0, 0)$ a saddle point such that the corresponding tangent plane intersects the surface along four curves, cf. Fig. 55.

Problem 44.1. Give an example of a surface which has a planar point P and lies entirely on one side of the tangent plane $E(P)$ at P , but has a curve in common with $E(P)$.

45. Formulae of Weingarten and Gauss. We first recall the following fact from the theory of curves: To any point P of a curve of class $r \geq 2$ at which $\kappa \neq 0$, we have associated the three orthogonal unit vectors

t, p, and b. Any vector bound at P can be represented as a linear combination of those vectors. In particular, if the derivatives \mathbf{t} , \mathbf{p} , and \mathbf{b} exist they permit such a representation; the corresponding formulae are called the formulae of Frenet.

In the theory of surfaces the situation is similar. At any regular point of a surface S : $\mathbf{x}(u^1, u^2)$ of class $r \geq 2$ we have the vectors

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2}, \quad \text{and} \quad \mathbf{n},$$

cf. Section 34. The partial derivatives of those vectors with respect to the coordinates u^1 and u^2 can be represented as a linear combination of those vectors. The corresponding representations are called the formulae of Weingarten and Gauss. Let us derive these basic formulae.



FIG. 56

From $\mathbf{n} \cdot \mathbf{n} = 1$ we have $\mathbf{n}_\alpha \cdot \mathbf{n} = 0$ ($\alpha = 1, 2$), where $\mathbf{n}_\alpha \equiv \partial \mathbf{n} / \partial u^\alpha$. Hence the vectors \mathbf{n}_1 and \mathbf{n}_2 lie in the tangent plane to the surface S . Since this plane is spanned by the vectors \mathbf{x}_1 and \mathbf{x}_2 we may set

$$\mathbf{n}_\alpha = c_\alpha^\gamma \mathbf{x}_\gamma, \quad (\alpha = 1, 2);$$

we have to determine the coefficients c_α^γ . If we take the scalar product of both sides and \mathbf{x}_σ we find

$$\mathbf{n}_\alpha \cdot \mathbf{x}_\sigma = c_\alpha^\gamma \mathbf{x}_\gamma \cdot \mathbf{x}_\sigma = c_\alpha^\gamma g_{\gamma\sigma}.$$

Using (38.4):

$$\mathbf{n}_\alpha \cdot \mathbf{x}_\sigma = -b_{\alpha\sigma}$$

and (32.7):

$$g_{\gamma\sigma} g^{\sigma\tau} = \delta_\gamma^\tau$$

we find $\mathbf{n}_\alpha \cdot \mathbf{x}_\sigma g^{\sigma\tau} = -b_{\alpha\sigma} g^{\sigma\tau} = -b_\alpha^\tau = c_\alpha^\lambda g_{\lambda\sigma} g^{\sigma\tau} = c_\alpha^\tau$,

that is,

$$c_\alpha^\tau = -b_\alpha^\tau.$$

We thus obtain the *formulae of Weingarten*

$$(45.1) \quad \mathbf{n}_\alpha \equiv \frac{\partial \mathbf{n}}{\partial u^\alpha} = -b_\alpha^\beta \mathbf{x}_\beta, \quad (\alpha = 1, 2),$$

where

$$(45.2) \quad b_\alpha^\beta = g^{\sigma\beta} b_{\alpha\sigma}.$$

If in particular the coordinates are chosen so that the coordinate curves are lines of curvature on the surface S then $g_{12} = b_{12} = 0$, cf. (41.6), and therefore, in consequence of (32.6) and (41.11),

$$(45.1') \quad \mathbf{n}_\alpha = -\frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} \mathbf{x}_\alpha \quad (\alpha = 1, 2)$$

(do not sum with respect to α)

or

$$\mathbf{x}_\alpha + R_\alpha \mathbf{n}_\alpha = 0, \quad (R_\alpha = 1/\kappa_\alpha; \alpha = 1, 2).$$

(Since both α 's are subscripts, $R_\alpha \mathbf{n}_\alpha$ is not summed.) These relations are known as the *formulae of O. Rodrigues* [1]; they are characteristic for the lines of curvature. We have thus obtained the following result:

Theorem 45.1. *For displacement in a principal direction,*

$$\kappa_n d\mathbf{x} + d\mathbf{n} = 0.$$

We now consider the partial derivatives

$$\mathbf{x}_{\alpha\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}$$

of the vectors \mathbf{x}_1 and \mathbf{x}_2 . These derivatives are linear combinations of all three vectors \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{n} since, in general, \mathbf{x}_1 and \mathbf{x}_2 are not unit vectors. We set

$$(45.3) \quad \mathbf{x}_{\alpha\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma + a_{\alpha\beta} \mathbf{n}$$

and determine first the coefficients $a_{\alpha\beta}$ and then the coefficients $\Gamma_{\alpha\beta}^\gamma$. We take the scalar product of \mathbf{n} and both sides of (45.3); since $\mathbf{x}_\gamma \cdot \mathbf{n} = 0$ and $\mathbf{x}_{\alpha\beta} \cdot \mathbf{n} = b_{\alpha\beta}$ we obtain

$$a_{\alpha\beta} = b_{\alpha\beta}.$$

In order to determine the coefficients $\Gamma_{\alpha\beta}^\gamma$ we take the scalar product of \mathbf{x}_λ and both sides of (45.3); since $\mathbf{x}_\lambda \cdot \mathbf{n} = 0$ we find

$$(45.4) \quad \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\lambda = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma \cdot \mathbf{x}_\lambda = \Gamma_{\alpha\beta}^\gamma g_{\gamma\lambda}.$$

Since $\mathbf{x}_\gamma \cdot \mathbf{x}^\kappa = \delta_\gamma^\kappa$ (where $\mathbf{x}^\kappa = g^{\rho\kappa} \mathbf{x}_\rho$, cf. Section 32) multiplication of (45.3) by \mathbf{x}^κ yields

$$(45.5) \quad \Gamma_{\alpha\beta}^\kappa = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\kappa = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\lambda g^{\lambda\kappa}.$$

So far we have determined all coefficients occurring in (45.3). We will now represent the coefficients $\Gamma_{\alpha\beta}^\kappa$ in terms of the components of the metric tensor and their derivatives. Since the scalar products $\mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\lambda$, cf. (45.4), will occur later it is advantageous to introduce a special notation. We set

$$(45.6) \quad \Gamma_{\alpha\beta\lambda} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\lambda.$$

These quantities are called *Christoffel symbols of the first kind*. Since each

of the indices may have the value 1 or 2 we have $2^3 = 8$ such symbols (in the theory of surfaces), namely Γ_{111} , Γ_{112} , Γ_{121} , etc., (45.4) now takes the form

$$(45.7) \quad \Gamma_{\alpha\beta\lambda} = g_{\gamma\lambda} \Gamma_{\alpha\beta}{}^\gamma,$$

and (45.5) takes the form

$$(45.8) \quad \Gamma_{\alpha\beta}{}^\kappa = g^{\lambda\kappa} \Gamma_{\alpha\beta\lambda}.$$

The quantities $\Gamma_{\alpha\beta}{}^\kappa$, which occur as coefficients in (45.3), are called *Christoffel symbols of the second kind*, cf. E. B. Christoffel [1]. The symbols $\Gamma_{\alpha\beta}{}^\gamma$ are of fundamental importance in the theory of surfaces as we shall see later. The transition from one kind of the symbols to the other can be effected by means of (45.7) and (45.8).

Since $x_{\alpha\beta} = x_{\beta\alpha}$ the Christoffel symbols are symmetric with respect to the first two indices,

$$(45.9) \quad \Gamma_{\alpha\beta\lambda} = \Gamma_{\beta\alpha\lambda}, \quad \Gamma_{\alpha\beta}{}^\kappa = \Gamma_{\beta\alpha}{}^\kappa.$$

If we differentiate $g_{\alpha\lambda} \equiv x_\alpha \cdot x_\lambda$, cf. (27.1), with respect to u^β we obtain

$$\frac{\partial g_{\alpha\lambda}}{\partial u^\beta} = x_{\alpha\beta} \cdot x_\lambda + x_\alpha \cdot x_{\lambda\beta}$$

and therefore, in consequence of (45.6), the important relations

$$(45.10) \quad \frac{\partial g_{\alpha\lambda}}{\partial u^\beta} = \Gamma_{\alpha\beta\lambda} + \Gamma_{\lambda\beta\alpha}$$

which we will often use. From this we have

$$(45.11) \quad \frac{\partial g_{\lambda\beta}}{\partial u^\alpha} = \Gamma_{\lambda\alpha\beta} + \Gamma_{\beta\alpha\lambda},$$

and

$$(45.12) \quad \frac{\partial g_{\beta\alpha}}{\partial u^\lambda} = \Gamma_{\beta\lambda\alpha} + \Gamma_{\alpha\lambda\beta}.$$

If we add (45.10) and (45.11) and from this sum subtract (45.12), taking into account (45.9) we find the following representation of the Christoffel symbols of the first kind in terms of the first partial derivatives of the components of the metric tensor:

$$(45.13) \quad \Gamma_{\alpha\beta\lambda} = \frac{1}{2} \left[\frac{\partial g_{\beta\lambda}}{\partial u^\alpha} + \frac{\partial g_{\lambda\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\lambda} \right].$$

We mention that the Christoffel symbols do not form a tensor; their transformation behaviour and further properties will be considered in Section 47.

We sum up the results obtained:

The second partial derivatives of the vector function $\mathbf{x}(u^1, u^2)$ by which a

surface can be represented, are linear combinations of the vectors $\mathbf{x}_\alpha \equiv \partial \mathbf{x} / \partial u^\alpha$, ($\alpha = 1, 2$), and \mathbf{n} ,

$$(45.14) \quad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma + b_{\alpha\beta} \mathbf{n}, \quad (\alpha = 1, 2, \beta = 1, 2).$$

These important relations are called the *formulae of Gauss*. The coefficients $\Gamma_{\alpha\beta}^\gamma$, defined by (45.8), are called Christoffel symbols of the second kind. The Christoffel symbols $\Gamma_{\alpha\beta\lambda}$ involved in (45.8) are functions of the components of the metric tensor, cf. (45.13).

We may write the formulae of Gauss in a different manner, as will be seen in Section 72.

Problem 45.1. Prove that a surface for which at every point all coefficients $b_{\alpha\beta}$ of the second fundamental form vanish is a plane.

46. Integrability conditions of the formulae of Weingarten and Gauss. Curvature tensors. Theorema egregium. We have noticed that the formulae of Weingarten and Gauss are the analogue of the formulae of Frenet. It was pointed out in Section 20 that if (continuous) functions $\kappa = \kappa(s)$ (> 0), and $\tau = \tau(s)$ are given there exists a curve whose arc length is s and whose curvature and torsion are determined by the functions $\kappa(s)$ and $\tau(s)$. Now the analogous question arises whether, if functions $g_{\alpha\beta}(u^1, u^2)$ and $b_{\alpha\beta}(u^1, u^2)$ (of a sufficiently high class) are given, there always exists a surface such that the given functions are the coefficients of the corresponding fundamental forms. That is, we ask whether for any given functions $g_{\alpha\beta}(u^1, u^2)$ and $b_{\alpha\beta}(u^1, u^2)$ (and corresponding Christoffel symbols) the partial differential equations (45.1) and (45.14) always have solutions. In the case of functions of the class $r \geq 3$ the answer to this problem is negative unless certain integrability conditions are satisfied which we will now derive. We will start from (45.14) and shall obtain the formulae of Mainardi–Codazzi, (46.3), and the relations (46.4) from which the Theorema egregium will follow. Formulae (45.1) lead also to the integrability conditions (46.3); that is, (46.3) and (46.4) constitute all integrability conditions of the formulae of Weingarten and Gauss. As is shown by means of the theory of partial differential equations, those conditions are also sufficient for the existence of solutions of (45.1) and (45.14) in the small, that is, in a suitable neighbourhood of any point of the domain under consideration.

Let $S: \mathbf{x}(u^1, u^2)$ be a surface of class $r \geq 3$. We must have

$$\frac{\partial \mathbf{x}_{\alpha\beta}}{\partial u^\lambda} = \frac{\partial \mathbf{x}_{\alpha\lambda}}{\partial u^\beta}, \quad (\alpha, \beta, \lambda = 1, 2),$$

which relations may be written in the form

$$\mathbf{x}_{\alpha\beta\lambda} = \mathbf{x}_{\alpha\lambda\beta}.$$

We determine $\mathbf{x}_{\alpha\beta\lambda}$. From (45.14) we obtain

$$\mathbf{x}_{\alpha\beta\lambda} = \frac{\partial \Gamma_{\alpha\beta}^{\kappa}}{\partial u^\lambda} \mathbf{x}_\kappa + \Gamma_{\alpha\beta}^\kappa \mathbf{x}_{\kappa\lambda} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} \mathbf{n} + b_{\alpha\beta} \mathbf{n}_\lambda$$

and from this, in consequence of (45.1) and (45.14),

$$\mathbf{x}_{\alpha\beta\lambda} = \frac{\partial \Gamma_{\alpha\beta}^{\kappa}}{\partial u^\lambda} \mathbf{x}_\kappa + \Gamma_{\alpha\beta}^\kappa (\Gamma_{\kappa\lambda}^\sigma \mathbf{x}_\sigma + b_{\kappa\lambda} \mathbf{n}) - b_{\alpha\beta} b_{\lambda}^\tau \mathbf{x}_\tau + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} \mathbf{n}.$$

If we interchange some terms and change some summation indices we obtain

$$(46.1) \quad \mathbf{x}_{\alpha\beta\lambda} = \left[\frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\lambda} + \Gamma_{\alpha\beta}^\kappa \Gamma_{\kappa\lambda}^\sigma - b_{\alpha\beta} b_{\lambda}^\sigma \right] \mathbf{x}_\sigma + \left[\Gamma_{\alpha\beta}^\rho b_{\rho\lambda} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} \right] \mathbf{n}.$$

If we interchange β and λ we find

$$(46.1') \quad \mathbf{x}_{\alpha\lambda\beta} = \left[\frac{\partial \Gamma_{\alpha\lambda}^\sigma}{\partial u^\beta} + \Gamma_{\alpha\lambda}^\kappa \Gamma_{\kappa\beta}^\sigma - b_{\alpha\lambda} b_{\beta}^\sigma \right] \mathbf{x}_\sigma + \left[\Gamma_{\alpha\lambda}^\rho b_{\rho\beta} + \frac{\partial b_{\alpha\lambda}}{\partial u^\beta} \right] \mathbf{n}.$$

The vectors \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{n} are linearly independent. In order that (46.1) be equal to (46.1'), that is,

$$(46.2) \quad \mathbf{x}_{\alpha\beta\lambda} - \mathbf{x}_{\alpha\lambda\beta} = 0$$

in this difference the coefficient of each of those three vectors must vanish. This means in the case of \mathbf{n} that

$$(46.3) \quad \Gamma_{\alpha\beta}^\rho b_{\rho\lambda} - \Gamma_{\alpha\lambda}^\rho b_{\rho\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} - \frac{\partial b_{\alpha\lambda}}{\partial u^\beta} = 0$$

must hold. These relations are called the *formulae of Mainardi-Codazzi*. They already occur implicitly in the famous 'Disquisitiones' of Gauss [1]. G. Mainardi [1] published relations from which follow the formulae (46.3) which are due to D. Codazzi [1].

(46.3) means

$$(46.3') \quad \begin{aligned} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} - \Gamma_{12}^1 b_{11} + (\Gamma_{11}^1 - \Gamma_{12}^2) b_{12} + \Gamma_{11}^2 b_{21} &= 0, \\ \frac{\partial b_{12}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} - \Gamma_{22}^1 b_{11} + (\Gamma_{12}^1 - \Gamma_{22}^2) b_{12} + \Gamma_{12}^2 b_{22} &= 0. \end{aligned}$$

Another form of (46.3) will occur in Section 72.

Equating to zero the coefficient of \mathbf{x}_σ in (46.2) (cf. 46.1), we obtain

$$(46.4) \quad R^\sigma_{\alpha\lambda\beta} = b_{\alpha\beta} b_{\lambda}^\sigma - b_{\alpha\lambda} b_{\beta}^\sigma$$

where

$$(46.5) \quad R^\sigma_{\alpha\lambda\beta} = \frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\lambda} - \frac{\partial \Gamma_{\alpha\lambda}^\sigma}{\partial u^\beta} + \Gamma_{\alpha\beta}^\kappa \Gamma_{\kappa\lambda}^\sigma - \Gamma_{\alpha\lambda}^\kappa \Gamma_{\kappa\beta}^\sigma.$$

As can be seen from (46.4) the quantities $R^\sigma_{\alpha\lambda\beta}$ are the components of a tensor of fourth order, contravariant of first order and covariant of third order. This tensor is called the *mixed Riemann curvature tensor*.

The inner product of this tensor and the metric tensor,

$$(46.6) \quad R_{\tau\alpha\lambda\beta} = g_{\sigma\tau} R^\sigma_{\alpha\lambda\beta}$$

is called the *covariant Riemann curvature tensor*; it is a covariant tensor of fourth order. (The components $R^\sigma_{\alpha\lambda\beta}$ and $R_{\tau\alpha\lambda\beta}$ are also known as *Riemann symbols of the first and second kind*, respectively.)

From (46.6) we obtain

$$g^{\gamma\tau} R_{\tau\alpha\lambda\beta} = g^{\gamma\tau} g_{\sigma\tau} R^\sigma_{\alpha\lambda\beta} = \delta_\sigma^\gamma R^\sigma_{\alpha\lambda\beta},$$

that is,

$$(46.6') \quad R^\gamma_{\alpha\lambda\beta} = g^{\gamma\tau} R_{\tau\alpha\lambda\beta}.$$

The covariant curvature tensor can be represented in a form similar to (46.5): We have

$$g_{\sigma\tau} \frac{\partial}{\partial u^\lambda} \Gamma_{\alpha\beta}^\sigma = \frac{\partial}{\partial u^\lambda} (g_{\sigma\tau} \Gamma_{\alpha\beta}^\sigma) - \Gamma_{\alpha\beta}^\sigma \frac{\partial g_{\sigma\tau}}{\partial u^\lambda} = \frac{\partial}{\partial u^\lambda} \Gamma_{\alpha\beta\tau} - \Gamma_{\alpha\beta}^\sigma (\Gamma_{\sigma\lambda\tau} + \Gamma_{\tau\lambda\sigma})$$

and $g_{\sigma\tau} \frac{\partial}{\partial u^\beta} \Gamma_{\alpha\lambda}^\sigma = \frac{\partial}{\partial u^\beta} \Gamma_{\alpha\lambda\tau} - \Gamma_{\alpha\lambda}^\sigma (\Gamma_{\sigma\beta\tau} + \Gamma_{\tau\beta\sigma}).$

Consequently, if we insert (46.5) into (46.6) we obtain

$$(46.6'') \quad R_{\tau\alpha\lambda\beta} = \frac{\partial}{\partial u^\lambda} \Gamma_{\alpha\beta\tau} - \frac{\partial}{\partial u^\beta} \Gamma_{\alpha\lambda\tau} + \Gamma_{\alpha\lambda}^\sigma \Gamma_{\tau\beta\sigma} - \Gamma_{\alpha\beta}^\sigma \Gamma_{\tau\lambda\sigma}.$$

From (46.4) and (46.6) follows

$$R_{\tau\alpha\lambda\beta} = g_{\sigma\tau} (b_{\alpha\beta} b_{\lambda\sigma} - b_{\alpha\lambda} b_{\beta\sigma}),$$

hence

$$(46.7) \quad R_{\tau\alpha\lambda\beta} = b_{\alpha\beta} b_{\lambda\tau} - b_{\alpha\lambda} b_{\beta\tau}.$$

As follows from this the first pair of indices and the second pair can be interchanged without altering the value of the components:

$$(46.8) \quad R_{\tau\alpha\lambda\beta} = R_{\lambda\beta\tau\alpha}.$$

Furthermore, the covariant curvature tensor is skew-symmetric with respect to the last two indices,

$$(46.9) \quad R_{\tau\alpha\lambda\beta} = -R_{\tau\alpha\beta\lambda},$$

as well as with respect to the first two indices,

$$(46.10) \quad R_{\tau\alpha\lambda\beta} = -R_{\alpha\tau\lambda\beta}$$

as follows from (46.7). The curvature tensor has n^4 components (where, in the case of two-dimensional Riemannian spaces, that is, in the case of surfaces, $n = 2$). In consequence of the skew symmetry (46.9) and (46.10) only those components are different from zero for which $\lambda \neq \beta$ and $\tau \neq \alpha$. Hence in the case of surfaces only four components are different from zero, namely

$$(46.11) \quad \begin{aligned} R_{1212} &= R_{2121} = b_{22} b_{11} - (b_{12})^2 = b, \\ R_{2112} &= R_{1221} = (b_{12})^2 - b_{22} b_{11} = -b, \end{aligned}$$

cf. (46.7).

The relations (46.11) are of fundamental importance: Since the Christoffel symbols depend only on the coefficients $g_{\alpha\beta}$ of the first fundamental form, the same holds for the components of the curvature tensor and, in consequence of (46.11) also for the discriminant b of the second fundamental form. In consequence of Definition 41.1 we thus obtain the fundamental

Theorem 46.1 (Theorema egregium, Gauss [1]). *The Gaussian curvature K of a surface is independent of the second fundamental form but depends only on the coefficients $g_{\alpha\beta}$ of the first fundamental form (and their first and second derivatives):*

$$(46.12) \quad K = \frac{R_{1212}}{g}.$$

The meaning of this important fact will be explained in Section 57, in connexion with the concepts of bending and isometric mapping.

By means of (46.6') and (46.11) we obtain the following expressions for the components of the mixed curvature tensor:

$$R^1_{112} = g^{11} R_{1112} + g^{12} R_{2112} = -g^{12}b,$$

etc., hence, in consequence of (32.6),

$$(46.11') \quad \begin{aligned} R^1_{112} &= -R^1_{121} = g_{12} \frac{b}{g}, & R^1_{212} &= -R^1_{221} = g_{22} \frac{b}{g}, \\ -R^2_{112} &= R^2_{121} = g_{11} \frac{b}{g}, & -R^2_{212} &= R^2_{221} = g_{12} \frac{b}{g}; \end{aligned}$$

all other components vanish. If, in particular, the coordinates u^1, u^2 are orthogonal ($g_{12} = 0$) then

$$(46.11'') \quad R^1_{212} = -R^1_{221} = \frac{b}{g_{11}}, \quad -R^2_{112} = R^2_{121} = \frac{b}{g_{22}},$$

that is, in this case only four components of the mixed curvature tensor are different from zero.

The meaning of the curvature tensors will be explained in Chapter VII, in connexion with parallel displacement and absolute differentiation.

From (46.8)–(46.12) we see that in the case of surfaces of vanishing curvature (for example, planes, cones, cylinders) all components of the curvature tensor are zero.

We will now derive a representation of the Gaussian curvature K in terms of the coefficients of the first fundamental form and Christoffel symbols.

Using (38.4) and (51.1) we find

$$b = (\mathbf{n}_1 \cdot \mathbf{x}_1)(\mathbf{n}_2 \cdot \mathbf{x}_2) - (\mathbf{n}_1 \cdot \mathbf{x}_2)(\mathbf{n}_2 \cdot \mathbf{x}_1) = (\mathbf{x}_1 \times \mathbf{x}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2).$$

Noting (34.1) and (41.9) we thus obtain

$$\sqrt{g} K = \frac{b}{\sqrt{g}} = \mathbf{n} \cdot (\mathbf{n}_1 \times \mathbf{n}_2).$$

We introduce the unit vector

$$\mathbf{m} = \mathbf{t} \times \mathbf{n},$$

where \mathbf{t} is a unit vector in the tangent plane of the surface. Then

$$\sqrt{g} K = (\mathbf{m} \times \mathbf{t}) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = (\mathbf{m} \cdot \mathbf{n}_1)(\mathbf{t} \cdot \mathbf{n}_2) - (\mathbf{m} \cdot \mathbf{n}_2)(\mathbf{t} \cdot \mathbf{n}_1).$$

If we differentiate $\mathbf{m} \cdot \mathbf{n} = 0$, $\mathbf{t} \cdot \mathbf{n} = 0$, and $\mathbf{m} \cdot \mathbf{m} = 1$ we have

$$\mathbf{m} \cdot \mathbf{n}_\alpha = -\mathbf{m}_\alpha \cdot \mathbf{n}, \quad \mathbf{t} \cdot \mathbf{n}_\alpha = -\mathbf{t}_\alpha \cdot \mathbf{n}, \quad \mathbf{m} \cdot \mathbf{m}_\alpha = 0, \quad (\alpha = 1, 2).$$

We may thus write

$$\begin{aligned} \sqrt{g} K &= (\mathbf{m}_1 \cdot \mathbf{n})(\mathbf{t}_2 \cdot \mathbf{n}) - (\mathbf{m}_2 \cdot \mathbf{n})(\mathbf{t}_1 \cdot \mathbf{n}) \\ &= \mathbf{m}_1 \cdot [(\mathbf{t}_2 \cdot \mathbf{n})\mathbf{n} + (\mathbf{t}_2 \cdot \mathbf{m})\mathbf{m}] - \mathbf{m}_2 \cdot [(\mathbf{t}_1 \cdot \mathbf{n})\mathbf{n} + (\mathbf{t}_1 \cdot \mathbf{m})\mathbf{m}] \\ &= \mathbf{m}_1 \cdot \mathbf{t}_2 - \mathbf{m}_2 \cdot \mathbf{t}_1. \end{aligned}$$

Hence

$$(46.13) \quad K = \frac{1}{\sqrt{g}} \{(\mathbf{t} \cdot \mathbf{m}_1)_2 - (\mathbf{t} \cdot \mathbf{m}_2)_1\}.$$

We now take the particular unit vector

$$\mathbf{m} = \frac{\mathbf{x}_1}{\sqrt{g_{11}}}.$$

$$\text{Then } \mathbf{m}_\alpha = \frac{\mathbf{x}_{1\alpha}}{\sqrt{g_{11}}} + \mathbf{x}_1 \left(\frac{1}{\sqrt{g_{11}}} \right)_\alpha, \quad (\alpha = 1, 2).$$

$$\text{Since } \mathbf{t} = \mathbf{n} \times \mathbf{m} = \mathbf{n} \times \frac{\mathbf{x}_1}{\sqrt{g_{11}}}$$

we thus obtain

$$(46.14) \quad K = \frac{1}{\sqrt{g}} \left\{ \left(\mathbf{t} \cdot \frac{\mathbf{x}_{11}}{\sqrt{g_{11}}} \right)_2 - \left(\mathbf{t} \cdot \frac{\mathbf{x}_{12}}{\sqrt{g_{11}}} \right)_1 \right\}.$$

Taking the scalar product of (45.14) and $\mathbf{n} \times \mathbf{x}_1$ and noting (34.1) we have

$$|\mathbf{n} \cdot \mathbf{x}_1 \cdot \mathbf{x}_{\alpha\beta}| = |\mathbf{n} \cdot \mathbf{x}_1 \cdot \mathbf{x}_2| \Gamma_{\alpha\beta}^2 = \mathbf{n} \cdot \mathbf{n} \sqrt{g} \Gamma_{\alpha\beta}^2.$$

Hence

$$(46.15) \quad |\mathbf{n} \cdot \mathbf{x}_1 \cdot \mathbf{x}_{\alpha\beta}| = \sqrt{g} \Gamma_{\alpha\beta}^2.$$

Since $\mathbf{t} = \mathbf{n} \times \mathbf{m}$ we thus obtain from (46.14) the following representation

$$(46.16) \quad K = -\frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial u^1} \left(\frac{\sqrt{g}}{g_{11}} \Gamma_{12}^2 \right) - \frac{\partial}{\partial u^2} \left(\frac{\sqrt{g}}{g_{11}} \Gamma_{11}^2 \right) \right\}.$$

By interchanging u^1 and u^2 in the above proof we can show that

$$(46.16') \quad K = \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial u^1} \left(\frac{\sqrt{g}}{g_{22}} \Gamma_{22}^1 \right) - \frac{\partial}{\partial u^2} \left(\frac{\sqrt{g}}{g_{22}} \Gamma_{21}^1 \right) \right\}.$$

If we start from (41.9) and use (38.8) we have

$$K = \frac{b}{g} = \frac{1}{g^2} \{ |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{11}| |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{22}| - |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{12}|^2 \}.$$

Carrying out the multiplication and expressing the elements of the resulting determinants in terms of the $g_{\alpha\beta}$ and their first and second derivatives we obtain another representation of the Gaussian curvature:

$$(46.17) \quad K = \frac{1}{g^2} \begin{vmatrix} g_{11} & g_{12} & \frac{\partial g_{12}}{\partial u^2} - \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} \\ g_{12} & g_{22} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^2} \\ \frac{1}{2} \frac{\partial g_{11}}{\partial u^1} & \left(\frac{\partial g_{12}}{\partial u^1} - \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} \right) & \left(-\frac{1}{2} \frac{\partial^2 g_{11}}{(\partial u^2)^2} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2} \frac{\partial^2 g_{22}}{(\partial u^1)^2} \right) \end{vmatrix} - \begin{vmatrix} g_{11} & g_{12} & \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} \\ g_{12} & g_{22} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} \\ \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} & 0 \end{vmatrix}$$

This representation is sometimes called the *equation of Gauss*.

If the coordinates are orthogonal we obtain from (46.16) or (46.17)

$$(46.17') \quad K = -\frac{1}{\sqrt{(g_{11}g_{22})}} \left\{ \frac{\partial}{\partial u^1} \left(\frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} \right) + \frac{\partial}{\partial u^2} \left(\frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial u^2} \right) \right\}.$$

Problem 46.1. Determine the components of the curvature tensors (1) of a surface of revolution $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2))$, (2) of a right conoid $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^1))$, cf. Section 28, (3) of a surface represented in the form $x_3 = F(x_1, x_2)$, (4) of the torus (43.1).

Problem 46.2. Prove (46.17).

47. Properties of the Christoffel symbols. The Christoffel symbols were introduced in Section 45, in connexion with the formulae of Gauss. Since these symbols will be frequently used we present a short collection of their most important properties.

The Christoffel symbols of the first kind are given by the expressions

$$(47.1) \quad \Gamma_{\alpha\beta\gamma} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\gamma = \frac{1}{2} \left[\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} \right],$$

cf. (45.6) and (45.13). From (47.1) we obtain the Christoffel symbols of the second kind,

$$(47.2) \quad \Gamma_{\alpha\beta}^\sigma = g^{\sigma\gamma} \Gamma_{\alpha\beta\gamma},$$

cf. (45.8). Using (47.1) and $\mathbf{x}^\sigma = g^{\sigma\gamma} \mathbf{x}_\gamma$ (cf. Section 32) we have

$$\Gamma_{\alpha\beta}^\sigma = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\sigma.$$

From (47.2) it follows that

$$g_{\gamma\sigma} \Gamma_{\alpha\beta}^\sigma = g_{\gamma\sigma} g^{\sigma\lambda} \Gamma_{\alpha\beta\lambda} = \delta_\gamma^\lambda \Gamma_{\alpha\beta\lambda} = \Gamma_{\alpha\beta\gamma},$$

that is,

$$(47.2') \quad \Gamma_{\alpha\beta\gamma} = g_{\gamma\sigma} \Gamma_{\alpha\beta}^\sigma,$$

cf. (45.7).

Since $\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha}$ the Christoffel symbols are symmetric with respect to the *first two indices*,

$$(47.3) \quad \Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}, \quad \Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma,$$

cf. (45.9). Consequently in the case of two-dimensional Riemannian spaces, that is, in the case of surfaces, there are six symbols of the first kind which are different from each other, namely

$$\Gamma_{111}, \quad \Gamma_{112}, \quad \Gamma_{121}, \quad \Gamma_{122}, \quad \Gamma_{221}, \quad \Gamma_{222}$$

and the same holds with respect to the symbols of the second kind.

The first partial derivatives of the coefficients $g_{\alpha\beta}$ of the first fundamental form can be represented as a sum of two Christoffel symbols of the first kind,

$$(47.4) \quad \frac{\partial g_{\alpha\beta}}{\partial u^\kappa} = \Gamma_{\alpha\kappa\beta} + \Gamma_{\beta\kappa\alpha} = g_{\lambda\beta} \Gamma_{\alpha\kappa}^\lambda + g_{\lambda\alpha} \Gamma_{\beta\kappa}^\lambda,$$

cf. (45.10). We will now obtain similar formulae for the components $g^{\alpha\beta}$ of the contravariant metric tensor. If we differentiate (32.7):

$g_{\alpha\rho} g^{\alpha\sigma} = \delta_\rho^\sigma$
with respect to u^κ we have

$$\frac{\partial g_{\alpha\rho}}{\partial u^\kappa} g^{\alpha\sigma} + g_{\alpha\rho} \frac{\partial g^{\alpha\sigma}}{\partial u^\kappa} = 0.$$

Taking the inner product of this expression and $g^{\lambda\rho}$ we obtain

$$\frac{\partial g_{\alpha\rho}}{\partial u^\kappa} g^{\alpha\sigma} g^{\lambda\rho} + \frac{\partial g^{\lambda\sigma}}{\partial u^\kappa} = 0.$$

Hence, by means of (47.4),

$$\frac{\partial g^{\lambda\sigma}}{\partial u^\kappa} = -\frac{\partial g_{\alpha\rho}}{\partial u^\kappa} g^{\alpha\sigma} g^{\lambda\rho} = -[\Gamma_{\alpha\kappa\rho} + \Gamma_{\rho\kappa\alpha}] g^{\alpha\sigma} g^{\lambda\rho}$$

and therefore, in consequence of (47.2),

$$(47.5) \quad \frac{\partial g^{\lambda\sigma}}{\partial u^\kappa} = -g^{\alpha\sigma} \Gamma_{\alpha\kappa}^\lambda - g^{\lambda\rho} \Gamma_{\rho\kappa}^\sigma.$$

For the derivatives of the discriminant g of the first fundamental form similar relations can be obtained. We have

$$\frac{\partial g}{\partial u^\gamma} = \frac{\partial}{\partial u^\gamma} (g_{11} g_{22} - (g_{12})^2) = \frac{\partial g_{11}}{\partial u^\gamma} g_{22} + g_{11} \frac{\partial g_{22}}{\partial u^\gamma} - 2g_{12} \frac{\partial g_{12}}{\partial u^\gamma}.$$

Using (32.6) we obtain from this

$$\frac{\partial g}{\partial u^\gamma} = g \left[g^{11} \frac{\partial g_{11}}{\partial u^\gamma} + g^{22} \frac{\partial g_{22}}{\partial u^\gamma} + 2g^{12} \frac{\partial g_{12}}{\partial u^\gamma} \right] = gg^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}$$

or, in consequence of (47.4) and (47.2),

$$\frac{\partial g}{\partial u^\gamma} = gg^{\alpha\beta} (\Gamma_{\alpha\gamma\beta} + \Gamma_{\beta\gamma\alpha}) = g(\Gamma_{\alpha\gamma}^\alpha + \Gamma_{\beta\gamma}^\beta).$$

Replacing the summation index β by α we find

$$(47.6) \quad \frac{\partial g}{\partial u^\gamma} = 2g \Gamma_{\alpha\gamma}^\alpha$$

or

$$(47.6') \quad \Gamma_{\alpha\gamma}^\alpha = \frac{1}{2g} \frac{\partial g}{\partial u^\gamma} = \frac{\partial}{\partial u^\gamma} (\log \sqrt{g}).$$

We will now give explicit formulae for the Christoffel symbols (in the case of surfaces). From (47.1) we obtain

$$(47.7) \quad \Gamma_{\alpha\alpha\alpha} = \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^\alpha}, \quad \Gamma_{\alpha\alpha\beta} = \frac{\partial g_{12}}{\partial u^\alpha} - \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^\beta}, \quad \Gamma_{\alpha\beta\alpha} = \Gamma_{\beta\alpha\alpha} = \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^\beta}, \\ (\alpha \neq \beta).$$

From (47.2) and (47.7) we obtain

$$(47.8) \quad \left. \begin{aligned} \Gamma_{\alpha\alpha}^{\alpha} &= \frac{1}{2g} \left[g_{\beta\beta} \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + g_{12} \left(\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} - 2 \frac{\partial g_{12}}{\partial u^{\alpha}} \right) \right] \\ \Gamma_{\alpha\alpha}^{\beta} &= \frac{1}{2g} \left[g_{\alpha\alpha} \left(2 \frac{\partial g_{12}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \right) - g_{12} \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} \right] \\ \Gamma_{\alpha\beta}^{\beta} &= \Gamma_{\beta\alpha}^{\beta} = \frac{1}{2g} \left[g_{\alpha\alpha} \frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} - g_{12} \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \right] \end{aligned} \right\} (\alpha \neq \beta, \text{ do not sum}).$$

(Since in the case of surfaces any index can only take the value 1 or 2 it is clear what is meant by ' $\alpha \neq \beta$ ' in the first line of (47.8).) In the particular case of orthogonal coordinates we have $g_{12} = 0$ and $g = g_{11}g_{22}$. Consequently in this case

$$(47.7') \quad \Gamma_{\alpha\alpha\alpha} = \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}}, \quad \Gamma_{\alpha\alpha\beta} = -\frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}, \quad \Gamma_{\alpha\beta\alpha} = \Gamma_{\beta\alpha\alpha} = \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}, \quad (\alpha \neq \beta)$$

and

$$(47.8') \quad \left. \begin{aligned} \Gamma_{\alpha\alpha}^{\alpha} &= \frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} (\log \sqrt{g_{\alpha\alpha}}) \\ \Gamma_{\alpha\alpha}^{\beta} &= -\frac{1}{2g_{\beta\beta}} \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \\ \Gamma_{\alpha\beta}^{\beta} &= \Gamma_{\beta\alpha}^{\beta} = \frac{1}{2g_{\beta\beta}} \frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} (\log \sqrt{g_{\beta\beta}}) \end{aligned} \right\} (\alpha \neq \beta, \text{ do not sum}).$$

We have already stressed that the Christoffel symbols are not components of a tensor. We may obtain the transformation law of these symbols by means of (45.14); we have

$$(47.9) \quad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^{\alpha} \partial u^{\beta}} = \Gamma_{\alpha\beta}^{\gamma} \mathbf{x}_{\gamma} + b_{\alpha\beta} \mathbf{n}$$

and

$$(47.10) \quad \mathbf{x}_{\lambda\epsilon} \equiv \frac{\partial^2 \mathbf{x}}{\partial \bar{u}^{\lambda} \partial \bar{u}^{\epsilon}} = \bar{\Gamma}_{\lambda\epsilon}^{\sigma} \mathbf{x}_{\sigma} + \bar{b}_{\lambda\epsilon} \bar{\mathbf{n}}$$

where $\bar{\Gamma}_{\lambda\epsilon}^{\sigma}$ are the Christoffel symbols of the second kind with respect to the coordinates \bar{u}^1, \bar{u}^2 . Since, on the other hand,

$$\mathbf{x}_{\alpha} = \mathbf{x}_{\lambda} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}}$$

and, by differentiation with respect to u^{β} ,

$$\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\lambda} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \mathbf{x}_{\sigma} \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}},$$

we obtain from (47.10)

$$(47.11) \quad x_{\alpha\beta} = [\bar{\Gamma}_{\lambda\epsilon}^{\sigma} x_{\bar{\sigma}} + \bar{b}_{\lambda\epsilon} \bar{n}] \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + x_{\bar{\sigma}} \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}}.$$

Since x_1 , x_2 , and n are linearly independent vectors for each of them the coefficient on the right-hand sides of (47.9) and (47.11) must be the same. We thus find

$$\Gamma_{\alpha\beta}^{\gamma} x_{\gamma} = \bar{\Gamma}_{\lambda\epsilon}^{\sigma} x_{\bar{\sigma}} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + x_{\bar{\sigma}} \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}}.$$

Since

$$x_{\gamma} = x_{\bar{\sigma}} \frac{\partial \bar{u}^{\sigma}}{\partial u^{\gamma}}$$

we finally obtain the transformation law of the Christoffel symbols of the second kind:

$$(47.12) \quad \Gamma_{\alpha\beta}^{\gamma} \frac{\partial \bar{u}^{\sigma}}{\partial u^{\gamma}} = \bar{\Gamma}_{\lambda\epsilon}^{\sigma} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}}.$$

In consequence of (29.3) these formulae are equivalent to

$$(47.12') \quad \Gamma_{\alpha\beta}^{\gamma} = \left[\bar{\Gamma}_{\lambda\epsilon}^{\sigma} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}} \right] \frac{\partial u^{\gamma}}{\partial \bar{u}^{\sigma}}.$$

Relations (47.12) are sometimes called the *formulae of Christoffel*.

By taking the inner product of both sides of (47.12') and

$$g_{\nu\rho} = \bar{g}_{\nu\tau} \frac{\partial \bar{u}^{\tau}}{\partial u^{\nu}} \frac{\partial \bar{u}^{\rho}}{\partial u^{\rho}}$$

we obtain

$$\Gamma_{\alpha\beta\rho} = g_{\gamma\rho} \Gamma_{\alpha\beta}^{\gamma} = \bar{g}_{\nu\tau} \left[\bar{\Gamma}_{\lambda\epsilon}^{\sigma} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}} \right] \frac{\partial u^{\gamma}}{\partial \bar{u}^{\sigma}} \frac{\partial u^{\nu}}{\partial u^{\gamma}} \frac{\partial u^{\tau}}{\partial u^{\rho}},$$

that is, the transformation law of the Christoffel symbols of the first kind is of the form

$$(47.13) \quad \Gamma_{\alpha\beta\rho} = \left[\bar{\Gamma}_{\lambda\epsilon\tau} \frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \bar{g}_{\sigma\tau} \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha} \partial u^{\beta}} \right] \frac{\partial u^{\gamma}}{\partial \bar{u}^{\rho}}.$$

In the plane there exist coordinate systems with respect to which the Christoffel symbols vanish everywhere. On an arbitrary surface we may introduce coordinate systems with respect to which, at an arbitrary fixed point P_0 , the Christoffel symbols vanish. Indeed, if u_0^{α} ($\alpha = 1, 2$) are the coordinates of P_0 with respect to an arbitrary allowable coordinate system and if $(\Gamma_{\beta\gamma}^{\alpha})_0$ are the values of the Christoffel symbols at P_0 then the coordinates $u^{*\alpha}$, $u^{*\beta}$ defined by

$$(47.14) \quad u^{\alpha} = u^{*\alpha} + u_0^{\alpha} - \frac{1}{2} (\Gamma_{\beta\gamma}^{\alpha})_0 u^{*\beta} u^{*\gamma}, \quad (\alpha = 1, 2),$$

have the desired property: P_0 is the origin of the $u^{*\alpha}$ -coordinate system,

and by differentiation we see that at P_0

$$(47.15) \quad (a) \frac{\partial u^\alpha}{\partial u^{*\sigma}} = \delta_\sigma^\alpha \quad \text{and} \quad (b) \frac{\partial^2 u^\alpha}{\partial u^{*\sigma} \partial u^{*\tau}} = -(\Gamma_{\sigma\tau}^\alpha)_0, \quad (u^{*1} = u^{*2} = 0).$$

As follows from (47.15 a), the coordinate transformation (47.14) is allowable in a neighbourhood of P_0 . By means of (47.15 b) and (47.12) we find that at P_0

$$\Gamma_{\sigma\tau}^* \frac{\partial u^\alpha}{\partial u^{*\rho}} = \Gamma_{\kappa\lambda}^\alpha \frac{\partial u^\kappa}{\partial u^{*\sigma}} \frac{\partial u^\lambda}{\partial u^{*\tau}} - (\Gamma_{\sigma\tau}^\alpha)_0, \quad (u^{*1} = u^{*2} = 0),$$

hence, by (47.15 a),

$$\Gamma_{\sigma\tau}^{*\alpha} = (\Gamma_{\sigma\tau}^\alpha)_0 - (\Gamma_{\sigma\tau}^\alpha)_0 = 0.$$

Problem 47.1. Determine the Christoffel symbols corresponding to polar coordinates in a plane.

Problem 47.2. Determine the Christoffel symbols of a surface of revolution represented in the form $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2))$.

Problem 47.3. Determine the Christoffel symbols of a surface represented in the form (24.3): $x_3 = F(x_1, x_2)$.

48. Umbilics. A point P of a surface S is called a *navel point* or *umbilic* if all normal sections of S at P have the same curvature at P . The point P is an umbilic if and only if at this point

$$(48.1) \quad b_{\alpha\beta} = \lambda(u^1, u^2)g_{\alpha\beta}, \quad (\alpha = 1, 2, \beta = 1, 2),$$

where the factor λ is independent of the direction of the normal section, cf. Section 41.

For example, if we rotate the curve $x_3 = x_1^2, x_2 = 0$ about the x_3 -axis we obtain a paraboloid of revolution which has an umbilic at $\mathbf{x} = (0, 0, 0)$. In the case of an ellipsoid of revolution the points of intersection of this surface and the axis of revolution are umbilics.

We will now consider surfaces whose points are all umbilics.

Theorem 48.1. *If every point of a surface S which has a representation of class $r \geq 3$ is an umbilic, S is a plane or a sphere.*

Proof. In order that every point of S be an umbilic (48.1) must hold at every point of S . (a) If $\lambda = 0$ the coefficients of the second fundamental form vanish identically, that is, S is a plane, cf. Problem 45.1. (b) Suppose now $\lambda \neq 0$. Since any point of S is assumed to be an umbilic, any curve on S is a line of curvature; hence from (45.1') and (48.1) we obtain

$$(48.2) \quad \mathbf{n}_\alpha = -\frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} \mathbf{x}_\alpha = -\lambda \mathbf{x}_\alpha \quad (\alpha = 1, 2 \text{ (do not sum)}).$$

From (48.2) it follows that

$$\mathbf{n}_{\alpha\beta} = -\lambda_\beta \mathbf{x}_\alpha - \lambda \mathbf{x}_{\alpha\beta}, \quad \lambda_\beta \equiv \frac{\partial \lambda}{\partial u^\beta}.$$

Analogously

$$\mathbf{n}_{\beta\alpha} = -\lambda_\alpha \mathbf{x}_\beta - \lambda \mathbf{x}_{\beta\alpha}.$$

By subtraction we find $\lambda_1 \mathbf{x}_2 - \lambda_2 \mathbf{x}_1 = \mathbf{0}$.

Since, at regular points of a surface, \mathbf{x}_1 and \mathbf{x}_2 are linearly independent vectors, $\lambda_1 = \lambda_2 = 0$ must hold, that is, $\lambda = \text{const.}$ Hence, by integrating (48.2) we obtain

$$\mathbf{n} = -\lambda \mathbf{x} + \mathbf{c},$$

where \mathbf{c} is a constant vector. Scalar multiplication yields

$$\mathbf{n} \cdot \mathbf{n} = (-\lambda \mathbf{x} + \mathbf{c}) \cdot (-\lambda \mathbf{x} + \mathbf{c}).$$

If we set $\lambda = -1/r$ and $\mathbf{c} = -\mathbf{x}_0/r$ then since $\mathbf{n} \cdot \mathbf{n} = 1$,

$$(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = r^2.$$

This is a representation of a sphere of radius r with centre at \mathbf{x}_0 .

GEODESIC CURVATURE AND GEODESICS

49. Geodesic curvature. We now will ask for a shortest arc on a surface S joining two arbitrary (fixed) points P_1 and P_2 . Of course there exist infinitely many arcs on S with endpoints P_1 and P_2 . The question is whether this set of arcs includes an arc C of minimum length and what conditions C must satisfy in order to be the shortest of all those arcs. Obviously this is a problem of calculus of variation. It was first considered by Johann Bernoulli (1697).

The problem becomes very simple if S is a plane. In this case a uniquely determined shortest arc joining P_1 and P_2 exists, namely, the segment of a straight line which has the endpoints P_1 and P_2 . On an arbitrary surface the curves of minimal length will play a role similar to that of straight lines in a plane.

Straight lines can be characterized by the property that their curvature vanishes identically. We first introduce the fundamental concept of geodesic curvature κ_g of a curve on a surface. In the following section we will study curves whose geodesic curvature vanishes identically; these curves are called geodesics. In Section 51 we shall prove that an arc of minimum length must *necessarily* be an arc of a geodesic. Finally we shall obtain sufficient conditions which must be satisfied in order that an arc C be the shortest of all arcs which join two given points P_1 and P_2 on a surface and lie in a neighbourhood of C .

We will now define the concept of geodesic curvature κ_g of a curve C on a surface S : $x(u^1, u^2)$ of class $r \geq 2$. We assume that C is given by a parametric representation $u^1 = u^1(s)$, $u^2 = u^2(s)$, where s is the arc length of C . Let P be an arbitrary point of C . We denote by C' the orthogonal projection of C on the tangent plane $E(P)$ to S at P , cf. Fig. 57. The *geodesic curvature* κ_g of C at P is defined as the curvature of the projected curve C' at P , taken with a suitable sign. The sign of κ_g is defined as follows: Suppose that the curvature of C' at P is not zero. Then κ_g has a positive sign if the centre of curvature of C' at P lies in the direction of the unit vector

$$(49.1) \quad \mathbf{e} = \mathbf{n} \times \mathbf{t},$$

where \mathbf{n} is the unit normal vector to S at P and \mathbf{t} is the unit tangent vector

to C at P ; κ_g has a negative sign if that centre of curvature lies in the direction opposite to the vector e . The sign of κ_g thus depends on the orientations of S and C . (In Fig. 57, κ_g is negative.)

The name 'geodesic curvature' was introduced by Liouville (1850).

According to (39.1) the normal curvature κ_n of S at P in the direction of C is given by

$$\kappa_n = \kappa \cos \gamma,$$

where γ is the angle between the unit normal vector n to S and the unit principal normal vector p to C at P . We now consider the cylinder Z which projects the curve C orthogonally on $E(P)$. The generating lines of

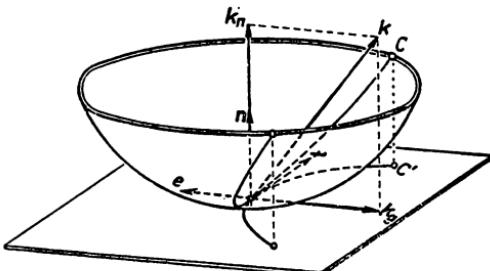


FIG. 57

Z have the direction of n . The normal to Z at P has the direction of e . Z intersects $E(P)$ orthogonally along C' . Hence the curvature of C' at P which, by definition, is $|\kappa_g|$ is the normal curvature of Z at P in the direction of the unit tangent vector t to C . In consequence of (39.1) we therefore have

$$(49.2) \quad |\kappa_g| = \kappa \sin \gamma.$$

We may represent the curvature vector $k = \kappa p$ of C at P as a sum of the normal curvature vector $k_n = \kappa_n n$ (cf. Section 39) and of the *geodesic curvature vector*

$$(49.3) \quad k_g = \kappa_g e,$$

$$(49.4) \quad k = k_n + k_g = \kappa_n n + \kappa_g e.$$

If we take the scalar product of (49.4) and e , in consequence of (49.1) we find

$$e \cdot k = e \cdot t = (n \times t) \cdot t = |t \cdot t \cdot n| = \kappa_g$$

or

$$(49.5) \quad \kappa_g = |\dot{x} \ddot{x} \cdot n|.$$

Clearly, in general, the geodesic curvature κ_g depends not only on the curve C but also on the surface S on which C lies, except for the case when $\dot{t} = \ddot{x} = 0$, and therefore $\kappa_g = 0$. Thus a circle C of radius r , when considered as a great circle on a sphere, has $\kappa_g = 0$, since in this

case the vectors \mathbf{t} , $\dot{\mathbf{t}}$, and \mathbf{n} lie in a plane (through the centre of the sphere). If, however, C lies in a plane, then $|\kappa_g| = \kappa = 1/r$.

There is a fundamental contrast between the normal curvature κ_n and the geodesic curvature κ_g . While κ_n depends on both fundamental forms, κ_g depends on the first one only.

Theorem 49.1. *The geodesic curvature κ_g of a curve C on a surface S depends on the first fundamental form of S only (and also, of course, on C).*

Proof. We have to find a representation of κ_g in terms of functions of the coefficients $g_{\alpha\beta}$ of the first fundamental form (of course, quantities which depend on the curve C will also occur in this representation). If we differentiate

$$\dot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha$$

with respect to the arc length s of C we obtain

$$\ddot{\mathbf{x}} = \mathbf{x}_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma + \mathbf{x}_\epsilon \ddot{u}^\epsilon.$$

From this, by (45.14) we find

$$(49.6) \quad \dot{\mathbf{x}} \times \ddot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha \times [(\Gamma_{\beta\gamma}^\epsilon \dot{u}^\beta \dot{u}^\gamma + \ddot{u}^\epsilon) \mathbf{x}_\epsilon + b_{\beta\gamma} \mathbf{n} \dot{u}^\beta \dot{u}^\gamma].$$

We have $\mathbf{x}_1 \times \mathbf{x}_2 = \sqrt{g} \mathbf{n}$, $\mathbf{x}_2 \times \mathbf{x}_1 = -\sqrt{g} \mathbf{n}$,

cf. (34.1). Furthermore, since $\mathbf{x}_\alpha \times \mathbf{x}_\alpha = \mathbf{0}$, $\mathbf{n} \times \mathbf{n} = \mathbf{0}$, and $\mathbf{n} \cdot \mathbf{n} = 1$, we finally obtain

$$\kappa_g = (\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) \cdot \mathbf{n} = \sqrt{g} (\Gamma_{\beta\gamma}^2 \dot{u}^1 \dot{u}^\beta \dot{u}^\gamma + \dot{u}^1 \dot{u}^2 - \Gamma_{\beta\gamma}^1 \dot{u}^2 \dot{u}^\beta \dot{u}^\gamma - \dot{u}^2 \dot{u}^1)$$

or

$$(49.7) \quad \kappa_g = \sqrt{g} [\Gamma_{11}^2 (\dot{u}^1)^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1) (\dot{u}^1)^2 \dot{u}^2 - (2\Gamma_{12}^1 - \Gamma_{22}^2) \dot{u}^1 (\dot{u}^2)^2 - \Gamma_{22}^1 (\dot{u}^2)^3 + \dot{u}^1 \dot{u}^2 - \dot{u}^1 \dot{u}^2].$$

This completes the proof of Theorem 49.1.

In (49.7) we may easily replace the arc length s by any allowable parameter t . Denoting by primes the derivatives with respect to t we have

$$\dot{u}^1 \dot{u}^2 - \dot{u}^1 \dot{u}^2 = (u^1' u^2'' - u^1'' u^2') t^3.$$

Analogously, all terms of (49.7) are multiplied by t^3 ; we therefore have a common factor t^3 on the right-hand side of (49.7) and have just to replace the dots by primes.

We apply (49.7) to the coordinate curves. On the curves $u^1 = \text{const}$ we have $\dot{u}^1 = 0$ and $\dot{u}^2 = 1/\sqrt{g_{22}}$. We thus obtain

$$(49.8 \text{ a}) \quad (\kappa_g)_{u^1=\text{const}} = -\Gamma_{22}^1 \frac{\sqrt{g}}{g_{22}^{1/2}}.$$

Analogously the geodesic curvature of the curves $u^2 = \text{const}$ is given by

the expression

$$(49.8 \text{ b}) \quad (\kappa_g)_{u^1=\text{const}} = \Gamma_{11}^2 \frac{\sqrt{g}}{g_{11}^{\frac{1}{4}}}.$$

If, in particular, the coordinates u^1, u^2 are orthogonal, by (47.8'):

$$\Gamma_{22}^{-1} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial u^1}, \quad \Gamma_{11}^{-2} = -\frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial u^2},$$

and therefore in this case

$$(49.9) \quad (a) \quad (\kappa_g)_{u^1=\text{const}} = \frac{1}{2g_{22}\sqrt{g_{11}}} \frac{\partial g_{22}}{\partial u^1} = \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial u^1} (\log \sqrt{g_{22}}), \quad (g_{12} = 0).$$

$$(b) \quad (\kappa_g)_{u^2=\text{const}} = -\frac{1}{2g_{11}\sqrt{g_{22}}} \frac{\partial g_{11}}{\partial u^2} = -\frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^2} (\log \sqrt{g_{11}})$$

50. Geodesics. A curve C on a surface S is called a *geodesic curve* or *geodesic* if its geodesic curvature κ_g vanishes identically.

Theorem 50.1. *Straight lines on any surface are geodesics. A curve C of class $r \geq 2$, not a straight line, is a geodesic on a surface S if and only if at each point of C at which the curvature κ of C is not zero the unit normal vector \mathbf{n} to S lies in the osculating plane O of C .*

Proof. We use (49.2). At points of C at which $\kappa = 0$, also $\kappa_g = 0$. At any other point we have: if \mathbf{n} lies in O then \mathbf{n} and \mathbf{p} lie in the same line and therefore κ_g vanishes, cf. (49.2). Conversely, if $\kappa_g = 0$ then, in consequence of (49.2), either κ must vanish or γ must have the values 0 or 2π , that is, \mathbf{n} must lie in the osculating plane O of C .

Along an asymptotic curve A on a surface S the osculating planes of A and the tangent planes to S coincide, while at any point of a geodesic on S those planes are orthogonal to each other. Furthermore, at any point of an asymptotic curve A the normal curvature κ_n of the surface (in the direction of A) vanishes, while at any point of a geodesic its geodesic curvature κ_g is zero. In the case of an asymptotic curve we have $|\kappa_g| = \kappa$, cf. (49.4).

Let us now consider the differential equation of the geodesics,

$$(50.1) \quad \kappa_g = 0$$

somewhat more in detail. Introducing in (49.7) u^1 as parameter we have

$$\dot{u}^1 \ddot{u}^2 - \dot{u}^1 \dot{u}^2 = \frac{d^2 u^2}{(du^1)^2} (u^1)^3$$

and (50.1) therefore takes the form

$$(50.2) \quad \frac{d^2 u^2}{(du^1)^2} - \Gamma_{22}^{-1} \left(\frac{du^2}{du^1} \right)^3 - (2\Gamma_{12}^{-1} - \Gamma_{22}^{-2}) \left(\frac{du^2}{du^1} \right)^2 + \\ + (2\Gamma_{12}^{-2} - \Gamma_{11}^{-1}) \frac{du^2}{du^1} + \Gamma_{11}^{-2} = 0.$$

This differential equation of second order is of the form

$$\frac{d^2u^2}{(du^1)^2} = h\left(u^1, u^2, \frac{du^2}{du^1}\right).$$

If we assume that the representations of the surface S and of the curve on S under consideration are of class $r \geq 3$ then h is a function of class $r-2 \geq 1$, and (50.2) has exactly one solution satisfying certain given initial conditions

$$u^2(u_0^1) = u_0^2, \quad \left. \frac{du^2}{du^1} \right|_{u^1=u_0^1} = \left(\frac{du^2}{du^1} \right)_0.$$

This means that at any point P on a surface S there is exactly one geodesic passing through P in a given direction. Furthermore, as will be seen in Section 53, there is exactly one geodesic through two given points of a (sufficiently small) portion of a surface.

In the large the latter property is true for the plane, but not in general. Thus on a sphere there are infinitely many great circles through two diametrically opposite points. On a cylinder of revolution the generating straight lines, circles, and circular helices are geodesics. Consequently, for two given points P and Q , not on the same circle, there are infinitely many helices through P and Q .

From (49.8) we immediately obtain

Theorem 50.2. *The coordinate curves $u^1 = \text{const}$ and $u^2 = \text{const}$ on a portion of a surface S : $\mathbf{x}(u^1, u^2)$ of class $r \geq 2$ are geodesics if and only if*

$$\Gamma_{22}^1 = 0 \quad \text{and} \quad \Gamma_{11}^2 = 0,$$

respectively.

Example. In the representation (25.2) of the sphere

$$\mathbf{x}(u^1, u^2) = (r \cos u^2 \cos u^1, r \cos u^2 \sin u^1, r \sin u^2)$$

the coordinates u^1, u^2 are orthogonal. We have $g_{11} = r^2 \cos^2 u^2$, $g_{22} = r^2$, $\partial g_{22} / \partial u^1 = 0$; therefore $\Gamma_{22}^1 = 0$, cf. (47.8), that is, the curves $u^1 = \text{const}$ are geodesics. Furthermore

$$(50.3) \quad \frac{\partial g_{11}}{\partial u^2} = -2r^2 \cos u^2 \sin u^2.$$

This expression vanishes if $u^2 = 0$, that is, $\Gamma_{11}^2 = 0$ along the equator which is the only geodesic of the curves $u^2 = \text{const}$. (At the poles $u^2 = \pm \frac{1}{2}\pi$ and also $\Gamma_{11}^2 = 0$ but (25.2) is not valid at these points.)

Furthermore we see from (49.9) that the coordinate curves of orthogonal coordinates are geodesics if and only if g_{11} does not depend on u^2 and g_{22} does not depend on u^1 .

Finally we mention an important application of Theorem 50.1. Let us determine the torsion $\tau(s)$ of a geodesic $G: \mathbf{x}(s)$ (not a straight line) on a surface $S: \mathbf{x}(u^1, u^2)$ in Euclidean space R_3 , where s denotes the arc length of G . According to Theorem 50.1 the principal normal of G and the normal of S coincide. If we choose a suitable orientation of the unit normal vector \mathbf{n} of S we thus have

$$\mathbf{p} = \mathbf{n},$$

where \mathbf{p} is the unit principal normal vector of G . Hence G has the unit binormal vector

$$\mathbf{b} = \dot{\mathbf{x}} \times \mathbf{p} = \dot{\mathbf{x}} \times \mathbf{n}.$$

By (15.1) we thus obtain

$$(50.4) \quad \dot{\mathbf{b}} = -\tau \mathbf{p} = -\tau \mathbf{n} = \ddot{\mathbf{x}} \times \mathbf{n} + \dot{\mathbf{x}} \times \dot{\mathbf{n}}.$$

Since $\ddot{\mathbf{x}}$ and \mathbf{n} are parallel vectors the first of the two vector products vanishes. Hence

$$(50.5) \quad \tau \mathbf{n} = \dot{\mathbf{n}} \times \dot{\mathbf{x}}.$$

If we take the scalar product of (50.5) and \mathbf{n} we thus find

$$(50.6) \quad \tau = |\mathbf{n} \cdot \dot{\mathbf{n}} \cdot \dot{\mathbf{x}}|.$$

Obviously this expression for the torsion of a geodesic G at a point P of S depends only on P and on the direction of the tangent to G at P . We may thus define: The *geodesic torsion* of a curve C on the surface S at a point P is the torsion of the geodesic through P which touches C at P . The geodesic torsion is thus given by (50.6) and will be denoted by τ_g .

From (50.6) and Theorem 45.1 we obtain

Theorem 50.3. *The geodesic torsion of a line of curvature vanishes identically.*

Furthermore we have

Theorem 50.4. *If a geodesic G on a surface S is a line of curvature on S then G is a plane curve. Every plane geodesic on a surface S is a line of curvature on S .*

A relation between the geodesic torsion τ_g and the torsion τ of a curve on a surface will be derived in Section 88.

Problem 50.1. Determine the geodesics of the plane by means of (50.2).

Problem 50.2. Prove that any circular helix C on a cylinder of revolution S is a geodesic on S .

51. Arcs of minimum length. Let P_1 and P_2 be any two (fixed) points on a surface S : $\mathbf{x}(u^1, u^2)$ of class $r \geq 2$. We assume that there exists an arc of class $\tilde{r} \geq 2$

$$C: \quad u^1 = F^1(t), \quad u^2 = F^2(t), \quad (t_1 \leq t \leq t_2)$$

with endpoints P_1 and P_2 such that C has minimum length among all arcs

$$C^*: \quad u^1 = F^{1*}(t), \quad u^2 = F^{2*}(t), \quad (t_1 \leq t \leq t_2)$$

on S which join P_1 and P_2 and lie in a neighbourhood of C . We will prove that C must be an arc of a geodesic. We set

$$(51.1) \quad F^{i*}(t) = F^i(t) + \epsilon v^i(t), \quad v^i(t_1) = v^i(t_2) = 0, \quad (i = 1, 2),$$

where $|\epsilon|$ is sufficiently small. Furthermore we assume that any function of t occurring is of class $r^* \geq 2$. According to (35.1) C has the length

$$s(C) = \int_{t_1}^{t_2} h(F^1, F^2, F^1', F^2') dt,$$

where

$$(51.2) \quad h = \sqrt{(g_{\alpha\beta} F^\alpha F^\beta)};$$

primes denote derivatives with respect to t . C^* has the length

$$s(C^*) = s(\epsilon) = \int_{t_1}^{t_2} h(F^{1*}, F^{2*}, F^{1*'}, F^{2*'}) dt.$$

We have $s(0) = s(C)$. In order that $s(\epsilon)$ have an extreme value for $\epsilon = 0$ we must necessarily have $\frac{ds}{d\epsilon}\Big|_{\epsilon=0} = 0$. Now

$$\frac{ds}{d\epsilon} = \int_{t_1}^{t_2} \left(\frac{\partial h}{\partial F^{i*}} \frac{\partial F^{i*}}{\partial \epsilon} + \frac{\partial h}{\partial F^{i*'}} \frac{\partial F^{i*'}}{\partial \epsilon} \right) dt$$

and therefore, in consequence of (51.1),

$$\frac{ds}{d\epsilon}\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \left(\frac{\partial h}{\partial F^i} v^i + \frac{\partial h}{\partial F^{i'}} v^{i'} \right) dt.$$

Integration by parts yields

$$\int_{t_1}^{t_2} \frac{\partial h}{\partial F^{i'}} v^{i'} dt = \frac{\partial h}{\partial F^{i'}} v^i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial h}{\partial F^{i'}} \right) v^i dt.$$

Since $v^i(t_1) = v^i(t_2) = 0$ the first term on the right-hand side vanishes. We thus obtain

$$\left. \frac{ds}{d\epsilon} \right|_{\epsilon=0} = \int_{t_1}^{t_2} \left[\frac{\partial h}{\partial F^i} - \frac{d}{dt} \left(\frac{\partial h}{\partial F^{i'}} \right) \right] v^i dt.$$

Since C is assumed to have minimum length compared with all neighbouring arcs, that is, since the preceding integral must vanish for any pair of functions v^i which vanish at t_1 and t_2 , we must have

$$(51.3) \quad \frac{\partial h}{\partial F^i} - \frac{d}{dt} \left(\frac{\partial h}{\partial F^{i'}} \right) = 0, \quad (i = 1, 2)$$

where h is defined by (51.2). Equations (51.3) are called the *Euler–Lagrange equations* of the variational problem under consideration. These equations must necessarily be satisfied in order that $C: u^1 = F^1(t), u^2 = F^2(t)$, compared with any above arc C^* in a neighbourhood of C , has minimum length.

We will now insert the function h into (51.3); instead of $F^i(t)$ we may simply write $u^i(t)$. We have

$$\frac{\partial h}{\partial u^\mu} = \frac{1}{2\sqrt{(g_{\alpha\beta} u^\alpha u^\beta)}} \frac{\partial g_{\nu\sigma}}{\partial u^\mu} u^\nu u^\sigma = \frac{1}{2s'} \frac{\partial g_{\nu\sigma}}{\partial u^\mu} u^\nu u^\sigma$$

and

$$\frac{\partial h}{\partial u^{\mu'}} = \frac{2g_{\mu\nu} u^{\nu'}}{2\sqrt{(g_{\alpha\beta} u^\alpha u^\beta)}} = \frac{1}{s'} g_{\mu\nu} u^{\nu'}.$$

Therefore $\frac{d}{dt} \left(\frac{\partial h}{\partial u^{\mu'}} \right) = -\frac{s''}{s'^2} g_{\mu\nu} u^{\nu'} + \frac{1}{s'} \left(\frac{\partial g_{\mu\nu}}{\partial u^\sigma} u^\nu u^\sigma + g_{\mu\nu} u^{\nu''} \right).$

(51.3) thus takes the form

$$\frac{1}{2} \frac{\partial g_{\nu\sigma}}{\partial u^\mu} u^\nu u^\sigma + \frac{s''}{s'} g_{\mu\nu} u^{\nu'} - \frac{\partial g_{\mu\nu}}{\partial u^\sigma} u^\nu u^\sigma - g_{\mu\nu} u^{\nu''} = 0.$$

Using (47.3), (47.4) and

$$\frac{\partial g_{\mu\nu}}{\partial u^\sigma} u^\nu u^\sigma = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial u^\nu} u^\sigma u^\nu + \frac{\partial g_{\mu\nu}}{\partial u^\sigma} u^\nu u^\sigma \right)$$

we obtain from this

$$\frac{s''}{s'} g_{\mu\nu} u^{\nu'} - \Gamma_{\nu\sigma\mu} u^\nu u^\sigma - g_{\mu\nu} u^{\nu''} = 0.$$

Taking the inner product of this expression and $g^{\mu\tau}$ we have

$$(51.4) \quad \frac{s''}{s'} u^{\tau'} - \Gamma_{\nu\sigma}{}^\tau u^\nu u^\sigma - u^{\tau''} = 0.$$

If we introduce the arc length s of C as parameter then, since $\dot{s} = 1$ and $\ddot{s} = 0$, the Euler–Lagrange equations (51.3) take the form

$$(51.5) \quad \ddot{u}^\tau + \Gamma_{\nu\sigma}^\tau \dot{u}^\nu \dot{u}^\sigma = 0, \quad (\tau = 1, 2).$$

We will prove that (51.5) is equivalent to

$$\kappa_g = |\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \cdot \mathbf{n}| = 0,$$

cf. (49.5). We have

$$(51.6) \quad \dot{\mathbf{x}} = \mathbf{x}_\tau \dot{u}^\tau, \quad \ddot{\mathbf{x}} = \mathbf{x}_\tau [\Gamma_{\nu\sigma}^\tau \dot{u}^\nu \dot{u}^\sigma + \ddot{u}^\tau] + \mathbf{n} b_{\nu\sigma} \dot{u}^\nu \dot{u}^\sigma,$$

cf. (49.6). Because of (51.5) the expression in brackets [...] vanishes, that is, $\ddot{\mathbf{x}}$ and \mathbf{n} are collinear. Consequently, $|\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \cdot \mathbf{n}| = 0$. We prove the converse, starting from the assumption $|\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \cdot \mathbf{n}| = 0$. Since $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ as well as $\dot{\mathbf{x}}$ and \mathbf{n} are linearly independent that determinant vanishes if and only if $\ddot{\mathbf{x}}$ and \mathbf{n} are linearly dependent, that is, in (51.6) the expression in brackets [...] must vanish. Consequently (51.5) must hold.

We sum up our result:

Theorem 51.1. *In order that an arc C of class $r \geq 2$ on a surface $S: \mathbf{x}(u^1, u^2)$ with end points P_1 and P_2 have minimum length among all neighbouring arcs on S which join P_1 and P_2 , C must necessarily satisfy the Euler–Lagrange equations (51.5). These equations are satisfied if and only if C is an arc of a geodesic on S .*

In the case of a plane the condition that C be an arc of a geodesic is also sufficient; indeed, in this case straight lines are the only geodesics, and any segment $P_1 P_2$ is the arc of minimum length joining its end points P_1 and P_2 .

The fact that the above condition cannot be sufficient in the case of any surface may be illustrated, for instance, on a sphere S . There is exactly one great circle G on S passing through two given points, P_1 and P_2 not diametrically opposite. But at most one of the two arcs of G with end points P_1 and P_2 can be an arc of minimum length joining P_1 and P_2 . A sufficient condition for an arc C to be an arc of minimum length will be stated in the following section.

Problem 51.1. Determine the solution of the first of the differential equations (51.5) in the case of a surface of revolution S and prove that the meridians of S are always geodesics.

52. Geodesic parallel coordinates. A one-parameter family of geodesics on a surface S is said to be a *field of geodesics* in a portion S' of S if through every point of S' there passes (just once) exactly one of those geodesics. The name ‘field of geodesics’ was introduced by Weierstrass.

For example, a family of parallel straight lines is a field of geodesics in a plane. A pencil of straight lines through a point P of a plane E is a field of geodesics in any portion of E which does not contain P . There does not exist a field of geodesics on the whole sphere since any two geodesics (great circles) intersect each other. The generating straight lines of a cylinder Z constitute a field of geodesics on Z .

We may introduce coordinates u^{1*}, u^{2*} on a sufficiently small portion S of a surface in the following manner: We choose a field of geodesics on S as coordinate curves $u^{2*} = \text{const.}$ and their orthogonal trajectories as coordinate curves $u^{1*} = \text{const.}$ These orthogonal coordinates are called *geodesic parallel coordinates*. (That we obtain an allowable coordinate system on S by the above procedure will be seen in connexion with Theorem 52.1.)

Since the coordinates u^{1*}, u^{2*} are orthogonal we have $g_{12}^* = 0$. Since the curves $u^{2*} = \text{const.}$ are geodesics ($\kappa_g = 0$), in consequence of (49.9 b) we have $\partial g_{11}^*/\partial u^{2*} = 0$, that is, g_{11}^* depends on u^{1*} only. We set

$$u^1 = \int_0^{u^{1*}} \sqrt{(g_{11}^*)} \, du^{1*}, \quad u^2 = u^{2*}.$$

This transformation leaves the curves $u^{2*} = \text{const.}$ unchanged and therefore also their orthogonal trajectories; of course the corresponding values $u^{1*} = \text{const.}$ will be transformed into other values $u^1 = \text{const.}$ The first fundamental form corresponding to the coordinates u^1, u^2 is given by the expression

$$(52.1) \quad ds^2 = (du^1)^2 + g_{22}(u^1, u^2)(du^2)^2,$$

which is characteristic for geodesic coordinates. Since in this case $g_{22} = g$, we may write simply

$$(52.1') \quad ds^2 = (du^1)^2 + g(du^2)^2.$$

If the first fundamental form is given by (52.1) the formula for the Gaussian curvature becomes very simple; we then obtain from (46.17)

$$(52.2) \quad K = -\frac{1}{\sqrt{g_{22}}} \frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2} = \frac{1}{4(g_{22})^2} \left(\frac{\partial g_{22}}{\partial u^1} \right)^2 - \frac{1}{2g_{22}} \frac{\partial^2 g_{22}}{(\partial u^1)^2}$$

or, since $g_{22} = g$,

$$(52.2') \quad K = -\frac{1}{\sqrt{g}} \frac{\partial^2 \sqrt{g}}{(\partial u^1)^2}.$$

(52.2) is advantageous, for instance, for the study of surfaces of constant Gaussian curvature, cf. Section 90, and other purposes.

From (52.1) we obtain the following

Theorem 52.1 (Gauss). *Let C be any curve on a surface S of class $r \geq 3$. When (sufficiently small) equal lengths are measured from C along the geodesics which intersect C orthogonally the locus of their end points is an orthogonal trajectory C^* of those geodesics.*

Proof. The condition ‘sufficiently small’ means that we consider the above geodesics only in a neighbourhood U of C where they form a field. The existence of such a neighbourhood is an immediate consequence of the fact that the geodesics satisfy the differential equation occurring in the preceding two sections. If we take on S coordinates u^1, u^2 so that the above geodesics are the curves $u^2 = \text{const.}$ and their orthogonal trajectories, one of which is C , are the curves $u^1 = \text{const.}$ then, in consequence of (52.1), the geodesic arcs between any two trajectories, say $u^1 = c_1$ and $u^1 = c_2$, have the length

$$\int_{c_1}^{c_2} du^1 = c_2 - c_1.$$

Since this length is independent of u^2 the arcs of all those geodesics between any two of their orthogonal trajectories have the same length. This completes the proof.

The curves C^* defined in Theorem 52.1 are called *geodesic parallels* of C .

In consequence of Theorem 52.1 we may introduce geodesic parallel coordinates u^{1*}, u^{2*} on a surface S as follows. We choose an arbitrary simple curve C on S to be the coordinate curve $u^{1*} = 0$; we take the geodesics orthogonal to C as coordinate curves $u^{2*} = \text{const.}$ and finally the geodesic parallels to C as curves $u^{1*} = \text{const.}$ The coordinates thus obtained are allowable in the neighbourhood U of C mentioned in the proof of Theorem 52.1.

We are now able to state a sufficient condition which a geodesic arc G must satisfy in order that G be the arc of minimum length between two given points on a surface.

Theorem 52.2. *Let G be a geodesic arc on a surface S with end points P_1 and P_2 . If we can choose a geodesic field F on S so that G becomes an arc of this field, and is the only geodesic arc in F joining P_1 and P_2 , then G is the arc of minimum length joining P_1 and P_2 , compared with any other arc which lies entirely in the field F and joins P_1 and P_2 .*

Proof. We assume that geodesic parallel coordinates u^1, u^2 have been introduced on S so that G is given by $u^2 = u_0^2 = \text{const.}$ An arbitrary arc $\mathbf{x}(u^1(t), u^2(t))$ ($t_1 \leq t \leq t_2$) which joins P_1 and P_2 and lies entirely in the

field F has the length

$$s = \int_{t_1}^{t_2} \sqrt{\{(u^1')^2 + g_{22}(u^2')^2\}} dt \geq \int_{t_1}^{t_2} u^1' dt = u_{(2)}^1 - u_{(1)}^1;$$

in this expression the equality sign holds if and only if $g_{22}(u^2')^2 = 0$, that is, since $g_{22} > 0$, if $du^2 = 0$. Then $u^2 = \text{const.}$, and the joining arc is a portion G . Cf. Fig. 58.

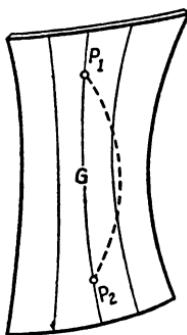


FIG. 58

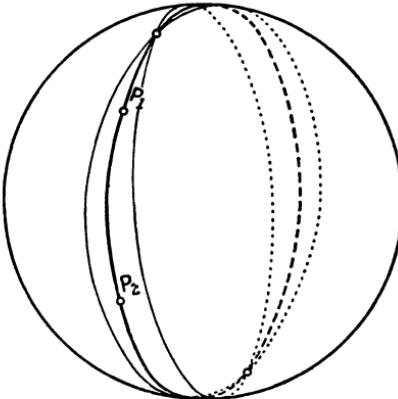


FIG. 59

Example. Two points P_1 and P_2 on a sphere, not diametrically opposite, determine a great circle G and divide G into two arcs. The shorter of these arcs can be embedded in a geodesic field and is therefore the arc of minimum length between P_1 and P_2 while the longer of those arcs of G cannot be embedded in a geodesic field, cf. Fig. 59.

53. Geodesic polar coordinates. In the preceding section we considered geodesic parallel coordinates which are a generalization of orthogonal parallel coordinates in a plane. We will now define geodesic polar coordinates which are a generalization of plane polar coordinates.

Through an arbitrary point P of a surface S of class $r \geq 3$ there passes exactly one geodesic in every direction. We will prove that those geodesics constitute a geodesic field in $U - P$, where U is a sufficiently small neighbourhood of P . For this purpose we introduce on S coordinates \tilde{u}^1, \tilde{u}^2 with origin at P so that $\tilde{g}_{\alpha\beta} = \delta_{\alpha\beta}$ at P . Let

$$\tilde{u}^\alpha = h^\alpha(s; c^1, c^2), \quad (\alpha = 1, 2),$$

be the solution of the differential equations (51.5) which satisfies the conditions

$$\tilde{u}^\alpha(0) = 0, \quad \dot{\tilde{u}}^\alpha(0) = c^\alpha, \quad (\alpha = 1, 2).$$

Depending on the choice of c^1, c^2 we obtain one of the geodesics through P , and P always corresponds to $s = 0$. If we set $c^1 = \cos u^2, c^2 = \sin u^2$ then u^2 is the angle between that geodesic and the curve $\tilde{u}^2 = 0$ at P , we have $(c^1)^2 + (c^2)^2 = 1$, and s is the arc length of that geodesic. We will prove that for any values of c^1 and c^2 the solution $h^\alpha(s; c^1, c^2)$ of (51.5) can be represented in the form $\phi^\alpha(sc^1, sc^2)$ ($\alpha = 1, 2$). The transformation $s = kt$ leaves the form of (51.5) unaffected. Hence if $h^\alpha(s; c^1, c^2)$ is a solution of (51.5), $h^\alpha(t; \tilde{c}^1, \tilde{c}^2)$ is also a solution of (51.5). Since

$$\frac{\partial \tilde{u}^\alpha}{\partial t} = \frac{\partial \tilde{u}^\alpha}{\partial s} k$$

or
$$\left. \frac{\partial \tilde{u}^\alpha}{\partial t} \right|_{t=0} = c^\alpha k, \quad (k \neq 0, \alpha = 1, 2),$$

we have

$$h^\alpha(s; c^1, c^2) = h^\alpha(t; c^1 k, c^2 k).$$

In particular for $t = 1$ we have

$$h^\alpha(k; c^1, c^2) = h^\alpha(1; c^1 k, c^2 k)$$

or, if we replace k by s ,

$$h^\alpha(s; c^1, c^2) = h^\alpha(1; c^1 s, c^2 s).$$

We set

$$h^\alpha(1; c^1 s, c^2 s) = \phi^\alpha(v^1, v^2), \quad (\alpha = 1, 2)$$

where $v^\alpha = c^\alpha s$. By means of the transformation

$$\tilde{u}^\alpha = \phi^\alpha(v^1, v^2), \quad (\alpha = 1, 2)$$

we now introduce v^1, v^2 as coordinates on S . This transformation is allowable in a (sufficiently small) neighbourhood U of P . Indeed, we have

$$c^\alpha = \tilde{u}^\alpha(0) = \left(\frac{\partial h^\alpha}{\partial s} \right)_{s=0}$$

and, on the other hand,

$$\left(\frac{\partial h^\alpha}{\partial s} \right)_{s=0} = \left(\frac{\partial \phi^\alpha}{\partial v^r} \frac{\partial v^r}{\partial s} \right)_{v^1=v^2=0} = \left(\frac{\partial \phi^\alpha}{\partial v^r} c^r \right)_{v^1=v^2=0},$$

hence, by comparison,
$$\left(\frac{\partial \phi^\alpha}{\partial v^r} \right)_{v^1=v^2=0} = \delta_r^\alpha.$$

Therefore at P the Jacobian J of the transformation has the value 1 and since J is continuous it cannot vanish in a certain neighbourhood U of P . Furthermore the coefficients $g_{\alpha\beta}^*$ and $\tilde{g}_{\alpha\beta}$ of the first fundamental forms

corresponding to v^1, v^2 and \tilde{u}^1, \tilde{u}^2 , respectively, have at P the same values,

$$g_{\alpha\beta}^*(0,0) = \delta_{\alpha\beta}.$$

v^1, v^2 are called *Riemann normal coordinates*. The geodesics through P have the representation $v^\alpha = c^\alpha s$; we now assume that $(c^1)^2 + (c^2)^2 = 1$, then s is the arc length of those geodesics. The values of s, c^1 , and c^2 corresponding to a given point $Q: (v^1, v^2) \neq (0,0)$ of U are uniquely determined, that is, there exists exactly one geodesic joining Q and P . Hence the geodesics under consideration constitute a geodesic field in $U - P$.

If we write (51.5) using the coordinates v^1, v^2 and insert $v^\alpha = c^\alpha s$ then, since $\dot{v}^\alpha = c^\alpha$ and $\ddot{v}^\alpha = 0$, we obtain

$$\Gamma_{\nu\sigma}^{*\tau} c^\nu c^\sigma = 0, \quad (\tau = 1, 2).$$

Since at P these equations must be satisfied for all values of c^α all Christoffel symbols $\Gamma_{\nu\sigma}^{*\tau}$ vanish at P ; hence all partial derivatives $\partial g_{\alpha\beta}^*/\partial u^\tau$ vanish also at P .

We set

$$v^1 = u^1 \cos u^2, \quad v^2 = u^1 \sin u^2.$$

The coordinates u^1, u^2 thus defined are called *geodesic polar coordinates on S with centre at P* , cf. Fig. 60. Since v^1, v^2 are allowable in U and the preceding transformation is allowable in $U - P$ the coordinates u^1, u^2 are allowable in $U - P$.

The coordinate curves $u^2 = \text{const.}$ are the geodesics through P . The corresponding first fundamental form is given by (52.1), and the curves $u^1 = \text{const.}$ are the orthogonal trajectories to the curves $u^2 = \text{const.}$; this can be seen in a similar manner as in the case of geodesic parallel coordinates. The curves $u^1 = \text{const.}$ are called *geodesic circles*; the corresponding value of u^1 is called the *radius* of such a circle.

In consequence of (52.1) we have $g_{11} = 1$ and $g_{12} = 0$. For later purposes it is necessary to investigate the behaviour of g_{22} at P . We have

$$dv^1 = \cos u^2 du^1 - u^1 \sin u^2 du^2, \quad dv^2 = \sin u^2 du^1 + u^1 \cos u^2 du^2.$$

If we insert these expressions into the first fundamental form $g_{\alpha\beta}^* dv^\alpha dv^\beta$ we find that $(du^2)^2$ has the coefficient

$$g_{22} = (u^1)^2(g_{11}^* \sin^2 u^2 - 2g_{12}^* \sin u^2 \cos u^2 + g_{22}^* \cos^2 u^2).$$

Since the first partial derivatives of $g_{\alpha\beta}^*$ vanish at P , cf. above, we obtain from this

$$(53.1) \quad \lim_{u^1 \rightarrow 0} g_{22} = 0, \quad \lim_{u^1 \rightarrow 0} \frac{\partial \sqrt{g_{22}}}{\partial u^1} = 1.$$

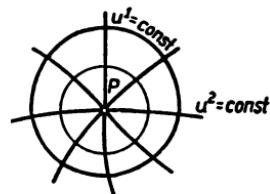


FIG. 60

From (52.2):

$$\frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2} = -K \sqrt{g_{22}}$$

we have

$$\frac{\partial^3 \sqrt{g_{22}}}{(\partial u^1)^3} = -K \frac{\partial \sqrt{g_{22}}}{\partial u^1} - \frac{\partial K}{\partial u^1} \sqrt{g_{22}},$$

and therefore

$$\lim_{u^1 \rightarrow 0} \frac{\partial^3 \sqrt{g_{22}}}{(\partial u^1)^3} = -\lim_{u^1 \rightarrow 0} K \equiv -K_0.$$

Since

$$\sqrt{\{g_{22}(u^1, u^2)\}} = \sqrt{\{g_{22}(0, u^2)\}} + u^1 \left(\frac{\partial \sqrt{g_{22}}}{\partial u^1} \right)_{u^1=0} + \frac{(u^1)^2}{2} \left(\frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2} \right)_{u^1=0} + \dots$$

we thus obtain

$$(53.2) \quad \sqrt{g_{22}} = u^1 - \frac{K_0(u^1)^3}{6} + \dots$$

or

$$(53.3) \quad g_{22} = (u^1)^2 - \frac{K_0(u^1)^4}{3} + \dots$$

where dots denote functions of u^1, u^2 which vanish at P of higher than third or fourth order, respectively.

Because of (52.2) the length S of a geodesic circle is thus given by the expression

$$S = \int_0^{2\pi} \sqrt{g_{22}} du^2 = 2\pi u_0^1 - \pi \frac{K_0}{3} (u_0^1)^3 + \dots$$

From this the formula of Puiseux and Bertrand (1848) for the Gaussian curvature K_0 at the centre P of the geodesic polar coordinate system,

$$(53.4) \quad K_0 = \lim_{u_0^1 \rightarrow 0} \frac{3(2u_0^1\pi - S)}{(u_0^1)^3\pi},$$

follows.

In consequence of (52.1) and (53.3) the Christoffel symbols with respect to geodesic polar coordinates vanish at the centre P , cf. (47.7) and (47.8).

Problem 53.1. Find a representation $\mathbf{x}(u^1, u^2)$ of the sphere where u^1, u^2 are geodesic polar coordinates.

54. Theorem of Gauss–Bonnet. Integral curvature. The Gauss–Bonnet theorem is one of the most important theorems of the theory of surfaces. It was first published by O. Bonnet (1819–92) in 1848, cf. Bonnet [1], but it was probably already known to Gauss. This theorem is an example of differential geometry in the large, cf. Section 2. Moreover, it is

analogous to Green's theorem and can be obtained from this theorem, cf. for example, G. Darboux [1, vol. 3].

The integral

$$\iint_S K \, dA$$

occurring in the Gauss-Bonnet theorem is called the *integral curvature* of the portion S of a surface under consideration.

Theorem 54.1 (Gauss-Bonnet). *Let S be a simply connected portion of a surface for which a representation $\mathbf{x}(u^1, u^2)$ of class $r \geq 3$ exists and whose boundary C is a simple closed curve which has a representation $\mathbf{x}(u^1(s), u^2(s))$ of class $r^* \geq 2$, where s is the arc length of C . Let κ_g be the geodesic curvature of C and let K be the Gaussian curvature of S . Then*

$$(54.1) \quad \int_C \kappa_g \, ds + \iint_S K \, dA = 2\pi,$$

where dA is the element of area of S ; the integration along C has to be carried out in such a sense that S stays on the left side.

Proof. We introduce on S allowable orthogonal coordinates u^1, u^2 in the following way: Let P be an arbitrary point of S , and let C_0 be a sufficiently small geodesic circle with centre at P . Then we assume: u^1, u^2 are geodesic polar coordinates in the interior of C_0 (cf. Section 53); the curve C is given by $u^1 = 1$, and on C the coordinate u^2 is equal to the arc length s .

The existence of such a coordinate system may be proved as follows. In Section 62 we shall see that under the above assumptions S can be mapped conformally into a plane E . Let S^* , C^* , and C_0^* be the images of S , C , and C_0 , respectively, in E . As is known from the theory of complex analytic functions, the portion of S^* bounded by C^* and C_0^* can be mapped conformally onto a plane ring bounded by two concentric circles. The circles concentric to these bounding circles correspond to a family of closed curves in S^* which cover S^* schlicht. The inverse images of these curves and their orthogonal trajectories on S , together with the geodesic circles in the interior of C_0 and their orthogonal geodesics through P , form the coordinate curves of the coordinates u^1, u^2 . According to (49.9 a) the geodesic curvature κ_g of C is then given by the expression

$$(54.2) \quad \kappa_g(C) = \frac{1}{\sqrt{(g_{11} g_{22})}} \frac{\partial \sqrt{g_{22}}}{\partial u^1}.$$

Since $u^2 = s$ on C we have $g_{22} = 1$ on C . We thus obtain from (54.2)

$$(54.3) \quad \int_C \kappa_g \, ds = \int_C \frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} \, du^2.$$

Let S_0 be the domain bounded by C_0 . From (46.17') we find

$$\begin{aligned} \iint_{S-S_0} K dA &= \iint_{S-S_0} K \sqrt{g} du^1 du^2 = - \iint_{S-S_0} \left[\frac{\partial}{\partial u^1} \left(\frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} \right) \right] du^1 du^2 - \\ &\quad - \iint_{S-S_0} \left[\frac{\partial}{\partial u^2} \left(\frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial u^2} \right) \right] du^1 du^2 \equiv J_1 + J_2. \end{aligned}$$

The integration with respect to u^2 has to be carried out along closed curves; hence $J_2 = 0$, and we therefore have

$$\iint_{S-S_0} K dA = J_1 = - \int_C \frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} du^2 + \int_{C_0} \frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} du^2;$$

hence, in consequence of (54.3),

$$(54.4) \quad \int_C \kappa_g ds + \iint_S K dA = \lim_{C_r \rightarrow P} \int_{C_r} \frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} du^2.$$

Since the integrals on the left-hand side of (54.4) exist the limit on the right-hand side also exists. In C_0 according to (52.1) $g_{11} = 1$ and according to (53.1)

$$\lim_{u^1 \rightarrow 0} \frac{\partial \sqrt{g_{22}}}{\partial u^1} = 1,$$

and (54.4) therefore becomes identical with (54.1).

The Gauss-Bonnet theorem can also be formulated for a simply-connected portion of a surface which is bounded by finitely many arcs so that at the end points the tangents to those arcs have different directions, that is, the boundary C may have cusps. In a neighbourhood of any cusp E we replace C by an arc C_r of a geodesic circle of radius r (cf. Section 53) whose tangents at the end points A, B coincide with that of C , cf. Fig. 61, p. 171. We then let $r \rightarrow 0$. We have

$$\lim_{r \rightarrow 0} \int_{C_r} \kappa_g d\sigma = \lim_{r \rightarrow 0} \int_{C_r} \frac{1}{r} r d\phi = \alpha$$

where $d\sigma$ is the linear element of C_r and C_r is an arc of a circle of radius r ; the angle α ($-\pi \leq \alpha \leq \pi$) is the directed angle of rotation of the tangent to C at E . Now if C has the cusps E_1, E_2, \dots, E_n with corresponding interior angles $\beta_\nu = \pi - \alpha_\nu$ ($\nu = 1, 2, \dots, n$), cf. Fig. 62, then instead of (54.1) we have

$$(54.5) \quad \int_C \kappa_g ds + \sum_{\nu=1}^n \alpha_\nu + \iint_S K dA = 2\pi$$

or, replacing α_ν by $\pi - \beta_\nu$,

$$(54.6) \quad (n-2)\pi + \int_C \kappa_g ds + \iint_S K dA = \sum_{\nu=1}^n \beta_\nu.$$

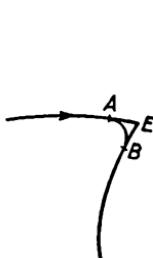


FIG. 61

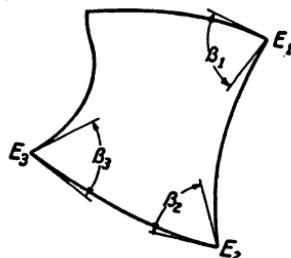


FIG. 62

A portion of a surface bounded by a simple connected broken line C of geodesic arcs is called a *geodesic polygon*. In this case we obtain from (54.6)

$$(54.7) \quad \sum_{\nu=1}^n \beta_\nu = (n-2)\pi + \iint_S K dA,$$

where n is the number of cusps of C .

If in particular the Gaussian curvature K ($\neq 0$) is constant, the area of the geodesic polygon S becomes

$$(54.8) \quad A(S) = \frac{1}{K} \left(\sum_{\nu=1}^n \beta_\nu - (n-2)\pi \right).$$

If $K = 0$ we obtain from (54.7) the well-known formula for the sum of the angles of an n -sided polygon in a plane. In the case $n = 3$ we find from (54.7): If at every point of a geodesic triangle the Gaussian curvature is positive the sum of the angles of that triangle is greater than π . (Example: geodesic triangles on a sphere.) If at every point of a geodesic triangle $K < 0$ the sum of the angles is less than π . (Example: geodesic triangles on a pseudosphere, cf. Section 93.) The deviation of the sum of the angles of a geodesic triangle from the value π is called its *excess*. According to (54.7) the excess is equal to the integral curvature of the triangle.

If we set $n = 0$ we obtain from (54.7)

Theorem 54.2. *On a surface S of class $r \geq 3$ for which the Gaussian curvature $K \leq 0$ everywhere there do not exist closed geodesics such that just one of them constitutes the entire boundary of a bounded portion of S .*

Example. A plane gives the simplest illustration of Theorem 54.2. Another example is the cylinder of revolution: The parallels are closed geodesics (circles) on the cylinder, but a single one of those geodesics is not the entire boundary of a bounded portion of the cylinder.

Since multiply connected pieces of surfaces can be cut so that they become simply connected, cf. Fig. 63, we may easily formulate the Gauss–Bonnet theorem for such pieces. The path of integration of the integral $\int \kappa_g ds$ has to be chosen as shown by the arrows (Fig. 63). Every section

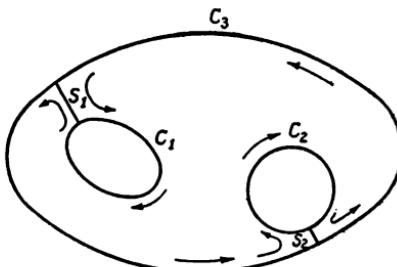


FIG. 63

S_1, S_2, \dots is passed twice, once in each of the two directions. Hence the integrals along those sections do not contribute to the above integral, but at the end points of each of those sections the tangent makes four rotations through an angle $\frac{1}{2}\pi$; from each section we thus have the contribution 2π .

Problem 54.1. Apply the Gauss–Bonnet theorem to a plane circular ring.

55. Application of the Gauss–Bonnet theorem to closed surfaces.

An important result of topology of surfaces is the theorem that any closed orientable surface (cf. Section 34) is homeomorphic (cf. Section 3) to one of the ∞^1 ‘normal types’ of closed orientable surfaces; these types can be obtained from a sphere by attaching $0, 1, \dots, p, \dots$ ‘handles’, respectively, cf. Fig. 66. The number p is called the *genus* of the surface. Any closed surface of genus 0 or 1 is homeomorphic to a sphere or a torus (cf. Section 43), respectively.

It is most remarkable that the integral curvature of a closed orientable surface of class $r \geq 3$ is completely independent of its particular shape but depends on its genus only. We have

Theorem 55.1. *Any closed orientable surface S of genus p for which a representation of class $r \geq 3$ exists has the integral curvature*

$$(55.1) \quad \iint_S K dA = 4\pi(1-p).$$

In particular the integral curvature of the torus vanishes.

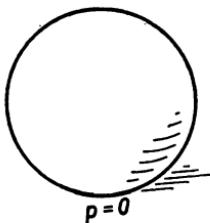


FIG. 64

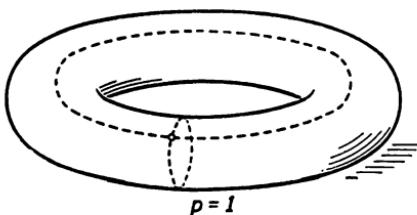
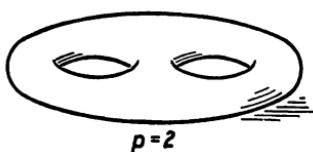
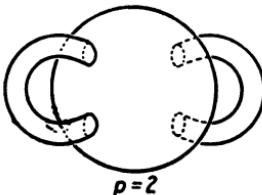


FIG. 65

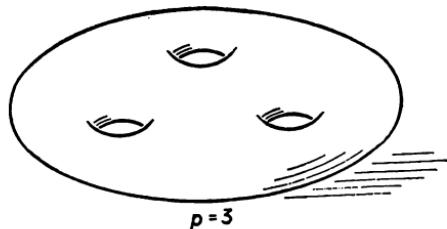


p = 2



p = 2

FIG. 66



p = 3

FIG. 67

Figs. 64–67. Normal types of closed orientable surfaces of genus $p = 0$ (Fig. 64), $p = 1$ (Fig. 65), $p = 2$ (Fig. 66), and $p = 3$ (Fig. 67).

Proof. (a) $p = 0$: We cut S into two pieces S_1 and S_2 in such a manner that the section C is a curve of the type indicated in Theorem 54.1. Then

$$(55.2) \quad \int_C \kappa_g \, ds + \iint_{S_i} K \, dA = 2\pi, \quad (i = 1, 2).$$

Obviously the sense of integration along C in the case of S_1 is opposite to that in the case of S_2 . By adding the two equations (55.2) we thus obtain

$$\iint_S K \, dA = 4\pi.$$

(b) $p \geq 1$: We choose any point P on the surface and cut the surface by means of $2p$ cuts starting from P and returning to P in such a manner that we obtain a simply connected surface and that at P the angle between

neighbouring cuts is equal to $2\pi/4p$. Since each cut serves twice as path of integration, once in each direction, the integrals $\int \kappa_g ds$ along the cuts do not contribute, but we have contributions from each of the $4p$ rotations of the tangent which takes place if at P we proceed from one cut to the next; on each of these occasions the tangent is rotated through an angle

$$\pi - \frac{2\pi}{4p} = \pi \left(1 - \frac{1}{2p}\right).$$

Therefore, according to the Gauss–Bonnet theorem,

$$4p\pi \left(1 - \frac{1}{2p}\right) + \iint_S K dA = 2\pi.$$

This completes the proof.

Problem 55.1. Determine the integral curvature of a sphere and of a torus by direct computation.

VI

MAPPINGS

56. Preliminaries. The concept of a mapping was introduced in Section 3. Let us now consider a one-to-one mapping

$$(56.1) \quad \tilde{u}^\alpha = h^\alpha(u^1, u^2), \quad (\alpha = 1, 2),$$

of a portion S of a surface with coordinates u^1, u^2 onto a portion S^* of a surface with coordinates \tilde{u}^1, \tilde{u}^2 . The mapping (56.1) is said to be an *allowable mapping* if the functions h^α are of class r (≥ 1) and if the corresponding Jacobian does not vanish.

We may introduce on S^* new coordinates u^{1*}, u^{2*} by using the mapping functions h^α for a coordinate transformation

$$(56.2) \quad \tilde{u}^1 = h^1(u^{1*}, u^{2*}), \quad \tilde{u}^2 = h^2(u^{1*}, u^{2*}).$$

Then the mapping of S onto S^* takes the simple form

$$u^{1*} = u^1, \quad u^{2*} = u^2,$$

that is, the values of the coordinates of any image point are the same as those of the corresponding inverse image point. We may briefly say that '*the coordinate systems on S and S^* are the same*'. In other words, the transformation (56.2) introduces the coordinate system on S as coordinate system on S^* .

Mappings of surfaces were first considered a long time ago, particularly in connexion with the problem of constructing maps of the globe for nautical or other purposes. Problems of that kind provided the impetus for much of the development of differential geometry. Several mapping problems were first solved empirically and the mathematical foundation for them was not developed until much later, cf. the remarks on the Mercator projection in Section 65. The whole development started with the investigation of special mapping problems by Jakob Bernoulli (1654–1705), Johann Bernoulli (1667–1748), Euler (1707–83), Lagrange (1736–1813), Monge (1746–1818), and others. But the development of a systematic theory of mapping is generally considered to have started with the publication of Gauss's paper on conformal mapping.

Of particular theoretical (and practical) interest are mappings which preserve certain geometric properties. The most important type is the isometric or length-preserving mapping which preserves the lengths of

every arc on a surface to be mapped. Further types which we will consider are the conformal or angle-preserving mappings which preserve the angle between every pair of intersecting curves on a surface, and the equiareal mappings which preserve the area of any portion of the surface to be mapped.

57. Isometric mapping. Bending. Concept of intrinsic geometry of a surface. An allowable mapping of a portion S of a surface onto a portion S^* of a surface is said to be *isometric* or *length-preserving* if the length of any arc on S^* is the same as that of its inverse image on S .

Necessary and sufficient conditions for a mapping to be isometric are as follows:

Theorem 57.1. *An allowable mapping of a portion S of a surface onto a portion S^* of a surface is isometric if and only if at corresponding points of S and S^* , when referred to the same coordinate systems on S and S^* (cf. Section 56), the coefficients of the first fundamental forms on S and S^* are the same, $g_{\alpha\beta} = g_{\alpha\beta}^*$ ($\alpha = 1, 2, \beta = 1, 2$).*

Proof. Let u^1, u^2 and u^{1*}, u^{2*} be the coordinates on S and S^* , respectively. Since those coordinate systems are assumed to be the same the mapping is given by $u^{\alpha*} = u^\alpha$, $\alpha = 1, 2$. If

$$u^\alpha = h^\alpha(t), \quad 0 \leq t \leq t_1, \quad (\alpha = 1, 2),$$

is an arbitrary arc C on S then its image C^* can be represented in the form

$$u^{\alpha*} = h^\alpha(t), \quad 0 \leq t \leq t_1, \quad (\alpha = 1, 2),$$

where $h^\alpha(t)$ are the same functions as before. An arc \tilde{C} : $u^\alpha = h^\alpha(t)$, $0 \leq t \leq t_0$ of C has the length

$$s(t_0) = \int_0^{t_0} \sqrt{(g_{\alpha\beta} h^\alpha' h^\beta') dt}, \quad (0 \leq t_0 \leq t_1)$$

and the corresponding image has the length

$$s^*(t_0) = \int_0^{t_0} \sqrt{(g_{\alpha\beta}^* h^\alpha' h^\beta') dt},$$

where primes indicate derivatives with respect to t . If for every point P of S and its image P^* of S^* we always have $g_{\alpha\beta} = g_{\alpha\beta}^*$ ($\alpha = 1, 2, \beta = 1, 2$), then $s(t_0) = s^*(t_0)$. Conversely, in order that $s(t_0) = s^*(t_0)$ hold for every value of t_0 , $0 \leq t_0 \leq t_1$, the integrands of the above integrals must be equal. In order that any arc on S^* have the same length as that of its inverse image on S the integrands must be equal for any pair of functions $h^1(t)$, $h^2(t)$

whatsoever. Consequently $g_{\alpha\beta} = g_{\alpha\beta}^* (\alpha = 1, 2, \beta = 1, 2)$ must hold. This completes the proof.

Two surfaces (or portions of surfaces) which can be mapped isometrically onto each other are said to be *isometric*.

Example. The cylinder $S: \mathbf{x}(u^1, u^2) = (\cos u^1, \sin u^1, u^2)$ can be mapped isometrically onto a strip of the plane $\mathbf{x}(u^{1*}, u^{2*}) = (u^{1*}, u^{2*}, 0)$; the mapping is given by

$$u^{1*} = u^1, \quad u^{2*} = u^2, \quad (0 \leq u^1 < 2\pi).$$

Since $g_{11} = g_{11}^* = 1, g_{12} = g_{12}^* = 0, g_{22} = g_{22}^* = 1$ the mapping is isometric. It may be visually accomplished by cutting the cylinder along the generating line $u^1 = 0$ and then bending it until it covers the strip $0 \leq u^1 < 2\pi$ of the plane.

We now state the following important

Theorem 57.2. *Isometric surfaces have the same Gaussian curvature at corresponding points. Corresponding curves on those surfaces have the same geodesic curvature at corresponding points.*

Proof. According to the theorema egregium (Theorem 46.1) the Gaussian curvature depends on the coefficients of the first fundamental form only; those coefficients are invariant under an isometric mapping, cf. Theorem 57.1. The statement on the geodesic curvature follows similarly from Theorem 49.1.

In consequence of Theorem 57.2 isometric surfaces must necessarily have the same Gaussian curvature at corresponding points. But this condition is, in general, not sufficient. It is sufficient in the case of surfaces of constant Gaussian curvature; for details see Section 91.

Furthermore we note

Theorem 57.3. *Let C be an arc of minimum length joining any two points P_1 and P_2 on a surface S . If S is mapped isometrically onto a surface S^* the image C^* of C is an arc of minimum length joining the image points P_1^* and P_2^* of P_1 and P_2 .*

Proof. If C^* were not an arc of minimum length between P_1^* and P_2^* then in a neighbourhood of C^* there would be an arc shorter than C^* ; the inverse image of that arc would lie in a neighbourhood of C and would be shorter than C since the mapping is isometric. This contradicts our assumption on C .

We will now explain the fundamental geometric significance of isometric mappings and introduce some concepts related to our previous considerations.

A continuous deformation of a surface (or of a portion of it) is called a *bending* if this deformation preserves the length of every arc on the surface. Surfaces which cannot be bent are said to be *rigid*. An example of a bending is illustrated by drawings in Section 81.

Surfaces which can be transformed into each other by bending are also called *applicable*. Of course, applicable surfaces are also isometric, since bending is length-preserving, by definition.

The concepts ‘applicable’ and ‘isometric’ are normally used in the same sense. But applicability is not an immediate consequence of isometry; we mention some results concerning local considerations without proof. In the case of surfaces of vanishing Gaussian curvature the two concepts are identical. Isometric surfaces of negative Gaussian curvature are applicable provided certain conditions of differentiability are satisfied. Analytical isometric surfaces of positive Gaussian curvature either are applicable to each other or one of them is applicable to the image of the other obtained by a reflection in a point or a plane.

Properties which remain unchanged under a bending are called *bending invariants*. Obviously a property is a bending invariant if it depends on the first fundamental form only, since this form remains unchanged under bending. Examples of bending invariants are the Gaussian and the geodesic curvature; counter-examples are the mean curvature and the normal curvature, since the latter two quantities also depend on the second fundamental form of a surface.

It is of basic importance that there is no difference in the measurement of lengths, angles, and areas on isometric surfaces, although the surfaces, when considered from the embedding space, may have an entirely different geometric shape.

In order to characterize this fact more precisely we call *intrinsic* or *absolute properties* of a surface all those which depend on the first fundamental form only but are independent of the second fundamental form. The totality of the intrinsic properties is said to constitute the *intrinsic geometry of a surface*.

Isometric surfaces have the same intrinsic geometry. Examples were considered in the preceding chapter, further ones will follow.

Plane and cylinder furnish a simple particular example. Portions of the latter are applicable to the plane. Consequently those surfaces have *locally* the same intrinsic properties and therefore the same local intrinsic geometry. But the global intrinsic geometries of those surfaces differ essentially from each other. For instance, there is exactly one geodesic (a straight line)

through two given points of the plane, but there are, in general, infinitely many geodesics through two points of a cylinder, cf. Section 50.

We recall that we have obtained the first fundamental form of a surface by means of the metric of the embedding Euclidean space. In this sense the metric of the space ‘induces’ a metric on a surface. But if the analytic formulae of the length, angle, and area in terms of the coefficients $g_{\alpha\beta}$ of the first fundamental form have been derived (cf. Sections 35 and 36) then, a metric tensor being given, one may use these formulae without considering the surface to be embedded in a space; the embedding becomes unessential.

The problem of finding all surfaces applicable to a given one was first investigated by Gauss and then later by Bour [1], and is known as the *problem of Bour*. Cf. also the important contributions to this problem by Bonnet [3] and Codazzi [2]. In the following we will consider the problem of Bour for the special case of surfaces of vanishing Gaussian curvature, that is, we will determine all surfaces applicable to a plane.

We finally mention

Theorem 57.4 (Bour). *For every right helicoid S there exist surfaces of revolution which can be mapped isometrically onto S .*

The following problem is called after Minding [1]. Given two surfaces with given first fundamental forms, to find whether there exists an isometric mapping of the one surface onto the other.

Problem 57.1. Prove Theorem 57.4.

58. Ruled surfaces, developable surfaces. The problem of finding surfaces isometric to a plane is of special importance. We will prove in Section 59 that a surface can be brought into isometric correspondence with a plane if and only if it is a developable surface. As a preparation for this we now introduce the concept of a ruled surface and of a developable surface.

A surface is called a *ruled surface* if it contains (at least) one one-parameter family of straight lines which can be chosen as coordinate curves on the surface. These straight lines are called *generators* of the ruled surface.

Consequently a ruled surface may be generated by a continuous motion of a straight line in space. Such a motion is completely determined if the path $\mathbf{y}(s)$ (with arc length s) of a point of the moving line is given and also the direction of the line for every value of s , for example, by a unit vector $\mathbf{z}(s)$. A ruled surface may therefore be represented in the form

$$(58.1) \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{z}(s), \quad (\dot{\mathbf{y}} \times \mathbf{z} \neq \mathbf{0}).$$

Obviously the coordinate t is the (directed) distance of the points of this surface from the curve $\mathbf{y}(s)$, measured along the corresponding generator.

Remark: In order that (58.1) represent a surface the vectors $\mathbf{x}_s \equiv \partial \mathbf{x} / \partial s$ and $\mathbf{x}_t \equiv \partial \mathbf{x} / \partial t$ must be linearly independent for every value of s . The curve $\mathbf{y}(s)$ is known as the *directrix* of the ruled surface. Clearly one may choose as directrix any curve on the surface which intersects every generator exactly once and at an angle different from zero; s may also be replaced by any other allowable parameter of the directrix. If the unit vector \mathbf{z} undergoes a translation and becomes bound at the origin O of the Cartesian coordinate system in space then it generates a curve on the unit sphere with centre O . This curve is called the *spherical indicatrix* of the ruled surface. Obviously the spherical indicatrix depends on the geometric shape of the ruled surface only but is independent of the special choice of the directrix. It may therefore sometimes be advantageous to choose the arc length of the spherical indicatrix as one of the coordinates on the ruled surface. But in doing so we exclude cylinders since for these surfaces the vector \mathbf{z} is constant and the spherical indicatrix therefore degenerates into a point.

Depending on the choice of the vectors \mathbf{y} and \mathbf{z} one may obtain different types of ruled surfaces; let us mention some examples:

If $\mathbf{z} = \mathbf{z}_0$ is a constant vector we have a cylinder. If

$$\mathbf{z} = \frac{\mathbf{c}_0 - \mathbf{y}}{|\mathbf{c}_0 - \mathbf{y}|}$$

then the ruled surface (58.1) is a cone with apex at the point \mathbf{c}_0 . To every twisted curve of class $r \geq 2$ with positive curvature we can associate three ruled surfaces by taking for \mathbf{z} the unit tangent vector \mathbf{t} , the unit principal normal vector \mathbf{p} , and the unit binormal vector \mathbf{b} , respectively. In this manner we obtain the '*tangent surface*'

$$(58.1') \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{t}(s),$$

the '*principal normal surface*'

$$(58.1'') \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{p}(s),$$

and the '*binormal surface*'

$$(58.1''') \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{b}(s)$$

respectively, of the curve $\mathbf{y}(s)$.

Other ruled surfaces are the right conoid (cf. Problem 28.2), the hyperbolic paraboloid

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_3 = 0,$$

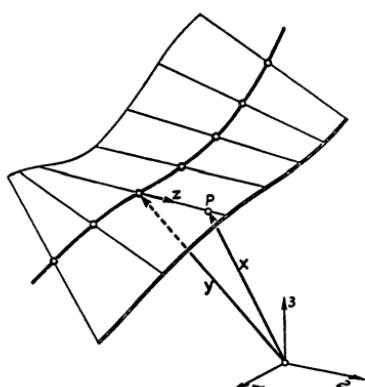


FIG. 68. Portion of a ruled surface

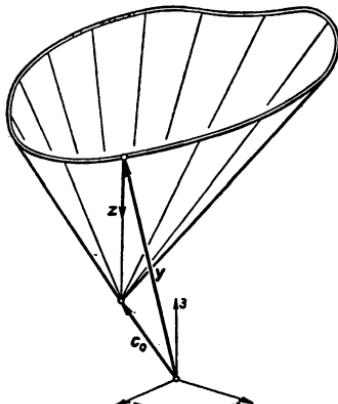


FIG. 69. Cone

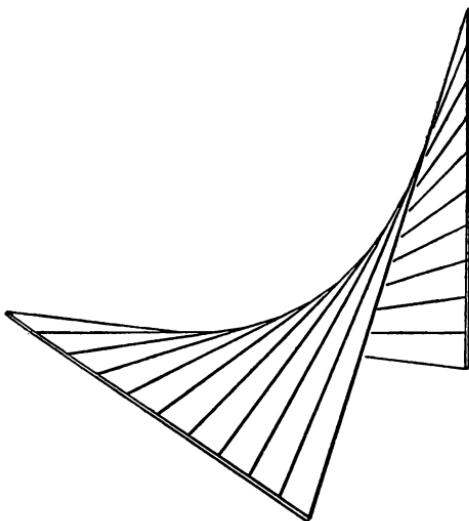


FIG. 70. Hyperbolic paraboloid

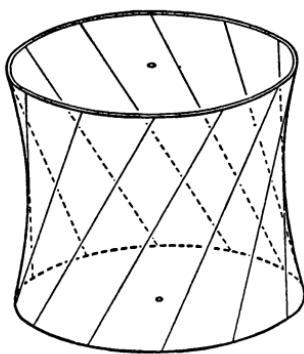


FIG. 71. Hyperboloid of one sheet

and the hyperboloid of one sheet

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1;$$

the last two surfaces are such that through every point there pass two straight lines which lie on the surface.

A *developable surface* is a special ruled surface with the property that it has the same tangent plane at all points on one and the same generator.

Theorem 58.1. *A ruled surface $\mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{z}(s)$ is a developable surface if and only if*

$$(58.2) \quad |\dot{\mathbf{y}} \ \mathbf{z} \ \dot{\mathbf{z}}| = 0,$$

where a dot denotes the derivative with respect to the arc length s of the curve $\mathbf{y}(s)$.

Proof. Let P_1 and P_2 be two points of an arbitrary (fixed) generator corresponding to $s = s_0$, and let P_1 and P_2 correspond to the respective values t_1 and t_2 . If the ruled surface is a developable surface then, by definition, its tangent planes at P_1 and P_2 coincide. Those planes are spanned by the vectors

$$(58.3) \quad \begin{aligned} \mathbf{x}_s(s_0, t_1) &= \dot{\mathbf{y}}(s_0) + t_1 \dot{\mathbf{z}}(s_0), & \mathbf{x}_s(s_0, t_2) &= \dot{\mathbf{y}}(s_0) + t_2 \dot{\mathbf{z}}(s_0), \\ \mathbf{x}_t(s_0, t_1) &= \mathbf{x}_t(s_0, t_2) = \mathbf{z}(s_0). \end{aligned}$$

Hence those three vectors must be linearly dependent,

$$(58.4) \quad |(\dot{\mathbf{y}} + t_1 \dot{\mathbf{z}}) \ (\dot{\mathbf{y}} + t_2 \dot{\mathbf{z}}) \ \mathbf{z}| = 0,$$

cf. Theorem 5.3. Applying to (58.4) the elementary rules of calculus of determinants we obtain (58.2). Conversely, starting from (58.2) we find (58.4) from which relation the coincidence of the tangent planes at the arbitrary points P_1 and P_2 on the same generator follows.

We next consider the three special ruled surfaces (58.1')–(58.1'') associated to a given curve $\mathbf{y}(s)$ with positive curvature.

Theorem 58.2. *The tangent surface is always a developable surface. The principal normal surface and the binormal surface are developable surfaces if and only if the corresponding curve is plane.*

Proof. In the case of the tangent surface we have $\mathbf{z} = \mathbf{t} = \dot{\mathbf{y}}$, cf. (58.1'); the determinant (58.2) vanishes since two columns are equal. In the case of the principal normal surface we have from (58.1'') and (58.2)

$$|\dot{\mathbf{y}} \ \mathbf{z} \ \dot{\mathbf{z}}| = |\dot{\mathbf{y}} \ \mathbf{p} \ \dot{\mathbf{p}}| = |\dot{\mathbf{y}} \ \mathbf{p} \ (-\kappa\dot{\mathbf{y}} + \tau\mathbf{b})| = |\dot{\mathbf{y}} \ \mathbf{p} \ \tau\mathbf{b}|,$$

cf. (15.1). Since the three vectors $\dot{\mathbf{y}}$, \mathbf{p} , and \mathbf{b} of the trihedron of the curve are linearly independent this determinant is zero if and only if $\tau \equiv 0$, that is, if and only if $\mathbf{y}(s)$ is a plane curve. If conversely $\mathbf{y}(s)$ is a plane curve the principal normal lies in the plane of the curve; the principal normal surface therefore is a plane. In the case of the binormal surface we have

$$|\dot{\mathbf{y}} \ \mathbf{z} \ \dot{\mathbf{z}}| = |\dot{\mathbf{y}} \ \mathbf{b} \ \dot{\mathbf{b}}| = |\dot{\mathbf{y}} \ \mathbf{b} \ -\tau\mathbf{p}|;$$

this determinant is zero if and only if $\tau \equiv 0$, that is, if and only if $\mathbf{y}(s)$ is a plane curve. Conversely, if $\mathbf{y}(s)$ is a plane curve \mathbf{b} is constant, hence $\mathbf{b} = \mathbf{0}$ and therefore also $|\dot{\mathbf{y}} \ \mathbf{b} \ \dot{\mathbf{b}}| = 0$; \mathbf{b} generates a cylinder which intersects the plane of $\mathbf{y}(s)$ along this curve.

In order that the tangents to a curve $\mathbf{y}(s)$ generate a surface T the vectors

$$\mathbf{x}_s = \dot{\mathbf{y}} + t\ddot{\mathbf{y}} \quad \text{and} \quad \mathbf{x}_t = \dot{\mathbf{y}}$$

must be linearly independent. Now $|\dot{\mathbf{y}}| = \kappa$, cf. (12.3). Hence those vectors are linearly independent if and only if $\kappa \neq 0$, but only at points for which $t \neq 0$. Along the curve $\mathbf{y}(s)$ ($t = 0$) (58.1') is never allowable. The curve $\mathbf{y}(s)$ is called the *edge of regression* of its tangent surface T . The tangent surface consists of two sheets corresponding to values $t \geq 0$ and $t \leq 0$, respectively, which have the curve $\mathbf{y}(s)$ in common. The fact that (58.1') is not allowable at the points of $\mathbf{y}(s)$ is therefore a consequence of the geometric shape of T and does not result from the special choice of the representation. Fig. 28 shows the tangent surface of the circular helix.

If the curve is plane the two sheets of the corresponding tangent surface coincide and lie in the plane of the curve.

In order to prove all those facts we choose an arbitrary point P_0 of $\mathbf{y}(s)$ (at which $\kappa_0 > 0$, $\tau_0 \neq 0$) and take as x_1 -, x_2 -, and x_3 -axes of a Cartesian coordinate system the tangent, principal normal, and binormal, respectively, of $\mathbf{y}(s)$ at P_0 . Then, in consequence of the canonical representation (18.4), the components y_1 , y_2 , and y_3 of the position vector $\mathbf{y}(s)$ take the form

$$y_1 = s - \frac{\kappa_0^2}{6}s^3 + \dots, \quad y_2 = \frac{\kappa_0}{2}s^2 + \frac{\dot{\kappa}_0}{6}s^3 + \dots, \quad y_3 = \frac{\kappa_0 \tau_0}{6}s^3 + \dots$$

where the index 0 denotes the value of the corresponding quantity at P_0 and the value $s = 0$ corresponds to P_0 . If we differentiate this representation with respect to s we obtain the components of the tangent vector $\dot{\mathbf{y}}(s)$. In consequence of (58.1') we thus have

$$(58.5) \quad \begin{aligned} x_1(s, t) &= s - \frac{\kappa_0^2}{6}s^3 + \dots + t\left(1 - \frac{\kappa_0^2}{2}s^2 + \dots\right), \\ x_2(s, t) &= \frac{\kappa_0}{2}s^2 + \frac{\dot{\kappa}_0}{6}s^3 + \dots + t\left(\kappa_0 s + \frac{\dot{\kappa}_0}{2}s^2 + \dots\right), \\ x_3(s, t) &= \frac{\kappa_0 \tau_0}{6}s^3 + \dots + t\left(\frac{\kappa_0 \tau_0}{2}s^2 + \dots\right). \end{aligned}$$

We now determine the curve of intersection of T and the normal plane N of the curve $\mathbf{y}(s)$ at P_0 . N is given by the equation $x_1 = 0$. Hence, from

the first of the three expressions (58.5), we obtain

$$(58.6) \quad t = -\frac{s - \frac{1}{6}\kappa_0^2 s^3 + \dots}{1 - \frac{1}{2}\kappa_0^2 s^2 + \dots} = -s - \frac{\kappa_0^2}{3}s^3 + \dots$$

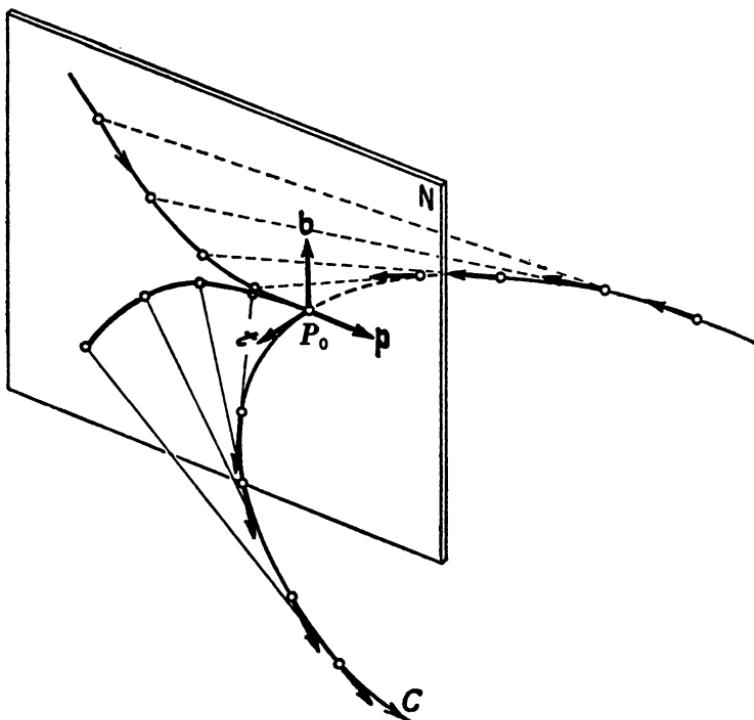


FIG. 72

If we insert this into the other two expressions of (58.5) we find the following representation of the curve of intersection of T and N :

$$(58.7) \quad x_2 = -\frac{\kappa_0}{2}s^2 + \dots, \quad x_3 = -\frac{\kappa_0 \tau_0}{3}s^3 + \dots$$

If we neglect the terms indicated by dots we have a semicubical parabola with a cusp at P_0 , cf. Fig. 72. The tangent surface T thus consists of two 'sheets' which have the curve $y(s)$ in common, and $y(s)$ is the edge of regression, as was stated above.

Any developable surface may be characterized as follows:

Theorem 58.3. Every developable surface $\mathbf{x}(s, t)$, cf. (58.1), can be subdivided into sufficiently small suitable s -intervals so that the portion of the surface corresponding to any of these intervals is a portion of either a plane, a cylinder, a cone, or a tangent surface.

Proof. If (58.1) is a developable surface (58.2) holds, that is, there are three functions $c(s)$, $d(s)$, and $e(s)$, not all zero for any value of s , such that

$$(58.8) \quad c\dot{\mathbf{y}} + d\mathbf{z} + e\dot{\mathbf{z}} = \mathbf{0};$$

we may assume the functions c , d , and e to be of class r if $\dot{\mathbf{y}}$ and $\dot{\mathbf{z}}$ are of that class.

Case 1. If $c(s) = 0$ for all values of s in an interval $I: s_0 < s < s_1$ then in I

$$(58.9) \quad d\mathbf{z} + e\dot{\mathbf{z}} = \mathbf{0},$$

where at every point of I at least one of the functions d and e is different from zero. Since \mathbf{z} is a unit vector, $\mathbf{z} \cdot \mathbf{z} = 1$ and therefore $\mathbf{z} \cdot \dot{\mathbf{z}} = 0$; (58.9) can be satisfied if and only if $\dot{\mathbf{z}} = \mathbf{0}$, that is, \mathbf{z} is constant. Hence, in this case we have a portion of a plane or of a cylinder.

Case 2. If $c(s) \neq 0$ in an interval $I': s_2 < s < s_3$ then (58.8) can be written in the form

$$(58.10) \quad \dot{\mathbf{y}} = \alpha\mathbf{z} + \beta\dot{\mathbf{z}}, \quad \alpha \equiv -\frac{d}{c}, \quad \beta \equiv -\frac{e}{c}.$$

We set

$$(58.11) \quad \mathbf{y} = \mathbf{y}^* + \beta\mathbf{z}, \quad \text{i.e. } \mathbf{y}^* = \mathbf{y} - \beta\mathbf{z}.$$

Differentiating this expression with respect to s and taking into account (58.10) we find

$$(58.12) \quad \dot{\mathbf{y}}^* = \dot{\mathbf{y}} - \beta\dot{\mathbf{z}} - \dot{\beta}\mathbf{z} = A\mathbf{z}, \quad A = \alpha - \dot{\beta}.$$

Case 2a. If $A = 0$ in I' then $\dot{\mathbf{y}}^* = \mathbf{0}$, hence $\mathbf{y}^* = \mathbf{y}_0^* = \text{const.}$ Consequently (58.1) is then of the form

$$\mathbf{x} = \mathbf{y}_0^* + (t + \beta)\mathbf{z};$$

this is a portion of a cone or of a plane.

Case 2b. If, however, $A \neq 0$ in a sub-interval I'' of I' then, in consequence of (58.12), in I''

$$\mathbf{z} = \frac{\dot{\mathbf{y}}^*}{A}$$

and therefore, by means of (58.11),

$$\mathbf{x} = \mathbf{y} + t\mathbf{z} = \mathbf{y}^* + v\dot{\mathbf{y}}^*, \quad v = (t + \beta)/A.$$

Thus in this case $\mathbf{x}(s, t)$ is a portion of the tangent surface of the curve $\mathbf{y}^*(s)$.

Developable surfaces are of particular importance because they are the only surfaces which can be mapped isometrically into a plane, as will be seen in the following section.

Problem 58.1. Find an analytic expression of the Gaussian curvature of a ruled surface and prove that a ruled surface is a developable surface if and only if its Gaussian curvature vanishes.

59. Spherical image of a surface. Third fundamental form. Isometric mapping of developable surfaces. In the theory of curves the spherical image enabled us to draw some conclusions on the geometric properties of a curve. Analogous possibilities exist also in the theory of surfaces.

We impose a parallel displacement to the unit normal vector $\mathbf{n}(u^1, u^2)$ to a surface S : $\mathbf{x}(u^1, u^2)$ and bind it at the origin O of the Cartesian coordinate system in space. Then the terminal point of the vector \mathbf{n} lies on the unit sphere S^* with centre O , and to every point of S there corresponds a certain point of S^* . The position vector of the image of a point of S is the corresponding unit normal vector \mathbf{n} bound at O . The mapping thus defined is called the *Gaussian spherical mapping* of S . The set of all image points is called the *spherical image* of S .

Let S be of class $r \geq 2$. Obviously the first fundamental form of the spherical image of S is given by the expression

$$(59.1) \quad ds^{*2} = c_{\alpha\beta} du^\alpha du^\beta, \quad c_{\alpha\beta} = \mathbf{n}_\alpha \cdot \mathbf{n}_\beta$$

and is sometimes called the *third fundamental form* of the surface S .

Hence we now have three fundamental forms altogether,

$$(59.2) \quad I = ds^2 = d\mathbf{x} \cdot d\mathbf{x}, \quad II = -d\mathbf{x} \cdot d\mathbf{n}, \quad III = ds^{*2} = d\mathbf{n} \cdot d\mathbf{n},$$

cf. (38.5) and (38.6).

By means of the formulae of Weingarten (45.1) we may represent the coefficients

$$(59.3) \quad c_{\alpha\beta} = \mathbf{n}_\alpha \cdot \mathbf{n}_\beta$$

of the third fundamental form in terms of the coefficients of the other ones,

$$c_{\alpha\beta} = b_\alpha^\gamma \mathbf{x}_\gamma \cdot b_\beta^\eta \mathbf{x}_\eta = b_\alpha^\gamma b_\beta^\eta g_{\gamma\eta} = b_\alpha^\gamma b_\beta^\gamma$$

hence

$$(59.4) \quad c_{\alpha\beta} = b_{\alpha\kappa} b_{\beta\gamma} g^{\kappa\gamma}.$$

In consequence of this representation $c_{\alpha\beta}$ is a covariant tensor of second order which is symmetric, $c_{\alpha\beta} = c_{\beta\alpha}$, as follows from (59.3). For the

discriminant c of the third fundamental form we find

$$c = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = \begin{vmatrix} b_{1\kappa} b_{1\gamma} g^{\kappa\gamma} & b_{1\kappa} b_{2\gamma} g^{\kappa\gamma} \\ b_{2\kappa} b_{1\gamma} g^{\kappa\gamma} & b_{2\kappa} b_{2\gamma} g^{\kappa\gamma} \end{vmatrix} = b^2 \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix}$$

hence, in consequence of (32.8),

$$(59.5) \quad c = \frac{b^2}{g}.$$

Furthermore

$$(59.6) \quad KI - 2HII + III = 0,$$

where K and H are the Gaussian and mean curvature, respectively. In order to prove (59.6) it suffices to give the proof with respect to a special coordinate system on a surface S under consideration, because K , H and the three fundamental forms are independent of the coordinates (more precisely: H may change its sign under a coordinate transformation, but in this case also II changes its sign, and the product HII is therefore invariant). We choose coordinates on S so that the coordinate curves are lines of curvature on S . Then $g_{12} = b_{12} = 0$, cf. (41.6) and K and H are given by (41.11). By means of the formulae (45.1') we find

$$c_{11} = \mathbf{n}_1 \cdot \mathbf{n}_1 = \left(\frac{b_{11}}{g_{11}} \right)^2 \mathbf{x}_1 \cdot \mathbf{x}_1 = \frac{(b_{11})^2}{g_{11}}, \quad c_{12} = 0, \quad c_{22} = \frac{(b_{22})^2}{g_{22}}.$$

By means of (41.11), (59.6) follows from those expressions.

Comparing in (59.6) the terms which contain $(du^1)^2$, $du^1 du^2$ and $(du^2)^2$, respectively, we immediately obtain the relations

$$(59.7) \quad c_{\alpha\beta} = 2Hb_{\alpha\beta} - Kg_{\alpha\beta}, \quad (\alpha = 1, 2, \beta = 1, 2).$$

From (59.5) we find the following geometrical interpretation of the Gaussian curvature:

Theorem 59.1 (O. Rodrigues [2]). *At any point of a surface S , the absolute value of the Gaussian curvature K of S is equal to the quotient of the element of area, dA^* , of the spherical image of S and of the corresponding element of area, dA , of S .*

Proof. By means of (36.8) and (41.9) we have

$$\frac{dA^*}{dA} = \frac{\sqrt{c}}{\sqrt{g}} = \frac{|b|}{\sqrt{g}} \frac{1}{\sqrt{g}} = |K|.$$

Remark. If we use the concept of an oriented element of area we may formulate Theorem 59.1 as follows: *If the spherical image S^* of a portion S of a surface is oriented so that the normal vectors to S and S^* have the same sense then at any point P of S the quotient dA^*/dA of the corresponding oriented elements of area equals the Gaussian curvature K of S at P .*

Theorem 59.1 is due to O. Rodrigues [2] and occurs later in the general theory of Gauss [1]. This theorem states a fact which is analogous to the one given by the relation (17.2).

The spherical image of a portion of a surface is, in general, a portion of a surface. However, in the case of a plane the unit normal vector is constant and the spherical image therefore degenerates into a point. If S is a developable surface then the unit normal vector \mathbf{n} is the same for all points on any generating straight line, hence the spherical image of a developable surface degenerates into a curve. Therefore, by Theorem 59.1, $K = 0$ must hold in this case. We can prove the following

Theorem 59.2. *A portion S of a surface of class $r \geq 2$ is a portion of a developable surface if and only if its Gaussian curvature K is zero everywhere.*

Proof. (a) The spherical image of a developable surface is a curve (see above), hence, in consequence of Theorem 59.1 $K = 0$ at any point. (b) If $K = 0$ then by means of (41.9) also $b = 0$. If $b_{\alpha\beta} = 0$ ($\alpha = 1, 2$, $\beta = 1, 2$) the surface is a plane, cf. Problem 45.1. If $b = 0$, that is,

$$(59.8) \quad b_{11} b_{22} = (b_{12})^2$$

but $b_{\alpha\beta}$ are not all zero then, by (59.8), the differential equation (39.4) of the asymptotic curves can be written in the form

$$(59.9) \quad (\sqrt{b_{11}} du^1 + \sqrt{b_{22}} du^2)^2 = 0.$$

That is, we have only one family of asymptotic curves as is the case for parabolic points. The coordinates on the portion S of the surface under consideration may be chosen so that the curves $u^2 = \text{const.}$ coincide with the asymptotic curves. Then on these curves $du^2 = 0$; hence, in consequence of (59.9), $b_{11} = 0$ and, in consequence of (59.8), also $b_{12} = 0$. By means of the formulae of Weingarten, (45.1) we obtain from this

$$\mathbf{n}_1 = -b_1^\beta \mathbf{x}_\beta = -(b_{11} g^{1\beta} + b_{12} g^{2\beta}) \mathbf{x}_\beta = 0,$$

that is, the unit normal vector \mathbf{n} is independent of u^1 and consequently \mathbf{n} does not vary along any asymptotic curve. If the asymptotic curves are straight lines then S is a portion of a developable surface which has those lines as generators. This is indeed true. Since $b_{11} = b_{12} = 0$, $b_{22} \neq 0$ we obtain from the first of the equations (46.3') that $\Gamma_{11}^2 = 0$, therefore (45.14) takes the form $\mathbf{x}_{11} = \Gamma_{11}^1 \mathbf{x}_1$, that is, the asymptotic curves are straight lines, cf. Problem 11.1. If all $b_{\alpha\beta}$ are zero along a straight line, S has flat points on that line; for developable surfaces this may happen and does not contradict Theorem 58.3.

Developable surfaces are important because of the following property.

Theorem 59.3. *A (sufficiently small) portion of a surface of class $r \geq 3$ can be mapped isometrically into a plane if and only if it is a portion of a developable surface.*

Proof. (a) The Gaussian curvature K is invariant with respect to isometric mappings, cf. Theorem 57.2. In the case of a plane we have $K = 0$. Hence in order that a portion S of a surface can be mapped isometrically into a plane we must necessarily have $K = 0$, that is, in consequence of Theorem 59.2, S must be a portion of a developable surface. (b) We will now prove that a (sufficiently small) portion S of a developable surface can be mapped isometrically into a plane. We introduce on S geodesic parallel coordinates u^1, u^2 so that the curve $u^1 = 0$ is a geodesic G , the curves $u^2 = \text{const.}$ are geodesics which intersect G at right angles, and the curves $u^1 = \text{const.}$ are geodesic parallels of G . As was pointed out in Section 52, by means of a suitable choice of u^1 the first fundamental form is then given by the expression (52.1):

$$ds^2 = (du^1)^2 + g_{22}(du^2)^2.$$

Moreover, if we take the arc length of G as coordinate u^2 then $ds = du^2$ on G , and therefore

$$(59.10) \quad g_{22}(0, u^2) = 1.$$

Since G is a geodesic, we have from (49.9 a)

$$(59.11) \quad \left. \frac{\partial g_{22}}{\partial u^1} \right|_{u^1=0} = 0.$$

In consequence of Theorem 59.2, $K = 0$ on S . Hence (52.2) takes the form

$$\frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2} = 0.$$

Integrating this relation we find

$$\sqrt{g_{22}} = c_1(u^2)u^1 + c_2(u^2).$$

It follows from (59.10) that $c_2 = 1$ and from (59.11) that $c_1 = 0$. Hence $g_{11} = 1, g_{12} = 0, g_{22} = 1$. If u^{1*}, u^{2*} are Cartesian coordinates in the image plane the mapping $u^{*\alpha} = u^\alpha$ ($\alpha = 1, 2$) is isometric, in consequence of Theorem 57.1.

We mention the following result of the preceding investigation: The orthogonal trajectories of geodesics on a developable surface are also geodesics on this surface if at least one of them is a geodesic. This property holds on developable surfaces only, since otherwise also surfaces other than

developable ones could be isometrically mapped into a plane, in contradiction to Theorems 57.2 and 59.2.

It was already known to L. Euler [4] and G. Monge [2] that there are developable surfaces other than cones or cylinders.

60. Conjugate directions. Conjugate families of curves. Developable surfaces contacting a surface. Following Dupin we call two directions at a point of a surface *conjugate directions* if the corresponding diameters of the indicatrix (42.5) are conjugate. Two families of curves which form a net are called *conjugate families* if at every point the directions of the tangents to those curves are conjugate.

We remember the definition of conjugate diameters of a conic section.

In the case of a circle every diameter can be defined as the locus of the midpoints of suitably directed parallel chords. The analogous locus is also called a diameter in the case of another (arbitrary) conic section. The diameters of an ellipse or of a hyperbola are segments or whole straight lines through the centres of these conic sections. (The diameters of a parabola are not segments but rays which are parallel to the axis of symmetry; we will exclude parabolas from the following considerations.) If a diameter d_1 bisects the chords parallel to a diameter d_2 then d_2 bisects the chords parallel to d_1 , and d_1 and d_2 are called *conjugate diameters* of the conic section under consideration. Let

$$(60.1) \quad a^{ik}x_i x_k = p = \text{const.}$$

be the equation of a conic section in the $x_1 x_2$ -plane. Let $\mathbf{x} = (x_1, x_2)$ be a point on a diameter d_1 and $\mathbf{y} = (y_1, y_2)$ a point on another diameter d_2 , cf. Fig. 73. d_1 and d_2 are conjugate if and only if \mathbf{y} is the midpoint of a chord parallel to d_1 , that is, if and only if for a suitably chosen constant $c \neq 0$ the points $(y_1 + cx_1, y_2 + cx_2)$ and $(y_1 - cx_1, y_2 - cx_2)$ lie on the conic section; hence we

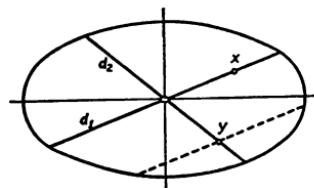


FIG. 73

must have

$$a^{ik}(y_i \pm cx_i)(y_k \pm cx_k) = p.$$

By subtracting one of these two equations from the other, since $c \neq 0$, we find

$$(60.2) \quad a^{ik}x_i y_k = 0.$$

Two diameters are conjugate if this relation is satisfied for any point (x_1, x_2) on one of them and any point (y_1, y_2) on the other one.

The intersection of a surface S and two planes parallel to the tangent plane at a point of S is given by the expression

$$b_{\alpha\beta} du^\alpha du^\beta = \pm 2\epsilon,$$

up to terms of third and higher order in $|du^1| + |du^2|$; here ϵ is the distance of those planes from the tangent plane, cf. (42.6). We therefore obtain immediately from (60.2) the following

Theorem 60.1. *At an elliptic or hyperbolic point P of a surface two directions $du^2:du^1$ and $\delta u^2:\delta u^1$ are conjugate if and only if at P*

$$(60.3) \quad b_{\alpha\beta} du^\alpha \delta u^\beta = 0,$$

that is,

$$b_{11} du^1 \delta u^1 + b_{12} (du^1 \delta u^2 + du^2 \delta u^1) + b_{22} du^2 \delta u^2 = 0$$

or, by means of (38.6),

$$(60.3') \quad d\mathbf{x} \cdot \delta \mathbf{n} = 0$$

or

$$(60.3'') \quad \delta \mathbf{x} \cdot d\mathbf{n} = 0.$$

From this theorem we obtain

Theorem 60.2. *The coordinate curves of a system of coordinates u^1, u^2 on a surface of class $r \geq 3$ are conjugate if and only if with respect to that system*

$$(60.4) \quad b_{12} = 0.$$

Proof. If the coordinate curves are conjugate then (60.3) holds for $du^1 = 0$ as well as for $\delta u^2 = 0$, that is, (60.4) must be true. Conversely, if $b_{12} = 0$ then (60.3) is satisfied for the coordinate curves.

From this theorem, by means of (41.3), we have

Theorem 60.3. *Apart from umbilics the families of the lines of curvature of a surface of class $r \geq 3$ are orthogonal and conjugate.*

In other words: Two conjugate families of curves constitute an orthogonal net if and only if the curves coincide with the lines of curvature on the surface under consideration.

Since the asymptotic curves are solutions of the differential equation $b_{\alpha\beta} du^\alpha du^\beta = 0$, cf. (39.4), that is, $d\mathbf{x} \cdot d\mathbf{n} = 0$, cf. (38.6), according to (60.3) they are 'self-conjugate'. This agrees with the fact that the asymptotes of the indicatrix of Dupin are self-conjugate diameters.

We will now investigate the properties of a developable surface contacting a surface S along a given curve of S . We shall use developable surfaces of that type as a tool in connexion with the consideration of displacement, cf. Section 77.

Theorem 60.4. Let $D: \mathbf{x}(s, t)$ be a developable surface contacting a surface $S: \mathbf{y}(u^1, u^2)$ along a curve $C: \mathbf{y}(s)$ on S . Then at every point of C the generator of D and the tangent to C are conjugate.

Proof. D can be represented in the form

$$(60.5) \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{z}(s)$$

where $\mathbf{z}(s)$ is a unit vector which has the direction of the generators of D . We differentiate (60.5) with respect to the arc length s of C ,

$$\frac{\partial \mathbf{x}}{\partial s} = \dot{\mathbf{y}} + t\dot{\mathbf{z}}.$$

If we take the scalar product of both sides of this relation and \mathbf{n} we find

$$\mathbf{n} \cdot \dot{\mathbf{y}} + t\mathbf{n} \cdot \dot{\mathbf{z}} = 0$$

because $\partial \mathbf{x} / \partial s \cdot \mathbf{n} = 0$. Now $\mathbf{n} \cdot \dot{\mathbf{y}} = 0$. Hence, for $t \neq 0$, we have

$$(60.6) \quad \mathbf{n} \cdot \dot{\mathbf{z}} = 0.$$

Differentiating $\mathbf{n} \cdot \mathbf{z} = 0$ with respect to s we obtain

$$\dot{\mathbf{n}} \cdot \mathbf{z} + \mathbf{n} \cdot \dot{\mathbf{z}} = 0;$$

hence, by means of (60.6) $\dot{\mathbf{n}} \cdot \mathbf{z} = 0$,

or $\mathbf{z} \cdot d\mathbf{n} = 0$.

Denoting by $\delta \mathbf{y}$ the direction of \mathbf{z} this becomes $d\mathbf{n} \cdot \delta \mathbf{y} = 0$; the directions $d\mathbf{y}$ and $\delta \mathbf{y}$ are conjugate.

We may also formulate this theorem in the following manner: If a point P of a surface S moves in a certain direction the tangent plane T to S at P revolves so that the direction of the motion of P and the direction of the axis of that revolution of T are conjugate.

From the proof of Theorem 60.4 it follows that if there exists a developable surface D contacting S along C then D is uniquely determined, since the direction of the vector \mathbf{z} is uniquely determined by the above relation $\dot{\mathbf{n}} \cdot \mathbf{z} = 0$. We prove that if the curvature of C is positive D exists. We choose a unit vector \mathbf{z} in the tangent plane to S along C so that $\dot{\mathbf{n}} \cdot \mathbf{z} = 0$ holds. We have $\mathbf{n} \cdot \mathbf{z} = 0$, $\dot{\mathbf{n}} \cdot \mathbf{z} + \mathbf{n} \cdot \dot{\mathbf{z}} = 0$ and therefore $\mathbf{n} \cdot \dot{\mathbf{z}} = 0$. Now $\dot{\mathbf{y}} \times \mathbf{z} = c\mathbf{n}$ where c is a scalar. Therefore $|\dot{\mathbf{y}} \times \mathbf{z}| \cdot |\dot{\mathbf{z}}| = (\dot{\mathbf{y}} \times \mathbf{z}) \cdot \dot{\mathbf{z}} = c\mathbf{n} \cdot \dot{\mathbf{z}} = 0$, that is, (58.2) is satisfied, and D is a developable surface. (If at a point P the direction of C is asymptotic the vectors $\dot{\mathbf{y}}$ and \mathbf{z} are linearly dependent; then the tangent plane to D at P is undetermined. In this case the tangent plane to S at P coincides with the tangent plane to D at the points of the

generator through P . If C is an asymptotic curve that fact is true for any of its points, and D is identical with the tangent surface of C .)

This completes our present study of developable surfaces; another important property of these surfaces will be considered in Section 86.

We finally include a table of important special coordinate systems on a surface, indicating those coefficients of the corresponding first and second fundamental forms which are zero. (Isotropic coordinates mentioned in this table will be introduced in Section 63.)

<i>Coordinate curves</i>	g_{11}	g_{12}	g_{22}	b_{11}	b_{12}	b_{22}	<i>Cf. formula</i>
orthogonal	0	(35.7)
conjugate	0	..	(60.4)
lines of curvature	0	0	..	(41.6)
asymptotic curves	0	..	0	(39.6)
isotropic . . .	0	..	0	(63.4)

61. Conformal mapping. The angle between two intersecting curves is, by definition, the angle between the tangents to these curves at the point of intersection, cf. Section 35.

An allowable mapping of a portion S of a surface onto a portion S^* of a surface is called *angle-preserving* or *conformal* if the angle of intersection of every arbitrary pair of intersecting arcs on S^* is the same as that of the corresponding inverse images on S at the corresponding point.

The name ‘conformal’ was introduced by Schubert [1].

Conformal mappings are important in cartography, cf. Section 65. It is therefore not surprising to find that special types of conformal mappings were already known a long time ago. For example, the stereographic projection (cf. Section 65) was introduced already by the ancient Greeks. A systematic theory of conformal mapping was developed by Euler, Lagrange, and especially Gauss [2].

Necessary and sufficient conditions for a mapping to be conformal are as follows:

Theorem 61.1. *An allowable mapping of a portion S of a surface onto a portion S^* of a surface is conformal if and only if, when on S and S^* the same coordinate systems have been introduced, the coefficients $g_{\alpha\beta}$ and $g_{\alpha\beta}^*$ of the first fundamental forms of S and S^* , respectively, are proportional,*

$$g_{\alpha\beta}^* = \eta(u^1, u^2)g_{\alpha\beta}, \quad \eta > 0, \quad (\alpha = 1, 2, \beta = 1, 2).$$

Proof. That this condition is sufficient follows immediately from (35.5) since the proportionality factor η occurs in the numerator of (35.5) as well

as in the denominator. We will now prove that the above condition is also necessary. Let a conformal mapping of S onto S^* be given which, in consequence of the special choice of the coordinates u^1, u^2 on S and u^{1*}, u^{2*} on S^* , takes the form $u^{\alpha*} = u^\alpha$ ($\alpha = 1, 2$), cf. Section 56. Let P be an arbitrary point of an arc C_1 on S and let $du^2:du^1$ be the direction of the tangent to C_1 at P . The image point P^* of P has the same values of the coordinates as P . The equations by which C_1 is represented on S represent also the image C_1^* of C_1 on S^* . The direction of the tangent to C_1^* at P^* is therefore given by the above value $du^2:du^1$. Consider now on S another arc C_2 intersecting C_1 at P and let $\delta u^2:\delta u^1$ be the direction of the tangent to C_2 at P . Then the angle α of intersection of those two arcs at P can be obtained from

$$(61.1) \quad \sin \alpha = \sqrt{g} \frac{|du^1 \delta u^2 - du^2 \delta u^1|}{ds \delta s}$$

where ds and δs are the linear elements of C_1 and C_2 , respectively, cf. (35.8). We denote by C_2^* the image of C_2 on S^* . The direction of the tangent to C_2^* at P^* is given by the above value $\delta u^2:\delta u^1$. The angle α^* between C_1^* and C_2^* at P^* can be obtained from

$$(61.2) \quad \sin \alpha^* = \sqrt{g^*} \frac{|du^1 \delta u^2 - du^2 \delta u^1|}{ds^* \delta s^*}$$

where ds^* and δs^* denote the linear elements of C_1^* and C_2^* , respectively. Since the mapping is assumed to be conformal, $\alpha = \alpha^*$. Hence, from (61.1) and (61.2),

$$(61.3) \quad \frac{\sqrt{g}}{ds \delta s} = \frac{\sqrt{g^*}}{ds^* \delta s^*}.$$

Taking as arc C_2 the (directed) coordinate curve $u^2 = \text{const.}$ through P , and as C_2^* , the coordinate curve $u^{2*} = \text{const.}$ through P^* then $\delta s = \sqrt{g_{11}} du^1$ and $\delta s^* = \sqrt{g_{11}^*} du^{1*} = \sqrt{g_{11}^*} du^1$. Consequently, by (61.3),

$$(61.4) \quad \frac{ds^*}{ds} = \frac{\sqrt{g^*} \sqrt{g_{11}}}{\sqrt{g} \sqrt{g_{11}^*}} = \sqrt{\{\eta(u^1, u^2)\}}$$

or

$$ds^* = \sqrt{\{\eta(u^1, u^2)\}} ds.$$

In this relation η depends on P only but is independent of the tangent direction under consideration. Since (61.4) is supposed to be true for any curve passing through P , that is, for every value of $du^2:du^1$ we must have $g_{\alpha\beta}^* = \eta(u^1, u^2) g_{\alpha\beta}$ ($\alpha = 1, 2, \beta = 1, 2$) at P . Now, since the mapping is assumed to be conformal at every point of S these relations hold everywhere on S .

From this theorem and Theorem 57.1 we immediately obtain the following

Theorem 61.2. *Every isometric mapping is conformal.*

Problem 61.1. Determine all the surfaces of revolution which can be mapped isometrically onto a given surface of revolution of variable Gaussian curvature.

62. Conformal mapping of surfaces into a plane. We consider an allowable mapping T of a portion S of a surface into a plane E and denote by u^1, u^2 Cartesian coordinates in E . We may introduce u^1, u^2 as coordinates on S . If in particular T is a conformal mapping then the first fundamental form on S with respect to u^1, u^2 is given by the expression

$$(62.1) \quad ds^2 = \eta(u^1, u^2)[(du^1)^2 + (du^2)^2], \quad \eta > 0;$$

this follows immediately from Theorem 61.1.

Definition 62.1. *Allowable coordinates u^1, u^2 on a surface S for which the corresponding first fundamental form is of the type (62.1) are called isothermal coordinates on S .*

As was seen in Section 59 the set of all surfaces which can be mapped isometrically into a plane is relatively small. The requirement of conformality is much less restrictive than isometry; we may consequently expect that the set of surfaces which can be mapped conformally into a plane will be much larger than the above set.

Let each of two surfaces S_1 and S_2 be mapped conformally onto a surface S_3 . If we associate each point of S_1 with that point of S_2 which has the same image point in S_3 we obtain a conformal mapping of S_1 onto S_2 . Let us assume that S_1 can be mapped conformally into a plane, say E_1 , and that S_2 can be mapped conformally into a plane, say E_2 . Then, if E_1 is mapped conformally onto E_2 in a general way, by composing the mappings we obtain the most general conformal mapping of S_1 onto S_2 .

Let u^1, u^2 and u^{1*}, u^{2*} be Cartesian coordinates in E_1 and E_2 , respectively. If $h(u), u = u^1 + iu^2$, is a function regular in a domain D_1 of the E_1 -plane and if D_2 denotes the range of values of h in the E_2 -plane, then

$$w \equiv u^{1*} + iu^{2*} = h(u), \quad u = u^1 + iu^2, \quad i = \sqrt{(-1)}, \quad \frac{dh}{du} \neq 0,$$

is a conformal mapping of D_1 onto D_2 . This is a well-known result of the theory of analytic functions of a complex variable.

$h(u)$ satisfies the Cauchy-Riemann differential equations

$$\frac{\partial u^{1*}}{\partial u^1} = \frac{\partial u^{2*}}{\partial u^2}, \quad \frac{\partial u^{1*}}{\partial u^2} = -\frac{\partial u^{2*}}{\partial u^1}.$$

By differentiating these equations and equating the mixed second partial derivatives we obtain the integrability conditions

$$\Delta u^{\alpha*} \equiv \frac{\partial^2 u^{\alpha*}}{(\partial u^1)^2} + \frac{\partial^2 u^{\alpha*}}{(\partial u^2)^2} = 0, \quad (\alpha = 1, 2).$$

The problem of determining a conformal mapping of a portion of a surface into a plane will lead to a more general form of the equation of Laplace. In order to solve this problem we have to prove that isothermal coordinates can be introduced on that portion. If we take these coordinates as Cartesian coordinates in the image plane we obtain the desired conformal mapping.

Theorem 62.1. *Any simply-connected portion S of a surface which has a representation of class $r \geq 3$ can be conformally mapped into a plane.*

Sketch of proof. It is sufficient to prove that S admits isothermal coordinates u^1, u^2 . Then, by introducing these coordinates as Cartesian coordinates in the image plane, we obtain the desired mapping. Let \tilde{u}^1, \tilde{u}^2 be the originally given allowable coordinates on S . We will prove the existence of an allowable transformation (cf. Section 24)

$$(62.2) \quad u^\alpha = u^\alpha(\tilde{u}^1, \tilde{u}^2), \quad (\alpha = 1, 2),$$

by which the isothermal coordinates u^1, u^2 are represented as functions of the originally given ones; if (62.2) is allowable its inverse exists, cf. Section 24. With respect to the isothermal coordinates, according to (62.1) we must have $g_{\alpha\beta} = \eta \delta_{\alpha\beta}$ and $g = \eta^2$. Consequently (62.2) must be of such a type that

$$\eta \delta_{\mu\nu} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{u}^\alpha}{\partial u^\mu} \frac{\partial \tilde{u}^\beta}{\partial u^\nu} \quad \text{and} \quad \delta^{\mu\nu}/\eta = \tilde{g}^{\alpha\beta} \frac{\partial u^\mu}{\partial \tilde{u}^\alpha} \frac{\partial u^\nu}{\partial \tilde{u}^\beta}, \quad \eta > 0.$$

Setting $\mu = 1$ and $\nu = 2$ we find from this

$$(62.3) \quad \tilde{g}^{\alpha\beta} \frac{\partial u^1}{\partial \tilde{u}^\alpha} \frac{\partial u^2}{\partial \tilde{u}^\beta} = 0.$$

Setting $\mu = \nu = 1$ and $\mu = \nu = 2$, respectively, and eliminating η we have

$$(62.4) \quad \tilde{g}^{\alpha\beta} \frac{\partial u^1}{\partial \tilde{u}^\alpha} \frac{\partial u^1}{\partial \tilde{u}^\beta} = \tilde{g}^{\sigma\tau} \frac{\partial u^2}{\partial \tilde{u}^\sigma} \frac{\partial u^2}{\partial \tilde{u}^\tau}.$$

If the functions (62.2) to be obtained satisfy the relations

$$(62.5) \quad \frac{\partial u^1}{\partial \tilde{u}^\alpha} = \tilde{g}_{\alpha\lambda} \tilde{\epsilon}^{\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \quad (\alpha = 1, 2)$$

(cf. Section 33) they also satisfy (62.3) and (62.4) as can be seen by inserting and by using the answer to Problem 33.2. The integrability condition of (62.5) is

$$\frac{\partial^2 u^1}{\partial \tilde{u}^1 \partial \tilde{u}^2} - \frac{\partial^2 u^1}{\partial \tilde{u}^2 \partial \tilde{u}^1} = 0,$$

which may also be written in the form

$$\sqrt{\tilde{g}} \tilde{\epsilon}^{\alpha\beta} \frac{\partial^2 u^1}{\partial \tilde{u}^\alpha \partial \tilde{u}^\beta} = 0.$$

Since $\sqrt{\tilde{g}} \tilde{\epsilon}^{\alpha\beta}$ is constant (0 or ± 1), by (62.5) we obtain from the preceding relation

$$(62.6) \quad \begin{aligned} \sqrt{\tilde{g}} \tilde{\epsilon}^{\alpha\beta} \frac{\partial}{\partial \tilde{u}^\beta} \left(\tilde{g}_{\alpha\kappa} \tilde{\epsilon}^{\kappa\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \right) &= \frac{\partial}{\partial \tilde{u}^\beta} \left(\sqrt{\tilde{g}} \tilde{g}_{\alpha\kappa} \tilde{\epsilon}^{\alpha\beta} \tilde{\epsilon}^{\kappa\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \right) \\ &= \frac{\partial}{\partial \tilde{u}^\beta} \left(\sqrt{\tilde{g}} \tilde{g}^{\beta\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \right) = 0, \end{aligned}$$

where the last expression is a consequence of the representation of the contravariant metric tensor given in Problem 33.2. Equation (62.6) is a linear partial differential equation with coefficients of (at least) class one. In every sufficiently small domain S_1 of S there exist solutions of that equation, which are not constant, as can be proved by successive approximation, cf. A. Korn [1]. The same thus holds for (62.5). A similar differential equation can be obtained from (62.5) by solving for $\partial u^2 / \partial \tilde{u}^\lambda$. For this purpose we have to take the inner product of (62.5) and $\tilde{g}_{\beta\sigma} \tilde{\epsilon}^{\alpha\beta}$. The right-hand side of (62.5) then takes the form

$$\tilde{g}_{\alpha\kappa} \tilde{g}_{\beta\sigma} \tilde{\epsilon}^{\alpha\beta} \tilde{\epsilon}^{\kappa\lambda} = \tilde{\epsilon}_{\kappa\sigma} \tilde{\epsilon}^{\kappa\lambda} = \delta_\sigma^\lambda.$$

Since $\tilde{\epsilon}^{\alpha\beta} = -\tilde{\epsilon}^{\beta\alpha}$ we thus obtain

$$(62.7) \quad \frac{\partial u^2}{\partial \tilde{u}^\sigma} \equiv -\tilde{g}_{\beta\sigma} \tilde{\epsilon}^{\beta\alpha} \frac{\partial u^1}{\partial \tilde{u}^\alpha}, \quad (\sigma = 1, 2).$$

Equations (62.5) and (62.7) are generalizations of the Cauchy–Riemann differential equations. From (62.1) we obtain

$$\eta^2 = g = \tilde{g} \left(\frac{\partial(\tilde{u}^1, \tilde{u}^2)}{\partial(u^1, u^2)} \right)^2,$$

cf. (28.4). We will now prove that the Jacobian of the transformation (62.2) is different from zero. From (62.5) we find

$$\frac{\partial(u^1, u^2)}{\partial(\tilde{u}^1, \tilde{u}^2)} = \tilde{g}_{1\kappa} \tilde{\epsilon}^{\kappa\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \frac{\partial u^2}{\partial \tilde{u}^2} - \tilde{g}_{2\kappa} \tilde{\epsilon}^{\kappa\lambda} \frac{\partial u^2}{\partial \tilde{u}^\lambda} \frac{\partial u^2}{\partial \tilde{u}^1}$$

or
$$\frac{\partial(u^1, u^2)}{\partial(\tilde{u}^1, \tilde{u}^2)} = \frac{1}{\sqrt{\tilde{g}}} \left[\tilde{g}_{22} \left(\frac{\partial u^2}{\partial \tilde{u}^1} \right)^2 - 2\tilde{g}_{12} \frac{\partial u^2}{\partial \tilde{u}^1} \frac{\partial u^2}{\partial \tilde{u}^2} + \tilde{g}_{11} \left(\frac{\partial u^2}{\partial \tilde{u}^2} \right)^2 \right].$$

This quadratic form is positive definite; consequently it vanishes only if both of the partial derivatives $\partial u^2 / \partial \bar{u}^\alpha$ ($\alpha = 1, 2$) are zero. But this cannot be true for a regular solution of (62.6). We may thus introduce isothermic coordinates in the domain S_1 of S and obtain a conformal mapping of S_1 into the plane. Local conformal mappings of other sufficiently small parts of S can be similarly obtained. A conformal mapping of S into the plane must be finally composed of all these different local mappings. The situation is as follows. Consider two parts of S , say S_1 and S_2 , which have a non-empty intersection D and which are mapped into the plane by two different local conformal mappings. D has two different images which are connected by a one-to-one conformal mapping T , cf. Fig. 74. The composition of the two local mappings consists in mapping the images of S_1 and S_2 onto a circular disk in such a manner that corresponding points of both images of D always coincide. The solution of this composition problem can be obtained by means of function theory. In this manner we gain (stepwise) a uniform conformal mapping of S into the plane. If the above procedure has led to infinitely many local mappings then difficulties arise which can be overcome by the theory of uniformization.

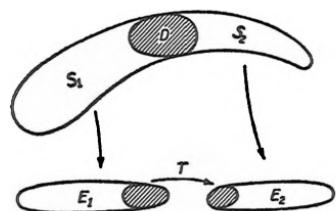


FIG. 74

composition of the two local mappings consists in mapping the images of S_1 and S_2 onto a circular disk in such a manner that corresponding points of both images of D always coincide. The solution of this composition problem can be obtained by means of function theory. In this manner we gain (stepwise) a uniform conformal mapping of S into the plane. If the above procedure has led to infinitely many local mappings then difficulties arise which can be overcome by the theory of uniformization.

Problem 62.1. A surface S is called a *surface of Liouville* if isothermic coordinates on S can be introduced so that the corresponding first fundamental form is of the type

$$ds^2 = (A + B)[(du^1)^2 + (du^2)^2]$$

where A depends on u^1 only while B depends on u^2 only. Prove that in the case of such a surface the differential equation of the geodesics can be completely integrated.

63. Isotropic curves and isothermic coordinates. We confine ourselves to real geometric objects and consequently, in general, also to real functions of real variables. Nevertheless, it is sometimes advantageous to introduce complex variables as a tool for the investigation of real surfaces, that is, to regard the real Euclidean space as being embedded in a complex Euclidean space in which the Cartesian coordinates x_j are complex numbers. This can be seen from this section; cf. also Section 83.

A real vector function $\mathbf{x}(t)$ where t is a real variable represents a real curve in space R_3 provided certain conditions (stated in Section 6) are satisfied. We now assume that the components $x_i(t)$ of $\mathbf{x}(t)$ are complex analytic functions of a complex variable $t = t_1 + it_2$, defined in a certain domain D of the complex t -plane; D is assumed to be the same for all three of those functions. Since t now depends on two real variables t_1 and t_2 the vector function $\mathbf{x}(t)$ represents a geometric configuration of ∞^2 points; this configuration is called a *complex curve* (or *imaginary curve*).

A curve is said to be an *isotropic curve* (or *minimal curve*) if the length of the arc between any two different points of the curve is zero. Hence a curve is isotropic if and only if

$$(63.1) \quad ds^2 = 0, \quad (\mathbf{x}' \neq 0).$$

Because of Theorem 28.1 Equation (63.1) cannot have real solutions but has two conjugate complex ones; isotropic curves are therefore always complex curves.

Isotropic curves were first considered by V. Poncelet [1], and later especially by S. Lie [1].

The first fundamental form can be represented as a product of two conjugate complex factors,

(63.2)

$$ds^2 = \left[\sqrt{g_{11}} du^1 + \frac{1}{\sqrt{g_{11}}} (g_{12} + i\sqrt{g}) du^2 \right] \left[\sqrt{g_{11}} du^1 + \frac{1}{\sqrt{g_{11}}} (g_{12} - i\sqrt{g}) du^2 \right].$$

For the expressions in brackets [...] integrating factors can be found. If $a_1(u^1, u^2)$ and $a_2(u^1, u^2)$ are real functions of u^1, u^2 and if $a_1 + ia_2$ is an integrating factor for the first bracket in (63.2) then $a_1 - ia_2$ is an integrating factor for the second one since the two expressions are conjugate. Hence there exist two functions $\tilde{u}^1(u^1, u^2)$ and $\tilde{u}^2(u^1, u^2)$ such that

$$(63.3) \quad \begin{aligned} d\tilde{u}^1 &= (a_1 + ia_2) \left[\sqrt{g_{11}} du^1 + \frac{1}{\sqrt{g_{11}}} (g_{12} + i\sqrt{g}) du^2 \right], \\ d\tilde{u}^2 &= (a_1 - ia_2) \left[\sqrt{g_{11}} du^1 + \frac{1}{\sqrt{g_{11}}} (g_{12} - i\sqrt{g}) du^2 \right], \end{aligned}$$

and the two families of isotropic curves on the surface S under consideration are the curves $\tilde{u}^1 = \text{const.}$ and $\tilde{u}^2 = \text{const.}$ Consequently by means of the transformation $\tilde{u}^\alpha = \tilde{u}^\alpha(u^1, u^2)$, $(\alpha = 1, 2)$,

the isotropic curves can be introduced as coordinate curves on S . By

comparing (63.2) and (63.3) we find that the first fundamental form then becomes

$$(63.4) \quad ds^2 = \frac{1}{a_1^2 + a_2^2} d\tilde{u}^1 d\tilde{u}^2.$$

We can now find isothermic coordinates on S in the following simple manner. From (63.3) we see that \tilde{u}^1 and \tilde{u}^2 are conjugate, that is, \tilde{u}^1 and \tilde{u}^2 are of the form

$$\tilde{u}^1 = v^1 + i v^2, \quad \tilde{u}^2 = v^1 - i v^2,$$

where v^1 and v^2 are real functions of u^1 and u^2 . Consequently,

$$d\tilde{u}^1 = dv^1 + i dv^2, \quad d\tilde{u}^2 = dv^1 - i dv^2,$$

and the corresponding first fundamental form is given by the expression

$$ds^2 = \frac{1}{a_1^2 + a_2^2} [(dv^1)^2 + (dv^2)^2].$$

That is, by means of the transformation

$$v^\alpha = v^\alpha(u^1, u^2), \quad (\alpha = 1, 2),$$

isothermic coordinates are introduced on S . Since v^1 and v^2 are real functions of u^1 and u^2 , the curves $v^\alpha = \text{const}$ ($\alpha = 1, 2$) are real curves.

Problem 63.1. Determine the isotropic curves of the unit sphere.

64. The Bergman metric. Conformal mapping plays an important role in function theory. The following basic theorem holds: Every analytic function $w = u + iv = f(z)$ of a complex variable $z = x + iy$ maps the domain D of the z -plane where $f(z)$ is defined onto the domain of the values of $f(z)$ in the w -plane, and this mapping is conformal at any point of D at which $df/dz \neq 0$.

We now ask whether and in what manner it is possible to introduce in a domain of the complex plane a metric which is invariant with respect to conformal mappings. This problem yields an example of the relations between differential geometry and function theory. We will use the method of the Bergman kernel function as an essential tool in our investigation. This method has the advantage that it can be extended also to mappings which are defined by analytic functions of several complex variables. For details of the whole theory see the book of S. Bergman, *Kernel Function and Conformal Mapping*, New York 1950; in what follows this book will be referred to as Be. We denote by Q the kernel function which is denoted by K in Bergman's book.

We must first introduce some concepts which will enter into our investigation.

A sequence $[\Phi_\alpha(z)]$ of complex, single-valued regular functions $\Phi_\alpha(z)$ of a complex variable $z = x+iy$ is called an *orthonormal system with respect to a domain B* of the complex plane if

$$(64.1) \quad J(\Phi_\alpha, \bar{\Phi}_\beta) \equiv \iint_B \Phi_\alpha(z) \overline{\Phi_\beta(z)} d\omega = \delta_{\alpha\beta} = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta) \end{cases}$$

where $d\omega = dx dy$ is the element of area and $\bar{\Phi}_\beta$ is the conjugate complex function of Φ_β . In the following we will always assume that the domain B is bounded. Instead of $J(\Phi_\alpha, \bar{\Phi}_\alpha)$ we write simply $J(\Phi_\alpha)$. For a general theory of orthogonal systems see also R. Courant and D. Hilbert [1, vol. 1] and G. Szegö [1].

The set of all functions which are regular and single-valued in B and for which the (Lebesgue) integral

$$(64.2) \quad \iint_B |f(z)|^2 d\omega$$

exists will be denoted as the *class $L^2(B)$* . If $f(z)$ is of this class and if $[\Phi_\alpha]$ is an orthonormal system with respect to B the numbers

$$(64.3) \quad a_\alpha = J(f, \bar{\Phi}_\alpha) = \iint_B f(z) \overline{\Phi_\alpha(z)} d\omega$$

are called the *Fourier coefficients* of $f(z)$ with respect to $[\Phi_\alpha]$. It can be proved that

$$(64.4) \quad \sum_{\alpha=1}^n c_\alpha \Phi_\alpha(z), \quad (n \text{ fixed}),$$

is the optimal approximation of $f(z)$ in the mean if and only if $c_\alpha = a_\alpha$ ($\alpha = 1, 2, \dots, n$). That is, if and only if $c_\alpha = a_\alpha$, the integral

$$(64.5) \quad J\left(f - \sum_{\alpha=1}^n c_\alpha \Phi_\alpha\right)$$

becomes minimum.

An orthonormal system is called *complete* if the Parseval formula

$$(64.6) \quad \sum_{\alpha=1}^{\infty} |a_\alpha|^2 = J(f)$$

holds. Hence by means of a complete orthonormal system the value of the integral (64.5) can be made arbitrarily small by increasing n , that is, by such a system $f(z)$ can be approximated in the mean to any degree of accuracy.

The function

$$(64.7) \quad Q_B(z, \bar{t}) = \sum_{\alpha=1}^{\infty} \Phi_\alpha(z) \overline{\Phi_\alpha(t)}$$

of the two complex variables z and t is called the *kernel* of the complete orthonormal system $\{\Phi_\alpha\}$. The series on the right-hand side of (64.7) converges in every closed subdomain B' of B ; the convergence is absolute and uniform in z for fixed t and is uniform in t for fixed z (Be 9). It can be proved that $Q_B(z, \bar{t})$ is independent of the special choice of the orthonormal system but depends on B only. $Q_B(z, \bar{t})$ is called the *Bergman kernel function of the domain B* .

Example. The functions

$$\Phi_\alpha(z) = \sqrt{\frac{\alpha}{\pi}} \frac{z^{\alpha-1}}{r^\alpha}, \quad (\alpha = 1, 2, \dots),$$

constitute a complete orthonormal system of the circular disk $|z| < r$. The corresponding kernel function is of the form

$$(64.8) \quad Q(z, \bar{t}) = \sum_{\alpha=1}^{\infty} \frac{\alpha}{\pi} \frac{(z\bar{t})^{\alpha-1}}{r^{2\alpha}} = \frac{r^2}{\pi(r^2 - z\bar{t})^2}.$$

The Bergman kernel function is related to the following minimum problem (Be 21): Determine a function $f(t)$ regular in a bounded domain B which at a fixed point $t = z$ of B has the value 1 and for which the integral

$$(64.9) \quad \iint_B |f(t)|^2 d\omega$$

is a minimum. The only function with those properties is

$$(64.10) \quad f(t) = \frac{Q_B(t, \bar{z})}{Q_B(z, \bar{z})};$$

the corresponding value of the integral (64.9) is $1/Q_B(z, \bar{z})$.

The Bergman metric which is invariant with respect to conformal mapping is given by the first fundamental form

$$(64.11) \quad ds_B^2 = Q_B(z, \bar{z}) |dz|^2 = Q_B(z, \bar{z})(dx^2 + dy^2).$$

Since $Q_B(z, \bar{z}) > 0$ this form is positive definite.

In order to prove the invariance property of that metric we consider an arbitrary regular function

$$\zeta = \xi + i\eta = F(z)$$

which maps the domain B conformally and one-to-one onto a domain B' of the ζ -plane. The Bergman metric in B' is determined by the first fundamental form

$$ds_{B'}^2 = Q_{B'}(\zeta, \bar{\zeta}) |\partial \zeta|^2,$$

where $Q_{B'}(\zeta, \bar{\zeta})$ denotes the Bergman kernel function of B' . We have to prove that

$$ds_{B'}^2 = ds_B^2.$$

Let $z = \Psi(\zeta)$ be the inverse of the mapping $\zeta = F(z)$. If $f(z)$ is a function of class $L^2(B)$ then $f(\Psi(\zeta)) d\Psi/d\zeta$ is a function of class $L^2(B')$, since we have

$$\begin{aligned} & \iint_{B'} |f(\Psi(\zeta))|^2 |\Psi'(\zeta)|^2 d\omega_\zeta \\ &= \iint_B |f(z)|^2 |\Psi'(\zeta)|^2 \frac{\partial(\xi, \eta)}{\partial(x, y)} d\omega_z = \iint_B |f(z)|^2 d\omega_z, \end{aligned}$$

where $d\omega_\zeta$ and $d\omega_z$ are the elements of area of B' and B , respectively. If $[\Phi_\alpha(z)]$ is a complete orthonormal system in B then $[\Psi_\alpha]$, where

$$\Psi_\alpha = \Phi_\alpha(\Psi(\zeta)) \Psi'(\zeta), \quad (\alpha = 1, 2, \dots),$$

is a complete orthonormal system in B' . Furthermore

$$\begin{aligned} Q_{B'}(\zeta, \bar{\tau}) &= \sum_{\alpha=1}^{\infty} \Psi_\alpha(\zeta) \overline{\Psi_\alpha(\tau)} = \sum_{\alpha=1}^{\infty} \Phi_\alpha(\Psi(\zeta)) \overline{\Phi_\alpha(\Psi(\tau))} \Psi'(\zeta) \overline{\Psi'(\tau)} \\ &= \Psi'(\zeta) \overline{\Psi'(\tau)} \sum_{\alpha=1}^{\infty} \Phi_\alpha(z) \overline{\Phi_\alpha(t)} = Q_B(z, \bar{t}) \frac{dz}{d\zeta} \overline{\left(\frac{dt}{d\tau} \right)}. \end{aligned}$$

Hence

$$ds_{B'}^2(\zeta) = Q_{B'}(\zeta, \bar{\zeta}) |d\zeta|^2 = Q_B(z, \bar{z}) \left| \frac{dz}{d\zeta} \right|^2 |d\zeta|^2 = ds_B^2(z).$$

Theorem 64.1. *If B' is a subdomain of a simply-connected schlicht domain B then, at every point of B' ,*

$$(64.12) \quad ds_{B'} \geq ds_B.$$

Proof. $1/Q_B(z, \bar{z})$ is the minimal value which the integral $\iint_B |f(t)|^2 d\omega$ can assume if all functions $f(t)$ which are of class $L^2(B)$ and have the value 1 at a fixed point $t = z$ of B are admitted. Hence

$$\frac{1}{Q_{B'}(z, \bar{z})} \leq \iint_{B'} \left| \frac{Q_B(t, \bar{z})}{Q_B(z, \bar{z})} \right|^2 d\omega \leq \iint_B \left| \frac{Q_B(t, \bar{z})}{Q_B(z, \bar{z})} \right|^2 d\omega = \frac{1}{Q_B(z, \bar{z})}$$

and consequently

$$Q_{B'}(z, \bar{z}) \leq Q_B(z, \bar{z}).$$

From this we obtain (64.12) by means of (64.11).

If, for instance, B is the interior of the unit circle then, according to (64.8),

$$(64.13) \quad ds_B = \frac{|dz|}{\sqrt{\pi(1 - |z|^2)}}.$$

In consequence of the Riemann mapping theorem B can be mapped conformally onto any simply-connected subdomain B' of the unit disk of the

complex w -plane. From Theorem 64.1 we thus obtain the well-known classical result

$$(64.14) \quad \frac{|dw|}{1-|w|^2} \leq \frac{|dz|}{1-|z|^2}.$$

From (64.11) we have in the case of the unit disk

$$g_{11} = g_{22} = \frac{1}{\pi(1-x^2-y^2)^2}, \quad g_{12} = 0;$$

since the coordinates $u^1 = x$ and $u^2 = y$ are orthogonal we may use (46.17') and obtain

$$\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial x} \right) = \frac{2(1+x^2-y^2)}{(1-x^2-y^2)^2}, \quad \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial y} \right) = \frac{2(1-x^2+y^2)}{(1-x^2-y^2)^2},$$

hence

$$(64.15) \quad K = -4\pi,$$

that is, with respect to the Bergman metric the Gaussian curvature of that disk has a constant negative value. Since the Bergman metric is invariant with respect to conformal mapping the above property of the Gaussian curvature holds for all simply-connected schlicht domains. In the case of multiply connected domains the Gaussian curvature with respect to the Bergman metric is not constant.

By using the above and further minimal properties of the Bergman kernel function we are able to find bounds for the value of the Gaussian curvature K at a point z of an arbitrary domain B . Let S_R and S_r be circular disks with radii R and r , respectively, with centres at the point z of B under consideration and let S_R contain B and B contain S_r , then

$$(64.16) \quad -4\pi \left(\frac{R}{r} \right)^4 \leq K \leq -4\pi \left(\frac{r}{R} \right)^4,$$

where K denotes the Gaussian curvature at the point z . The proof of (64.16) and further interesting properties of the invariant Bergman metric can be found in Bergman's book and in the original literature quoted therein.

65. Conformal mapping of a sphere into a plane. Stereographic and Mercator projection. Mappings of a sphere into a plane are of basic interest especially because of their application to the practical problem of constructing maps of the globe. Problems of that type were considered by Lambert [1] in 1772, Euler [5], [6], Lagrange [2], Gauss [3], and others.

It is known that the globe may be considered as a sphere of radius 6371 km. Since the Gaussian curvature of the plane is zero while a sphere of radius r has the Gaussian curvature $K = 1/r^2$ it is clear that no portion

of the sphere can be mapped isometrically into a plane, cf. Theorem 57.2. Hence there cannot exist a map of a portion of the globe which is (exactly) isometric. This fact can be proved in an elementary manner, without using Theorem 57.2; cf. Kreyszig [1]. But we can find conformal and equiareal mappings of a sphere into a plane; some of them possess additional properties which are important for practical reasons. Clearly the choice of the mapping depends on the purpose of the map to be constructed and also on the location and size of the portion of the globe to be mapped.

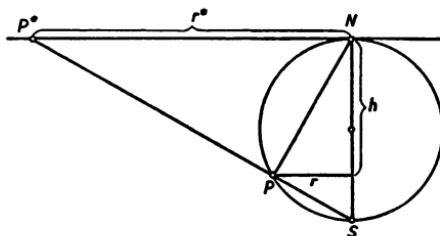


FIG. 75. Stereographic projection

We mention that the globe can be approximated more exactly by an ellipsoid of revolution with radius 6378 km at the equator and with distance of the poles from the centre of 6357 km.

A first important example of a conformal mapping of a sphere into a plane is the *stereographic projection*. This mapping was most likely discovered by Hipparch (160 B.C.); the name ‘stereographic projection’ was introduced by d’Aguillon [1] in 1613.

The stereographic projection is obtained by projecting the points of the sphere from a point S on the sphere, called the *centre of projection*, on the tangent plane to the sphere at the point diametrically opposite to S . By this projection the whole sphere (with the exception of the point S) is mapped one-to-one onto the (open) plane. In Fig. 76 the south pole of the globe has been chosen as centre of projection. The images of the circles parallel to the equator are concentric circles with centre at the North pole $N^* = N$; the images of the meridians are straight lines through N^* . The inverse image of an arbitrary straight line G^* is a circle on the sphere passing through S since the projecting straight lines corresponding to G^* lie in a plane which intersects the sphere along a circle.

Stereographic projection is conformal

We may prove this as follows. Let C_1 , and C_2 be any two curves on the sphere F intersecting at a point P . Let T_1 and T_2 , respectively, be the

tangents to C_1 and C_2 at P . We denote by C_1^* , C_2^* , and P^* respectively, the images of C_1 , C_2 , and P , and by S_α^* the tangents to C_α^* ($\alpha = 1, 2$) at P^* . The inverse images S_1 and S_2 of those tangents are circles which intersect at P and S , the angle of intersection being the same at both points. The tangents to S_1 and S_2 at S are parallel to S_1^* and S_2^* , respectively. The angle of intersection of these tangents at S is therefore equal to that of

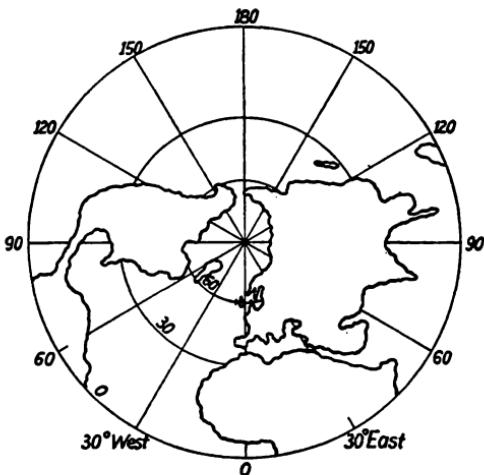


FIG. 76. Stereographic projection

S_1^* and S_2^* ; hence the same holds for the tangents to S_1 and S_2 at P which coincide with the tangents T_1 and T_2 to C_1 and C_2 at P . This completes the proof.

Stereographic projection has the following interesting property. It maps every circle on the sphere F onto a circle (or a straight line) in the plane. This fact can be proved as follows. We choose Cartesian coordinates x_1, x_2, x_3 in space such that the point N of F becomes the origin and the positive rays of the x_1 - and x_2 -axis coincide with those of the Cartesian x_1^*, x_2^* -coordinate system in the image plane. For the sake of simplicity we choose the diameter of the sphere F to be 1; then S coincides with the unit point on the x_3 -axis. If P ($\neq S$) is an arbitrary point of F and P^* its image then (cf. Fig. 75)

$$r^{*2} = x_1^{*2} + x_2^{*2}, \quad h = x_3, \quad x_3:r = r^*:1, \quad r:(1-x_3) = r^*:1;$$

hence
$$r = \frac{r^*}{1+r^{*2}}, \quad x_3 = \frac{r^{*2}}{1+r^{*2}}.$$

Setting

$$x_1^* = r^* \cos \alpha, \quad x_1 = r \cos \alpha,$$

$$x_2^* = r^* \sin \alpha, \quad x_2 = r \sin \alpha,$$

we have $x_1 = \frac{x_1^*}{a}, \quad x_2 = \frac{x_2^*}{a}, \quad x_3 = \frac{x_1^{*2} + x_2^{*2}}{a},$

where

$$a = 1 + x_1^{*2} + x_2^{*2},$$

and conversely

$$x_1^* = \frac{x_1}{c}, \quad x_2^* = \frac{x_2}{c}, \quad r^{*2} = \frac{x_3}{c},$$

where

$$c = 1 - x_3.$$

By inserting these expressions into the equation of a circle in the image plane,

$$A(x_1^{*2} + x_2^{*2}) + Bx_1^* + Cx_2^* + D = 0,$$

we see that the corresponding inverse image is represented by the linear equation

$$Ax_3 + Bx_1 + Cx_2 + D(1 - x_3) = 0.$$

Hence this inverse image is a plane curve on the sphere, that is, a circle.

We mention, without giving the proof, that the stereographic projection is the only conformal mapping of a sphere onto a plane which maps circles onto circles; this result was first obtained by Lagrange [2].

The *Mercator projection* is another important conformal mapping of the sphere onto the plane. This mapping was introduced in 1569 by G. Mercator, latinized for Kremer (1512–94), who did not indicate a mathematical procedure of construction; this was not communicated until much later by Halley (1695).

Using the representation (25.2) of the sphere we have

$$ds^2 = r^2[\cos^2 u^2 (du^1)^2 + (du^2)^2] = r^2 \cos^2 u^2 [(du^1)^2 + \sec^2 u^2 (du^2)^2].$$

We set $dx_1^* = du^1, \quad dx_2^* = \sec u^2 du^2 \quad (|u^2| < \frac{1}{2}\pi)$

or

$$(65.1) \quad x_1^* = u^1, \quad x_2^* = \log \tan\left(\frac{u^2}{2} + \frac{\pi}{4}\right) \quad (|u^2| < \frac{1}{2}\pi)$$

and introduce x_1^*, x_2^* as Cartesian coordinates in the image plane F^* . Then the Mercator projection of the sphere represented by (25.2) onto the plane F^* is accomplished by (65.1). Since the coefficients of the first fundamental form of the sphere and of the plane are proportional, the mapping is conformal.

The images of the meridians $u^1 = c_1 = \text{const.}$ are the straight lines $x_1^* = c_1 = \text{const.}$ The circles $u^2 = c_2 = \text{const.}$ parallel to the 'equator' $u^2 = 0$ are mapped onto straight lines

$$x_2^* = \log \tan\left(\frac{c_2}{2} + \frac{\pi}{4}\right).$$

The mapping is satisfactory near the equator but gives an exaggerated impression of the dimensions of Arctic regions, cf. Fig. 77.

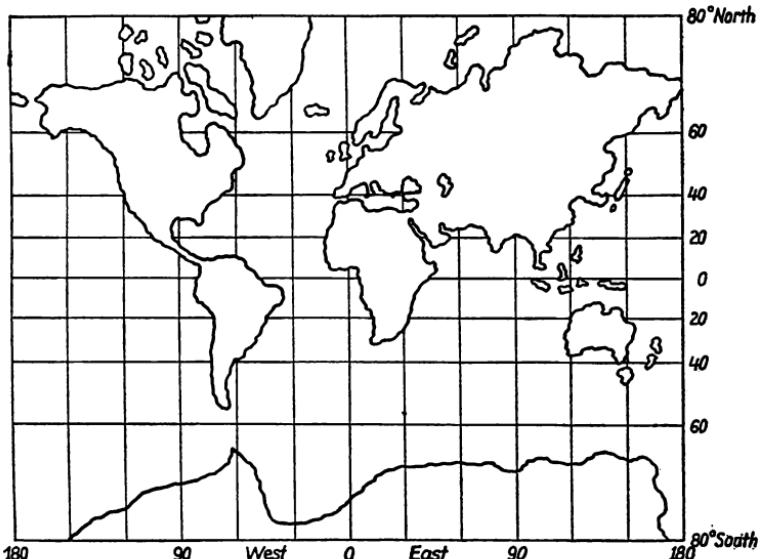


FIG. 77. Mercator projection

Curves of constant direction (for instance, continually north-east) on the globe are called *loxodromes*. Special cases of these curves are the meridians and parallels; the general type of a loxodrome is a curve which winds around the poles an infinity of times and approaches them more and more.

The images of loxodromes are straight lines; this fact can be proved as follows. By definition, any loxodrome intersects every meridian at the same angle. Since the Mercator projection is conformal, the image of the loxodrome intersects the images of the meridians also at a constant angle. Now those images are parallel straight lines; this completes the proof.

66. Equiareal mapping. An allowable mapping of a portion S of a surface onto a portion S^* of a surface is called *equiareal* if every part of S is mapped onto a part of S^* which has the same area.

The necessary and sufficient conditions for a mapping to be equiareal are as follows:

Theorem 66.1. *An allowable mapping of a portion S of a surface onto a portion S^* of a surface is equiareal if and only if at corresponding points the discriminants g and g^* of the first fundamental form of S and S^* , when referred to the same coordinate systems on S and S^* , are equal.*

Proof. The proof is analogous to that of Theorem 57.1; instead of the integral of the arc length one has to consider the integral (36.7) by which the area is defined.

Furthermore we have

Theorem 66.2. (1) *Every isometric mapping is equiareal.* (2) *Every equiareal and conformal mapping is isometric.*

Proof. (1) follows immediately from the preceding theorem in combination with Theorem 57.1. (2) If the mapping under consideration is conformal then by Theorem 61.1 at corresponding points, when referred to the same coordinate systems on both surfaces,

$$g_{\alpha\beta}^* = \eta(u^1, u^2) g_{\alpha\beta}.$$

From this we have $g^* = \eta^2 g$. In consequence of Theorem 66.1 $g^* = g$ hence $\eta^2 = 1$. Now since $g_{11} > 0$ as well as $g_{11}^* > 0$, $\eta = +1$ and consequently $g_{\alpha\beta}^* = g_{\alpha\beta}$, that is, the mapping is isometric, cf. Theorem 57.1.

We will now study equiareal mappings of surfaces of revolution into a plane. We obtain a surface of revolution by rotating a curve C about a straight line in space; that line is called the axis of revolution of that surface.

We rotate a curve $C: x_1 = p(u^2) (> 0), x_2 = 0, x_3 = q(u^2)$ about the x_3 -axis of the Cartesian coordinate system in space. The surface of revolution S thus obtained can be represented in the form

$$(66.1) \quad \mathbf{x}(u^1, u^2) = (p(u^2)\cos u^1, p(u^2)\sin u^1, q(u^2)).$$

The curves $u^1 = \text{const.}$ are called *meridians* and the curves $u^2 = \text{const.}$ *parallels* of S .

Theorem 66.3. *A mapping*

$$(66.2) \quad x_1^* = x_1^*(u^1, u^2), \quad x_2^* = x_2^*(u^1, u^2)$$

of the surface S , represented by (66.1), into a plane S^ with Cartesian coordinates x_1^*, x_2^* is equiareal if and only if the corresponding Jacobian \bar{D} satisfies*

$$\bar{D}^2 = p^2(p'^2 + q'^2),$$

where primes denote derivatives with respect to u^2 .

Proof. On S we have $g_{11} = p^2$, $g_{12} = 0$, and $g_{22} = p'^2 + q'^2$ and therefore

$$(66.3) \quad g = p^2(p'^2 + q'^2).$$

The discriminant of the first fundamental form of the image plane S^* has the value 1. By means of (66.2) we introduce in S^* the coordinates u^1, u^2 of S . Then that discriminant is multiplied by \bar{D}^2 , cf. (28.4). Hence afterwards $g = g^*$, that is, the mapping is equiareal; cf. Theorem 66.1.

Important applications of this theorem will be considered in the following section.

67. Equiareal mapping of spheres into planes. Mappings of Lambert, Sanson, and Bonne. The practical importance of mappings of the sphere into the plane has already been discussed in Section 65. We will now apply Theorem 66.3 to some interesting types of equiareal mappings of a sphere S into a plane S^* with Cartesian coordinates x_1^*, x_2^* .

Setting in (66.1)

$$(67.1) \quad p(u^2) = r \cos u^2, \quad q(u^2) = r \sin u^2,$$

we obtain the representation (25.2) of the sphere S of radius r with centre at the origin of the Cartesian coordinates in space. In consequence of Theorem 66.3 the Jacobian \bar{D} of an equiareal mapping of S into the plane S^* must now be equal to $\pm r^2 \cos u^2$.

Lambert [1] made the first general investigations of mathematical cartography; the so-called *Lambert projection*, however, was already known much earlier.

In terms of the coordinates just introduced [cf. (66.1), (67.1)] the Lambert projection is of the form

$$(67.2) \quad x_1^* = r u^1, \quad x_2^* = r \sin u^2, \quad (-\pi < u^1 \leq \pi, -\frac{1}{2}\pi \leq u^2 \leq \frac{1}{2}\pi).$$

Since the corresponding Jacobian is

$$\bar{D} = r^2 \cos u^2$$

the Lambert projection is equiareal, cf. Theorem 66.3. The image of the sphere is a rectangle whose sides have the lengths $2\pi r$ and $2r$. The mapping which is one-to-one, with the exception of the ‘poles’ of the sphere, can be geometrically obtained by projecting every point of S on a cylinder S^* in contact with S along the ‘equator’, by means of projecting lines parallel to the plane of the equator and passing through the axis of rotation of S^* ; by cutting S^* along the line corresponding to $u^1 = \pi$ and bending it into the plane we obtain the desired image of the sphere S . The images of the

parallels and meridians are straight lines; the latter are equidistant, the distance between the former decreases with increasing latitude.

We now consider the *mapping of Sanson*, which Sanson obtained from Lambert and used for construction of maps (1650). This mapping is given by

$$(67.3) \quad x_1^* = ru^1 \cos u^2, \quad x_2^* = ru^2, \quad (-\pi < u^1 \leq \pi, -\frac{1}{2}\pi \leq u^2 \leq \frac{1}{2}\pi).$$

It is equiareal since the corresponding Jacobian $\bar{D} = r^2 \cos u^2$. The images of the parallels $u^2 = \text{const.}$ are parallel straight lines $x_2^* = \text{const.}$ The

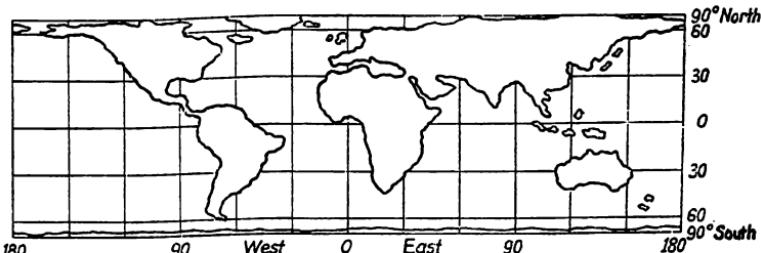


FIG. 78. Lambert projection

distance of those images is the same as the (spherical) distance of the corresponding inverse images. The length of any segment of the images of the parallels equals the length of the corresponding inverse image.

We finally consider the *mapping of Bonne*, introduced by Bonne in the eighteenth century and frequently used in cartography. It is defined by

$$(67.4) \quad r^* = r(\frac{1}{2}\pi - u^2), \quad \alpha^* = \frac{u^1 \cos u^2}{\frac{1}{2}\pi - u^2},$$

where u^1, u^2 are the coordinates on the sphere S , defined by (25.2), and r^*, α^* are polar coordinates in the image plane S^* .

The discriminant of the first fundamental form of S^* is $\tilde{g} = r^{*2}$. When u^1, u^2 are introduced as coordinates in S^* , in consequence of (28.4) \tilde{g} is multiplied by the square of the Jacobian

$$\bar{D} = \frac{r \cos u^2}{\frac{1}{2}\pi - u^2}.$$

We therefore obtain

$$g^* = r^2(\frac{1}{2}\pi - u^2)^2 \bar{D}^2 = r^4 \cos^2 u^2 = g,$$

that is, the mapping of Bonne is equiareal.

The images of the parallels $u^2 = u_0^2 = \text{const.}$ are arcs of the circles $r_0^* = r(\frac{1}{2}\pi - u_0^2)$. The image point of the North pole is the centre of these

circles, cf. Fig. 80. The length of an arc of the image of a parallel is equal to the length of the corresponding inverse image.

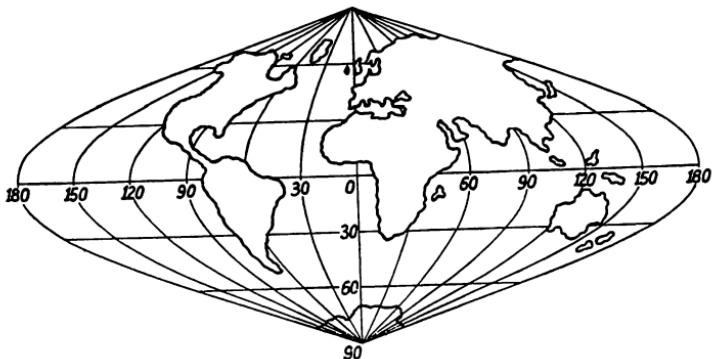


FIG. 79. Sanson projection

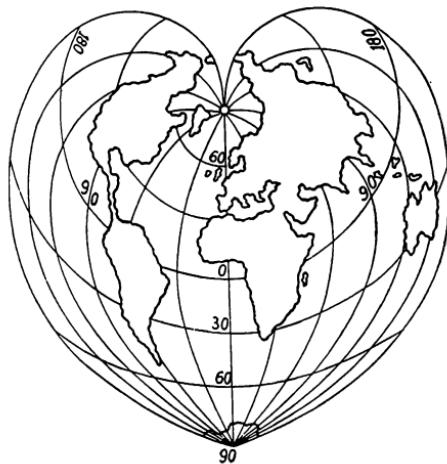


FIG. 80. Bonne projection

Problem 67.1. Investigate the mapping of Collignon [1]

$$x_1^* = \frac{2\sqrt{2}}{\sqrt{\pi}} r u^1 \sin\left(\frac{\pi}{4} - \frac{u^2}{2}\right), \quad x_2^* = r \sqrt{\pi} \left(1 - \sqrt{2} \sin\left(\frac{\pi}{4} - \frac{u^2}{2}\right)\right).$$

68. Conformal mapping of the Euclidean space. We consider the Euclidean space R_3 and assume that Cartesian coordinates have been introduced in this space. Let

$$(68.1) \quad y_i = h_i(x_1, x_2, x_3), \quad (i = 1, 2, 3),$$

be functions of class r (≥ 1) which are defined in one and the same domain B of R_3 and whose Jacobian is different from zero everywhere in B . By means of (68.1) to any point P of B with position vector $\mathbf{x} = (x_1, x_2, x_3)$ there is associated a point P^* with position vector $\mathbf{y} = (y_1, y_2, y_3)$. B is thus mapped onto a certain domain B^* of R_3 . If, in particular, B and B^* are both identical with the entire space the mapping is called a *mapping of R_3 onto itself*.

A one-to-one mapping of class r which maps a domain B of R_3 onto a domain B^* of the same space is called *angle-preserving* or *conformal* if, for any pair of intersecting curves, the angle of intersection of the images is equal to that of the corresponding inverse images at the corresponding point. We shall restrict our consideration to conformal mappings for which the image of any surface of class $r \geq 3$ is of class $r^* \geq 3$. Such a mapping will be called *admissible*.

Conformal mapping of Euclidean space differs from that of the plane in many aspects. For the plane there does not exist a theorem analogous to the Theorem 68.2 of Liouville; the image of a circle in the plane, when being mapped conformally, is not a circle in general. While in the plane any regular function $f(z)$ gives rise to a conformal mapping (cf. Section 64) we shall find that there are very few types of conformal mappings of space. It will be proved that any admissible conformal mapping of Euclidean space can be decomposed into inversions; cf. Theorem 68.3.

This means that inversions, that is, mappings of the form

$$\mathbf{y} = r^2 \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}},$$

are the only non-trivial admissible conformal mappings of the space. We will first consider some properties of those mappings; without great loss of generality we may set $r = 1$, that is, we consider the mapping

$$(68.2) \quad \mathbf{y} = \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}, \quad (\mathbf{x} \neq 0),$$

A transformation is called *involutory* if two successive applications of it result in the identity transformation. Inversion is involutory since when (68.2) and then another inversion $\mathbf{z} = \mathbf{y}/\mathbf{y} \cdot \mathbf{y}$ are applied we obtain

$$\mathbf{z} = \frac{\mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} = \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \frac{(\mathbf{x} \cdot \mathbf{x})^2}{\mathbf{x} \cdot \mathbf{x}} = \mathbf{x}.$$

Since in (68.2) the denominator $\mathbf{x} \cdot \mathbf{x}$ is a positive scalar, a point in space and its image lie on the same ray of a straight line passing through the

origin O of the coordinate system. Furthermore, from (68.2),

$$\mathbf{y} \cdot \mathbf{y} = \frac{1}{\mathbf{x} \cdot \mathbf{x}}$$

that is, the product of the distances from O of an arbitrary point and of its image is always equal to 1.

The fixed points of the transformation (68.2) can be obtained from the condition $\mathbf{y} = \mathbf{x}$, that is, from

$$\mathbf{x} = \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \quad \text{or} \quad \mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2 = 1.$$

Consequently the image point of any point P on the unit sphere with centre O coincides with P . The inversion is therefore also known as a *reflection in a sphere* (of radius r ; in our case $r = 1$).

The origin O of the coordinate system is called the *pole of the inversion*; it has no image point. But if we close the space at infinity by just one point Q , then O and Q correspond to each other, and the inversion becomes a one-to-one mapping in the whole space.

A sphere through O inverts into a plane, its points being transformed according to stereographic projection. Other spheres invert into spheres. Straight lines and planes may conveniently be regarded as circles and spheres through the point at infinity, Q . Indeed, let

$$a_0 \mathbf{x} \cdot \mathbf{x} + \sum_{i=1}^3 a_i x_i + a_4 = 0$$

be a representation of a sphere. Then we obtain from (68.2)

$$a_0 + \sum_{i=1}^3 a_i y_i + a_4 \mathbf{y} \cdot \mathbf{y} = 0.$$

Inversion is conformal

This can be proved as follows. Let $\mathbf{x}(s)$ and $\bar{\mathbf{x}}(\sigma)$ be any two intersecting curves and $\mathbf{y}(s^*)$ and $\bar{\mathbf{y}}(\sigma^*)$ the respective images; s , σ , s^* , and σ^* denote arc lengths. Then the derivatives \mathbf{x}_s , $\bar{\mathbf{x}}_\sigma$, \mathbf{y}_{s^*} , and $\bar{\mathbf{y}}_{\sigma^*}$ are all unit vectors. Hence the angles of intersection of the curves and their images are equal if

$$(68.3) \quad \mathbf{x}_s \cdot \bar{\mathbf{x}}_\sigma = \mathbf{y}_{s^*} \cdot \bar{\mathbf{y}}_{\sigma^*}.$$

We will prove (68.3). From (68.2) we have

$$(68.4) \quad \mathbf{y}_{s^*} = \mathbf{y}_s \frac{ds}{ds^*} = \frac{\mathbf{x}_s(\mathbf{x} \cdot \mathbf{x}) - \mathbf{x}(2\mathbf{x} \cdot \mathbf{x}_s)}{(\mathbf{x} \cdot \mathbf{x})^2} \frac{ds}{ds^*}.$$

Since $\mathbf{x}_s \cdot \mathbf{x}_s = 1$ and $\mathbf{y}_s \cdot \mathbf{y}_s = 1$ we obtain from (68.4)

$$\frac{ds}{ds^*} = \mathbf{x} \cdot \mathbf{x}.$$

Analogously

$$(68.5) \quad \bar{\mathbf{y}}_s = \frac{\bar{\mathbf{x}}_s(\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}) - \bar{\mathbf{x}}(2\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}_s)}{\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}}.$$

By taking the scalar product of (68.4) and (68.5) and using the fact that $\bar{\mathbf{x}} = \mathbf{x}$ at the point of intersection we obtain (68.3).

We now explain the concept of curvilinear coordinates in space; these coordinates resemble curvilinear coordinates in a plane, such as polar coordinates. Curvilinear coordinates u^1, u^2, u^3 can be introduced by a transformation of the form

$$(68.6) \quad x_i = h_i(u^1, u^2, u^3), \quad (i = 1, 2, 3);$$

we assume that the functions h_i are defined in the whole space and are of class r (≥ 1). Furthermore, the Jacobian of (68.6),

$$D = \frac{\partial(x_1, x_2, x_3)}{\partial(u^1, u^2, u^3)} \equiv \begin{vmatrix} \frac{\partial h_1}{\partial u^1} & \frac{\partial h_1}{\partial u^2} & \frac{\partial h_1}{\partial u^3} \\ \frac{\partial h_2}{\partial u^1} & \frac{\partial h_2}{\partial u^2} & \frac{\partial h_2}{\partial u^3} \\ \frac{\partial h_3}{\partial u^1} & \frac{\partial h_3}{\partial u^2} & \frac{\partial h_3}{\partial u^3} \end{vmatrix},$$

is assumed to be not identically zero. Then, in a neighbourhood of every point at which $D \neq 0$, there exists the inverse transformation

$$(68.6') \quad u^1 = u(x_1, x_2, x_3), \quad u^2 = v(x_1, x_2, x_3), \quad u^3 = w(x_1, x_2, x_3).$$

The surfaces

$$u^1 = u = \text{const.}, \quad u^2 = v = \text{const.}, \quad u^3 = w = \text{const.},$$

are called coordinate surfaces of the $u^1 u^2 u^3$ coordinate-system. These surfaces constitute a system of three families of ∞^1 surfaces. This system is called a *triply orthogonal system* if the three surfaces of the system passing through any point are mutually orthogonal. The surfaces under consideration form a triply orthogonal system if and only if the vectors

$$\mathbf{x}_j = \left(\frac{\partial x_1}{\partial u^j}, \frac{\partial x_2}{\partial u^j}, \frac{\partial x_3}{\partial u^j} \right), \quad (j = 1, 2, 3),$$

satisfy the orthogonality conditions

$$(68.7) \quad \mathbf{x}_j \cdot \mathbf{x}_k = 0, \quad j \neq k, \quad (j = 1, 2, 3; k = 1, 2, 3).$$

Examples. The coordinate planes of Cartesian coordinates form a triply orthogonal system. The coordinate surfaces of cylindrical coordinates

$$u^1 = \arctan \frac{x_2}{x_1}, \quad u^2 = +\sqrt{(x_1^2 + x_2^2)}, \quad u^3 = x_3$$

(cf. Fig. 81) form a triply orthogonal system in space with the exception of the x_3 -axis; the same is true for polar coordinates in space, cf. Problem 68.2.

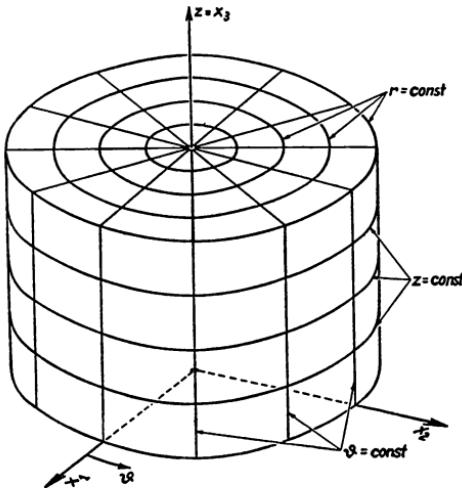


FIG. 81

To a given one-parameter family of (non-intersecting) curves in a plane there always exist orthogonal trajectories. In space, however, the analogous fact with respect to surfaces is not always true. Let an orthogonal system F of two families of surfaces be given. Then there exists a family G of surfaces whose surfaces intersect every surface of F orthogonally if and only if the curve of intersection of any two surfaces of F is a line of curvature on both surfaces. We have in fact the following

Theorem 68.1 (Dupin, 1813). *The curve of intersection of any pair of surfaces of class $r \geq 2$ of a triply orthogonal system is a line of curvature on both surfaces.*

Proof. Let H be a triply orthogonal system of surfaces

$$u^1 = u = \text{const.}, \quad u^2 = v = \text{const.}, \quad u^3 = w = \text{const.},$$

cf. (68.6), satisfying (68.7). It suffices to consider one of the three families of surfaces, for instance, the surfaces

$$W: w = \text{const.},$$

since the situation with respect to the other two families U : $u = \text{const.}$ and V : $v = \text{const.}$ is the same. u and v are coordinates on the surfaces of W . $u = \text{const.}$ and $v = \text{const.}$ are the curves of intersection of W with the surfaces of U and V , respectively. We have to prove that these curves of intersection are lines of curvature of the surfaces of W , that is,

$$g_{12} \equiv \mathbf{x}_1 \cdot \mathbf{x}_2 = 0 \quad \text{and} \quad b_{12} \equiv \mathbf{x}_{12} \cdot \mathbf{n} = 0,$$

where the indices denote partial derivatives and \mathbf{n} is the unit normal vector to a surface of W under consideration. In consequence of the assumed orthogonality $g_{12} = 0$. Differentiating (68.7) we obtain

$$(68.8) \quad \mathbf{x}_{13} \cdot \mathbf{x}_2 + \mathbf{x}_1 \cdot \mathbf{x}_{23} = 0,$$

$$(68.9) \quad \mathbf{x}_{21} \cdot \mathbf{x}_3 + \mathbf{x}_2 \cdot \mathbf{x}_{31} = 0,$$

$$(68.10) \quad \mathbf{x}_{32} \cdot \mathbf{x}_1 + \mathbf{x}_3 \cdot \mathbf{x}_{12} = 0.$$

Subtracting (68.8) from (68.10) we find

$$\mathbf{x}_{13} \cdot \mathbf{x}_2 = \mathbf{x}_3 \cdot \mathbf{x}_{12}.$$

Inserting this into (68.9) we have

$$(68.11) \quad \mathbf{x}_{12} \cdot \mathbf{x}_3 = 0.$$

(68.7) means that \mathbf{x}_3 is orthogonal to \mathbf{x}_1 and \mathbf{x}_2 , that is, to the surface under consideration. (68.11) is therefore equivalent to $b_{12} = 0$.

A theorem related to Theorem 68.1 will be considered in Section 88.

Using Theorem 68.1 we can prove the following

Theorem 68.2 (Liouville). *The image of every sphere with respect to any admissible conformal mapping of Euclidean space is again a sphere. (Here planes have to be considered as special cases of spheres, namely spheres through Q .)*

Proof. We consider an arbitrary sphere K_0 with centre M and will prove that the image K_0^* of K_0 with respect to any admissible conformal mapping in space is a sphere. For this purpose we choose a one-parameter family \mathfrak{K} of concentric spheres with centre M ; obviously K_0 is a surface of this family. We take a pencil \mathfrak{E} of planes E with axis G through M and a one-parameter family \mathfrak{A} of coaxial circular cones A with axis G . \mathfrak{K} , \mathfrak{E} , and \mathfrak{A} form a triply orthogonal system \mathfrak{F}_0 except on G . We denote by $T(\gamma)$ ($0 \leq \gamma \leq \pi$) a continuous rotation of the whole family \mathfrak{K} such that the axis of rotation passes through M and an arbitrary point P_0 of K_0 not on G ; γ is the angle of rotation; the families \mathfrak{E} and \mathfrak{A} are kept fixed. In consequence of this rotation the curve of intersection of K_0 and the plane $E = E_0$ through P_0 assumes every direction in the tangent plane of K_0 .

at P_0 . The rotating family $\mathfrak{R} = \mathfrak{R}(\gamma)$ and the fixed families \mathfrak{E} and \mathfrak{U} constitute a triply orthogonal system $\mathfrak{F} = \mathfrak{F}(\gamma)$, except on G . We consider now the image $\mathfrak{F}^*(\gamma)$ of $\mathfrak{F}(\gamma)$ with respect to any admissible conformal mapping. In consequence of conformality $\mathfrak{F}^*(\gamma)$ is a triply orthogonal system (except for the image of G) for every value of γ . The continuous rotation $T(\gamma)$ of K_0 corresponds to a continuous motion of the image K_0^* of K_0 which leaves the image P_0^* of P_0 fixed. In consequence of that motion the curve of intersection of K_0^* with the image of the plane E_0 assumes every direction in the tangent plane of K_0^* at P_0^* . We conclude from Theorem 68.1 that all those curves of intersection of different directions are lines of curvature on K_0^* , that is, P_0^* is a navel point of K_0^* . Since P_0 was arbitrarily chosen the image K_0^* of K_0 consists of navel points only; it is of class $r^* \geq 3$ and thus is a sphere (or a plane), cf. Theorem 48.1.

Theorem 68.3. *Every admissible conformal mapping K of the space is a composition of (at most five) inversions.*

Proof. Let P^* be the image of an arbitrary point P with respect to any admissible conformal mapping K of the space and, at the same time, with respect to a suitable translation T . Then $T^{-1}(K(P)) = P$. Let R be an inversion with pole P . By R a plane, not passing through P , is transformed into a sphere K_1 passing through P . $T^{-1}K$ is an admissible conformal mapping; in consequence of Theorem 68.2 it maps K_1 onto a sphere K_2 passing through P . R transforms K_2 into a plane. Hence by the transformation $A = RT^{-1}KR$ the arbitrary above plane is mapped onto a plane so that the image point of any point at a finite distance from the origin is a point at a finite distance from the origin; as is known A must therefore be an affine transformation. Since A is conformal it must be a dilatation. We thus obtain $K = TRAR$. Now, translation is the composition of reflections in two parallel planes. Reflection is inversion in a sphere of infinite radius. Dilatation is the composition of inversions in two concentric spheres. This completes the proof.

Problem 68.1. Using (9.7) and (68.6) determine the linear element corresponding to arbitrary curvilinear coordinates in space.

Problem 68.2. Let x_1, x_2, x_3 be Cartesian coordinates in space. Setting

$$x_1 = u^1 \sin u^2 \cos u^3, \quad x_2 = u^1 \sin u^2 \sin u^3, \quad x_3 = u^1 \cos u^2$$

we obtain spherical coordinates u^1, u^2, u^3 . Investigate the Jacobian of this transformation and the coordinate surfaces of these coordinates.

VII

ABSOLUTE DIFFERENTIATION AND PARALLEL DISPLACEMENT

69. Concept of absolute differentiation. In the present chapter an introduction to the concepts of absolute differentiation and parallel displacement of vectors will be given. We shall see that these two fundamental ideas in modern differential geometry are closely related to each other. They are important especially in Einstein's theory of relativity.

We first consider the problem which leads us to formulate the definition of the absolute differentiation and later (in Section 76) the problem which leads us to introduce the displacement of Levi-Civita.

We start from the following

Fact. The first partial derivatives of a tensor of first or higher order, whose components are differentiable functions of the coordinates on a surface (for example, the metric tensor), do not form a tensor.

In order to understand this fact we may consider a tensor a^β of first order whose components are differentiable functions of the coordinates u^1, u^2 on a surface. Differentiating its law of transformation,

$$(69.1) \quad \bar{a}^\alpha = a^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta},$$

cf. (29.4) with respect to \bar{u}^ρ we find

$$(69.2) \quad \frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} = \frac{\partial a^\beta}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\alpha}{\partial u^\beta} + a^\kappa \frac{\partial^2 \bar{u}^\alpha}{\partial u^\kappa \partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho}.$$

If the second double sum on the right-hand side of (69.2) were zero, (69.2) would be of the form (30.4), that is, we would have a tensor in this case, namely a mixed tensor of second order. This holds if

$$(69.3) \quad \frac{\partial^2 \bar{u}^\alpha}{\partial u^\kappa \partial u^\sigma} = 0, \quad (\alpha, \kappa, \sigma = 1, 2),$$

that is, if the transformation $\bar{u}^\alpha(u^1, u^2)$ ($\alpha = 1, 2$) is linear. Transformations of Cartesian coordinates in the plane into Cartesian coordinates are of that type. But on surfaces or in Riemannian spaces of higher dimension the allowable transformations are not linear, in general, and consequently (69.3) will not hold.

Laws of transformation analogous to (69.2) hold for the first partial

derivatives of tensors of higher order; but these laws are even more complicated than (69.2).

Since derivatives of tensors figure in many investigations the corresponding formalism would be very difficult. We thus come to the following

Problem. A differentiation process must be defined in such a manner that the derivative (in the sense of this process) of any tensor is again a tensor.

This requirement means that the new differentiation to be defined should be independent of the coordinate system; it will consequently be called *absolute differentiation*.

We shall, of course, require that in the case of Euclidean spaces, when parallel coordinates have been introduced, absolute differentiation shall be identical with the usual partial differentiation.

In the following section we will define the absolute differentiation of tensors of first order, that is, of vectors. By generalizing this definition in a natural and very simple manner we shall later obtain the definition of the absolute differentiation of tensors of arbitrary order.

We will restrict the dimension n of the Riemannian space to the value $n = 2$, that is, we will consider surfaces only, for the sake of uniformity in this book. This restriction is not essential, however; all the investigations and results contained in this chapter are also valid for n -dimensional Riemannian spaces of an arbitrary higher dimension n ; the only difference is then that all the summations to be done in consequence of the summation convention are carried through from 1 to n instead of from 1 to 2 only. Cf. also the remarks contained in Sections 27 and 29.

70. Absolute differentiation of tensors of first order. The reasons for generalizing the usual partial differentiation have been outlined in the preceding section. We will now define the absolute differentiation of a tensor of first order.

For this purpose we first consider a contravariant vector a^β whose components are differentiable functions of the coordinates u^1, u^2 on a surface. If an allowable transformation $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$ ($\alpha = 1, 2$) is carried out, a^β behaves according to (29.4), that is,

$$\bar{a}^\alpha = a^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta}.$$

By taking the partial derivative with respect to \bar{u}^σ we obtain the relations (69.2):

$$\frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\sigma} = \frac{\partial a^\beta}{\partial u^\sigma} \frac{\partial \bar{u}^\alpha}{\partial \bar{u}^\beta} + a^\kappa \frac{\partial^2 \bar{u}^\alpha}{\partial u^\kappa \partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\beta}$$

which were already considered in the preceding section. We can now replace the second derivatives occurring in these relations by means of the transformation formula (47.12) of the Christoffel symbols of the second kind; we find

$$(70.1) \quad \frac{\partial^2 \bar{u}^\alpha}{\partial u^\kappa \partial u^\sigma} = \Gamma_{\kappa\sigma}^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta} - \bar{\Gamma}_{\tau\epsilon}^\alpha \frac{\partial \bar{u}^\tau}{\partial u^\kappa} \frac{\partial \bar{u}^\epsilon}{\partial u^\sigma}.$$

By inserting (70.1) into (69.2) we obtain

$$\frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} = \frac{\partial a^\beta}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\alpha}{\partial u^\beta} + a^\kappa \left(\Gamma_{\kappa\sigma}^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta} - \bar{\Gamma}_{\tau\epsilon}^\alpha \frac{\partial \bar{u}^\tau}{\partial u^\kappa} \frac{\partial \bar{u}^\epsilon}{\partial u^\sigma} \right) \frac{\partial u^\sigma}{\partial \bar{u}^\rho}.$$

In consequence of $a^\kappa \frac{\partial \bar{u}^\tau}{\partial u^\kappa} \frac{\partial \bar{u}^\epsilon}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} = \bar{a}^\tau \delta_\rho^\epsilon$

we have $\frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} = \left(\frac{\partial a^\beta}{\partial u^\sigma} + a^\kappa \Gamma_{\kappa\sigma}^\beta \right) \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\alpha}{\partial u^\beta} - \bar{a}^\tau \bar{\Gamma}_{\tau\rho}^\alpha$

or

$$(70.2) \quad \frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} + \bar{a}^\tau \bar{\Gamma}_{\tau\rho}^\alpha = \left(\frac{\partial a^\beta}{\partial u^\sigma} + a^\kappa \Gamma_{\kappa\sigma}^\beta \right) \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\alpha}{\partial u^\beta}.$$

Using the notations

$$(70.3) \quad \bar{a}^\alpha{}_\rho = \frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} + \bar{a}^\tau \bar{\Gamma}_{\tau\rho}^\alpha$$

and

$$(70.3') \quad a^\beta{}_\sigma = \frac{\partial a^\beta}{\partial u^\sigma} + a^\kappa \Gamma_{\kappa\sigma}^\beta$$

we may write (70.2) in the form

$$(70.4') \quad \bar{a}^\alpha{}_\rho = a^\beta{}_\sigma \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \frac{\partial u^\sigma}{\partial \bar{u}^\rho}.$$

If we multiply both sides of this relation by $\frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\rho}{\partial u^\nu}$ and sum with respect to α and ρ then in consequence of (29.3) we obtain

$$\bar{a}^\alpha{}_\rho \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\rho}{\partial u^\nu} = a^\beta{}_\sigma \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\rho}{\partial u^\nu} = a^\beta{}_\sigma \delta_\beta^\mu \delta_\nu^\alpha$$

and therefore

$$(70.4) \quad a^\mu{}_\nu = \bar{a}^\alpha{}_\rho \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\rho}{\partial u^\nu}.$$

Hence the quantities $a^\beta{}_\sigma$ are the components of a mixed tensor of second order with respect to the coordinates u^1, u^2 . This tensor is called the *covariant derivative of the contravariant vector a^β with respect to the tensor $g_{\alpha\beta}$* . The covariant derivative is a tensor which has one more covariant index

than the originally given one; this is the reason for the name ‘covariant derivative’. The covariant derivative is denoted by a covariant index preceded by a comma. By the terminology *covariant derivative with respect to the tensor* $g_{\alpha\beta}$ we indicate the fact that the coefficients in (70.3), that is, the Christoffel symbols, are functions of the metric tensor $g_{\alpha\beta}$. (Since only the transformation behaviour (47.12) but no other property of the Christoffel symbols entered into our considerations we may replace these functions by other functions with the same law of transformation and thus obtain a generalized absolute differentiation, cf. Section 78.)

The covariant derivative differs from the ordinary one only by a sum of linear terms,

$$a^\kappa \Gamma_{\kappa\alpha}{}^\beta.$$

In Euclidean spaces, when Cartesian coordinates have been introduced, $\Gamma_{\kappa\alpha}{}^\beta = 0$, that is, in this case the covariant derivative becomes identical with the usual partial derivative. The above defined absolute differentiation thus satisfies all the conditions stated in the preceding section. All further investigations are more or less repetitions of what we have just done; the method of reasoning is essentially the same as that used above.

We will define the covariant derivative of a covariant vector a_β . We have

$$\bar{a}_\alpha = a_\beta \frac{\partial u^\beta}{\partial \bar{u}^\alpha},$$

cf. (29.5). By differentiating these relations with respect to \bar{u}^ρ we have

$$(70.5) \quad \frac{\partial \bar{a}_\alpha}{\partial \bar{u}^\rho} = \frac{\partial a_\beta}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} + a_\kappa \frac{\partial^2 u^\kappa}{\partial \bar{u}^\alpha \partial \bar{u}^\rho}.$$

From (47.12) we obtain

$$\frac{\partial^2 u^\kappa}{\partial \bar{u}^\alpha \partial \bar{u}^\rho} = \bar{\Gamma}_{\alpha\rho}{}^\tau \frac{\partial u^\kappa}{\partial \bar{u}^\tau} - \Gamma_{\beta\sigma}{}^\kappa \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho},$$

which relations enable us to eliminate the second derivatives from (70.5),

$$\frac{\partial \bar{a}_\alpha}{\partial \bar{u}^\rho} = \frac{\partial a_\beta}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} + a_\kappa \left(\bar{\Gamma}_{\alpha\rho}{}^\tau \frac{\partial u^\kappa}{\partial \bar{u}^\tau} - \Gamma_{\beta\sigma}{}^\kappa \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \right).$$

By means of

$$a_\kappa \frac{\partial u^\kappa}{\partial \bar{u}^\tau} = \bar{a}_\tau$$

we obtain from the preceding relations the formulae

$$(70.6) \quad \frac{\partial \bar{a}_\alpha}{\partial \bar{u}^\rho} - \bar{a}_\tau \bar{\Gamma}_{\alpha\rho}{}^\tau = \left(\frac{\partial a_\beta}{\partial u^\sigma} - a_\kappa \Gamma_{\beta\sigma}{}^\kappa \right) \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha},$$

which are analogous to (70.2). Using the notations

$$(70.7) \quad \bar{a}_{\alpha,\rho} = \frac{\partial \bar{a}_\alpha}{\partial \bar{u}^\rho} - \bar{a}_\tau \bar{\Gamma}_{\alpha\rho}^\tau$$

and

$$(70.7') \quad a_{\beta,\sigma} = \frac{\partial a_\beta}{\partial u^\sigma} - a_\kappa \Gamma_{\beta\sigma}^\kappa$$

we may write (70.6) in the form

$$(70.8) \quad \bar{a}_{\alpha,\rho} = a_{\beta,\sigma} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho}.$$

That is, the quantities $a_{\beta,\sigma}$ are the components of a covariant tensor of second order (with respect to the coordinates u^1, u^2); this tensor is called the *covariant derivative of the vector a_β with respect to the tensor $g_{\alpha\beta}$* .

The covariant derivative of a covariant vector differs from the usual partial derivative by a sum of linear terms which are preceded by a negative sign. (In the case of the covariant derivative of a contravariant vector a positive sign appeared in the formulae.)

71. Absolute differentiation of tensors of arbitrary order. By a simple generalization of the investigations carried out in the preceding section we shall now obtain the definition of the absolute derivative of a tensor of higher order; the different steps of our reasoning will be exactly the same as those of that section.

We have to start from the transformation law of an arbitrary tensor, that is, from (30.5). To every index of the tensor there corresponds, in every term of the multiple sum on the right-hand side of (30.5), exactly one first order partial derivative. For example, the transformation law of a mixed tensor of second order is given by

$$\bar{a}_\alpha^\tau = a_\beta^\lambda \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\tau}{\partial u^\lambda};$$

it involves two such partial derivatives in every term of the double sum on the right-hand side. The first step consists in differentiating (30.5) once with respect to one of the coordinates \bar{u}^1, \bar{u}^2 . When doing so each of the aforementioned derivatives yields one second order partial derivative in each term on the right-hand side. In our example we thus obtain

$$(71.1) \quad \frac{\partial \bar{a}_\alpha^\tau}{\partial \bar{u}^\rho} = \frac{\partial a_\beta^\lambda}{\partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\tau}{\partial u^\lambda} + a_\mu^\nu \left(\frac{\partial^2 u^\mu}{\partial \bar{u}^\alpha \partial \bar{u}^\rho} \frac{\partial \bar{u}^\tau}{\partial u^\nu} + \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial^2 \bar{u}^\tau}{\partial u^\nu \partial u^\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \right).$$

The second step consists in eliminating each of these second derivatives by means of the transformation law (47.12) of the Christoffel symbols, cf. (70.1). In this manner we obtain in our example instead of the bracket on the right-hand side of (71.1) the expression

$$\left[\bar{\Gamma}_{\alpha\rho}^{\gamma} \frac{\partial u^\mu}{\partial \bar{u}^\gamma} - \bar{\Gamma}_{\beta\rho}^{\mu} \frac{\partial u^\delta}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \right] \frac{\partial \bar{u}^\tau}{\partial u^\nu} + \left[\bar{\Gamma}_{\nu\sigma}^{\zeta} \frac{\partial \bar{u}^\tau}{\partial u^\zeta} - \bar{\Gamma}_{\epsilon\lambda}^{\tau} \frac{\partial \bar{u}^\epsilon}{\partial u^\nu} \frac{\partial \bar{u}^\lambda}{\partial u^\sigma} \right] \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho}.$$

We have now to order the expression thus obtained. All sums of products which involve Christoffel symbols $\bar{\Gamma}_{\alpha\beta}^\gamma$ corresponding to the coordinates \bar{u}^1, \bar{u}^2 are transposed to the left-hand side of the expression under consideration. When doing so all tensor components which occur in those products and correspond to the coordinates u^1, u^2 are transformed into tensor components corresponding to \bar{u}^1, \bar{u}^2 ; this can be done by means of (30.5) and leads to terms which no longer involve first partial derivatives. Consequently no such partial derivatives appear on the left-hand side afterwards. Since

$$a_\mu^\nu \frac{\partial u^\mu}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\tau}{\partial u^\nu} = \bar{a}_\gamma^\tau$$

and $a_\mu^\nu \frac{\partial \bar{u}^\epsilon}{\partial u^\nu} \frac{\partial \bar{u}^\lambda}{\partial \bar{u}^\sigma} \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\rho} = \bar{a}_\alpha^\epsilon \delta_\rho^\lambda$,

we thus obtain in our example

$$(71.2) \quad \frac{\partial \bar{a}_\alpha^\tau}{\partial \bar{u}^\rho} - \bar{a}_\gamma^\tau \bar{\Gamma}_{\alpha\rho}^\gamma + \bar{a}_\alpha^\epsilon \bar{\Gamma}_{\epsilon\rho}^\tau = \left[\frac{\partial a_\beta^\zeta}{\partial u^\sigma} - a_\mu^\zeta \Gamma_{\beta\sigma}^\mu + a_\beta^\nu \Gamma_{\nu\sigma}^\zeta \right] \frac{\partial u^\sigma}{\partial \bar{u}^\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\tau}{\partial u^\zeta}.$$

Also in the general case to every index of the tensor there corresponds exactly one multiple sum of additional terms on each side of the expression obtained by the above ordering process; every one of these sums has a positive sign if it corresponds to a contravariant index and a negative one if it corresponds to a covariant one; we have already come across a special case of this fact in the preceding section. In the case of our example it follows from (71.2) that the quantities

$$(71.3') \quad a_{\beta,\sigma}^\zeta = \frac{\partial a_\beta^\zeta}{\partial u^\sigma} - a_\mu^\zeta \Gamma_{\beta\sigma}^\mu + a_\beta^\nu \Gamma_{\nu\sigma}^\zeta$$

are the components of a tensor of third order, covariant of second order and contravariant of first order; this tensor is called the covariant derivative of the tensor a_β^ζ with respect to the tensor $g_{\alpha\beta}$. In the general case of a tensor $a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s}$ of order $r+s$, covariant of order r and contravariant

of order s , we find that the quantities

$$(71.3) \quad a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s, \sigma} = \frac{\partial a_{\alpha_1 \dots \alpha_r}}{\partial u^\sigma}{}^{\beta_1 \dots \beta_s} - \sum_{\gamma=1}^r a_{\alpha_1 \dots \alpha_{\gamma-1} \kappa \alpha_{\gamma+1} \dots \alpha_r}{}^{\beta_1 \dots \beta_s} \Gamma_{\alpha_\gamma \sigma}{}^\kappa + \\ + \sum_{\delta=1}^s a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_{\delta-1} \lambda \beta_{\delta+1} \dots \beta_s} \Gamma_{\lambda \sigma}{}^{\beta_\delta}$$

are the components of a tensor of order $r+s+1$, covariant of order $r+1$, contravariant of order s ; this tensor is called the *covariant derivative of the tensor $a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s}$ with respect to the tensor $g_{\alpha\beta}$* .

In consequence of the covariant differentiation the number of the covariant indices increases by one.

The covariant derivative of a differentiable tensor of order zero, that is, of a scalar $\phi(u^1, u^2)$, is defined by

$$(71.4) \quad \phi_{,\sigma} = \frac{\partial \phi}{\partial u^\sigma}.$$

This covariant vector $\phi_{,\sigma}$ is called the *gradient* of the scalar ϕ , cf. Section 29.

We finally mention two special cases of (71.3) which occur frequently. In the case of a covariant tensor of second order, $a_{\alpha\beta}$, the covariant derivative takes the form

$$(71.3'') \quad a_{\alpha\beta,\sigma} = \frac{\partial a_{\alpha\beta}}{\partial u^\sigma} - a_{\gamma\beta} \Gamma_{\alpha\sigma}{}^\gamma - a_{\alpha\gamma} \Gamma_{\beta\sigma}{}^\gamma.$$

The covariant derivative of a contravariant tensor of second order, $a^{\alpha\beta}$, is given by

$$(71.3''') \quad a^{\alpha\beta}_{,\sigma} = \frac{\partial a^{\alpha\beta}}{\partial u^\sigma} + a^{\gamma\beta} \Gamma_{\gamma\sigma}{}^\alpha + a^{\alpha\gamma} \Gamma_{\gamma\sigma}{}^\beta.$$

The covariant derivative of a mixed tensor of second order is of the form (71.3').

72. Further properties of absolute differentiation. As can be immediately seen from (71.3), the covariant differentiation of sums and products is governed by the same formal rules as those that govern the usual differentiation:

A sum of tensors (which, by the definition of addition of tensors, must be of the same type) may be differentiated term by term, for instance,

$$(72.1) \quad (a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s} + b_{\kappa_1 \dots \kappa_r}{}^{\lambda_1 \dots \lambda_s})_{,\sigma} = a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s, \sigma} + b_{\kappa_1 \dots \kappa_r}{}^{\lambda_1 \dots \lambda_s, \sigma}.$$

The covariant differentiation of a product of arbitrary tensors is governed by the elementary product rule, for example,

$$(c_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_r} d_{\pi_1 \dots \pi_r}{}^{\rho_1 \dots \rho_s})_{,\sigma} = c_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_r, \sigma} d_{\pi_1 \dots \pi_r}{}^{\rho_1 \dots \rho_s} + c_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_s} d_{\pi_1 \dots \pi_r}{}^{\rho_1 \dots \rho_s, \sigma}.$$

For example, in the case of a product of the tensors $a_{\alpha\beta}$ and $b^{\rho\sigma}$ we obtain the covariant derivative in the form

$$(a_{\alpha\beta} b^{\rho\sigma})_{,\kappa} = \frac{\partial}{\partial u^\kappa} (a_{\alpha\beta} b^{\rho\sigma}) - b^{\rho\sigma} (a_{\gamma\beta} \Gamma_{\alpha\kappa}{}^\gamma + a_{\alpha\gamma} \Gamma_{\beta\kappa}{}^\gamma) + a_{\alpha\beta} (b^{\gamma\sigma} \Gamma_{\gamma\kappa}{}^\rho + b^{\rho\gamma} \Gamma_{\gamma\kappa}{}^\sigma),$$

hence

$$(72.2) \quad (a_{\alpha\beta} b^{\rho\sigma})_{,\kappa} = a_{\alpha\beta,\kappa} b^{\rho\sigma} + a_{\alpha\beta} b^{\rho\sigma,\kappa}.$$

In (72.2) if we set $\beta = \rho$ we obtain the covariant derivative of the inner product of the given tensors in the form

$$(a_{\alpha\rho} b^{\rho\sigma})_{,\kappa} = a_{\alpha\rho,\kappa} b^{\rho\sigma} + a_{\alpha\rho} b^{\rho\sigma,\kappa}.$$

The tensors $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\delta_\alpha{}^\beta$ behave with respect to covariant differentiation in the same manner as constants with respect to the usual differentiation. Indeed, we have the following

Theorem 72.1 (Ricci). *The covariant derivatives of the tensors $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\delta_\alpha{}^\beta$ are zero tensors.*

Proof. In consequence of (71.3'') we obtain

$$g_{\alpha\beta,\kappa} = \frac{\partial g_{\alpha\beta}}{\partial u^\kappa} - g_{\gamma\beta} \Gamma_{\alpha\kappa}{}^\gamma - g_{\alpha\gamma} \Gamma_{\beta\kappa}{}^\gamma = \frac{\partial g_{\alpha\beta}}{\partial u^\kappa} - (\Gamma_{\alpha\kappa\beta} + \Gamma_{\beta\kappa\alpha}),$$

hence, by (47.4), $g_{\alpha\beta,\kappa} = 0$. Furthermore, by (71.3'''),

$$g^{\alpha\beta,\kappa} = \frac{\partial g^{\alpha\beta}}{\partial u^\kappa} + g^{\gamma\beta} \Gamma_{\gamma\kappa}{}^\alpha + g^{\alpha\gamma} \Gamma_{\gamma\kappa}{}^\beta;$$

in consequence of (47.5) these expressions vanish. Finally we obtain from

$$(71.3') \quad \delta_\alpha{}^\beta,\kappa = \frac{\partial \delta_\alpha{}^\beta}{\partial u^\kappa} + \delta_\alpha{}^\gamma \Gamma_{\gamma\kappa}{}^\beta - \delta_\gamma{}^\beta \Gamma_{\alpha\kappa}{}^\gamma = \Gamma_{\alpha\kappa}{}^\beta - \Gamma_{\alpha\kappa}{}^\beta = 0.$$

From (71.4) we obtain the second covariant derivative of a scalar $\phi(u^1, u^2)$ of class $r \geq 2$ in the form

$$\phi_{,\sigma\lambda} = \frac{\partial^2 \phi}{\partial u^\sigma \partial u^\lambda} - \frac{\partial \phi}{\partial u^\tau} \Gamma_{\sigma\lambda}{}^\tau.$$

Consequently we may write the formulae of Gauss (45.14) in the form

$$(72.3) \quad x_{i,\alpha\beta} = b_{\alpha\beta} n_i, \quad (i = 1, 2, 3),$$

and the formulae (46.3) of Mainardi–Codazzi in the form

$$(72.4) \quad b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}.$$

73. Interchange of the order of absolute differentiation. The Ricci identity. The value of the second covariant derivative depends on the order of differentiation. Of course the order becomes immaterial if the Christoffel symbols vanish identically and covariant differentiation is then

reduced to the usual partial differentiation as is the case when a surface admits Cartesian coordinates.

The order of covariant differentiation is also immaterial in the case of the second derivative of a scalar. We first prove the following

Theorem 73.1. *The first covariant derivative of a covariant vector is a symmetric covariant tensor of second order if and only if that vector is the gradient of a scalar of class $r \geq 2$.*

Proof. The covariant derivative $a_{\beta,\sigma}$ of a covariant vector a_β is symmetric if and only if

$$(73.1) \quad a_{\beta,\sigma} - a_{\sigma,\beta} = \frac{\partial a_\beta}{\partial u^\sigma} - \frac{\partial a_\sigma}{\partial u^\beta}$$

vanishes. Here the form of the right-hand side of (73.1) is a consequence of the symmetry of the Christoffel symbols, cf. (47.3); the quantity on the right-hand side is the rotation of the vector a_β ; it vanishes if and only if a_β is the gradient of a scalar. Denoting this scalar by ϕ we have

$$a_\beta = \phi_\beta,$$

and (73.1) becomes equivalent to

$$(73.2) \quad \phi_{,\beta\sigma} - \phi_{,\sigma\beta} = \frac{\partial}{\partial u^\sigma} \left(\frac{\partial \phi}{\partial u^\beta} \right) - \frac{\partial}{\partial u^\beta} \left(\frac{\partial \phi}{\partial u^\sigma} \right) = 0.$$

We will now study the general case of the second covariant derivative of a tensor of arbitrary order; in this investigation the curvature tensor will figure.

Differentiating covariantly with respect to u^μ the covariant derivative (cf. (70.7'))

$$(73.3) \quad a_{\beta,\sigma} = \frac{\partial a_\beta}{\partial u^\sigma} - a_\eta \Gamma_{\beta\sigma}^\eta$$

of an arbitrary covariant vector a_β (of class $r \geq 2$) we obtain

$$a_{\beta,\sigma\mu} = \frac{\partial}{\partial u^\mu} a_{\beta,\sigma} - a_{\nu,\sigma} \Gamma_{\beta\mu}^\nu - a_{\beta,\nu} \Gamma_{\sigma\mu}^\nu,$$

that is,

$$(73.4) \quad a_{\beta,\sigma\mu} = \frac{\partial^2 a_\beta}{\partial u^\mu \partial u^\sigma} - \frac{\partial a_\eta}{\partial u^\mu} \Gamma_{\beta\sigma}^\eta - a_\eta \frac{\partial}{\partial u^\mu} \Gamma_{\beta\sigma}^\eta - \left(\frac{\partial a_\nu}{\partial u^\sigma} - a_\eta \Gamma_{\nu\sigma}^\eta \right) \Gamma_{\beta\mu}^\nu - \left(\frac{\partial a_\beta}{\partial u^\nu} - a_\eta \Gamma_{\beta\nu}^\eta \right) \Gamma_{\sigma\mu}^\nu.$$

$a_{\beta,\mu\sigma}$ can be determined analogously. When the expression $a_{\beta,\sigma\mu} - a_{\beta,\mu\sigma}$ is formed a great number of terms cancel out in pairs. The result is

$$a_{\beta,\sigma\mu} - a_{\beta,\mu\sigma} = a_\eta \left[\frac{\partial}{\partial u^\sigma} \Gamma_{\beta\mu}^\eta - \frac{\partial}{\partial u^\mu} \Gamma_{\beta\sigma}^\eta + \Gamma_{\nu\sigma}^\eta \Gamma_{\beta\mu}^\nu - \Gamma_{\nu\mu}^\eta \Gamma_{\beta\sigma}^\nu \right],$$

or, in consequence of (46.5),

$$(73.5) \quad a_{\beta,\sigma\mu} - a_{\beta,\mu\sigma} = a_\eta R^\eta_{\beta\sigma\mu}.$$

In the case of a covariant tensor of second order we obtain similarly

$$a_{\alpha\beta,\sigma\mu} - a_{\alpha\beta,\mu\sigma} = a_{\eta\beta} R^\eta_{\alpha\sigma\mu} + a_{\alpha\eta} R^\eta_{\beta\sigma\mu}.$$

In the case of an arbitrary covariant tensor $a_{\alpha_1 \dots \alpha_r}$ we therefore find

$$(73.6) \quad a_{\alpha_1 \dots \alpha_r, \sigma\mu} - a_{\alpha_1 \dots \alpha_r, \mu\sigma} = \sum_{p=1}^r a_{\alpha_1 \dots \alpha_{p-1} \eta \alpha_{p+1} \dots \alpha_r} R^\eta_{\alpha_p \sigma\mu}.$$

From this result we obtain the corresponding identities for contravariant tensors by taking suitable inner products. Since $g^{\alpha\beta}_{,\sigma} = 0$, cf. Theorem 72.1, in the case of a contravariant vector a^α we find the relation

$$a^\alpha_{,\sigma\mu} = (g^{\alpha\beta} a_\beta)_{,\sigma\mu} = (g^{\alpha\beta} a_{\beta,\sigma})_{,\mu} = g^{\alpha\beta} a_{\beta,\sigma\mu}.$$

Applying (73.5) to this expression we obtain

$$a^\alpha_{,\sigma\mu} - a^\alpha_{,\mu\sigma} = g^{\alpha\beta} (a_{\beta,\sigma\mu} - a_{\beta,\mu\sigma}) = g^{\alpha\beta} a_\eta R^\eta_{\beta\sigma\mu}.$$

Since $R^\eta_{\beta\sigma\mu} = g^{\eta\tau} R_{\tau\beta\sigma\mu}$ and $R_{\tau\beta\sigma\mu} = -R_{\beta\tau\sigma\mu}$,

cf. (46.6') and (46.10), we finally have the identity

$$a^\alpha_{,\sigma\mu} - a^\alpha_{,\mu\sigma} = g^{\alpha\beta} a_\eta g^{\eta\tau} R_{\tau\beta\sigma\mu} = g^{\alpha\beta} a^\tau R_{\tau\beta\sigma\mu} = -g^{\alpha\beta} a^\tau R_{\beta\tau\sigma\mu};$$

hence

$$(73.7) \quad a^\alpha_{,\sigma\mu} - a^\alpha_{,\mu\sigma} = -a^\tau R^\alpha_{\tau\sigma\mu}.$$

In the case of a contravariant tensor of second order we obtain analogously

$$(73.8) \quad a^{\alpha\beta}_{,\sigma\mu} - a^{\alpha\beta}_{,\mu\sigma} = -a^{\tau\beta} R^\alpha_{\tau\sigma\mu} - a^{\alpha\tau} R^\beta_{\tau\sigma\mu}.$$

From (73.6) and (73.8), in the case of an arbitrary mixed tensor $a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$ (of class $r \geq 2$) we obtain

$$(73.9) \quad a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s, \sigma\mu} - a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s, \mu\sigma} = \sum_{p=1}^r [a_{\alpha_1 \dots \alpha_{p-1} \eta \alpha_{p+1} \dots \alpha_r}^{\beta_1 \dots \beta_s} R^\eta_{\alpha_p \sigma\mu} - a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_{p-1} \eta \beta_{p+1} \dots \beta_s} R^\eta_{\alpha_p \sigma\mu}]$$

$$- \sum_{\tau=1}^s a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_{\tau-1} \eta \beta_{\tau+1} \dots \beta_s} R^\eta_{\alpha_p \sigma\mu}.$$

The relation (73.9) is called the *Ricci identity*, cf. G. Ricci-Curbastro and T. Levi-Civita [1].

74. Bianchi identities. We shall now prove the following

Theorem 74.1. *The covariant derivatives of the curvature tensors (46.5) and (46.6) satisfy the following identities of Bianchi:*

$$(74.1 \text{ a}) \quad R^\sigma_{\alpha\eta\beta,\tau} + R^\sigma_{\alpha\tau\eta,\beta} + R^\sigma_{\alpha\beta\tau,\eta} = 0,$$

$$(74.1 \text{ b}) \quad R_{\alpha\eta\beta,\tau} + R_{\alpha\tau\eta,\beta} + R_{\alpha\beta\tau,\eta} = 0.$$

Proof. In the case of two-dimensional Riemannian spaces, that is surfaces, this theorem is a trivial consequence of (46.9). We will now prove the theorem for general values of $n \geq 3$. In Section 47 we showed that on every surface there exist coordinate systems of such a type that the Christoffel symbols vanish at an arbitrary (fixed) point P_0 . It follows from the proof that the same fact also holds in n -dimensional Riemannian spaces, $n > 2$. With respect to such a coordinate system the representation (46.5) of the mixed curvature tensor at P_0 reduces to

$$R^\sigma_{\alpha\eta\beta} = \frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\eta} - \frac{\partial \Gamma_{\alpha\eta}^\sigma}{\partial u^\beta}.$$

Furthermore, at P_0 the covariant derivative reduces to the ordinary one; indeed, since the Christoffel symbols vanish at P_0 , by differentiating (46.5) we obtain

$$R^\sigma_{\alpha\eta\beta,\tau} = \frac{\partial R^\sigma_{\alpha\eta\beta}}{\partial u^\tau} = \frac{\partial^2 \Gamma_{\alpha\beta}^\sigma}{\partial u^\eta \partial u^\tau} - \frac{\partial^2 \Gamma_{\alpha\eta}^\sigma}{\partial u^\beta \partial u^\tau}.$$

From this it follows that (74.1 a) holds at P_0 and with respect to the above special coordinate system. Since the left-hand side of (74.1 a) is a sum of components of tensors and therefore a component of a tensor, it is zero at P_0 in every coordinate system. Since P_0 was arbitrarily chosen (74.1 a) holds at any point of the space under consideration. We will now prove (74.1 b). Theorem 72.1 is valid not only for surfaces but also for Riemannian spaces of higher dimension as can be immediately seen from the proof. Hence, by (46.6'),

$$R^\sigma_{\alpha\eta\beta,\tau} = (g^{\sigma\lambda} R_{\lambda\alpha\eta\beta})_\tau = g^{\sigma\lambda} R_{\lambda\alpha\eta\beta,\tau}.$$

Using this relation we obtain (74.1 b) from (74.1 a).

75. Differential parameters of Beltrami. Let $\phi(u^1, u^2)$ and $\psi(u^1, u^2)$ be scalar functions of class $r \geq 2$ of the coordinates u^1, u^2 on a surface. The gradients

$$\phi_\alpha \equiv \frac{\partial \phi}{\partial u^\alpha}, \quad \psi_\alpha \equiv \frac{\partial \psi}{\partial u^\alpha}$$

are covariant vectors, cf. Section 71. Hence the expression

$$(75.1) \quad \nabla(\phi, \psi) = g^{\alpha\beta} \phi_{,\alpha} \psi_{,\beta} = g^{\alpha\beta} \frac{\partial \phi}{\partial u^\alpha} \frac{\partial \psi}{\partial u^\beta}$$

is an invariant with respect to allowable transformations of the coordinates. ∇ is called *differentiator of the first order* or *first differential parameter of Beltrami* [1]; it was already used by Gauss and Lamé. Instead of $\nabla(\phi, \phi)$ we write briefly $\nabla\phi$,

$$(75.2) \quad \nabla\phi = g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} = g^{\alpha\beta}\frac{\partial\phi}{\partial u^\alpha}\frac{\partial\phi}{\partial u^\beta}.$$

This expression is the square of the length of the gradient of ϕ as can be seen from (35.3). In the case of Cartesian coordinates in the plane we have $g^{11} = g^{22} = g = 1$, $g^{12} = 0$ and therefore

$$\nabla\phi = \left(\frac{\partial\phi}{\partial u^1}\right)^2 + \left(\frac{\partial\phi}{\partial u^2}\right)^2.$$

If $\phi(u^1, u^2)$ is a function of u^1, u^2 , not a constant, and if du^α/dt are the components of a contravariant vector tangent to a curve of the family $\phi(u^1, u^2) = \text{const.}$ then

$$\frac{\partial\phi}{\partial u^\alpha}\frac{du^\alpha}{dt} = 0,$$

that is, at any point of the curves $\phi(u^1, u^2) = \text{const.}$ the gradient $\partial\phi/\partial u^\alpha$ is orthogonal to the corresponding curve.

Let us now consider two families of curves $\phi(u^1, u^2) = \text{const.}$ and $\psi(u^1, u^2) = \text{const.}$ which form a net. Since the gradients are orthogonal to the corresponding curves they make the same angle γ as the curves; we have

$$\cos\gamma = \frac{\nabla(\phi, \psi)}{\sqrt{(\nabla\phi)\nabla\psi}}.$$

We thus obtain the following

Theorem 75.1. *If two families of curves*

$$\phi(u^1, u^2) = \text{const.} \quad \text{and} \quad \psi(u^1, u^2) = \text{const.}$$

form a net, this net is orthogonal if and only if at every point

$$(75.3) \quad \nabla(\phi, \psi) = 0.$$

We will now mention the following connexion between the differential parameters of Beltrami and geodesic coordinates (cf. Section 52): We consider a family of curves $\phi(u^1, u^2) = \text{const.}$ which at every point satisfy $\nabla\phi = 1$ and which, together with another family of curves

$$\psi(u^1, u^2) = \text{const.},$$

form an orthogonal net. Then, by the preceding theorem, $\nabla(\phi, \psi) = 0$. We now introduce new coordinates \tilde{u}^1, \tilde{u}^2 defined by

$$\tilde{u}^1 = \phi(u^1, u^2), \quad \tilde{u}^2 = \psi(u^1, u^2)$$

whose coordinate curves are the curves

$$\phi(u^1, u^2) = \text{const.} \quad \text{and} \quad \psi(u^1, u^2) = \text{const.}$$

In these new coordinates

$$\nabla \tilde{u}^1 = \tilde{g}^{11} = \nabla \phi = 1$$

and therefore also $\tilde{g}_{11} = 1/\tilde{g}^{11} = 1$. Hence the corresponding first fundamental form is given by the expression

$$ds^2 = (d\tilde{u}^1)^2 + \tilde{g}_{22}(d\tilde{u}^2)^2,$$

that is, the \tilde{u}^1, \tilde{u}^2 -coordinate system is geodesic, cf. (52.1).

By contraction of the covariant derivative $a^\alpha_{,\beta}$ of a contravariant vector a^α we obtain the scalar $a^\alpha_{,\alpha}$ which is called the *divergence* of the vector a^α ,

$$(75.4) \qquad \text{div } a^\alpha = a^\alpha_{,\alpha}.$$

Using (70.3'):

$$a^\alpha_{,\alpha} = \frac{\partial a^\alpha}{\partial u^\alpha} + a^\nu \Gamma_{\nu\alpha}^\alpha$$

and (47.6'):

$$\Gamma_{\nu\alpha}^\alpha = \frac{\partial}{\partial u^\nu} \log \sqrt{g} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial u^\nu}$$

$$\text{we find} \qquad a^\alpha_{,\alpha} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\nu} (\sqrt{g} a^\nu) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\nu} (\sqrt{g} g^{\mu\nu} a_\mu).$$

If, in particular, a_μ is the gradient of a scalar ϕ , that is, $a_\mu = \partial\phi/\partial u^\mu$ we obtain the invariant

$$(75.5) \qquad \Delta\phi \equiv \text{div grad } \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\nu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial\phi}{\partial u^\mu} \right).$$

Δ is called *differentiator of the second order* or *second differential parameter of Beltrami*. In the case of Cartesian coordinates in the plane we have $g^{11} = g^{22} = g = 1$ and $g^{12} = 0$, and therefore

$$(75.5') \qquad \Delta\phi = \frac{\partial^2\phi}{(\partial u^1)^2} + \frac{\partial^2\phi}{(\partial u^2)^2},$$

that is, the Laplacian in its simplest form. We may write (75.5) also as follows. In consequence of (75.4) we obviously have

$$\Delta\phi = \left(g^{\mu\nu} \frac{\partial\phi}{\partial u^\mu} \right)_{,\nu}.$$

Therefore, by (71.4) and Theorem 72.1,

$$(75.5'') \qquad \Delta\phi = g^{\mu\nu} \phi_{,\mu\nu}.$$

76. Definition of the displacement of Levi-Civit  . This chapter is devoted to the concepts of absolute differentiation and displacement.

In Section 69 we mentioned the importance of these concepts in modern differential geometry and their application in the theory of relativity. We then proceeded to the discussion of the problem which gave rise to absolute differentiation. In Sections 70–75 the definition and basic properties of absolute differentiation were stated.

We will now formulate the problem which gives rise to the displacement of Levi-Civita [1]. A generalization of the displacement of Levi-Civita will be considered in Section 78. Instead of *displacement* the name *parallelism* is often used, in the same sense.

The problem consists of comparing directions of vectors tangent to a surface at different points of it. That is, we want to compare the direction of two vectors in a surface (cf. Section 32) bound at different points of it, say P and Q . This is impossible, *a priori*. In order to make such a comparison we must first define a displacement, that is, a rule according to which a vector in a surface can be displaced from one point of the surface to another.

The simplest displacement is defined by the requirement that the components of every vector remain unchanged, that is, the vector displaced from P to Q is, by definition, that vector bound at Q which has the same components as the one originally given at P . The two vectors at P and Q are called *parallel* to each other *with respect to this displacement*. Obviously this simplest displacement and the concept of parallelism resulting from it correspond to the case when the surface under consideration is a plane in which parallel coordinates, for instance Cartesian coordinates, have been introduced.

The transition from one given Cartesian coordinate system to another is governed by a transformation of the form (3.1) (summation from 1 to 2 only), and vectors then behave according to (4.2). The coefficients figuring in the transformation formulae are constant, they do not depend on the points of the plane under consideration. Hence two vectors bound at two different points P and Q which are parallel with respect to the originally given coordinate system and with respect to the displacement under consideration are also parallel with respect to the transformed coordinate system. Consequently every vector can be displaced in a unique manner from any point of the plane to any other one. In the case of Euclidean coordinate systems it is therefore sensible to introduce and use the concept of a free vector, that is, a vector whose initial point can be arbitrarily chosen, cf. Section 4. The concept of parallelism resulting from this simplest type of displacement is the elementary one.

It is impossible, however, to use this simple displacement on an arbitrary surface. The reason is simply that vectors in an arbitrary surface, when an allowable coordinate transformation is imposed, undergo a transformation of the form (29.4) and (29.5), respectively. In consequence of such a transformation the components of a vector are multiplied by partial derivatives which, in general, are not constants but are functions of the coordinates on the surface, since the allowable transformations of coordinates on an arbitrary surface are not linear in general. Consequently two vectors in a surface which are bound at different points and whose corresponding components with respect to a certain coordinate system are equal will, in general, lose this property if the coordinate system is transformed.

We could consider the possibility of using the concept of parallelism of the space in which a surface is embedded, but this would not lead to a reasonable result. In fact, when displacing a vector in a surface S , given at a point P , from P to a point Q of S we would not even obtain a vector in S at Q since the tangent planes to S at P and Q are, in general, not parallel to each other. Moreover, the concepts of parallelism and angular magnitude on the surface would depend on the special embedding and consequently change under isometric mappings, which mappings do not affect the first fundamental form, that is, the intrinsic geometry of the surface. We have therefore to define a displacement valid for any surface in another manner.

The definition of displacement of a vector in an arbitrary surface and the resulting concept of parallelism will, in contrast to the above elementary displacement in a plane, have the following very essential restriction: The displacement will depend on the choice of an arc on a surface S along which a vector is displaced from a point P to a point Q on S .

We will of course require that in the case of Cartesian coordinates the new concept of displacement reduces to the elementary one.

Let $\mathbf{x}(u^1, u^2)$ be an allowable representation of an arbitrary surface S of class $r \geq 3$. We denote by a^α a given vector bound at a point P of S and we wish to displace this vector to another point Q of S . For this purpose we choose on S an arbitrary (fixed) curve C : $\mathbf{x}(s) = \mathbf{x}(u^1(s), u^2(s))$ passing through P and Q and reason as follows:

Parallel vectors in a plane make the same angle with any geodesic, that is, with any straight line. In order to obtain the desired definition of displacement on an arbitrary surface we use as a starting-point the analogous requirement, namely: If the above curve C through P and Q , in particular, is a geodesic the displacement along this special curve C should be carried

out so that at every point of C the displaced vector $a^\alpha(s)$ in the surface S makes the same angle with the unit tangent vector to C .

This means we first consider the *special case* of displacing $a^\alpha(s)$ along a geodesic G : $u^\alpha = u^\alpha(s)$ ($\alpha = 1, 2$), on a surface S : $x(u^1, u^2)$; here s is the arc length of G . Let $a^\alpha(s)$ be unit vectors of class $\tilde{r} \geq 1$ in S at points of G and not tangent to G . We have

$$(76.1) \quad g_{\alpha\nu} a^\alpha a^\nu = 1.$$

We shall say that the vectors $a^\alpha(s)$ are parallel in the sense of the displacement to be defined (and with respect to G) if the angle between $a^\alpha(s)$ and the unit tangent vector $x_\alpha \dot{u}^\alpha$ to G is the same for all values of s . This holds if and only if the scalar product $g_{\alpha\nu} a^\alpha \dot{u}^\nu$ is constant, that is

$$\frac{d}{ds} (g_{\alpha\nu} a^\alpha \dot{u}^\nu) = 0$$

or $\frac{\partial g_{\alpha\nu}}{\partial u^\sigma} \dot{u}^\sigma a^\alpha \dot{u}^\nu + g_{\alpha\nu} \dot{a}^\alpha \dot{u}^\nu + g_{\alpha\tau} a^\alpha \dot{u}^\tau = 0.$

Replacing the derivative of the metric tensor by means of (47.4) we obtain

$$(76.2) \quad ((\Gamma_{\alpha\sigma\nu} + \Gamma_{\nu\sigma\alpha}) a^\alpha \dot{u}^\sigma + g_{\alpha\nu} \dot{a}^\alpha) \dot{u}^\nu + g_{\alpha\tau} a^\alpha \dot{u}^\tau = 0.$$

Since G is a geodesic, in consequence of (51.5) we have

$$g_{\alpha\tau} a^\alpha (\dot{u}^\tau + \Gamma_{\nu\sigma}^\tau \dot{u}^\sigma \dot{u}^\nu) = 0.$$

Now

$$\Gamma_{\nu\sigma\alpha} = g_{\alpha\tau} \Gamma_{\nu\sigma}^\tau$$

and

$$a^\alpha \Gamma_{\alpha\sigma\nu} = a^\beta \Gamma_{\beta\sigma\nu} = a^\beta g_{\alpha\nu} \Gamma_{\beta\sigma}^\alpha.$$

Hence, by (70.3'), (76.2) becomes

$$(76.2') \quad g_{\alpha\nu} (\dot{a}^\alpha + a^\beta \Gamma_{\beta\sigma}^\alpha \dot{u}^\sigma) \dot{u}^\nu = 0.$$

We will prove that the vector

$$(76.3) \quad \frac{\delta a^\alpha}{\delta s} \equiv \dot{a}^\alpha + a^\beta \Gamma_{\beta\sigma}^\alpha \dot{u}^\sigma$$

is the null vector. In consequence of Theorem 72.1 we obtain from (76.1)

$$(76.4) \quad g_{\alpha\nu} \frac{\delta a^\alpha}{\delta s} a^\nu = 0.$$

Hence if the vector $\delta a^\alpha / \delta s$ is not the null vector it must be orthogonal to G , cf. (76.2'), as well as to a^α , cf. (76.4). But this is impossible since the direction of a^α is arbitrary. We therefore have

$$\frac{\delta a^\alpha}{\delta s} = 0,$$

that is,

$$(76.5) \quad \frac{da^\alpha}{ds} + a^\beta \Gamma_{\beta\sigma}^\alpha u^\sigma = 0, \quad (\alpha = 1, 2).$$

It is now natural to define also by (76.5) the *displacement of a vector a^α along any arbitrary curve C on S* . If C is given, that is, is defined analytically by a representation $u^1 = u^1(s)$, $u^2 = u^2(s)$, then the equations (76.5) admit a solution $a^1(s)$, $a^2(s)$ uniquely determined by an arbitrary vector a^α at a point of C . The vectors $a^\alpha(s)$ thus obtained are called *parallel with respect to C* (and with respect to the displacement just defined). The displacement defined in this manner is called the *displacement of Levi-Civita*.

Of course, instead of the arc length s we may introduce an arbitrary allowable parameter t ; then (76.5) takes the form

$$(76.5') \quad a^{\alpha'} + a^\beta \Gamma_{\beta\sigma}^\alpha u^{\sigma'} = 0, \quad (\alpha = 1, 2),$$

where the prime denotes the derivative with respect to t .

Since

$$a^\alpha = g^{\alpha\nu} a_\nu,$$

and $g^{*\alpha\beta} \equiv \det(g^{\alpha\beta}) \neq 0$ we obtain from (76.5') the differential equations of the displacement of a covariant vector a_ν in the form

$$(76.5'') \quad a'_\nu - a_\gamma \Gamma_{\nu\sigma}^\gamma u^{\sigma'} = 0, \quad (\nu = 1, 2).$$

In the case of Cartesian coordinates in the plane the Christoffel symbols vanish identically. In this case (76.5') reduces therefore to $a^{\alpha'} = 0$ ($\alpha = 1, 2$), that is, the components of the vector remain constant when the vector is displaced, and the displacement becomes independent of the choice of the curve C . In other words, in this particular case the displacement of Levi-Civita becomes identical with the above elementary displacement (which is as it should be).

We have already stressed that in general the displacement of Levi-Civita depends on the choice of the above curve C passing through the points P and Q of the surface S under consideration.

Theorem 76.1. *The displacement of Levi-Civita, defined by (76.5), is independent of the choice of a curve C (see above) if and only if the surface under consideration is a developable surface.*

Proof. If the solution of (76.5) is independent of the choice of the curve C then

$$(76.6) \quad a^\alpha_{,\alpha} = 0$$

must hold. The integrability condition of (76.6) is obtained from (73.7),

$$(76.7) \quad a^\gamma R_{\gamma\sigma}^\alpha = 0, \quad (\alpha = 1, 2).$$

If solutions of (76.6) corresponding to arbitrary initial conditions should exist, (76.7) must be valid for arbitrary a^α and therefore $R_{\gamma\sigma}^\alpha = 0$ must hold. Hence, in consequence of (46.6) and (46.12), we must have $K = 0$, that is, the surface is a developable one, cf. Theorem 59.2. Conversely, since for a plane the displacement is independent of the choice of a curve and since planes and developable surfaces are isometric, the same is true for every developable surface.

In the following we will give a simple geometric interpretation of the displacement of Levi-Civit , assuming and using the fact that the surface S is embedded in three-dimensional Euclidean space.

77. Further properties of the displacement of Levi-Civit . As can be seen from (76.5) the displacement of Levi-Civit  is an intrinsic property of a surface (cf. Section 57), that is, it is independent of the embedding of the surface in space. In consequence of Theorem 57.1 we thus have the following

Theorem 77.1. *If vectors are parallel with respect to a curve C on a surface S they are also parallel with respect to the image of C under an isometric mapping of S .*

The displacement is independent of the special choice of the coordinates on a surface since the left-hand side of (76.5) is a tensor.

When setting $a^\alpha = du^\alpha/dt$ in (76.5) we see that the tangents to a geodesic G are parallel with respect to G . Obviously this is a generalization of the property of constancy of direction of a straight line in a plane.

Another important property of the displacement of Levi-Civit  is the following constancy of the scalar product:

Theorem 77.2. *If two vector fields $a^\mu(t)$ and $b^\mu(t)$ are generated by displacing each of two given vectors $a^\mu(t_0)$ and $b^\mu(t_0)$ along one and the same curve C : $u^\alpha = u^\alpha(t)$, ($\alpha = 1, 2$), on a surface S : $x(u^1, u^2)$ then at every point of C the scalar product $g_{\mu\nu}(t)a^\mu(t)b^\nu(t)$ has the same value.*

Proof. We have to prove that the derivative of the above scalar product with respect to t , namely

$$\frac{\partial g_{\alpha\nu}}{\partial u^\beta} u^\beta a^\alpha b^\nu + g_{\mu\nu} (a^\mu b^\nu + a^\nu b^\mu),$$

vanishes. In consequence of (76.5') we have

$$\begin{aligned} g_{\mu\nu}(a^\mu b^\nu + a^\nu b^\mu) &= -g_{\mu\nu}(a^\alpha \Gamma_{\alpha\beta}^\mu b^\nu + a^\mu b^\alpha \Gamma_{\alpha\beta}^\nu) u^\beta \\ &= -(a^\alpha \Gamma_{\alpha\beta}^\mu b^\nu + a^\mu b^\alpha \Gamma_{\alpha\beta}^\nu) u^\beta = -a^\alpha b^\nu (\Gamma_{\alpha\beta\nu} + \Gamma_{\nu\beta\alpha}) u^\beta. \end{aligned}$$

From this, in consequence of (47.4), the statement of the theorem follows.

Setting in particular $b^\mu(t) = a^\mu(t)$ we have as an immediate consequence of Theorem 77.2 the following

Theorem 77.3. *The length of a vector a^α in a surface remains unchanged under a displacement of Levi-Civit  . The angle between two vectors in a surface bound at the same point remains the same if those vectors undergo a displacement along the same curve.*

Consequently if two vectors which are bound at the same point P and have different directions at P undergo a displacement along the same curve C they can never, at any point of C , simultaneously assume the same direction; at any point of C we obtain a one-to-one correspondence between any set of vectors displaced along C and the originally given ones.

In the case that a surface is embedded in Euclidean space R_3 we can obtain a simple geometric interpretation of the displacement of Levi-Civit  . For this purpose we will use

Theorem 77.4. *Let C be a curve on a surface in Euclidean space R_3 . Suppose that vectors along C are given and are differentiable functions of the arc length s of C . Then these vectors are parallel with respect to C in the sense of the displacement of Levi-Civit   if and only if the derivative of these vectors (considered as vectors in R_3) with respect to s is the null vector or is normal to the surface.*

Proof. Every vector a^μ in a surface $\mathbf{x}(u^1, u^2)$, considered as a vector in the embedding space, can be represented in the form

$$\mathbf{v} = (v_1, v_2, v_3) = a^\mu \mathbf{x}_\mu.$$

If we differentiate this expression with respect to the arc length s of the curve C we obtain

$$\dot{\mathbf{v}} = \dot{a}^\mu \mathbf{x}_\mu + a^\gamma \mathbf{x}_{\gamma\eta} \dot{u}^\eta$$

or, in consequence of (45.14),

$$(77.1) \quad \dot{\mathbf{v}} = (\dot{a}^\mu + a^\gamma \Gamma_{\gamma\eta}^\mu \dot{u}^\eta) \mathbf{x}_\mu + a^\gamma b_{\gamma\eta} \dot{u}^\eta \mathbf{n}.$$

If $\dot{\mathbf{v}}$ is the null vector or has the direction of the normal to the surface then the expression in brackets (...) must vanish, that is, (76.5) is satisfied, and vice versa.

Let S_1 and S_2 be two surfaces which touch each other along a curve C , that is, by definition, they have the curve C in common and along C

common tangent planes and therefore also common normals. Using Theorem 77.4 we thus obtain

Theorem 77.5. *If two surfaces S_1 and S_2 touch each other along a curve C then vectors in S_1 which are parallel with respect to C are also parallel with respect to C when considered as vectors in S_2 .*

These theorems describe the displacement of Levi-Civita from the standpoint of the embedding space. We thus may carry out the displacement of a vector in a surface S , embedded in Euclidean space, as follows. Let C be any (fixed) curve on S along which a given vector should be displaced. We choose a developable surface D touching S along C , cf. Section 60. According to Theorem 59.3 we can map D isometrically into a plane. We displace the given vector along the image C^* of C in the elementary sense and map onto S the vectors thus obtained. In consequence of Theorem 77.5 these vectors are parallel with respect to C and in the sense of the displacement of Levi-Civita.

Let us consider the following

Example. Let C be a circle of radius

$$r_c = r \sin \phi$$

on a sphere S of radius r , cf. Fig. 82, p. 239. Let a vector at a point P of C be given; for the sake of simplicity we suppose that vector to be tangent to C . We displace this vector from P along C back to P . The developable surface D touching S along C is a cone (or a cylinder if C is a great circle on S), cf. Fig. 82. The apex of D is at the distance

$$m = r_c \sec \phi = r \tan \phi$$

from C . We cut the cone D along the generator through P and bend it in a plane, cf. Fig. 83. Since C and its plane image have the length

$$l = 2\pi r_c = 2\pi r \sin \phi$$

the angle β at the apex of the plane image of D (cf. Fig. 83) has the magnitude

$$\beta = \frac{2\pi r \sin \phi}{r \tan \phi} = 2\pi \cos \phi.$$

The displaced vector and the tangent to C at P make therefore the angle

$$\delta = 2\pi(1 - \cos \phi).$$

This angle depends on ϕ , that is, on the choice of C . In particular $\delta = 0$ if and only if C is a great circle on S , and in this case the displaced vector coincides with the originally given one. Indeed this must be so since great

circles are geodesics on the sphere and the tangents to a geodesic G are parallel with respect to G (in the sense of the displacement of Levi-Civit  ).

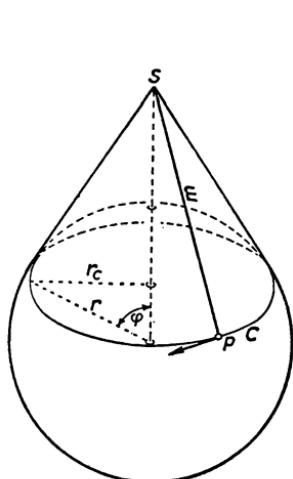


FIG. 82

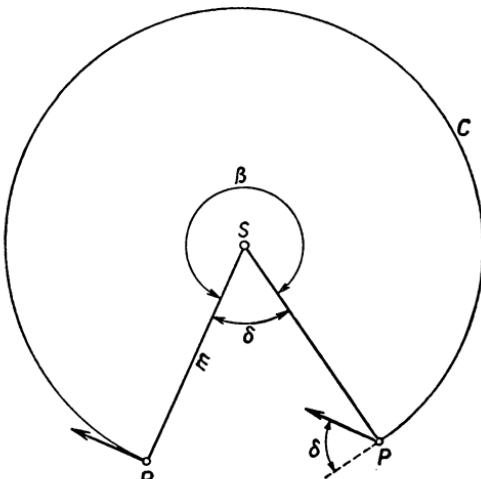


FIG. 83

78. A more general definition of absolute differentiation and displacement. The absolute differentiation and the corresponding displacement of Levi-Civit   can be generalized. In the following we shall study the basic ideas of a generalization which will be obtained by replacing the Christoffel symbols occurring in the covariant derivative and in the differential equation (76.5) by other suitable functions. By this procedure the absolute differentiation and displacement will become independent of the metric on the surface which is defined by the tensor $g_{\alpha\beta}$. It is of interest to note that the generalized theory was developed not only for purely mathematical reasons but also for its usefulness in solving certain problems of modern physics, as can be seen from the basic papers of Eddington, Einstein, Eisenhart, Schouten, Veblen, H. Weyl, and others.

In Section 70 when establishing the definition of absolute differentiation the fundamental idea consisted of the elimination of second derivatives by means of Christoffel symbols of the second kind. That elimination process could be carried out in consequence of the transformation behaviour (47.12) of the Christoffel symbols. No other property of those symbols was used. In particular, the fact that those symbols are functions of the metric tensor $g_{\alpha\beta}$ was completely unessential in that investigation.

We may therefore replace the Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ by other functions $G_{\alpha\beta}^\gamma$ whose transformation behaviour is analogous to (47.12), namely

$$(78.1) \quad G_{\alpha\beta}^\gamma \frac{\partial \bar{u}^\sigma}{\partial u^\gamma} = \bar{G}_{\eta\epsilon}^\sigma \frac{\partial \bar{u}^\eta}{\partial u^\alpha} \frac{\partial \bar{u}^\epsilon}{\partial u^\beta} + \frac{\partial^2 \bar{u}^\sigma}{\partial u^\alpha \partial u^\beta}$$

and then define a more general absolute differentiation which is independent of the metric on the surface.

Instead of (70.3) and (70.3') we have then to consider the quantities

$$(78.2) \quad \bar{a}^\alpha_{;\rho} = \frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} + \bar{a}^\eta \bar{G}_{\eta\rho}^\alpha$$

and

$$(78.2') \quad a^\beta_{;\sigma} = \frac{\partial a^\beta}{\partial u^\sigma} + a^\kappa G_{\kappa\sigma}^\beta,$$

respectively. These quantities satisfy the relations

$$(78.3) \quad \bar{a}^\alpha_{;\rho} = a^\beta_{;\sigma} \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \frac{\partial u^\sigma}{\partial \bar{u}^\rho}, \quad a^\beta_{;\sigma} = \bar{a}^\alpha_{;\rho} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\rho}{\partial u^\sigma}$$

which are analogous to (70.4). Thus these quantities are the components of a mixed tensor of second order. Conversely, if (78.3) holds for any contravariant vector a^β it follows from (69.2) that (78.1) must be satisfied. If we take any set of functions $\bar{G}_{\eta\epsilon}^\sigma$ of the coordinates \bar{u}^1, \bar{u}^2 on a surface equations (78.1) determine the corresponding functions in any other allowable coordinates u^1, u^2 on that surface so that (78.2) and (78.2') define the components of the same tensor in the two coordinate systems under consideration. This tensor is called the *covariant derivative of the vector a^β with respect to the functions $G_{\alpha\beta}^\gamma$* . We denote this generalized covariant derivative by a subscript preceded by a semicolon, in contrast to the covariant derivative with respect to $g_{\alpha\beta}$ which is denoted by a subscript preceded by a comma.

The displacement of Levi-Civita, defined by (76.5)

$$\dot{a}^\beta + a^\gamma \Gamma_{\gamma\sigma}^\beta \dot{u}^\sigma = 0, \quad (\beta = 1, 2)$$

corresponds to covariant differentiation with respect to $g_{\alpha\beta}$. Analogously the covariant differentiation with respect to $G_{\alpha\beta}^\gamma$ leads to a *displacement with respect to $G_{\alpha\beta}^\gamma$* , defined by

$$(78.4) \quad \dot{a}^\beta + a^\gamma G_{\gamma\sigma}^\beta \dot{u}^\sigma = 0, \quad (\beta = 1, 2).$$

If we choose on a surface S an arbitrary (fixed) curve C : $u^\alpha = u^\alpha(t)$ ($\alpha = 1, 2$) then the solutions $a^\beta(t)$ of (78.4) corresponding to certain initial conditions, that is, to a vector in S given at a point of C , form a vector field

along C ; the vectors $a^\beta(t)$ are said to be *parallel with respect to C and with respect to the displacement defined by $G_{\alpha\beta}^\gamma$* .

The displacement with respect to $G_{\alpha\beta}^\gamma$ is called *symmetric* if the functions $G_{\alpha\beta}^\gamma$ are symmetric in α and β , i.e. $G_{\alpha\beta}^\gamma = G_{\beta\alpha}^\gamma$; otherwise it is called *asymmetric*. Since we now are able to consider also asymmetric displacements new ideas enter into the whole theory.

The laws of covariant differentiation of sums and products stated in Section 72 are valid also for covariant differentiation with respect to $G_{\alpha\beta}^\gamma$. Of course Theorem 72.1 is no longer valid since the metric tensor $g_{\alpha\beta}$ does not play a special role in the differentiation with respect to $G_{\alpha\beta}^\gamma$.

In Section 73, the curvature tensor $R^\eta{}_{\beta\sigma\mu}$ must be replaced by another tensor of fourth order, contravariant of first order, covariant of third order,

$$S^\eta{}_{\beta\sigma\mu} = \frac{\partial}{\partial u^\sigma} G_{\beta\mu}^\eta - \frac{\partial}{\partial u^\mu} G_{\beta\sigma}^\eta + G_{\nu\sigma}^\eta G_{\beta\mu}^\nu - G_{\nu\mu}^\eta G_{\beta\sigma}^\nu,$$

called the *curvature tensor* corresponding to the differentiation with respect to $G_{\alpha\beta}^\gamma$.

Setting

$$(78.5) \quad V_{\alpha\beta}^\gamma = \frac{1}{2}(G_{\alpha\beta}^\gamma + G_{\beta\alpha}^\gamma)$$

and

$$(78.6) \quad W_{\alpha\beta}^\gamma = \frac{1}{2}(G_{\alpha\beta}^\gamma - G_{\beta\alpha}^\gamma)$$

we obtain a decomposition of the functions $G_{\alpha\beta}^\gamma$ into a symmetric and a skew-symmetric part,

$$(78.7) \quad G_{\alpha\beta}^\gamma = V_{\alpha\beta}^\gamma + W_{\alpha\beta}^\gamma.$$

From (78.1) we immediately find

$$(78.8) \quad G_{\alpha\beta}^\gamma - G_{\beta\alpha}^\gamma = (\tilde{G}_{\eta\epsilon}^\sigma - \tilde{G}_{\epsilon\eta}^\sigma) \frac{\partial \bar{u}^\eta}{\partial u^\alpha} \frac{\partial \bar{u}^\epsilon}{\partial u^\beta} \frac{\partial u^\gamma}{\partial \bar{u}^\sigma},$$

that is, the skew-symmetric parts $W_{\alpha\beta}^\gamma$ of the functions $G_{\alpha\beta}^\gamma$ are the components of a tensor. From (78.5) and (78.1) we see that the symmetric parts $V_{\alpha\beta}^\gamma$ have a transformation law which is obtained from (78.1) by replacing $G_{\alpha\beta}^\gamma$ and $\tilde{G}_{\eta\epsilon}^\sigma$ by $V_{\alpha\beta}^\gamma$ and $\bar{V}_{\eta\epsilon}^\sigma$ respectively. This fact enables us to define a covariant differentiation with respect to $V_{\alpha\beta}^\gamma$.

The tangents to a geodesic G on a surface S are parallel with respect to G in the sense of the displacement of Levi-Civit . Under displacement with respect to $G_{\alpha\beta}^\gamma$ other curves will have this property, depending on the choice of the functions $G_{\alpha\beta}^\gamma$ or, more exactly, on their symmetric parts $V_{\alpha\beta}^\gamma$ only.

So far we have investigated the basic idea of a generalization of covariant differentiation and displacement. We finally mention some further properties; the proofs and further details can be found, for instance, in the books of L. P. Eisenhart [1], and H. Weyl [1].

The covariant differentiation and displacement with respect to $G_{\alpha\beta}^\gamma$ have some specific properties which result from the fact that the functions $G_{\alpha\beta}^\gamma$ are not necessarily symmetric with respect to α and β . For example, vectors which are parallel with respect to a displacement defined by $G_{\alpha\beta}^\gamma$ are also parallel with respect to a displacement defined by the functions

$$(78.9) \quad G_{\alpha\beta}^{*\gamma} = G_{\alpha\beta}^\gamma + 2\delta_\alpha^\gamma a_\beta,$$

where a_β is an arbitrary covariant vector. By (78.9) we see that at most one of the two sets of functions $G_{\alpha\beta}^\gamma$ and $G_{\alpha\beta}^{*\gamma}$ can be symmetric. The concepts of parallelism corresponding to two different symmetric displacements are always different from each other. It can be proved that two displacements, one of which is symmetric, define the same parallelism with respect to any curve if and only if the skew-symmetric parts of the functions which define the other displacement are of the form

$$(78.10) \quad W_{\alpha\beta}^\gamma = \delta_\alpha^\gamma b_\beta - \delta_\beta^\gamma b_\alpha,$$

where b_β is an arbitrary covariant vector, and the symmetric parts are the same.

VIII

SPECIAL SURFACES

79. Definition and simple properties of minimal surfaces. A surface S of class $r \geq 2$ whose mean curvature

$$H = \frac{1}{2}b_{\alpha}^{\alpha} = \frac{1}{2}b_{\alpha\beta}g^{\alpha\beta},$$

cf. (41.10), is zero at every point of S is called a *minimal surface*.

This name will be justified in the next section. We first state a remarkable property of the spherical image of a minimal surface.

Theorem 79.1. *A surface of class $r \geq 3$ is a minimal surface or a sphere if and only if its spherical mapping (cf. Section 59) is conformal.*

Proof. We consider an arbitrary minimal surface S . Since $H = 0$ we obtain from (59.7)

$$(79.1) \quad c_{\alpha\beta} = -Kg_{\alpha\beta},$$

where $c_{\alpha\beta}$ are the coefficients of the third fundamental form. This form plays the role of the first fundamental form of the spherical image of S . We see from (79.1) that the coefficients $c_{\alpha\beta}$ and $g_{\alpha\beta}$ are proportional. Therefore, in consequence of Theorem 61.1, the spherical mapping of S is conformal. That the spherical mapping of a sphere is also conformal is a trivial fact. We now consider any surface whose spherical mapping is conformal. Then, by Theorem 61.1 we must have

$$c_{\alpha\beta} = \eta(u^1, u^2)g_{\alpha\beta}.$$

Hence (59.6) takes the form

$$(79.2) \quad (K + \eta)g_{\alpha\beta} du^{\alpha}du^{\beta} = 2Hb_{\alpha\beta} du^{\alpha}du^{\beta}.$$

We may assume that the second fundamental form does not vanish identically, otherwise the surface would be a plane and the spherical image would degenerate into a point. For $K = -\eta$ we obtain $H = 0$, the surface is a minimal surface. Assuming $H \neq 0$ we obtain another solution of (79.2), namely

$$\frac{K + \eta}{2H} = \frac{b_{\alpha\beta} du^{\alpha}du^{\beta}}{g_{\alpha\beta} du^{\alpha}du^{\beta}} = \kappa_n.$$

Hence at every point of the surface the normal curvature is the same for all directions, that is, the surface consists of umbilics only and is therefore a sphere, cf. Theorem 48.1. There are no solutions of (79.2) other than those indicated before.

We stress the following very simple property:

Theorem 79.2. *Minimal surfaces cannot have elliptical points.*

Proof. Since $H = 0$ we obtain from (41.10) the relation $\kappa_1 = -\kappa_2$ and, in consequence of (41.9), $K = \kappa_1 \kappa_2 \leq 0$.

Since $K = b/g$ and $g > 0$, at any point of a minimal surface also $b \leq 0$.

80. Surfaces of smallest area. We will prove that if there is a portion S of a surface of minimum area among all portions of surfaces bounded by the same (fixed) closed space curve then S must necessarily be a portion of a minimal surface.

By that fact the name ‘minimal surface’ will be justified to a certain extent. Minimal surfaces are sometimes defined as surfaces with the property that any portion of them bounded by a closed curve C has the smallest area, compared with any portion of a surface bounded by C . But this definition by means of a minimum condition of the calculus of variation is not always satisfactory. We therefore preferred the definition by the property of identically vanishing mean curvature H . The situation is analogous to that in the case of geodesics: An arc of shortest length joining two fixed points on a surface must necessarily be an arc of a geodesic, but this condition is not always sufficient, cf. Section 51.

Theorem 80.1. *Let C be a simple closed curve. If among all portions of surfaces of class $r \geq 2$ bounded by C there exists a portion $\mathbf{x}(u^1, u^2)$ of minimum area then $\mathbf{x}(u^1, u^2)$ is necessarily a portion of a minimal surface.*

Proof. We compare the portion $\mathbf{x}(u^1, u^2)$ with portions

$$\mathbf{x}^*(u^1, u^2, \epsilon) = \mathbf{x}(u^1, u^2) + \epsilon \lambda(u^1, u^2) \mathbf{n}$$

in a neighbourhood of $\mathbf{x}(u^1, u^2)$. In this representation \mathbf{n} is the unit normal vector to $\mathbf{x}(u^1, u^2)$ and ϵ is a number of sufficiently small absolute value. In order that all these portions be bounded by the same simple closed curve C we must require that $\lambda(u^1, u^2)$ vanishes at all points of C . We have

$$\mathbf{x}_\alpha^* = \mathbf{x}_\alpha + \epsilon (\lambda_\alpha \mathbf{n} + \lambda \mathbf{n}_\alpha),$$

where $\lambda_\alpha = \partial \lambda / \partial u^\alpha$. Hence, by (27.1) and (38.4),

$$g_{\alpha\beta}^* \equiv \mathbf{x}_\alpha^* \cdot \mathbf{x}_\beta^* = g_{\alpha\beta} - 2\epsilon \lambda b_{\alpha\beta} + \dots$$

where the dots denote terms of order ϵ^2 and higher order. From this we obtain

$$g^* \equiv g_{11}^* g_{22}^* - (g_{12}^*)^2 = g - 2\epsilon \lambda (g_{11} b_{22} - 2g_{12} b_{12} + g_{22} b_{11}) + \dots$$

or, because of (41.10),

$$g^* = g(1 - 4\epsilon\lambda H + \dots).$$

Consequently

$$\sqrt{g^*} = \sqrt{g}(1 - 2\epsilon\lambda H + \dots).$$

According to (36.7) the area $A(\epsilon)$ of $\mathbf{x}^*(u^1, u^2, \epsilon)$ is given by the expression

$$(80.1) \quad A(\epsilon) = \iint_B \sqrt{g^*} du^1 du^2 = \iint_B \sqrt{g} du^1 du^2 - 2\epsilon \iint_B \lambda H \sqrt{g} du^1 du^2 + \dots$$

where B denotes the domain in the $u^1 u^2$ -plane corresponding to the range of values of the vector function $\mathbf{x}(u^1, u^2)$ in the interior of C . If

$$\mathbf{x}(u^1, u^2) \equiv \mathbf{x}^*(u^1, u^2, 0)$$

among all portions $\mathbf{x}^*(u^1, u^2, \epsilon)$ of surfaces is to be of minimum area we must necessarily have

$$\left. \frac{dA(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0.$$

Consequently the second integral on the right-hand side of (80.1) must vanish for any portion $\mathbf{x}^*(u^1, u^2, \epsilon)$, that is, for any choice of the function $\lambda(u^1, u^2)$. We therefore must have $H = 0$. This completes the proof.

In 1866 Plateau illustrated by his experiments that portions of minimal surfaces can be realized as a soap film by dipping a wire in the form of a closed curve into a soap solution. The problem of finding a minimal surface is therefore also known as the ‘problem of Plateau’, cf. T. Rado [3].

Theorem 80.2. *If a portion S of a minimal surface of class $r \geq 3$ is bounded by a simple closed curve C of length L and if the area $A(S)$ of S is a minimum then*

$$A(S) \leq \frac{L^2}{4\pi}.$$

Proof (W. Blaschke). We consider a cone with apex at an arbitrary fixed point on C whose generators pass through C . Let S^* be the portion of that cone bounded by C . S^* is developable and can therefore be mapped isometrically into a plane, cf. Theorem 59.3. Isometric mappings are equiareal, cf. Theorem 66.2. As is well known, among all portions of a plane bounded by a simple closed curve of length L the circular disk has the maximum area, given by $A = \pi r^2 = L^2/4\pi$. Since S is assumed to have the minimum area $A(S)$ we must have $A(S) \leq A(S^*) \leq A$. From this the above inequality follows.

We finally mention some facts of the history of minimal surfaces. The problem of determining portions of surfaces of minimum area with given

boundary was first investigated by Lagrange [1]. Meusnier discovered the fact that for a portion of a surface which solves that problem, we must always have $H = 0$. He discovered also the first special minimal surfaces, namely the catenoid and the helicoid, cf. Meusnier [1]. Further progress was obtained by Enneper [1] and Weierstrass [1]. The results of Weierstrass are important especially because of the fact that on the one hand they permit the investigations of problems on minimal surfaces in a general manner and on the other hand they make it possible to discover new types of minimal surfaces. We will study the so-called formulae of Weierstrass in Section 82. Further progress in the theory of minimal surfaces was made by Riemann, Schwarz, Lie, and Ribaucour. T. Radó [1], [2] and J. Douglas [1] obtained the following general solution of the problem of Plateau: Given a simple closed Jordan curve C there exists at least one portion of a minimal surface bounded by C .

81. Examples of minimal surfaces. Surfaces of revolution and right conoids were introduced in Section 28. We will now see that each of those two classes of surfaces includes exactly one type of minimal surface, namely the catenoid and the right helicoid. These two types of surface which look entirely different in the embedding space are even isometric; we will see that a catenoid when cut along a meridian can be isometrically mapped into a suitably chosen right helicoid.

The principal normal surface of a circular helix (cf. Section 8) is called a *right helicoid*; this surface can be represented in the form

$$(81.1) \quad \mathbf{x}(u^{1*}, u^{2*}) = (u^{2*} \cos u^{1*}, u^{2*} \sin u^{1*}, cu^{1*})$$

and is therefore a special right conoid, cf. the solution of Problem 28.2.

Theorem 81.1. *Right helicoids are the only right conoids which are minimal surfaces.*

Proof. From the general local representation of the right conoids,

$$\mathbf{x}(u^{1*}, u^{2*}) = (u^{2*} \cos u^{1*}, u^{2*} \sin u^{1*}, h(u^{1*})),$$

we obtain

$$g_{11}^* = (u^{2*})^2 + h'^2, \quad g_{12}^* = 0, \quad g_{22}^* = 1, \quad g^* = (u^{2*})^2 + h'^2,$$

where a prime denotes the derivative with respect to u^{1*} . From (34.1) we find the following representation of the corresponding unit normal vector

$$\mathbf{n} = \left(-\frac{h' \sin u^{1*}}{\sqrt{g^*}}, \frac{h' \cos u^{1*}}{\sqrt{g^*}}, -\frac{u^{2*}}{\sqrt{g^*}} \right).$$

Hence, because of (38.2),

$$b_{11}^* = -\frac{u^{2*}h''}{\sqrt{g^*}}, \quad b_{12}^* = \frac{h'}{\sqrt{g^*}}, \quad b_{22}^* = 0$$

and therefore, by (41.10),

$$H = -\frac{u^{2*}h''}{2g^{*\frac{1}{2}}}.$$

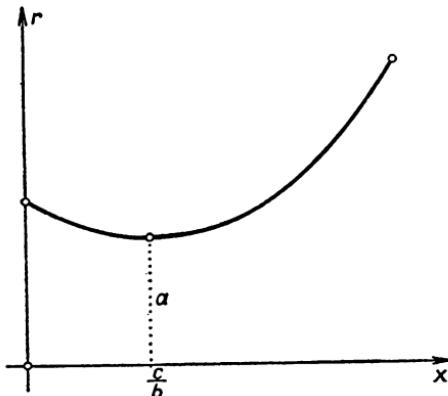


FIG. 84. Catenary

In order that the right conoid be a minimal surface ($H = 0$) we must have $h'' = 0$, that is, h must be a linear function of u^{1*} . This means that the right conoid must be a right helicoid.

We now consider surfaces of revolution. The curve which a perfectly flexible chain of homogeneous mass distribution with fixed end points assumes under the action of gravity is called a *catenary*, cf. Fig. 84. As follows from the differential equation corresponding to this problem of mechanics the catenary can be represented in the form

$$r(x) = a \cosh(bx - c).$$

When the catenary is rotated about the x -axis it generates a surface of revolution which is called a *catenoid*, cf. Fig. 85.

Theorem 81.2. *Catenoids are the only surfaces of revolution which are minimal surfaces.*

Proof. Every surface of revolution (not a cylinder) whose axis is the x_3 -axis of the Cartesian coordinate system in space can be (locally) represented in the form

$$x_3 = q(r), \quad r^2 = x_1^2 + x_2^2,$$

or, by writing u^2 instead of r and choosing a parametric representation, in the form

$$(81.2) \quad \mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2)).$$

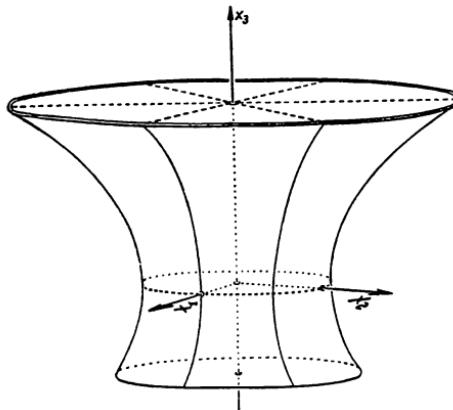


FIG. 85. Catenoid

Consequently

$$g_{11} = (u^2)^2, \quad g_{12} = 0, \quad g_{22} = 1 + h'^2, \quad g = (u^2)^2(1 + h'^2),$$

where a prime denotes the derivative with respect to u^2 . Furthermore we obtain

$$b_{11} = -\frac{u^2 h'}{\sqrt{(1+h'^2)}}, \quad b_{12} = 0, \quad b_{22} = -\frac{h''}{\sqrt{(1+h'^2)}},$$

cf. Problem 40.1. Since $g_{12} = 0$ and $b_{12} = 0$ the coordinate curves are lines of curvature on the surface of revolution, cf. (41.6). Consequently, by (41.11),

$$H = \frac{1}{2} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right) = -\frac{1}{2} \left(\frac{h'}{u^2(1+h'^2)^{\frac{3}{2}}} + \frac{h''}{(1+h'^2)^{\frac{3}{2}}} \right).$$

If $h(u^2)$ is constant we obtain a plane parallel to the $x_1 x_2$ -plane; this case is not of interest. If $h' \neq 0$, the condition $H = 0$ implies

$$\frac{h''h'}{h'^2(1+h'^2)} = -\frac{1}{u^2}.$$

We set $h'^2 = w$. Then $2h'h'' = w''$ and, from the preceding relation,

$$\frac{w'}{w(1+w)} = -\frac{2}{u^2}$$

or

$$(81.3) \quad \left(\frac{1}{w} - \frac{1}{1+w} \right) dw = -\frac{2}{u^2} du^2.$$

Integrating this equation we find

$$\log \frac{w}{1+w} = -2(\log u^2 - c) = \log \frac{c^2}{(u^2)^2}.$$

This equation is equivalent to

$$\frac{h'^2}{1+h'^2} = \frac{c^2}{(u^2)^2},$$

that is,

$$(81.4) \quad h' = \frac{c}{\sqrt{\{(u^2)^2 - c^2\}}}.$$

Integrating once more we finally obtain

$$(81.5) \quad h = c \cosh^{-1} \frac{u^2}{c} - \tilde{c},$$

that is,

$$u^2 = c \cosh \left(\frac{x_3 + \tilde{c}}{c} \right).$$

This means that the meridians of our surface of revolution of vanishing mean curvature are catenaries and consequently the surface is a catenoid.

We will finally investigate the isometry of the right helicoids and the catenoids.

Theorem 81.3. *The catenoid defined by (81.2) and (81.5) can be isometrically mapped onto a portion of the right helicoid (81.1).*

Proof. The first fundamental form corresponding to (81.1) is given by the expression

$$(81.6) \quad ds^{*2} = [(u^{2*})^2 + c^2](du^{1*})^2 + (du^{2*})^2.$$

For the catenoid we have

$$(81.7) \quad ds^2 = (u^2)^2(du^1)^2 + \frac{(u^2)^2}{(u^2)^2 - c^2}(du^2)^2,$$

cf. (81.4). Setting $u^{1*} = u^1$, $u^{2*} = \sqrt{\{(u^2)^2 - c^2\}}$

we have $du^{1*} = du^1$, $du^{2*} = \frac{u^2}{\sqrt{\{(u^2)^2 - c^2\}}} du^2$,

and the first fundamental form (81.6) becomes identical with (81.7). By means of these relations we introduce u^1, u^2 as coordinates in the right helicoid. The mapping under which any point of the right helicoid corresponds to the point of the conoid with the same values of the coordinates u^1, u^2 is then isometric, in consequence of Theorem 57.1.

This mapping can be realized by bending, cf. Fig. 86: We cut the catenoid along the meridian $u^1 = 0$ and bend it until it coincides with a pitch of the right helicoid under consideration. Since the curves $u^1 = \text{const.}$ are meridians of the catenoid and the curves $u^{1*} = \text{const.}$ are generators of the right helicoid the meridians are mapped onto straight lines on the right

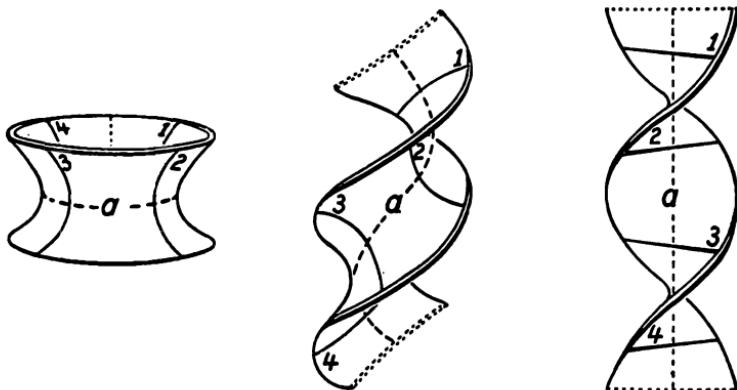


FIG. 86

helicoid. The images of the parallels $u^2 = u_0^2 = \text{const.}$ of the catenoid are arcs of the circular helices

$$u^{2*} = u_0^{2*} = \sqrt{(u_0^2)^2 - c^2} = \text{const.};$$

in particular, the smallest of the parallels, $u^2 = c$, is mapped into the axis of the right helicoid, cf. Fig. 86.

It can be proved that all the surfaces occurring in connexion with the bending of the catenoid into the right helicoid are minimal surfaces; they are called minimal surfaces of Scherk, cf. H. F. Scherk [1].

Problem 81.1. Find a representation of the right helicoids which are isometric to a catenoid. (Cf. Problem 57.1.)

Problem 81.2. Investigate the behaviour of the smallest circle of the catenoid when the bending represented in Fig. 86 is performed (use the results of Problems 57.1 and 81.1).

82. Relations between function theory and minimal surfaces.
The formulae of Weierstrass. We will now study an interesting relation between minimal surfaces and function theory, which was discovered by Weierstrass; we mentioned already in Section 80 that these results had considerable influence on the development of the theory of minimal surfaces.

We start from the following

Theorem 82.1. *Let S be a simply-connected minimal surface, given by a representation $\mathbf{x}(u^1, u^2)$ of class $r \geq 3$, where u^1, u^2 are isothermic coordinates on S (cf. Section 62). Then the components x_i of the vector function \mathbf{x} are harmonic functions, that is, solutions of Laplace's equation $\Delta x_i = 0$ or, in vector notation,*

$$\Delta \mathbf{x} \equiv \mathbf{x}_{11} + \mathbf{x}_{22} = \mathbf{0}.$$

Proof. The existence of isothermic coordinates u^1, u^2 on S follows from Theorem 62.1, and we have

$$(82.1) \quad ds^2 = \eta(u^1, u^2)[(du^1)^2 + (du^2)^2], \quad (\eta > 0),$$

cf. Definition 62.1, and $g = \eta^2$. Consequently

$$g^{11} = g^{22} = \frac{1}{\eta}.$$

Therefore, by (38.2) and (41.10),

$$H = \frac{1}{2}g^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2\eta}(b_{11} + b_{22}) = \frac{1}{2\eta}(\mathbf{x}_{11} + \mathbf{x}_{22}) \cdot \mathbf{n} = \frac{1}{2\eta}\mathbf{n} \cdot \Delta \mathbf{x}.$$

Since S is a minimal surface, i.e. $H = 0$, we have $\mathbf{n} \cdot \Delta \mathbf{x} = 0$; the vectors \mathbf{n} and $\Delta \mathbf{x}$ are orthogonal. From (82.1) we have $g_{11} = g_{22}$, that is,

$$(82.2) \quad \mathbf{x}_1 \cdot \mathbf{x}_1 = \mathbf{x}_2 \cdot \mathbf{x}_2,$$

cf. (27.1). By differentiating (82.2) with respect to u^1 we obtain

$$(82.3) \quad \mathbf{x}_1 \cdot \mathbf{x}_{11} = \mathbf{x}_2 \cdot \mathbf{x}_{12}.$$

Furthermore

$$(82.4) \quad g_{12} \equiv \mathbf{x}_1 \cdot \mathbf{x}_2 = 0.$$

By differentiating this with respect to u^2 we find

$$\mathbf{x}_{12} \cdot \mathbf{x}_2 + \mathbf{x}_1 \cdot \mathbf{x}_{22} = 0.$$

Hence, by (82.3),

$$\mathbf{x}_1 \cdot \mathbf{x}_{11} = -\mathbf{x}_1 \cdot \mathbf{x}_{22}$$

or

$$\mathbf{x}_1 \cdot \Delta \mathbf{x} = 0;$$

the vector $\Delta \mathbf{x}$ is also orthogonal to the vector \mathbf{x}_1 . By differentiating (82.2) with respect to u^2 and (82.4) with respect to u^1 we similarly find $\mathbf{x}_2 \cdot \Delta \mathbf{x} = 0$. Therefore $\Delta \mathbf{x} = \mathbf{0}$. This completes the proof.

Every function of two real variables which is harmonic in a simply-connected domain is the real part of a suitable analytic function. Thus we obtain a relation between minimal surfaces and function theory. We set

$$u = u^1 + iu^2, \quad i = \sqrt{(-1)}$$

and

$$\mathbf{z} = \mathbf{x} + iy.$$

Then

$$\mathbf{x}(u^1, u^2) = \mathbf{x}(u) = \Re \mathbf{z}(u),$$

where $\Re z$ is the real part of z . In consequence of the Cauchy-Riemann differential equations

$$x_1 = y_2, \quad x_2 = -y_1$$

we obtain

$$\frac{dz}{du} = x_1 - ix_2,$$

and therefore

$$\frac{dz}{du} \cdot \frac{dz}{du} = x_1 \cdot x_1 - 2ix_1 \cdot x_2 - x_2 \cdot x_2 = g_{11} - 2ig_{12} - g_{22}.$$

Hence from the assumption that u^1, u^2 are isothermic coordinates (cf. Theorem 82.1) it follows that

$$(82.5) \quad \frac{dz}{du} \cdot \frac{dz}{du} = 0.$$

Conversely, if $x(u)$ is harmonic, and the corresponding analytic function $z(u)$ satisfies (82.5), then $x(u) = x(u^1, u^2)$ represents a minimal surface, and u^1, u^2 are isothermic coordinates. Indeed, from (82.5) we obtain $g_{11} = g_{22}$ and $g_{12} = 0$; that is, u^1, u^2 are isothermic coordinates. Since $\Delta x = 0$ we thus find $H = \frac{1}{2\eta} n \cdot \Delta x = 0$.

Let $p(u)$ and $q(u)$ be arbitrary functions regular in the domain under consideration. If we set

$$(82.6) \quad \frac{dz_1}{du} = p^2 - q^2, \quad \frac{dz_2}{du} = i(p^2 + q^2), \quad \frac{dz_3}{du} = 2pq$$

then (82.5) is satisfied, since

$$\frac{dz}{du} \cdot \frac{dz}{du} = (p^2 - q^2)^2 - (p^2 + q^2)^2 + 4p^2q^2 = 0.$$

When integrating (82.6) we obtain the following representation of a minimal surface of class $r \geq 3$:

$$(82.7) \quad \begin{aligned} x_1 &= \Re z_1 = \Re \int (p^2 - q^2) du, \\ x_2 &= \Re z_2 = \Re \int i(p^2 + q^2) du, \\ x_3 &= \Re z_3 = \Re \int 2pq du. \end{aligned}$$

Let us assume $p \neq 0$ and $(q/p)' \neq 0$, that is, $pq' - p'q \neq 0$. (If $p \equiv 0$ or $(q/p)' \equiv 0$ we would obtain a plane.) We set $t = q/p$. Then

$$du = \frac{p^2}{q'p - p'q} dt.$$

Setting

$$F(t) = \frac{p^4}{q'p - p'q}$$

we obtain from (82.7) the *formulae of Weierstrass* (1866)

$$(82.8) \quad \begin{aligned} x_1 &= \Re \int (1-t^2) F(t) dt, \\ x_2 &= \Re \int i(1+t^2) F(t) dt, \\ x_3 &= \Re \int 2t F(t) dt. \end{aligned}$$

These formulae enable us to associate a minimal surface $\mathbf{x}(t^1, t^2)$ to every regular function $F(t) \not\equiv 0$ of the complex variable $t = t^1 + it^2$.

We finally mention that (82.8) can also be written in a form free of integrals. Setting $F = h''$ and integrating by parts we find

$$x_1 = \Re \left[\int h'''(t) dt - \int t^2 h''(t) dt \right] = \Re [h''(t) - t^2 h''(t) + 2 \int t h''(t) dt],$$

etc., hence

$$x_1 = \Re [(1-t^2)h'' + 2th' - 2h],$$

$$x_2 = \Re [i(1+t^2)h'' - 2ith' + 2ih],$$

$$x_3 = \Re [2th'' - 2h'].$$

Problem 82.1. Prove that if $F = i/2t^2$ the right helicoid $\tan x_3 = x_1/x_2$ is obtained from (82.8).

83. Minimal surfaces as translation surfaces with isotropic generators. A surface which can be represented in the form

$$(83.1) \quad \mathbf{x}(u^1, u^2) = \mathbf{y}(u^1) + \mathbf{z}(u^2) \quad \left(\frac{dy}{du^1} \times \frac{dz}{du^2} \neq 0 \right)$$

is called a *translation surface*.

For every fixed value of u^1 , (83.1) is a representation of a curve; the curves corresponding to different values of u^1 are congruent. For every fixed value of u^2 , (83.1) represents also a curve and the curves thus obtained are also congruent. A translation surface can therefore be generated by a suitable translation of a curve of one of the above two families of curves. Each of those curves is called a *generator* of the translation surface under consideration.

A simple example of a translation surface is the cylinder of revolution

$$\mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2) = \mathbf{y}(u^1) + \mathbf{z}(u^2),$$

$$\text{where } \mathbf{y}(u^1) = (r \cos u^1, r \sin u^1, 0), \quad \mathbf{z}(u^2) = (0, 0, u^2).$$

This surface can be generated either by a translation of a straight line parallel to the x_3 -axis and always passing through the circle $x_1^2 + x_2^2 = r^2$, $x_3 = 0$, or by a translation of this circle such that its centre is always a point of the x_3 -axis.

If we restrict our considerations to analytic minimal surfaces we have

Theorem 83.1 (S. Lie [1]). *A surface is a minimal surface if and only if it can be represented as a translation surface whose generators are isotropic curves.*

Proof. As was shown when the isotropic curves are introduced as coordinate curves on a surface,

$$(83.2) \quad g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = 0, \quad g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = 0,$$

cf. (63.4). Hence, by (41.10)

$$H = \frac{b_{12}}{g_{12}}.$$

Since a minimal surface is a surface of vanishing mean curvature H we obtain from this

$$b_{12} = \mathbf{x}_{12} \cdot \mathbf{n} = 0,$$

cf. (38.2); that is, the vector \mathbf{x}_{12} is orthogonal to the unit normal vector \mathbf{n} of the surface. By differentiation of (83.2),

$$\mathbf{x}_1 \cdot \mathbf{x}_{12} = 0, \quad \mathbf{x}_2 \cdot \mathbf{x}_{12} = 0,$$

we see that \mathbf{x}_{12} must also be orthogonal to \mathbf{x}_1 and \mathbf{x}_2 ; hence $\mathbf{x}_{12} = 0$. This relation holds if and only if \mathbf{x} is a sum of functions each of which depends either on u^1 or on u^2 only. Consequently \mathbf{x} must be of the form (83.1). Furthermore we have

$$(83.3) \quad g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = \frac{dy}{du^1} \cdot \frac{dy}{du^1} = 0, \quad g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = \frac{dz}{du^2} \cdot \frac{dz}{du^2} = 0.$$

Conversely, from (83.1) and (83.3) it follows that $H = 0$. This completes the proof.

We replace (83.1) by

$$(83.4) \quad \mathbf{x}(u^1, u^2; \alpha) = e^{i\alpha} \mathbf{y}(u^1) + e^{-i\alpha} \mathbf{z}(u^2), \quad (i = \sqrt{(-1)})$$

where α is a real parameter. Then if \mathbf{y} and \mathbf{z} are conjugate and (83.3) is satisfied we obtain a family of real minimal surfaces which are called *associated minimal surfaces*.

Theorem 83.2. *Each pair of associated minimal surfaces can be mapped isometrically onto each other in such a way that at corresponding points the tangent planes to both surfaces are parallel.*

Proof. The first fundamental form

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = 2g_{12} du^1 du^2 = 2 \frac{dy}{du^1} \cdot \frac{dz}{du^2} du^1 du^2,$$

corresponding to (83.4), is independent of α . In consequence of Theorem

57.1 the surfaces (83.4) thus are isometric. Furthermore, the unit normal vector

$$\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g}}$$

of these surfaces, cf. (34.1), is also independent of α , since in the representation (5.7) of the vector product $\mathbf{x}_1 \times \mathbf{x}_2$ one column involves a common factor $e^{i\alpha}$ and the other a common factor $e^{-i\alpha}$. Hence at corresponding points the tangent planes of the surfaces are parallel.

Two surfaces of the family (83.4) in whose representations the values of α differ by π can be obtained from each other by reflection in the origin of the Cartesian coordinate system in space. For instance, the surface corresponding to $\alpha = \pi$ is the image of the surface (83.1) under this reflection.

Two surfaces of the family (83.4) whose parametric values differ by $\frac{1}{2}\pi$ are called *adjoint minimal surfaces*. For example,

$$(83.5) \quad \mathbf{x}(u^1, u^2; \frac{1}{2}\pi) = iy(u^1) - iz(u^2)$$

is the adjoint of (83.1). Each surface of the family (83.4) can be represented by the surfaces (83.1) and (83.5) in the form

$$\mathbf{x}(u^1, u^2; \alpha) = e^{i\alpha}\mathbf{y} + e^{-i\alpha}\mathbf{z} = \mathbf{x}(u^1, u^2; 0)\cos \alpha + \mathbf{x}(u^1, u^2; \frac{1}{2}\pi)\sin \alpha$$

as can immediately be seen. For fixed u^1, u^2 and variable α the terminal point of the vector $\mathbf{x}(u^1, u^2; \alpha)$ generates an ellipse in the plane spanned by the vectors $\mathbf{x}(u^1, u^2; 0)$ and $\mathbf{x}(u^1, u^2; \frac{1}{2}\pi)$. This means that a minimal surface admits a continuous deformation such that every point of the surface moves along an ellipse and the surface is a minimal surface at every stage of this deformation.

We finally mention the following

Theorem 83.3. *If S and S^* are adjoint minimal surfaces the asymptotic curves of S correspond to the lines of curvature of S^* and vice versa.*

Problem 83.1. Prove Theorem 83.3.

84. Modular surfaces of analytic functions. Let

$$(84.1) \quad h(z) = \sum_{\alpha=0}^{\infty} a_{\alpha}(z-z_0)^{\alpha}$$

be an analytic function of a complex variable $z = x+iy$. We set

$$(84.2) \quad h(z) = U+iV = We^{i\gamma}$$

that is, we denote by U and V , respectively, the real and the imaginary part of $h(z)$ and by W and γ , respectively, its modulus and phase. Moreover, we introduce the following notations

$$(84.3) \quad a_{\alpha} = A_{\alpha} e^{i\beta_{\alpha}}, \quad z-z_0 = re^{i\phi},$$

cf. (84.1). Let x_1, x_2, x_3 be Cartesian coordinates in space R_3 . Then

$$(84.4) \quad \mathbf{x}(x, y) = (x, y, W(x, y))$$

is a representation of a surface in R_3 which is called the *modular surface* or *surface of absolute value* of the analytic function $h(z)$.

Examples of such surfaces of higher transcendental functions (gamma function, elliptic functions, cylinder functions, Riemann zeta function, etc.) can be found in the book *Tables of Functions* by E. Jahnke and F. Emde (New York, 1945). On the modular surface of a function the curves of constant absolute value W and of constant phase γ can be described; in this manner we obtain a useful illustration of the behaviour of the function for complex values of the argument. We may say that modular surfaces are analogues of the usual graphs of a real function of a real variable. These surfaces are therefore of interest also in applied mathematics.

The investigation of these surfaces by means of the methods of differential geometry, as it will be presented in the following pages, was made by E. Ullrich and his pupils, cf. E. Ullrich [1], J. Zaat [1].

We will now investigate some basic problems which will yield a general characterization of the modular surfaces.

We start with the consideration of rays which are tangent to a point of a surface of absolute value; we call these rays *half-tangents*.

Theorem 84.1. *At every point z_0 which is not at infinity and is a point of regularity of $h(z)$ other than a simple zero the half-tangents to the corresponding surface of absolute value form a tangent plane. The tangent in the direction*

$$(84.5) \quad \phi^* = \beta_0 - \beta_1$$

makes the maximum angle with the $x_1 x_2$ -plane, compared with all other tangents at z_0 ; this angle is

$$(84.6) \quad \psi^* = \arctan |h'(z_0)|,$$

cf. (84.1) and (84.3). At every multiple zero of $h(z)$ the tangent plane exists and is parallel to the $x_1 x_2$ -plane. If z_0 is a simple zero of $h(z)$ the half-tangents at z_0 form a cone whose generators make the angle ψ^ , cf. (84.6), with the $x_1 x_2$ -plane.*

Proof. The problem consists in determining at z_0 the derivative in a certain direction of a function of two real variables; we denote this derivative by $|h(z_0)|'_\phi$, cf. (84.3). Since

$$f(z) = a_0 + a_1(z - z_0) + o(r) = a_0 + a_1 r e^{i\phi} + o(r),$$

cf. (84.1), we obtain

$$h(z) = \bar{a}_0 + \bar{a}_1 r e^{-i\phi} + o(r)$$

and, if $a_0 \neq 0$, therefore

$$\begin{aligned} W^2 &= h\bar{h} = A_0^2 + 2\Re(a_0\bar{a}_1 e^{-i\phi})r + o(r) \\ &= A_0^2 + 2A_0 A_1 r \cos(\beta_0 - \beta_1 - \phi) + o(r). \end{aligned}$$

Hence $W = A_0 \left(1 + \frac{A_1}{A_0} r \cos(\beta_0 - \beta_1 - \phi) \right) + o(r).$

We thus find

$$(84.7) \quad \begin{aligned} |h(z_0)|'_\phi &= W_r(r, \phi)|_{r=0} = A_1 \cos(\beta_0 - \beta_1 - \phi) \\ &= |h'(z_0)| \cos(\beta_0 - \beta_1 - \phi). \end{aligned}$$

From this the statement involving (84.5) follows. (84.7) is the tangent of the angle ψ between the $x_1 x_2$ -plane and that half-tangent whose orthogonal projection in the $x_1 x_2$ -plane corresponds to the angle ϕ . From this (84.6) follows. If z_0 is a simple zero of $h(z)$ then $a_0 = 0$ and

$$|h(z_0)|'_\phi = |a_1| = A_1;$$

in this case the half-tangents form a cone at this point. If z_0 is a multiple zero of $h(z)$ we have $a_0 = a_1 = 0$, and the modular surface has a tangent plane parallel to the $x_1 x_2$ -plane.

We will now determine the first fundamental form of the surface (84.4). We have $W(x, y) = W(r \cos \phi, r \sin \phi)$. Hence, for all values of ϕ ,

$$(84.8) \quad W_r = W_x \cos \phi + W_y \sin \phi.$$

On the other hand, in consequence of (84.7),

$$\begin{aligned} W_r|_{z_0} &= A_1 \cos(\beta_0 - \beta_1 - \phi) \\ &= A_1 \{ \cos(\beta_0 - \beta_1) \cos \phi + \sin(\beta_0 - \beta_1) \sin \phi \}. \end{aligned}$$

Comparing this expression and (84.8) we find

$$W_x = A_1 \cos(\beta_0 - \beta_1), \quad W_y = A_1 \sin(\beta_0 - \beta_1).$$

We set $Q = h'/h$. Then

$$Q(z_0) = \frac{a_1}{a_0} = \frac{A_1}{A_0} e^{i(\beta_1 - \beta_0)}.$$

Hence at z_0 $W_x = |h|\Re Q$, $W_y = -|h|\Im Q$,

where $\Re Q$ and $\Im Q$ are the real and imaginary part of Q , respectively. By (27.1) we obtain from this

$$(84.9) \quad g_{11} = 1 + |h|^2(\Re Q)^2, \quad g_{12} = -|h|^2\Re Q \Im Q, \quad g_{22} = 1 + |h|^2(\Im Q)^2$$

and $g = 1 + |h|^2[(\Re Q)^2 + (\Im Q)^2];$

that is, $g = 1 + |h'|^2,$

cf. (28.2).

We now investigate the Gaussian curvature of the surface (84.4). From (38.8) and (84.4) we find

$$b_{11} = \frac{1}{\sqrt{g}} \begin{vmatrix} 1 & 0 & W_x \\ 0 & 1 & W_y \\ 0 & 0 & W_{xx} \end{vmatrix} = \frac{W_{xx}}{\sqrt{g}}, \quad \text{etc.}$$

In consequence of (84.9) we thus obtain

$$(84.10) \quad b = b_{11} b_{22} - (b_{12})^2 = \frac{W_{xx} W_{yy} - W_{xy}^2}{1 + |h'|^2}.$$

We have to determine the second derivatives occurring in (84.10). Differentiating

$$(84.10') \quad W_x - iW_y = |h|(\Re Q + i\Im Q) = |h| \frac{h'}{h} = WQ$$

with respect to x , we obtain, in consequence of $\partial z/\partial x = 1$,

$$W_{xx} - iW_{xy} = W_x Q + WQ_x = WQ(\Re Q - Q) + WS = -iWQ\Im Q + WS,$$

where $S = \frac{h''}{h}$.

Differentiating (84.10') with respect to y we find, in consequence of $\partial z/\partial y = i$,

$$W_{xy} - iW_{yy} = -iWQ\Re Q + iWS.$$

Comparing the real and imaginary parts on both sides of the two relations we obtain

$$W_{xx} = W[(\Im Q)^2 + \Re S], \quad W_{xy} = W[\Re Q\Im Q - \Im S], \quad W_{yy} = W[(\Re Q)^2 - \Re S]$$

and from this

$$W_{xx} W_{yy} - W_{xy}^2 = W^2(\Re S[(\Re Q)^2 - (\Im Q)^2] + 2\Re Q\Im Q\Im S) - |h''|^2.$$

Now

$$\begin{aligned} \Re \frac{h'^2}{h''h} &= \Re \frac{Q^2}{S} = \Re \frac{[(\Re Q)^2 - (\Im Q)^2 + 2i\Re Q\Im Q](\Re S - i\Im S)}{|S|^2} \\ &= \frac{\Re S[(\Re Q)^2 - (\Im Q)^2] + 2\Re Q\Im Q\Im S}{|h''|^2/W^2} \end{aligned}$$

and consequently

$$W_{xx} W_{yy} - W_{xy}^2 = |h''|^2 \left(\Re \frac{h'^2}{h''h} - 1 \right).$$

By (41.9), (84.9), and (84.10) the Gaussian curvature of the modular surface of an analytic function $h(z)$ is given by the expression

$$(84.11) \quad K = \frac{b}{g} = |h''|^2 \frac{\Re(h'^2/h''h) - 1}{(1 + |h'|^2)^2}.$$

From this formula we can draw several important conclusions. We first consider the case that $h''(z) = 0$ at a certain point. We set

$$(84.12) \quad h(z+\zeta) = a_0 + a_1 \zeta + a_2 \zeta^2 + \dots + a_n \zeta^n + \dots$$

Let $h(z) = 0$, then $a_0 = 0$. Since simple zeros are singular points of the modular surface we may assume $h(z) = h'(z) = h''(z) = 0$. Therefore $a_0 = a_1 = a_2 = 0$. If a_k is the first non-vanishing coefficient of the expansion (84.12) then

$$\frac{h'^2}{h''h} = \frac{k}{k-1} + o(1), \quad (k > 2).$$

Consequently at the point z under consideration the value of

$$(84.13) \quad \Re \frac{h'^2}{h''h} - 1$$

is finite. We thus obtain the following result: At all points which are at least triple zeros of $h(z)$ the Gaussian curvature of the corresponding modular surface vanishes. It can be proved that if $a_0 \neq 0$ and $a_2 = 0$, (84.13) has also a finite value at the point z under consideration. From (84.11) we thus obtain the following

Theorem 84.2. *If, at a point, $h'' \neq 0$ and*

$$(84.14) \quad \Re \frac{h'^2}{h''h}$$

is greater than 1 or less than 1, the corresponding point of the modular surface of $h(z)$ is respectively elliptic or hyperbolic. If $h'' = 0$, or if the expression (84.14) has the value 1, the corresponding point of the modular surface of $h(z)$ is parabolic.

Parabolic points corresponding to $h'' = 0$ are isolated, in general, while the other parabolic points form certain curves which we call ‘parabolic curves’ of the modular surface. In special cases a point corresponding to $h'' = 0$ may lie on a parabolic curve. If $h''(z) \equiv 0$ then $h(z) = az + c$; the corresponding surface is a cone or a plane; it consists of parabolic points only (except for the apex of the cone).

From Theorem 84.2 we obtain

Theorem 84.3. *In a (sufficiently small) neighbourhood of a pole at a point $z (\neq \infty)$ the modular surface of $h(z)$ consists of hyperbolic points only.*

Proof. Without loss of generality we may assume the pole to be at $z = 0$. We denote by k its order. Then

$$h(z) = a_{-k} z^{-k} + \dots, \quad h'(z) = -ka_{-k} z^{-k-1} + \dots, \\ h''(z) = k(k+1)a_{-k} z^{-k-2} + \dots, \quad (a_{-k} \neq 0)$$

and therefore

$$\frac{h'^2}{h''h} = \frac{k}{k+1} + o(1)$$

is a power series which at $z = 0$ has the value $k/(k+1)$. From this and Theorem 84.2 the statement follows.

We now consider the curvature of the surface in the neighbourhood of a point which corresponds to a zero of $h(z)$.

Theorem 84.4. *If z_0 ($\neq \infty$) is a multiple zero of $h(z)$ the points of the modular surface of $h(z)$ which correspond to z , $0 < |z - z_0| < \epsilon$, $\epsilon > 0$ and sufficiently small, are elliptic. If z_0 is a simple zero of $h(z)$ then on the portion of that surface corresponding to $0 < |z - z_0| < \epsilon$ there exist $2k-2$ angular regions whose points are elliptic or hyperbolic points. k is the exponent in the first non-vanishing term after the linear term in the power series development of $h(z)$ at z_0 .*

Proof. Without loss of generality we may assume the zero of $h(z)$ to be at the point $z = 0$. If $z = 0$ is a multiple zero, $h(z)$ is of the form

$$h(z) = a_m z^m + \dots, \quad (a_m \neq 0, m \geq 2).$$

Therefore

$$h'(z) = m a_m z^{m-1} + \dots, \quad h''(z) = m(m-1) a_m z^{m-2} + \dots,$$

and

$$\frac{h'^2}{h''h} = \frac{m}{m-1} + o(1).$$

From the fact that in a sufficiently small neighbourhood of the point $z = 0$ the real part of this expression is greater than 1 the statement concerning multiple zeros follows. If $z = 0$ is a simple zero of $h(z)$ then $h(z)$ is of the form $h(z) = a_1 z + a_k z^k + \dots$ ($a_1 \neq 0, a_k \neq 0$). Therefore

$$h'(z) = a_1 + k a_k z^{k-1} + \dots, \quad h''(z) = k(k-1) a_k z^{k-2} + \dots$$

and

$$\Re \frac{h'^2}{h''h} = \Re \left[\frac{a_1 z^{1-k}}{k(k-1)a_k} \{1 + o(1)\} \right] = \frac{A_1 r^{1-k}}{k(k-1)A_k} \cos(\beta_1 - \beta_k - (k-1)\phi) \{1 + o(1)\},$$

cf. (84.3). Since this expression increases arbitrarily if $r \rightarrow 0$ the type of the curvature depends on the sign of the cosine. The values ϕ for which

$$\beta_1 - \beta_k - (k-1)\phi = \frac{1}{2}(2n+1)\pi$$

that is,

$$\phi = \frac{\beta_1 - \beta_k}{k-1} - \frac{(2n+1)\pi}{2(k-1)}, \quad (k \geq 2)$$

correspond to the zeros of the cosine; these are the directions tangent to the parabolic curves at $z = 0$. These curves are the boundaries of portions of the neighbourhood of $z = 0$ containing either only hyperbolic points or only elliptic points of the surface. This completes the proof.

As simple examples we finally consider the functions $\sin \pi z$ and

$$\operatorname{cosec} \pi z = \frac{1}{\sin \pi z},$$

cf. Figs. 87, 88, and 89. The orthogonal projection of the modular surface of $\text{cosec } \pi z$ onto the xy -plane can be obtained from that of $\sin \pi z$ (cf. Fig. 88) by replacing W by $1/W$ and γ by $-\gamma$. The zeros of $\sin \pi z$ at $z = 0, \pm 1, \pm 2, \dots$

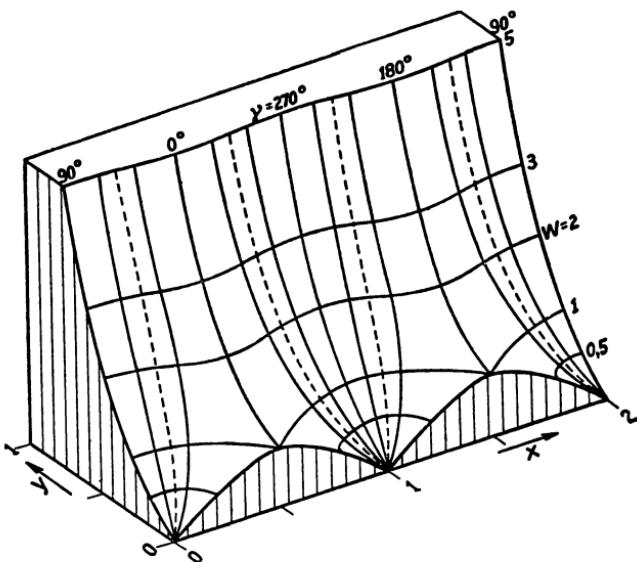


FIG. 87. Modular surface of $\sin \pi z$

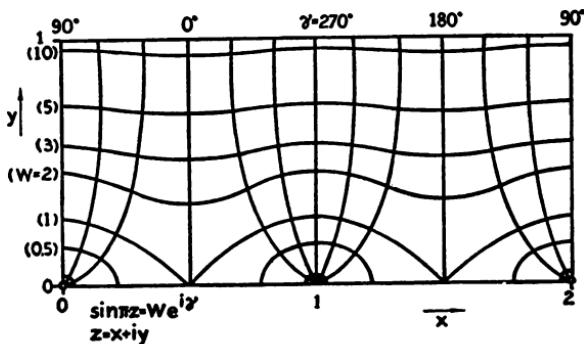


FIG. 88. Orthogonal projection of the modular surface of $\sin \pi z$ into the xy -plane

correspond to poles of $\text{cosec } \pi z$ at these points. The parabolic curves (dotted lines in Fig. 87) of $\sin \pi z$ are the solutions of the equation $\Re(\cot^2 \pi z) = -1$; the parabolic curves of $\text{cosec } \pi z$ are obtained from the

equation $\Re[\cos^2 \pi z / (1 + \cos^2 \pi z)] = 1$. These curves are the boundaries of portions of the surfaces consisting of hyperbolic or of elliptic points only.

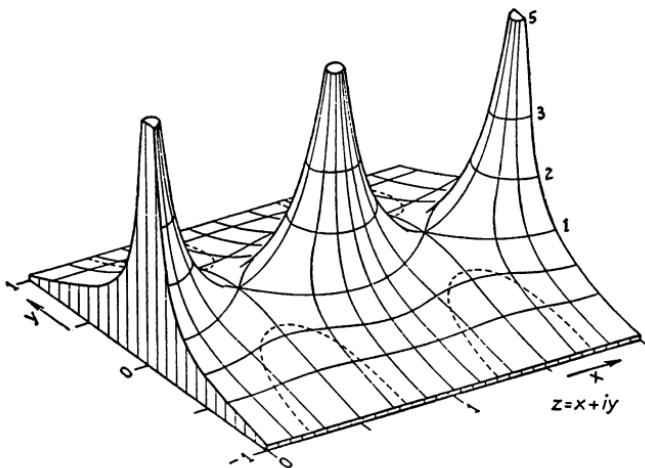


FIG. 89. Modular surface of $\text{cosec } \pi z$

From 84.11 it follows that the analytic functions whose modular surfaces are developable are the exponential functions $h(z) = \exp(az+b)$, and the powers $h(z) = (az+b)^{1+ik}$, where a , b , and k are constants and k is real.

85. Envelope of a one-parameter family of surfaces. We now consider families of surfaces without singular points and their envelopes. In this section we start with arbitrary surfaces (without singular points). The results obtained in this section will be applied to the particular case of planes in the following sections. In these investigations we will reconsider developable surfaces from another point of view.

In order to introduce the concept of a family of surfaces we use representations of the form (24.5): $G(x_1, x_2, x_3) = 0$; we remember that at every point at least one of the three first partial derivatives of G is different from zero. We write for short

$$G(x_i) = 0.$$

For the sake of simplicity we start from a particular example. A sphere of radius r whose centre lies on the x_3 -axis at a distance t from the origin of the Cartesian coordinate system in space can be represented in the form

$$G(x_i) = x_1^2 + x_2^2 + (x_3 - t)^2 - r^2 = 0.$$

If we now assume t to be variable we obtain a *one-parameter family of surfaces*; t is the parameter of this family which in our particular case consists of congruent spheres with centres on the x_3 -axis; every value of t corresponds to exactly one of these surfaces.

We now consider a one-parameter family F of arbitrary surfaces which we may represent in the form

$$(85.1) \quad G(x_i, t) = 0.$$

We assume that (85.1), considered as a function of t , is of class $r \geq 2$, and that for every fixed value of t (85.1) is a representation of a surface of class r , which satisfies the conditions stated in connexion with (24.5).

We assume that a surface S of the family F corresponding to an arbitrary (fixed) value of t intersects every surface corresponding to a value $t+h$ ($|h|$ sufficiently small). This intersection $C^*(h)$ can be represented in the form

$$(85.2) \quad (a) \quad G(x_i, t) = 0, \quad (b) \quad G(x_i, t+h) = 0.$$

Since C^* lies also on the surface $G(x_i, t+h) - G(x_i, t) = 0$ we may replace (85.2 b) by

$$(85.3) \quad \frac{G(x_i, t+h) - G(x_i, t)}{h} = 0.$$

If h tends to zero, (85.3) tends to

$$(85.4) \quad \frac{\partial G(x_i, t)}{\partial t} = 0.$$

The point set C represented by

$$(85.5) \quad G(x_i, t) = 0, \quad \frac{\partial G(x_i, t)}{\partial t} = 0,$$

is called the *characteristic* of the surface S with respect to the family F .

Remark: Under the assumption

$$(85.5') \quad \lim_{\substack{x_i \rightarrow x_i^0 \\ h \rightarrow 0}} \frac{G(x_i, t+h) - G(x_i, t)}{h} = \left. \frac{\partial G(x_i, p)}{\partial p} \right|_{\substack{x_i = x_i^0 \\ p = t}}$$

the set C contains every point of the set to which the points of the intersection $C^*(h)$ tend if h tends to zero. However, C will contain, in general, also other points. It may even be that neighbouring surfaces of the family do not intersect at all but nevertheless (85.5) determines a characteristic on each of the surfaces.

Example 1. The equation $x_3 = x_2^3$ represents a cylinder Z which intersects the $x_2 x_3$ -plane along a cubic parabola and whose generators are parallel

to the x_1 -axis. By a translation of Z in the direction of the x_2 -axis we obtain a family of congruent cylinders which can be represented in the form

$$G(x_i, t) = (x_2 - t)^3 - x_3 = 0.$$

We have

$$\frac{\partial G}{\partial t} = -3(x_2 - t)^2.$$

Hence every surface has a characteristic, given by $x_2 = t$, $x_3 = 0$, although there does not exist an intersection of any two surfaces at all.

If the characteristics of the family F exist and if their totality obtained by letting t assume all possible values generates a surface, this surface is called the *envelope* of the one-parameter family F . (Of course the envelope may also consist of several surfaces.)

In general the parameter t can be eliminated from (85.5) and a representation of the envelope of the form $F(x_i) = 0$ can be obtained in this manner.

There are different possible cases which we may illustrate by the following simple examples.

Example 2. In the case of the above family of spheres we have

$$\frac{\partial G}{\partial t} = -2(x_3 - t) = 0.$$

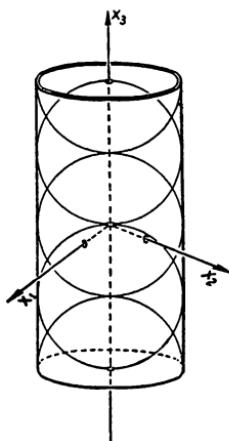


FIG. 90

Consequently the characteristic of each of these spheres is a circle of radius r in a plane parallel to the $x_1 x_2$ -plane passing through the centre of the corresponding sphere. From the preceding relation we obtain $t = x_3$. Inserting this value in the representation of the family we obtain the following representation of the envelope of the family:

$$F(x_i) = x_1^2 + x_2^2 - r^2 = 0.$$

This envelope is a cylinder of revolution of radius r with the x_3 -axis as axis of revolution, cf. Fig. 90.

Example 3. A pencil of planes passing through the x_1 -axis can be represented in the form

$$G(x_i, t) = x_3 - tx_2 = 0$$

(except for the plane $x_2 = 0$). As follows from

$$\frac{\partial G}{\partial t} = -x_2 = 0$$

all characteristics coincide with the x_1 -axis, $x_2 = 0$, $x_3 = 0$, that is, they do not generate a surface. Cf. Fig. 91.

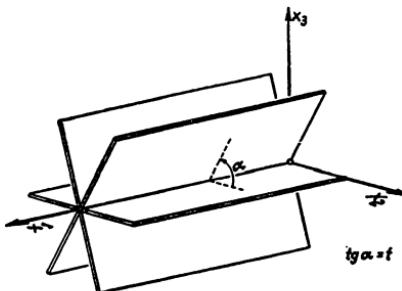


FIG. 91

Example 4. A family of spheres of radius t and centre at the point $x_3 = t$ of the x_3 -axis can be represented in the form

$$G(x_i, t) = x_1^2 + x_2^2 + (x_3 - t)^2 - t^2 = 0.$$

We have $\frac{\partial G}{\partial t} = -2x_3 = 0$.

The system of these two equations has the only solution $x_1 = x_2 = x_3 = 0$. The characteristics all degenerate into a point and they all coincide, cf. Fig. 92.

Example 5. Concentric spheres

$$x_1^2 + x_2^2 + x_3^2 - t^2 = 0$$

do not possess (real) characteristics at all.

We will now introduce the concept of characteristic points of a surface. Let us assume that the characteristic of the surface S : $G(x_i, t) = 0$ (t arbitrary and fixed) intersects every surface $S(k)$: $G(x_i, t+k) = 0$ of a family F , where $|k|$ is sufficiently small. Obviously these points of intersection are determined by

$$G(x_i, t) = 0, \quad \frac{\partial G(x_i, t)}{\partial t} = 0, \quad G(x_i, t+k) = 0.$$

Since

$$G(x_i, t+k) = G(x_i, t) + k \frac{\partial G(x_i, t)}{\partial t} + \frac{k^2}{2} \frac{\partial^2 G(x_i, t+\theta k)}{\partial t^2} \quad (0 < \theta < 1),$$

the last equation can be replaced by

$$\frac{\partial^2 G(x_i, t+\theta k)}{\partial t^2} = 0 \quad (0 < \theta < 1).$$

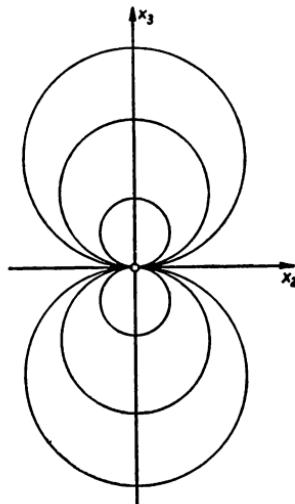


FIG. 92

Hence if k tends to zero we obtain

$$(85.6) \quad G(x_i, t) = 0, \quad \frac{\partial G(x_i, t)}{\partial t} = 0, \quad \frac{\partial^2 G(x_i, t)}{\partial t^2} = 0.$$

The points determined by (85.6) are called the *characteristic* points of the surface S with respect to the family F .

Under an assumption similar to (85.5') the set of characteristic points will contain the limit points of those points of intersection; it may contain also other points of the surface. It may be that the surfaces of a family do not intersect the characteristics of neighbouring surfaces but, nevertheless, (85.6) determines characteristic points on every surface.

Example 6. In the case of the family

$$G(x_i, t) = \{x_1 - (x_2 - t)^3\}^2 - x_3 = 0$$

we obtain $\frac{\partial G}{\partial t} = 6\{x_1 - (x_2 - t)^3\}(x_2 - t)^2 = 0.$

Hence there exist two families of characteristics, namely the family $x_1 = (x_2 - t)^3$, $x_3 = 0$ and the family $x_2 = t$, $x_3 = x_1^2$. The envelope thus consists of the $x_1 x_2$ -plane and the parabolic cylinder $x_3 = x_1^2$. Each point of this cylinder is a characteristic point of a surface of the family, as follows from

$$\frac{\partial^2 G}{\partial t^2} = 6(x_2 - t)\{5(x_2 - t)^3 - 2x_1\} = 0$$

and the preceding two equations.

If characteristic points exist and if they generate a curve (or several curves) when t varies, this curve lies on the envelope of the family F and is called the *edge of regression* of this envelope.

Example 7. A family of spheres of constant radius r with centres on a curve C : $\mathbf{y}(s)$ can be represented in the form

$$G(x_i, s) = \{\mathbf{x} - \mathbf{y}(s)\} \cdot \{\mathbf{x} - \mathbf{y}(s)\} - r^2 = 0.$$

We find $\frac{\partial G}{\partial s} = -2\dot{\mathbf{y}} \cdot (\mathbf{x} - \mathbf{y}) = 0$

where the dot denotes the derivative with respect to the arc length s of the curve C . Hence at every point of a characteristic the corresponding vector $\mathbf{x} - \mathbf{y}$ is orthogonal to the unit tangent vector $\dot{\mathbf{y}}$ of the curve. Consequently the characteristic of every sphere S is a circle in the normal

plane of the curve C at the centre of S . The envelope of a family of spheres is called a *canal surface*. Furthermore if the curvature κ of C is positive, we have

$$\frac{\partial^2 G}{\partial s^2} = -2\ddot{y} \cdot (\mathbf{x} - \mathbf{y}) + 2 = -2\kappa \mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) + 2 = 0,$$

cf. (15.1), where \mathbf{p} is the unit principal normal vector to the curve C . Hence

$$\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) - \rho = 0,$$

that is, the edge of regression of the envelope consists of the points of intersection of the polar axes of C and the corresponding sphere of the family. If $\rho > r$ the edge of regression does not exist. If $\rho < r$ every sphere has two characteristic points P_1, P_2 ; consequently the edge of regression consists of two curves which have in common those points at which $\rho = r$. If, for instance, we take as curve C a circle K_0 of radius $r_0 > r$ we obtain a torus, cf. Section 43; this surface has no edge of regression. If $r_0 \leq r$ the edge of regression degenerates into two points which coincide if $r_0 = r$.

The name 'envelope' will be justified by the following

Theorem 85.1. *If the envelope E of the family F , cf. (85.1), exists then at every point of a characteristic of a surface S of F the tangent planes of S and E coincide, that is, the envelope touches every surface along the corresponding characteristic.*

Proof. The vector $(\partial G / \partial x_i)$ does not vanish and lies in the normal of the corresponding surface, cf. the remarks in connexion with formulae (24.5) and (26.6). Let $\mathbf{x} = \mathbf{x}(u^1, u^2)$ be a representation of the envelope E under consideration. Then the characteristics are determined by

$$t = t(u^1, u^2) = \text{const.}$$

According to (85.5) we have

$$G\{x_i(u^1, u^2), t(u^1, u^2)\} = 0, \quad \frac{\partial G\{x_i(u^1, u^2), t(u^1, u^2)\}}{\partial t} = 0.$$

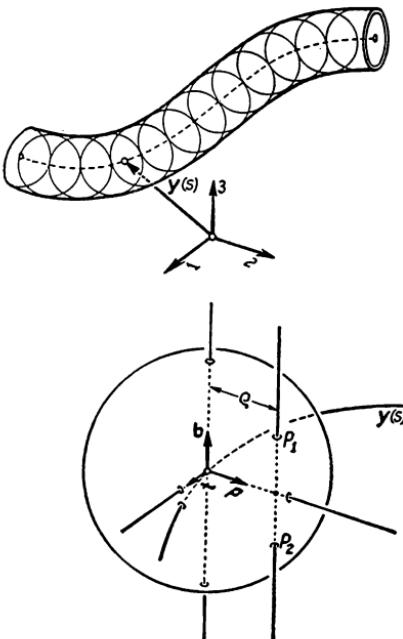


FIG. 93

From this we obtain

$$(85.7) \quad \frac{\partial G}{\partial x_i} \frac{\partial x_i}{\partial u^\alpha} + \frac{\partial G}{\partial t} \frac{\partial t}{\partial u^\alpha} = \frac{\partial G}{\partial x_i} \frac{\partial x_i}{\partial u^\alpha} = 0, \quad (\alpha = 1, 2).$$

Hence the vector $(\partial G / \partial x_i)$ lies also in the normal of the envelope. This completes the proof.

We now state an important property of the edge of regression of the envelope.

Theorem 85.2. *If the envelope E of the family (85.1) exists and has an edge of regression R then at every point of R the tangents to R and to the corresponding characteristic coincide.*

Proof. Let $\mathbf{x} = \mathbf{x}(v)$ be a representation of the edge of regression R . Then $t = t(v)$ is determined. From

$$G\{x_i(v), t(v)\} = 0, \quad \frac{\partial G\{x_i(v), t(v)\}}{\partial t} = 0, \quad \frac{\partial^2 G\{x_i(v), t(v)\}}{\partial t^2} = 0,$$

cf. (85.6), we find

$$\frac{\partial G}{\partial x_i} \frac{dx_i}{dv} + \frac{\partial G}{\partial t} \frac{dt}{dv} = \frac{\partial G}{\partial x_i} \frac{dx_i}{dv} = 0, \quad \frac{\partial^2 G}{\partial t \partial x_i} \frac{dx_i}{dv} + \frac{\partial^2 G}{\partial t^2} \frac{dt}{dv} = \frac{\partial^2 G}{\partial t \partial x_i} \frac{dx_i}{dv} = 0.$$

Hence the vectors $(\partial G / \partial x_i)$ and $(\partial^2 G / \partial t \partial x_i)$ lie in the normal plane of R . Consequently, if the vector

$$(85.8) \quad \left(\frac{\partial G}{\partial x_i} \right) \times \left(\frac{\partial^2 G}{\partial t \partial x_i} \right)$$

is not the null vector it is tangent to R . Let $\mathbf{x} = \mathbf{x}(u)$ be the characteristic corresponding to a fixed value of t . Then from

$$G\{x_i(u), t\} = 0, \quad \frac{\partial G\{x_i(u), t\}}{\partial t} = 0,$$

cf. (85.5), we obtain the relations

$$\frac{\partial G}{\partial x_i} \frac{dx_i}{du} = 0, \quad \frac{\partial^2 G}{\partial t \partial x_i} \frac{dx_i}{du} = 0.$$

Hence the vector (85.8) is also tangent to the characteristic at the points of the edge of regression. This completes the proof.

Problem 85.1. Obviously one of the two principal curvatures of a canal surface of spheres of constant radius is constant. Prove that a surface with this property is always a canal surface of spheres of constant radius.

86. Developable surfaces as envelopes of families of planes. We will now apply the results obtained in the preceding section to the important

particular case of one-parameter families of planes. Such a family can be represented in the form

$$(86.1) \quad G(\mathbf{x}, t) = \mathbf{a}(t) \cdot \mathbf{x} + c(t) = 0,$$

where $\mathbf{a}(t)$ is a unit vector normal to the corresponding plane. If

$$\frac{d\mathbf{a}}{dt} \equiv \mathbf{a}' \equiv \mathbf{0}$$

then the family consists of parallel planes and does not have an envelope; we will exclude this case from our further investigation and even assume that $\mathbf{a}' \neq \mathbf{0}$ for all values of t . From (86.1) we obtain

$$(86.2) \quad \frac{\partial G}{\partial t} = \mathbf{a}' \cdot \mathbf{x} + c' = 0$$

and

$$(86.3) \quad \frac{\partial^2 G}{\partial t^2} = \mathbf{a}'' \cdot \mathbf{x} + c'' = 0$$

where primes denote derivatives with respect to t . In consequence of the results obtained in the preceding section the characteristics of the family (86.1) are the intersections of the planes defined by (86.1) and (86.2); these are straight lines. If the determinant

$$(86.4) \quad D = |\mathbf{a} \ \mathbf{a}' \ \mathbf{a}''|$$

does not vanish identically the system of the equations (86.1)–(86.3) has a solution, say

$$(86.5) \quad \mathbf{x} = \mathbf{h}(t);$$

if this is a curve, it is the edge of regression of the envelope of the family (86.1). In consequence of Theorems 85.1 and 85.2 the envelope is the tangent surface of this edge of regression. If, however, $\mathbf{h}(t) = \mathbf{h}_0 = \text{const.}$ then, by (86.1),

$$\mathbf{a} \cdot \mathbf{h}_0 + c = 0,$$

that is, every plane of the family (86.1) passes through the point \mathbf{h}_0 ; consequently, in this particular case the envelope is a cone with apex at \mathbf{h}_0 or, if the planes of the family (86.1) have a straight line in common the envelope degenerates into this straight line. In this degenerate case the planes form a pencil and all characteristics coincide. Hence the line of intersection of all planes (86.1) with the plane (86.2) must be independent of t and must therefore be identical with the straight line defined by (86.2) and (86.3). This case occurs if and only if the rank of the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & c \\ a'_1 & a'_2 & a'_3 & c' \\ a''_1 & a''_2 & a''_3 & c'' \end{pmatrix}$$

is less than 3. If the rank of M is equal to 3 and $D = 0$ we obtain a cylinder as envelope; this can be proved as follows: The determinant D vanishes if and only if the vectors \mathbf{a} , \mathbf{a}' , and \mathbf{a}'' are linearly dependent, that is, lie in the same plane. We denote by \mathbf{m} the unit normal vector to this plane. Then, if $D = 0$,

$$(86.6) \quad (\text{a}) \quad \mathbf{m} \cdot \mathbf{a} = 0, \quad (\text{b}) \quad \mathbf{m} \cdot \mathbf{a}' = 0, \quad (\text{c}) \quad \mathbf{m} \cdot \mathbf{a}'' = 0.$$

If we differentiate (86.6 a), in consequence of (86.6 b) we find

$$(86.7) \quad \mathbf{m}' \cdot \mathbf{a} = 0.$$

If we differentiate (86.6 b) in consequence of (86.6 c) we have

$$(86.8) \quad \mathbf{m}' \cdot \mathbf{a}' = 0.$$

From (86.6 a) and (86.7) we see that \mathbf{a} is orthogonal to \mathbf{m} as well as to \mathbf{m}' . From (86.6 b) and (86.8) it follows that the same is true with respect to \mathbf{a}' . Since \mathbf{m} is a unit vector, \mathbf{m}' cannot be parallel to \mathbf{m} . In order that the equations (86.6 a), (86.6 b), (86.7), and (86.8) be consistent, $\mathbf{m}' = 0$ must be true, that is, $\mathbf{m} = \mathbf{m}_0 = \text{const}$. Hence, by (86.6 a) for all values of t the normal vector \mathbf{a} to the planes (86.1) is orthogonal to the constant vector \mathbf{m}_0 . This means that the planes (86.1) and consequently their characteristics also are parallel to \mathbf{m}_0 . From this we obtain the statement that if the rank of M is equal to 3 and $D = 0$ the envelope of (86.1) is a cylinder.

We sum up our results as follows:

Theorem 86.1. *If the envelope of a one-parameter family of planes is a surface then it is always a developable one.*

Furthermore we have

Theorem 86.2. *If the planes of the family (86.1) are not parallel, if there does not exist a point common to all planes and if $D \neq 0$, cf. (86.5), then the envelope of this family is a tangent surface, namely the tangent surface of its edge of regression.*

87. Envelope of the osculating, normal, and rectifying planes of a curve, polar surface. As was shown in Section 19, to every point of a curve $\mathbf{y}(s)$ with non-vanishing curvature κ we can associate a polar axis. The polar axis can be represented in the form

$$(87.1) \quad \mathbf{x}(t) = \mathbf{y} + \rho \mathbf{p} + t \mathbf{b}, \quad (\kappa > 0)$$

cf. (19.10), where $\rho = 1/\kappa$ is the radius of curvature of $\mathbf{y}(s)$; \mathbf{p} and \mathbf{b} are the unit principal and unit binormal vectors, and the real variable t is the parameter of the representation (87.1). The polar axis is the locus of the centres of the spheres which have a contact of second order (at least) with $\mathbf{y}(s)$ at

the corresponding point. Furthermore we found that, in general, to every point of $\mathbf{y}(s)$ there is exactly one sphere which has a contact of third order with $\mathbf{y}(s)$ at that point; the centre of this sphere has the position vector

$$(87.2) \quad \mathbf{x} = \mathbf{y} + \rho \mathbf{p} + \frac{\dot{\rho}}{\tau} \mathbf{b}, \quad (\kappa > 0, \tau \neq 0).$$

In general the polar axes of a curve generate a surface which is called the *polar surface* of this curve. In consequence of (87.1) the polar surface of a curve $\mathbf{y}(s)$ can be represented in the form

$$(87.1') \quad \mathbf{x}(s, t) = \mathbf{y}(s) + \rho(s) \mathbf{p}(s) + t \mathbf{b}(s), \quad (\kappa > 0);$$

the arc length s of $\mathbf{y}(s)$ and the parameter t are the coordinates on this surface.

Example. The polar surface of a plane curve is a cylinder because the unit binormal vector of such a curve is constant. The polar surface of a spherical curve (cf. Problem 19.3) is a cone with apex at the centre of the sphere on which the curve lies. The polars of a circle coincide, they are orthogonal to the plane of the circle and pass through its centre; hence these polars do not generate a surface.

Theorem 87.1. *The envelope E of the normal planes to a curve C of class $r \geq 2$ with positive curvature κ is identical with the polar surface of this curve. If C is of class $r^* \geq 3$ and if its torsion τ does not vanish then E has an edge of regression R , and R is the locus of the centres of the osculating spheres of the curve C .*

Proof. Let $\mathbf{t}(s)$, $\mathbf{p}(s)$, and $\mathbf{b}(s)$ be the vectors of the moving trihedron of $C: \mathbf{y}(s)$. The normal planes to C are orthogonal to the unit tangent vector $\mathbf{t}(s)$. Hence we can represent the family of these planes in the form

$$(87.3) \quad G(\mathbf{x}, s) = [(\mathbf{x} - \mathbf{y}(s)) \cdot \mathbf{t}(s)] = 0,$$

cf. (13.2 a). By means of (15.1) we find

$$\frac{\partial G}{\partial s} = -\mathbf{t} \cdot \mathbf{t} - (\mathbf{x} - \mathbf{y}) \cdot \kappa \mathbf{p} = 0$$

or, since $\mathbf{t} \cdot \mathbf{t} = 1$ and $\kappa = 1/\rho > 0$,

$$(87.4) \quad (\mathbf{x} - \mathbf{y}) \cdot \mathbf{p} = \rho.$$

In consequence of (87.3) we can represent the vector $\mathbf{x} - \mathbf{y}$ in the form

$$(87.5) \quad \mathbf{x} - \mathbf{y} = q \mathbf{p} + t \mathbf{b}.$$

Inserting (87.5) into (87.4) we find $q = \rho$. Consequently (87.5) becomes identical with (87.1). Hence the characteristics of the family of the normal

planes to C are the polar axes of C , and the envelope of the family under consideration is identical with the polar surface of C . In order to determine the edge of regression of the envelope we differentiate (87.4) with respect to s ,

$$-\mathbf{t} \cdot \mathbf{p} + (\mathbf{x} - \mathbf{y}) \cdot \dot{\mathbf{p}} = \dot{\rho}.$$

In consequence of (87.3), (15.1), and $\mathbf{t} \cdot \mathbf{p} = 0$ we obtain from this relation

$$(87.6) \quad (\mathbf{x} - \mathbf{y}) \cdot \tau \mathbf{b} = \dot{\rho}.$$

Inserting (87.1') into (87.6) and taking $\mathbf{p} \cdot \mathbf{b} = 0$, $\mathbf{b} \cdot \mathbf{b} = 1$ into account we have

$$t = \frac{\dot{\rho}}{\tau}, \quad (\tau \neq 0).$$

When inserting this (87.1') takes the form (87.2); hence the edge of regression of the envelope of the normal planes of C is the locus of the centres of the osculating spheres of C .

We may also formulate the second statement of Theorem 87.1 in the following manner: The polar surface of a curve C is the tangent surface of the curve generated by the centres of the osculating spheres of C .

By comparing (87.1') and (22.3) we see that the evolutes of a curve lie on its polar surface; through every point of this surface there passes one and only one evolute.

By similar considerations of the envelopes of the osculating planes and of the rectifying planes we obtain the following results:

Theorem 87.2. *The envelope of the osculating planes of a curve of class $r \geq 3$ with non-vanishing curvature and torsion is identical with the tangent surface of this curve.*

Theorem 87.3. *Every generator of the envelope of the rectifying planes of a curve of class $r \geq 3$ with non-vanishing curvature passes through a point of the curve and has the direction of the corresponding vector of Darboux.*

The envelope of the rectifying planes of a curve is known as its *rectifying developable surface*; this name is justified by the following fact: If this surface is bent into a plane the curve becomes a straight line, that is, is ‘rectified’.

Problem 87.1. Prove Theorem 87.2.

Problem 87.2. Prove Theorem 87.3.

Problem 87.3. Prove that if the rectifying developable surface is mapped isometrically in the plane the corresponding curve becomes a straight line.

88. Centre surfaces of a surface. We have seen in Section 58 that we can associate three ruled surfaces to every given space curve. One of them, the tangent surface, is always developable while the other two, the principal normal surface and the binormal surface, are developable if and only if the curve is plane. Similarly to every curve $C: \mathbf{x}(s)$ on a surface $S: \mathbf{x}(u^1, u^2)$ we can associate the ruled surface

$$(88.1) \quad \mathbf{y}(s, t) = \mathbf{x}(s) + t\mathbf{n}(s)$$

generated by the normals to S along C . In (88.1) s is the arc length of C and $\mathbf{n}(s)$ is the unit normal vector to S at the points of C .

If C is an arbitrary curve on S , (88.1) will not be a developable surface, in general. We may illustrate this fact by the simple example of a circular helix $C: \mathbf{x}(t) = (r \cos t, r \sin t, ct)$ on the cylinder of revolution $S: \mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2)$. Since at every point of C the unit normal vector \mathbf{n} to S lies in the corresponding principal normal to the helix, the surface (88.1) is identical with the principal normal surface of C which is not a developable surface, cf. Theorem 58.2.

Theorem 88.1. *The normals to a surface $S: \mathbf{x}(u^1, u^2)$ corresponding to a curve $\mathbf{x}(s)$ of class $r \geq 2$ on S generate a developable surface if and only if C is a line of curvature on S .*

Proof. The normals to S at the points of C generate a ruled surface R which can be represented in the form (88.1). By (58.2) R is a developable surface if and only if

$$(88.2) \quad |\dot{\mathbf{x}} \cdot \mathbf{n} \cdot \dot{\mathbf{n}}| = 0.$$

We assume the coordinates u^1, u^2 to be chosen so that the coordinate curves are lines of curvature on S . Then

$$(88.3) \quad \dot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha, \quad \dot{\mathbf{n}} = \mathbf{n}_\alpha \dot{u}^\alpha = -\kappa_\alpha \mathbf{x}_\alpha \dot{u}^\alpha,$$

cf. (45.1'). Inserting these expressions in (88.2) and simplifying the determinant we obtain

$$|\dot{\mathbf{x}} \cdot \mathbf{n} \cdot \dot{\mathbf{n}}| = |\mathbf{x}_\alpha \dot{u}^\alpha \quad \kappa_\beta \mathbf{x}_\beta \dot{u}^\beta \quad \mathbf{n}| = |\mathbf{x}_\alpha \dot{u}^\alpha \quad (\kappa_2 - \kappa_1) \mathbf{x}_2 \dot{u}^2 \quad \mathbf{n}|;$$

hence

$$(88.4) \quad |\dot{\mathbf{x}} \cdot \mathbf{n} \cdot \dot{\mathbf{n}}| = \dot{u}^1 \dot{u}^2 (\kappa_2 - \kappa_1) |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{n}| = 0.$$

Since \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{n} are linearly independent vectors the determinant on the right-hand side of (88.4) does not vanish. Hence, in order that (88.4) be satisfied, we must have $\dot{u}^1 = 0$, that is, $u^1 = \text{const.}$ or $\dot{u}^2 = 0$, that is, $u^2 = \text{const.}$ or $\kappa_2 - \kappa_1 = 0$. In consequence of the choice of the coordinates u^1, u^2 the first conditions mean that C is a line of curvature on S . The third condition is satisfied if and only if the corresponding point is a navel point,

cf. Sections 41 and 48; at these points the normal curvature is the same for all directions. In an analogous manner the converse can be proved.

The lines of curvature form an orthogonal net, except at navel points, cf. Theorem 41.2. Hence, in general, every normal to S is a generator of two of the developable surfaces characterized in Theorem 88.1 which intersect orthogonally. The net of the lines of curvature consists of two families

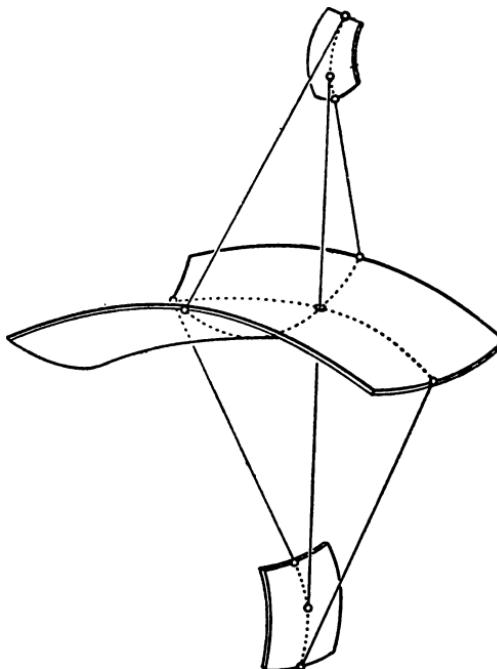


FIG. 94. Portions of the centre surfaces of a portion of a surface whose Gaussian curvature is negative.

of curves. We consider one of those families. To every curve of this family there corresponds one of the above developable surfaces. If each of these surfaces has an edge of regression, and if these edges of regression generate a surface, this surface is called a *centre surface* of the given surface S . Analogously another centre surface of S corresponds to the other family of lines of curvature of S ; hence there are two centre surfaces of S , cf. Fig. 94.

In order to justify the name 'centre surface' we prove the following

Theorem 88.2. *The centre surfaces of a surface are the loci of the centre of the circles of principal curvature.*

Proof. The normals of a surface S corresponding to a line of curvature on S generate a developable surface, cf. Theorem 88.1 which may be represented in the form (88.1). We have to determine the edge of regression G of this developable surface, that is, we have to find a function $t = t(s)$ of class $r \geq 1$ which corresponds to G . If such a function has been determined we can represent G in the form

$$(88.5) \quad \mathbf{y}(s) = \mathbf{x}(s) + t(s)\mathbf{n}(s).$$

Differentiating (88.5) with respect to s we obtain the vector

$$(88.6) \quad \dot{\mathbf{y}} = \dot{\mathbf{x}} + t\mathbf{n} + t\dot{\mathbf{n}}$$

which is tangent to G . If $\dot{\mathbf{n}} = \mathbf{0}$, that is, $\mathbf{n} = \text{const.}$ then (88.1) is a plane or a cylinder and has no edge of regression; we may exclude this case from our further consideration. We assume that the coordinates on S have been chosen so that the coordinate curves are lines of curvature on S and that, in particular, the family of lines of curvature under consideration are the curves $u^2 = \text{const.}$ Then, by (45.1'),

$$\dot{\mathbf{n}} = \mathbf{n}_1 \dot{u}^1 = -\frac{1}{R_1} \mathbf{x}_1 \dot{u}^1, \quad (R_1 = 1/\kappa_1).$$

Since $\dot{\mathbf{x}} = \mathbf{x}_1 \dot{u}^1$ on the lines $u^2 = \text{const.}$, (88.6) takes the form

$$(88.7) \quad \dot{\mathbf{y}} = \left(1 - \frac{t}{R_1}\right) \mathbf{x}_1 \dot{u}^1 + t\mathbf{n}.$$

The generators of the developable surface have the same direction as the unit normal vectors $\mathbf{n}(s)$ and are tangent to G . Consequently the vectors $\dot{\mathbf{y}}$ and \mathbf{n} are linearly dependent. Hence, in (88.7), $1 - (t/R_1) = 0$ must be true, that is, $t = R_1$. Thus (88.5) takes the form

$$(88.8) \quad \mathbf{y} = \mathbf{x} + R_1 \mathbf{n}.$$

Analogous reasoning in the case of the other family of lines of curvature leads to $t = R_2$. This completes the proof of Theorem 88.2.

We will now investigate some further properties of the lines of curvature and centre surfaces and start with a preliminary consideration.

Let $C: \mathbf{x}(s)$ be a curve on a surface $S: \mathbf{x}(u^1, u^2)$. Let $\mathbf{t} = \dot{\mathbf{x}}$ be the unit tangent vector, \mathbf{p} the unit principal normal vector, τ the torsion, and τ_g the geodesic torsion of C (cf. Section 50). We consider any fixed point P of C . Let γ denote the directed angle from the vector \mathbf{p} at P to the unit normal vector \mathbf{n} of S at P , angles being measured from \mathbf{p} towards the unit binormal vector \mathbf{b} to C at P . By (50.5) we have

$$(88.9) \quad \tau_g \mathbf{n} = \dot{\mathbf{x}} \times \ddot{\mathbf{x}}.$$

The unit binormal vector to C is

$$\mathbf{b} = \dot{\mathbf{x}} \times \mathbf{p},$$

$$\text{cf. (13.1), and } \cos \gamma = \mathbf{p} \cdot \mathbf{n}, \quad \sin \gamma = \mathbf{b} \cdot \mathbf{n}.$$

By differentiating this last relation and using (13.1), (15.1), and (88.9) we have

$$\dot{\gamma} \cos \gamma = \dot{\mathbf{b}} \cdot \mathbf{n} + \mathbf{b} \cdot \dot{\mathbf{n}} = -\tau \mathbf{p} \cdot \mathbf{n} + (\dot{\mathbf{x}} \times \mathbf{p}) \cdot \dot{\mathbf{n}} = (-\tau + \tau_g) \cos \gamma.$$

If the direction of C at P is not an asymptotic direction then $\cos \gamma \neq 0$, and we obtain the following formula of Bonnet:

$$(88.10) \quad \tau_g = \tau + \dot{\gamma}.$$

Theorem 88.3 (O. Bonnet [2]). (a) If two surfaces of class $r \geq 2$ intersect along a curve C which is a line of curvature on one of these surfaces and if the angle of intersection is constant then C is a line of curvature also on the other surface. (b) If two surfaces of class $r \geq 2$ intersect along a curve C which is a line of curvature on both surfaces then the angle of intersection is constant.

Proof. (a) Let γ and γ^* be the normal angles of C for the two surfaces. We have

$$\gamma - \gamma^* = \text{const.}$$

and therefore

$$(88.11) \quad \dot{\gamma} = \dot{\gamma}^*,$$

where the dot denotes the derivative with respect to the arc length s of the curve. Let τ_g and τ_g^* be the geodesic torsions of C for the two surfaces. From (88.10) and (88.11) we have

$$\tau_g = \tau_g^*.$$

From this and Theorem 50.3 the statement (a) follows.

(b) Using the above notations we have from (88.10)

$$\dot{\gamma} + \tau = 0, \quad \dot{\gamma}^* + \tau = 0$$

or

$$\dot{\gamma} - \dot{\gamma}^* = 0.$$

Hence

$$\gamma - \gamma^* = \text{const.}$$

Thus the surfaces intersect at a constant angle.

Theorem 88.4 (Joachimsthal [1]). *The angle of intersection of a surface S of class $r \geq 3$ and a plane or sphere is constant if and only if the curve of intersection is a line of curvature on S .*

In general the principal curvatures $1/R_1$ and $1/R_2$ of a surface are independent of each other. A surface whose principal curvatures satisfy a relation of the form

$$(88.12) \quad F(R_1, R_2) = 0$$

is called a *surface of Weingarten*, cf. Weingarten [1]. Minimal surfaces are surfaces of Weingarten, for example.

Theorem 88.5. *The centre surfaces (corresponding to R_1 , for instance) of the surfaces of Weingarten of class $r \geq 3$ defined by one and the same relation (88.12) can be mapped isometrically onto one and the same surface of revolution.*

Problem 88.1. Determine the coefficients of the first fundamental form of the centre surfaces of a surface S in the case where the coordinate curves are lines of curvature on S .

Problem 88.2. Give another proof of Theorem 88.3 (use Theorem 88.1). Prove Theorem 88.4.

Problem 88.3. Prove Theorem 88.5.

89. Parallel surfaces. If segments of constant length are laid off along the normals to a surface S (all along the positive rays of the normals or all along the negative ones) the locus of their end points in general is a surface S^* which is called a *parallel surface of S* .

Obviously if $\mathbf{x}(u^1, u^2)$ is a representation of S ,

$$(89.1) \quad \mathbf{x}^*(u^1, u^2) = \mathbf{x}(u^1, u^2) + a\mathbf{n}(u^1, u^2)$$

is a representation of S^* . In (89.1) \mathbf{n} is the unit normal vector to S and a is a constant. There exist infinitely many parallel surfaces of a given surface, each of which corresponds to a certain value of a .

Parallel planes, concentric spheres, coaxial cylinders, and tori generated by a rotation of concentric circles about the same axis are simple examples of parallel surfaces. In all these cases the surfaces have the same normals. We will prove that this holds for any parallel surfaces.

Theorem 89.1. *Two surfaces of class $r \geq 2$ are parallel surfaces if and only if they have common normals.*

Proof. We start from the representation (89.1) assuming that a is a variable and prove that if S and S^* have the same normals, a must be constant, that is, S^* must be a parallel surface of S ; we will see that a can be any constant, that is, all parallel surfaces of S are obtained in this manner. Since S and S^* are assumed to have the same normal, $\mathbf{n} \cdot \mathbf{x}_a^* = 0$

must be true, and so

$$(89.2) \quad \mathbf{n} \cdot (\mathbf{x}_\alpha + a_\alpha \mathbf{n} + a \mathbf{n}_\alpha) = 0, \quad a_\alpha \equiv \frac{\partial a}{\partial u^\alpha}, \quad (\alpha = 1, 2).$$

Since the vectors \mathbf{x}_α and \mathbf{n}_α are orthogonal to \mathbf{n} , from (89.2) we obtain $a_\alpha = 0$, hence a is a constant whose value may be arbitrarily chosen. Every surface S^* parallel to S therefore has the same normal as S .

This property could have been taken as a definition of a surface parallel to a given one. To this extent the concept of parallel surfaces is an analogue of that of Bertrand curves.

From Theorem 88.1 we find that the lines of curvature on parallel surfaces correspond to each other, that is, these curves have the same parametric representation on all parallel surfaces if the coordinates u^1, u^2 have been chosen according to (89.1). Moreover, it follows immediately from the investigation in the preceding section that parallel surfaces have the same centre surfaces.

Theorem 89.2 (O. Bonnet [4]). *Let S be a surface of constant Gaussian curvature $K = 1/r^2$. Then there are two surfaces parallel to S which have the constant mean curvature $H = \pm 1/2r$; the (directed) distances of these surfaces from S are $\mp r$, respectively.*

Proof. The centre surfaces of a surface are the loci of the centres of principal curvature. Parallel surfaces have the same centre surfaces; consequently, if $\kappa_1 = 1/R_1$ and $\kappa_2 = 1/R_2$ are the principal curvatures of a surface S then the principal curvatures of a surface S^* parallel to S at the distance r are given by the expressions

$$\kappa_\alpha^* = \frac{1}{R_\alpha^*} = \frac{1}{R_\alpha - r}, \quad (\alpha = 1, 2).$$

By hypothesis $R_1 R_2 = 1/K = r^2$; consequently S^* has the mean curvature

$$H^* = \frac{1}{2}(\kappa_1^* + \kappa_2^*) = \frac{1}{2}\left(\frac{1}{R_1 - r} + \frac{1}{R_2 - r}\right) = -\frac{1}{2r} = \text{const.}$$

In an analogous manner we find that the mean curvature of the surface parallel to S at the (directed) distance $-r$ is

$$H = \frac{1}{2r} = \text{const.}$$

As an immediate consequence of this theorem we obtain

Theorem 89.3. *To a given surface of constant mean curvature there exists a parallel surface of constant mean curvature and another one of positive constant Gaussian curvature.*

As follows from (89.1) the coefficients of the first and second fundamental forms of a surface parallel to a given surface can be represented in terms of the three fundamental forms of the latter:

$$(89.3) \quad g_{\alpha\beta}^* = g_{\alpha\beta} - 2ab_{\alpha\beta} + a^2c_{\alpha\beta}, \quad b_{\alpha\beta}^* = b_{\alpha\beta} - ac_{\alpha\beta},$$

cf. (27.1), (38.4), and (59.3).

90. Surfaces of constant Gaussian curvature. Surfaces of constant Gaussian curvature $K \neq 0$ are of great interest since the intrinsic geometry of these surfaces exhibits properties similar to that of the plane. We will later prove that this intrinsic geometry is locally identical with the so-called non-Euclidean geometry which has been developed from quite another point of view; cf. Section 96.

We first consider the surfaces of constant Gaussian curvature in general. We introduce geodesic polar coordinates u^1, u^2 on the surface S under consideration. (Depending on S , these coordinates may be valid on the entire surface or on a portion of S only.) Then the first fundamental form is given by (52.1),

$$ds^2 = (du^1)^2 + g_{22}(u^1, u^2)(du^2)^2,$$

where

$$(90.1) \quad g_{22}(0, u^2) = 0 \quad \text{and} \quad \left. \frac{\partial \sqrt{g_{22}}}{\partial u^1} \right|_{u^1=0} = 1,$$

cf. (53.1). Furthermore, by (52.2),

$$K = -\frac{1}{\sqrt{g_{22}}} \frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2}.$$

This is a linear differential equation of second order which can be written in the form

$$(90.2) \quad \frac{\partial^2 B}{(\partial u^1)^2} + KB = 0, \quad B = \sqrt{g_{22}} = \sqrt{g}.$$

We consider the solutions of this equation for constant K .

(A) If $K = 0$ the solution of (90.2) is

$$B = C_1(u^2)u^1 + C_2(u^2).$$

From (90.1) we have $C_1 = 1, C_2 = 0$. Hence, in this case

$$(90.3) \quad ds^2 = (du^1)^2 + (u^1)^2(du^2)^2, \quad (K = 0)$$

as it must be for polar coordinates in the plane.

(B) If $K > 0$ then

$$B = C_1(u^2)\cos(\sqrt{K}u^1) + C_2(u^2)\sin(\sqrt{K}u^1)$$

is the solution of (90.2). In consequence of (90.1) we have $C_1 = 0$ and $C_2 = 1/\sqrt{K}$. Therefore

$$g_{22} = B^2 = \frac{1}{K} \sin^2(\sqrt{K} u^1)$$

and

$$(90.4) \quad ds^2 = (du^1)^2 + \frac{1}{K} \sin^2(\sqrt{K} u^1) (du^2)^2, \quad (K > 0).$$

(C) In the case of negative constant Gaussian curvature $K < 0$ the solution of (90.2) takes the form

$$B = C_1(u^2) \cosh\{\sqrt{(-K)} u^1\} + C_2(u^2) \sinh\{\sqrt{(-K)} u^1\},$$

where the hyperbolic functions are defined as follows:

$$\cosh x = \cos ix = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = -i \sin ix = \frac{1}{2}(e^x - e^{-x}),$$

as is well known. In consequence of (90.1) we have $C_1 = 0$ and $C_2 = 1/\sqrt{(-K)}$. Hence

$$(90.5) \quad ds^2 = (du^1)^2 + \frac{1}{-K} \sinh^2\{\sqrt{(-K)} u^1\} (du^2)^2, \quad (K < 0),$$

cf. H. S. M. Coxeter [1, § 15.5]. Planes and spheres are simple examples of surfaces of constant Gaussian curvature $K = 0$ and $K > 0$, respectively. Examples of surfaces of negative constant Gaussian curvature will be considered in Section 93.

Problem 90.1. Introduce coordinates on a sphere so that the first fundamental form is given by (90.4).

91. Isometric mapping of surfaces of constant Gaussian curvature. The concept of isometric mapping and related concepts were introduced in Section 57. The Gaussian curvature K is an invariant under isometric mapping, cf. Theorem 57.2. Hence at corresponding points of two isometric surfaces the Gaussian curvature must necessarily be the same. However, in general, this condition is not sufficient, as can be seen from the following

Example (Staeckel [1] and Wangerin [1]). If the logarithmic curve $x_3 = \log x_1$ is rotated about the x_3 -axis it generates a surface of revolution which can be represented in the form

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, \log u^1), \quad (u^1 > 0).$$

The corresponding first fundamental form is given by the expression

$$(91.1) \quad ds^2 = \left(1 + \frac{1}{(u^1)^2}\right) (du^1)^2 + (u^1)^2 (du^2)^2.$$

This surface has the Gaussian curvature

$$(91.2) \quad K = -\frac{1}{(1+(u^1)^2)^2}.$$

The right helicoid (cf. Section 81) can be represented in the form

$$\mathbf{x}(u^{1*}, u^{2*}) = (u^{1*} \cos u^{2*}, u^{1*} \sin u^{2*}, u^{2*}).$$

To this representation there corresponds the first fundamental form

$$(91.3) \quad ds^{*2} = (du^{1*})^2 + \{1+(u^{1*})^2\}(du^{2*})^2,$$

and the surface has the Gaussian curvature

$$(91.4) \quad K^* = -\frac{1}{(1+(u^{1*})^2)^2}.$$

The mapping $u^{\alpha*} = u^\alpha$ ($\alpha = 1, 2$) is such that at corresponding points the Gaussian curvature of both surfaces is the same, but this mapping is not isometric, cf. Theorem 57.1. We will prove that there does not exist an isometric mapping of one of those surfaces onto the other. In consequence of (91.2) and (91.4) the most general mapping under which the Gaussian curvature at corresponding points is the same must satisfy the condition

$$1+(u^{1*})^2 = \pm(1+(u^1)^2).$$

Hence, either

$$(91.5) \quad (a) \quad u^{1*} = \pm u^1 \quad \text{or} \quad (b) \quad u^{1*} = \sqrt{-2-(u^1)^2}$$

must be true while u^{2*} can be an arbitrary function of u^1 and u^2 such that its first partial derivative with respect to u^2 does not vanish,

$$(91.6) \quad u^{2*} = \phi(u^1, u^2), \quad \phi_2 \neq 0.$$

where $\phi_\alpha \equiv \partial\phi/\partial u^\alpha$. By inserting (91.5 a) and (91.6) into (91.3) we obtain

$$ds^{*2} = (du^1)^2 + [1+(u^1)^2](\phi_1 du^1 + \phi_2 du^2)^2.$$

In order that this expression should be identical with (91.1) the conditions

$$(91.7) \quad (a) \quad 1+[1+(u^1)^2](\phi_1)^2 = 1+\frac{1}{(u^1)^2}, \quad (b) \quad \phi_1 \phi_2 = 0,$$

$$(c) \quad [1+(u^1)^2](\phi_2)^2 = (u^1)^2$$

should be satisfied. From (91.6) and (91.7 b) we have $\phi_1 = 0$. Applying this relation to (91.7 a) we obtain a contradiction. If we insert (91.5 b) and (91.6) into (91.3) we find

$$ds^{*2} = -\frac{(u^1)^2}{2+(u^1)^2}(du^1)^2 - [1+(u^1)^2](\phi_1 du^1 + \phi_2 du^2)^2.$$

In order that this expression should become identical with (91.1) the relations

$$-\frac{(u^1)^2}{2+(u^1)^2} - [1+(u^1)^2](\phi_1)^2 = 1 + \frac{1}{(u^1)^2}, \quad \phi_1 \phi_2 = 0,$$

$$-[1+(u^1)^2](\phi_2)^2 = (u^1)^2$$

should hold; in this case we also obtain a contradiction.

Surfaces of constant Gaussian curvature K have the important property that if K is equal the surfaces are isometric. In the case $K = 0$ this fact has already been stated, cf. Theorem 59.3. In the other two cases, $K > 0$ and $K < 0$ it is an immediate consequence of (90.4) and (90.5): The coefficients of those fundamental forms are completely determined by K ; hence if two surfaces have the same constant curvature K the first fundamental forms are also the same. In consequence of the fact that (90.3)–(90.5) were obtained by using geodesic coordinates which, in general, are allowable on a certain portion of a surface we may formulate our result as follows:

Theorem 91.1 (F. Minding [1]). *Two (sufficiently small) portions of surfaces of class $r \geq 3$ having the same constant Gaussian curvature are isometric.*

92. Spherical surfaces of revolution. A surface of positive constant Gaussian curvature is called a *spherical surface*. A surface of negative constant Gaussian curvature is called a *pseudospherical surface*.

In this and the following section we shall consider spherical and pseudospherical surfaces of revolution. In consequence of Theorem 91.1 the intrinsic geometry of surfaces of the same constant Gaussian curvature is locally identical. However, the geometric form of these surfaces as viewed from the embedding space may be different as we will now see.

We start from a representation of a surface S of revolution of the form

$$(92.1) \quad \mathbf{x}(\bar{u}^1, \bar{u}^2) = (\bar{u}^1 \cos \bar{u}^2, \bar{u}^1 \sin \bar{u}^2, h(\bar{u}^1)).$$

The corresponding first fundamental form is given by the expression

$$(92.2) \quad ds^2 = (1+h'^2)(d\bar{u}^1)^2 + (\bar{u}^1)^2(d\bar{u}^2)^2,$$

where a prime denotes the derivative with respect to \bar{u}^1 . We now introduce geodesic parallel coordinates u^1, u^2 on S so that the first fundamental form becomes

$$ds^2 = (du^1)^2 + g_{22}(u^1, u^2)(du^2)^2,$$

cf. (52.1), and the coordinate curve $u^1 = 0$ is a geodesic. Then the curves $u^2 = \text{const.}$ are geodesics orthogonal to the curve $u^1 = 0$ and the curves

$u^1 = \text{const.}$ are geodesic parallels of the curve $u^1 = 0$. We set

$$K = \frac{1}{c^2} = \text{const.} \quad \text{and} \quad \sqrt{g_{22}} = B.$$

Then the differential equation (90.2) takes the form

$$(92.3) \quad \frac{\partial^2 B}{(\partial u^1)^2} + \frac{B}{c^2} = 0.$$

Since c is a constant (92.3) has the solution

$$\sqrt{g_{22}} = B = C_1(u^2) \cos \frac{u^1}{c} + C_2(u^2) \sin \frac{u^1}{c}.$$

Since the geodesic curvature of the curve $u^1 = 0$ vanishes identically we obtain from (49.9 a)

$$\left. \frac{\partial B}{\partial u^1} \right|_{u^1=0} = 0, \quad \text{and therefore } C_2 = 0.$$

If u^2 is chosen proportional to the arc length σ of the curve $u^1 = 0$, that is, $u^2 = \sigma/\lambda$, the first fundamental form becomes

$$(92.4) \quad ds^2 = (du^1)^2 + \lambda^2 \cos^2 \frac{u^1}{c} (du^2)^2$$

where λ is a constant, not zero. (92.2) and (92.4) become identical if

$$(92.5) \quad (1+h'^2)(d\bar{u}^1)^2 = (du^1)^2, \quad \bar{u}^1 = \lambda \cos \frac{u^1}{c}, \quad \bar{u}^2 = u^2.$$

From this it follows that

$$h' = \sqrt{\left(\left(\frac{du^1}{d\bar{u}^1} \right)^2 - 1 \right)}$$

and
$$\frac{d\bar{u}^1}{du^1} = -\frac{\lambda}{c} \sin \frac{u^1}{c};$$

hence, altogether

$$(92.6) \quad x_3 = \int \sqrt{\left(1 - \frac{\lambda^2}{c^2} \sin^2 \frac{u^1}{c} \right)} du^1.$$

If $\lambda \neq c$ this integral is elliptic; its function values can be found in the book *Tables of Functions* by E. Jahnke and F. Emde (New York, 1945). The surface of revolution under consideration is now represented in the form

$$(92.7) \quad x(u^1, u^2) = \left(\lambda \cos \frac{u^1}{c} \cos u^2, \lambda \cos \frac{u^1}{c} \sin u^2, \int \sqrt{\left(1 - \frac{\lambda^2}{c^2} \sin^2 \frac{u^1}{c} \right)} du^1 \right),$$

$(c = 1/\sqrt{K}, K > 0, \lambda > 0),$

where u^1, u^2 are the above geodesic parallel coordinates. In order that $K > 0$ the meridians of the surface must be of such a form that their unit

principal normal vector when bound at a point of the meridian is directed towards the axis of revolution. We obtain three different types of spherical surfaces of revolution corresponding to the values $\lambda = c$, $\lambda > c$, and $\lambda < c$, respectively.

I. If $\lambda = c$, (92.7) takes the form

$$(92.8) \quad \mathbf{x}(u^1, u^2) = \left(c \cos \frac{u^1}{c} \cos u^2, c \cos \frac{u^1}{c} \sin u^2, c \sin \frac{u^1}{c} \right).$$

This is a representation of a sphere of radius c with centre at the origin of the coordinate system in space.

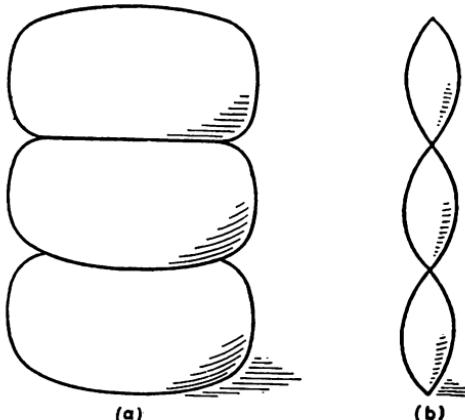


FIG. 95. (a) Hyperbolic, and (b) elliptic spherical surface of revolution

II. If in (92.7) $\lambda > c$ we obtain a surface which is called a *hyperbolic spherical surface of revolution*, cf. Fig. 95 a. In this case x_3 is a periodic function; the surface consists of a succession of congruent portions called *zones* each of which is bounded by two circles parallel to the $x_1 x_2$ -plane. The coordinate u^1 is the radius of the circles parallel to the $x_1 x_2$ -plane. From (92.5) we have $u^1 \leqslant \lambda$. In order that x_3 be real we must have

$$\sin^2 \frac{u^1}{c} \leqslant \frac{c^2}{\lambda^2}$$

in (92.6). By (92.5) we obtain from this

$$(92.9) \quad u^1 \geqslant \sqrt{(\lambda^2 - c^2)}.$$

Hence $u^1 = \sqrt{(\lambda^2 - c^2)}$ is the radius of the above circles by which every zone is bounded. The maximum value, $u^1 = \lambda$, corresponds to the circle in the middle of every zone.

III. If in (92.7) $\lambda < c$ we obtain a surface which is called an *elliptic spherical surface of revolution*. In this case the radii of the circles parallel to the x_1x_2 -plane can have every value of u^1 between 0 and λ . x_3 is again a periodic function; each zone of the surface is now bounded by two points, cf. Fig. 95 b.

We finally mention the isometric mappings of spherical surfaces into each other, cf. Theorems 57.1 and 91.1. Let S_1 and S_2 be two such surfaces having the same (constant) Gaussian curvature corresponding to values $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. The first fundamental forms are

$$(S_1): \quad ds^2 = (du^1)^2 + \lambda_1^2 \cos^2 \frac{u^1}{c} (du^2)^2$$

and $(S_2): \quad ds^{*2} = (du^{1*})^2 + \lambda_2^2 \cos^2 \frac{u^{1*}}{c} (du^{2*})^2.$

Setting $du^{1*} = du^1, \quad \lambda_2 du^{2*} = \lambda_1 du^2$

these two expressions become identical; hence the isometric mapping of S_1 into S_2 is given by the formulae

$$(92.10) \quad u^{1*} = u^1, \quad u^{2*} = \frac{\lambda_1}{\lambda_2} u^2.$$

The image of the portion of S_1 consisting of all points with coordinates u^1, u^2 ,

$$a \leq u^1 \leq b, \quad 0 \leq u^2 < 2\pi,$$

consists of all points of S_2 with coordinates u^{1*}, u^{2*} ,

$$a \leq u^{1*} \leq b, \quad 0 \leq u^{2*} < 2\pi \frac{\lambda_1}{\lambda_2}.$$

Consequently if $\lambda_1 > \lambda_2$ certain points of S_2 correspond to several points of S_1 . If however, $\lambda_1 < \lambda_2$ then the portion of S_2 , defined by $a \leq u^{1*} \leq b, 0 \leq u^{2*} < 2\pi$, also contains points which are not points of the image of the above portion of S_1 . Under this mapping meridians correspond to meridians as follows from (92.10).

93. Pseudospherical surfaces of revolution. A surface of negative constant Gaussian curvature is called a *pseudospherical surface*.

We will now investigate pseudospherical surfaces of revolution and shall obtain three different types. We set

$$K = -\frac{1}{c^2} (< 0).$$

Then the differential equation (90.2) has the solution

$$(93.1) \quad \sqrt{g_{22}} = B = C_1(u^2) \cosh \frac{u^1}{c} + C_2(u^2) \sinh \frac{u^1}{c}.$$

I. We introduce geodesic parallel coordinates u^1, u^2 so that the coordinate curve $u^1 = 0$ is a curve of constant geodesic curvature $1/c$, the coordinate curves $u^2 = \text{const.}$ are geodesics orthogonal to the curve $u^1 = 0$, and the coordinate curves $u^1 = \text{const.}$ are geodesic parallels of the curve $u^1 = 0$. Since $g_{11} = 1$ we have from (49.9a)

$$(\kappa_g)_{u^1=0} = \frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{22}}}{\partial u^1} = \frac{1}{c}, \quad (c > 0).$$

From (93.1) we obtain $\sqrt{g_{22}}(0, u^2) = C_1(u^2)$

$$\text{and } \left. \frac{\partial \sqrt{g_{22}}}{\partial u^1} \right|_{u^1=0} = \frac{C_2(u^2)}{c}.$$

Consequently, $C_1 = C_2$. If we choose u^2 proportional to the arc length σ of the curve $u^1 = 0$, that is $u^2 = \sigma/\lambda$, then, since

$$\cosh \frac{u^1}{c} + \sinh \frac{u^1}{c} = e^{u^1/c},$$

we have

$$(93.2) \quad ds^2 = (du^1)^2 + \lambda^2 e^{2u^1/c} (du^2)^2,$$

where $\lambda (\neq 0)$ is a constant. (92.2) and (93.2) become identical if

$$(93.3) \quad \bar{u}^1 = \lambda e^{u^1/c}, \quad \bar{u}^2 = u^2.$$

Then

$$(93.4) \quad x_3 = \int \sqrt{\left(1 - \frac{\lambda^2}{c^2} e^{2u^1/c}\right)} du^1.$$

Inserting these expressions into (92.1) we obtain a representation of pseudo-spherical surfaces of revolution which are called parabolic pseudospherical surfaces of revolution or *pseudospheres*, cf. Fig. 98 b, p. 289. From (93.3) we have

$$u^1 = c \log \frac{\bar{u}^1}{\lambda}, \quad du^1 = \frac{c}{\bar{u}^1} d\bar{u}^1$$

and therefore

$$(93.5) \quad x_3 = \int \frac{c}{\bar{u}^1} \sqrt{\left(1 - \frac{(\bar{u}^1)^2}{c^2}\right)} d\bar{u}^1.$$

If we set $\bar{u}^1 = c \cos \alpha$ then $d\bar{u}^1 = -c \sin \alpha d\alpha$ and, by a suitable choice of the sign of the square root,

$$x_3 = c \int \frac{\sin^2 \alpha}{\cos \alpha} d\alpha = c \left[\log \tan \left(\frac{\alpha}{2} + \frac{\pi}{4} \right) - \sin \alpha \right].$$

Together with $\bar{u}^1 = c \cos \alpha$ this is a parametric representation of the meridian of the pseudosphere. We may write this more elegantly

$$\bar{u}^1 = c \operatorname{sech} t, \quad x_3 = c(t - \tanh t),$$

where $t = \log(\sec \alpha - \tan \alpha)$.

This curve is known as a *tractrix*. The tractrix, first considered by G. W. Leibniz, in 1693, is the path of a moving heavy particle P which is connected with a point Q by a thread of constant length c ; Q is moving along a straight line which does not contain P , cf. Fig. 96.

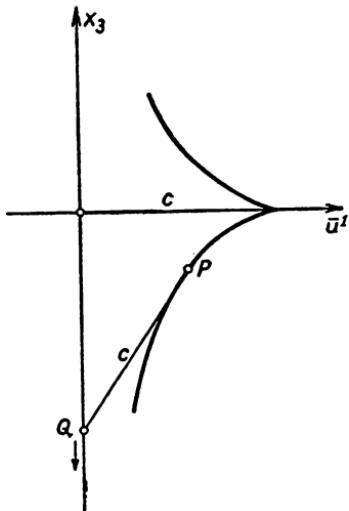


FIG. 96. Tractrix

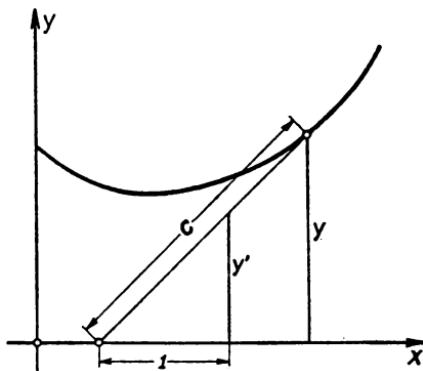


FIG. 97

Indeed, we can easily see that the segment of the tangent to the tractrix between the x_3 -axis and the point P always has the constant length c : Since $dx_3/d\bar{u}^1 = \tan \alpha$ we see that α is the angle between the tangent to the meridians of the pseudosphere and the x_1x_2 -plane. For every plane curve $y(x)$ the length c of the segment of the tangent between the point of contact and the x -axis is given by the formula

$$c = \frac{y}{y'} \sqrt{1+y'^2},$$

where the prime denotes the derivative with respect to x , cf. Fig. 97. Hence

$$y' = \frac{y}{\sqrt{c^2-y^2}}$$

or
$$x = \pm \int \frac{c}{y} \sqrt{\left(1 - \frac{y^2}{c^2}\right)} dy;$$

this expression corresponds to (93.5).

II. We choose geodesic parallel coordinates u^1, u^2 so that the curve $u^1 = 0$ is a geodesic and the curves $u^2 = \text{const.}$ are geodesics orthogonal to the curve $u^1 = 0$. Then, since

$$(\kappa_\rho)_{u^1=0} = 0,$$

we have

$$\left. \frac{\partial B}{\partial u^1} \right|_{u^1=0} = 0,$$

in the same manner as in the preceding section. Hence, in (93.1), $C_2 = 0$. Consequently by a suitable choice of the coordinate u^2 , the first fundamental form becomes

$$(93.6) \quad ds^2 = (du^1)^2 + \lambda^2 \cosh^2 \frac{u^1}{c} (du^2)^2,$$

where $\lambda (\neq 0)$ denotes a constant. (93.6) and (92.2) are identical if

$$(93.7) \quad (1+h'^2)(d\bar{u}^1)^2 = (du^1)^2, \quad \bar{u}^1 = \lambda \cosh \frac{u^1}{c}, \quad \bar{u}^2 = u^2.$$

Then

$$x_3 = \int \sqrt{\left(1 - \frac{\lambda^2}{c^2} \sinh^2 \frac{u^1}{c}\right)} du^1.$$

Inserting these expressions into (92.1) we obtain a representation of a surface which is called a *hyperbolic pseudospherical surface of revolution*; cf. Fig. 98a, p. 289. The function x_3 is an elliptic integral (cf. the preceding section) whose values are real if

$$\sinh^2 \frac{u^1}{c} \leq \frac{c^2}{\lambda^2}.$$

Consequently the values of the radius \bar{u}^1 are restricted to the interval

$$\lambda \leq \bar{u}^1 \leq \sqrt{(\lambda^2 + c^2)}.$$

Since x_3 is periodic the surface consists of a succession of congruent zones. These zones are spool-like; the circles by which every zone is bounded have the radius $\bar{u}^1 = \sqrt{(\lambda^2 + c^2)}$ while the radius of the circle in the middle of every zone is $\bar{u}^1 = \lambda$. From (93.7) we obtain

$$(93.8) \quad \frac{dx_3}{d\bar{u}^1} = \sqrt{\left(\frac{c^2 + \lambda^2 - (\bar{u}^1)^2}{(\bar{u}^1)^2 - \lambda^2}\right)}.$$

Thus at the points of the boundary of the zones the tangent plane of the surface is parallel to the $x_1 x_2$ -plane while in the middle of every zone it is parallel to the axis of revolution; the maximum parallels are cuspidal edges of the surface.

III. We finally choose geodesic polar coordinates u^{1*}, u^{2*} . Then, from (53.1):

$$g_{22}^*(0, u^{2*}) = 0, \quad \frac{\partial \sqrt{g_{22}^*}}{\partial u^{1*}} \Big|_{u^{1*}=0} = 1$$

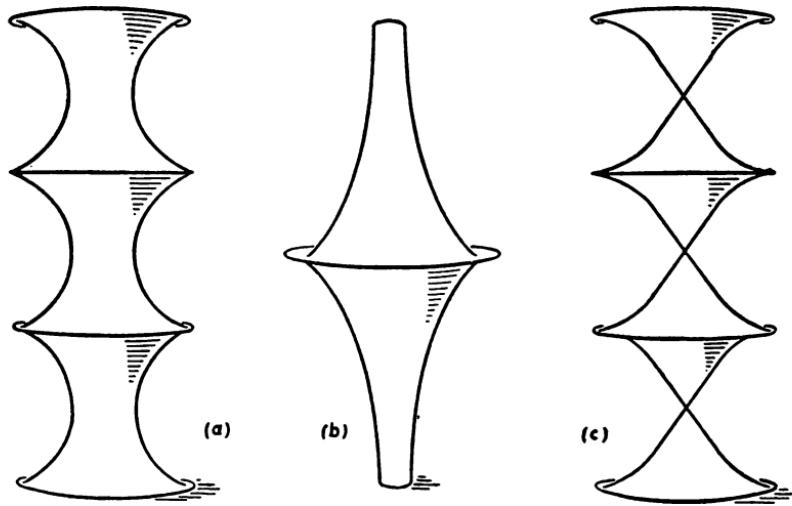


FIG. 98. (a) Hyperbolic, (b) parabolic, and (c) elliptic pseudo-spherical surfaces of revolution

in (93.1) we obtain $C_1 = 0$ and $C_2 = c$. Consequently

$$ds^2 = (du^{1*})^2 + c^2 \sinh^2 \frac{u^{1*}}{c} (du^{2*})^2.$$

If we set

$$u^1 = u^{1*}, \quad u^2 = \frac{c}{\lambda} u^{2*}$$

we have

$$(93.9) \quad ds^2 = (du^1)^2 + \lambda^2 \sinh^2 \frac{u^1}{c} (du^2)^2;$$

this becomes identical with (92.2) if

$$(93.10) \quad \bar{u}^1 = \lambda \sinh \frac{u^1}{c}, \quad \bar{u}^2 = u^2, \quad (1 + h'^2)(d\bar{u}^1)^2 = (du^1)^2.$$

Then

$$x_3 = \int \sqrt{\left(1 - \frac{\lambda^2}{c^2} \cosh^2 \frac{u^1}{c}\right)} du^1.$$

If we insert these expressions into (92.1) we obtain a representation of a surface which is known as an *elliptic pseudospherical surface of revolution*; cf. Fig. 98 c. x_3 is again an elliptic integral, hence the surface consists of

congruent zones. For real values of the argument the hyperbolic cosine is greater than or equal to 1. Consequently, in order that x_3 be real we have to require that $\lambda^2 \leq c^2$. Hence the radius varies in the interval

$$0 \leq \bar{u}^1 \leq \sqrt{(c^2 - \lambda^2)}.$$

The direction of the tangent to the meridians can be obtained from (93.10),

$$\frac{dx_3}{d\bar{u}^1} = \sqrt{\left(\frac{c^2 - \lambda^2 - (\bar{u}^1)^2}{(\bar{u}^1)^2 + \lambda^2} \right)};$$

at the boundaries of the zones the tangent is parallel to the $x_1 x_2$ -plane, cf. Fig. 98 (c).

94. Geodesic mapping. In order to consider the geodesic mapping of surfaces of constant Gaussian curvature we first state some general facts on geodesic mappings.

An allowable mapping (cf. Section 56) of a portion S of a surface onto a portion S^* of a surface is said to be *geodesic* if to each geodesic arc on either portion there corresponds a geodesic arc on the other.

Theorem 94.1. *An allowable mapping of a portion S of a surface onto a portion S^* of a surface is geodesic if and only if the Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ on S and $\Gamma_{\alpha\beta}^{*\gamma}$ on S^* (with respect to the same Gaussian coordinates on S and S^*) are connected by the relations*

$$(94.1) \quad \Gamma_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^{*\gamma} + \delta_\alpha^\gamma \frac{\partial h}{\partial u^{\beta*}} + \delta_\beta^\gamma \frac{\partial h}{\partial u^{\alpha*}}, \quad h \equiv \frac{1}{6} \log \frac{g}{g^*},$$

where g^* and g are the discriminants of the first fundamental forms of S^* and S , respectively.

Proof. We start from the differential equations (51.4) of the geodesics on S ,

$$u^\tau s'' - (\Gamma_{\nu\sigma}^\tau u^\nu u^\sigma + u^\tau'') s' = 0, \quad (\tau = 1, 2)$$

and eliminate s' and s'' ,

$$(94.2) \quad u^1'(u^{2''} + \Gamma_{\nu\sigma}^{*2} u^\nu u^\sigma) - u^{2'}(u^{1''} + \Gamma_{\nu\sigma}^{*1} u^\nu u^\sigma) = 0.$$

Similarly, the differential equation of the geodesics on S^* is

$$(94.3) \quad u^{1*'}(u^{2**} + \Gamma_{\nu\sigma}^{*2} u^{*\nu} u^{*\sigma}) - u^{2*'}(u^{1**} + \Gamma_{\nu\sigma}^{*1} u^{*\nu} u^{*\sigma}) = 0$$

where u^{1*} , u^{2*} are the coordinates on S^* . The coordinates on S and S^* are assumed to be the same, that is, the mapping under consideration is defined by $u^{1*} = u^1$, $u^{2*} = u^2$. If we subtract (94.3) from (94.2) we obtain

$$(94.4) \quad [u^1'(\Gamma_{\nu\sigma}^{*2} - \Gamma_{\nu\sigma}^{*2}) - u^{2'}(\Gamma_{\nu\sigma}^{*1} - \Gamma_{\nu\sigma}^{*1})] u^\nu u^\sigma = 0.$$

In consequence of (47.12) we have

$$\Gamma_{\nu\sigma}^{\alpha} - \Gamma_{\nu\sigma}^{*\alpha} = (\bar{\Gamma}_{\eta\epsilon}^{\kappa} - \bar{\Gamma}_{\eta\epsilon}^{*\kappa}) \frac{\partial \bar{u}^{\eta}}{\partial u^{\nu}} \frac{\partial \bar{u}^{\epsilon}}{\partial u^{\sigma}} \frac{\partial u^{\alpha}}{\partial \bar{u}^{\kappa}}.$$

Hence the expressions

$$(94.5) \quad d_{\nu\sigma}^{\alpha} = \Gamma_{\nu\sigma}^{\alpha} - \Gamma_{\nu\sigma}^{*\alpha}$$

are the components of a tensor of third order, contravariant of first order and covariant of second order. The equation (94.4) must be satisfied identically, otherwise only three but not ∞^1 geodesics would pass through every point of S and S^* . In consequence of (94.5) we may write (94.4) in the form

$$(u^{1'} d_{\nu\sigma}^{1'} - u^{2'} d_{\nu\sigma}^{2'}) u^{\nu} u^{\sigma'} = 0.$$

This is identically satisfied if

$$d_{11}^1 = 2d_{12}^2, \quad d_{11}^2 = d_{22}^1 = 0, \quad d_{22}^2 = 2d_{12}^1.$$

These conditions can also be written in the form

$$3d_{\nu\sigma}^{\alpha} - \delta_{\nu}^{\alpha} d_{\sigma\beta}^{\beta} - \delta_{\sigma}^{\alpha} d_{\nu\beta}^{\beta} = 0$$

or, in consequence of (94.5),

$$(94.6) \quad \Gamma_{\nu\sigma}^{\alpha} = \Gamma_{\nu\sigma}^{*\alpha} + \frac{1}{3}(\delta_{\nu}^{\alpha} d_{\sigma\beta}^{\beta} + \delta_{\sigma}^{\alpha} d_{\nu\beta}^{\beta}),$$

as can easily be verified. If we insert (94.6) into (94.2) we obtain (94.3). That is, (94.6) is the necessary and sufficient condition for the mapping under consideration to be geodesic. Since

$$d_{\sigma\beta}^{\beta} = \Gamma_{\sigma\beta}^{\beta} - \Gamma_{\sigma\beta}^{*\beta} = \frac{\partial}{\partial u^{\sigma}} (\log \sqrt{g} - \log \sqrt{g^*}) = \frac{1}{2} \frac{\partial}{\partial u^{\sigma}} \log \frac{g}{g^*},$$

$$d_{\nu\beta}^{\beta} = \frac{1}{2} \frac{\partial}{\partial u^{\nu}} \log \frac{g}{g^*},$$

cf. (47.6'), we see that (94.1) and (94.6) are identical.

From the theorems 57.1 and 94.1 we obtain

Theorem 94.2. *Any isometric mapping is geodesic.*

This also follows immediately from Theorem 57.2.

95. Geodesic mapping of surfaces of constant Gaussian curvature. E. Beltrami discovered the following interesting

Theorem 95.1. *If a portion S of a surface can be mapped geodesically onto a portion S^* of a surface of constant Gaussian curvature K^* the Gaussian curvature K of S must also be constant.*

Proof. We use Theorem 94.1 and the corresponding notations. We assume that S is geodesically mapped onto S^* . Then, in consequence of (94.1), we find a relation between the curvature tensor (46.5) of S ,

$$R_{\mu\eta\nu}^\sigma = \frac{\partial}{\partial u^\eta} \Gamma_{\mu\nu}^\sigma - \frac{\partial}{\partial u^\nu} \Gamma_{\mu\eta}^\sigma + \Gamma_{\mu\nu}^\kappa \Gamma_{\kappa\eta}^\sigma - \Gamma_{\mu\eta}^\kappa \Gamma_{\kappa\nu}^\sigma$$

and the curvature tensor $R^{*\sigma}_{\mu\eta\nu}$ of S^* . By inserting (94.1) we obtain

$$(95.1) \quad R_{\mu\eta\nu}^\sigma = R^{*\sigma}_{\mu\eta\nu} + \delta_\nu^\sigma \left(\frac{\partial^2 h}{\partial u^\mu \partial u^\eta} - \frac{\partial h}{\partial u^\kappa} \Gamma_{\mu\eta}^{*\kappa} - \frac{\partial h}{\partial u^\mu} \frac{\partial h}{\partial u^\eta} \right) - \delta_\eta^\sigma \left(\frac{\partial^2 h}{\partial u^\mu \partial u^\nu} - \frac{\partial h}{\partial u^\kappa} \Gamma_{\mu\nu}^{*\kappa} - \frac{\partial h}{\partial u^\mu} \frac{\partial h}{\partial u^\nu} \right).$$

This is equivalent to

$$(95.2) \quad R_{\mu\eta\nu}^\sigma = R^{*\sigma}_{\mu\eta\nu} + \delta_\nu^\sigma (h_{,\mu\eta} - h_{,\mu} h_{,\eta}) - \delta_\eta^\sigma (h_{,\mu\nu} - h_{,\mu} h_{,\nu}),$$

where the indices preceded by a comma indicate covariant derivatives with respect to the metric tensor $g_{\alpha\beta}^*$ of S^* ; for the sake of simplicity the first partial derivatives of h are also written in the symbolism of absolute differentiation; this is justified by the fact that in the case of a scalar the covariant derivative and the usual partial one are the same. By contraction with respect to σ and ν we have

$$(95.3) \quad R_{\mu\eta} = R_{\mu\eta}^* + h_{,\mu\eta} - h_{,\mu} h_{,\eta}.$$

The quantities $R_{\mu\eta} = R_{\mu\eta\sigma}^\sigma$ and $R_{\mu\eta}^* = R_{\mu\eta\sigma}^{*\sigma}$ are the components of the so-called *Ricci tensor* of S and S^* , respectively. These quantities are closely related to the Gaussian curvature (46.12):

$$K^* = \frac{b^*}{g^*} = \frac{R_{1212}^*}{g^*};$$

by (46.11) we obtain immediately

$$(95.4) \quad R_{\mu\eta\nu}^{*\sigma} = g^{*\sigma\alpha} R_{\alpha\mu\eta\nu}^* = K^* (\delta_\eta^\sigma g_{\mu\nu}^* - \delta_\nu^\sigma g_{\mu\eta}^*).$$

By contraction with respect to σ and ν we find

$$R_{\mu\eta}^* = -K^* g_{\mu\eta}^*$$

since $\delta_\nu^\sigma = 2$. Analogously

$$R_{\mu\eta} = -K g_{\mu\eta}$$

where $g_{\mu\eta}$ are the components of the metric tensor of S . Hence (95.3) becomes

$$(95.5) \quad h_{,\mu\eta} = h_{,\mu} h_{,\eta} + K^* g_{\mu\eta}^* - K g_{\mu\eta}.$$

Using the assumption that K^* is constant we will prove that the first partial derivatives of K with respect to u^1 and u^2 vanish, that is, that K must be constant. In order to obtain those derivatives it is natural to take the

covariant derivative of (95.5) with respect to $g_{\alpha\beta}^*$. Since $g_{\mu\eta,\kappa}^* = 0$, cf. Theorem 72.1, we obtain in this manner

$$(95.6) \quad h_{,\mu\eta\kappa} = h_{,\mu\kappa} h_{,\eta} + h_{,\mu} h_{,\eta\kappa} - K_{,\kappa} g_{\mu\eta} - K g_{\mu\eta,\kappa}.$$

We now determine the covariant derivative $g_{\mu\eta,\kappa}$. From (47.4) we have

$$\frac{\partial g_{\mu\eta}}{\partial u^\kappa} = g_{\sigma\eta} \Gamma_{\mu\kappa}^\sigma + g_{\mu\sigma} \Gamma_{\eta\kappa}^\sigma,$$

hence, by (94.1),

$$\frac{\partial g_{\mu\eta}}{\partial u^\kappa} = g_{\sigma\eta} (\Gamma_{\mu\kappa}^{*\sigma} + \delta_\mu^\sigma h_{,\kappa} + \delta_\kappa^\sigma h_{,\mu}) + g_{\mu\sigma} (\Gamma_{\eta\kappa}^{*\sigma} + \delta_\eta^\sigma h_{,\kappa} + \delta_\kappa^\sigma h_{,\eta}).$$

Consequently, by (71.3''),

$$g_{\mu\eta,\kappa} = \frac{\partial g_{\mu\eta}}{\partial u^\kappa} - g_{\sigma\eta} \Gamma_{\mu\kappa}^{*\sigma} - g_{\mu\sigma} \Gamma_{\eta\kappa}^{*\sigma} = 2g_{\mu\eta} h_{,\kappa} + g_{\kappa\eta} h_{,\mu} + g_{\mu\kappa} h_{,\eta}.$$

Inserting this expression into (95.6) and replacing the expressions $K g_{\mu\eta}$, etc., by means of (95.5) we obtain

$$h_{,\mu\eta\kappa} = 2(h_{,\mu} h_{,\eta\kappa} + h_{,\eta} h_{,\kappa\mu} + h_{,\kappa} h_{,\mu\eta}) - 4h_{,\mu} h_{,\eta} h_{,\kappa} - K^*(2g_{\mu\eta}^* h_{,\kappa} + g_{\eta\kappa}^* h_{,\mu} + g_{\kappa\mu}^* h_{,\eta}) - K_{,\kappa} g_{\mu\eta}.$$

From this it follows that

$$(95.7) \quad h_{,\mu\eta\kappa} - h_{,\mu\kappa\eta} = K^*(g_{\mu\kappa}^* h_{,\eta} - g_{\mu\eta}^* h_{,\kappa}) - K_{,\kappa} g_{\mu\eta} + K_{,\eta} g_{\mu\kappa}.$$

On the other hand, from (73.5) and (95.4),

$$h_{,\mu\eta\kappa} - h_{,\mu\kappa\eta} = h_{,\sigma} R^{*\sigma}_{\mu\eta\kappa} = h_{,\sigma} K^*(\delta_\eta^\sigma g_{\mu\kappa}^* - \delta_\kappa^\sigma g_{\mu\eta}^*) = K^*(g_{\mu\kappa}^* h_{,\eta} - g_{\mu\eta}^* h_{,\kappa}).$$

In consequence of (95.7) we obtain from this

$$g_{\mu\kappa} K_{,\eta} - g_{\mu\eta} K_{,\kappa} = 0.$$

Since the determinant g of the coefficients of this system of linear equations is different from zero the only solutions are the trivial ones, $K_{,\alpha} = 0$ ($\alpha = 1, 2$). Hence K is a constant. This completes the proof.

96. Surfaces of constant Gaussian curvature and non-Euclidean geometry. We conclude our study of surfaces of constant Gaussian curvature $K \neq 0$ by proving the fact that the intrinsic geometry of these surfaces—at least locally—is identical with non-Euclidean geometry. We first make some remarks on non-Euclidean geometry; further details can be seen in the appropriate literature, for instance, H. S. M. Coxeter [1].

In his famous thirteen books Euclid (325 B.C.) investigates the foundations of geometry. He starts with twenty-three definitions which are followed by five postulates and five axioms. The fifth axiom is as follows: ‘If a straight line meet two other straight lines so as to make the two

interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.' This axiom is equivalent to the statement that there is only one straight line G_1 parallel to a given line G and passing through a point P not on G . The other axioms are much more self-evident than this one. Hence it was conjectured even a long time ago that the fifth axiom was not an axiom but a theorem, that is, a consequence of the others. Many attempts have been made by a great number of outstanding mathematicians to deduce this 'theorem' from the other axioms. Since no attempt turned out to be successful, Gauss drew the conclusion that the fifth axiom could not be proved at all. In 1792 he started working on the foundations of a geometry in which this axiom is replaced by a contrary assumption, and by 1816 at the latest he had gained a clear idea of such a geometry. He did not publish anything on this matter. N. Lobatšewskij (1793–1856) was the first to state publicly (in 1829) the existence of a geometry which is independent of the fifth axiom of Euclid. Independently J. Bolyai (1802–60) published similar results (in 1832). Both investigations were concerned with the so-called hyperbolic non-Euclidean geometry. This geometry is—at least locally—identical with the intrinsic geometry of the surfaces of *negative* constant Gaussian curvature, as was shown by the investigations of E. Beltrami [3]. The importance of this paper consists especially in the fact that it makes evident that the hyperbolic non-Euclidean geometry can never lead to contradictions, since contradictions of this kind would also cause contradictions in the intrinsic geometry of the above surfaces. Since those surfaces are embedded in three-dimensional Euclidean space R_3 , the aforementioned contradictions would lead to contradictions in Euclidean geometry.

Hyperbolic non-Euclidean geometry can briefly be characterized by the fact that there are (at least) two straight lines passing through a point and parallel to a given line, and the sum of the angles of a triangle is less than π . Another type, the so-called elliptic non-Euclidean geometry, was first investigated by B. Riemann [1]. In this geometry there does not exist a straight line parallel to a given one, and the sum of the angles of a triangle is greater than π . The intrinsic geometry of any surface of positive constant Gaussian curvature is—at least locally—identical with elliptic non-Euclidean geometry.

A clear insight into the intrinsic geometry of a pseudospherical surface S can be obtained by a certain conformal mapping (H. Poincaré, 1854–1912) of S into a plane under which mapping the geodesics on S correspond to

certain semicircles in the plane. In order to introduce this mapping we first prove the following

Theorem 96.1. *If a surface S can be mapped conformally into a plane E so that the geodesics of S correspond to circles in E , the Gaussian curvature K of S must be constant.*

Proof. We assume that the same coordinates have been introduced on S and E . Then, since the mapping of S into E is assumed to be conformal, at corresponding points the coefficients of the first fundamental form of S and E are proportional, cf. Theorem 61.1. Hence the images of the isotropic curves (cf. Section 63) of S are isotropic curves in E . We introduce the isotropic curves on S as coordinate curves and denote by u^1, u^2 the corresponding coordinates. Then, in consequence of (63.4),

$$g_{11} = g_{22} = g^{11} = g^{22} = 0, \quad g = -(g_{12})^2.$$

Furthermore, from (47.1) and (47.2),

$$\Gamma_{112} = \frac{\partial g_{12}}{\partial u^1}, \quad \Gamma_{221} = \frac{\partial g_{12}}{\partial u^2},$$

$$\Gamma_{11}{}^1 = \frac{1}{g_{12}} \frac{\partial g_{12}}{\partial u^1} = \frac{\partial}{\partial u^1} \log g_{12}, \quad \Gamma_{22}{}^2 = \frac{1}{g_{12}} \frac{\partial g_{12}}{\partial u^2} = \frac{\partial}{\partial u^2} \log g_{12}$$

while all other Christoffel symbols are zero. Hence the differential equation (50.2) of the geodesics takes the form

$$(96.1) \quad u^{2''} + A(u^{2'})^2 + Bu^{2'} = 0$$

where

$$(96.2) \quad A = \frac{\partial}{\partial u^2} \log g_{12}, \quad B = -\frac{\partial}{\partial u^1} \log g_{12}$$

and a prime denotes the derivative with respect to u^1 . From (46.17) we find

$$K = \frac{1}{(g_{12})^4} \begin{vmatrix} 0 & g_{12} & \frac{\partial g_{12}}{\partial u^2} \\ g_{12} & 0 & 0 \\ 0 & \frac{\partial g_{12}}{\partial u^1} & \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} \end{vmatrix} = -\frac{1}{(g_{12})^3} \left[g_{12} \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{\partial g_{12}}{\partial u^1} \frac{\partial g_{12}}{\partial u^2} \right]$$

or

$$(96.3) \quad K = -\frac{1}{g_{12}} \frac{\partial^2 \log g_{12}}{\partial u^1 \partial u^2}.$$

In order to have the same coordinates on S and E we proceed in E from Cartesian coordinates x_1, x_2 to isotropic ones,

$$(96.4) \quad u^1 = x_1 + ix_2, \quad u^2 = x_1 - ix_2.$$

Then the equation of a circle,

$$x_1^2 + x_2^2 + a_1 x_1 + a_2 x_2 + a_3 = 0$$

takes the form

$$(96.5) \quad u^1 u^2 + a u^1 + b u^2 + d = 0.$$

In order to compare (96.1) and (96.5) we write (96.5) in the form of a differential equation. By successive differentiation with respect to u^1 we obtain

$$(96.6) \quad u^{2'} = -\frac{u^2 + a}{u^1 + b}, \quad u^{2''} = 2 \frac{u^2 + a}{(u^1 + b)^2}, \quad u^{2'''} = -6 \frac{u^2 + a}{(u^1 + b)^3}.$$

Hence the circles in E can be represented by the differential equation

$$(96.7) \quad u^{2'''} = \frac{3}{2} \frac{(u^{2''})^2}{u^{2'}};$$

this representation does not involve the coefficients a, b , and d . In order to compare (96.7) and (96.1) we differentiate (96.1) with respect to u^1 , finding

$$u^{2'''} + \left(\frac{\partial A}{\partial u^2} u^{2'} + \frac{\partial A}{\partial u^1} \right) (u^{2'})^2 + 2A u^{2'} u^{2''} + \left(\frac{\partial B}{\partial u^2} u^{2'} + \frac{\partial B}{\partial u^1} \right) u^{2'} + B u^{2''} = 0.$$

If we replace $u^{2''}$ by means of (96.1) we obtain

$$(96.8) \quad u^{2'''} + \left(\frac{\partial A}{\partial u^2} - 2A^2 \right) (u^{2'})^3 + \left(\frac{\partial A}{\partial u^1} - 3AB + \frac{\partial B}{\partial u^2} \right) (u^{2'})^2 + \left(\frac{\partial B}{\partial u^1} - B^2 \right) u^{2'} = 0.$$

Since we have the same coordinate systems on S and E and since we assumed the images of the geodesics on S to be circles in E , (96.7) must be identically satisfied when we insert $u^{2''}$ according to (96.8) and $u^{2''}$ according to (96.1). From this we obtain conditions for A and B . We find

$$\left(\frac{\partial A}{\partial u^2} - \frac{A^2}{2} \right) (u^{2'})^3 + \left(\frac{\partial A}{\partial u^1} + \frac{\partial B}{\partial u^2} \right) (u^{2'})^2 + \left(\frac{\partial B}{\partial u^1} + \frac{B^2}{2} \right) u^{2'} = 0.$$

From (96.2) we see that the coefficient of $(u^{2'})^2$ vanishes. The vanishing of the other two coefficients leads to the conditions

$$(96.9) \quad \frac{\partial}{\partial u^2} \left(\frac{1}{A} \right) = -\frac{1}{2}, \quad \frac{\partial}{\partial u^1} \left(\frac{1}{B} \right) = \frac{1}{2}$$

because we may assume that $A \neq 0$ and $B \neq 0$. By integrating the preceding expressions we obtain

$$\frac{1}{A} = -\frac{1}{2}\{u^2 + C_1(u^1)\}, \quad \frac{1}{B} = \frac{1}{2}\{u^1 + C_2(u^2)\}$$

or, in consequence of (96.2),

$$\frac{\partial}{\partial u^2} \log g_{12} = -\frac{2}{u^2 + C_1(u^1)}, \quad \frac{\partial}{\partial u^1} \log g_{12} = -\frac{2}{u^1 + C_2(u^2)}.$$

If we integrate once more we find

$$\log g_{12} = -2 \log\{u^2 + C_1(u^1)\} - 2 \log C_3(u^1) = -2 \log[C_3(u^1)\{u^2 + C_1(u^1)\}]$$

$$\text{and } \log g_{12} = -2 \log[C_4(u^2)\{u^1 + C_2(u^2)\}],$$

respectively. In order that these expressions be identical the argument of the logarithm on the right-hand side must be a bilinear form in u^1 and u^2 ,

$$(96.10) \quad \log g_{12} = -2 \log(\alpha_{12} u^1 u^2 + \alpha_1 u^1 + \alpha_2 u^2 + \alpha_0).$$

Inserting this in (96.3) we find that the Gaussian curvature becomes a constant

$$(96.11) \quad K = 2(\alpha_0 \alpha_{12} - \alpha_1 \alpha_2).$$

This completes the proof of Theorem 96.1.

We now consider the aforementioned mapping of Poincaré which enables us to gain an intuitive understanding of the hyperbolic non-Euclidean geometry. We set in (96.10)

$$\alpha_{12} = 0, \quad \alpha_1 = ic, \quad \alpha_2 = -ic, \quad \alpha_0 = 0.$$

Then, by (96.4), the first fundamental form is given by the expression

$$(96.12) \quad ds^2 = 2g_{12} du^1 du^2 = -\frac{2}{c^2(u^1 - u^2)^2} du^1 du^2 = \frac{dx_1^2 + dx_2^2}{2c^2 x_2^2}$$

and, in consequence of (96.3),

$$K = -2c^2 \quad (< 0).$$

The differential equation (96.1) of the geodesics on S takes the form

$$(96.13) \quad u^{2''} + \frac{2}{u^1 - u^2} [(u^{2'})^2 + u^{2'}] = 0.$$

If we insert in (96.13) the derivatives of u^2 according to (96.6) we find that the images of the geodesics on S are those circles (96.5) in E for which $a = b$, that is, which have the equation

$$u^1 u^2 + a(u^1 + u^2) + d = 0$$

or, in terms of the coordinates x_1, x_2 ,

$$x_1^2 + x_2^2 + 2ax_1 + d = 0,$$

cf. (96.4). Hence the centre of each of these circles lies on the x_1 -axis. (The limit cases of these circles, that is, the straight lines $x_1 = \text{const.}$ also figure among the images of the geodesics of S^* .) We restrict the mapping of Poincaré to the upper half plane ($x_2 \geq 0$); in this manner we obtain the desired intuition of the hyperbolic non-Euclidean geometry. The half-plane $x_2 \geq 0$, when the metric defined by (96.12) has been introduced, is called the half-plane of Poincaré, cf. H. Poincaré [1].

The image of a geodesic on S passing through two points \tilde{P} and \tilde{Q} is the semicircle $H(P, Q)$ with centre on the x_1 -axis and passing through the image points P and Q of \tilde{P} and \tilde{Q} , respectively.

In analogy with the displacements in the Euclidean plane (cf. Section 3) there are transformations under which the first fundamental form (96.12) is invariant. These are the so called '*linear transformations*'

$$(96.14) \quad u^{\mu*} = \frac{\alpha u^\mu + \beta}{\gamma u^\mu + \delta}, \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} > 0 \quad (\mu = 1, 2, \text{ do not sum})$$

with real coefficients α, β, γ , and δ . This transformation group is considered in function theory; the transformations are conformal and transform circles into circles (cf. Sections 61 and 65).

In order to determine the non-Euclidean distance between two arbitrary points P and Q in the image plane E , we first impose on E a non-Euclidean motion under which the semicircle $H(P, Q)$ is transformed into the positive ray of the x_2 -axis. We denote by u_a^1 and u_b^1 ($> u_a^1$) the points of intersection of $H(P, Q)$ and the x_1 -axis, cf. Fig. 99. Then this transformation is of the form

$$u^{1*} = \frac{u_a^1 - u^1}{u^1 - u_b^1}.$$

The non-Euclidean distance between P and Q is preserved under this transformation and is afterwards given by the expression

$$s = \frac{1}{c\sqrt{2}} \left| \int_{x_2^*(P)}^{x_2^*(Q)} \frac{dx_2}{x_2} \right| = \frac{1}{c\sqrt{2}} \left| \log \frac{x_2^*(Q)}{x_2^*(P)} \right|.$$

Apart from the factor $1/c\sqrt{2}$ this is nothing but the cross-ratio of the points P, Q and the two points of intersection of $H(P, Q)$ and the x_1 -axis. Indeed the cross-ratio of four points on a circle is invariant under the above linear transformations, as can be shown.

If P or Q tends to one of the aforementioned points of intersection, s tends to infinity. Consequently the x_1 -axis represents the improper line with respect to the metric defined by (96.12); this fact can also immediately be seen from (96.12). Through every point P^* of E there passes a bundle B of semicircles with centres on the x_1 -axis. Let H^* be a circle with centre on the x_1 -axis and which does not pass through P^* . Obviously B contains

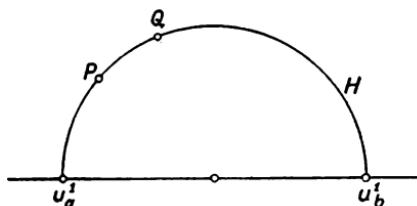


FIG. 99

infinitely many semicircles which intersect H^* at proper points, exactly two semicircles 'parallel' to H^* each of which has an improper point in common with H^* , and infinitely many 'hyper-parallel' semicircles which do not intersect H^* at all. The analogous property therefore also holds for the geodesics of the pseudospherical surface which has been mapped onto the half-plane of Poincaré under consideration.

E. Beltrami's geodesic mapping of pseudospherical surfaces in a plane can be obtained from Poincaré's mapping. We choose a sphere K^* with centre M at an arbitrary point of the improper line $G: x_2 = 0$ of the Poincaré half-plane E and project E stereographically into K^* (cf. Section 65), choosing as the centre of this projection one of the two points of intersection of K^* and the normal to E passing through M . The image of G is a great circle G^* of K^* ; the image k^* of any semicircle k with centre on G is again a semicircle (cf. Section 65) whose plane is orthogonal to the plane of G^* . The original pseudospherical surface S which has been mapped onto E is now mapped conformally into K^* , and the image of any geodesic on S is a semicircle k^* . We now project the image of S on K^* orthogonally into the plane of G^* ; under this projection the image of any semicircle k^* is a segment of a straight line. This is the desired geodesic mapping of S into the plane, and the image of S is the circular disk with boundary G^* .

Spherical surfaces can also be mapped geodesically into the plane. Since all these surfaces corresponding to a certain value of K are isometric, cf. Theorems 91.1 and 94.2, it suffices to construct a geodesic mapping of a sphere K into a plane. Such a mapping can be obtained by a central projection of K onto a tangent plane E of K where the centre M of K is the

centre of projection; in this manner a hemisphere of K is mapped onto E . This mapping is geodesic because every plane passing through M intersects K along a great circle and E along a straight line.

In consequence of Theorem 95.1 we therefore have the following result:

Theorem 96.2 (E. Beltrami [2]). *The portions of surfaces of constant Gaussian curvature are the only ones which can be mapped geodesically into the plane.*

Problem 96.1. Prove that the metric defined by (96.12) is invariant with respect to the linear mappings (96.14).

Problem 96.2. Prove that a function $u^2(u^1)$ is a rational function whose numerator and denominator are linear functions if and only if $u^2(u^1)$ satisfies a differential equation of the form (96.7).

Problem 96.3. Prove that on a surface of constant Gaussian curvature the geodesic circles (cf. Section 53) are curves of constant geodesic curvature.

ANSWERS TO PROBLEMS

Problem 3.1. The transformations (a) constitute a group, and so do the transformations (b), (c), and (d). The opposite orthogonal transformations do not constitute a group since by composing two such transformations we obtain a direct orthogonal transformation. In detail: (a) The inverse of a translation is a translation of the same magnitude but in opposite direction. If we compose any two translations we obtain a translation. A similar reasoning holds in case (b). Case (c): By composing two direct congruent transformations

$$T_1: \bar{x}_i = \sum_{k=1}^3 \bar{a}_{ik} \bar{x}_k + \bar{b}_i, \quad \sum_{j=1}^3 \bar{a}_{ji} \bar{a}_{jk} = \delta_{ik}, \quad \det(\bar{a}_{ik}) = 1$$

and $T_2: \bar{x}_k = \sum_{p=1}^3 a_{kp} x_p + b_k, \quad \sum_{j=1}^3 a_{ji} a_{jk} = \delta_{ik}, \quad \det(a_{ik}) = 1$

we obtain a direct congruent transformation. Indeed we find

$$\bar{x}_i = \sum_{k=1}^3 \bar{a}_{ik} \left(\sum_{p=1}^3 a_{kp} x_p + b_k \right) + \bar{b}_i = \sum_{p=1}^3 a_{ip}^* x_p + b_i^*$$

where

$$(1) \quad a_{ip}^* = \sum_{k=1}^3 \bar{a}_{ik} a_{kp}, \quad b_i^* = \sum_{k=1}^3 \bar{a}_{ik} b_k + \bar{b}_i.$$

$$\text{Now } \sum_{j=1}^3 a_{ji}^* a_{jk}^* = \sum_{j=1}^3 \left[\left(\sum_{h=1}^3 \bar{a}_{jh} a_{hi} \right) \left(\sum_{p=1}^3 \bar{a}_{jp} a_{pk} \right) \right] = \sum_{h=1}^3 \sum_{p=1}^3 \delta_{hp} a_{hi} a_{pk} = \sum_{h=1}^3 a_{hi} a_{hk} = \delta_{ik}.$$

Furthermore $\det(a_{ik}^*) = \det(\bar{a}_{ik}) \det(a_{ik}) = 1$.

By definition the *product of two square matrices* $A = (a_{ik})$ and $B = (b_{ik})$ is a matrix $C = (c_{ik})$ where

$$c_{ik} = \sum_{p=1}^3 a_{ip} b_{pk}.$$

If we compare this expression and (1) we see that we may obtain the matrix of the coefficients of a composite transformation by multiplying the matrices of the coefficients of the given transformations according to the preceding definition of matrix multiplication. The determination of the inverse of a linear transformation

$$\bar{x}_i = \sum_{k=1}^3 a_{ik} x_k$$

therefore becomes identical with the problem of inverting the matrix $A = (a_{ik})$ of the coefficients. The inverse of A is given by the expression

$$A^{-1} = \frac{(A_{ki})}{\det(a_{ik})}$$

where A_{ki} is the algebraic complement of the element a_{ki} of A . In order that A^{-1} exist we must have $\det(a_{ik}) \neq 0$; in this case A is said to be *non-singular*. The inverse A^{-1} of an orthogonal matrix $A = (a_{ik})$ is equal to its *transpose* $A' = (a_{ki})$; obviously A' is also an orthogonal matrix. The inverse of a direct congruent transformation

$$(2) \quad x_i = \sum_{k=1}^3 \bar{a}_{ik} \bar{x}_k + \bar{b}_i, \quad \sum_{j=1}^3 \bar{a}_{ji} \bar{a}_{jk} = \delta_{ik}, \quad \det(\bar{a}_{ik}) = 1$$

can be obtained by multiplying (2) by the algebraic complement \bar{A}_{ip} of the element a_{ip} , with respect to the matrix (\bar{a}_{ij}) and summation with respect to i ,

$$\sum_{i=1}^3 x_i \bar{A}_{ip} = \sum_{i=1}^3 \left[\sum_{k=1}^3 \bar{a}_{ik} \bar{A}_{ip} \bar{x}_k + \bar{b}_i \bar{A}_{ip} \right] = \sum_{k=1}^3 \bar{a}_{kp} \bar{x}_k + \sum_{i=1}^3 \bar{b}_i \bar{A}_{ip} = \bar{x}_p + \sum_{i=1}^3 \bar{b}_i \bar{A}_{ip}$$

or $\bar{x}_p = \sum_{i=1}^3 \bar{A}_{ip} x_i + B_p, \quad B_p = - \sum_{i=1}^3 \bar{b}_i \bar{A}_{ip}.$

Hence the inverse of a direct congruent transformation is a direct congruent transformation. Case (d) follows immediately from case (c) if we set $\bar{b}_i = 0$ ($i = 1, 2, 3$) and $b_k = 0$ ($k = 1, 2, 3$).

Problem 5.1. The equation of a plane

$$a_1^* x_1 + a_2^* x_2 + a_3^* x_3 = b^*$$

can be written in the form $\mathbf{a}^* \cdot \mathbf{x} = b^*$ where $\mathbf{a}^* = (a_1^*, a_2^*, a_3^*)$ and $\mathbf{x} = (x_1, x_2, x_3)$. If we set

$$a_i = \frac{a_i^*}{|\mathbf{a}^*|} \quad (i = 1, 2, 3), \quad \text{and} \quad b = \frac{b^*}{|\mathbf{a}^*|}$$

we obtain

$$\mathbf{a} \cdot \mathbf{x} = b;$$

the unit vector \mathbf{a} is orthogonal to the plane, and $|b|$ is the distance of the plane from the origin of the coordinate system.

Problem 5.2. Under a direct congruent transformation (3.1) the components of a vector behave according to (4.2). Let $|\mathbf{a} \mathbf{b} \mathbf{c}|$ be the mixed product under consideration. Then if we impose the transformation (4.2) we obtain

$$\begin{aligned} |\bar{\mathbf{a}} \bar{\mathbf{b}} \bar{\mathbf{c}}| &= \begin{vmatrix} \sum_{k=1}^3 a_{1k} a_k & \sum_{k=1}^3 a_{1k} b_k & \sum_{k=1}^3 a_{1k} c_k \\ \sum_{k=1}^3 a_{2k} a_k & \sum_{k=1}^3 a_{2k} b_k & \sum_{k=1}^3 a_{2k} c_k \\ \sum_{k=1}^3 a_{3k} a_k & \sum_{k=1}^3 a_{3k} b_k & \sum_{k=1}^3 a_{3k} c_k \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det(a_{ik}) |\mathbf{a} \mathbf{b} \mathbf{c}| \end{aligned}$$

or

$$|\bar{\mathbf{a}} \bar{\mathbf{b}} \bar{\mathbf{c}}| = |\mathbf{a} \mathbf{b} \mathbf{c}|.$$

Problem 5.3. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ be any four given vectors in R_3 . By definition these vectors are linearly dependent if and only if there exist real numbers a, b, c, k , not all zero, such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + k\mathbf{z} = 0$. This vector equation is equivalent to a system of three linear homogeneous equations in the four unknowns a, b, c, k . Since the number of equations is smaller than the number of unknowns that system has a non-trivial solution. This proves that those four vectors are linearly dependent.

Problem 7.1. (a) Every number t , $0 < t \leq 1$ can be uniquely represented by an infinite decimal,

$$(1) \quad t = 0, c_1 c_2 \dots c_n \dots .$$

For example, $\frac{1}{3} = 0,125 = 0,124\ 999\dots$. Let c_{n_1}, c_{n_2}, \dots be the non-vanishing digits in (1) taken in the order of their occurrence in (1). The image point of t is that point (x_1, x_2) of the square $0 < x_1 \leq 1$, $0 < x_2 \leq 1$ whose coordinates are

$$x_1 = 0, c_1 \dots c_{n_1} c_{n_2+1} \dots c_{n_3} \dots, \quad x_2 = 0, c_{n_1+1} \dots c_{n_2} c_{n_3+1} \dots .$$

The mapping thus defined is one-to-one, since any point (x_1, x_2) of the above square has exactly one inverse image t . For example, the point

$$x_1 = 0,04601\dots, \quad x_2 = 0,7000302\dots$$

has the inverse image $t = 0,047600030102\dots$. Since neither x_1 nor x_2 have ultimately only zeros we always obtain an infinite decimal. Our problem illustrates the fact that the dimension of a geometric configuration is not necessarily invariant with respect to one-to-one mappings. The discovery of this fact by G. Cantor (1845–1918) in 1873, Cantor [1], had far-reaching consequences for the development of geometry. The earliest general proof of the fact that topological mappings preserve the dimension was given by L. E. J. Brouwer [1] in 1911.

(b) A unique and continuous mapping of a segment onto a square was first given by G. Peano [1] in 1890; an image of a segment with respect to such a mapping is therefore known as a *Peano curve*; it is not an arc of a curve in the sense of Definition 6.1. In order to prove the statement of Problem 7.1 (b) we subdivide the closed square

$$Q: \quad 0 < x_1 \leq 1, \quad 0 < x_2 \leq 1$$

into four squares q which, together with their boundaries will be denoted by $0, 1, 2, 3$. We subdivide each of those squares into four squares qr which, together with their boundaries, will be denoted by $q0, q1, q2, q3$, cf. Fig. 100. We subdivide again, etc. We number the squares so that if we pass through all of them in the order of increasing numbers we obtain a polygon which has no multiple points. That such a choice of the numbers is possible can be proved by induction: For $n = 1$ the statement is true. Let S and T be consecutive squares obtained by the n th subdivision. S and T have a side in common. When carrying out the $(n+1)$ th subdivision we can number the four squares, into which S is subdivided, in such a manner that the highest number is assigned to one of the two squares which have a part of the boundary in common with the boundary of T . This successive subdivision has the following properties. Any point P of the square Q is a point of every square of at least one sequence $q_1, q_1 q_2, q_1 q_2 q_3, \dots$, and every such sequence of squares converges to a point of Q . Furthermore, two numbers

$$t_a = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \dots \quad \text{and} \quad t_b = \frac{b_1}{4} + \frac{b_2}{4^2} + \frac{b_3}{4^3} + \dots \quad (0 \leq a_i \leq 3) \\ (0 \leq b_i \leq 3)$$

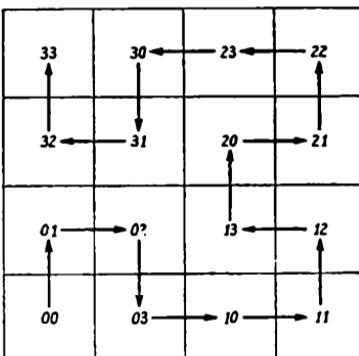


FIG. 100

are identical if and only if either $a_i = b_i$ for all i or $a_i = b_i$ for $i < n$, $a_n + 1 = b_n$, $a_i = 3$, $b_i = 0$ for $i > n$ (or $b_n + 1 = a_n$, $b_i = 3$, $a_i = 0$ for $i > n$). If two numbers t_a and t_b are equal the corresponding sequences of squares $a_1, a_1 a_2, a_1 a_2 a_3, \dots$ and $b_1, b_1 b_2, b_1 b_2 b_3, \dots$ converge to the same point of Q . The mapping is unique. It is also continuous; this can be seen as follows. If for two numbers

$$t_k = \frac{k_1}{4} + \frac{k_2}{4^2} + \dots + \frac{k_{n_0}}{4^{n_0}} + R_1 = K + R_1, \quad t_m = \frac{m_1}{4} + \frac{m_2}{4^2} + \dots + \frac{m_{n_0}}{4^{n_0}} + R_2 = M + R_2$$

the relation $|t_k - t_m| < 1/4^{n_0}$ holds and if the corresponding image points did not lie in the same or in two successive squares of the n_0 th subdivision then

$$|K - M| \geq \frac{2}{4^{n_0}}$$

should hold. But, since $0 < R_i < 1/4^{n_0}$ ($i = 1, 2$) we should have

$$|t_k - t_m| \geq |K - M| - |R_1 - R_2| \geq \frac{1}{4^{n_0}};$$

this contradicts our assumption. The mapping under consideration is not one-to-one, since a point on the boundary of one of the squares has several inverse images.

Problem 8.1. We have to eliminate the parameter t . The functions $x_1(t)$, $x_2(t)$ may be written in the form $F(x_1, t) = 0$ and $G(x_2, t) = 0$. If F and G are polynomials in t we can always find the desired representation by equating to zero the resultant of these two polynomials. In the first case we obtain $x_1^3 + x_2^3 - 3x_1 x_2 = 0$, $x_3 = 0$, and in the second case we find $4b^2 x_1^2 + a^2 x_2^2 - a^2 b^2 = 0$, $x_3 = 0$.

Problem 8.2. Spiral of Archimedes:

$$x_2 - x_1 \tan \frac{\sqrt{(x_1^2 + x_2^2)}}{a} = 0.$$

Cissoid: $x_1(x_1^2 + x_2^2) - 2cx_2 = 0$. Conchoid: $(x_1 - a)^2(x_1^2 + x_2^2) - c^2 x_1^2 = 0$. Trisectrix: We have $\cos \frac{1}{3}\phi = a/r$. Now $\cos \phi = 4\cos^3 \frac{1}{3}\phi - 3\cos \frac{1}{3}\phi$, hence in our case $\cos \phi = 4a^3/r^3 - 3a/r$ and therefore $(x_1 + 3a)(x_1^2 + x_2^2) - 4a^3 = 0$.

Problem 9.1. From (8.4) we obtain $\mathbf{x}(s) = (r \cos(s/w), r \sin(s/w), cs/w)$ where

$$w = \sqrt{r^2 + c^2}.$$

Problem 9.2. We have $\mathbf{x}' = (1, \sinh(t/a), 0)$, $\mathbf{x}' \cdot \mathbf{x}' = 1 + \sinh^2(t/a) = \cosh^2(t/a)$ and therefore

$$s(t) = \int_0^t \cosh \frac{u}{a} du = a \sinh \frac{t}{a},$$

where $s = 0$ corresponds to $t = 0$.

Problem 9.3. From (8.2): $\mathbf{x}(t) = (a \cos t, b \sin t, 0)$ we obtain

$$\mathbf{x}' = (-a \sin t, b \cos t, 0), \quad \mathbf{x}' \cdot \mathbf{x}' = a^2 \sin^2 t + b^2 \cos^2 t$$

and therefore

$$s = a \int_0^{2\pi} \sqrt{(1 - \epsilon^2 \cos^2 t)} dt \quad \text{where} \quad \epsilon = \frac{1}{a} \sqrt{(a^2 - b^2)};$$

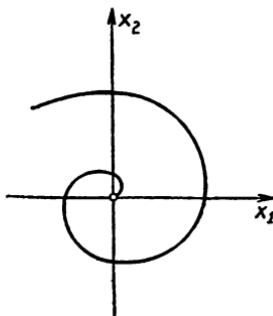


FIG. 101. Spiral of Archimedes

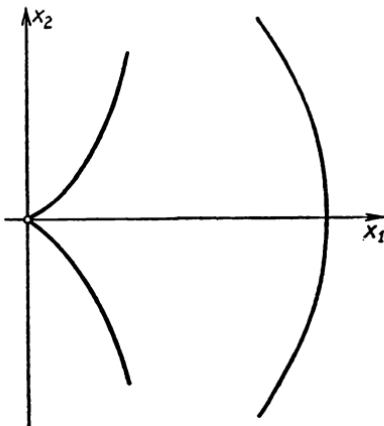


FIG. 102. Cissoid of Diocles

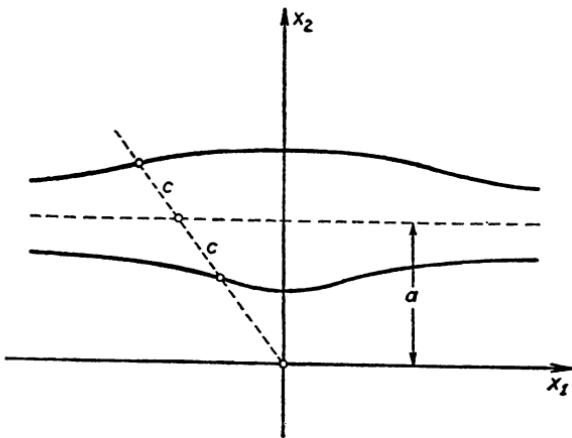


FIG. 103. Conchoid of Nicomedes

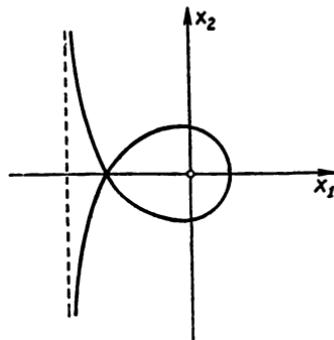


FIG. 104. Trisectrix of Maclaurin

ϵ is called the numerical eccentricity of the ellipse. Using the binomial theorem we find

$$(1 - \epsilon^2 \cos^2 t)^{\frac{1}{2}} = 1 - \frac{1}{2} \epsilon^2 \cos^2 t - \frac{1}{2 \cdot 4} \epsilon^4 \cos^4 t - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \epsilon^6 \cos^6 t - \dots;$$

this expansion converges uniformly in the interval $0 \leq t \leq 2\pi$; we may therefore integrate term by term. Since

$$\int_0^{2\pi} \cos^{2n} t dt = \frac{2\pi}{2^{2n}} \binom{2n}{n} = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}, \quad (n = 1, 2, \dots),$$

we find $s = 2\pi a \left[1 - \left(\frac{1}{2}\right)^2 \epsilon^2 - \frac{1}{3} \left(\frac{1.3}{2.4}\right)^2 \epsilon^4 - \frac{1}{5} \left(\frac{1.3.5}{2.4.6}\right)^2 \epsilon^6 - \dots \right]$.

If $b = a$ then $\epsilon = 0$ and we obtain $s = 2\pi a$ as the arc length of a circle of radius a .

Problem 9.4. It suffices to construct a broken line of chords whose vertices, in their natural order, correspond to increasing (or decreasing) values of t and whose length is infinite. Without loss of generality we set $t_1 = 2/\pi$. We let the vertices P_n of the broken line of chords coincide with the points of the extreme values of the function $\sin 1/t$, that is, the points corresponding to the values $t = 2/(2n+1)\pi$. At those points $|x_2(t)| = t$ and the sign of $x_2(t)$ alternates. The length of the chord $P_{n-1}P_n$ is greater than $4/(2n+1)\pi$, hence the length of the broken line of chords between the points P_0 and P_N is greater than

$$\frac{2}{\pi} \sum_{n=1}^N \frac{1}{n+1};$$

as is known, if N tends to infinity the value of this sum tends to infinity.

Problem 10.1. The curve is a quadratic parabola. We have $\mathbf{x}' = (1, 2t, 0)$ and therefore $\mathbf{y}(v) = (t+v, t^2+2vt, 0)$, cf. (10.3'), in particular at the point corresponding to $t = 1$: $\mathbf{y}(v) = (1+v, 1+2v, 0)$. At the point of intersection the component y_2 of \mathbf{y} must vanish; from this we have $v = -\frac{1}{2}$ and therefore $y_1 = \frac{1}{2}$; that is, the tangent under consideration intersects the x_1 -axis at the point $x_1 = \frac{1}{2}$.

Problem 10.2. From (10.3') we obtain (1) $\mathbf{y}(v) = \mathbf{x} + v\mathbf{x}' = (t+v, Bt(t+2v), Ct^{n-1}(t+nv))$. Since the curve to be determined lies in the x_1x_2 -plane we must have $y_3 = 0$ from which $v = -t/n$ follows. If we insert this into (1) we obtain the representation $\mathbf{z}(t) = [(1-1/n)t, B(1-2/n)t^2, 0]$ of the curve of intersection which, for any value of n ($\neq 0$), is always a quadratic parabola.

Problem 10.3. The circles can be represented in the form $\mathbf{x}(t) = (\cos t, \sin t, 0)$ and $\mathbf{x}^*(t) = (1+\cos t, -1+\sin t, 0)$, respectively. Hence

$$\mathbf{x}' = \mathbf{x}^{**} = (-\sin t, \cos t, 0).$$

The tangents to the circles can therefore be represented in the form

$$\mathbf{y}(v) = \mathbf{x} + v\mathbf{x}' = (\cos t - v \sin t, \sin t + v \cos t, 0),$$

$$\mathbf{y}(v^*) = \mathbf{x}^* + v^*\mathbf{x}^{**} = (1 + \cos t - v^* \sin t, -1 + \sin t + v^* \cos t, 0).$$

From $y_1 = y_1^*$ and $y_2 = y_2^*$ we obtain $(v^* - v)\sin t = 1$ and $(v^* - v)\cos t = 1$, respectively, hence $\sin t = \cos t$ or $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$. Hence the lines $x_2 = -x_1 \pm \sqrt{2}$, $x_3 = 0$ are tangent to both circles.

Problem 11.1. Since $\mathbf{x}'(t) \neq 0$ we may start from a relation of the form

$$x_i'' = c(t)x_i', \quad (i = 1, 2, 3).$$

If we set $x_i' = v_i$ that relation takes the form $v_i'/v_i = c(t)$. We integrate twice, $v_i = C_i e^{\int c dt}$, $x_i(t^*) = C_i t^* + c_i$, where $t^*(t) = \int e^{\int c dt} dt$ ($i = 1, 2, 3$). This is a parametric representation of a straight line.

Problem 12.1. We have $\dot{\mathbf{x}} = \frac{\mathbf{x}'}{|\mathbf{x}'|} = \frac{\mathbf{x}'}{\sqrt{(\mathbf{x}' \cdot \mathbf{x}')}}$ and therefore

$$\ddot{\mathbf{x}} = \frac{d\dot{\mathbf{x}}}{dt} \dot{t} = \frac{d\dot{\mathbf{x}}}{dt} \frac{1}{s'} = \frac{1}{(\mathbf{x}' \cdot \mathbf{x}')^2} [\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}') - \mathbf{x}'(\mathbf{x}' \cdot \mathbf{x}'')].$$

From this, (12.6), and (12.7),

$$\mathbf{z} = \mathbf{x} + \frac{\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}')^2 - \mathbf{x}'(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}' \cdot \mathbf{x}'')}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}.$$

Problem 12.2. We set $w^2 = r^2 + c^2$. We have $s = wt$, cf. Section 9, and therefore $\mathbf{x}(s) = (r \cos(s/w), r \sin(s/w), c(s/w))$, $\dot{\mathbf{x}} = (-r(w^2/c^2)\cos t, -r(w^2/c^2)\sin t, 0)$. Since $\rho = w^2/r$,

cf. Section 12, we thus have from (12.6) $\mathbf{z} = \mathbf{x} + \rho^2 \ddot{\mathbf{x}} = (-c^2/r)\cos t, -(c^2/r)\sin t, ct$. This is a representation of a circular helix on a cylinder Z^* of radius c^2/r . If $c = r$, Z^* coincides with the cylinder Z on which the given helix lies. If $c > r$, Z^* is exterior to Z ; if $c < r$, Z^* is interior to Z .

Problem 13.1. We choose the respective arc lengths s and s^* of the curves as parameters. Then, by a suitable orientation, the unit tangent vectors are identical, $\mathbf{t} \equiv \mathbf{t}^*$. Hence

$$\kappa \mathbf{p} = \frac{d\mathbf{t}}{ds} = \frac{dt^*}{ds^*} \frac{ds^*}{ds} = \kappa^* p^* \frac{ds^*}{ds},$$

that is, $\mathbf{p} = \pm \mathbf{p}^*$. The principal normals of the curves thus have the same direction. Hence, the binormals at corresponding points also have the same direction.

Problem 14.1. We have $\tau = \rho^2 |\dot{\mathbf{x}} \times \ddot{\mathbf{x}}|$ and

$$\dot{\mathbf{x}} = \frac{\mathbf{x}'}{s'}, \quad \ddot{\mathbf{x}} = \frac{\mathbf{x}''}{s'^2} + \dots, \quad \dddot{\mathbf{x}} = \frac{\mathbf{x}'''}{s'^3} + \dots$$

where the dots denote terms which can be made to vanish by the rules of calculus of determinants when $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$, and $\dddot{\mathbf{x}}$ are inserted into the determinant. In consequence of (12.7) we obtain

$$\tau = \frac{(\mathbf{x}' \cdot \mathbf{x}')^3}{s'^6} \frac{|\mathbf{x}' \mathbf{x}'' \mathbf{x}'''|}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}.$$

Since $s'^2 = \mathbf{x}' \cdot \mathbf{x}'$ this is identical with the second expression in (14.4) from which the first one follows by means of the identity of Lagrange.

Problem 14.2. Since $\mathbf{x}' = (1, t, \frac{1}{2}t^2)$, $\mathbf{x}'' = (0, 1, t)$, $\mathbf{x}''' = (0, 0, 1)$ the mixed product in (14.4) has the value 1. We find $\tau = (1 + \frac{1}{2}t^2)^{-\frac{3}{2}}$.

Problem 15.1. We set $\mathbf{t} = \mathbf{a}_1$, $\mathbf{p} = \mathbf{a}_2$, $\mathbf{b} = \mathbf{a}_3$, and

$$(1) \quad \dot{\mathbf{a}}_i = \sum_{k=1}^3 c_{ik} \mathbf{a}_k, \quad (i = 1, 2, 3)$$

and determine the coefficients c_{ik} . The vectors \mathbf{a}_k are orthogonal,

$$(2) \quad \mathbf{a}_i \cdot \mathbf{a}_p = \delta_{ip} = \begin{cases} 0 & (i \neq p) \\ 1 & (i = p). \end{cases}$$

If we differentiate (2) with respect to the arc length s we obtain

$$(3) \quad \dot{\mathbf{a}}_i \cdot \mathbf{a}_p = -\mathbf{a}_i \cdot \dot{\mathbf{a}}_p.$$

On the other hand, if we take the scalar product of (1) and \mathbf{a}_p , in consequence of (2) we have $\dot{\mathbf{a}}_i \cdot \mathbf{a}_p = c_{ip}$, hence, in consequence of (3), $c_{ip} = -c_{pi}$, that is, the matrix (c_{ik}) is skew-symmetric. From this and the Frenet formulae $\dot{\mathbf{t}} = \kappa \mathbf{p}$ and $\dot{\mathbf{b}} = -\tau \mathbf{p}$ we obtain $\dot{\mathbf{p}} = -\kappa \mathbf{t} + \tau \mathbf{b}$.

Problem 16.1. Obviously, the direction of the vector of Darboux \mathbf{d} in space is constant if its derivative $\dot{\mathbf{d}} = \dot{\tau} \mathbf{t} + \tau \dot{\mathbf{t}} + \kappa \dot{\mathbf{b}} + \kappa \dot{\mathbf{b}} = \dot{\tau} \mathbf{t} + \kappa \mathbf{b}$ is proportional to \mathbf{d} , that is, $\dot{\mathbf{d}}$ and \mathbf{d} are linearly dependent,

$$\mathbf{d} \times \dot{\mathbf{d}} = (\tau \mathbf{t} + \kappa \mathbf{b}) \times (\dot{\tau} \mathbf{t} + \kappa \mathbf{b}) = (\dot{\tau}\kappa - \tau\kappa) \mathbf{p} = 0,$$

cf. Theorem 5.2 and formula (5.10). From this we have $\dot{\tau}\kappa - \tau\kappa = 0$ or $\kappa^2 \frac{d(\tau)}{d\theta(\kappa)} = 0$.

In consequence of Theorem 15.1 we thus obtain the result that the vector of Darboux of a curve C at all points of C has the same direction in space if and only if C is a general helix.

Problem 17.1. The tangent indicatrix is a point if the curve is a straight line. The binormal indicatrix is a point if the curve is plane. A spherical image is a closed curve if the generating vector is a periodic function of the parameter. The corresponding curve is not necessarily closed as can be seen, for instance, from a circular helix, cf. Problem 17.2.

Problem 17.2. We set $w^2 = r^2 + c^2$. We have

$$\mathbf{t} = \dot{\mathbf{x}} = \left(-\frac{r}{w} \sin t, \frac{r}{w} \cos t, \frac{c}{w} \right),$$

$$\mathbf{p} = \rho \ddot{\mathbf{x}} = \frac{w^2}{r} \dot{\mathbf{x}} = (-\cos t, -\sin t, 0),$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{p} = \left(\frac{c}{w} \sin t, -\frac{c}{w} \cos t, \frac{r}{w} \right).$$

The spherical images are parallel circles. The principal normal indicatrix lies in the $x_1 x_2$ -plane; the other two images are small circles which coincide if $c = r$.

Problem 18.1. Let C be any curve and $\mathbf{x}(s)$ its representation with respect to the coordinate system defined by (18.3), where s is the arc length of C . The points of C corresponding to the values 0 and s of the parameter determine a chord which has the length

$$h = \sqrt{\left(\sum_{i=1}^3 x_i^2(s) \right)}.$$

From (18.4) we find

$$h^2 = \left(s - \frac{\kappa_0}{6} s^3 + \dots \right)^2 + \left(\frac{\kappa_0}{2} s^2 + \frac{\kappa_0}{6} s^4 + \dots \right)^2 + \left(\frac{\kappa_0 \tau_0}{6} s^3 + \dots \right)^2;$$

hence
$$h^2 = s^2 - \frac{\kappa_0^2}{12} s^4 + \dots, \quad h = s \left(1 - \frac{\kappa_0^2}{24} s^2 + \dots \right)$$

and consequently

$$s - h = \frac{\kappa_0^2}{24} s^3 + \dots .$$

Problem 19.1. The surface is a sphere S of radius r with centre M at the origin of the coordinate system in space. The curve is a straight line C in the $x_1 x_2$ -plane which is parallel to the x_2 -axis and lies at the distance r from that axis. If we insert the representation of C into that of S we find

$$p(s) \equiv G\{x_1(s), x_2(s), x_3(s)\} = s^2 + r^2 - r^2 = 0,$$

that is, the point $P: s = 0$ of C is also a point of S . We have $dp/ds = 2s$, $d^2 p/ds^2 = 2$; the contact of C with S at P is therefore of first order. For example, the circle

$$x_1^2 + x_2^2 - r^2 = 0, \quad x_3 = 0,$$

which lies on S has a contact of first order with C at P .

Problem 19.2. We may represent a plane in the form

$$(1) \quad \mathbf{a} \cdot \mathbf{z} + d = 0,$$

cf. Problem 5.1. Let $\mathbf{z} = \mathbf{x}_0$ be the point at which the curve C has a contact with the plane (1). In order that this contact be of second order we must have

$$(2) \quad (a) \quad \mathbf{a} \cdot \mathbf{x}_0 + d = 0, \quad (b) \quad \mathbf{a} \cdot \mathbf{z}'|_{\mathbf{z}=\mathbf{x}_0} = 0, \quad (c) \quad \mathbf{a} \cdot \mathbf{z}''|_{\mathbf{z}=\mathbf{x}_0} = 0,$$

cf. Lemma 19.2. We denote by \mathbf{x}'_0 and \mathbf{x}''_0 the derivatives of \mathbf{z} at the point \mathbf{x}_0 . In consequence of (2b) and (2c) the vector \mathbf{a} is orthogonal to \mathbf{x}'_0 and \mathbf{x}''_0 , that is, if $\mathbf{x}'_0 \times \mathbf{x}''_0 \neq 0$, the vector \mathbf{a} is of the form

$$(3) \quad \mathbf{a} = \mathbf{c}(\mathbf{x}'_0 \times \mathbf{x}''_0),$$

where c is a real number. The vectors \mathbf{x}'_0 and \mathbf{x}''_0 span the osculating plane S_0 of the curve at the point \mathbf{x}_0 . Because of (3) the normal vector \mathbf{a} of the plane under consideration is orthogonal to S_0 . This completes the proof.

Problem 19.3. Let $\mathbf{x}(s)$ be a representation of the curve C under consideration where s is the arc length of C . In order that the centres of the osculating spheres be independent of s we must have $\dot{\mathbf{a}} = \mathbf{0}$, that is,

$$\dot{\mathbf{a}} = \dot{\mathbf{x}} + \dot{\rho}\mathbf{p} + \rho\dot{\mathbf{p}} + \frac{d}{ds}\left(\frac{\dot{\rho}}{\tau}\right)\mathbf{b} + \frac{\dot{\rho}}{\tau}\mathbf{b} = \left[\rho\tau + \frac{d}{ds}\left(\frac{\dot{\rho}}{\tau}\right)\right]\mathbf{b} = \mathbf{0},$$

cf. (19.12) and (15.1). From this we have $\rho\tau + \frac{d}{ds}\left(\frac{\dot{\rho}}{\tau}\right) = 0$. From (19.13) thus follows

$$\frac{d(R_s^2)}{ds} = 2\frac{\dot{\rho}}{\tau}\left[\rho\tau + \frac{d}{ds}\left(\frac{\dot{\rho}}{\tau}\right)\right],$$

that is, the radius of the osculating spheres is constant, all osculating spheres of C coincide. Hence C lies on a sphere. Curves of this type are called *spherical curves*.

Problem 21.1. From the given equation we find $s = \rho/a - b/a$. Therefore, in consequence of (21.1),

$$\alpha = \int_{s_0}^s \frac{ds}{\rho(s)} = \int_{\rho_0}^\rho \frac{1}{\rho} \frac{ds}{d\rho} d\rho = \frac{1}{a} \log \frac{\rho}{\rho_0} \quad \text{or} \quad \rho = \rho_0 e^{a\alpha}.$$

From (21.1) we have $\rho = ds/d\alpha$. Therefore

$$dx_1 = ds \cos \alpha = \rho \cos \alpha d\alpha = \rho_0 e^{a\alpha} \cos \alpha d\alpha, \quad dx_2 = \rho_0 e^{a\alpha} \sin \alpha d\alpha.$$

Consequently, $d(x_1 + ix_2) = \rho_0 e^{(a+i)\alpha} d\alpha$, and, by integration, $x_1 + ix_2 = \rho_0 e^{(a+i)\alpha}/(a+i)$. From this, $r^2 = x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2) = \rho_0^2 e^{2a\alpha}/(a+i)(a-i)$

or $r(\alpha) = \rho_0 e^{a\alpha}/\sqrt{(a^2+1)}$. This is a logarithmic spiral.

Problem 21.2. We set $x_1 = t$. Then $x_2 = c \cosh(t/c)$ and $\mathbf{x}' = (1, \sinh(t/c), 0)$,

$$\mathbf{x}'' = \left(0, \frac{1}{c} \cosh \frac{t}{c}, 0\right), \quad \mathbf{x}' \cdot \mathbf{x}' = 1 + \sinh^2 \frac{t}{c} = \cosh^2 \frac{t}{c}, \quad \mathbf{x}' \cdot \mathbf{x}'' = \frac{1}{c^2} \cosh^2 \frac{t}{c},$$

$$s = c \sinh \frac{t}{c}, \quad s' = \cosh \frac{t}{c}, \quad s'' = \frac{1}{c} \sinh \frac{t}{c}.$$

From this and (12.7) it follows that $\rho = c \cosh^2(t/c)$ or $\rho = s^2/c + c$, $\tau = 0$.

Problem 22.1. Two different evolutes of the same curve correspond to different values of $\alpha(s)$. Since $\alpha(s)$ is the angle between the vector \mathbf{a} and the unit binormal vector \mathbf{b} the statement follows.

Problem 23.1. Since the unit tangent vectors \mathbf{t} and \mathbf{t}^* are orthogonal to the common principal normal to the curves, \mathbf{t}^* is of the form

$$(1) \quad \mathbf{t}^* = \pm (\mathbf{t} \cos \alpha + \mathbf{b} \sin \alpha)$$

where α is the (constant) angle between \mathbf{t} and \mathbf{t}^* . If we take the vector product of (1) and \mathbf{p}^* we have

$$\mathbf{b}^* = \mathbf{t}^* \times \mathbf{p}^* = (\mathbf{t} \cos \alpha + \mathbf{b} \sin \alpha) \times \mathbf{p} = \mathbf{b} \cos \alpha - \mathbf{t} \sin \alpha.$$

If we differentiate this with respect to s we find

$$(2) \quad \frac{d\mathbf{b}^*}{ds} = \frac{d\mathbf{b}^*}{ds} \frac{ds}{ds} = -\tau^* \mathbf{p}^* \frac{ds^*}{ds} = -(\tau \cos \alpha + \kappa \sin \alpha) \mathbf{p}.$$

Since $\mathbf{p}^* = \pm \mathbf{p}$, in consequence of (23.11) we obtain from (2)

$$(3) \quad \tau^* \frac{ds^*}{ds} = \pm(\tau \cos \alpha + \kappa \sin \alpha) = \pm \frac{\sin \alpha}{a}.$$

From $\mathbf{x}^*(s) = \mathbf{x}(s) + a\mathbf{p}(s)$ we obtain

$$(4) \quad \mathbf{t}^* = \frac{d\mathbf{x}^*}{ds} \frac{ds}{ds^*} = (\mathbf{t} + a\dot{\mathbf{p}}) \frac{ds}{ds^*} = [(1 - a\kappa)\mathbf{t} + a\tau \mathbf{b}] \frac{ds}{ds^*}.$$

If we compare the coefficients of \mathbf{b} in (1) and (4) we obtain $ds^*/ds = \pm a\tau/\sin \alpha$. If we insert this into (3) we find the desired result

$$\tau\tau^* = \frac{\sin^2 \alpha}{a^2} = \text{const.}$$

Problem 25.1. (a) is a representation of the x_2x_3 -plane. Since the corresponding matrix

$$J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is of rank 2 the representation is allowable for every value of (u^1, u^2) . (b) is a representation of the plane passing through the x_3 -axis and intersecting the x_1x_2 -plane along the line $x_1 = x_2$. The corresponding matrix J is of rank 2. (c) is a representation of the cylinder of revolution which has radius a and the x_3 -axis as axis of revolution.

The corresponding matrix

$$J = \begin{pmatrix} -a \sin u^1 & 0 \\ a \cos u^1 & 0 \\ 0 & 1 \end{pmatrix}$$

is of rank 2.

Problem 25.2. The cylinder can be represented in the form

$$\mathbf{x}(u^1, u^2) = (h_1(u^1), h_2(u^1), h_3(u^1) + u^2), \quad u^1 = s;$$

the curve C corresponds to $u^2 = 0$. We have

$$J = \begin{pmatrix} h'_1 & 0 \\ h'_2 & 0 \\ h'_3 & 1 \end{pmatrix},$$

where a prime denotes the derivative with respect to u^1 . J is of rank 2 provided that C is not a straight line parallel to the x_3 -axis; in this case we do not obtain a surface.

Problem 25.3. (a) Ellipsoid: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} - 1 = 0$.

(b) Elliptic paraboloid: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - x_3 = 0$.

(c) Hyperbolic paraboloid: $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_3 = 0$.

(d) Hyperboloid of two sheets: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} + 1 = 0$.

The curves $u^1 = \text{const.}$ are (a), (b), (d) ellipses, (c) hyperbolas. The curves $u^2 = \text{const.}$ are (a) ellipses, (b), (c) parabolas, (d) hyperbolas.

Problem 27.1. We have $x_1 = r \cos \phi$, $x_2 = r \sin \phi$ where x_1 , x_2 are Cartesian coordinates and r, ϕ are polar coordinates. Setting $r = u^1$, $\phi = u^2$ we have

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, 0),$$

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = (u^1)^2, \quad ds^2 = (du^1)^2 + (u^1)^2(du^2)^2$$

or, written in r and ϕ ,

$$ds^2 = dr^2 + r^2 d\phi^2.$$

Problem 27.2. $g_{11} = h_1^2 + h_2^2$, $g_{12} = 0$, $g_{22} = 1$, where a prime denotes the derivative with respect to u^1 . Therefore $ds^2 = (h_1^2 + h_2^2)(du^1)^2 + (du^2)^2$. If we set $h_1 = r \cos u^1$, $h_2 = r \sin u^1$ we obtain a representation of a cylinder of revolution of radius r with x_3 -axis as axis of revolution; to this representation there corresponds the first fundamental form $ds^2 = r^2(du^1)^2 + (du^2)^2$.

Problem 27.3. We set $x_1 = u^1$, $x_2 = u^2$; then (24.3) takes the form

$$\mathbf{x}(u^1, u^2) = (u^1, u^2, F(u^1, u^2)).$$

We find $g_{11} = 1 + F_1^2$, $g_{12} = F_1 F_2$, $g_{22} = 1 + F_2^2$ where $F_\alpha \equiv \partial F / \partial u^\alpha$. Hence the first fundamental form, written in x_1 , x_2 , is given by the expression

$$ds^2 = \left[1 + \left(\frac{\partial x_3}{\partial x_1} \right)^2 \right] dx_1^2 + 2 \frac{\partial x_3}{\partial x_1} \frac{\partial x_3}{\partial x_2} dx_1 dx_2 + \left[1 + \left(\frac{\partial x_3}{\partial x_2} \right)^2 \right] dx_2^2.$$

Problem 27.4. Let x_1 , x_2 , x_3 be Cartesian coordinates. Then

$$(1) \quad ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

Setting $x_1 = u^1 \cos u^3 \cos u^2$, $x_2 = u^1 \cos u^2 \sin u^3$, $x_3 = u^1 \sin u^1$ we obtain spherical coordinates u^1 , u^2 , u^3 , and from (1) it follows that

$$ds^2 = \sum_{\alpha=1}^3 \left(\sum_{\beta=1}^3 \frac{\partial x_\alpha}{\partial u^\beta} du^\beta \right)^2 = (du^1)^2 + (u^1)^2(du^2)^2 + (u^1)^2 \cos^2 u^2 (du^3)^2.$$

Problem 28.1. A surface of revolution S , not a cylinder (cf. Problem 27.2), can be (locally) represented in the form $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2))$. This representation is allowable when $u^2 > 0$ and $dh/du^2 < \infty$, that is, at any point of the rotating curve C which does not lie on the axis A of S or at which the tangent to C is not parallel to A . The curves $u^1 = \text{const.}$ are called meridians, the curves $u^2 = \text{const.}$ are called parallels; the former are the curves of intersection of S and planes passing through the x_3 -axis, the latter are circles parallel to the $x_1 x_2$ -plane. We have

$$ds^2 = (u^2)^2(du^1)^2 + \left[1 + \left(\frac{dh}{du^2} \right)^2 \right] (du^2)^2.$$

Problem 28.2. We choose coordinates in space so that the straight line G_0 coincides with the x_3 -axis. Then G is parallel to the $x_1 x_2$ -plane, and a right conoid S generated by G can be (locally) represented in the form

$$\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^1)).$$

This representation is allowable at any point at which $dh/du^1 < \infty$ and $(u^2)^2 + (dh/du^1)^2 > 0$. The coordinate curves $u^1 = \text{const.}$ are straight lines; each value of u^1 corresponds to a certain position of G in space. Any coordinate line $u^2 = \text{const.}$ is the locus of all points of S which are at the same distance from G_0 . We have

$$ds^2 = \left[(u^2)^2 + \left(\frac{dh}{du^1} \right)^2 \right] (du^1)^2 + (du^2)^2.$$

If in particular h is a linear function of u^1 , say $h = cu^1 + d$, the corresponding right conoid is called a *helicoid* (cf. Section 81); in this case the curves $u^2 = \text{const.}$ are circular helices.

Problem 28.3. Let \bar{g} be the discriminant of the first fundamental form corresponding to the coordinates \bar{u}^1 , \bar{u}^2 . According to (28.4) we have $\bar{g} = \left[\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 g$

and therefore $\bar{g} = \left[\frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 \bar{g} = \left[\frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(\bar{u}^1, \bar{u}^2)} \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 g.$

Now, because of the chain rule,

$$\frac{\partial u^\alpha}{\partial \bar{u}^\gamma} = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial \bar{u}^\gamma}$$

(where, in consequence of the summation convention, we have to sum with respect to β from 1 to 2). Consequently

$$\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^\beta} \frac{\partial \bar{u}^\beta}{\partial \bar{u}^2} \end{vmatrix} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix} \begin{vmatrix} \frac{\partial \bar{u}^1}{\partial \bar{u}^1} & \frac{\partial \bar{u}^1}{\partial \bar{u}^2} \\ \frac{\partial \bar{u}^2}{\partial \bar{u}^1} & \frac{\partial \bar{u}^2}{\partial \bar{u}^2} \end{vmatrix}$$

that is, $\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} = \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(\bar{u}^1, \bar{u}^2)}.$

This important relation is called the *multiplication theorem* for Jacobians; it is the analogue of the chain rule of differential calculus and is valid also in the general case of systems of n functions (of class $r \geq 1$) of n variables, as can be seen from the above reasoning. From this theorem we have

$$\bar{g} = \left[\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 g = \left[\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 g.$$

Problem 28.4. This is an immediate consequence of the multiplication theorem for Jacobians, cf. Problem 28.3; we have

$$1 = \frac{\partial(u^1, u^2)}{\partial(u^1, u^2)} = \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)}.$$

Problem 28.5. The intersection M of a helicoid S and a plane passing through the axis A of S is called a *meridian* of S . The meridians are congruent curves. Consequently, any general helicoid can be generated by a plane curve. We choose as axis A the x_1 -axis of the coordinates in space. Then M can be represented in the form $x_3 = G(u^1)$ where u^1 is the distance of the points of M from A . We assume that at the start of the motion M lies in the $x_1 x_3$ -plane. Let u^2 denote the angle of rotation. The displacement of M has the direction of the x_3 -axis and is proportional to u^2 . We may thus represent S in the form

$$\mathbf{x}(u^1, u^2) = [u^1 \cos u^2, u^1 \sin u^2, c u^2 + G(u^1)].$$

If $c \neq 0$ this local representation is allowable at any point at which the tangent to M is not parallel to A . The curves $u^1 = \text{const.}$ are circular helices; the curves $u^2 = \text{const.}$ are the meridians of the helicoid.

Problem 31.1. $a^{\alpha\beta} b_{\alpha\beta} = a^{11} b_{11} + a^{12} b_{12} + a^{21} b_{21} + a^{22} b_{22}.$

Since $a^{\alpha\beta}$ is skew-symmetric, $a^{11} = a^{22} = 0$, $a^{21} = -a^{12}$. Since $b_{\alpha\beta}$ is symmetric, $b_{12} = b_{21}$. Therefore $a^{\alpha\beta} b_{\alpha\beta} = 0$.

Problem 31.2. Let us denote by $\bar{d}_{\lambda_1 \dots \lambda_r}^{\lambda_1 \dots \lambda_s}$ the tensor (with respect to the coordinates \bar{u}^ν) occurring in (30.5). If we insert into (30.5) the relation analogous to (30.5),

$$\bar{d}_{\lambda_1 \dots \lambda_r}^{\lambda_1 \dots \lambda_s} = \bar{d}_{\eta_1 \dots \eta_r}^{\eta_1 \dots \eta_s} \frac{\partial \bar{u}^{\eta_1}}{\partial \bar{u}^{\lambda_1}} \dots \frac{\partial \bar{u}^{\eta_r}}{\partial \bar{u}^{\lambda_r}} \frac{\partial \bar{u}^{\lambda_1}}{\partial \bar{u}^{\eta_1}} \dots \frac{\partial \bar{u}^{\lambda_s}}{\partial \bar{u}^{\eta_s}}$$

then, in consequence of

$$\frac{\partial \bar{u}^{\lambda\rho}}{\partial \bar{u}^{\kappa\rho}} \frac{\partial u^{\alpha\rho}}{\partial \bar{u}^{\lambda\rho}} = \frac{\partial u^{\alpha\rho}}{\partial \bar{u}^{\kappa\rho}}, \quad \frac{\partial \bar{u}^{\lambda\sigma}}{\partial \bar{u}^{\eta\sigma}} \frac{\partial u^{\eta\sigma}}{\partial u^{\beta\sigma}} = \frac{\partial \bar{u}^{\lambda\sigma}}{\partial u^{\beta\sigma}}, \quad (\rho = 1, 2, \dots, r),$$

we immediately obtain the statement.

Problem 33.1. From (33.2) we obtain

$$g_{\mu\tau} g_{\nu\sigma} \epsilon^{\mu\nu} = \epsilon_{\alpha\beta} g_{\mu\tau} g^{\mu\alpha} g_{\nu\sigma} g^{\nu\beta} = \epsilon_{\alpha\beta} \delta_{\tau}{}^{\alpha} \delta_{\sigma}{}^{\beta},$$

that is,

$$(1) \quad \epsilon_{\tau\sigma} = \epsilon^{\mu\nu} g_{\mu\tau} g_{\nu\sigma}.$$

If we insert the components of the contravariant ϵ -tensor into (1) we find

$$\epsilon_{\tau\sigma} = \frac{1}{\sqrt{g}} (g_{1\tau} g_{2\sigma} - g_{2\tau} g_{1\sigma}),$$

hence $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = \sqrt{g}$, $\epsilon_{21} = -\sqrt{g}$.

Problem 33.2. Using (33.4) and the relation (1) of the answer to Problem 33.1 we obtain $\epsilon^{\alpha\gamma} \epsilon^{\mu\nu} g_{\mu\beta} g_{\nu\gamma} g^{\beta\kappa} = \delta_{\beta}{}^{\alpha} g^{\beta\kappa} = g^{\alpha\kappa}$. In consequence of (32.7) we find from this the desired representation $g^{\alpha\kappa} = \epsilon^{\alpha\gamma} \epsilon^{\kappa\nu} g_{\nu\gamma}$.

Problem 33.3. We have to raise a subscript of $g_{\alpha\beta}$ by means of the contravariant metric tensor. We thus obtain (32.7), that is, the mixed δ -tensor may be considered as the mixed metric tensor.

Problem 33.4. Since contraction has to be carried out with respect to all indices the result is a scalar; namely in each case the number 2.

Problem 35.1. In the answer to Problem 28.5 the curves $u^1 = \text{const.}$ are helices on the helicoid S . We determine the orthogonal trajectories of those helices by introducing new orthogonal coordinates $u^{1*} = u^1$, $u^{2*} = F(u^1, u^2)$. To the coordinates u^1, u^2 there corresponds the first fundamental form

$$ds^2 = (1 + G'^2)(du^1)^2 + 2cG' du^1 du^2 + [(u^1)^2 + c^2](du^2)^2$$

$$\text{or} \quad ds^2 = \left[1 + \frac{(u^1)^2 G'^2}{(u^1)^2 + c^2} \right] (du^1)^2 + [(u^1)^2 + c^2] \left(\frac{cG'}{(u^1)^2 + c^2} du^1 + du^2 \right)^2.$$

$$\text{We set} \quad \frac{cG'}{(u^1)^2 + c^2} du^1 + du^2 = \eta \, du^{2*} \quad (\eta = \text{const.}),$$

that is, we make the transformation

$$u^{1*} = u^1, \quad u^{2*} = \frac{1}{\eta} \left[c \int \frac{G'}{(u^1)^2 + c^2} du^1 + u^2 \right] \quad (\eta \neq 0).$$

Then we have

$$ds^2 = \left[1 + \frac{(u^{1*})^2 G'^2}{(u^{1*})^2 + c^2} \right] (du^{1*})^2 + \eta^2 [(u^{1*})^2 + c^2] (du^{2*})^2,$$

where a prime denotes the derivative with respect to u^{1*} . The coordinates u^{1*}, u^{2*} are orthogonal, cf. Theorem 35.1. The curves $u^{1*} = \text{const.}$ are the helices under consideration, and the curves $u^{2*} = \text{const.}$ are the orthogonal trajectories of those helices.

Problem 37.1. $g_{11} = 1 + a^2$, $g_{12} = 0$, $g_{22} = (u^1)^2$, $dA = \sqrt{(1+a^2)u^1 du^1 du^2}$. The value $x_3 = B$ corresponds to $u^1 = B/a$. We thus find $A = (B^2/a^2)\pi\sqrt{1+a^2}$.

Problem 37.2. We set $x_1 = u^1$, $x_2 = u^2$. Then (24.3) takes the form

$$\mathbf{x}(u^1, u^2) = [u^1, u^2, F(u^1, u^2)].$$

$$\text{We have} \quad g_{11} = 1 + \left(\frac{\partial F}{\partial u^1} \right)^2, \quad g_{12} = \frac{\partial F}{\partial u^1} \frac{\partial F}{\partial u^2}, \quad g_{22} = 1 + \left(\frac{\partial F}{\partial u^2} \right)^2.$$

Hence, replacing u^1 and u^2 by x_1 and x_2 , respectively, we obtain

$$dA = \sqrt{1 + \left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2} dx_1 dx_2.$$

Problem 37.3. We use the representation occurring in the answer to Problem 28.1. Then $dA = u^2 \sqrt{1 + (dh/du^2)^2} du^1 du^2$. In the case of a sphere of radius r we have $x_2^2 + (u^2)^2 = r^2$, hence $h = \sqrt{r^2 - (u^2)^2}$. The area of a hemisphere is therefore given by the integral

$$A = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{r-\epsilon} \int_0^{2\pi} \frac{u^2 r}{\sqrt{r^2 - (u^2)^2}} du^1 du^2 = 2\pi r \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{r-\epsilon} \frac{u^2}{\sqrt{r^2 - (u^2)^2}} du^2 = 2\pi r^2.$$

Problem 38.1. We set $x_1 = u^1$, $x_2 = u^2$, i.e. $\mathbf{x}(u^1, u^2) = [u^1, u^2, F(u^1, u^2)]$. Then $\mathbf{x}_1 = (1, 0, F_{u^1})$, $\mathbf{x}_2 = (0, 1, F_{u^2})$, $\mathbf{x}_{\alpha\beta} = (0, 0, F_{u^\alpha u^\beta})$, $g = 1 + F_{u^1}^2 + F_{u^2}^2$,

$$b_{\alpha\beta} = \frac{1}{\sqrt{g}} \begin{vmatrix} 1 & 0 & F_{u^1} \\ 0 & 1 & F_{u^2} \\ 0 & 0 & F_{u^\alpha u^\beta} \end{vmatrix} = \frac{F_{u^\alpha u^\beta}}{\sqrt{1 + F_{u^1}^2 + F_{u^2}^2}},$$

cf. (38.8) where $F_{u^\alpha} \equiv \partial F / \partial u^\alpha$, etc. We replace u^1, u^2 again by x_1, x_2 , respectively; the second fundamental form is then given by the expression

$$[1 + F_{x_1}^2 + F_{x_2}^2]^{-\frac{1}{2}} [F_{x_1 x_1} (dx_1)^2 + 2F_{x_1 x_2} dx_1 dx_2 + F_{x_2 x_2} (dx_2)^2], \quad F_{x_\alpha} \equiv \frac{\partial F}{\partial x_\alpha}, \text{ etc.}$$

Problem 38.2. Let $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$ ($\alpha = 1, 2$) be an allowable coordinate transformation which preserves the sense of the unit normal vector \mathbf{n} . We denote by $\bar{b}_{\alpha\beta}$ the coefficients of the second fundamental form with respect to the coordinates \bar{u}_1, \bar{u}_2 . Then, because of (38.4) and (29.5),

$$\bar{b}_{\alpha\beta} \equiv \mathbf{x}_{\alpha\beta} \cdot \mathbf{n} = -\mathbf{x}_\alpha \cdot \mathbf{n}_\beta = -\mathbf{x}_\mu \cdot \mathbf{n}_\nu \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\nu}{\partial \bar{u}^\beta} = b_{\mu\nu} \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\nu}{\partial \bar{u}^\beta},$$

where $\mathbf{x}_\alpha = \partial \mathbf{x} / \partial \bar{u}^\alpha$, etc. Indeed these relations are of the form (30.3).

Problem 39.1. Using the representation (25.4):

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, au^1)$$

we have $g_{11} = 1 + a^2$, $g_{12} = 0$, $g_{22} = (u^1)^2$,

$$\mathbf{n} = \frac{1}{u^1 \sqrt{1 + a^2}} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos u^2 & \sin u^2 & a \\ -u^1 \sin u^2 & u^1 \cos u^2 & 0 \end{vmatrix}$$

where \mathbf{e}_i is a unit vector in the direction of the positive ray of the x_i -axis. We thus find

$$\mathbf{n} = \left(-\frac{a \cos u^2}{w}, -\frac{a \sin u^2}{w}, \frac{1}{w} \right), \quad (w \equiv \sqrt{1 + a^2})$$

that is, the normal direction to the cone is independent of u^1 . Since the curves $u^2 = \text{const.}$ are the generating straight lines of the cone, \mathbf{n} is constant along any of those generators; hence the tangent planes to the cone at the points of any generating straight line coincide.

Problem 39.2. The intersection of the paraboloid S and the plane E : $x_3 = x_3^{(0)}$ is a circle of radius $\rho = \sqrt{(x_3^{(0)})^2 - r_0^2}$. This is not a normal section of S ; the angle γ between E and the normal to S can be obtained from the relation

$$\tan(\frac{1}{2}\pi - \gamma) = \tan \alpha = 2r_0.$$

Hence $\cos \gamma = \sin \alpha = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}} = \frac{2r_0}{\sqrt{1 + 4r_0^2}}$.

According to Meusnier's theorem we thus obtain

$$R = \frac{\rho}{\cos \gamma} = \frac{1}{2} / (1 + 4r_0^2).$$

The centre of curvature of the normal section under consideration lies on the axis of revolution.

Problem 39.3. The generating straight lines are asymptotic curves of the cylinder S , cf. Theorem 39.3. We will prove that those curves are the only asymptotic curves of S . We represent S in the form $\mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2)$. Then

$$\mathbf{x}_{11} = (-r \cos u^1, -r \sin u^1, 0), \quad \mathbf{x}_{12} = \mathbf{0}, \quad \mathbf{x}_{22} = \mathbf{0}.$$

Therefore $b_{12} = b_{22} = 0$, and (39.4) thus takes the form $b_{11}(du^1)^2 = 0$. Since

$$b_{11} = -\frac{r^2}{\sqrt{g}} \neq 0$$

we must have $du^1 = 0$, that is, $u^1 = \text{const}$. Since this is the only solution of that equation the above straight lines are the only asymptotic curves on the cylinder.

Problem 40.1. We may locally represent any surface of revolution S which is not a cylinder (cf. Problem 39.3), in the form $\mathbf{x}(u^1, u^2) = [u^2 \cos u^1, u^2 \sin u^1, h(u^2)]$, cf. Problem 28.1. Since $g = (u^2)^2(1+h'^2)$ we find from (38.8)

$$b_{11} = \frac{1}{\sqrt{g}} \begin{vmatrix} -u^2 \sin u^1 & u^2 \cos u^1 & 0 \\ \cos u^1 & \sin u^1 & h' \\ -u^2 \cos u^1 & -u^2 \sin u^1 & 0 \end{vmatrix} = -\frac{u^2 h'}{\sqrt{(1+h'^2)}},$$

$b_{12} = 0$, $b_{22} = -h''/\sqrt{(1+h'^2)}$, where primes denote derivatives with respect to u^2 . That is, the asymptotic curves are solutions of the differential equation

$$u^2 h'(du^1)^2 + h''(du^2)^2 = 0.$$

We have $b = u^2 h' h''/(1+h'^2)$. If $h' = 0$ then $b = 0$, that is, points of S at which the tangent plane is orthogonal to the axis of revolution are parabolic points. Points at which $h' h''$ is positive are elliptic, and points at which $h' h''$ is negative are hyperbolic. If $h'' \equiv 0$ then $h = cu^2 + d$; in this case we obtain a cone of revolution.

Problem 41.1. In the particular case of a cylinder of revolution the theorem holds as can easily be seen. Any other surface of revolution can be (locally) represented in the form $\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2))$. Then the meridians and parallels are coordinate curves, cf. Problem 28.1. We have $g_{12} = 0$ and, since

$$|\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_{12}| = 0,$$

also $b_{12} = 0$. From this and (41.6) Theorem 41.5 follows.

Problem 41.2. Using (25.2) we have $g = r^4 \cos^2 u^2$. Any normal section has the curvature $|\kappa_n| = 1/r$. Hence $K = 1/r^2$ and therefore $b = Kg = r^2 \cos^2 u^2$, cf. (41.9); of course this result can also be obtained from (38.4) by direct calculation.

Problem 44.1. The surface $x_3 = x_2^4$ or $\mathbf{x}(u^1, u^2) = (u^1, u^2, (u^2)^4)$ is of the desired type. Since $b_{11} = b_{12} = 0$ and $b_{22} = 12(u^2)^2/\sqrt{g}$ all points of the x_1 -axis are flat points.

Problem 45.1. If $b_{\alpha\beta} = 0$ then $b_{\alpha\beta}^\beta = b_{\alpha\gamma} g^{\gamma\beta} = 0$ and, because of (45.1), also $n_\alpha = 0$ ($\alpha = 1, 2$); the normal vector to the surface is constant which means that the surface is a plane.

Problem 46.1. (1) In the case of a cylinder of revolution we have $R_{1212} = b = 0$; hence all $R_{\alpha\beta\gamma\eta}$ and $R_{\alpha\beta\gamma\eta}$ vanish. Any other surface of revolution can be (locally)

represented in the form $\mathbf{x}(u^1, u^2) = (u^1 \cos u^1, u^1 \sin u^1, h(u^2))$. Since $g_{11} = (u^1)^2$, $g_{12} = 0$, $g_{22} = 1 + h'^2$, $g = (u^1)^2(1 + h'^2)$, $R_{1212} = b = u^1 h' h'' / (1 + h'^2)$ (cf. Problem 40.1) we obtain from (46.11')

$$R^1_{212} = -R^1_{221} = \frac{h' h''}{u^2(1+h'^2)}, \quad -R^2_{112} = R^2_{121} = \frac{u^2 h' h''}{(1+h'^2)^2};$$

all other components of the mixed curvature tensor vanish (primes indicate derivatives with respect to u^2). (2) Using the representation

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^1, u^1 \sin u^1, h(u^1)),$$

cf. the answer to Problem 28.2, we have

$$\begin{aligned} g_{11} &= (u^1)^2 + h'^2, & g_{12} &= 0, & g_{22} &= 1, & g &= (u^1)^2 + h'^2, \\ b_{11} &= -\frac{u^2 h''}{\sqrt{g}}, & b_{12} &= \frac{h'}{\sqrt{g}}, & b_{22} &= 0, & R_{1212} &= b = -\frac{h'^2}{g} \end{aligned}$$

and therefore

$$R^1_{212} = -R^1_{221} = -\frac{h'^2}{[(u^1)^2 + h'^2]^2}, \quad -R^2_{112} = R^2_{121} = -\frac{h'^2}{(u^1)^2 + h'^2};$$

all other components vanish. (Primes denote derivatives with respect to u^1). (3) Using the notations $F_\alpha = \partial F / \partial x_\alpha$ and $F_{\alpha\beta} = \partial^2 F / \partial x_\alpha \partial x_\beta$ we obtain

$$\begin{aligned} g_{11} &= 1 + F_1^2, & g_{12} &= F_1 F_2, & g_{22} &= 1 + F_2^2, & g &= 1 + F_1^2 + F_2^2, & b_{\alpha\beta} &= \frac{F_{\alpha\beta}}{\sqrt{g}}, \\ R_{1212} &= b = \frac{F_{11} F_{22} - F_{12}^2}{1 + F_1^2 + F_2^2}, \end{aligned}$$

hence $R^1_{112} = -R^1_{121} = \frac{F_1 F_2 (F_{11} F_{22} - F_{12}^2)}{(1 + F_1^2 + F_2^2)^2}$, etc.

(4) We have

$$\begin{aligned} g_{11} &= (r_0 + r \cos u^2)^2, & g_{12} &= 0, & g_{22} &= r^2, & g &= r^2(r_0 + r \cos u^2)^2, \\ R_{1212} &= b = r(r_0 + r \cos u^2) \cos u^2, \end{aligned}$$

cf. Section 43. Consequently

$$R^1_{212} = -R^1_{221} = \frac{r \cos u^2}{r_0 + r \cos u^2}, \quad -R^2_{112} = R^2_{121} = \frac{(r_0 + r \cos u^2) \cos u^2}{r};$$

all other components vanish.

Problem 46.2. We have

$$K = g^{-2}(|\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{11}| |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{22}| - |\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_{12}|^2).$$

If we carry out the multiplication and use (27.1) we find

$$(1) \quad K = g^{-2} \left\{ \begin{array}{ccc} g_{11} & g_{12} & \mathbf{x}_1 \cdot \mathbf{x}_{22} \\ g_{21} & g_{22} & \mathbf{x}_2 \cdot \mathbf{x}_{22} \\ \mathbf{x}_{11} \cdot \mathbf{x}_1 & \mathbf{x}_{11} \cdot \mathbf{x}_2 & \mathbf{x}_{11} \cdot \mathbf{x}_{22} \end{array} \right| - \left| \begin{array}{ccc} g_{11} & g_{12} & \mathbf{x}_1 \cdot \mathbf{x}_{12} \\ g_{21} & g_{22} & \mathbf{x}_2 \cdot \mathbf{x}_{12} \\ \mathbf{x}_{12} \cdot \mathbf{x}_1 & \mathbf{x}_{12} \cdot \mathbf{x}_2 & \mathbf{x}_{12} \cdot \mathbf{x}_{12} \end{array} \right| \right\}.$$

We have

$$(2) \quad \frac{\partial g_{\alpha\beta}}{\partial u^\sigma} = \mathbf{x}_{\alpha\sigma} \cdot \mathbf{x}_\beta - \mathbf{x}_\alpha \cdot \mathbf{x}_{\beta\sigma}$$

cf. (45.10) and (45.6), and thus in particular

$$(3) \quad \frac{1}{2} \frac{\partial g_{\alpha\alpha}}{\partial u^\sigma} = \mathbf{x}_{\alpha\sigma} \cdot \mathbf{x}_\alpha.$$

From (2) it follows that

$$\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} = \mathbf{x}_{112} \cdot \mathbf{x}_2 + \mathbf{x}_{11} \cdot \mathbf{x}_{22} + \mathbf{x}_{12} \cdot \mathbf{x}_{12} + \mathbf{x}_1 \cdot \mathbf{x}_{122}.$$

From (3) we obtain

$$\frac{1}{2} \frac{\partial^2 g_{11}}{(\partial u^1)^2} = \mathbf{x}_{122} \cdot \mathbf{x}_1 + \mathbf{x}_{11} \cdot \mathbf{x}_{12}, \quad \frac{1}{2} \frac{\partial^2 g_{22}}{(\partial u^1)^2} = \mathbf{x}_{112} \cdot \mathbf{x}_2 + \mathbf{x}_{12} \cdot \mathbf{x}_{11}.$$

Hence, by subtraction

$$(4) \quad \mathbf{x}_{11} \cdot \mathbf{x}_{22} - \mathbf{x}_{12} \cdot \mathbf{x}_{11} = -\frac{1}{2} \frac{\partial^2 g_{11}}{(\partial u^1)^2} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2} \frac{\partial^2 g_{22}}{(\partial u^1)^2}.$$

Since the upper left-hand subdeterminants of second order of the two determinants in (1) are equal, we obtain (46.17) by means of (2)-(4).

Problem 47.1. Let x_1, x_2, x_3 be Cartesian coordinates in space. Then u^1, u^2 , defined by $x_1 = u^1 \cos u^2, x_2 = u^1 \sin u^2$, are polar coordinates in the plane $x_3 = 0$. We have

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = (u^1)^2, \quad \Gamma_{122} = \Gamma_{212} = u^1, \quad \Gamma_{221} = -u^1,$$

$$\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = 1/u^1, \quad \Gamma_{22}{}^1 = -u^1;$$

all other Christoffel symbols vanish.

Problem 47.2. We have

$$g_{11} = (u^2)^2, \quad g_{12} = 0, \quad g_{22} = 1 + h'^2, \quad \Gamma_{112} = -u^2, \quad \Gamma_{121} = \Gamma_{211} = u^2,$$

$$\Gamma_{222} = h'h'', \quad \Gamma_{11}{}^2 = -u^2/(1 + h'^2), \quad \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = 1/u^2, \quad \Gamma_{22}{}^2 = h'h''/(1 + h'^2);$$

all other Christoffel symbols vanish.

Problem 47.3. We use the notations $F_\alpha = \partial F / \partial u^\alpha$, $F_{\alpha\beta} = \partial^2 F / \partial u^\alpha \partial u^\beta$ and write the given representation in the form $\mathbf{x}(u^1, u^2) = (u^1, u^2, F(u^1, u^2))$. Then

$$\mathbf{x}_1 = (1, 0, F_1), \quad \mathbf{x}_2 = (0, 1, F_2), \quad \mathbf{x}_{\alpha\beta} = (0, 0, F_{\alpha\beta}),$$

$$g_{11} = 1 + F_1^2, \quad g_{12} = F_1 F_2, \quad g_{22} = 1 + F_2^2.$$

Therefore $\Gamma_{\alpha\beta\gamma} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\gamma = F_{\alpha\beta} F_\gamma \equiv \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \frac{\partial F}{\partial x_\gamma}$,

cf. (45.6). From this, (32.6), and (45.8) we obtain

$$\begin{aligned} \Gamma_{\alpha\beta}{}^1 &= g^{11} \Gamma_{\alpha\beta 1} + g^{12} \Gamma_{\alpha\beta 2} = \frac{1}{g} [(1 + F_2^2) F_{\alpha\beta} F_1 - F_1 F_2 F_{\alpha\beta} F_2] \\ &= \frac{1}{g} F_{\alpha\beta} F_1 \equiv \frac{1}{g} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \frac{\partial F}{\partial x_1}, \end{aligned}$$

and similarly

$$\Gamma_{\alpha\beta}{}^2 = \frac{1}{g} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \frac{\partial F}{\partial x_2}.$$

Problem 50.1. We choose Cartesian coordinates in the plane. Then all Christoffel symbols vanish and (50.2) reduces to $d^2 u^2 / (du^1)^2 = 0$. Integrating twice we have $u^2 = au^1 + c$; this is a representation of a straight line.

Problem 50.2. Let $\mathbf{x}(t) = (r \cos t, r \sin t, ct)$ be any circular helix on a cylinder of revolution $\mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2)$. The vector $\mathbf{x}'' = (-r \cos t, -r \sin t, 0)$ lies in the osculating plane, cf. (11.2). The unit normal vector $\mathbf{n} = (\cos u^1, \sin u^1, 0)$ to the cylinder (cf. (34.1)) lies also in the corresponding osculating plane of the helix; hence helices are geodesics of the cylinder, cf. Theorem 50.1.

Problem 51.1. Cylinders of revolution have been considered just before. Any other surface of revolution can be locally represented in the form

$$\mathbf{x}(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, h(u^2)).$$

We thus obtain $g_{11} = (u^2)^2$, $g_{12} = 0$, $g_{22} = 1 + h'^2$ where a prime denotes the derivative with respect to u^2 . Furthermore, $\Gamma_{12}^{-1} = \Gamma_{21}^{-1} = 1/u^2$; all other Christoffel symbols with the superscript 1 vanish. The first of the two equations (51.5) thus takes the form

$$\ddot{u}^1 + \frac{2}{u^2} \dot{u}^1 \dot{u}^2 = 0 \quad \text{or} \quad \frac{\dot{u}^1}{u^1} = -2 \frac{\dot{u}^2}{u^2}.$$

Integrating once we have $\log u^1 = -2 \log u^2 + c^*$, or

$$\dot{u}^1 = \frac{c}{(u^2)^2},$$

where c^* and c are constants. From this we obtain

$$(du^1)^2 = \frac{c^2}{(u^2)^4} ds^2 = \frac{c^2}{(u^2)^4} [(u^2)^2 (du^1)^2 + (1 + h'^2)(du^2)^2].$$

Integrating once more we have

$$u^1 = \pm c \int \sqrt{\left(\frac{1 + h'^2}{(u^2)^4 - c^2(u^2)^2} \right)} du^2.$$

Setting $c = 0$ we have $u^1 = 0$, that is, the meridian $u^1 = 0$ is a geodesic. Hence all meridians are geodesics.

Problem 53.1. Since the geodesics of a sphere are its great circles, the coordinates u^1, u^2 occurring in (25.2) are geodesic coordinates. If we write

$$\mathbf{x}(u^{1*}, u^{2*}) = (r \cos u^{1*} \cos u^{2*}, r \cos u^{1*} \sin u^{2*}, r \sin u^{1*})$$

we have

$$ds^2 = r^2 [(du^{1*})^2 + \cos^2 u^{1*} (du^{2*})^2].$$

Setting $u^1 = ru^{1*}$, $u^2 = u^{2*}$ the first fundamental form is of the type (52.1). The curves $u^2 = \text{const.}$ form a geodesic field, except at the poles. In the representation

$$\mathbf{x}(u^1, u^2) = \left(r \cos u^2 \sin \frac{u^1}{r}, r \sin u^2 \sin \frac{u^1}{r}, r \cos \frac{u^1}{r} \right)$$

u^1, u^2 are geodesic polar coordinates with centre at the 'North pole' P of the sphere, and u^1 is the arc length of the geodesics $u^2 = \text{const.}$, measured from P .

Problem 54.1. Let the circular ring be bounded by the circles C_1 and C_2 whose respective radii are r_1 and r_2 . We cut the ring along a segment with end points on those circles. Since the ring is plane, $K = 0$. Hence

$$\int_{C_1} \kappa_g ds + \int_{C_2} \kappa_g ds + 2\pi = 2\pi.$$

For plane curves $|\kappa_g| = \kappa$ and therefore in our case

$$\int_{C_1} \kappa_g ds + \int_{C_2} \kappa_g ds = \int_{\alpha=0}^{2\pi} \frac{1}{r_1} r_1 d\alpha - \int_{\alpha=0}^{2\pi} \frac{1}{r_2} r_2 d\alpha = 0;$$

this is in agreement with the preceding relation.

Problem 55.1. A sphere S of radius r has the area $A = 4\pi r^2$ and the constant Gaussian curvature $K = 1/r^2$. Therefore

$$\iint_S K \, dA = \frac{1}{r^2} \iint_S dA = \frac{A}{r^2} = 4\pi.$$

In the case of the torus we have from (43.1)

$$dA = r(r_0 + r \cos u^2) \, du^1 du^2 \quad \text{and} \quad K = \frac{\cos u^2}{r(r_0 + r \cos u^2)}.$$

cf. Section 43, and therefore

$$\iint_S K \, dA = \int_0^{2\pi} \int_0^{2\pi} \cos u^2 \, du^1 du^2 = 0.$$

Problem 57.1. We consider a surface of revolution

$$R: \quad \mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, h(u^1)).$$

Then

$$(1) \quad ds^2 = (1+h'^2)(du^1)^2 + (u^1)^2(du^2)^2.$$

Let S be the helicoid to be mapped onto R . We introduce on S the orthogonal coordinates u^{1*}, u^{2*} , defined in the answer to Problem 35.1. Then

$$ds^{*2} = \left[1 + \frac{(u^{1*})^2 G'^2}{(u^{1*})^2 + c^2} \right] (du^{1*})^2 + \eta^2 [(u^{1*})^2 + c^2] (du^{2*})^2.$$

We set

$$(2) \quad u^2 = u^{2*}, \quad (3) \quad (u^1)^2 = \eta^2 [(u^{1*})^2 + c^2], \quad (4) \quad (1+h'^2) \left(\frac{du^1}{du^{1*}} \right)^2 = 1 + \frac{(u^{1*})^2 G'^2}{(u^{1*})^2 + c^2}.$$

If the right helicoid, that is, the function G , is given we obtain from (3) expressions for u^{1*} and du^1/du^{1*} ; inserting these expressions into (4) and integrating we find the function h , that is, the meridian of the surface of revolution R which can be mapped isometrically onto S . If R is given the corresponding isometric right helicoid can be determined in a similar manner. The isometric mapping is given by (2) and (3). Since in the above expressions the constant η as well as the constant of integration may be arbitrarily chosen, for any given right helicoid there exist infinitely many isometric surfaces of revolution and vice versa. It follows from (2) and (3) that the images of the parallels $u^1 = \text{const.}$ of the surface of revolution are the helices $u^{1*} = \text{const.}$ An example of this mapping will be considered in Problem 81.1.

Problem 58.1. Using (58.1) we have $\mathbf{x}_s = \dot{\mathbf{y}} + t\dot{\mathbf{z}}$, $\mathbf{x}_t = \mathbf{z}$ and, since s is the arc length of the curve $\mathbf{y}(s)$ and \mathbf{z} is a unit vector,

$$g_{11} \equiv \mathbf{x}_s \cdot \mathbf{x}_s = 1 + 2t\dot{\mathbf{y}} \cdot \dot{\mathbf{z}} + t^2\dot{\mathbf{z}} \cdot \dot{\mathbf{z}}, \quad g_{12} \equiv \mathbf{x}_s \cdot \mathbf{x}_t = \dot{\mathbf{y}} \cdot \mathbf{z},$$

$$g_{22} \equiv \mathbf{x}_t \cdot \mathbf{x}_t = 1, \quad g = 1 + 2t\dot{\mathbf{y}} \cdot \dot{\mathbf{z}} + t^2\dot{\mathbf{z}} \cdot \dot{\mathbf{z}} - (\dot{\mathbf{y}} \cdot \mathbf{z})^2.$$

Hence, from (34.1), $\mathbf{n} = \frac{\mathbf{x}_s \times \mathbf{x}_t}{\sqrt{g}} = \frac{1}{\sqrt{g}} (\dot{\mathbf{y}} \times \mathbf{z} + t\dot{\mathbf{z}} \times \mathbf{z}).$

Since $\mathbf{x}_{tt} = 0$ also $b_{22} = \mathbf{x}_{tt} \cdot \mathbf{n} = 0$, cf. (38.8). Since $\mathbf{x}_{st} = \dot{\mathbf{z}}$ we obtain

$$K = \frac{b}{g} = - \left[\frac{|\dot{\mathbf{y}} \mathbf{z} \dot{\mathbf{z}}|}{g} \right]^2.$$

K vanishes if and only if (58.2) is satisfied.

Problem 61.1. For any surface of revolution the Gaussian curvature is the same at all points of any parallel. Since the Gaussian curvature of the given surface of revolution S is assumed to be not constant the images of the parallels of the surface

of revolution S^* to be obtained must be the parallels of S , cf. Theorem 57.2. Since an isometric mapping is conformal, cf. Theorem 61.2, the meridians of S and S^* must also correspond to each other. Let the x_3 -axis be the axis of revolution of S and let $\mathbf{x}(u^1) = ((p(u^1), 0, q(u^1)))$ be a representation of the meridian M of S which lies in the $x_1 x_3$ -plane; let u^1 be the arc length of M . Then S can be represented in the form

$$\mathbf{x}(u^1, u^2) = (p(u^1) \cos u^2, p(u^1) \sin u^2, q(u^1)).$$

S^* may be represented in the form

$$\mathbf{x}^*(u^{1*}, u^{2*}) = (p^*(u^{1*}) \cos u^{2*}, p^*(u^{1*}) \sin u^{2*}, q^*(u^{1*}))$$

where u^{1*} is the arc length of the meridians of S^* . Then we have

$$ds^2 = (du^1)^2 + p^2(du^2)^2 \quad \text{and} \quad ds^{*2} = (du^{1*})^2 + p^{*2}(du^{2*})^2,$$

respectively. The meridians on both surfaces correspond to each other, and so do the parallels. Consequently u^{1*} depends on u^1 only while u^{2*} depends on u^2 only. Comparing the first fundamental forms and using Theorem 57.1 we have

$$(du^{1*})^2 = (du^1)^2,$$

that is, $u^{1*} = \pm u^1 + d$; we may set $d = 0$ and take the positive sign, i.e. $u^{1*} = u^1$; furthermore $p^{*2}(du^{2*})^2 = p^2(du^2)^2$ or

$$\frac{p(u^1)}{p^*(u^1)} = \pm \frac{du^{2*}}{du^2}.$$

The left-hand side of this relation depends on u^1 only while the right-hand side depends on u^2 only; we may thus set $p^* = cp$, $du^{2*} = \pm(1/c) du^2$, where $c (\neq 0)$ is a constant. If we choose the positive sign and set the constant of integration equal to zero we have from this $u^{2*} = u^2/c$. (The negative sign would correspond to a reflection, and the constant of integration corresponds to a rotation about the axis.) Since u^{1*} is the arc length of the meridians we have $p^{*2} + q^{*2} = 1$ and therefore

$$(1) \quad q^* = \int \sqrt{(1 - c^2 p'^2)} du^1.$$

S^* thus has the representation

$$\mathbf{x}^*(u^1, u^2) = \left(cp \cos \frac{u^2}{c}, cp \sin \frac{u^2}{c}, \int \sqrt{(1 - c^2 p'^2)} du^1 \right).$$

If we set $c = 1$ this representation is identical with that of S . A continuous variation of the value of c corresponds to a continuous deformation, that is, a bending. Hence for a given surface of revolution there exists a one-parameter family of isometric surfaces of revolution.

Problem 62.1. The coordinates are orthogonal. We have $g_{11} = g_{22} = \sqrt{g} = A + B$ and from (47.8')

$$2\sqrt{g}\Gamma_{12}^2 = 2\sqrt{g}\Gamma_{11}^1 = -2\sqrt{g}\Gamma_{22}^1 = A', \quad -2\sqrt{g}\Gamma_{11}^2 = 2\sqrt{g}\Gamma_{12}^1 = 2\sqrt{g}\Gamma_{22}^2 = B'$$

where a prime denotes the derivative with respect to that variable on which the corresponding function depends. From (49.7) we thus find

$$\frac{1}{2}[(\dot{u}^1)^2 + (\dot{u}^2)^2](A' \dot{u}^2 - B' \dot{u}^1) + (A + B)(\dot{u}^1 \dot{u}^2 - \dot{u}^1 \dot{u}^2) = 0.$$

This is equivalent to

$$\frac{d}{ds} \left[\frac{A(\dot{u}^2)^2 - B(\dot{u}^1)^2}{(\dot{u}^1)^2 + (\dot{u}^2)^2} \right] = 0.$$

If we integrate we obtain

$$\frac{A(\dot{u}^1)^2 - B(\dot{u}^2)^2}{(\dot{u}^1)^2 + (\dot{u}^2)^2} = c \quad \text{or} \quad \frac{(\dot{u}^1)^2}{A-c} = \frac{(\dot{u}^2)^2}{B+c}$$

where c is a constant of integration. Integrating once more we have

$$\int \frac{du^1}{\sqrt{(A-c)}} - \int \frac{du^2}{\sqrt{(B+c)}} = c^*.$$

Problem 63.1. If we choose the representation

$$\mathbf{x}(u^1, u^2) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1)$$

we have

$$ds^2 = (du^1)^2 + \cos^2 u^1 (du^2)^2 = (du^1 + i \cos u^1 du^2)(du^1 - i \cos u^1 du^2),$$

and the differential equations of the isotropic curves are

$$du^1 + i \cos u^1 du^2 = 0 \quad \text{and} \quad du^1 - i \cos u^1 du^2 = 0.$$

The function $\sec u^1$ is an integrating factor. We thus obtain

$$d\tilde{u}^1 = \sec u^1 du^1 + i du^2, \quad d\tilde{u}^2 = \sec u^1 du^1 - i du^2,$$

and by integration

$$\tilde{u}^1 = \log \tan(\tfrac{1}{2}u^1 + \tfrac{1}{4}\pi) + iu^2, \quad \tilde{u}^2 = \log \tan(\tfrac{1}{2}u^1 + \tfrac{1}{4}\pi) - iu^2.$$

The curves $u^1 = \text{const}$ and $u^2 = \text{const}$ are the isotropic curves on the unit sphere. We may represent these curves in the form

$$\frac{u^{1*}}{u^{2*}} = e^{\pm iu^2} \tan(\tfrac{1}{2}u^1 + \tfrac{1}{4}\pi) = (\cos u^2 \pm i \sin u^2) \tan(\tfrac{1}{2}u^1 + \tfrac{1}{4}\pi) = \text{const.}$$

Then $\sqrt{(u^{1*}u^{2*})} = \tan(\tfrac{1}{2}u^1 + \tfrac{1}{4}\pi), \quad -i \frac{u^{1*}-u^{2*}}{u^{1*}+u^{2*}} = \tan u^2,$

$$\text{or} \quad \sin u^1 = \frac{u^{1*}u^{2*}-1}{u^{1*}u^{2*}+1}, \quad \cos u^1 = \frac{2\sqrt{(u^{1*}u^{2*})}}{u^{1*}u^{2*}+1},$$

$$\sin u^2 = -\frac{i}{2} \frac{u^{1*}-u^{2*}}{\sqrt{(u^{1*}u^{2*})}}, \quad \cos u^2 = \frac{1}{2} \frac{u^{1*}+u^{2*}}{\sqrt{(u^{1*}u^{2*})}}.$$

From this we obtain

$$\mathbf{x}(u^{1*}, u^{2*}) = \left[\frac{u^{1*}+u^{2*}}{u^{1*}u^{2*}+1}, -i \frac{u^{1*}-u^{2*}}{u^{1*}u^{2*}+1}, \frac{u^{1*}u^{2*}-1}{u^{1*}u^{2*}+1} \right].$$

This is a representation of the unit sphere; the coordinate curves $u^{1*} = \text{const}$ and $u^{2*} = \text{const}$ are the isotropic curves on this sphere. The ratio

$$\frac{\partial x_1}{\partial u^{2*}} : \frac{\partial x_2}{\partial u^{2*}} : \frac{\partial x_3}{\partial u^{2*}} = [1 - (u^{1*})^2] : i[1 + (u^{1*})^2] : 2u^{1*}$$

is independent of u^{2*} ; the ratio

$$\frac{\partial x_1}{\partial u^{1*}} : \frac{\partial x_2}{\partial u^{1*}} : \frac{\partial x_3}{\partial u^{1*}} = [1 - (u^{2*})^2] : [-i(1 + (u^{2*})^2)] : 2u^{2*}$$

is independent of u^{1*} ; consequently the isotropic curves of the unit sphere are (imaginary) straight lines.

Problem 67.1. Since $\partial x_3^*/\partial u^1 = 0$ we have

$$\bar{D} = \frac{\partial x_1^*}{\partial u^1} \frac{\partial x_2^*}{\partial u^2} = 2r^2 \sin(\tfrac{1}{2}\pi - \tfrac{1}{2}u^2) \cos(\tfrac{1}{2}\pi - \tfrac{1}{2}u^2) = r^2 \sin(\tfrac{1}{2}\pi - u^2) = r^2 \cos u^2.$$

The mapping is equiareal. The North pole $u^2 = \tfrac{1}{2}\pi$, cf. (66.1), (67.1) is mapped onto

the point $x_1^* = 0, x_2^* = r\sqrt{\pi}$. The South pole corresponds to a segment of the straight line $x_2^* = -\sqrt{\pi}(\sqrt{2}-1)r$; this segment has the length $4r\sqrt{(2\pi)}$. The meridians are mapped onto segments of straight lines as can easily be seen. The image of the sphere is an isosceles triangle whose base and apex correspond to the South pole and North pole, respectively. The parallels are mapped onto segments which are parallel to the base.

Problem 68.1. In the case of Cartesian coordinates we have

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

Using (68.6) we thus obtain

$$(2) \quad ds^2 = \sum_{i=1}^3 \left(\frac{\partial x_i}{\partial u^1} du^1 + \frac{\partial x_i}{\partial u^2} du^2 + \frac{\partial x_i}{\partial u^3} du^3 \right)^2$$

or

$$(3) \quad ds^2 = g_{\alpha\beta} du^\alpha du^\beta \quad \text{where } g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta;$$

here we have to sum with respect to α and β (independently) from 1 to 3. If the coordinates are orthogonal, $g_{\alpha\beta} = 0, \alpha \neq \beta$ ($\alpha, \beta = 1, 2, 3$), cf. (68.7) and therefore

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2.$$

Problem 68.2. We have

$$D = \begin{vmatrix} \sin u^2 \cos u^3 & u^1 \cos u^2 \cos u^3 & -u^1 \sin u^2 \sin u^3 \\ \sin u^2 \sin u^3 & u^1 \cos u^2 \sin u^3 & u^1 \sin u^2 \cos u^3 \\ \cos u^2 & -u^1 \sin u^2 & 0 \end{vmatrix} = (u^1)^2 \sin u^2.$$

If $u^1 = 0$ or $u^2 = 0$ then $D = 0$, that is, D vanishes at every point of the x_3 -axis. (68.7) is satisfied, as can easily be verified. Consequently the coordinate surfaces form a triply orthogonal system, except along the x_3 -axis.

Problem 81.1. If $u^1 = a \cosh(x_3/a)$ is a representation of the meridian of the catenoid this surface may be represented in the form

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, a \cosh^{-1}(u^1/a)).$$

Then

$$(\alpha) \quad ds^2 = \frac{(u^1)^2}{(u^1)^2 - a^2} (du^1)^2 + (u^1)^2 (du^2)^2.$$

We may use the answer to Problem 57.1; from equation (3), cf. Problem 57.1, we obtain

$$\frac{du^1}{du^{1*}} = \eta^* \frac{u^{1*}}{u^1}.$$

Comparing (1) and (α) we have

$$1 + h'^2 = \frac{(u^1)^2}{(u^1)^2 - a^2}.$$

Hence if we replace u^1 according to (3), the left-hand side of (4) takes the form

$$\frac{\eta^4(u^{1*})^2}{\eta^2[(u^{1*})^2 + c^2] - a^2}.$$

(4) can be solved for G' ; we set $\eta = 1$ and $a^2 - c^2 = k^2$; then

$$\begin{aligned} G &= k \int \sqrt{\left(\frac{(u^{1*})^2 + c^2}{(u^{1*})^2[(u^{1*})^2 - k^2]} \right)} du^{1*} \\ &= k \log[\sqrt{(u^{1*})^2 + c^2} + \sqrt{(u^{1*})^2 - k^2}] - c \tan^{-1} \frac{k}{c} \sqrt{\left(\frac{(u^{1*})^2 + c^2}{(u^{1*})^2 - k^2} \right)}, \end{aligned}$$

where we have set the constant of integration equal to zero; (this constant corresponds to a displacement of the right helicoid parallel to the x_3 -axis). We thus obtain the following representation of the right helicoids which are isometric with the given catenoid (cf. Problems 28.5 and 35.1):

$$(1) \quad \mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, cu^2 + G(u^1)),$$

where G is defined by the preceding formula. The pitch $2\pi c$ can be arbitrarily chosen. If we choose $c = 0$ we obtain the given catenoid. If c increases, the pitch increases. If $c = a$, that is, $k = 0$, we obtain the right helicoid

$$\mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, cu^2).$$

This proves that a catenoid, which has been cut along a meridian, can be bent onto a pitch of the right helicoid.

Problem 81.2. The smallest circle C is given by $u^1 = a$ (cf. the notations in the answer to the preceding problem). If we insert this into equation (3) of the answer to Problem 57.1 and if we set $\eta = 1$ we obtain $(u^1)^2 = a^2 - c^2$. Equation (1), cf. Problem 88.1, thus takes the form

$$(1) \quad \mathbf{x}(u^2) = (\sqrt{(a^2 - c^2)} \cos u^2, \sqrt{(a^2 - c^2)} \sin u^2, cu^2 + k_0);$$

we may set $k_0 = 0$. The image of C is a pitch of a circular helix on a cylinder of revolution of radius $r = \sqrt{(a^2 - c^2)}$; the length $2\pi a$ of the image is independent of c . The bending illustrated in Fig. 86 corresponds to an increase of c from 0 to a ; in the course of this bending the radius r of the above cylinder decreases and ultimately tends to zero, that is, the image of C ultimately is a segment of a straight line.

Problem 82.1. We have

$$x_1 = \Re \int (1-t^2) \frac{i}{2t^2} dt = \Re \left[-\frac{i}{2} \left(\frac{1+t^2}{t} \right) \right].$$

If we insert $t = t_1 + it_2$ we obtain

$$-\frac{i}{2} \left(\frac{1+t^2}{t} \right) = -\frac{i}{2(t_1^2+t_2^2)} [1+t_1^2+2it_1t_2-t_2^2](t_1-it_2)$$

hence $x_1 = \frac{t_2}{2} \left(1 - \frac{1}{t_1^2+t_2^2} \right)$ and similarly $x_2 = -\frac{t_1}{2} \left(1 - \frac{1}{t_1^2+t_2^2} \right)$.

Therefore

$$(1) \quad \frac{x_1}{x_2} = -\frac{t_2}{t_1}.$$

Finally

$$x_3 = \Re \int \frac{i}{t} dt = \Re[i \log t] = \Re \left[i \left(\log |t| + i \tan^{-1} \frac{t_2}{t_1} \right) \right] = -\tan^{-1} \frac{t_2}{t_1}$$

or, using (1), $\tan(-x_3) = -\tan x_3 = \frac{t_2}{t_1} = -\frac{x_1}{x_2}$.

Problem 83.1. We represent the minimal surface S in the form

$$\mathbf{x}(u^1, u^2) = \mathbf{y}(u^1) + \mathbf{z}(u^2).$$

Then, according to (83.5), the adjoint minimal surface S^* can be represented in the form $\mathbf{x}^*(u^1, u^2) = i\mathbf{y}(u^1) - i\mathbf{z}(u^2)$. Since $b_{12} \equiv \mathbf{x}_{12} \cdot \mathbf{n} = 0$, in consequence of (39.4) the asymptotic curves of S are determined by the equation

$$(1) \quad b_{11}(du^1)^2 + b_{22}(du^2)^2 = 0.$$

Since $g_{11}^* = g_{22}^* = 0$, cf. (83.3) the lines of curvature of S^* are determined by

$$(2) \quad g_{12}^* [b_{22}^*(du^2)^2 - b_{11}^*(du^1)^2] = 0,$$

cf. (41.2'). Now

$$b_{11}^* \equiv \mathbf{x}_{11}^* \cdot \mathbf{n} = i \frac{d^2 \mathbf{y}}{(du^1)^2} \cdot \mathbf{n} = i b_{11}, \quad b_{22}^* \equiv \mathbf{x}_{22}^* \cdot \mathbf{n} = -i \frac{d^2 \mathbf{z}}{(du^2)^2} \cdot \mathbf{n} = -i b_{22}.$$

By comparing (1) and (2) we thus obtain the statement.

Problem 85.1. Let S : $\mathbf{x}(u^1, u^2)$ be the surface under consideration; we assume that, for instance, $\kappa_1 = 1/R_1 = \text{const}$ ($\neq 0$) and that the coordinates u^1, u^2 were chosen so that the coordinate curves are lines of curvature on S . The centre of curvature corresponding to κ_1 has the position vector $\mathbf{y} = \mathbf{x} + R_1 \mathbf{n}$. Because of (45.1') we obtain from this

$$\mathbf{y}_1 \equiv \frac{\partial \mathbf{y}}{\partial u^1} = \mathbf{x}_1 + R_1 \mathbf{n}_1 = 0,$$

the position of that centre of curvature depends on u^2 only, that is, is the same for all points of any (fixed) curve $u^2 = \text{const}$. Consequently the totality of all those centres of curvature form a curve C : $\mathbf{y}(u^2)$. We have

$$(\mathbf{x} - \mathbf{y}) \cdot \mathbf{y}_2 = -R_1 \mathbf{n} \cdot (\mathbf{x}_2 + R_1 \mathbf{n}_2);$$

this expression vanishes since the vectors \mathbf{x}_2 and \mathbf{n}_2 are orthogonal to \mathbf{n} . This means that the vector $\mathbf{x} - \mathbf{y}$ is orthogonal to the vector \mathbf{y}_2 ; the vector \mathbf{y}_2 is tangent to the curve C and is constant if $u^2 = \text{const}$. Hence the curves $u^2 = \text{const}$ are plane; the plane of such a curve passes through the corresponding centre of curvature; since

$$|\mathbf{x} - \mathbf{y}| = |R_1| = \text{const},$$

any of those curves is a circle with the same radius whose plane is the normal plane to the curve C at the corresponding point. We thus obtain a canal surface.

Problem 87.1. Since the unit binormal vector \mathbf{b} to the curve $\mathbf{y}(s)$ is orthogonal to the osculating plane, the family of those planes can be represented in the form

$$(1) \quad G(\mathbf{x}, s) = [\mathbf{x} - \mathbf{y}(s)] \cdot \mathbf{b}(s) = 0;$$

s denotes the arc length of $\mathbf{y}(s)$. In consequence of (85.5) the characteristics of the family can be represented by (1) and $\partial G / \partial s = -\mathbf{t} \cdot \mathbf{b} + (\mathbf{x} - \mathbf{y}) \cdot \dot{\mathbf{b}} = 0$ or, since $\tau \neq 0$,

$$(2) \quad (\mathbf{x} - \mathbf{y}) \cdot \mathbf{p} = 0.$$

It follows from (1) and (2) that the vector $\mathbf{x} - \mathbf{y}$ is orthogonal to \mathbf{b} as well as to the unit principal normal vector \mathbf{p} , that is, $\mathbf{x} - \mathbf{y}$ lies in the tangent to $\mathbf{y}(s)$, and

$$(3) \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{t}(s)$$

is a representation of the envelope of the osculating planes. The formulae (3) and (58.1') are identical; the tangent surface of a curve is the envelope of the family of the osculating planes. If we differentiate (2) with respect to s we find

$$-\mathbf{t} \cdot \mathbf{p} + (\mathbf{x} - \mathbf{y}) \cdot \dot{\mathbf{p}} = 0.$$

From this, (1), $\kappa > 0$, and (15.1) we have $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{t} = 0$. Hence, because of (3), $\mathbf{x} = \mathbf{y}$, that is, the curve $\mathbf{y}(s)$ is the edge of regression of the envelope of the osculating planes.

Problem 87.2. The family of the rectifying planes of a curve $\mathbf{y}(s)$ can be represented in the form

$$(1) \quad G(\mathbf{x}, s) = [\mathbf{x} - \mathbf{y}(s)] \cdot \mathbf{p}(s) = 0,$$

cf. (13.2); s denotes the arc length of $\mathbf{y}(s)$. Consequently the vector $\mathbf{x} - \mathbf{y}$ is of the form

$$(2) \quad \mathbf{x} - \mathbf{y} = \alpha \mathbf{t} + \beta \mathbf{b}.$$

According to (85.5) the characteristics of the family (1) can be represented by (1) and $\partial G / \partial s = -\mathbf{t} \cdot \mathbf{p} + (\mathbf{x} - \mathbf{y}) \cdot \dot{\mathbf{p}} = 0$ or, because of (15.1),

$$(3) \quad (\mathbf{x} - \mathbf{y}) \cdot (-\kappa \mathbf{t} + \tau \mathbf{b}) = 0.$$

By inserting (2) into (3) we find $-\alpha \kappa + \beta \tau = 0$ or $\alpha = c\tau, \beta = c\kappa$, where c is a constant. Consequently $\mathbf{x} - \mathbf{y} = c(\tau \mathbf{t} + \kappa \mathbf{b}) = cd$ where d is the vector of Darboux, cf. (16.5). We thus obtain $\mathbf{x}(s, t) = \mathbf{y}(s) + td(s)$; this completes the proof of Theorem 87.3.

Problem 87.3. At any point of the curve $\mathbf{y}(s)$ under consideration the osculating plane is orthogonal to the rectifying plane; the normal vector to the rectifying developable surface R lies in the corresponding osculating plane of $\mathbf{y}(s)$. In consequence of Theorem 50.1 $\mathbf{y}(s)$ thus is a geodesic on R . Since the geodesic curvature of a curve is invariant under an isometric mapping (cf. Theorem 57.2), $\mathbf{y}(s)$ is mapped onto a geodesic in the image plane, that is, onto a straight line.

Problem 88.1. The centre surface of a surface \mathbf{x} corresponding to the principal curvature $1/R_1$ can be represented in the form $\mathbf{y} = \mathbf{x} + R_1 \mathbf{n}$, cf. (88.8). We choose coordinates u^1, u^2 on \mathbf{x} so that the coordinate curves are lines of curvature. Then, in consequence of (45.1'), the partial derivatives $\mathbf{y}_a = \mathbf{x}_a + (\partial R_1 / \partial u^a) \mathbf{n} + R_1 \mathbf{n}_a$ take the form

$$\mathbf{y}_1 = \frac{\partial R_1}{\partial u^1} \mathbf{n}, \quad \mathbf{y}_2 = \left(1 - \frac{R_1}{R_2}\right) \mathbf{x}_2 + \frac{\partial R_1}{\partial u^2} \mathbf{n}.$$

Hence the coefficients g_{ab}^* of the first fundamental form of the centre surface under consideration are given by the expressions

$$g_{11}^* = \left(\frac{\partial R_1}{\partial u^1}\right)^2, \quad g_{12}^* = \frac{\partial R_1}{\partial u^1} \frac{\partial R_1}{\partial u^2}, \quad g_{22}^* = \left(1 - \frac{R_1}{R_2}\right)^2 g_{22} + \left(\frac{\partial R_1}{\partial u^2}\right)^2.$$

Similar expressions can be obtained for the coefficients of the first fundamental form of the centre surface corresponding to $1/R_2$.

Problem 88.2. Let S_1 denote that surface on which C is a line of curvature. The normals to S_1 along C generate a developable surface T_1 , cf. Theorem 88.1. Let G_1 be the edge of regression of T_1 . The surface T_1 is the tangent surface of G_1 . Since C is orthogonal to the generators of T_1 , C is an involute of G_1 , that is, G_1 is an evolute of C . The corresponding tangents to two involutes of the same curve intersect at a constant angle, cf. Theorem 22.1. If we rotate the generators until they assume the normal direction to S_2 we obtain a ruled surface T_2 . By assumption the angle of rotation is the same for all those generators, hence G_1 is transformed into another evolute of C , say G_2 , and T_2 is the tangent surface of G_2 , that is, T_2 is a developable surface. From this and Theorem 88.1 it follows that C is also a line of curvature on the other of the two given surfaces. The second statement of Theorem 88.3 can be proved in a similar manner: According to Theorem 88.1 the normals to the given surfaces at the points of C generate two developable surfaces T_1 and T_2 , that is, tangent surfaces of two different evolutes G_1 and G_2 of C ; the corresponding tangents to G_1 and G_2 , that is, the corresponding generators of T_1 and T_2 , intersect at a constant angle; hence the surfaces possess the same property. Since any curve on a sphere or in a plane is a line of curvature, Theorem 88.4 is an immediate consequence of Theorem 88.3.

Problem 88.3. Using Theorem 57.1 we have to prove that the first fundamental forms of the centre surfaces S of surfaces of Weingarten W , satisfying the same

relation $F(R_1, R_2) = 0$, can be transformed so that they become identical with the first fundamental form of a surface of revolution R represented by

$$(1) \quad \mathbf{x}(u^1, u^2) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)).$$

We assume that S corresponds, for instance, to the principal curvature $1/R_1$ of W . Let

$$x_1 = r(u^1), \quad x_3 = h(u^1)$$

be a representation of the meridian M of R in the $x_1 x_3$ -plane and let u^1 be the arc length of M . Then $r'^2 + h'^2 = 1$, where the prime denotes the derivative with respect to u^1 , and $ds^2 = (du^1)^2 + r^2(du^2)^2$. We choose coordinates on W so that the coordinate curves are lines of curvature. Then

$$(2) \quad R_1 = g_{11}/b_{11}, \quad R_2 = g_{22}/b_{22},$$

cf. (41.11), and

$$(3) \quad \frac{\partial b_{11}}{\partial u^2} = \frac{1}{2} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right) \frac{\partial g_{11}}{\partial u^2}, \quad \frac{\partial b_{22}}{\partial u^1} = \frac{1}{2} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right) \frac{\partial g_{22}}{\partial u^1},$$

cf. (46.3'), (41.6), and (47.8'). Now, by (2),

$$(4) \quad \frac{\partial b_{22}}{\partial u^1} = \frac{\partial}{\partial u^1} \left(\frac{g_{22}}{R_2} \right) = \frac{1}{R_2} \frac{\partial g_{22}}{\partial u^1} - \frac{g_{22}}{R_2^2} \frac{\partial R_2}{\partial u^1}.$$

By inserting (2) into (3) we have

$$(5) \quad \frac{\partial b_{22}}{\partial u^1} = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\partial g_{22}}{\partial u^1}.$$

Suppose that $R_1 - R_2 \neq 0$. Then from (4) and (5) we obtain

$$\frac{g_{22}}{R_2} \frac{\partial R_2}{\partial u^1} = \frac{\partial g_{22}}{\partial u^1} - R_2 \frac{\partial b_{22}}{\partial u^1} = \frac{R_1 - R_2}{2R_1} \frac{\partial g_{22}}{\partial u^1}.$$

Therefore $\frac{\partial}{\partial u^1} \log \sqrt{g_{22}} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial u^1} = \frac{R_1}{R_2(R_1 - R_2)} \frac{\partial R_2}{\partial u^1}$

and from this

$$\frac{\partial}{\partial u^1} \log \left[\sqrt{g_{22}} \left(1 - \frac{R_1}{R_2} \right) \right] = \frac{\partial}{\partial u^1} \left[\log \sqrt{g_{22}} + \log \left(1 - \frac{R_1}{R_2} \right) \right] = \frac{1}{R_1 - R_2} \frac{\partial R_1}{\partial u^1}.$$

If we insert R_2 (as a function of R_1) into this expression and integrate we obtain

$$(6) \quad g_{22} \left(1 - \frac{R_1}{R_2} \right)^2 = e^{2 \int dR_1 / (R_1 - R_2)}.$$

Using the answer to Problem 88.1 we find the first fundamental form of the centre surface,

$$ds^2 = \left(\frac{\partial R_1}{\partial u^1} \right)^2 (du^1)^2 + 2 \frac{\partial R_1}{\partial u^1} \frac{\partial R_1}{\partial u^2} du^1 du^2 + \left[\left(1 - \frac{R_1}{R_2} \right)^2 g_{22} + \left(\frac{\partial R_1}{\partial u^2} \right)^2 \right] (du^2)^2$$

or $ds^2 = (dR_1)^2 + g_{22} \left(1 - \frac{R_1}{R_2} \right)^2 (du^2)^2, \quad dR_1 = \frac{\partial R_1}{\partial u^1} du^1 + \frac{\partial R_1}{\partial u^2} du^2.$

In consequence of (6) we thus obtain

$$(7) \quad ds^2 = (dR_1)^2 + e^{2 \int dR_1 / (R_1 - R_2)} (du^2)^2.$$

Since W is a surface of Weingarten, R_2 is a function of R_1 only. Hence the coefficient of $(du^2)^2$ in (7) depends only on R_1 and is the same for all surfaces of Weingarten defined by the same relation $F(R_1, R_2) = 0$. Consequently, the centre surfaces under consideration are isometric and the corresponding first fundamental form (7) is of the type of that of a surface of revolution. This completes the proof.

Problem 90.1. The representation

$$\mathbf{x}(u^1, u^2) = \left(r \cos u^2 \sin \frac{u^1}{r}, r \sin u^2 \sin \frac{u^1}{r}, r \cos \frac{u^1}{r} \right)$$

is of the desired type. The 'North pole' P corresponds to $u^1 = 0$, and the 'equator' corresponds to $u^1 = \frac{1}{2}\pi r$. P is the centre of these geodesic polar coordinates; we have $g_{11} = 1$, $g_{12} = 0$, $g_{22} = r^2 \sin^2(u^1/r)$, $ds^2 = (du^1)^2 + r^2 \sin^2(u^1/r) (du^2)^2$; since $K = 1/r^2$, this is in agreement with (90.4).

Problem 96.1. We will prove that if we apply (96.14) to (96.12) we obtain

$$(1) \quad -\frac{2 du^1 du^2}{c^2(u^1 - u^2)^2} = -\frac{2 du^{1*} du^{2*}}{c^2(u^{1*} - u^{2*})^2}.$$

From (96.14) it follows that

$$du^{\mu*} = \frac{D du^\mu}{(\gamma u^\mu + \delta)^2}, \quad (\mu = 1, 2), \quad u^{1*} - u^{2*} = \frac{D(u^1 - u^2)}{(\gamma u^1 + \delta)(\gamma u^2 + \delta)}, \quad D \equiv \alpha\delta - \beta\gamma.$$

By inserting this we see that (1) holds.

Problem 96.2. According to (96.5) u^2 is a function of the type indicated in the enunciation of the problem and satisfies (96.7). In order to prove the converse we have to solve (96.7) which is equivalent to

$$3 \frac{u^{2''}}{u^{2'}} - 2 \frac{u^{2''}}{u^{2''}} = 0.$$

By integration we find, $3 \log u^{2'} - 2 \log u^{2''} = c^*$ or

$$\frac{u^{2''}}{(u^{2'})^{\frac{3}{2}}} = c.$$

We integrate once more, and find $-2(u^{2'})^{-\frac{1}{2}} = cu^1 + d$, or

$$u^{2'} = \frac{4}{(cu^1 + d)^2}.$$

We integrate once more, and have

$$u^2 = -\frac{4}{c(cu^1 + d)} + a;$$

this completes the proof.

Problem 96.3. We consider a surface S of constant Gaussian curvature and introduce on S geodesic polar coordinates u^1, u^2 (cf. Section 53). Then the curves $u^1 = \text{const}$ are geodesic circles. Since $g_{11} = 1$ these curves have the geodesic curvature

$$(\kappa_g)_{u^1=\text{const}} = \frac{\partial}{\partial u^1} (\log \sqrt{g_{22}}),$$

cf. (49.9 a). In order that the geodesic curvature of any of these curves be a constant, that is, be independent of u^2 , we must have

$$\frac{\partial^2}{\partial u^1 \partial u^2} (\log \sqrt{g_{22}}) = 0,$$

that is, $\log \sqrt{g_{22}}$ must be a sum of two functions each of which depends on one of the two variables u^1, u^2 only. That is, $\sqrt{g_{22}}$ must be a product of two such functions. As can be seen from (90.3)–(90.5) this holds in the case of surfaces of constant Gaussian curvature (g_{22} even depends on u^1 only).

COLLECTION OF FORMULAE

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(Sections 3–5) Vectors in Euclidean space R_3

Notation: $\mathbf{a} = (a_1, a_2, a_3)$, etc.; a_1, a_2, a_3 are the *components* of the vector \mathbf{a} with respect to Cartesian coordinates in R_3 .

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Sum of two vectors \mathbf{a} and \mathbf{b} : vector with components

$$(5.2) \quad c_i = a_i + b_i.$$

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Scalar product of two vectors \mathbf{a} and \mathbf{b} :

$$(5.3) \quad \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^3 a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

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Length of a vector \mathbf{a} :

$$(5.4) \quad |\mathbf{a}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})}.$$

12

Unit vector: vector of length 1.

9

Angle α between two vectors \mathbf{a} and \mathbf{b} :

$$(5.6) \quad \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{(\mathbf{a} \cdot \mathbf{a})}\sqrt{(\mathbf{b} \cdot \mathbf{b})}}.$$

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Orthogonality of two vectors \mathbf{a} and \mathbf{b} : $\mathbf{a} \cdot \mathbf{b} = 0$.

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Vector product of two vectors \mathbf{a} and \mathbf{b} :

$$(5.7) \quad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & a_1 & b_1 \\ \mathbf{e}_2 & a_2 & b_2 \\ \mathbf{e}_3 & a_3 & b_3 \end{vmatrix}$$

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where \mathbf{e}_i is a unit vector in the (positive) direction of the i th coordinate axis of the Cartesian coordinate system in space.

$$(5.8) \quad |\mathbf{v}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha$$

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where α is the angle between the vectors \mathbf{a} and \mathbf{b} .

Linear dependence of two vectors \mathbf{a} and \mathbf{b} :

$$\mathbf{a} \times \mathbf{b} = 0.$$

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Mixed product (determinant) of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$(5.13) \quad |\mathbf{a} \ \mathbf{b} \ \mathbf{c}| = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

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Linear dependence of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$|\mathbf{a} \ \mathbf{b} \ \mathbf{c}| = 0.$$

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(Sections 6–23) Curves

Parametric representation:

$$(6.1) \quad \mathbf{x} = \mathbf{x}(t), \quad \mathbf{x}' \equiv \frac{d\mathbf{x}}{dt} \neq \mathbf{0}. \quad 17$$

Arc length:

$$(9.1) \quad s = \int \sqrt{(\mathbf{x}' \cdot \mathbf{x}') dt}; \quad 26$$

linear element:

$$(9.7) \quad ds = \sqrt{d\mathbf{x} \cdot d\mathbf{x}}. \quad 28$$

*Moving trihedron:**Unit tangent vector*

$$(10.2) \quad \mathbf{t} = \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}'}{|\mathbf{x}'|}. \quad 30$$

Unit principal normal vector

$$(12.2) \quad \mathbf{p} = \frac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|} = \frac{\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|}, \quad (\dot{\mathbf{t}} \neq \mathbf{0}). \quad 34$$

Unit binormal vector

$$(13.1) \quad \mathbf{b} = \mathbf{t} \times \mathbf{p}. \quad 36$$

Osculating plane: spanned by \mathbf{t} and \mathbf{p} ,

$$(11.2) \quad |(\mathbf{z}-\mathbf{x}) \quad \mathbf{x}' \quad \mathbf{x}''| = 0 \quad (\mathbf{x}' \times \mathbf{x}'' \neq \mathbf{0}). \quad 33$$

$$(13.2 \text{ c}) \quad (\mathbf{z}-\mathbf{x}) \cdot \mathbf{b} = 0 \quad 37$$

Normal plane: spanned by \mathbf{p} and \mathbf{b} ,

$$(13.2 \text{ a}) \quad (\mathbf{z}-\mathbf{x}) \cdot \mathbf{t} = 0. \quad 37$$

Rectifying plane: spanned by \mathbf{b} and \mathbf{t} ,

$$(13.2 \text{ b}) \quad (\mathbf{z}-\mathbf{x}) \cdot \mathbf{p} = 0, \quad (\ddot{\mathbf{x}} \neq \mathbf{0}). \quad 37$$

*Curvature and torsion:**Curvature:*

$$(12.3), (12.7) \quad \kappa = |\dot{\mathbf{t}}| = \sqrt{(\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}})} = \frac{\sqrt{\{(\mathbf{x}' \cdot \mathbf{x}')(x'' \cdot x'') - (x' \cdot x'')^2\}}}{(x' \cdot x')} \quad 35$$

Radius of curvature

$$(12.4) \quad \rho = \frac{1}{\kappa} \quad (\kappa > 0). \quad 34$$

Centre of curvature M

$$(12.6) \quad \mathbf{z} = \mathbf{x} + \rho \mathbf{p}, \quad (\kappa > 0). \quad 35$$

Osculating circle: circle of radius ρ in the osculating plane, with centre at M .

Torsion:

(14.2)–(14.4)

$$\tau = -\mathbf{p} \cdot \dot{\mathbf{b}} = \frac{|\dot{\mathbf{x}} \ddot{\mathbf{x}} \ddot{\mathbf{x}}|}{\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}} = \frac{|\mathbf{x}' \mathbf{x}'' \mathbf{x}'''|}{(\mathbf{x}' \cdot \mathbf{x}')(x'' \cdot x'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}. \quad 38$$

Polar axis: straight line through M and orthogonal to the osculating plane. The polar axis is the locus of the centres of the spheres which have a contact of (at least) second order with the curve at the corresponding point. Representation:

(19.10) $\mathbf{a}(\beta) = \mathbf{x} + \rho \mathbf{p} + \beta \mathbf{b}, \quad (\kappa > 0).$

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Osculating sphere S : centre

(19.12) $\mathbf{a} = \mathbf{x} + \rho \mathbf{p} + \frac{\dot{\rho}}{\tau} \mathbf{b}, \quad (\kappa > 0, \quad \tau \neq 0),$

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radius

(19.13) $R_S = \sqrt{\left\{ \rho^2 + \left(\frac{\dot{\rho}}{\tau} \right)^2 \right\}}. \quad 55$

S has a contact of (at least) third order with the curve at the corresponding point.

Formulae of Frenet:

(15.1) $\mathbf{t} = \kappa \mathbf{p}, \quad \dot{\mathbf{p}} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \dot{\mathbf{b}} = -\tau \mathbf{p}, \quad (\kappa > 0),$

41

Vector of Darboux:

(16.5) $\mathbf{d} = \tau \mathbf{t} + \kappa \mathbf{b}.$

45

Spherical images: Linear elements

tangent indicatrix

(17.1 a) $ds_T^2 = \kappa^2 ds^2, \quad 46$

principal normal indicatrix

(17.1 b) $ds_P^2 = (\kappa^2 + \tau^2) ds^2, \quad 46$

binormal indicatrix

(17.1c) $ds_B^2 = \tau^2 ds^2, \quad 46$

hence

(17.2) $\kappa = \frac{ds_T}{ds}, \quad |\tau| = \frac{ds_B}{ds}; \quad 46$

Equation of Lancret

(17.3) $ds_P^2 = ds_T^2 + ds_B^2. \quad 47$

Canonical representation:

$$(18.4) \quad x_1 = s - \frac{\kappa_0^2}{6} s^3 + \dots, \quad x_2 = \frac{1}{2} \kappa_0 s^2 + \frac{\dot{\kappa}_0}{6} s^3 + \dots,$$

$$x_3 = \frac{\kappa_0 \tau_0}{6} s^3 + \dots.$$

48

Natural equations: Given two continuous functions $\kappa = \kappa(s)$ (> 0) and $\tau = \tau(s)$ ($0 \leq s \leq a$) of a real variable s , there exists an arc C : $\mathbf{x}(s)$ ($0 \leq s \leq a$) whose arc length is s and whose curvature and torsion are given by the above functions; C is uniquely determined except for its position in space. $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the natural equations of C .

56

Involutes and evolutes:

Involutes of a curve $\mathbf{x}(s)$: curves which intersect the tangents of $\mathbf{x}(s)$ at right angles. Representation:

$$(22.2) \quad \mathbf{z}(s) = \mathbf{x}(s) + (c - s)\mathbf{t}(s),$$

65

where c is an arbitrary constant.

Evolutes of a curve $\mathbf{x}(s)$: curves for which $\mathbf{x}(s)$ is an involute.

Representation:

$$(22.3) \quad \mathbf{y}(s) = \mathbf{x}(s) + \rho(s)[\mathbf{p}(s) + \mathbf{b}(s) \cot \alpha(s)], \quad \alpha(s) = \int_0^s \tau(\sigma) d\sigma + k^*.$$

The evolutes lie on the polar surface of $\mathbf{x}(s)$.

67

Bertrand curves: curves with common principal normals. Given a plane curve C there exist curves C^* such that C and C^* are Bertrand curves; given a twisted curve C there exists a curve C^* such that C and C^* are Bertrand curves if and only if the curvature and torsion of C satisfy a relation of the form

$$(23.2) \quad c_1 \kappa(t) + c_2 \tau(t) = 1,$$

68

where c_1 and c_2 are constants.

(Sections 27, 29–33) Tensors

Summation convention : If in a product an index occurs twice, once as superscript and once as subscript, summation has to be carried out with respect to that index from 1 to 2; the summation sign will be omitted.

84

Transformation law of a tensor of order $r+s$, covariant of order r , contravariant of order s :

$$(30.5) \quad \bar{a}_{\gamma_1 \dots \gamma_r}^{\kappa_1 \dots \kappa_s} = a_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s} \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\gamma_1}} \dots \frac{\partial u^{\alpha_r}}{\partial \bar{u}^{\gamma_r}} \frac{\partial \bar{u}^{\kappa_1}}{\partial u^{\beta_1}} \dots \frac{\partial \bar{u}^{\kappa_s}}{\partial u^{\beta_s}}. \quad 97$$

Superscript: contravariant index; *subscript:* covariant index. 98

Important particular cases of (30.5):

contravariant tensor of first order (contravariant vector)

$$(29.4) \quad \bar{a}^\beta = a^\alpha \frac{\partial u^\beta}{\partial u^\alpha}, \quad 90$$

covariant tensor of first order (covariant vector)

$$(29.5) \quad \bar{a}_\beta = a_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \quad 92$$

contravariant tensor of second order

$$(30.2) \quad \bar{a}^{\gamma\kappa} = a^{\alpha\beta} \frac{\partial \bar{u}^\gamma}{\partial u^\alpha} \frac{\partial \bar{u}^\kappa}{\partial u^\beta}, \quad 94$$

covariant tensor of second order

$$(30.3) \quad \bar{a}_{\gamma\kappa} = a_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\kappa}, \quad 96$$

mixed tensor of second order

$$(30.4) \quad \bar{a}_\gamma{}^\kappa = a_\alpha{}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial \bar{u}^\kappa}{\partial u^\beta}. \quad 96$$

Sum of tensors: The tensors to be added must be of the same type; add the components whose indices have the same values; the result is a tensor of the same type as that of the given tensors.

99

Product of two arbitrary tensors $a_{\alpha_1 \dots \alpha_p}^{\beta_1 \dots \beta_q}$ and $b_{\gamma_1 \dots \gamma_r}^{\eta_1 \dots \eta_s}$:
tensor with components

$$h_{\alpha_1 \dots \alpha_p \gamma_1 \dots \gamma_r}^{\beta_1 \dots \beta_q \eta_1 \dots \eta_s} = a_{\alpha_1 \dots \alpha_p}^{\beta_1 \dots \beta_q} b_{\gamma_1 \dots \gamma_r}^{\eta_1 \dots \eta_s}. \quad 99$$

The contravariant and covariant orders of the product tensor equal the sum of the respective orders of the given tensors.

Contraction of a mixed tensor: Equate a subscript and a superscript and sum with respect to that pair of indices; the result is a tensor of order 2 less, the contravariant and covariant orders being each one less than for the original tensor.

100

Inner product of two tensors: Multiplication followed by contraction with respect to two indices each of which belongs to one of the two tensors.

100

(Sections 24–28, 32, 34–37) Concept of a surface. First fundamental form

Parametric representation of a surface:

$$(24.1), (24.2) \quad \mathbf{x} = \mathbf{x}(u^1, u^2), \quad \left(\frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \equiv \mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0} \right). \quad 72$$

The real variables u^1, u^2 are called *coordinates* on the surface.

First fundamental form:

$$(27.1), (27.2) \quad ds^2 = g_{\alpha\beta} du^\alpha du^\beta, \quad g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \quad g_{\alpha\beta} = g_{\beta\alpha}. \quad 82$$

Discriminant: $g \equiv \det(g_{\alpha\beta}) = g_{11}g_{22} - (g_{12})^2$. At regular points, $g_{11} > 0$, $g > 0$, and the first fundamental form is positive definite. 86

Behaviour with respect to coordinate transformations: The quantities $g_{\alpha\beta}$ are the components of a covariant tensor of second order (*metric tensor, fundamental tensor*),

$$(28.3) \quad \bar{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial u^\mu}{\partial \bar{u}^\alpha} \frac{\partial u^\nu}{\partial \bar{u}^\beta}, \quad \text{cf. (30.3).} \quad 86$$

$$(28.4) \quad \bar{g} = \left[\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right]^2 g. \quad 87$$

Contravariant metric tensor $g^{\alpha\beta}$:

$$(32.6) \quad g^{11} = \frac{g_{22}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g}, \quad 104$$

$$(32.7) \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta.$$

$$(32.8) \quad g^* \equiv \det(g^{\alpha\beta}) = \frac{1}{g}. \quad 104$$

Further special tensors:

$$\text{Mixed } \delta\text{-tensor: } \delta_\alpha^\beta = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta) \end{cases} \quad 106$$

Covariant ϵ -tensor of second order:

$$(33.1) \quad \epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = \sqrt{g}. \quad 106$$

Contravariant ϵ -tensor of second order:

$$(33.3) \quad \epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = \frac{1}{\sqrt{g}}; \quad 106$$

$$(33.4) \quad \epsilon^{\alpha\gamma} \epsilon_{\beta\gamma} = \delta_\beta^\alpha. \quad 107$$

Tangent plane: Representation:

$$(26.3) \quad \mathbf{y}(q^1, q^2) = \mathbf{x} + q^1 \mathbf{x}_1 + q^2 \mathbf{x}_2 \quad 80$$

or

$$(26.4) \quad |(\mathbf{y} - \mathbf{x}) \quad \mathbf{x}_1 \quad \mathbf{x}_2| = 0, \quad (\mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0}) \quad 81$$

or, if the surface is represented in the form $G(x_1, x_2, x_3) = 0$,
 $(\partial G / \partial x_i \neq 0$ for at least one value of $i = 1, 2, 3$),

$$(26.6) \quad \sum_{i=1}^3 (y_i - x_i) \frac{\partial G}{\partial x_i} = 0. \quad 82$$

Vector in a surface S: vector \mathbf{v} bound at a point P of S and in the tangent plane $E(P)$ to S at P . Representation (32.1) $\mathbf{v} = a^\alpha \mathbf{x}_\alpha$. The quantities a^α are called the *contravariant components* of \mathbf{v} ; the quantities $a_\alpha = \mathbf{x}_\alpha \cdot \mathbf{v}$ are called the *covariant components* of \mathbf{v} . 101

$$(32.4) \quad a_\alpha = g_{\alpha\beta} a^\beta, \quad (32.5) \quad a^\beta = g^{\alpha\beta} a_\alpha. \quad 104$$

Unit normal vector n to a surface S: $\mathbf{x}(u^1, u^2)$:

$$(34.1) \quad \mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{+\sqrt{g}}, \quad (\mathbf{x}_1 \times \mathbf{x}_2 \neq 0). \quad 107$$

Curves on a surface S: $\mathbf{x}(u^1, u^2)$:

Parametric representation: $u^\alpha = u^\alpha(t)$. 79

Arc length: (35.1) $s = \int \sqrt{(g_{\alpha\beta} u^\alpha u^\beta)} dt$. 109

Angle γ between two intersecting curves $u^\alpha = h^\alpha(t)$, $u^\alpha = h^{\alpha*}(t)$, ($\alpha = 1, 2$) *on a surface:*

$$(35.5) \quad \cos \gamma = \frac{g_{\alpha\beta} h^\alpha h^{\beta*}}{\sqrt{(g_{\mu\nu} h^\mu h^\nu)} \sqrt{(g_{\sigma\tau} h^{\sigma*} h^{\tau*})}}. \quad 110$$

These curves intersect orthogonally if and only if $g_{\alpha\beta} h^\alpha h^{\beta*} = 0$.

Angle ω between the coordinate curves:

$$(35.6) \quad \cos \omega = \frac{g_{12}}{\sqrt{(g_{11} g_{22})}}. \quad 110$$

Orthogonal coordinate system: (35.7) $g_{12} = 0$. 110

Area of a portion of a surface:

$$(36.7) \quad A = \iint_U \sqrt{g} du^1 du^2. \quad 114$$

Element of area:

$$(36.8) \quad dA = \sqrt{g} du^1 du^2. \quad 114$$

(Sections 38–44, 48) *Second fundamental form. Gaussian and mean curvature of a surface*

Second fundamental form: (38.2)–(38.4), (38.8) $b_{\alpha\beta} du^\alpha du^\beta$, 119

$$b_{\alpha\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{n} = -\mathbf{x}_\alpha \cdot \mathbf{n}_\beta = \frac{|\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_{\alpha\beta}|}{\sqrt{g}}, \quad b_{\alpha\beta} = b_{\beta\alpha}. \quad 120$$

Discriminant (40.1) $b = \det(b_{\alpha\beta}) = b_{11} b_{22} - (b_{12})^2$. 125

Normal, principal, Gaussian, and mean curvature:

Normal curvature

$$(39.2) \quad \kappa_n = \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}; \quad 122$$

$|\kappa_n|$ is the curvature of a normal section of the surface in the corresponding direction $du^2:du^1$.

Meusnier's formula:

$$(39.3) \quad \rho = R \cos \gamma \quad (R = 1/\kappa_n); \quad 122$$

γ is the angle between the unit principal normal vector of the curve (with the radius of curvature ρ) and the unit normal vector to the surface under consideration.

Navel point (umbilic): the normal curvature κ_n is independent of the direction. 128

Principal curvatures κ_1 and κ_2 : extreme values of the normal curvature. κ_1 and κ_2 are the roots of the equation

$$(41.8) \quad \kappa_n^2 - 2H\kappa_n + K = 0. \quad 130$$

Principal directions: directions corresponding to κ_1 and κ_2 . At any point which is not a navel point the principal directions are orthogonal; these directions are determined by the roots of

$$(41.2'') \quad \begin{vmatrix} (du^2)^2 & -du^1 du^2 & (du^1)^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix} = 0. \quad 129$$

Euler's formula:

$$(42.1) \quad \kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha;$$

α is the angle between the normal section and the principal direction corresponding to κ_1 .

Gaussian curvature:

$$(41.9), (46.12) \quad K \equiv \kappa_1 \kappa_2 = \frac{b}{g} = \frac{R_{1212}}{g}. \quad 131$$

145

Theorema egregium (Gauss): K depends on the first fundamental form only (Theorem 46.1). 145

Types of points of a surface: elliptic point: $b > 0$ (hence $K > 0$), parabolic point: $b = 0$ (hence $K = 0$), hyperbolic point: $b < 0$ (hence $K < 0$).

Mean curvature:

$$(41.10) \quad H \equiv \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}b_{\alpha\beta}g^{\alpha\beta}. \quad 131$$

Indicatrix of Dupin: elliptic point: ellipse; parabolic point: two

parallel straight lines; hyperbolic point: two hyperbolas. The principal axes of the indicatrix correspond to the principal directions on the surface. <i>Conjugate directions</i> on a surface: The corresponding diameters of the indicatrix of Dupin are conjugate.	133						
<i>Asymptotic direction</i> ; direction for which $\kappa_n = 0$. At hyperbolic (parabolic, elliptic) points there are 2 (1, no) real asymptotic directions.	122						
<i>Asymptotic curve</i> : curve whose direction is asymptotic at any of its points. Differential equation (39.4) $b_{\alpha\beta} du^\alpha du^\beta = 0$. Any straight line on a surface is an asymptotic curve. At the points of an asymptotic curve at which $\kappa > 0$ the osculating plane of the curve and the tangent plane to the surface coincide.	123						
<i>Line of curvature</i> : curve whose direction is a principal direction at any of its points. A curve C is a line of curvature if and only if the normals to the surface along C generate a developable surface (Theorem 88.1). If the curve of intersection of two surfaces is a line of curvature on both surfaces the angle of intersection is constant (Theorem 88.3). The intersection of any two surfaces of a triply orthogonal system is a line of curvature on both surfaces (Theorem 68.1, Dupin). The lines of curvature form an orthogonal net (except at navel points) (Theorem 41.2). The coordinate curves on a surface are lines of curvature if and only if	129 273 276 216 130						
(41.6) $g_{12} = 0, \quad b_{12} = 0;$ in this case	130						
(41.11) $\kappa_1 = \frac{b_{11}}{g_{11}}, \quad \kappa_2 = \frac{b_{22}}{g_{22}}, \quad K = \frac{b_{11} b_{22}}{g_{11} g_{22}}, \quad H = \frac{1}{2} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right).$	132						
<i>Special coordinates</i> on a surface: The following coefficients of the first and second fundamental form are zero:							
Coordinate curves	g_{11}	g_{12}	g_{22}	b_{11}	b_{12}	b_{22}	Cf.
orthogonal	0	(35.7)
conjugate	0	..	(60.4)
lines of curvature .	..	0	0	..	(41.6)
asymptotic curves	0	..	0	(39.6)
isotropic . .	0	..	0	(63.4)
(Isotropic curves: $ds^2 = 0, \mathbf{x}' \neq \mathbf{0}$.)							199
Isothermic coordinates: if $ds^2 = \eta(u^1, u^2)[(du^1)^2 + (du^2)^2]$.							195

(Sections 45–47) Christoffel symbols, curvature tensors,
formulae of Weingarten and Gauss*Christoffel symbols**of the first kind:*

$$(47.1) \quad \Gamma_{\alpha\beta\gamma} \equiv \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\gamma = \frac{1}{2} \left(\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} \right), \quad 148$$

$$(47.4) \quad \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} = \Gamma_{\alpha\gamma\beta} + \Gamma_{\beta\gamma\alpha}. \quad 148$$

of the second kind:

$$(47.2) \quad \Gamma_{\alpha\beta}^\sigma = g^{\sigma\alpha} \Gamma_{\alpha\beta}, \quad (47.2') \quad \Gamma_{\alpha\beta\gamma} = g_{\gamma\sigma} \Gamma_{\alpha\beta}^\sigma. \quad 148$$

Symmetry relations:

$$(47.3) \quad \Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}, \quad \Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma. \quad 148$$

Transformation behaviour:

$$(47.12') \quad \Gamma_{\alpha\beta}^\gamma = \left(\bar{\Gamma}_{\lambda\epsilon}^\sigma \frac{\partial \bar{u}^\lambda}{\partial u^\alpha} \frac{\partial \bar{u}^\epsilon}{\partial u^\beta} + \frac{\partial^2 \bar{u}^\sigma}{\partial u^\alpha \partial u^\beta} \right) \frac{\partial u^\gamma}{\partial \bar{u}^\sigma} \quad 151$$

$$(47.13) \quad \Gamma_{\alpha\beta\rho} = \left(\bar{\Gamma}_{\lambda\epsilon\tau} \frac{\partial \bar{u}^\lambda}{\partial u^\alpha} \frac{\partial \bar{u}^\epsilon}{\partial u^\beta} + \bar{g}_{\sigma\tau} \frac{\partial^2 \bar{u}^\sigma}{\partial u^\alpha \partial u^\beta} \right) \frac{\partial u^\rho}{\partial \bar{u}^\tau}.$$

Formulae of Weingarten:

$$(45.1) \quad \mathbf{n}_\alpha \equiv \frac{\partial \mathbf{n}}{\partial u^\alpha} = -b_{\alpha\sigma} g^{\sigma\beta} \mathbf{x}_\beta \quad (\alpha = 1, 2). \quad 139$$

Formulae of Gauss:

$$(45.14) \quad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma + b_{\alpha\beta} \mathbf{n} \quad (\alpha, \beta = 1, 2). \quad 142$$

Formulae of Mainardi–Codazzi:

$$(46.3) \quad \Gamma_{\alpha\beta}^\rho b_{\rho\lambda} - \Gamma_{\alpha\lambda}^\rho b_{\rho\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} - \frac{\partial b_{\alpha\lambda}}{\partial u^\beta} = 0. \quad 143$$

Mixed Riemann curvature tensor:

$$(46.5) \quad R^\sigma_{\alpha\lambda\beta} = \frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\lambda} - \frac{\partial \Gamma_{\alpha\lambda}^\sigma}{\partial u^\beta} + \Gamma_{\alpha\beta}^\kappa \Gamma_{\kappa\lambda}^\sigma - \Gamma_{\alpha\lambda}^\kappa \Gamma_{\kappa\beta}^\sigma. \quad 144$$

Covariant Riemann curvature tensor:

$$(46.6) \quad R_{\tau\alpha\lambda\beta} = g_{\sigma\tau} R^\sigma_{\alpha\lambda\beta}, \quad 144$$

$$(46.6') \quad R'_{\alpha\lambda\beta} = g^{\tau\gamma} R_{\tau\alpha\lambda\beta}, \quad 144$$

$$(46.8)–(46.10) \quad R_{\tau\alpha\lambda\beta} = R_{\lambda\beta\tau\alpha}, \quad R_{\tau\alpha\lambda\beta} = -R_{\alpha\beta\lambda\tau}, \quad 144$$

$$R_{\tau\alpha\lambda\beta} = -R_{\alpha\tau\lambda\beta}.$$

$$(46.11) \quad R_{1212} = R_{2121} = -R_{2112} = -R_{1221} = b; \quad 145$$

all other components are zero.

(Sections 49–55) Geodesics

Geodesic curvature of a curve C: $\mathbf{x}(s)$ on a surface S: $\mathbf{x}(u^1, u^2)$:

$$(49.5) \quad \kappa_g = |\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \cdot \mathbf{n}|. \quad 155$$

$|\kappa_g|$ is the curvature of the curve C' which is obtained by orthogonal projection of C onto the tangent plane to S at the point P under consideration. Geodesic curvature of the coordinate curves of an orthogonal coordinate system:

$$(49.9) \quad \begin{aligned} (\kappa_g)_{u^1=\text{const}} &= \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial u^1} (\log \sqrt{g_{22}}), \\ (\kappa_g)_{u^2=\text{const}} &= -\frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^2} (\log \sqrt{g_{11}}), \end{aligned} \quad (g_{12} = 0). \quad 157$$

Geodesic: curve C on a surface for which $\kappa_g \equiv 0$. Differential equations

$$(51.5) \quad \ddot{u}^\tau + \Gamma_{\nu\sigma}^\tau \dot{u}^\nu \dot{u}^\sigma = 0 \quad (\tau = 1, 2). \quad 162$$

(Dots denote derivatives with respect to the arc length s of C .)

An arc of shortest length joining two (fixed) points on a surface must necessarily be an arc of a geodesic (Theorem 51.1).

Geodesic parallel coordinates: The coordinate curves are the geodesics of a geodesic field and their orthogonal trajectories. First fundamental form

$$(52.1) \quad ds^2 = (du^1)^2 + g_{22}(u^1, u^2)(du^2)^2; \quad 163$$

then

$$(52.2) \quad K = -\frac{1}{\sqrt{g_{22}}} \frac{\partial^2 \sqrt{g_{22}}}{(\partial u^1)^2}. \quad 163$$

Geodesic polar coordinates with centre P: The coordinate curves are the geodesics through P and their orthogonal trajectories.

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Gauss–Bonnet theorem:

$$(54.1) \quad \int_C \kappa_g ds + \sum_\nu \alpha_\nu + \iint_S K dA = 2\pi; \quad 169$$

α_ν is the angle of rotation of the tangent at the ν th vertex of the boundary C of the simply connected portion S of a surface.

$\iint_S K dA$ is called the *integral curvature* of S . If G is a closed orientable surface of class $r \geq 3$ and of genus p then

$$(55.1) \quad \iint_G K dA = 4\pi(1-p). \quad 172$$

(Sections 56–68, 94–96) Mappings

Let T be a one-to-one mapping of a portion S of a surface onto a portion S^* of a surface. T is said to be

- (a) isometric, if corresponding arcs on S and S^* have the same length, 176
- (b) conformal, if corresponding pairs of intersecting curves on S and S^* make the same angle, 193
- (c) equiareal, if the areas of corresponding parts of S and S^* are equal, 208
- (d) geodesic, if the geodesics on S and S^* correspond to each other. 290

Corresponding necessary and sufficient conditions (under the assumption that the coordinate systems u^1, u^2 on S and u^{1*}, u^{2*} on S^* are the same, that is, the mapping is given by $u^{*\alpha} = u^\alpha$ ($\alpha = 1, 2$)):

- | | |
|--|--------------------------|
| <ul style="list-style-type: none"> (a) $g_{\alpha\beta}^* = g_{\alpha\beta}$ (Theorem 57.1) (b) $g_{\alpha\beta}^* = \eta(u^1, u^2)g_{\alpha\beta}$ (Theorem 61.1) (c) $g^* = g$ (Theorem 66.1) (d) $\Gamma_{\alpha\beta}^{*\gamma} = \Gamma_{\alpha\beta}^{\gamma} + \delta_{\alpha}^{\gamma} \frac{\partial h}{\partial u^{\beta}} + \delta_{\beta}^{\gamma} \frac{\partial h}{\partial u^{\alpha}}$ | 176
193
209
290 |
| }
at corresponding points | |
| $h = \frac{1}{6} \log \frac{g^*}{g}$ (Theorem 94.1) | |

Intrinsic geometry of a surface: deals with those properties which depend on the first fundamental form only but are independent of the second one ('intrinsic' or 'absolute' properties). Examples: Gaussian curvature K , geodesic curvature κ_g of a curve on a surface. Isometric surfaces have the same local intrinsic geometry. In order that two portions of surfaces be isometric it is necessary that at corresponding points the Gaussian curvature be the same. For surfaces of constant Gaussian curvature this condition is also sufficient, cf. Theorem 91.1.

Further properties: An isometric mapping is conformal (Theorem 61.2) and equiareal (Theorem 66.2); a mapping which is both conformal and equiareal is isometric. Developable surfaces are the only surfaces which can be mapped isometrically into a plane (Theorem 59.3). Hence a sphere cannot be mapped isometrically into a plane; for instance, the stereographic and the

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177

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195

209

189

Mercator projection are conformal mappings and the mappings of Lambert, Sanson, Bonne, and Collignon are equiareal mappings of spheres into planes. 205
210

Spherical image of a surface: generated by the unit normal vector \mathbf{n} when bound at a fixed point in space. The first fundamental form

$$(59.1) \quad ds^{*2} = \mathbf{n}_\alpha \cdot \mathbf{n}_\beta du^\alpha du^\beta \equiv c_{\alpha\beta} du^\alpha du^\beta \quad 186$$

of the spherical image of a surface S is called the third fundamental form of S . Let dA and dA^* be the elements of area of a surface S and its spherical image, respectively. Then $|K| = \frac{dA^*}{dA}$. 187

Conformal mapping of space R_3 into itself: The images of spheres are spheres (Theorem 68.2, Liouville). Any conformal mapping of R_3 onto itself is a composition of inversions (Theorem 68.3). 217
218

(Sections 69–78) **Absolute differentiation and parallel displacement of Levi-Civita**

The first partial derivatives of a tensor of first or higher order do not form a tensor. 219

The covariant derivative of a tensor $a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s}$, defined by

$$(71.3) \quad a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s, \sigma} = \frac{\partial a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_s}}{\partial u^\sigma} - \sum_{\gamma=1}^r a_{\alpha_1 \dots \alpha_{\gamma-1} \kappa \alpha_{\gamma+1} \dots \alpha_r}{}^{\beta_1 \dots \beta_s} \Gamma_{\alpha_\gamma \sigma}{}^\kappa + \sum_{\delta=1}^s a_{\alpha_1 \dots \alpha_r}{}^{\beta_1 \dots \beta_{\delta-1} \lambda \beta_{\delta+1} \dots \beta_s} \Gamma_{\lambda \sigma}{}^{\beta_\delta} \quad 225$$

is a tensor of order $r+s+1$, covariant of order $r+1$, contravariant of order s .

Important particular cases: Covariant derivative of a

contravariant vector a^β : (70.3') $a^\beta, \sigma = \frac{\partial a^\beta}{\partial u^\sigma} + a^\gamma \Gamma_{\gamma \sigma}{}^\beta$, 221

covariant vector a_β : (70.7') $a_{\beta, \sigma} = \frac{\partial a_\beta}{\partial u^\sigma} - a_\gamma \Gamma_{\beta \sigma}{}^\gamma$. 223

Covariant differentiation of sums and products of tensors is governed by the rules of the usual differential calculus. The result of repeated covariant differentiation in general depends on the order of differentiation. The covariant derivatives of the tensors $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\delta_\alpha{}^\beta$ are zero tensors. (Theorem 72.1, Ricci.) 225
226
226

Beltrami's differential parameters:

$$(75.1) \quad \nabla(\phi, \psi) = g^{\alpha\beta} \frac{\partial\phi}{\partial u^\alpha} \frac{\partial\psi}{\partial u^\beta}, \quad 229$$

$$(75.5) \quad \Delta\phi \equiv \operatorname{div} \operatorname{grad} \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\beta} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial\phi}{\partial u^\alpha} \right) = g^{\alpha\beta} \phi_{,\alpha\beta}. \quad 231$$

Parallel displacement of Levi-Civit  :

Differential equations of the displacement of a vector a^α in a surface along a curve C : $u^\alpha = u^\alpha(t)$ ($\alpha = 1, 2$):

$$(76.5') \quad \frac{da^\alpha}{dt} + a^\beta \Gamma_{\beta\sigma}{}^\alpha \frac{du^\sigma}{dt} = 0 \quad (\alpha = 1, 2). \quad 235$$

Vectors $a^\alpha(t)$ which satisfy (76.5') are called parallel in the sense of the displacement of Levi-Civit   and with respect to C . For any surface, not a developable one, the displacement depends on C .

235

The scalar product of two vectors which undergo a displacement along the same curve is constant and so are the angle between these vectors and also their lengths. The tangent vectors to a geodesic G are parallel with respect to G .

236

(Sections 79–96) Special surfaces

Ruled surface: generated by a one-parameter family of straightlines.

Representation:

$$(58.1) \quad \mathbf{x}(s, t) = \mathbf{y}(s) + t\mathbf{z}(s). \quad 179$$

Developable surface: special ruled surface; the tangent plane at all points of any of its generators is the same. Types: planes, cylinders, cones, tangent surfaces. (58.1) is a developable surface if and only if

$$(58.2) \quad |\dot{\mathbf{y}} \ \mathbf{z} \ \dot{\mathbf{z}}| = 0. \quad 182$$

A surface is developable if and only if $K \equiv 0$.

188

Minimal surface: surface whose mean curvature H vanishes identically. If a portion of a surface has the smallest area among all portions which are bounded by the same closed curve it is a portion of a minimal surface. Catenoids are the only minimal surfaces of revolution. Right helicoids are the only right conoids which are minimal surfaces. Formulae of Weierstrass: Let $F(t)$, $t = t^1 + it^2$ be any regular function. Then $\mathbf{x}(t^1, t^2)$ where

243

244

246

$$(82.8) \quad x_1 = \Re \int (1-t^2)F(t) dt, \quad x_2 = \Re \int i(1+t^2)F(t) dt,$$

$$x_3 = \Re \int 2tF(t) dt, \quad 253$$

represents a minimal surface.

One-parameter family F of surfaces: Representation

$$(85.1) \quad G(x_i, t) = 0. \quad 263$$

Characteristic of a surface S of F:

$$(85.5) \quad G(x_i, t) = 0, \quad \frac{\partial G(x_i, t)}{\partial t} = 0. \quad 263$$

Envelope: surface generated by the characteristics. At any point of a characteristic of a surface S of the family F the envelope E and the surface S have the same tangent plane. If the envelope of a family of planes exists it is a developable surface. The tangent surface of a curve C is the envelope of the osculating planes of C. The polar surface of C, generated by the polar axes, is the envelope of the normal planes of C; the envelope of the rectifying planes is called the rectifying developable surface. 264
267
270
272

Surfaces of constant Gaussian curvature: $K > 0$ 'spherical surfaces' 282
(example: spheres), $K < 0$ 'pseudospherical surfaces' (example: pseudospheres). Their intrinsic geometry is (at least locally) non-Euclidean. Surfaces of constant Gaussian curvature are 285
293
the only surfaces which can be mapped geodesically into a plane. 300

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