Pythagorean Triples in Uniquef Factorization Domains

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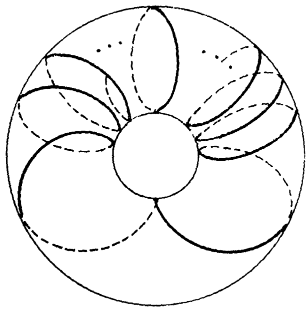
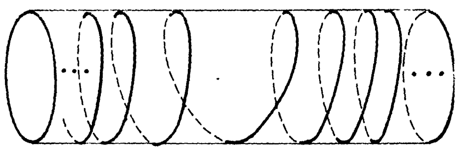
compactification of $\mathbb{R}$:



$c\mathbb{R} \times K$

If one uses $c\mathbb{R} = [0,1]$, the two point compactification, then $e\mathbb{R}$ is the "two circle compactification":



$c\mathbb{R} \times K$

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PYTHAGOREAN TRIPLES IN UNIQUE FACTORIZATION DOMAINS

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In two MONTHLY notes [4] and [5], Sexhauer has determined the primitive Pythagorean triples for a certain class of unique factorization domains. The aim here is to characterize Pythagorean triples in an arbitrary unique factorization domain.

Throughout this note, $D \neq (0)$ will be a unique factorization domain with field of quotients K. A Pythagorean triple in D is a triple (a,b,c) of elements of D satisfying

$$(1) \qquad a^2 + b^2 = c^2.$$

It is easy to verify that if $u, v, w \in D$, then (a,b,c) and (b,a,c), where

(2) $$a = w(u^2 - v^2), \quad b = 2wuv, \quad \text{and } c = w(u^2 + v^2),$$

are Pythagorean triples in D.

Not every Pythagorean triple is of this form if D is of characteristic 2 or if 2 is neither a unit nor a prime in D. In fact, if D has characteristic 2, it is easy to see that the Pythagorean triples are those of the form $(a, b, a + b)$, where $a, b \in D$. Also, if D is a ring such that $0 \neq 2 = pq$, where $p, q \in D$ are non-units, then $(p + 2, q + 2, p + q + 2)$ is a Pythagorean triple in D. But it cannot be of the form (2) since $2 \nmid p+2$ and $2 \nmid q + 2$.

In general, if f, u, and v are arbitrary elements of D and if d is a factor of 2 relatively prime to f such that $d \mid u^2 \pm v^2$, then (a, b, c), where

(3) $$a = \frac{f(u^2 - v^2)}{d}, \quad b = \frac{2fuv}{d}, \quad \text{and } c = \frac{f(u^2 + v^2)}{d},$$

can be verified to be a Pythagorean triple. The theorem is the converse.

THEOREM. *If $D \neq (0)$ is a unique factorization domain of characteristic not* 2, *then every Pythagorean triple is of the form* (3). *If, in addition, the element* 2 *of D is either prime or invertible in D, then every Pythagorean triple is of the form* (2).

Proof. Let (a, b, c) be a Pythagorean triple in D. Since the case where $c - a = 0$ is trivial, we assume that $c - a \neq 0$. Then we write $c - a = gh^2$, where $g, h \in D$ and g is square-free. Define $v = h$, $u = hb/(c - a)$ and $f/d = g/2$, where $d \mid 2$ and $(f, d) = 1$. A computation using $a^2 + b^2 = c^2$ shows that these values of $f, d, u,$ and v satisfy equation (3). It follows that $a + c = 2fu^2/d = gu^2$, so that $gu^2 \in D$. Since g is square free and $u \in K$, the field of quotients of D, it follows that $u \in D$. Also, since $(f, d) = 1$ and $a, c \in D$, equation (3) implies $d \mid u^2 \pm v^2$. Hence (a, b, c) is of the form (3).

Now suppose 2 is a unit or a prime in D. If $2 \mid g$, then $f/d = g/2 \in D$ so that (a, b, c) is of the form (2). If $2 \nmid g$, define $w = g$, $u_1 = (u + v)/2$, and $v_1 = (u - v)/2$. Then using equation (3), it is easy to see that $a = 2wu_1v_1$, $b = w(u_1^2 - v_1^2)$, and $c = w(u_1^2 + v_1^2)$. Therefore $2wu_1^2 = c + b \in D$ and $2wv_1^2 = c - b \in D$. Now $2w$ is square free since $2 \nmid w$, and $u_1, v_1 \in K$; consequently, $u_1, v_1 \in D$. Hence (b, a, c) is of the form (2) and the proof is complete.

The theorem implies that the Pythagorean triples in each of the following cases are all of the form (2):

(a) $D = Z$, the ring of ordinary integers.

(b) $D = K$, a field of characteristic not 2.

(c) $D = K[x_1, \cdots, x_n]$, where K is as in (b) or is a unique factorization domain like Z, where 2 is prime or invertible.

(d) $D = K[[x_1, \cdots, x_n]]$ (power series), where K is regular and satisfies either of the two conditions in (c).

(e) D is the ring of integers of an algebraic number field of class number 1, in which 2 is prime. For example, the cubic field of $x^3 + x + 1 = 0$.

For proofs of the facts that the rings in (c) and (d) have unique factorization, the reader is referred to Zariski and Samuel [6] and Samuel [3]. It is these two cases that motivated this work in light of Greenleaf [1], and Gross [2].

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RESEARCH PROBLEMS

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In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

DO SELF-INTERSECTIONS CHARACTERIZE CURVES OF CONSTANT WIDTH?

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A convex curve, the boundary of a compact convex body in the Euclidean plane, has **constant width** if the distance between parallel support lines to the body is the same for all directions. On a curve of constant width w any two points at distance w lie on parallel support lines, and the chord joining them is perpendicular to the lines. Every normal to a curve of constant width is a double-normal, and this property characterizes the curves. For curves of constant width, diameters always intersect in the interior of the curve or on the curve itself. Further properties can be found in [1], [2], [4], [5], [6], [11], and [12].

For any two convex curves S_1 and S_2, we define $\alpha(S_1, S_2)$ to be the number of components of $S_1 \cap S_2$. We assume in all cases that the curves are so situated that $\alpha(S_1, S_2) > 1$, so that in particular we rule out cases where the two curves coincide or are externally tangent. In the case of two curves of constant width w, the function α can never take on odd values, although it can become infinite [10].