Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 162 (2004) 365–392

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Characterization and construction of helical polynomial space curves

Rida T. Farouki[a,*], Chang Yong Han[a], Carla Manni[b], Alessandra Sestini[c]

[a] *Department of Mechanical and Aeronautical Engineering, University of California, Davis, CA 95616, USA*
[b] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, Roma 00133, Italy*
[c] *Dipartimento di Energetica, Università di Firenze, Via Lombroso 6/17, Firenze 50134, Italy*

Received 27 March 2003; received in revised form 14 July 2003

Abstract

Helical space curves are characterized by the property that their unit tangents maintain a constant inclination with respect to a fixed line, the axis of the helix. Equivalently, a helix exhibits a circular tangent indicatrix, and constant curvature/torsion ratio. If a polynomial space curve is helical, it must be a Pythagorean-hodograph (PH) curve. The quaternion representation of spatial PH curves is used to characterize and construct helical curves. Whereas all spatial PH cubics are helical, the helical PH quintics form a proper subset of all PH quintics. Two types of PH quintic helix are identified: (i) the "monotone-helical" PH quintics, in which a scalar quadratic factors out of the hodograph, and the tangent exhibits a consistent sense of rotation about the axis; and (ii) general helical PH quintics, which possess irreducible hodographs, and may suffer reversals in the sense of tangent rotation. First-order Hermite interpolation is considered for both helical PH quintic types. The helicity property offers a means of fixing the residual degrees of freedom in the general PH quintic Hermite interpolation problem, and yields interpolants with desirable shape features.

Keywords: Pythagorean-hodograph curves; Quaternions; Tangent indicatrix; Rational quartic; Curvature; Torsion; Helix; Hermite interpolation; Energy integral

1. Introduction

Pythagorean-hodograph (PH) space curves are polynomial parametric curves $\mathbf{r}(\xi)=(x(\xi),y(\xi),z(\xi))$ characterized by the property that their derivatives or *hodographs* $\mathbf{r}'(\xi)=(x'(\xi),y'(\xi),z'(\xi))$ satisfy

* Corresponding author. Tel.: +1-530-752-1779; fax: +1-530-572-4158.
E-mail addresses: farouki@ucdavis.edu (R.T. Farouki), cyhan@ucdavis.edu (C.Y. Han), manni@mat.uniroma2.it (C. Manni), sestini@de.unifi.it (A. Sestini).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.08.030

the Pythagorean condition

$$x'^2(\xi) + y'^2(\xi) + z'^2(\xi) = \sigma^2(\xi) \tag{1}$$

for some polynomial $\sigma(\xi)$. This endows PH space curves with the capability for *exact* measurement of arc length, a feature that is especially advantageous in formulating real-time interpolators to drive multi-axis CNC machines at constant or variable speeds along curved paths [13,17,32].

Algorithms to construct and manipulate PH curves are a basic necessity for their use in design and manufacturing applications. Hermite interpolation—i.e., the construction of smooth curve segments matching given end points and derivatives—is a common approach to satisfying this requirement. The first-order Hermite interpolation problem for spatial PH quintics has been thoroughly studied [11] using the quaternion model [5]. For given initial and final points $\mathbf{p}_i, \mathbf{p}_f$ and derivatives $\mathbf{d}_i, \mathbf{d}_f$ the Hermite interpolants comprise, in general, a two-parameter family, and their shape properties can be sensitively dependent upon the choice of these free parameters. Thus, a quantitative and geometrically motivated criterion to fix these free parameters, so as to ensure Hermite interpolants of desirable shape, is needed.

In this paper, our focus is on the family of *helical* PH quintics. A *helix*—or "curve of constant slope"—exhibits a fixed ratio of curvature to torsion along its length [23]. It is known [16] that all spatial PH cubics are helical, but the helical PH quintic space curves are a proper subset of all spatial PH quintics. We give a complete characterization of the helical PH quintics, and study their use as Hermite interpolants using the quaternion model.

Our plan for this paper is as follows. In Section 2 we review the definitions and characteristic properties of helices and PH curves, using the quaternion representation. Characterizations and construction algorithms for a class of "simple" helical PH curves, the *monotone-helical* PH quintics (for which the tangent maintains a fixed sense of rotation about the axis), are then derived in Section 3. A more general class of helical PH quintics allows reversals in the sense of the tangent rotation: a sufficient condition, in terms of linear dependence of the quaternion coefficients, for PH quintics to belong to this class is given in Section 4, and Hermite interpolation algorithms are developed. Finally, Section 5 summarizes our results and makes some concluding remarks.

2. Characterization of helical PH quintics

Before presenting detailed descriptions and construction algorithms for the helical PH quintics, we review some basic properties of helices, their relation to PH curves, and the possible types of helical PH quintics.

2.1. Properties of a helix

A (general) helix is a curve whose unit tangent $\mathbf{t}$ maintains a constant angle ψ with a fixed line in space, the *axis* of the helix. If $\mathbf{a}$ is a unit vector along the axis, we have

$$\mathbf{t} \cdot \mathbf{a} = \cos\psi = \text{constant}, \tag{2}$$

and hence a helix is also called a "curve of constant slope." Note that any *planar* curve is, trivially, a helix—with axis $\mathbf{a}$ orthogonal to the plane of the curve, and $\psi = \pi/2$. Henceforth it will be understood that, when we speak of a helix, we mean specifically a *spatial* curve. It can be shown, using the Frenet–Serret equations [23,24,30], that condition (2) is equivalent to the requirement that the curvature κ and torsion τ along a helix satisfy [1]

$$\kappa/\tau = \tan\psi = \text{constant}. \tag{3}$$

The familiar circular helix, which lies on a cylinder of revolution, corresponds to the case where (3) holds because κ and τ are individually constant.

The *tangent indicatrix* or "spherical image" of a curve—i.e., the locus on the unit sphere traced by the tip of the tangent $\mathbf{t}$, considered to emanate from the origin—offers another characterization. Any curve whose tangent indicatrix is a (small) circle on the unit sphere is a helix [24]. The axis vector $\mathbf{a}$ identifies the center of the circular tangent indicatrix on the sphere.

If we choose coordinates with z along the axis and x, y perpendicular to the axis, a helix can be parameterized in the form

$$x = x(s), \quad y = y(s), \quad z(s) = s\cos\psi.$$

A further characterization for helices is expressed [24] by the condition

$$(\mathbf{r}^{(2)} \times \mathbf{r}^{(3)}) \cdot \mathbf{r}^{(4)} \equiv 0,$$

where $\mathbf{r}^{(k)}$ denotes the kth arc-length derivative of the curve.

For any helix, the instantaneous motion of the unit tangent amounts to a rotation about the axis. For a general helix, the *sense* of this rotation may exhibit reversals at points that have "stationary" tangents, and the tangent indicatrix is doubly traced in the vicinity of such points. Hence, we define a *monotone-helical curve* as a helix that maintains a fixed sense of tangent rotation, and a one-to-one correspondence between points and tangents—i.e., its tangent indicatrix is a singly traced circle.

2.2. Pythagorean-hodograph curves

In this paper, we employ the quaternion representation [2] of spatial PH curves, introduced by Choi et al. [5]. Given a quaternion polynomial

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\mathbf{i} + p(\xi)\mathbf{j} + q(\xi)\mathbf{k}, \tag{4}$$

the product

$$\mathbf{r}'(\xi) = \mathcal{A}(\xi)\mathbf{i}\,\mathcal{A}^*(\xi) \tag{5}$$

defines a spatial PH, with components of the form

$$\begin{aligned} x'(\xi) &= u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi), \\ y'(\xi) &= 2\,[u(\xi)q(\xi) + v(\xi)p(\xi)], \\ z'(\xi) &= 2\,[v(\xi)q(\xi) - u(\xi)p(\xi)], \\ \sigma(\xi) &= u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi) \end{aligned} \tag{6}$$

[1] For space curves, κ is by definition non-negative, but τ is a signed quantity. Hence, the constant in (3) may exhibit a change of sign at certain special points where $\kappa = \tau = 0$.

[2] See Appendix A for a brief review of the algebra of quaternions.

satisfying [7] the Pythagorean condition (1). Moreover, this form is invariant [10] with respect to spatial rotations. If the polynomial (4) is of degree m, integrating (5) yields a spatial PH curve $\mathbf{r}(\xi)$ of degree $n = 2m + 1$.

Note that (1) is also satisfied if we multiply each of the terms (6) by any polynomial $h(\xi)$. We choose $h(\xi) \equiv 1$ henceforth, since real roots of $h(\xi)$ may incur *cusps* on the curve. Note, however, that this choice does not guarantee that $\gcd(x', y', z') = \text{constant}$ when $\gcd(u, v, p, q) = \text{constant}$—see Section 3.1.

A spatial PH quintic is defined [5,10,11] by a quadratic polynomial

$$\mathscr{A}(\xi) = \mathscr{A}_0(1-\xi)^2 + \mathscr{A}_1 2(1-\xi)\xi + \mathscr{A}_2\xi^2 \tag{7}$$

with quaternion Bernstein coefficients

$$\mathscr{A}_r = a_r + \mathbf{a}_r = a_r + a_{rx}\mathbf{i} + a_{ry}\mathbf{j} + a_{rz}\mathbf{k}, \quad r = 0, 1, 2. \tag{8}$$

The Bézier control points $\mathbf{p}_k = x_k\mathbf{i} + y_k\mathbf{j} + z_k\mathbf{k}$ of the resulting PH quintic

$$\mathbf{r}(\xi) = \sum_{k=0}^{5} \mathbf{p}_k \binom{5}{k} (1-\xi)^{5-k}\xi^k$$

are then given by the formulae

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \tfrac{1}{5}\,\mathscr{A}_0\mathbf{i}\,\mathscr{A}_0^*,\\
\mathbf{p}_2 &= \mathbf{p}_1 + \tfrac{1}{10}(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_1^* + \mathscr{A}_1\mathbf{i}\,\mathscr{A}_0^*),\\
\mathbf{p}_3 &= \mathbf{p}_2 + \tfrac{1}{30}(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + 4\,\mathscr{A}_1\mathbf{i}\,\mathscr{A}_1^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*),\\
\mathbf{p}_4 &= \mathbf{p}_3 + \tfrac{1}{10}(\mathscr{A}_1\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_1^*),\\
\mathbf{p}_5 &= \mathbf{p}_4 + \tfrac{1}{5}\,\mathscr{A}_2\mathbf{i}\,\mathscr{A}_2^*,
\end{aligned} \tag{9}$$

where $\mathbf{p}_0$ is an arbitrary integration constant.

The circular helix is clearly a *transcendental* curve, since it may intersect a plane in an infinite number of points. For practical design purposes, we are usually concerned with the use of *polynomial* curves. There is an intimate connection between polynomial curves, helices, and PH curves:

Lemma 1. *If a polynomial space curve is helical, it must be a PH curve.*

Proof. The unit tangent to a space curve $\mathbf{r}(\xi)$ is given by $\mathbf{t}(\xi) = \mathbf{r}'(\xi)/|\mathbf{r}'(\xi)|$, and hence the helix condition (2) may be written as

$$\mathbf{a} \cdot \mathbf{r}'(\xi) = \cos\psi\,|\mathbf{r}'(\xi)|.$$

The left-hand side of this equation is evidently a polynomial in ξ, if $\mathbf{r}(\xi)$ is a polynomial curve. However, the right-hand side can be a polynomial only if $\mathbf{r}(\xi)$ is a PH curve, since only PH curves have polynomial speed $|\mathbf{r}'(\xi)|$. $\square$

Although a polynomial curve *must* be a PH curve in order to be helical, not all PH curves are helical. As shown in [16], all PH cubics are helical, but there are non-helical PH quintics. An algebraic condition for the helicity of PH curves may be derived from the characterization (3). Recall [23] that the curvature and torsion of a space curve are given by

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \quad \text{and} \quad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}. \tag{10}$$

Thus, for a helix we must have

$$|\mathbf{r}' \times \mathbf{r}''|^3 = \tan\psi\, \sigma^3\, (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''', \tag{11}$$

where $\sigma = |\mathbf{r}'|$. Now a spatial PH curve satisfies [9] the condition

$$|\mathbf{r}' \times \mathbf{r}''|^2 = \sigma^2 \rho,$$

where the polynomial ρ is defined by

$$\begin{aligned}\rho = |\mathbf{r}''|^2 - \sigma'^2 = 4\,[&2(uv' - u'v)(pq' - p'q) \\ &+ (up' - u'p)^2 + (uq' - u'q)^2 + (vp' - v'p)^2 + (vq' - v'q)^2]\end{aligned}$$

and $\deg(\rho) = 2n - 6$ for a degree-n PH curve. Hence, for a PH curve, the helicity condition (11) becomes

$$\rho^{3/2} = \tan\psi\, (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''.$$

Since the right-hand side is evidently a polynomial, this can only be satisfied when ρ is the *perfect square* of a polynomial. For PH quintics, in particular, the right-hand side is of degree 6, and satisfaction of this condition requires that ρ be the perfect square of a quadratic polynomial.

2.3. Morphology of helical PH quintics

We may classify the possible types of helical PH quintics as follows. Without loss of generality, we take the helical axis in the positive x-direction—i.e., $\mathbf{a} = (1, 0, 0)$—and let ψ be the constant angle that the tangent makes with this axis. Then defining Eq. (2) of the helix becomes

$$u^2 + v^2 - p^2 - q^2 = \cos\psi(u^2 + v^2 + p^2 + q^2).$$

This is equivalent to

$$\frac{p^2 + q^2}{u^2 + v^2} = \frac{1 - \cos\psi}{1 + \cos\psi} = \tan^2\frac{\psi}{2} = t^2,$$

where $0 \leqslant \psi \leqslant \pi$, but we may exclude the degenerate cases $\psi = 0$, $\pi/2$, or π (so that $0 < t < \infty$, but $t \neq 1$).

Now we can re-arrange the above equation to yield either

$$(p - tu)(p + tu) = (tv - q)(tv + q) \tag{12}$$

or

$$(p - tv)(p + tv) = (tu - q)(tu + q). \tag{13}$$

The analysis of these equations is essentially the same, so we concentrate on the first. Now for a PH quintic helix, u, v, p, q are real quadratic polynomials, i.e., members of the unique factorization domain (UFD) $\mathbb{R}[\xi]$.

Consider first the case where Eq. (12) is actually of the form $0=0$. This means that $p=\pm tu$ and $q=\pm tv$, and hence

$$\begin{aligned}
x' &= u^2+v^2-p^2-q^2=(1-t^2)(u^2+v^2),\\
y' &= 2(uq+vp)=2t(\pm uv\pm uv),\\
z' &= 2(vq-up)=2t(\mp u^2\pm v^2),\\
\sigma &= u^2+v^2+p^2+q^2=(1+t^2)(u^2+v^2).
\end{aligned}$$

Note that all four sign combinations are possible in the expressions for y', z'. The tangent indicatrix $\mathbf{t}=(x',y',z')/\sigma$ thus has the components

$$\begin{aligned}
x'/\sigma &= \frac{1-t^2}{1+t^2}=\text{constant},\\
y'/\sigma &= \frac{2t}{1+t^2}\,\frac{\pm uv\pm uv}{u^2+v^2},\\
z'/\sigma &= \frac{2t}{1+t^2}\,\frac{\mp u^2\pm v^2}{u^2+v^2}.
\end{aligned}$$

If we choose unlike signs in y'/σ and like signs in z'/σ, the tangent indicatrix degenerates to a single point, $(\cos\psi, 0, \sin\psi)$. We discount this case, in which the PH quintic degenerates to a straight line. However, if we choose like signs in y'/σ and unlike signs in z'/σ, the tangent indicatrix is a circle—this circle is doubly traced, since u and v are quadratic polynomials.

Now suppose that each of the factors in (12) is non-zero and irreducible. Since $\mathbb{R}[\xi]$ is a UFD, we must have

$$p-tu=\pm(tv-q),\qquad p+tu=\pm(tv+q)$$

or

$$p-tu=\pm(tv+q),\qquad p+tu=\pm(tv-q).$$

One can easily see that these equations also reduce to the previous case, i.e., the tangent indicatrix is a single point or a doubly traced circle. Likewise, when the left- and right-hand sides of (12) each have just one irreducible factor ($p-tu$ and $tv-q$, say), we again obtain the above equations.

The only remaining case is that in which all the terms in (12) are products of linear factors. Since $\mathbb{R}[\xi]$ is a UFD, there are at most four distinct linear factors, which we denote by a,b,c,d. Suppose that

$$p-tu=ab,\qquad p+tu=cd.$$

To obtain results that differ from the preceding cases, we must have different combinations for $tv\pm q$, say

$$tv-q=ad,\qquad tv+q=bc.$$

Then the tangent indicatrix becomes

$$x'/\sigma = \frac{1-t^2}{1+t^2} = \text{constant},$$

$$y'/\sigma = \frac{2t}{1+t^2}\,\frac{2bd}{b^2+d^2},$$

$$z'/\sigma = \frac{2t}{1+t^2}\,\frac{b^2-d^2}{b^2+d^2}.$$

Since b and d are real, linear polynomials, this defines a singly traced circle.

In summary, there are three types of helical PH quintics. We ignore the first, in which the tangent indicatrix degenerates to a single point and the PH quintic is a straight line. The second and third types, in which the tangent indicatrix is, respectively a singly and doubly traced circle, [3] will be treated in Sections 3 and 4. The characterization of helical PH quintic types is evidently equivalent to determining the nature of the parameterization of circular arcs on the unit sphere. Rational representations of circular arcs are fundamental in computer aided geometric design, and have been discussed in detail by a number of authors—see, for example, [2,6,18,26].

3. Monotone-helical PH quintics

We consider first the helical PH quintics with rational quadratic (rather than quartic) tangent indicatrix. This degree reduction arises from a cancellation of factors common to the hodograph components (6).

3.1. Common factors of hodograph components

The tangent indicatrix of a degree-n PH curve is, in general, a rational curve of degree $n-1$ on the unit sphere. When the PH curve is helical, the tangent indicatrix is a circle (possibly multiply traced) on the unit sphere.

Now in (6) we usually choose polynomials with $\gcd(u,v,p,q) = \text{constant}$, since common real roots of these four polynomials incur cusps on the curve. However, we shall see below that

$$\gcd(u,v,p,q) = \text{constant} \quad \not\Rightarrow \quad \gcd(x',y',z') = \text{constant}$$

in (6)—the hodograph components may exhibit common quadratic factors, with complex conjugate roots, even if $\gcd(u,v,p,q) = \text{constant}$.

A degree-n PH curve is *monotone-helical* if x', y', z', σ possess a non-constant common factor, whose cancellation causes the tangent indicatrix to become a rational quadratic on the unit sphere—i.e., a singly traced circle. To identify the monotone-helical PH curves, we first write the hodograph

[3] Note that the singly or doubly traced property refers to the *entire* tangent indicatrix, for $-\infty < \xi < +\infty$. A doubly traced indicatrix may seem singly traced when restricted, for example, to the standard parameter interval $\xi \in [0,1]$.

components (6) in terms of complex polynomials $u \pm \mathrm{i}v$ and $p \pm \mathrm{i}q$ as

$$x' = u^2 + v^2 - p^2 - q^2 = (u + \mathrm{i}v)(u - \mathrm{i}v) - (p + \mathrm{i}q)(p - \mathrm{i}q),$$

$$y' = 2(uq + vp) = \mathrm{i}[(u - \mathrm{i}v)(p - \mathrm{i}q) - (u + \mathrm{i}v)(p + \mathrm{i}q)],$$

$$z' = 2(vq - up) = -[(u - \mathrm{i}v)(p - \mathrm{i}q) + (u + \mathrm{i}v)(p + \mathrm{i}q)]. \tag{14}$$

Hence, $y' = z' = 0$ if and only if $(u - \mathrm{i}v)(p - \mathrm{i}q) = (u + \mathrm{i}v)(p + \mathrm{i}q) = 0$. Coupled with the condition $(u + \mathrm{i}v)(u - \mathrm{i}v) = (p + \mathrm{i}q)(p - \mathrm{i}q)$ obtained from $x' = 0$, we deduce that

$$x' = y' = z' = 0 \quad \Leftrightarrow \quad u + \mathrm{i}v = p - \mathrm{i}q = 0 \quad \text{or } u - \mathrm{i}v = p + \mathrm{i}q = 0.$$

Furthermore, one can verify that multiple roots of either $u + \mathrm{i}v = p - \mathrm{i}q = 0$ or $u - \mathrm{i}v = p + \mathrm{i}q = 0$ are also multiple roots of $x' = y' = z' = 0$, with the same multiplicity. This implies that

$$\gcd(x', y', z') = \gcd(u + \mathrm{i}v, p - \mathrm{i}q) \cdot \gcd(u - \mathrm{i}v, p + \mathrm{i}q).$$

When u, v, p, q are real polynomials, $\gcd(u + \mathrm{i}v, p - \mathrm{i}q) = \overline{\gcd(u - \mathrm{i}v, p + \mathrm{i}q)}$, and we may write

$$\gcd(x', y', z') = |\gcd(u + \mathrm{i}v, p - \mathrm{i}q)|^2.$$

Thus, $\gcd(x', y', z')$ is evidently a real polynomial of *even* degree, whose roots occur only in complex conjugate pairs.

In the case of PH quintics, $\gcd(x', y', z')$ may be of degree 0, 2, or 4. The degree 0 case is generic, while in the degree 4 case the tangent indicatrix is just a single point (the PH quintic becomes a straight line). It is the degree 2 case that interests us: the tangent indicatrix reduces to a rational quadratic curve on the unit sphere—which is necessarily planar, and thus circular, so the degree 2 case *always* defines a monotone-helical PH quintic.

In view of the preceding arguments, $\gcd(x', y', z')$ will be quadratic for a PH quintic when we have polynomials (with real ξ values) of the form

$$u + \mathrm{i}v = \zeta(\xi - \alpha)(\xi - \beta), \quad p - \mathrm{i}q = \eta(\xi - \alpha)(\xi - \gamma),$$

$$u - \mathrm{i}v = \bar{\zeta}(\xi - \bar{\alpha})(\xi - \bar{\beta}), \quad p + \mathrm{i}q = \bar{\eta}(\xi - \bar{\alpha})(\xi - \bar{\gamma}), \tag{15}$$

where $\alpha, \beta, \gamma, \zeta, \eta$ are complex values. Thus, to construct a monotone-helical PH quintic, we select $\alpha, \beta, \gamma, \zeta, \eta$ and substitute them into the expressions

$$u(\xi) = \mathrm{Re}(\zeta)\,\xi^2 - \mathrm{Re}(\zeta(\alpha + \beta))\,\xi + \mathrm{Re}(\zeta\alpha\beta),$$

$$v(\xi) = \mathrm{Im}(\zeta)\,\xi^2 - \mathrm{Im}(\zeta(\alpha + \beta))\,\xi + \mathrm{Im}(\zeta\alpha\beta),$$

$$p(\xi) = \mathrm{Re}(\eta)\,\xi^2 - \mathrm{Re}(\eta(\alpha + \gamma))\,\xi + \mathrm{Re}(\eta\alpha\gamma),$$

$$q(\xi) = -\mathrm{Im}(\eta)\,\xi^2 + \mathrm{Im}(\eta(\alpha + \gamma))\,\xi - \mathrm{Im}(\eta\alpha\gamma).$$

Example 1. The complex numbers $\alpha = 1 + 2\mathrm{i}$, $\beta = 3 - \mathrm{i}$, $\gamma = 2 + \mathrm{i}$, $\zeta = 1 + \mathrm{i}$, $\eta = -2 - \mathrm{i}$ yield the four quadratic polynomials

$$u(\xi) = \xi^2 - 3\xi, \quad v(\xi) = \xi^2 - 5\xi + 10,$$

$$p(\xi) = -2\xi^2 + 3\xi + 5, \quad q(\xi) = \xi^2 - 9\xi + 10,$$

which define the Pythagorean hodograph

$$
\begin{aligned}
x'(\xi) &= -3\xi^4 + 14\xi^3 - 36\xi^2 + 50\xi - 25,\\
y'(\xi) &= -2\xi^4 + 2\xi^3 - 14\xi^2 - 50\xi + 100,\\
z'(\xi) &= 6\xi^4 - 46\xi^3 + 1386\xi^2 - 250\xi + 200,\\
\sigma(\xi) &= 7\xi^4 - 46\xi^3 + 144\xi^2 - 250\xi + 225.
\end{aligned}
$$

By construction, these hodograph components possess the common factor

$$\gcd(u + \mathrm{i}v, p - \mathrm{i}q) \cdot \gcd(u - \mathrm{i}v, p + \mathrm{i}q) = (\xi - 1 - 2\mathrm{i})(\xi - 1 + 2\mathrm{i}) = \xi^2 - 2\xi + 5$$

and hence the tangent indicatrix reduces to the rational quadratic

$$\mathbf{t}(\xi) = \frac{(-3\xi^2 + 8\xi - 5, -2\xi^2 - 2\xi + 20, 6\xi^2 - 34\xi + 40)}{7\xi^2 - 32\xi + 45}.$$

Now setting $\zeta = r\exp(\mathrm{i}\phi)$ and $\eta = s\exp(\mathrm{i}\theta)$ in (15) and substituting into (14), it becomes apparent that the hodograph components depend only on the *difference* $\varphi = \phi - \theta$. Thus, we obtain

$$
\begin{aligned}
x' &= |\xi - \alpha|^2[r^2|\xi - \beta|^2 - s^2|\xi - \gamma|^2],\\
y' &= 2\,rs\,|\xi - \alpha|^2\mathrm{Im}[\mathrm{e}^{\mathrm{i}\varphi}(\xi - \beta)(\xi - \bar{\gamma})],\\
z' &= -2\,rs\,|\xi - \alpha|^2\mathrm{Re}[\mathrm{e}^{\mathrm{i}\varphi}(\xi - \beta)(\xi - \bar{\gamma})]
\end{aligned}
$$

as the most general hodograph form for a monotone-helical PH quintic. The real values r, s, φ and complex values α, β, γ amount to 9 scalar freedoms, as compared to 12 for general PH quintics. Since interpolation of first-order spatial Hermite data involves the satisfaction of 9 scalar equations, we might expect the monotone-helical PH quintics to be capable of solving the general first-order Hermite interpolation problem.

The above complex representation for the monotone-helical PH quintics, arising from our desire to understand the occurrence of factors common to x', y', z' when $\gcd(u, v, p, q) = \text{constant}$, is rather inconvenient for Hermite interpolation. Knowing the circumstances that give $\gcd(x', y', z') \neq \text{constant}$, we now derive an equivalent quaternion representation.

3.2. First-order Hermite interpolation

We begin by noting that the imaginary unit "i" is algebraically identical to the element "**i**" of the quaternion basis. Using the relations $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, we may write (4) and its conjugate as

$$
\begin{aligned}
\mathcal{A} &= u + v\mathbf{i} + p\mathbf{j} + q\mathbf{k} = (u + v\mathbf{i}) + \mathbf{j}\,(p - q\mathbf{i}),\\
\mathcal{A}^* &= u - v\mathbf{i} - p\mathbf{j} - q\mathbf{k} = (u - v\mathbf{i}) - (p + q\mathbf{i})\mathbf{j}.
\end{aligned}
\tag{16}
$$

Now suppose that

$$\gcd(u+v\mathbf{i},p-q\mathbf{i})=f+g\mathbf{i} \quad \text{and} \quad \gcd(u-v\mathbf{i},p+q\mathbf{i})=f-g\mathbf{i}$$

are non-constant polynomials, so we can write

$$u+v\mathbf{i}=(a+b\mathbf{i})(f+g\mathbf{i}), \quad p-q\mathbf{i}=(c-d\mathbf{i})(f+g\mathbf{i}),$$

$$u-v\mathbf{i}=(f-g\mathbf{i})(a-b\mathbf{i}), \quad p+q\mathbf{i}=(f-g\mathbf{i})(c+d\mathbf{i})$$

for suitable polynomials $a \pm b\mathbf{i}$ and $c \pm d\mathbf{i}$ (note that, since they involve only the basis element $\mathbf{i}$, we are free to choose the ordering of the above products). Substituting these expressions into (16), we obtain

$$\mathscr{A}=[(a+b\mathbf{i})+\mathbf{j}(c-d\mathbf{i})](f+g\mathbf{i}), \quad \mathscr{A}^*=(f-g\mathbf{i})[(a-b\mathbf{i})-(c+d\mathbf{i})\mathbf{j}]$$

and forming the Pythagorean hodograph $\mathbf{r}'=\mathscr{A}\mathbf{i}\mathscr{A}^*$ yields

$$\mathbf{r}'=(a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k})(f+g\mathbf{i})\mathbf{i}(f-g\mathbf{i})(a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}).$$

Now since we have

$$(f+g\mathbf{i})\mathbf{i}(f-g\mathbf{i})=(f^2+g^2)\mathbf{i},$$

this hodograph simplifies to

$$\mathbf{r}'=(f^2+g^2)\mathscr{B}\mathbf{i}\mathscr{B}^*, \tag{17}$$

where the "reduced" quaternion polynomial

$$\mathscr{B}=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$$

and its conjugate are defined by

$$\mathscr{B}=\frac{\mathscr{A}}{\gcd(u+v\mathbf{i},p-q\mathbf{i})}, \quad \mathscr{B}^*=\frac{\mathscr{A}^*}{\gcd(u-v\mathbf{i},p+q\mathbf{i})}.$$

The above divisions amount to extracting common factors from the terms in parentheses in (16)—on the right for $\mathscr{A}$, and on the left for $\mathscr{A}^*$.

In the case of a monotone-helical PH quintic, the polynomials f and g in (17) are linear, and so is the quaternion polynomial $\mathscr{B}$. Thus, the hodograph of a monotone-helical PH quintic is just the product of a non-negative scalar quadratic polynomial with the hodograph of a PH cubic. Monotone helicity is an intrinsic property of the latter—the non-negative quadratic polynomial simply serves to modulate the length of the hodograph vector.

To perform Hermite interpolation with monotone-helical PH quintics, we consider a PH quintic with hodograph given by

$$\mathbf{r}'(\xi)=[b_0(1-\xi)^2+b_1 2(1-\xi)\xi+b_2\xi^2]\,\mathscr{B}(\xi)\mathbf{i}\mathscr{B}^*(\xi), \tag{18}$$

where

$$\mathscr{B}(\xi)=\mathscr{B}_0(1-\xi)+\mathscr{B}_1\xi$$

is a linear quaternion polynomial. Integration of (18) yields the control points

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \tfrac{1}{5}\, b_0\, \mathscr{B}_0 \mathbf{i}\, \mathscr{B}_0^*,\\
\mathbf{p}_2 &= \mathbf{p}_1 + \tfrac{1}{20}\, [2b_1\, \mathscr{B}_0 \mathbf{i}\, \mathscr{B}_0^* + b_0(\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^*)],\\
\mathbf{p}_3 &= \mathbf{p}_2 + \tfrac{1}{30}\, [b_2\, \mathscr{B}_0 \mathbf{i}\, \mathscr{B}_0^* + 2b_1\, (\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^*) + b_0\, \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_1^*],\\
\mathbf{p}_4 &= \mathbf{p}_3 + \tfrac{1}{20}\, [b_2(\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^*) + 2b_1\, \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_1^*],\\
\mathbf{p}_5 &= \mathbf{p}_4 + \tfrac{1}{5}\, b_2\, \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_1^*.
\end{aligned} \tag{19}$$

We wish to interpolate the first-order Hermite data

$$\mathbf{r}(0) = \mathbf{p}_i, \quad \mathbf{r}'(0) = \mathbf{d}_i \quad \text{and} \quad \mathbf{r}(1) = \mathbf{p}_f, \quad \mathbf{r}'(1) = \mathbf{d}_f \tag{20}$$

using such a curve. Interpolation of the end derivatives yields the equations

$$b_0\, \mathscr{B}_0 \mathbf{i}\, \mathscr{B}_0^* = \mathbf{d}_i \quad \text{and} \quad b_2\, \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_1^* = \mathbf{d}_f \tag{21}$$

while the condition $\int_0^1 \mathbf{r}'(\xi)\mathrm{d}\xi = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$ gives

$$\begin{aligned}
&(12b_0 + 6b_1 + 2b_2)\, \mathscr{B}_0 \mathbf{i}\, \mathscr{B}_0^*\\
&\quad + (3b_0 + 4b_1 + 3b_2)\, (\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^*)\\
&\quad + (2b_0 + 6b_1 + 12b_2)\, \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_1^* = 60\Delta\mathbf{p}.
\end{aligned} \tag{22}$$

Now in (18) we may assume, without loss of generality, that the quadratic is monic, so that $b_2 - 2b_1 + b_0 = 1$. Eliminating b_1 and invoking (21), Eq. (22) may then be re-written as

$$\begin{aligned}
&b_0 b_2 (5b_0 + 5b_2 - 2)(\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^*)\\
&\quad = 60 b_0 b_2 \Delta\mathbf{p} - b_2(15b_0 + 5b_2 - 3)\mathbf{d}_i - b_0(5b_0 + 15b_2 - 3)\mathbf{d}_f.
\end{aligned} \tag{23}$$

Writing $\mathbf{d}_i = |\mathbf{d}_i|(\lambda_i, \mu_i, \nu_i)$ and $\mathbf{d}_f = |\mathbf{d}_f|(\lambda_f, \mu_f, \nu_f)$ the general solutions to (21) are

$$\mathscr{B}_0 = \sqrt{\frac{(1+\lambda_i)|\mathbf{d}_i|}{2b_0}} \left(-\sin\phi_0 + \cos\phi_0 \mathbf{i} + \frac{\mu_i \cos\phi_0 + \nu_i \sin\phi_0}{1+\lambda_i}\mathbf{j} + \frac{\nu_i \cos\phi_0 - \mu_i \sin\phi_0}{1+\lambda_i}\mathbf{k} \right), \tag{24}$$

$$\mathscr{B}_1 = \sqrt{\frac{(1+\lambda_f)|\mathbf{d}_f|}{2b_2}} \left(-\sin\phi_1 + \cos\phi_1 \mathbf{i} + \frac{\mu_f \cos\phi_1 + \nu_f \sin\phi_1}{1+\lambda_f}\mathbf{j} + \frac{\nu_f \cos\phi_1 - \mu_f \sin\phi_1}{1+\lambda_f}\mathbf{k} \right) \tag{25}$$

and hence, with $\phi = \phi_1 - \phi_0$, we may write

$$\mathscr{B}_0 \mathbf{i}\, \mathscr{B}_1^* + \mathscr{B}_1 \mathbf{i}\, \mathscr{B}_0^* = \sqrt{\frac{|\mathbf{d}_i|\,|\mathbf{d}_f|}{b_0 b_2}}\, (c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}),$$

where

$$c_x = \frac{[(1+\lambda_i)(1+\lambda_f) - (\mu_i\mu_f + v_i v_f)]\cos\phi - (\mu_i v_f - \mu_f v_i)\sin\phi}{\sqrt{(1+\lambda_i)(1+\lambda_f)}},$$

$$c_y = \frac{[(1+\lambda_f)\mu_i + (1+\lambda_i)\mu_f]\cos\phi + [(1+\lambda_i)v_f - (1+\lambda_f)v_i]\sin\phi}{\sqrt{(1+\lambda_i)(1+\lambda_f)}},$$

$$c_z = \frac{[(1+\lambda_f)v_i + (1+\lambda_i)v_f]\cos\phi + [(1+\lambda_f)\mu_i - (1+\lambda_i)\mu_f]\sin\phi}{\sqrt{(1+\lambda_i)(1+\lambda_f)}}. \tag{26}$$

Substituting the above expressions into (23), we obtain

$$\sqrt{b_0 b_2}(5b_0 + 5b_2 - 2)\sqrt{|\mathbf{d}_i|\,|\mathbf{d}_f|}\,(c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k})$$
$$= 60 b_0 b_2 \Delta\mathbf{p} - b_2(15b_0 + 5b_2 - 3)\mathbf{d}_i - b_0(5b_0 + 15b_2 - 3)\mathbf{d}_f. \tag{27}$$

This vector equation is equivalent to three scalar equations in b_0, b_2, and ϕ. Once these equations have been solved, we can obtain $\mathscr{B}_0$ and $\mathscr{B}_1$ by choosing ϕ_0 freely and setting $\phi_1 = \phi + \phi_0$ in (24) and (25). The control points of the monotone-helical PH quintic Hermite interpolant are then given by (19).

In order for the quadratic in (18) to be non-negative, the solution of (27) must yield positive b_0, b_2 values, and must also have a negative discriminant. This is equivalent, under the assumption that it is monic, to the constraint

$$b_2^2 + b_0^2 - 2b_0 b_2 - 2b_0 - 2b_2 + 1 < 0. \tag{28}$$

This constraint excludes a small region of the positive quadrant in the (b_0, b_2) plane from consideration, bounded by a parabolic arc from $(1,0)$ to $(0,1)$ with vertex at $(\frac{1}{4}, \frac{1}{4})$. However, the following example shows that Eq. (27), subject to constraint (28), does not always admit a solution.

Example 2. Consider the Hermite data

$$\Delta\mathbf{p} = (L, 0, 0), \quad \mathbf{d}_i = \frac{(0,1,-1)}{\sqrt{2}}, \quad \mathbf{d}_f = \frac{(0,1,1)}{\sqrt{2}}.$$

From the z component of (27), we deduce that either $b_2 = b_0$ or $b_2 = \frac{3}{5} - b_0$. If $b_2 = b_0$, the x component of (27) gives

$$\cos\phi - \sin\phi = \frac{b_0}{5b_0 - 1}\,30L.$$

This implies that, for large positive values of L, the quantity $b_0 = b_2$ can be made arbitrarily close to 0, which violates constraint (28). Similarly, if $b_2 = \frac{3}{5} - b_0$, then (b_0, b_2) must approach $(\frac{3}{5}, 0)$ or $(0, \frac{3}{5})$ in order to satisfy the x component of (27) for large positive L—again, this violates (28).

Since it appears that the monotone-helical PH quintics are insufficiently flexible to interpolate *arbitrary* Hermite data, we turn our attention to the general helical PH quintics. In this case, interpolants to arbitrary data may be constructed with little more effort than the solution of a quartic equation.

4. General helical PH quintics

We now investigate helical PH quintics whose hodograph components satisfy $\gcd(x', y', z') =$ constant. We call such curves "general helical PH quintics" (since the monotone-helical PH quintics comprise a lower-dimension subset of the set all helical PH quintics). To identify the general helical PH quintics, we examine the behavior of the tangent indicatrix.

4.1. Tangent indicatrix of PH quintics

The tangent indicatrix of a PH quintic may be expressed as a rational quartic

$$\mathbf{t}(\xi) = \frac{\mathbf{r}'(\xi)}{|\mathbf{r}'(\xi)|} = \frac{\displaystyle\sum_{k=0}^{4} w_k \mathbf{t}_k \binom{4}{k} (1-\xi)^{4-k} \xi^k}{\displaystyle\sum_{k=0}^{4} w_k \binom{4}{k} (1-\xi)^{4-k} \xi^k} \tag{29}$$

with weights $w_0, \ldots, w_4$ and control points $\mathbf{t}_0, \ldots, \mathbf{t}_4$ given by

$$\begin{aligned}
w_0 &= \mathscr{A}_0 \mathscr{A}_0^*, \\
w_1 &= \tfrac{1}{2}\,(\mathscr{A}_0 \mathscr{A}_1^* + \mathscr{A}_1 \mathscr{A}_0^*), \\
w_2 &= \tfrac{1}{6}\,(\mathscr{A}_0 \mathscr{A}_2^* + 4\,\mathscr{A}_1 \mathscr{A}_1^* + \mathscr{A}_2 \mathscr{A}_0^*), \\
w_3 &= \tfrac{1}{2}\,(\mathscr{A}_1 \mathscr{A}_2^* + \mathscr{A}_2 \mathscr{A}_1^*), \\
w_4 &= \mathscr{A}_2 \mathscr{A}_2^*
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
w_0 \mathbf{t}_0 &= \mathscr{A}_0 \mathbf{i}\, \mathscr{A}_0^*, \\
w_1 \mathbf{t}_1 &= \tfrac{1}{2}\,(\mathscr{A}_0 \mathbf{i}\, \mathscr{A}_1^* + \mathscr{A}_1 \mathbf{i}\, \mathscr{A}_0^*), \\
w_2 \mathbf{t}_2 &= \tfrac{1}{6}\,(\mathscr{A}_0 \mathbf{i}\, \mathscr{A}_2^* + 4\,\mathscr{A}_1 \mathbf{i}\, \mathscr{A}_1^* + \mathscr{A}_2 \mathbf{i}\, \mathscr{A}_0^*), \\
w_3 \mathbf{t}_3 &= \tfrac{1}{2}\,(\mathscr{A}_1 \mathbf{i}\, \mathscr{A}_2^* + \mathscr{A}_2 \mathbf{i}\, \mathscr{A}_1^*), \\
w_4 \mathbf{t}_4 &= \mathscr{A}_2 \mathbf{i}\, \mathscr{A}_2^*.
\end{aligned} \tag{31}$$

In terms of the scalar and vector parts of the quaternion coefficients (8), the control points of the tangent indicatrix are

$$\begin{aligned}
\mathbf{t}_0 &= \frac{(a_0^2 - |\mathbf{a}_0|^2)\mathbf{i} + 2a_{0x}\mathbf{a}_0 - 2a_0 \mathbf{i} \times \mathbf{a}_0}{a_0^2 + |\mathbf{a}_0|^2}, \\
\mathbf{t}_1 &= \frac{(a_0 a_1 - \mathbf{a}_0 \cdot \mathbf{a}_1)\mathbf{i} + a_{1x}\mathbf{a}_0 + a_{0x}\mathbf{a}_1 - \mathbf{i} \times (a_1 \mathbf{a}_0 + a_0 \mathbf{a}_1)}{a_0 a_1 + \mathbf{a}_0 \cdot \mathbf{a}_1}, \\
\mathbf{t}_2 &= 2\,\frac{(a_1^2 - |\mathbf{a}_1|^2)\mathbf{i} + 2a_{1x}\mathbf{a}_1 - 2a_1 \mathbf{i} \times \mathbf{a}_1}{2(a_1^2 + |\mathbf{a}_1|^2) + a_2 a_0 + \mathbf{a}_2 \cdot \mathbf{a}_0} \\
&\quad + \frac{(a_2 a_0 - \mathbf{a}_2 \cdot \mathbf{a}_0)\mathbf{i} + a_{0x}\mathbf{a}_2 + a_{2x}\mathbf{a}_0 - \mathbf{i} \times (a_0 \mathbf{a}_2 + a_2 \mathbf{a}_0)}{2(a_1^2 + |\mathbf{a}_1|^2) + a_2 a_0 + \mathbf{a}_2 \cdot \mathbf{a}_0},
\end{aligned}$$

$$\mathbf{t}_3 = \frac{(a_1a_2 - \mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{i} + a_{2x}\mathbf{a}_1 + a_{1x}\mathbf{a}_2 - \mathbf{i} \times (a_2\mathbf{a}_1 + a_1\mathbf{a}_2)}{a_1a_2 + \mathbf{a}_1 \cdot \mathbf{a}_2},$$

$$\mathbf{t}_4 = \frac{(a_2^2 - |\mathbf{a}_2|^2)\mathbf{i} + 2a_{2x}\mathbf{a}_2 - 2a_2\mathbf{i} \times \mathbf{a}_2}{a_2^2 + |\mathbf{a}_2|^2}. \tag{32}$$

Now for any $\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2$ expressions (29)–(31) define a rational quartic on the unit sphere. A rational quartic may be either the *complete* intersection of two quadric surfaces (in which case it is the *base curve* of a one-parameter family or "pencil" of quadrics) or one component of a composite intersection of a quadric with a higher-order surface [28,29,31]. These are called *quartics of the first kind* and *quartics of the second kind*, respectively.

To identify the nature of the quartic curve defined by (29)–(31), consider the general quadric equation

$$\begin{bmatrix} W & X & Y & Z \end{bmatrix} \begin{bmatrix} a & f & h & k \\ f & b & g & l \\ h & g & c & m \\ k & l & m & d \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} = 0$$

in homogeneous coordinates (W, X, Y, Z). Substituting the four polynomials $W(\xi)$, $X(\xi)$, $Y(\xi)$, $Z(\xi)$ of degree 4 that define (29) into the above, we obtain a polynomial of degree 8 in ξ. This polynomial must vanish identically if the quartic curve lies on the quadric surface. Setting its coefficients equal to zero yields a system of nine homogeneous linear equations in the ten quantities $a, \ldots, m$. There is a unique solution[4] if the matrix of this system has rank 9, and the quartic is then of the second kind. If it is only of rank 8, however, a one-parameter family of solutions exists, and the quartic is then of the first kind—i.e., it is the base curve of a pencil of quadrics. By use of a computer algebra system, it can be verified that the matrix is generically of rank 8 and hence the quartic defined by (29)–(31) is of the first kind.

In order to be a rational curve, a quartic of the first kind must be singular, i.e., it must have a double point [28,29,31]—which may be either a self-intersection, a cusp, or an isolated real point [21]. The double point of the indicatrix indicates the existence of two distinct curve points with the same tangent vector (or a point with "stationary" tangent in the case of a cusp).

4.2. Condition for helicity of PH quintics

A special case arises if the tangent indicatrix degenerates into a planar curve—namely, a *doubly traced circle*. This circumstance, characterized by the fact that each curve point has a corresponding point with the same tangent, identifies the (general) helical PH quintics.

Proposition 1. *A sufficient condition for the hodograph (5) to yield a helical PH quintic is that the quaternions $\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2$ in (7) are linearly dependent.*

[4] We consider all (non-zero) multiples of a given set of coefficients $a, \ldots, m$ to constitute a single unique solution, since such coefficients define the same quadric surface.

Proof. If $\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2$ are linearly dependent, we can write

$$\mathscr{A}_1 = c_0\mathscr{A}_0 + c_2\mathscr{A}_2 \tag{33}$$

for suitable scalars c_0, c_2. Under this supposition Eqs. (30) and (31) become

$$\begin{aligned}
w_0 &= \mathscr{A}_0\mathscr{A}_0^*,\\
w_1 &= c_0\mathscr{A}_0\mathscr{A}_0^* + \tfrac{1}{2}\,c_2(\mathscr{A}_0\mathscr{A}_2^* + \mathscr{A}_2\mathscr{A}_0^*),\\
w_2 &= \tfrac{1}{6}\,[4c_0^2\mathscr{A}_0\mathscr{A}_0^* + (1+4c_0c_2)(\mathscr{A}_0\mathscr{A}_2^* + \mathscr{A}_2\mathscr{A}_0^*) + 4c_2^2\mathscr{A}_2\mathscr{A}_2^*],\\
w_3 &= \tfrac{1}{2}\,c_0(\mathscr{A}_0\mathscr{A}_2^* + \mathscr{A}_2\mathscr{A}_0^*) + c_2\mathscr{A}_2\mathscr{A}_2^*,\\
w_4 &= \mathscr{A}_2\mathscr{A}_2^*,
\end{aligned} \tag{34}$$

$$\begin{aligned}
w_0\mathbf{t}_0 &= \mathscr{A}_0\mathbf{i}\,\mathscr{A}_0^*,\\
w_1\mathbf{t}_1 &= c_0\mathscr{A}_0\mathbf{i}\,\mathscr{A}_0^* + \tfrac{1}{2}\,c_2(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*),\\
w_2\mathbf{t}_2 &= \tfrac{1}{6}\,[4c_0^2\mathscr{A}_0\mathbf{i}\,\mathscr{A}_0^* + (1+4c_0c_2)(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*) + 4c_2^2\mathscr{A}_2\mathbf{i}\,\mathscr{A}_2^*],\\
w_3\mathbf{t}_3 &= \tfrac{1}{2}\,c_0(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*) + c_2\mathscr{A}_2\mathbf{i}\,\mathscr{A}_2^*,\\
w_4\mathbf{t}_4 &= \mathscr{A}_2\mathbf{i}\,\mathscr{A}_2^*.
\end{aligned} \tag{35}$$

Now each point of the tangent indicatrix (29) may be interpreted as a weighted sum of the control points $\mathbf{t}_0, \dots, \mathbf{t}_4$ and thus $\mathbf{t}(\xi)$ is a planar locus if and only if these five control points are coplanar. Specifically, $\mathbf{t}(\xi)$ is a circle if $\mathbf{t}_0, \dots, \mathbf{t}_4$ are coplanar, since circles are the only plane curves on the unit sphere. To demonstrate coplanarity of $\mathbf{t}_0, \dots, \mathbf{t}_4$ under supposition (33), we examine the volumes of the tetrahedra defined by taking two distinct subsets of four points. Writing $\mathbf{t}_r = (t_{rx}, t_{ry}, t_{rz})$ two such volumes are

$$\begin{vmatrix} t_{0x} & t_{0y} & t_{0z} & 1 \\ t_{1x} & t_{1y} & t_{1z} & 1 \\ t_{2x} & t_{2y} & t_{2z} & 1 \\ t_{3x} & t_{3y} & t_{3z} & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} t_{1x} & t_{1y} & t_{1z} & 1 \\ t_{2x} & t_{2y} & t_{2z} & 1 \\ t_{3x} & t_{3y} & t_{3z} & 1 \\ t_{4x} & t_{4y} & t_{4z} & 1 \end{vmatrix} \tag{36}$$

and by use of a computer algebra system, one can verify symbolically[5] that these determinants vanish for arbitrary $\mathscr{A}_0, \mathscr{A}_2$ and c_0, c_2. Hence, condition (33) yields a circular tangent indicatrix—i.e., a helical PH quintic. □

It should be noted that, although Proposition 1 was proved solely on the basis of planarity of the tangent indicatrix, the curves that satisfy the linear dependence condition (33) are general helical PH quintics—the monotone-helical PH quintics discussed in Section 3 do not, in general, satisfy this condition.

[5] An alternative proof, that does not rely on computer algebra, is given in Appendix B.

Remark 1. Condition (33) can be interpreted geometrically in terms of the *generalized stereographic projection* introduced in [7], which maps a point of three-dimensional projective space with homogeneous coordinates (u, v, p, q) to the point on the unit sphere with coordinates

$$\left(\frac{u^2+v^2-p^2-q^2}{u^2+v^2+p^2+q^2}, \frac{2(uq+vp)}{u^2+v^2+p^2+q^2}, \frac{2(vq-up)}{u^2+v^2+p^2+q^2}\right).$$

Eq. (7) defines a straight line if condition (33) holds, and we interpret the quaternion components as three-dimensional homogeneous coordinates. According to [7, Lemma 3.5], the generalized stereographic projection maps lines in three-dimensional projective space to circles on the unit sphere.

The helical PH quintics defined by (33) for given quaternions $\mathscr{A}_0$, $\mathscr{A}_2$ and different scalars c_0, c_2 are closely related, as shown in the following result.

Proposition 2. *For a helical PH quintic with the quaternion coefficients* (8), *where* $\mathscr{A}_1 = c_0\mathscr{A}_0 + c_2\mathscr{A}_2$, *the axis* $\mathbf{a}$ *and angle* ψ *in the helix condition*

$$\mathbf{t}(\xi)\cdot\mathbf{a} \equiv \cos\psi \tag{37}$$

are independent of c_0, c_2 *and are given by*

$$\mathbf{a} = \frac{a_0\mathbf{a}_2 - a_2\mathbf{a}_0 + \mathbf{a}_0\times\mathbf{a}_2}{|a_0\mathbf{a}_2 - a_2\mathbf{a}_0 + \mathbf{a}_0\times\mathbf{a}_2|} \tag{38}$$

and

$$\cos\psi = \frac{a_0a_{2x} - a_2a_{0x} - a_{0y}a_{2z} + a_{0z}a_{2y}}{|a_0\mathbf{a}_2 - a_2\mathbf{a}_0 + \mathbf{a}_0\times\mathbf{a}_2|}. \tag{39}$$

Proof. For the tangent indicatrix (29) with control points and weights given by (34) and (35), condition (37) is equivalent to

$$\sum_{k=0}^{4} w_k(\mathbf{t}_k\cdot\mathbf{a} - \cos\psi)\binom{4}{k}(1-\xi)^{4-k}\xi^k \equiv 0,$$

which implies that

$$w_k(\mathbf{t}_k\cdot\mathbf{a} - \cos\psi) = 0, \quad k = 0,\ldots,4. \tag{40}$$

This amounts to five homogeneous linear equations in four unknowns, namely, $\cos\psi$ and the components a_x, a_y, a_z of $\mathbf{a}$ (which also satisfy $a_x^2 + a_y^2 + a_z^2 = 1$). By use of a computer algebra system, one can easily verify that (38) and (39) define a solution of this system for any c_0 and c_2 (see also Appendix B). □

Example 3. With the choices $\mathscr{A}_0 = 5 + \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathscr{A}_2 = -2 - 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $c_0 = 1$, $c_2 = -3$ in (33), we obtain the polynomials

$$u(\xi) = -19\xi^2 + 12\xi + 5, \quad v(\xi) = -22\xi^2 + 18\xi + 1,$$

$$p(\xi) = 15\xi^2 - 12\xi - 1, \quad q(\xi) = -31\xi^2 + 24\xi + 3,$$

which yield the Pythagorean hodograph

$$x'(\xi) = -341\xi^4 + 600\xi^3 - 270\xi^2 - 12\xi + 16,$$
$$y'(\xi) = 518\xi^4 - 588\xi^3 - 206\xi^2 + 252\xi + 28,$$
$$z'(\xi) = 1934\xi^4 - 2988\xi^3 + 770\xi^2 + 300\xi + 16,$$
$$\sigma(\xi) = 2031\xi^4 - 3096\xi^3 + 738\xi^2 + 324\xi + 36.$$

Since $\gcd(x', y', z') = \text{constant}$, the tangent indicatrix $\mathbf{t} = (x', y', z')/\sigma$ is a rational quartic in this case, corresponding to a doubly traced circle—from Proposition 2, we can verify that it lies in the plane defined by

$$15\,x'(\xi) - 3\,y'(\xi) + 15\,z'(\xi) = 11\,\sigma(\xi).$$

In Proposition 1, the sufficient condition on the quaternion coefficients to yield a general helical PH quintic is attractive, on account of its simplicity. In Appendix B we give a *sufficient-and-necessary* condition for the helicity of any PH quintic, which amounts to augmenting (33) with an alternative, non-linear condition—in fact, the latter transpires to be equivalent to the conditions given in Section 3 for monotone-helical PH quintics.

4.3. First-order Hermite interpolants

Interpolation of the Hermite data (20) yields [11] the system of equations

$$\mathcal{A}_0\mathbf{i}\,\mathcal{A}_0^* = \mathbf{d}_i, \quad \mathcal{A}_2\mathbf{i}\,\mathcal{A}_2^* = \mathbf{d}_f, \tag{41}$$

$$\begin{aligned}&(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)\mathbf{i}(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)^* \\ &\quad = 120\Delta\mathbf{p} - 15(\mathbf{d}_i + \mathbf{d}_f) + 5(\mathcal{A}_0\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\mathbf{i}\,\mathcal{A}_0^*)\end{aligned} \tag{42}$$

for the quaternion unknowns $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ (where $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$). Upon solving these equations, the PH quintic control points $\mathbf{p}_k = x_k\mathbf{i} + y_k\mathbf{j} + z_k\mathbf{k}$ are given by formulae (9), where we take $\mathbf{p}_0 = \mathbf{p}_i$ (and $\mathbf{p}_5 = \mathbf{p}_f$ by construction).

Writing $\mathbf{d}_i = |\mathbf{d}_i|(\lambda_i, \mu_i, \nu_i)$ and $\mathbf{d}_f = |\mathbf{d}_f|(\lambda_f, \mu_f, \nu_f)$, we have [11]:

$$\mathcal{A}_0 = \sqrt{\tfrac{1}{2}(1+\lambda_i)|\mathbf{d}_i|}\left(-\sin\phi_0 + \cos\phi_0\mathbf{i} + \frac{\mu_i\cos\phi_0 + \nu_i\sin\phi_0}{1+\lambda_i}\mathbf{j} + \frac{\nu_i\cos\phi_0 - \mu_i\sin\phi_0}{1+\lambda_i}\mathbf{k}\right), \tag{43}$$

$$\mathcal{A}_2 = \sqrt{\tfrac{1}{2}(1+\lambda_f)|\mathbf{d}_f|}\left(-\sin\phi_2 + \cos\phi_2\mathbf{i} + \frac{\mu_f\cos\phi_2 + \nu_f\sin\phi_2}{1+\lambda_f}\mathbf{j} + \frac{\nu_f\cos\phi_2 - \mu_f\sin\phi_2}{1+\lambda_f}\mathbf{k}\right), \tag{44}$$

where ϕ_0 and ϕ_2 are free angular variables. To obtain a helical PH quintic, we now substitute (33) into (42) and obtain

$$[1+3(c_0+c_2)+4c_0c_2]\,(\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^*+\mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*)$$
$$=30\Delta\mathbf{p}-(6+6c_0+4c_0^2)\mathbf{d}_i-(6+6c_2+4c_2^2)\mathbf{d}_f. \tag{45}$$

Now from (43) and (44) we may write

$$\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^*+\mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*=\sqrt{|\mathbf{d}_i|\,|\mathbf{d}_f|}\,(c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}) \tag{46}$$

where, with $\phi=\phi_2-\phi_0$, the vector (c_x,c_y,c_z) is again given by (26).

The vector equation (45) amounts to three scalar equations in c_0, c_2, ϕ. Once this system is solved, we can choose either ϕ_0 or ϕ_2 arbitrarily, and fix the other one from the difference $\phi=\phi_2-\phi_0$. Substituting these values into (43) and (44), we can determine $\mathscr{A}_1$ from (33) using the computed c_0, c_2 values.

To solve (45), we set $k_0=c_0+\frac{3}{4}$ and $k_2=c_2+\frac{3}{4}$, allowing us to express this equation in the simpler form

$$\mathbf{d}_i\,16k_0^2+\mathbf{d}_f\,16k_2^2+(\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)(16k_0k_2-5)=\mathbf{h}, \tag{47}$$

where $\mathbf{h}=120\Delta\mathbf{p}-15(\mathbf{d}_i+\mathbf{d}_f)$, and $\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^*+\mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*=\mathbf{u}\cos\phi+\mathbf{v}\sin\phi$, the vectors $\mathbf{u}$ and $\mathbf{v}$ being defined by (26) and (46).

If we regard the coordinate components of (47) as a system of three linear equations in $16k_0^2$, $16k_2^2$, $16k_0k_2-5$, Cramer's rule gives the solutions

$$16k_0^2=\frac{(\mathbf{h},\mathbf{d}_f,\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)}, \tag{48}$$

$$16k_2^2=\frac{(\mathbf{d}_i,\mathbf{h},\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)}, \tag{49}$$

$$16k_0k_2-5=\frac{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h})}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{u}\cos\phi+\mathbf{v}\sin\phi)}, \tag{50}$$

where $(\mathbf{a},\mathbf{b},\mathbf{c})$ denotes the scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Clearly, these solutions must satisfy the compatibility condition

$$(16k_0^2)(16k_2^2)=[(16k_0k_2-5)+5]^2.$$

By substituting into this condition and writing $\cos\phi=(1-t^2)/(1+t^2)$ and $\sin\phi=2t/(1+t^2)$, where $t=\tan\frac{1}{2}\phi$, we obtain a quartic equation

$$d_4t^4+d_3t^3+d_2t^2+d_1t+d_0=0 \tag{51}$$

in t, with coefficients given by

$$d_4=(\mathbf{h},\mathbf{d}_f,\mathbf{u})\,(\mathbf{d}_i,\mathbf{h},\mathbf{u})-(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}-5\mathbf{u})^2,$$

$$d_3=-2\,(\mathbf{h},\mathbf{d}_f,\mathbf{u})\,(\mathbf{d}_i,\mathbf{h},\mathbf{v})-2\,(\mathbf{d}_i,\mathbf{h},\mathbf{u})\,(\mathbf{h},\mathbf{d}_f,\mathbf{v})-20\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{v})\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}-5\mathbf{u}),$$

$$d_2 = -2\,(\mathbf{h},\mathbf{d}_f,\mathbf{u})\,(\mathbf{d}_i,\mathbf{h},\mathbf{u}) + 4\,(\mathbf{h},\mathbf{d}_f,\mathbf{v})\,(\mathbf{d}_i,\mathbf{h},\mathbf{v})$$
$$\quad - 100\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{v})^2 - 2\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}-5\mathbf{u})\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}+5\mathbf{u}),$$

$$d_1 = 2\,(\mathbf{h},\mathbf{d}_f,\mathbf{v})\,(\mathbf{d}_i,\mathbf{h},\mathbf{u}) + 2\,(\mathbf{d}_i,\mathbf{h},\mathbf{v})\,(\mathbf{h},\mathbf{d}_f,\mathbf{u}) - 20\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{v})\,(\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}+5\mathbf{u}),$$

$$d_0 = (\mathbf{h},\mathbf{d}_f,\mathbf{u})\,(\mathbf{d}_i,\mathbf{h},\mathbf{u}) - (\mathbf{d}_i,\mathbf{d}_f,\mathbf{h}+5\mathbf{u})^2.$$

This quartic admits a closed-form solution by means of Ferrari's method (see Appendix C)—it may have four, two, or zero real roots. For each real root t, we can evaluate the expressions on the right-hand sides of (48)–(50) by setting $\cos\phi = (1-t^2)/(1+t^2)$ and $\sin\phi = 2t/(1+t^2)$. The expressions for $16k_0^2$ and $16k_2^2$ must both yield non-negative quantities for a valid solution. If they are both positive, they each yield two values of equal magnitude and opposite sign. However, not all four combinations of signs are permissible—compatibility with (50) allows pairs of like sign *or* unlike sign, but not both. Thus each real root of (51) yields at most two feasible pairs of (k_0,k_2) values, and there is apparently an even number (between 0 and 8) of distinct helical PH quintic interpolants for given first-order Hermite data.

To construct the helical PH quintic that corresponds to a valid solution t,k_0,k_2, we take $\phi_0 = 0$ and $\phi_2 = 2\tan^{-1}t$ in (43) and (44), and use $c_0 = k_0 - \frac{3}{4}$ and $c_2 = k_2 - \frac{3}{4}$ in (33): the control points are then given by expressions (9). Because of the highly non-linear nature of the problem, it is rather difficult to derive a priori arguments concerning the number of distinct interpolants for a given set of Hermite data. However, we have observed empirically through numerous test cases that the quartic (51) generically has four real roots—of which two yield positive values for $16k_0^2$ and $16k_2^2$, and the other two yield negative values. There are thus, in general, *four* distinct helical PH quintic interpolants to first-order spatial Hermite data, in agreement [15] with the case of planar PH quintics [6]—which are (trivially) helical. Furthermore, since the two helical PH quintics corresponding to a particular root t of (51) share the same angle ϕ, the quaternions (43) and (44) are identical for these two PH quintics—they differ only in $\mathscr{A}_1$. Thus, by Proposition 2, these two PH quintics share the same helical axis $\mathbf{a}$ and angle ψ.

4.4. Choosing the "best" interpolant

As in the planar case [15], we need a means to select the "best" interpolant among the four solutions. In [11] we used the integral

$$\mathscr{E} = \int_0^1 \omega^2 |\mathbf{r}'|\,\mathrm{d}\xi. \tag{52}$$

as a measure of "fairness" for a space curve $\mathbf{r}(\xi)$, where the *total curvature* is given in terms of the curvature and torsion (10) by $\omega = \sqrt{\kappa^2+\tau^2}$.

The integral (52) represents the strain energy stored when an elastic rod is bent and twisted to assume the shape of the curve $\mathbf{r}(\xi)$, with the twisting being defined by the Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$—the instantaneous rotation rate of the Frenet frame is characterized [23] by the *Darboux vector*,

[6] See also [20,25] for further details on planar Hermite interpolation by PH curves.

$\mathbf{d}=\tau\mathbf{t}+\kappa\,\mathbf{b}$. Other "adapted" orthonormal frames (with the tangent as one component) exist on space curves [3], however, and are preferable in many applications. For example, the *rotation-minimizing frame* [4,9,19,22,34] is used in motion design, animation, and swept surface constructions. The energy associated with the rotation-minimizing frame is

$$\mathscr{E}_{\mathrm{RMF}} = \int_0^1 \kappa^2\,|\mathbf{r}'|\mathrm{d}\xi, \tag{53}$$

and this is the *least possible* energy[7] of adapted orthonormal frames along a given space curve. We therefore define the "best" among the four interpolants to be the curve with the least value for (53).

4.5. Illustrative examples

We now present some examples of the construction of first-order helical PH quintic Hermite interpolants, using the methods described above.

Example 4. For Hermite data $\mathbf{p}_i=(0,0,0)$, $\mathbf{d}_i=(1,0,1)$ and $\mathbf{p}_f=(1,1,1)$, $\mathbf{d}_f=(0,1,1)$, the quartic (51) has the four approximate roots

$$t_1=-0.059419,\quad t_2=-1.761857,\quad t_3=19.411014,\quad t_4=0.661850.$$

The roots t_1 and t_3 are rejected, since they do not yield non-negative values for k_0^2 and k_2^2. Root t_2 yields the pair of feasible solutions

$$(k_0,k_2,\phi)=(\mp 1.705395,\pm 1.705395,-2.109108)$$

for which the helical axis and angle are given by

$$\mathbf{a}=(-0.309913,-0.309913,-0.898837),\quad \cos\psi=-0.854715.$$

The curvature/torsion ratio has magnitude $|\kappa/\tau|=0.607333$ for both curves (κ/τ changes sign, since the curves contain an inflection: see Fig. 1, upper). Similarly, root t_4 yields the pair of feasible solutions

$$(k_0,k_2,\phi)=(\mp 1.850380,\mp 1.850380,1.169321)$$

for which the helical axis and angle are given by

$$\mathbf{a}=(-0.354664,-0.354664,0.865117),\quad \cos\psi=0.862515.$$

These curves have no inflections (see Fig. 1, lower)—the curvature/torsion ratio is $\kappa/\tau=-0.586692$ in the first case, and $\kappa/\tau=0.586692$ in the second.

The four helical PH quintics have energy values $\mathscr{E}_{\mathrm{RMF}}=322.40$, 322.40, 89.17, and 1.31. As in the planar case [15], it is essential that the least-energy curve be chosen as the "good" solution, since the other three solutions may exhibit rather contorted shapes (see Fig. 1).

[7] It also agrees with the energy integral [8] for planar PH curves.

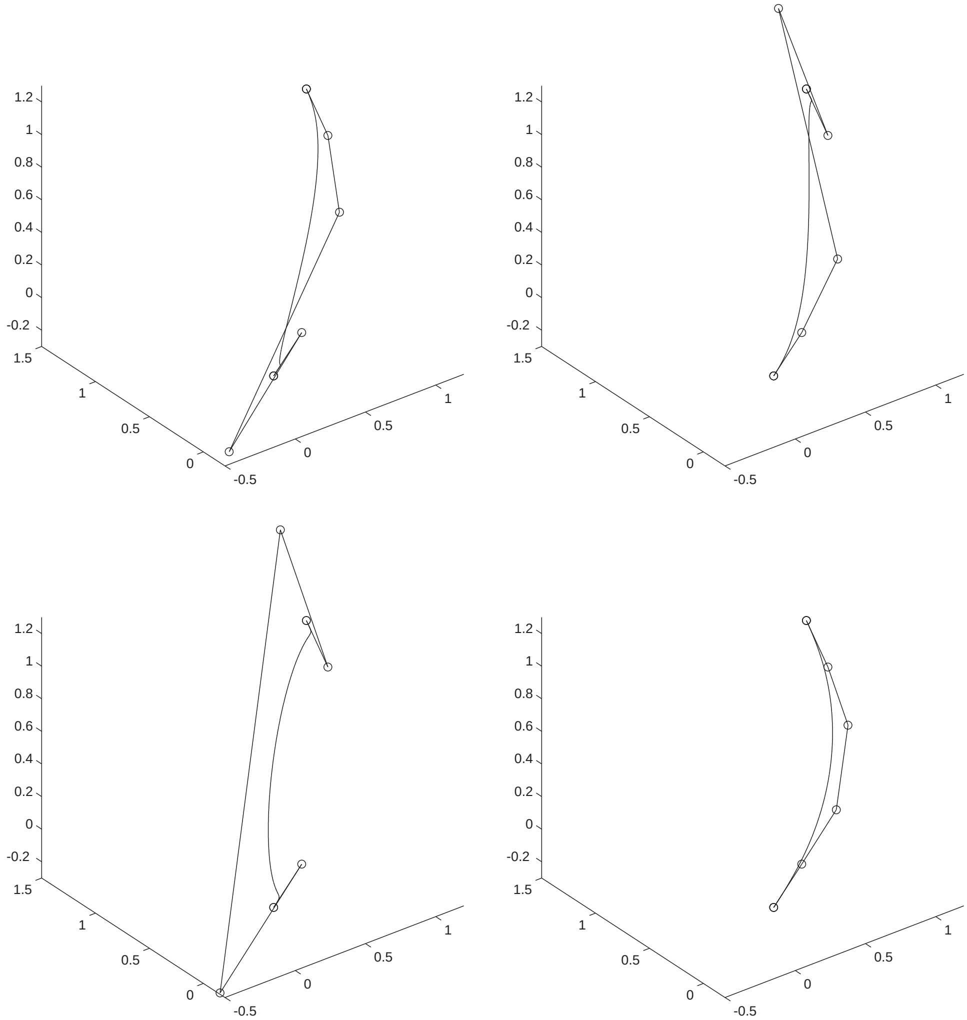

Fig. 1. The four PH quintic Hermite interpolants for the data $\mathbf{p}_i = (0, 0, 0)$, $\mathbf{p}_f = (1, 1, 1)$ and $\mathbf{d}_i = (1, 0, 1)$, $\mathbf{d}_f = (0, 1, 1)$ of Example 4. Upper: the two curves defined by the root t_2. Lower: the two curves defined by the root t_4.

In Fig. 2 we compare the "good" helical PH quintic interpolants with the solutions presented in [11] using the *ad hoc* choice $\phi_0 = \phi_2 = -\frac{1}{2}\pi$ in the general Hermite interpolation algorithm. Both examples use the end points $\mathbf{p}_i = (0, 0, 0)$ and $\mathbf{p}_f = (1, 1, 1)$. The first example is for end derivatives $\mathbf{d}_i = (-0.8, 0.3, 1.2)$ and $\mathbf{d}_f = (0.5, -1.3, -1.0)$—in this case, the ad hoc solution is serendipitously quite good, and has energy $\mathscr{E}_{\mathrm{RMF}} = 8.89$ only slightly larger than that of the helical solution, $\mathscr{E}_{\mathrm{RMF}} = 8.69$. The second example has end derivatives $\mathbf{d}_i = (0.4, -1.5, -1.2)$ and $\mathbf{d}_f = (-1.2, -0.6, -1.2)$ and exhibits a more pronounced discrepancy between the helical and ad hoc solutions: the former has energy $\mathscr{E}_{\mathrm{RMF}} = 16.72$; the latter $\mathscr{E}_{\mathrm{RMF}} = 50.51$—clearly, the "S" shape of this curve is an extravagance not warranted by the Hermite data.

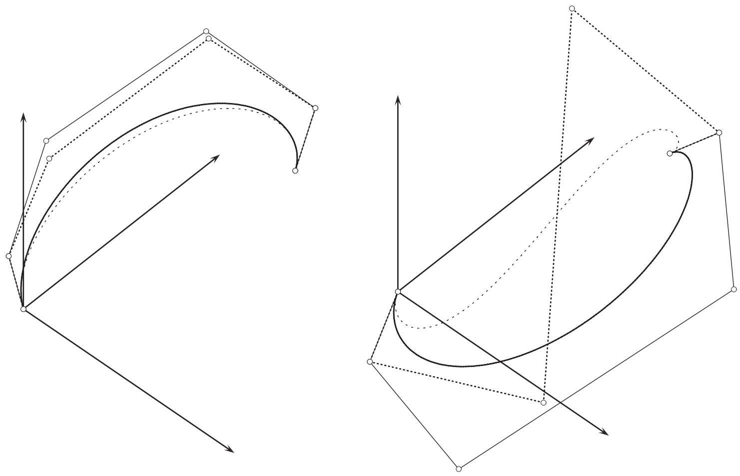

Fig. 2. PH quintics interpolating Hermite data $\mathbf{p}_i = (0,0,0)$, $\mathbf{p}_f = (1,1,1)$ and $\mathbf{d}_i = (-0.8, 0.3, 1.2)$, $\mathbf{d}_f = (0.5, -1.3, -1.0)$ and $\mathbf{d}_i = (0.4, -1.5, -1.2)$, $\mathbf{d}_f = (-1.2, -0.6, -1.2)$ on the left and the right. The solid curves and control polygons are the "good" helical PH quintics, and the dashed curves and control polygons correspond to the choice $\phi_0 = \phi_2 = -\frac{1}{2}\pi$ in the algorithm of [11].

5. Closure

Although all spatial PH cubics are helical curves, the helical PH quintics form a proper subset of all spatial PH quintics. Two principal types of helical PH quintic exist. The "monotone-helical" PH quintics exhibit a degree reduction in the rational tangent indicatrix from 4 to 2, due to the presence of a common quadratic factor among the hodograph components—for these curves, the tangent maintains a consistent sense of rotation about the helical axis. The general helical PH quintics, on the other hand, have a rational quartic tangent indicatrix (corresponding to a doubly traced circle), and hence may exhibit reversals in the sense of tangent rotation about the axis.

With both types of helical PH quintic, first-order Hermite interpolation incurs a three-dimensional vector equation with a non-linear dependence on three scalar unknowns. For the monotone-helical PH quintics, interpolants do not exist for all Hermite data sets. For the general helical PH quintics, however, there is strong empirical evidence that solutions exist for arbitrary data, and their construction may be reduced to solving a quartic equation. As in the planar case [15], there are in general four distinct interpolants to given Hermite data, and the "good" solution must be selected by a minimum energy criterion. The resulting interpolant typically has excellent shape properties, especially compared to interpolants resulting from ad hoc choices for the two free parameters in the general PH quintic Hermite interpolation scheme [11].

Helical curves are intimately related to PH curves, since the imposition of helicity upon a non-planar polynomial curve coerces it to be a PH curve. Thus, the algorithms described herein serve not only to provide convenient constructions for spatial PH curves without left-over degrees of freedom, but also to allow the "constant slope" property of helical curves to be reconciled with the ubiquitous Bernstein–Bézier representation.

A challenging open problem is to find construction schemes for C^2 helical PH quintic splines in $\mathbb{R}^3$, extending the methods presented in [1,12,14].

Acknowledgements

The first author was supported in part by the National Science Foundation, under grants CCR-9902669, CCR-0202179, and DMS-0138411. The second author was supported in part by KOSEF through the Statistical Research Center for Complex Systems at Seoul National University.

Appendix A. review of quaternion algebra

Quaternions are "four-dimensional numbers" of the form

$$\mathscr{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad \text{and} \quad \mathscr{B} = b + b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}, \tag{A.1}$$

where the "basis elements" 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Here 1 is the usual real unit; its product with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ leaves them unchanged. Preserving the order of terms in products, we deduce from the above that

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{A.2}$$

Thus, since the products of the basis elements are non-commutative, we have $\mathscr{A}\mathscr{B} \neq \mathscr{B}\mathscr{A}$ in general. Quaternion multiplication is associative, however—so that $(\mathscr{A}\mathscr{B})\mathscr{C} = \mathscr{A}(\mathscr{B}\mathscr{C})$ for any three quaternions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$.

The sum of the two quaternions (A.4) is simply

$$\mathscr{A} + \mathscr{B} = (a + b) + (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}, \tag{A.3}$$

and using relations (A.2), the product is given by

$$\begin{aligned} \mathscr{A}\mathscr{B} = {} & (ab - a_xb_x - a_yb_y - a_zb_z) \\ & + (ab_x + ba_x + a_yb_z - a_zb_y)\mathbf{i} \\ & + (ab_y + ba_y + a_zb_x - a_xb_z)\mathbf{j} \\ & + (ab_z + ba_z + a_xb_y - a_yb_x)\mathbf{k}. \end{aligned} \tag{A.4}$$

The notations of three-dimensional vector analysis furnish a useful shorthand for quaternion operations. Regarding $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as unit vectors in a Cartesian coordinate system, we interpret $\mathscr{A}$ as comprising "scalar" and "vector" parts,[8] a and $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, and we write $\mathscr{A} = (a, \mathbf{a})$. All real numbers and three-dimensional vectors are subsumed as "pure scalar" and "pure vector" quaternions, of

[8] Also known as the "real" and "imaginary" parts—the square of a "pure imaginary" quaternion is always a negative real number.

the form $(a, \mathbf{0})$ and $(0, \mathbf{a})$, respectively—for brevity, we often denote such quaternions as simply a and $\mathbf{a}$.

Writing $\mathscr{A} = (a, \mathbf{a})$ and $\mathscr{B} = (b, \mathbf{b})$ in lieu of (A.4), the sum (A.3) and product (A.4) may be more compactly expressed [27] as

$$\mathscr{A} + \mathscr{B} = (a + b, \mathbf{a} + \mathbf{b}),$$

$$\mathscr{A}\,\mathscr{B} = (\, ab - \mathbf{a} \cdot \mathbf{b},\, a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b}),$$

where the usual rules for vector sums and dot and cross products are invoked. Every quaternion $\mathscr{A} = (a, \mathbf{a})$ has a *conjugate*, $\mathscr{A}^* = (a, -\mathbf{a})$, and a *magnitude* equal to the non-negative real number $|\mathscr{A}|$ defined by

$$|\mathscr{A}|^2 = \mathscr{A}^*\mathscr{A} = \mathscr{A}\mathscr{A}^* = a^2 + |\mathbf{a}|^2. \tag{A.5}$$

One can readily verify that the conjugates of products satisfy the rule

$$(\mathscr{A}\,\mathscr{B})^* = \mathscr{B}^*\mathscr{A}^*. \tag{A.6}$$

If $|\mathscr{A}| = 1$, we say that $\mathscr{A}$ is a *unit* quaternion. The unit quaternions form a (non-commutative) *group* under multiplication, since the product of two unit quaternions is always a unit quaternion. Unit quaternions are necessarily of the form $\mathscr{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\mathbf{n})$ for some angle θ and unit vector $\mathbf{n}$.

For any pure vector quaternion $\mathbf{v}$ and unit quaternion $\mathscr{U}$, the quaternion product $\mathscr{U}\mathbf{v}\,\mathscr{U}^*$ always yields a pure vector quaternion, that corresponds to a rotation of $\mathbf{v}$ through angle θ about the axis defined by $\mathbf{n}$ [27]. Note also that the unit quaternion $-\mathscr{U} = (-\cos\frac{1}{2}\theta, -\sin\frac{1}{2}\theta\mathbf{n})$ specifies a rotation through $2\pi - \theta$ about $-\mathbf{n}$, and thus has the same effect as $\mathscr{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$.

Appendix B. Proposition 1—alternative proof

Consider four quaternions, defined in terms of $\mathscr{A}_0$ and $\mathscr{A}_2$ by

$$\mathscr{Q}_0 = (q_0, \mathbf{q}_0) = (\mathscr{A}_0\mathscr{A}_0^*, \mathscr{A}_0\mathbf{i}\,\mathscr{A}_0^*),$$

$$\mathscr{Q}_2 = (q_2, \mathbf{q}_2) = (\mathscr{A}_2\mathscr{A}_2^*, \mathscr{A}_2\mathbf{i}\,\mathscr{A}_2^*),$$

$$\mathscr{R} = (r, \mathbf{r}) = (\mathscr{A}_0\mathscr{A}_2^* + \mathscr{A}_2\mathscr{A}_0^*, \mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*),$$

$$\mathscr{S} = (s, \mathbf{s}) = (-\mathscr{A}_0\mathbf{i}\,\mathscr{A}_2^* + \mathscr{A}_2\mathbf{i}\,\mathscr{A}_0^*, \mathscr{A}_0\mathscr{A}_2^* - \mathscr{A}_2\mathscr{A}_0^*).$$

With some algebra, it is possible to show that these quaternions are linearly independent if and only if the four quaternions $\mathscr{A}_0$, $\mathscr{A}_0\mathbf{i}$, $\mathscr{A}_2$, $\mathscr{A}_2\mathbf{i}$ are linearly independent. Assuming their linear independence, we can write $\mathscr{A}_1$ as the linear combination

$$\mathscr{A}_1 = \mathscr{A}_0\,(c_0 + \gamma_0\mathbf{i}) + \mathscr{A}_2\,(c_2 + \gamma_2\mathbf{i}). \tag{B.1}$$

for suitable scalars c_0, γ_0 and c_2, γ_2. This gives

$$\mathscr{A}_0 \mathbf{i} \mathscr{A}_1^* + \mathscr{A}_1 \mathbf{i} \mathscr{A}_0^* = 2c_0\mathbf{q}_0 + c_2\mathbf{r} + \gamma_2\mathbf{s},$$

$$\mathscr{A}_1 \mathbf{i} \mathscr{A}_1^* = (c_0^2+\gamma_0^2)\mathbf{q}_0 + (c_2^2+\gamma_2^2)\mathbf{q}_2 + (c_0c_2+\gamma_0\gamma_2)\mathbf{r} + (c_0\gamma_2 - c_2\gamma_0)\mathbf{s}, \tag{B.2}$$

$$\mathscr{A}_1 \mathbf{i} \mathscr{A}_2^* + \mathscr{A}_2 \mathbf{i} \mathscr{A}_1^* = 2c_2\mathbf{q}_2 + c_0\mathbf{r} - \gamma_0\mathbf{s}$$

and

$$\mathscr{A}_0 \mathscr{A}_1^* + \mathscr{A}_1 \mathscr{A}_0^* = c_0 q_0 + c_2 r + \gamma_2 s,$$

$$\mathscr{A}_1 \mathscr{A}_1^* = (c_0^2+\gamma_0^2)q_0 + (c_2^2+\gamma_2^2)q_2 + (c_0c_2+\gamma_0\gamma_2)r + (c_0\gamma_2 - c_2\gamma_0)s, \tag{B.3}$$

$$\mathscr{A}_1 \mathscr{A}_2^* + \mathscr{A}_2 \mathscr{A}_1^* = 2c_2 q_2 + c_0 r - \gamma_0 s.$$

Now when the spatial Pythagorean hodograph defined by (5) and (7) satisfies the helix condition (2), the system of five homogeneous linear equations (40) in the four unknowns $\cos\psi$ and a_x, a_y, a_z (the components of $\mathbf{a}$) must hold. Thus, from (29)–(31) and (B.2)–(B.3), we infer that a PH curve is a helix if and only if the 4×5 matrix

$$\begin{bmatrix} q_0 & c_2 r + \gamma_2 s & (4c_0c_2 + 4\gamma_0\gamma_2 + 1)r + 4(c_0\gamma_2 - c_2\gamma_0)s & c_0 r - \gamma_0 s & q_2 \\ \mathbf{q}_0 & c_2\mathbf{r} + \gamma_2\mathbf{s} & (4c_0c_2 + 4\gamma_0\gamma_2 + 1)\mathbf{r} + 4(c_0\gamma_2 - c_2\gamma_0)\mathbf{s} & c_0\mathbf{r} - \gamma_0\mathbf{s} & \mathbf{q}_2 \end{bmatrix}$$

has rank $\leqslant 3$. Since we assume that the four quaternions $(q_0, \mathbf{q}_0)$, $(r, \mathbf{r})$, $(s, \mathbf{s})$, $(q_2, \mathbf{q}_2)$ are linearly independent, the rank of this matrix can be modified only by changing the coefficients c_0, γ_0 and c_2, γ_2. Specifically, it is clear that the rank is ≤ 3 if and only if the central 4×3 sub-matrix

$$\begin{bmatrix} c_2 r + \gamma_2 s & (4c_0c_2 + 4\gamma_0\gamma_2 + 1)r + 4(c_0\gamma_2 - c_2\gamma_0)s & c_0 r - \gamma_0 s \\ c_2\mathbf{r} + \gamma_2\mathbf{s} & (4c_0c_2 + 4\gamma_0\gamma_2 + 1)\mathbf{r} + 4(c_0\gamma_2 - c_2\gamma_0)\mathbf{s} & c_0\mathbf{r} - \gamma_0\mathbf{s} \end{bmatrix} \tag{B.4}$$

has rank 1 (note that, under our hypothesis, this sub-matrix cannot vanish).

Proposition B.1. *When the quaternion $\mathscr{A}_1$ is expressed in terms of $\mathscr{A}_0$, $\mathscr{A}_2$ in form* (B.1), *the PH quintic defined by* (4) *and* (5) *is helical if and only if one of the following conditions holds*:

$$\gamma_0 = \gamma_2 = 0 \quad \text{or} \quad 4(c_0 + \gamma_0\mathbf{i})(c_2 + \gamma_2\mathbf{i}) = 1. \tag{B.5}$$

Proof. From the preceding discussion, we know that the PH quintic is helical if and only if the sub-matrix (B.4) is of rank 1, i.e., its three columns define parallel 4-vectors. This is obviously true under the first of conditions (B.5). Also, when $(\gamma_0, \gamma_2) \neq (0, 0)$, the second of conditions

(B.5) yields

$$\frac{c_2}{\gamma_2} = -\frac{c_0}{\gamma_0} = \frac{1 + 4c_0c_2 + 4\gamma_0\gamma_2}{4(c_0\gamma_2 - c_2\gamma_0)},$$

which is also equivalent to parallelism of the columns of (B.4). □

Proposition 1 of Section 4, with real coefficients c_0 and c_2 in (33), corresponds to the first of conditions (B.5). Moreover, one can easily verify that the quadratic quaternion polynomial (7) has the factorization

$$\mathscr{A}(\xi) = [\mathscr{B}_0(1-\xi) + \mathscr{B}_1\xi]\,[(f_0 + g_0\mathbf{i})(1-\xi) + (f_1 + g_1\mathbf{i})\xi],$$

where f_0, g_0, f_1, g_1 are real constants, if and only if the second condition in (B.5) holds. Hence, this second condition is equivalent to the characterization of monotone-helical PH quintics obtained in Section 3.

Remark B.1. Under mapping by the generalized stereographic projection [7], the pre-image of a circle on the unit sphere is either (i) a line, or (ii) a conic lying on a certain quadric surface. These two cases correspond to the first and second of conditions (B.5)—i.e., to the general helical PH quintics and the monotone-helical PH quintics, respectively.

Appendix C. Ferrari's method for quartics

Ferrari's method [33] computes the four roots of the quartic equation

$$t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 = 0 \tag{C.1}$$

as follows. Let z be a real root of the *resolvent* cubic equation

$$z^3 + c_2z^2 + c_1z + c_0 = 0 \tag{C.2}$$

with $c_2 = -a_2$, $c_1 = a_1a_3 - 4a_0$, and $c_0 = 4a_2a_0 - a_1^2 - a_3^2a_0$. Then the roots of (C.1) are the same as the roots of the two quadratic equations

$$t^2 + (\tfrac{1}{2}a_3 \pm E)\,t + (\tfrac{1}{2}z \pm F) = 0, \tag{C.3}$$

where we define

$$E = \tfrac{1}{2}\sqrt{a_3^2 + 4(z - a_2)} \quad \text{and} \quad F = \frac{a_3z - 2a_1}{4E}. \tag{C.4}$$

The roots of the cubic (C.2) may be obtained by Cardano's method [33]. Set

$$Q = \frac{3c_1 - c_2^2}{9}, \quad R = \frac{9c_1c_2 - 27c_0 - 2c_2^3}{54}, \quad \Delta = Q^3 + R^2,$$

and let S be any of the three complex values specified by

$$S^3 = R + \sqrt{\Delta}. \tag{C.5}$$

Then, writing

$$A = S - \frac{Q}{S} \quad \text{and} \quad B = S + \frac{Q}{S}, \tag{C.6}$$

the roots of (C.2) are given by

$$z = \begin{cases} -\frac{1}{3}c_2 + A, \\ -\frac{1}{3}c_2 - \frac{1}{2}A + \frac{1}{2}\sqrt{3}\mathrm{i}\,B, \\ -\frac{1}{3}c_2 - \frac{1}{2}A - \frac{1}{2}\sqrt{3}\mathrm{i}\,B. \end{cases} \tag{C.7}$$

One root is real and the other two are complex conjugates if $\Delta > 0$; all three roots are real and distinct if $\Delta < 0$; and when $\Delta = 0$ there is a multiple root.

References

[1] G. Albrecht, R.T. Farouki, Construction of C^2 Pythagorean hodograph interpolating splines by the homotopy method, Adv. Comput. Math. 5 (1996) 417–442.

[2] C. Bangert, H. Prautzsch, Circle and sphere as rational splines, Neural Parallel Sci. Comput. 5 (1997) 153–161.

[3] R.L. Bishop, There is more than one way to frame a curve, Amer. Math. Monthly 82 (1975) 246–251.

[4] H.I. Choi, C.Y. Han, Euler–Rodrigues frames on spatial Pythagorean-hodograph curves, Comput. Aided Geom. Design 19 (2002) 603–620.

[5] H.I. Choi, D.S. Lee, H.P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, Adv. Comput. Math. 17 (2002) 5–48.

[6] J.J. Chou, Higher order Bézier circles, Comput. Aided Design 27 (1995) 303–309.

[7] R. Dietz, J. Hoschek, B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, Comput. Aided Geom. Design 10 (1993) 211–229.

[8] R.T. Farouki, The elastic bending energy of Pythagorean hodograph curves, Comput. Aided Geom. Design 13 (1996) 227–241.

[9] R.T. Farouki, Exact rotation-minimizing frames for spatial Pythagorean-hodograph curves, Graphical Models 64 (2002) 382–395.

[10] R.T. Farouki, M. al-Kandari, T. Sakkalis, Structural invariance of spatial Pythagorean hodographs, Comput. Aided Geom. Design 19 (2002) 395–407.

[11] R. T. Farouki, M. al-Kandari, T. Sakkalis, Hermite interpolation by rotation-invariant spatial Pythagorean-hodograph curves, Adv. Comput. Math. 17 (2002) 369–383.

[12] R.T. Farouki, B.K. Kuspa, C. Manni, A. Sestini, Efficient solution of the complex quadratic tridiagonal system for C^2 PH quintic splines, Numer. Algorithms 27 (2001) 35–60.

[13] R.T. Farouki, J. Manjunathaiah, D. Nicholas, G.-F. Yuan, S. Jee, Variable feedrate CNC interpolators for constant material removal rates along Pythagorean-hodograph curves, Comput. Aided Geom. Design 30 (1998) 631–640.

[14] R.T. Farouki, C. Manni, A. Sestini, Spatial C^2 PH quintic splines, in: T. Lyche et al. (Eds.) Curve and Surface Design: St. Malo 2002, Nashboro Press, 2003, pp. 123–146.

[15] R.T. Farouki, C.A. Neff, Hermite interpolation by Pythagorean hodograph quintics, Math. Comput. 64 (1995) 1589–1609.

[16] R.T. Farouki, T. Sakkalis, Pythagorean-hodograph space curves, Adv. Comput. Math. 2 (1994) 41–66.

[17] R.T. Farouki, S. Shah, Real-time CNC interpolators for Pythagorean-hodograph curves, Comput. Aided Geom. Design 13 (1996) 583–600.

[18] J.-C. Fiorot, P. Jeannin, I. Cattiaux-Huillard, The circle as a smoothly joined BR-curve on [0,1], Comput. Aided Geom. Design 14 (1997) 313–323.

[19] H. Guggenheimer, Computing frames along a trajectory, Comput. Aided Geom. Design 6 (1989) 77–78.

[20] B. Jüttler, Hermite interpolation by Pythagorean hodograph curves of degree seven, Math. Comput. 70 (2001) 1089–1111.

[21] B. Jüttler, W. Wang, The shape of spherical quartics, Comput. Aided Geom. Design 20 (2003), to appear.
[22] F. Klok, Two moving coordinate frames for sweeping along a 3D trajectory, Comput. Aided Geom. Design 3 (1986) 217–229.
[23] E. Kreyszig, Differential Geometry, University of Toronto Press, 1959.
[24] M. Lipschutz, Theory and Problems of Differential Geometry, McGraw-Hill, New York, 1969.
[25] H.P. Moon, R.T. Farouki, H.I. Choi, Construction and shape analysis of PH quintic Hermite interpolants, Comput. Aided Geom. Design 18 (2001) 93–115.
[26] L. Piegl, W. Tiller, A menagerie of rational B-spline circles, IEEE Comput. Graph. Appl. 9 (5) (1989) 48–56.
[27] J. Roe, Elementary Geometry, Oxford University Press, 1993.
[28] V. Snyder, C.H. Sisam, Analytic Geometry of Space, Henry Holt & Co., New York, 1914.
[29] D.M.Y. Sommerville, Analytical Geometry of Three Dimensions, Cambridge University Press, 1951.
[30] D.J. Struik, Lectures on Classical Differential Geometry, Dover Publications (reprint), New York, 1988.
[31] H.G. Telling, The Rational Quartic Curve in Space of Three and Four Dimensions, Cambridge University Press, Cambridge, 1936.
[32] Y-F. Tsai, R.T. Farouki, B. Feldman, Performance analysis of CNC interpolators for time-dependent feedrates along PH curves, Comput. Aided Geom. Design 18 (2001) 245–265.
[33] J.V. Uspensky, Theory of Equations, McGraw-Hill, New York, 1948.
[34] W. Wang, B. Joe, Robust computation of the rotation minimizing frame for sweep surface modelling, Comput. Aided Design 29 (1997) 379–391.