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Clifford algebra, spin representation, and rational parameterization of curves and surfaces *

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The Pythagorean hodograph (PH) curves are characterized by certain Pythagorean n-tuple identities in the polynomial ring, involving the derivatives of the curve coordinate functions. Such curves have many advantageous properties in computer aided geometric design. Thus far, PH curves have been studied in 2- or 3-dimensional Euclidean and Minkowski spaces. The characterization of PH curves in each of these contexts gives rise to different combinations of polynomials that satisfy further complicated identities. We present a novel approach to the Pythagorean hodograph curves, based on Clifford algebra methods, that unifies all known incarnations of PH curves into a single coherent framework. Furthermore, we discuss certain differential or algebraic geometric perspectives that arise from this new approach.

1. Introduction

In this paper, we present an approach that unifies all known incarnations of the so-called *Pythagorean hodograph curves* into a single coherent framework, through the use of Clifford algebra. As we shall see, Pythagorean hodograph (PH) curves are characterized by a certain Pythagorean n-tuple identity in the polynomial ring, relating derivatives of the curve coordinate functions. Thus far, the PH curves have been studied in 2- or 3-dimensional Euclidean and Minkowski spaces. The characterization of PH curves in each of these contexts involves rather different combinations of certain sets of polynomials. Currently, the diverse algebraic forms for PH curves in spaces with different dimensions and metrics are something of an enigma, that suggest the presence of a deeper underlying structure.

The approach expounded in this paper reveals that all the different forms of PH curves can be expressed via a certain map, which we shall call the *PH representation*

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map, that is a natural extension of the usual twisted adjoint representation of the spin group to a larger set of the even Clifford algebra. This PH representation map provides a framework that unifies all known instances of the PH curves (it transpires that this map is the same as the so-called Kustaanheimo–Stiefel transformation [11] in physics).

We also study the PH representation map in the context of 4-dimensional Minkowski space. In this case, the PH representation map approach reveals a new and unexpected phenomenon that is carefully analyzed in this paper. The most salient consequence is that *every* space-like polynomial curve in the 4-dimensional Minkowski space is a Pythagorean hodograph curve. This result is in marked contrast to the situation in 2 or 3 dimensions, where only certain special curves qualify for Pythagorean hodograph status.

The PH representation map can also be studied from a more geometrical (homogeneous space) viewpoint, involving the Pauli and Dirac matrices that appear in relativistic quantum mechanics. This leads naturally to some interesting geometrical interpretations. For example, the PH representation map corresponding to a 3-dimensional PH curve turns out to be nothing other than the celebrated Hopf fibration, or rather a singular foliation of the Hopf fibration (we were informed by Helmut Pottmann that this fact is already known to experts in the field). The Minkowski space setting also has similar homogeneous space interpretations.

The problem of Pythagorean hodograph curves that we address herein is purely mathematical. In fact, it can be regarded as belonging to the realm of algebra, but with a strong geometrical flavor and consequences. Nevertheless, this problem has roots in the practical field of computer aided geometric design (CAGD). In this field, the specification of curves and surfaces by rational parameterizations is strongly preferred. However, many important geometrical objects that arise naturally in CAGD are not ordinarily rational. One such example is the representation of the *offsets* to given a plane curve, i.e., the loci of points at given fixed distances from a given curve: in general, the offsets to a rational curve are not rational loci. To address this problem, it has been common practice to invoke various offset curve approximation schemes. Such schemes often incur significant penalties in terms of accuracy, efficiency, and reliability. Thus, if possible, it is preferable to introduce curves whose offsets are guaranteed *a priori* to be rational curves.

Toward this goal, Farouki and Sakkalis introduced a class of polynomial curves called the *Pythagorean hodograph curves*. A plane polynomial curve $\gamma(t) = (x(t), y(t))$ is a Pythagorean hodograph (PH) curve if its derivative (hodograph) satisfies

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2 \tag{1}$$

for some polynomial $\sigma(t)$. If this condition is satisfied, the unit normal vector field along the curve is a rational function in the parameter t, and hence the offset curves are rational. In their seminal paper [16], Farouki and Sakkalis found necessary and sufficient conditions for (1) to hold. Namely, equation (1) holds if and only if there exist polynomials $u(t)$, $v(t)$, $w(t)$ such that

$$\begin{aligned} x'(t) &= w(t)\{u(t)^2 - v(t)^2\}, \\ y'(t) &= w(t)\{2u(t)v(t)\}, \\ \sigma(t) &= w(t)\{u(t)^2 + v(t)^2\}. \end{aligned} \tag{2}$$

Later, Farouki studied PH curves from a complex analysis viewpoint, and clarified earlier results using conformal mapping [14]. Pottmann further developed the theory by giving a characterization for all rational curves and surfaces that possess rational offsets [25]. Together with his co-authors, he introduced a wealth of related sophisticated techniques that have proven very fruitful. In particular, the work concerning rational representation of canal surfaces is closely tied to our ideas on PH curves in 4-dimensional Minkowski space, which we present in section 6 below.

The 3-dimensional generalization of Pythagorean hodograph curves was developed by Farouki and Sakkalis [17] and Dietz et al. [12]. They independently obtained more-or-less similar conditions for space curves to have polynomial speed functions, and called them PH space curves. In fact, the condition of Farouki and Sakkalis can be regarded as a special case of that of Dietz et al. This condition is similar in spirit to that for plane PH curves obtained by Farouki and Sakkalis, except that the case of PH space curves is more complicated, and the algebra becomes more involved. PH space curves are useful in various contexts, such as the study of pipe surfaces, rational frames, and rational curves on the unit three-sphere.

A polynomial curve $\gamma(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ is called a *Pythagorean hodograph (PH) space curve* if its hodograph $\gamma'(t)$ satisfies the condition

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma^2(t)$$

for some real polynomial $\sigma(t)$. According to Dietz et al. [12], a polynomial curve $\gamma(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ has a *primitive* Pythagorean hodograph, i.e.,

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma^2(t) \quad \text{and} \quad \gcd\bigl(x'(t), y'(t), z'(t)\bigr) = 1$$

for some polynomial $\sigma(t)$ if and only if there exist four polynomials $u(t)$, $v(t)$, $w(t)$, $\rho(t)$ such that the hodograph $\gamma'(t)$ and the speed $\sigma(t)$ satisfy

$$\begin{aligned} \sigma(t) &= \pm\bigl(u^2(t) + v^2(t) + w^2(t) + \rho^2(t)\bigr), \\ x'(t) &= u^2(t) - v^2(t) - w^2(t) + \rho^2(t), \\ y'(t) &= 2u(t)v(t) + 2\rho(t)w(t), \\ z'(t) &= 2u(t)w(t) - 2\rho(t)v(t). \end{aligned} \tag{3}$$

Meanwhile, the first and third authors together with colleagues worked on the medial axis transform of planar domains. The medial axis transform of a planar domain is defined to be the set of pairs consisting of centers and radii of the circles maximally inscribed in the domain. Our earlier paper [5] deals with fundamental mathematical aspects of the medial axis transform. It is proved therein that, under rather mild assumptions on the domain boundary, the medial axis transform is a finite geometric graph embedded in $\mathbb{R}^3$.

An algorithm to compute the medial axis transform was presented in [6]. This algorithm comprises two major ingredients. The first is topological in nature. A typical domain can be quite complicated and the medial axis – being the cut locus of the boundary – may therefore be very intricate. The key tool in [5,6] is the so-called *domain decomposition lemma*, which enables one to decompose a complicated domain into a union of relatively simple subdomains, each of which can be dealt with independently. Each subdomain is then decomposed into even simpler ones, and so on recursively until the whole domain has been decomposed into a union of very simple "fundamental" domains, that can be handled very easily.

The essential point is that the algorithm in [6] shows how to keep track of the decomposition process, in such a manner that topological information is fully preserved. This part corresponds to the global step. The remaining step is the problem of how to represent each of the fundamental domains or basic "building blocks". This problem is essentially local in nature. If one knows exactly the medial axis transform (or, for that matter, the medial axis transform of each fundamental domain), one can reconstruct the domain boundary by means of the envelope formula. Let $\gamma(t) = (x(t), y(t), r(t))$ be the medial axis transform of a fundamental domain. Interpreting $\gamma(t)$ as a one-parameter family of circles with centers $(x(t), y(t))$ and radii $r(t)$, the envelope curve $(\tilde{x}(t), \tilde{y}(t))$ can be computed by the formula

$$\tilde{x}(t) = x(t) + r(t)\frac{-r'(t)x'(t) \mp \sqrt{x'(t)^2 + y'(t)^2 - r'(t)^2}\,y'(t)}{x'(t)^2 + y'(t)^2},$$
$$\tilde{y}(t) = y(t) + r(t)\frac{-r'(t)y'(t) \pm \sqrt{x'(t)^2 + y'(t)^2 - r'(t)^2}\,x'(t)}{x'(t)^2 + y'(t)^2}.$$

This opens up the possibility of treating the medial axis transform, rather than the boundary of the domain itself, as a primary geometric object. The domain boundary is regarded as a derived quantity, that must be constructed. This kind of perspective is not new. For example, when dealing with a canal surface (the envelope surface of a one-parameter family of spheres in $\mathbb{R}^3$), the spine curve is the medial axis (i.e., the locus of centers of maximally inscribed spheres in the domain enclosed by the canal surface) – from which, together with the radius information, the canal surface can be constructed. We will return to this point in section 6. For the moment, let us first deal with the 2-dimensional case, and assume that a curve $\gamma(t) = (x(t), y(t), r(t))$ in $\mathbb{R}^3$ is given. Using the above envelope formula, its envelope curve is rational if

$$x'(t)^2 + y'(t)^2 - r'(t)^2 = \sigma(t)^2 \tag{4}$$

for some polynomial $\sigma(t)$. This approach is also well-suited to the study of offset curves. Namely, once a domain is specified by a one-parameter family of circles, say $\gamma(t) = (x(t), y(t), r(t))$, its offset curve at distance $\pm\delta$ is the envelope curve constructed from the curve $\gamma_{\pm\delta}(t) = (x(t), y(t), r(t) \pm \delta)$ in $\mathbb{R}^3$. Obviously, the rationality does not change even if a constant amount is added to or subtracted from the radius. Note that, when δ is large, the offset curves obtained in this manner may change their topological character. This requires a careful analysis – see [3,4].

Note that the expression $x'(t)^2 + y'(t)^2 - r'(t)^2$ in (4) is the squared norm of the tangent vector $\gamma'(t)$ of the curve $\gamma(t)$ when $\mathbb{R}^3$ is viewed as the Minkowski space with the usual Minkowski inner product – i.e., the nondegenerate bilinear form with signature $+ + -$. This observation led the third author to the study of the space curve $\gamma(t) = (x(t), y(t), r(t))$ satisfying (4) in the Minkowski space setting. He called the curve $\gamma(t)$ a *Minkowski Pythagorean hodograph (MPH) curve* if it satisfies (4). He then obtained a necessary and sufficient condition for $\gamma(t)$ to be an MPH curve [22]. Namely, it is proved that a polynomial curve $\gamma(t) = (x(t), y(t), r(t))$ in $\mathbb{R}^{2,1}$ is an MPH curve with $x'(t)^2 + y'(t)^2 - r'(t)^2 = \sigma^2(t)$ if and only if there exist four polynomials $u(t), v(t), w(t), \rho(t)$ satisfying the relations

$$\begin{aligned} \sigma(t) &= \pm\big(u^2(t) - v^2(t) - w^2(t) + \rho^2(t)\big), \\ x'(t) &= -2u(t)\rho(t) - 2v(t)w(t), \\ y'(t) &= u^2(t) + v^2(t) - w^2(t) - \rho^2(t), \\ r'(t) &= 2u(t)v(t) + 2\rho(t)w(t). \end{aligned} \tag{5}$$

For reasons that will become clear in the Clifford algebra formalism, we still call such a curve in the Minkowski space a Pythagorean hodograph (PH) curve. Hence, the class of PH curves encompasses the MPH curves; and we sometimes use the term Minkowski Pythagorean hodograph (MPH) curve only to emphasize the fact it is a PH curve in the Minkowski space.

One can go further and ask similar questions about PH curves in the 4-dimensional Minkowski space $\mathbb{R}^{3,1}$ (or the even higher-dimensional Minkowski space $\mathbb{R}^{n,1}$, or the Euclidean space $\mathbb{R}^n$). In the case of 4-dimensional space, a drastically different picture emerges. In 3-dimensional space, be it Euclidean or Minkowski, the norm of any $\mathbf{x} \in \mathcal{C}\ell^+(3)$ or $\mathcal{C}\ell^+(2,1)$ is a scalar. This is due to the fact that any such nonzero element $\mathbf{x}$ is decomposable, i.e., it is a product of an even number of vectors up to scalar multiple. In dimension 4, however, not every element $\mathbf{x}$ is decomposable, and this complicates matters. In fact, for $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$ or $\mathcal{C}\ell^+(4)$, the norm $N(\mathbf{x})$ of $\mathbf{x}$ is not a scalar, but a sum of a scalar (i.e., a real number) and a pseudo-scalar (i.e., a real multiple of the top degree element $\mathbf{e}_{1234} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$). This necessitates embracing a more general definition of the PH representation map (for more detail on this point, see section 2.3). The resulting PH condition for the space-like curve $\gamma(t) = (x(t), y(t), z(t), r(t))$ in the 4-dimensional Minkowski space is closely tied to the identity

$$x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2 - r'(t)^2 = f_1(t)^2 + f_2(t)^2$$

for some polynomials $f_1(t)$ and $f_2(t)$. This in turn is closely related to rational parameterization of the canal surface in $\mathbb{R}^3$ that is the envelope of the one-parameter family of spheres specified by $\gamma(t)$. We return to this in section 6, and show how our result ties in with that of Peternell and Pottmann [23].

An issue that arises naturally from all these considerations is the question of whether there is a deeper coherent structure underlying all the different incarnations of the PH curves. We shall answer the question affirmatively in this paper. As alluded to at the beginning of this section, we claim that there is an underlying algebraic structure which, when fully utilized, yields a natural and unifying framework for the study of PH curves. Furthermore, this framework makes the algebraic manipulations significantly easier.

The algebraic structure we claim as a correct one is the Clifford algebra. Examples of the Clifford algebra include the complex field, the quaternions, and a host of other interesting algebraic structures that reflect in an algebraic manner the geometric (metric) structure of the underlying vector space.

Informally, the PH relations (2) and (3) are squaring operations, although it may seem difficult to discern what is being squared, and in what manner. One of our key contributions in this paper of is to pin down these different squaring operations as instances of a single operation in the Clifford algebras, which we call the PH representation map. It proves to be an extension of the usual twisted adjoint representation of the spin group to a larger set of the even Clifford algebra.

Besides the algebraic aspects, the PH representation map has a wealth of corresponding geometric structures. In dimension 2, it is easily seen to be the branched double covering map $z \mapsto z^2$ of the complex plane $\mathbb{C}$. In the 3-dimensional Euclidean space $\mathbb{R}^3$, it turns out that the PH representation map is a family of (singular foliations of) the celebrated Hopf fibration maps. We shall expound on this in section 4 to illuminate its geometric meaning. This geometric insight enables us to formulate an independent *geometric* proof of the Dietz–Hoschek–Jüttler theorem. First, the Clifford algebra $\mathcal{C}\ell^+(3)$ is seen to be isomorphic to the quaternion algebra generated by the complex multiples of Pauli spin matrices; and by writing out the matrices, $\mathrm{Spin}(3)$ is seen to be isomorphic to $\mathrm{SU}(2)$. We then cast the PH representation map in the homogeneous space framework, and show that it is exactly the Hopf fibration that gives $\mathrm{SU}(2)/U(1) = S^2$. From this picture emerges the usual algebro-geometric form of the Hopf fibration $\pi : S^3 \subset \mathbb{C}^2 \to \mathbb{C}\mathrm{P}^1$ given by $\pi(z_1, z_2) = [z_1, z_2]$ for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$.

The study of Pythagorean hodograph curves in Minkowski space follows along more-or-less the same line of reasoning. In this case, however, a direct algebro-geometric analog is cumbersome to describe, whereas the group-theoretic approach of casting things in the homogeneous space framework is quite natural. One technical difference in this case is that we have to choose a good matrix representation of the full Clifford algebra $\mathcal{C}\ell(2, 1)$, or $\mathcal{C}\ell(3, 1)$, then everything flows naturally. One can easily see the parallels of the results in section 4 with those in sections 5 and 6.

2. Clifford algebra and PH representation map

2.1. *Clifford algebra*

In this section, we shall review some basic ideas of Clifford algebra that are relevant to our subsequent discussions. Since the Clifford algebra is a well-known mathematical object, we only comment on certain facts mainly to fix the conventions and notations. The interested reader may consult standard literature on the subject for further details.

Let V be a real vector space, and let Q be a nondegenerate quadratic form on V. Simply put, the Clifford algebra $\mathcal{C}\ell(V, Q)$ is an algebra that allows multiplication between elements of V, and recursively, between elements of $\mathcal{C}\ell(V, Q)$, subject to the relation that $v^2 = -Q(v)$ for any $v \in V$. Its formal definition can be given as follows. Let $\mathcal{T}(V) = \sum_{r=0}^{\infty} \bigotimes^r V$ denote the graded tensor algebra over V, and $\mathcal{S}(V)$ be the ideal generated by all elements of the form $v \otimes v + Q(v)$ for $v \in V$. Then the Clifford algebra $\mathcal{C}\ell(V, Q)$ is defined by the quotient algebra

$$\mathcal{C}\ell(V, Q) = \mathcal{T}(V)/\mathcal{S}(V).$$

Therefore, the Clifford algebra $\mathcal{C}\ell(V, Q)$ is an algebra that is generated by the element of the vector space V and the identity element 1, subject to the relations $v^2 = -Q(v)$ for $v \in V$. Then for all $v, w \in V$,

$$vw + wv = -2\langle v, w\rangle,$$

where $\langle\cdot, \cdot\rangle$ is the symmetric bilinear form obtained by polarizing Q, i.e.,

$$2\langle v, w\rangle = Q(v + w) - Q(v) - Q(w).$$

Remark 2.1. The standard diagonalization theorem on quadratic forms says that quadratic forms are classified by their signatures. Thus if Q is a quadratic form with p positive eigenvalues and q negative eigenvalues on the vector space V of dimension $n = p + q$, all Clifford algebras $\mathcal{C}\ell(V, R)$ for the quadratic form R with the same signature as Q are isomorphic, and thus they can be unambiguously denoted by $\mathcal{C}\ell(p, q)$. If $q = 0$ (i.e., $n = p$), we denote $\mathcal{C}\ell(V, Q) = \mathcal{C}\ell(n)$. The reader must be warned that there are two different conventions with regard to this notation. We have followed the convention of Lawson and Michelson [19], whereas others take the opposite convention. Crumeyrolle [10], for example, uses $\mathcal{C}\ell(p, q)$ to mean $\mathcal{C}\ell(q, p)$ in our case. The physics literature tends to use the convention adopted in [10].

A first nontrivial example of the Clifford algebra is $\mathcal{C}\ell(\mathbb{R}^2, Q)$, where $Q(v) = |v|^2$. By remark 2.1, we denote this Clifford algebra by $\mathcal{C}\ell(2)$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $\mathbb{R}^2$. Then $\mathcal{C}\ell(2)$ is generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ with relations:

$$\mathbf{e}_i^2 = -1 \quad \text{for } i = 1, 2 \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.$$

Therefore, $\mathcal{C}\ell(2)$ is an algebra over $\mathbb{R}$ of dimension 4 with basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$, where $\mathbf{e}_{12}$ denotes $\mathbf{e}_1\mathbf{e}_2$. Thus, the Clifford algebra $\mathcal{C}\ell(2)$ is decomposed into vector spaces as

$$\mathcal{C}\ell(2) = \mathcal{C}\ell^0(2) \oplus \mathcal{C}\ell^1(2) \oplus \mathcal{C}\ell^2(2),$$

where $\mathcal{C}\ell^0(2) = \mathbb{R}$, $\mathcal{C}\ell^1(2) = \mathrm{span}_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{C}\ell^2(2) = \mathbb{R}\mathbf{e}_{12}$. An element $\mathbf{x} \in \mathcal{C}\ell(2)$ can be expressed as $\mathbf{x} = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_{12}$ for $x_i \in \mathbb{R}$. It is easy to check that $\mathcal{C}\ell(2)$ is isomorphic to the quaternion algebra $\mathbb{H}$; and $\mathcal{C}\ell^+(2) = \mathcal{C}\ell^0(2) \oplus \mathcal{C}\ell^2(2) = \mathrm{span}_{\mathbb{R}}\{1, \mathbf{e}_{12}\}$ is isomorphic to the complex number field $\mathbb{C}$.

We define the norm and the conjugation in the Clifford algebra $\mathcal{C}\ell(V, Q)$ through two involutions. The *main involution* $\alpha : \mathcal{C}\ell(V, Q) \to \mathcal{C}\ell(V, Q)$ is defined as the algebra extension of the map $\alpha : V \to V$ with $\alpha(v) = -v$. And the other involution, called *reversion*, $\cdot^t : \mathcal{C}\ell(V, Q) \to \mathcal{C}\ell(V, Q)$, is defined from the reversion map on the tensor algebra $\mathcal{T}(V)$, which is a map $(v_1 \otimes \cdots \otimes v_k)^t = v_k \otimes \cdots \otimes v_1$. Since the reversion map on the tensor algebra preserves the ideal $\mathcal{S}(V)$, we can induce the reversion map on $\mathcal{C}\ell(V, Q)$. Then, the *conjugation* $\overline{\mathbf{x}}$ of $\mathbf{x} \in \mathcal{C}\ell(V, Q)$ is defined by

$$\overline{\mathbf{x}} = \alpha(\mathbf{x})^t = \alpha\big(\mathbf{x}^t\big).$$

And we define the *norm* $N(\mathbf{x})$ as $N(\mathbf{x}) = \mathbf{x}\overline{\mathbf{x}}$.

In the previous example, for an element $\mathbf{x} \in \mathcal{C}\ell(2)$, these maps look like

$$\begin{aligned} \alpha(\mathbf{x}) &= x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 + x_3\mathbf{e}_{12}, \\ \mathbf{x}^t &= x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - x_3\mathbf{e}_{12}, \\ \overline{\mathbf{x}} &= x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_{12}, \\ N(\mathbf{x}) &= x_0^2 + x_1^2 + x_2^2 + x_3^2. \end{aligned}$$

Since $\alpha : \mathcal{C}\ell(V, Q) \to \mathcal{C}\ell(V, Q)$ is an involution, its eigenspace decomposition decomposes $\mathcal{C}\ell(V, Q)$ into two subspaces $\mathcal{C}\ell^+(V, Q)$ and $\mathcal{C}\ell^-(V, Q)$. Thus

$$\mathcal{C}\ell(V, Q) = \mathcal{C}\ell^+(V, Q) \oplus \mathcal{C}\ell^-(V, Q),$$

where

$$\mathcal{C}\ell^{\pm}(V, Q) = \big\{\mathbf{x} \in \mathcal{C}\ell(V, Q) \colon \alpha(\mathbf{x}) = \pm\mathbf{x}\big\}.$$

$\mathcal{C}\ell^+(V, Q)$ is a subalgebra called the even Clifford algebra, or the *even part* of $\mathcal{C}\ell(V, Q)$, and $\mathcal{C}\ell^-(V, Q)$ is not a subalgebra, but is still called the *odd part*.

Also, the *multiplicative group of units* $\mathcal{C}\ell^\times(V, Q)$ in the Clifford algebra is defined by

$$\mathcal{C}\ell^\times(V, Q) = \big\{\mathbf{x} \in \mathcal{C}\ell(V, Q) \colon \exists \mathbf{y} \in \mathcal{C}\ell(V, Q) \text{ such that } \mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x} = 1\big\}.$$

The multiplicative group of units acts as a group of automorphisms of $\mathcal{C}\ell(V, Q)$ with the *twisted adjoint representation* $\chi : \mathcal{C}\ell^\times(V, Q) \to \mathrm{Aut}(\mathcal{C}\ell(V, Q))$ such that for $\mathbf{x} \in \mathcal{C}\ell^\times(V, Q)$, $\chi(\mathbf{x}) \in \mathrm{Aut}(\mathcal{C}\ell(V, Q))$ is defined by

$$\chi(\mathbf{x})(\mathbf{y}) = \alpha(\mathbf{x})\mathbf{y}\mathbf{x}^{-1}$$

for $\mathbf{y} \in \mathcal{C}\ell(V, Q)$.

If χ is restricted to $V \subset \mathcal{C}\ell(V, Q)$, its image $\chi(V)$ in $\mathrm{Aut}(\mathcal{C}\ell(V, Q))$ is the *orthogonal* group $\mathrm{O}(V, Q)$. Let v and w be vectors in V. And assume $Q(v) \neq 0$. Then we can decompose w into $\lambda v + v^\perp$ for some $\lambda \in \mathbb{R}$, where $v^\perp$ is a vector perpendicular to v. An explicit computation shows that $\chi(v)$ is the reflection with respect to the perpendicular hyperspace of v. Thus it is in fact given by

$$\chi(v)(w) = \alpha(v)wv^{-1} = -v\big(\lambda v + v^\perp\big)\frac{v}{-Q(v)} = -\lambda v + v^\perp.$$

(Note that $v^2 = -Q(v)$ and $vv^\perp = -v^\perp v$.)

The *Clifford group* $\Gamma(V, Q)$ is a subgroup of $\mathcal{C}\ell^{\times}(V, Q)$ defined by

$$\Gamma(V, Q) = \left\{\mathbf{x} \in \mathcal{C}\ell^{\times}(V, Q)\colon\ \chi(\mathbf{x})(v) \in V \text{ for any } v \in V\right\}.$$

The Clifford group has two important subgroups: the Pin group and the Spin group. The *Pin group* $\mathrm{Pin}(V, Q)$ is the subgroup of $\Gamma(V, Q)$ generated by the elements $v \in V$ with $Q(v) = \pm 1$. And the *Spin group* $\mathrm{Spin}(V, Q)$ is defined by

$$\mathrm{Spin}(V, Q) = \mathrm{Pin}(V, Q) \cap \mathcal{C}\ell^{+}(V, Q).$$

Also, the group $\Gamma^{+}(V, Q) = \Gamma(V, Q) \cap \mathcal{C}\ell^{+}(V, Q)$ is called the *even Clifford group*. We adopt the notation of remark 2.1 to denote $\Gamma(V, Q) = \Gamma(n)$ if Q is positive definite on $\mathbb{R}^n$, and $\Gamma(V, Q) = \Gamma(n, 1)$ if Q is the Lorentzian quadratic form of signature $++\cdots+-$ on the Minkowski space $\mathbb{R}^{n,1}$.

In the previous example of $\mathcal{C}\ell(2)$, the Clifford group $\Gamma(2)$ is the same as the multiplicative group $\mathcal{C}\ell^{\times}(2)$ itself.

It is well known that the twisted adjoint representation χ is a double covering map from $\mathrm{Pin}(V, Q)$ (respectively, $\mathrm{Spin}(V, Q)$) to $\mathrm{O}(V, Q)$ (respectively, $\mathrm{SO}(V, Q)$).

2.2. *PH representation map*

The primary objective of this section is to define the PH representation map, which is a fundamental tool throughout this paper. We will show that all previous incarnations of the Pythagorean hodograph curves are unified into a single coherent framework by this PH representation map.

The PH representation map is based on the twisted adjoint representation of the spin group, but it is the twisted adjoint representation extended to a certain subset, which we call $\Lambda(V, Q)$, of the even Clifford algebra $\mathcal{C}\ell^{+}(V, Q)$. If $\mathbf{x}$ is an element of $\mathrm{Spin}(V, Q)$, it can be expressed as a product of an even number of unit vectors in V, i.e., $\mathbf{x} = v_1 \cdots v_{2k}$ for $v_i \in V$, $Q(v_i) = \pm 1$. If we restrict the twisted adjoint representation χ to $\mathrm{Spin}(V, Q)$, χ can be expressed in a simpler form: i.e., $\chi(\mathbf{x})(w) = \alpha(\mathbf{x}) w \mathbf{x}^{-1} = \mathbf{x} w \overline{\mathbf{x}}$ for $w \in V$.

We define a new map $T\colon \mathcal{C}\ell^{+}(V, Q) \to \mathrm{End}(\mathcal{C}\ell(V, Q))$ by

$$T(\mathbf{x})(\mathbf{y}) = \mathbf{x}\mathbf{y}\overline{\mathbf{x}},$$

for $\mathbf{x} \in \mathcal{C}\ell^{+}(V, Q)$ and $\mathbf{y} \in \mathcal{C}\ell(V, Q)$. If we restrict T to $\mathrm{Spin}(V, Q)$, T is the twisted adjoint representation χ. But, in general, it is not guaranteed that $T(\mathbf{x})(V) \subset V$ for all $\mathbf{x} \in \mathcal{C}\ell(V, Q)$ as the following example shows.

Example 2.2. For $\mathbf{x} = \mathbf{e}_{23} + \mathbf{e}_{45} \in \mathcal{C}\ell^{+}(5)$,

$$\begin{aligned}
\mathbf{x}\mathbf{e}_1\overline{\mathbf{x}} &= (\mathbf{e}_{23} + \mathbf{e}_{45})\mathbf{e}_1(\mathbf{e}_{23} + \mathbf{e}_{45}) \\
&= \mathbf{e}_{23}\mathbf{e}_1\mathbf{e}_{23} + \mathbf{e}_{45}\mathbf{e}_1\mathbf{e}_{45} + \mathbf{e}_{23}\mathbf{e}_1\mathbf{e}_{45} + \mathbf{e}_{45}\mathbf{e}_1\mathbf{e}_{23} \\
&= -2\mathbf{e}_1 + 2\mathbf{e}_{12345} \notin V.
\end{aligned}$$

So we need to restrict T to those $\mathbf{x}$ such that $T(\mathbf{x})(V) \subset V$. For that purpose, let $\Lambda(V, Q)$ be the subset of $\mathcal{C}\ell^{+}(V, Q)$ such that

$$\Lambda(V, Q) = \{\mathbf{x} \in \mathcal{C}\ell^{+}(V, Q)\colon\ T(\mathbf{x})(v) \in V \text{ for any } v \in V\}.$$

Clearly, $\mathrm{Spin}(V, Q)$ is a subset of $\Lambda(V, Q)$, and $\Lambda(V, Q)$ plays the role of the Clifford group $\Gamma(V, Q)$ for the PH representation map, even though $\Lambda(V, Q)$ is not a (multiplicative) group (it is a subalgebra of $\mathcal{C}\ell^{+}(V, Q)$). We adopt the notation convention of remark 2.1 to denote $\Lambda(V, Q) = \Lambda(n)$ if Q is positive definite on $\mathbb{R}^n$, and $\Lambda(V, Q) = \Lambda(n, 1)$ if Q is the Lorentzian quadratic form of signature $++\cdots+-$ on the Minkowski space $\mathbb{R}^{n,1}$.

We now define the PH representation map.

Definition 2.3. For a fixed unit vector a in V, the *PH representation map* $T_a : \Lambda(V, Q) \to V$ is defined by

$$T_a(\mathbf{x}) = T(\mathbf{x})(a) = \mathbf{x}a\overline{\mathbf{x}}$$

for any $\mathbf{x} \in \Lambda(V, Q)$. We call a the *base (vector)* of the PH representation map T_a. When we fix $\mathbf{x}$, we can define an action $R_{\mathbf{x}}$ from V to V by

$$R_{\mathbf{x}}(\mathbf{v}) = T_a(\mathbf{x}) = \mathbf{x}\mathbf{v}\overline{\mathbf{x}}$$

for $v \in V$.

Let $V[t]$ be the V-valued polynomials given by

$$V[t] = \{\gamma : \mathbb{R} \to V\colon\ \text{Each coordinate function of } \gamma(t) \text{ is a polynomial}\}.$$

And similarly we define $V(t)$ as the V-valued rational functions.

Then $\Lambda(V, Q)[t]$ is defined to be the $\Lambda(V, Q)$-valued polynomial. Similarly, $\Lambda(V, Q)(t)$ is defined to be the $\Lambda(V, Q)$-valued rational functions. When there is no danger of confusion we use $\Lambda[t]$ instead of $\Lambda(V, Q)[t]$, and use $\Lambda(t)$ instead of $\Lambda(V, Q)(t)$.

The PH representation map T_a naturally induces a map $T_a : \Lambda(V, Q)[t] \to V[t]$ by

$$T_a(\mathbf{x}(t)) = T(\mathbf{x}(t))(a) = \mathbf{x}(t)a\overline{\mathbf{x}(t)}$$

for each t (we abuse notation to employ T_a for either case). Similarly we can define the rational map $T_a : \Lambda(V, Q)(t) \to V(t)$.

In the above definition we can choose a to be any unit vector, although we usually choose one of the orthonormal basis vectors of V. The algebraic representation generated by T_a may be somewhat different for different choices of the vector a. But since it does not change the algebraic structure itself, we will choose a in the subsequent section so as to make the results of our presentation coincide with previously published formulae.

It is perhaps instructive to compare T_a with $\chi_a : \Gamma(V, Q) \to V$, which is a map defined as $\chi_a(\mathbf{x}) = \chi(\mathbf{x})(a)$ for $\mathbf{x} \in \Gamma(V, Q)$. The basic difference between the PH representation map T_a and the map χ_a is that T_a brings out the norm of $\mathbf{x}$, whereas χ_a ignores it by canceling out. Another difference is that the domain of the definition of T_a is generally bigger. Note, however, that T_a and χ_a are identical maps if restricted to $\mathrm{Spin}(V, Q)$.

2.3. PH characterization problem

Given $\mathbf{x}(t) \in \Lambda[t]$, we can define a curve $\gamma(t) \in V[t]$ by

$$\gamma'(t) = T_a\big(\mathbf{x}(t)\big). \tag{6}$$

Clearly $\gamma(t)$ is well defined up to rigid motion in $\mathrm{O}(V, Q)$.

Definition 2.4. Any curve $\gamma(t) \in V[t]$ satisfying (6) for *some* $\mathbf{x}(t) \in \Lambda[t]$ is called a Pythagorean hodograph (PH) curve in (V, Q).

This definition of PH curve has merit, in that it can be stated regardless of the dimension of V or the signature of the quadratic form Q. However, the danger with this kind of definition is that it may be too general to mean anything concrete. The process of making concrete sense out of the definition, and showing that it coincides with the accepted definition of PH curves when specialized to familiar situations, such as 2-dimensional Euclidean space or 3-dimensional Euclidean or Minkowski space, is one of the main tasks of this paper. We call this the PH characterization problem:

Definition 2.5 (PH characterization problem).

- Step 1
  Characterize $\Lambda(V, Q)$.
- Step 2 (PH curve definition)
  This step is done by definition 2.4. Namely, a curve $\gamma(t) \in V[t]$ is a PH curve if $\gamma'(t) = T_a(\mathbf{x}(t))$ for some $\mathbf{x}(t) \in \Lambda(V, Q)$.
- Step 3 (single polynomial identity)
  Definition 2.4 gives the following polynomial identity

$$\begin{aligned}\big\langle \gamma'(t), \gamma'(t)\big\rangle = Q\big(\gamma'(t)\big) = \big|\gamma(t)\big|_Q^2 &= -\big(T_a(\mathbf{x})\big)\overline{\big(T_a(\mathbf{x})\big)} \\ &= -\mathbf{x}(t)a\overline{\mathbf{x}(t)}\mathbf{x}(t)a\overline{\mathbf{x}(t)}.\end{aligned} \tag{7}$$

  Characterize this polynomial identity.
- Step 4 (Characterization step)
  Given a curve $\gamma(t) \in V[t]$ satisfying the polynomial identity characterized in step 3, show that there is $\mathbf{x}(t) \in \Lambda[t]$ such that

$$\gamma'(t) = T_a\big(\mathbf{x}(t)\big).$$

The rest of the paper will be devoted to studying the PH characterization problem for the spaces $V = \mathbb{R}^2,\ V = \mathbb{R}^3,\ V = \mathbb{R}^{2,1}$ and $V = \mathbb{R}^{3,1}$. In all these cases, it will be shown without too much difficulty that $\Lambda(V, Q) = V$. There is nothing to do for step 2. Step 3 is not hard either. In section 3, we show that the polynomial identity turns out to be the identity

$$\langle \gamma'(t), \gamma'(t) \rangle = x'(t)^2 + y'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$; this is exactly the condition of Farouki and Sakkalis [17]. In section 4, it is easily shown that the polynomial relation is such that there exists a polynomial $\sigma(t)$ satisfying the relation

$$\langle \gamma'(t), \gamma'(t) \rangle = x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2.$$

This relation is exactly the condition of Dietz et al. [12]. In section 5, we similarly show that, when $V = \mathbb{R}^{2,1}$, the polynomial identity becomes the equality

$$\langle \gamma'(t), \gamma'(t) \rangle = x'(t)^2 + y'(t)^2 - z'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$; this condition was discovered by the third author [22]. Finally, in chapter 6 we prove that, when $V = \mathbb{R}^{3,1}$, the polynomial identity becomes the equality

$$\langle \gamma'(t), \gamma'(t) \rangle = x'(t)^2 + y'(t)^2 + z'(t)^2 - r'(t)^2 = f_1(t)^2 + f_2(t)^2$$

for some polynomials $f_1(t)$ and $f_2(t)$. Unlike the previous polynomial identities, this one is newly discovered in this paper and is due to our general analysis using the Clifford algebra setting.

The most difficult part of the PH characterization problem is step 4. In this paper, instead of relying on the earlier results of Farouki and Sakkalis [17], Dietz et al. [12], or the third author [22], we shall give independent and more geometrical proofs. A wealth of geometrical insights are obtained as byproducts along the way.

3. Two-dimensional Pythagorean hodographs: branched double covering

In 2-dimensional Euclidean space, Pythagorean hodographs possess a relatively simple algebraic structure. We briefly point out some essential features related to the algebraic properties of 2-dimensional PH curves. We also show how to interpret this algebraic structure from the perspective of our unifying PH representation map.

3.1. *Farouki–Sakkalis revisited*

A 2-dimensional Pythagorean hodograph (PH) curve $\gamma(t) = (x(t), y(t))$ is a polynomial curve whose hodograph $\gamma'(t) = (x'(t), y'(t))$ satisfies $x'(t)^2 + y'(t)^2 = \sigma(t)^2$ for some polynomial $\sigma(t)$.

Kubota [20] proved (2) in any unique factorization domain. But in the polynomial ring $\mathbb{R}[t]$, we can give a simpler and more direct proof. Assume $\gamma(t)$ is a PH curve in $\mathbb{R}^2$. The speed $\sigma(t)$ can be factored into

$$\sigma(t) = \prod_{i=1}^{l}(t-\alpha_i)\big(t-\overline{\alpha_i}\big) \cdot \prod_{j=1}^{m}(t-a_j)^2 \cdot \prod_{k=1}^{n}(t-b_k)$$

for some $\alpha_i \in \mathbb{C}\backslash\mathbb{R}$ and $a_j, b_k \in \mathbb{R}$. Define the real polynomial $w(t)$ by

$$w(t) = \prod_{j=1}^{m}(t-a_j)^2 \cdot \prod_{k=1}^{n}(t-b_k).$$

Then the relation $x'(t)^2+y'(t)^2 = (x'(t)+\mathrm{i}y'(t))(x'(t)-\mathrm{i}y'(t)) = \sigma(t)^2$ can be rewritten as

$$\frac{x'(t)+\mathrm{i}y'(t)}{w(t)}\frac{x'(t)-\mathrm{i}y'(t)}{w(t)} = \left(\prod_{i=1}^{l}(t-\alpha_i)\big(t-\overline{\alpha_i}\big)\right)^2.$$

Since $(x'+\mathrm{i}y')/w$ and $(x'-\mathrm{i}y')/w$ have no real roots, upon exchanging α_i and $\overline{\alpha_i}$ if necessary, we may assume α_i's are roots of $(x'+\mathrm{i}y')/w$ and $\overline{\alpha_i}$'s are roots of $(x'-\mathrm{i}y')/w$. Thus, we have

$$\frac{x'(t)+\mathrm{i}y'(t)}{w(t)} = \left(\prod_{i=1}^{l}(t-\alpha_i)\right)^2.$$

Therefore, upon writing

$$u(t)+\mathrm{i}v(t) = \prod_{i=1}^{l}(t-\alpha_i),$$

the relation (2) is proved.

When $x'(t)$ and $y'(t)$ have no common factors, $\gamma(t)$ is called *primitive*, in which case $w(t)$ should be a nonzero scalar. In this case, the representation becomes particularly simple. Namely, we can put $w(t)$ in (2) identically equal to 1, and thus

$$x'(t)+\mathrm{i}y'(t) = \big(u(t)+\mathrm{i}v(t)\big)^2,$$

which reduces (2) to

$$\begin{aligned} x'(t) &= u(t)^2 - v(t)^2, \\ y'(t) &= 2u(t)v(t), \\ \sigma(t) &= u(t)^2 + v(t)^2. \end{aligned} \tag{8}$$

This representation can be understood by the complex conformal map $z \mapsto z^2$, where $z(t) = u(t) + \mathrm{i}v(t)$. The map $z \mapsto z^2$ is a double covering map of the complex plane $\mathbb{C}$ over itself except at the origin, which is a branch point. Note also that the map

$z \mapsto z^2$ sends the circle of radius r onto the circle of radius r^2 while doubling the angle. This simple observation will be carefully exploited when we deal with the space PH curve in section 4.

3.2. Clifford algebra formalism

In this section, we show how to recast the map $z \mapsto z^2$ in a Clifford algebra framework using our PH representation map. Consider the map T on $\mathcal{C}\ell^+(2)$. Since for any $\mathbf{x} \in \mathcal{C}\ell^+(2)$, $\mathbf{x}$ can be written as $a + b\mathbf{e}_{12}$ for some real numbers a and b, $T(\mathbf{x})$ applied to $\mathbf{e}_k$ gives

$$\begin{aligned} T(\mathbf{x})(\mathbf{e}_k) &= (a + b\mathbf{e}_{12})\mathbf{e}_k(a - b\mathbf{e}_{12}) = \left(a^2 - b^2\right)\mathbf{e}_k - 2ab\mathbf{e}_k\mathbf{e}_{12} \\ &= \begin{cases} \left(a^2 - b^2\right)\mathbf{e}_1 + 2ab\mathbf{e}_2 & \text{if } k = 1, \\ -2ab\mathbf{e}_1 + \left(a^2 - b^2\right)\mathbf{e}_2 & \text{if } k = 2. \end{cases} \end{aligned}$$

Thus, $T(\mathbf{x})(\mathbf{e}_k) \in \mathbb{R}^2$, which implies that $\Lambda(2) = \mathcal{C}\ell^+(2)$.

Now consider the PH representation map $T_{\mathbf{e}_1} : \mathcal{C}\ell^+(2) \to \mathbb{R}^2$, where e_1 is a base as defined in definition 2.3. Suppose $\mathbf{x}(t)$ is a polynomial curve in $\mathcal{C}\ell^+(\mathbb{R}^2)$, that is, $\mathbf{x}(t) = u(t) + v(t)\mathbf{e}_{12}$ for $u(t), v(t) \in \mathbb{R}[t]$, then

$$\begin{aligned} T_{\mathbf{e}_1}\left(\mathbf{x}(t)\right) &= \mathbf{x}(t)\mathbf{e}_1\overline{\mathbf{x}(t)} = \left\{u(t) + v(t)\mathbf{e}_{12}\right\}\mathbf{e}_1\left\{u(t) - v(t)\mathbf{e}_{12}\right\} \\ &= \left(u(t)^2 - v(t)^2\right)\mathbf{e}_1 + 2u(t)v(t)\mathbf{e}_2. \end{aligned}$$

Note that by identifying $\mathcal{C}\ell^+(2)$ with $\mathbb{C}$ by the example in section 2.1, and $\mathbb{R}^2$ also with $\mathbb{C}$, our PH representation map $T_{\mathbf{e}_1}$ gives the same result as in the previous section. For further reference, it is a good idea to illustrate, by recasting this simple case into a Clifford algebra framework, the nature of our strategy in this paper in studying the Pythagorean hodograph problem.

The idea is as follows. First, suppose $\mathbf{x}(t) = u(t) + v(t)\mathbf{e}_{12}$ is a polynomial curve in $\mathcal{C}\ell^+(2)$, and define a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$ by the relation

$$T_{\mathbf{e}_1}\left(\mathbf{x}(t)\right) = x'(t)\mathbf{e}_1 + y'(t)\mathbf{e}_2.$$

Obviously, $\gamma(t)$ is well defined up to a rigid motion in $\mathbb{R}^2$. Now check that the magnitude $\sigma(t)$ of the hodograph of $\gamma(t)$ is given by

$$\sigma(t)^2 = x'(t)^2 + y'(t)^2 = T_{\mathbf{e}_1}\left(\mathbf{x}(t)\right)\overline{T_{\mathbf{e}_1}\left(\mathbf{x}(t)\right)}.$$

It is easy to see that $\sigma(t)^2 = N(\mathbf{x}(t))^2$, where the norm $N(\mathbf{x}(t))$ of $\mathbf{x}(t)$ is obtained as

$$N\left(\mathbf{x}(t)\right) = \mathbf{x}(t)\overline{\mathbf{x}(t)} = \left\{u(t) + v(t)\mathbf{e}_{12}\right\}\left\{u(t) - v(t)\mathbf{e}_{12}\right\} = u(t)^2 + v(t)^2.$$

It is now a trivial matter to see that $\gamma(t)$ is a PH curve in $\mathbb{R}^2$.

The important part is the characterization problem suggested in section 2.3. The result in the previous section can be simply rephrased in the context of the PH representation map in the following theorem.

Theorem 3.1 (Solution of PH characterization problem for $\mathbb{R}^2$). Let $\gamma(t) = (x(t), y(t))$ be a polynomial curve in $\mathbb{R}^2$ satisfying

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$. Then $\gamma(t)$ is a PH curve – i.e., there exists a polynomial curve $\mathbf{x}(t) = u(t) + v(t)\mathbf{e}_{12} \in \mathcal{C}\ell^+(2)$ such that $\gamma'(t) = T_{\mathbf{e}_1}(\mathbf{x}(t))$.

4. Hopf fibration and 3-dimensional Pythagorean hodographs

The 3-dimensional Pythagorean hodograph curves have a quadratic algebraic structure that is vaguely similar to that of 2-dimensional PH curves. But the 3-dimensional case is more complicated and has several different, but equivalent, interpretations. Dietz et al. [12] obtained the representation formula based on the algebraic fact that the polynomial rings $\mathbb{R}[t]$ and $\mathbb{C}[t]$ are unique factorization domains (Farouki and Sakkalis obtained a similar representation formula [17]). In this section, we recast the PH representation map in a Clifford algebra framework. This PH representation map has a wealth of geometric structure. We present two equivalent geometric viewpoints that eventually tie it to the celebrated Hopf fibration.

Let a polynomial space curve $\gamma(t) = (x(t), y(t), z(t))$ satisfy $x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$ for some polynomial $\sigma(t)$ and also satisfy the primitive condition that $x'(t)$, $y'(t)$, $z'(t)$ have no common factors. Then Dietz et al. have proved that for this $\gamma(t)$, there exist polynomials $p_0(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$ satisfying

$$\begin{aligned}
x'(t) &= 2p_0(t)p_2(t) + 2p_1(t)p_3(t),\\
y'(t) &= -2p_0(t)p_1(t) + 2p_2(t)p_3(t),\\
z'(t) &= p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2,\\
\sigma(t) &= p_0(t)^2 + p_1(t)^2 + p_2(t)^2 + p_3(t)^2.
\end{aligned} \tag{9}$$

Note that (9) is in fact (3) but with appropriate symbol permutation. We choose (9) in order to conform to the mathematically well-accepted convention for the Hopf map. In [27], Wallner and Pottmann describe the control structure of rational curves on the unit three sphere by using this Hopf map.

The formula (9) can be succinctly expressed using the quaternions $\mathbb{H}$. Let $\mathbf{p}(t)$ be a polynomial $p_0(t) + p_1(t)\mathbf{i} + p_2(t)\mathbf{j} + p_3(t)\mathbf{k}$ in $\mathbb{H}[t]$. By direct computation, one can check that

$$\begin{aligned}
\mathbf{p}(t)\mathbf{k}\overline{\mathbf{p}(t)} &= \mathbf{i}\big(2p_0(t)p_2(t) + 2p_1(t)p_3(t)\big)\\
&\quad + \mathbf{j}\big(-2p_0(t)p_1(t) + 2p_2(t)p_3(t)\big)\\
&\quad + \mathbf{k}\big(p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2\big)\\
&= \mathbf{i}x'(t) + \mathbf{j}y'(t) + \mathbf{k}z'(t),
\end{aligned} \tag{10}$$

and also that $\sigma(t) = N(\mathbf{p}(t)) = \mathbf{p}(t)\overline{\mathbf{p}(t)} = p_0(t)^2 + p_1(t)^2 + p_2(t)^2 + p_3(t)^2$.

Let us now describe the above map in the complex notation in the form geared toward the Hopf map description in sections 4.2 and 4.3. First, identify any complex number $z = (a, b) = a + \mathrm{i}b$ with the quaternion $a + \mathbf{k}b$; and a pair (z_1, z_2) of complex numbers with the quaternion $\mathbf{p} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ by the following identity:

$$\mathbf{p} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = (p_0 + p_3\mathbf{k}) + \mathbf{j}(p_2 + p_1\mathbf{k}) = z_1 + \mathbf{j}z_2,$$

where $z_1 = p_0 + p_3\mathbf{k}$ and $z_2 = p_2 + p_1\mathbf{k}$.

Check, using $\mathbf{j}z = \overline{z}\mathbf{j}$, $\mathbf{i}z = \overline{z}\mathbf{i}$ and $\mathbf{k}z = z\mathbf{k}$, that

$$\begin{aligned}\mathbf{p}\mathbf{k}\overline{\mathbf{p}} &= (z_1 + \mathbf{j}z_2)\mathbf{k}\big(\overline{z_1} - \mathbf{j}z_2\big) = 2z_1\overline{z_2}\mathbf{i} + \big(|z_1|^2 - |z_2|^2\big)\mathbf{k} \\ &= 2\,\mathrm{Re}\,\big(z_1\overline{z_2}\big)\mathbf{i} + 2\,\mathrm{Im}\,\big(z_1\overline{z_2}\big)\mathbf{j} + \big(|z_1|^2 - |z_2|^2\big)\mathbf{k} \\ &= \big(2z_1\overline{z_2}, |z_1|^2 - |z_2|^2\big) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3, \end{aligned} \tag{11}$$

where the last equality is due to the identification $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. Note that this is a complex notation for the celebrated Hopf map [1]. The interplay between this geometric picture of the Hopf map and the algebraic nature of the PH representation map is the main theme of the subsequent sections.

4.1. Clifford algebra formalism

We now rewrite the PH representation in terms of the Clifford algebra $\mathcal{C}\ell^+(3)$. It is well known that $\mathcal{C}\ell^+(3)$ is isomorphic to the quaternion algebra $\mathbb{H}$ by an algebra isomorphism φ such that $\varphi(\mathbf{e}_{23}) = \mathbf{i}$, $\varphi(\mathbf{e}_{31}) = \mathbf{j}$ and $\varphi(\mathbf{e}_{12}) = \mathbf{k}$.

Choose $\mathbf{e}_3$ to be the base vector of the PH representation map. One can easily show that $\Lambda(3)$ is the whole even Clifford algebra $\mathcal{C}\ell^+(3)$ as in the two dimensional case. So the PH representation map is

$$T_{\mathbf{e}_3}(\mathbf{x}) = \mathbf{x}\mathbf{e}_3\overline{\mathbf{x}} \quad \text{for } \mathbf{x} \in \mathcal{C}\ell^+(3).$$

This map is also known as Kustaanheimo–Stiefel transformation in physics [11]. Let $\mathbf{p}(t) = p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12}$ be in $\mathcal{C}\ell^+(3)[t]$, and check that

$$\begin{aligned} T_{\mathbf{e}_3}\big(\mathbf{p}(t)\big) &= p(t)\mathbf{e}_3\overline{p(t)} \\ &= \big(2p_0(t)p_2(t) + 2p_1(t)p_3(t)\big)\mathbf{e}_1 + \big(-2p_0(t)p_1(t) + 2p_2(t)p_3(t)\big)\mathbf{e}_2 \\ &\quad + \big(p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2\big)\mathbf{e}_3 \\ &= x'(t)\mathbf{e}_1 + y'(t)\mathbf{e}_2 + z'(t)\mathbf{e}_3, \end{aligned} \tag{12}$$

which gives an expression identical to (10).

4.2. Spin representation and homogeneous space

It is well known that $\mathcal{C}\ell(3)$ is algebra isomorphic to the 2×2 matrix algebra $\mathrm{M}(2, \mathbb{C})$. And $\mathcal{C}\ell^+(3)$ is isomorphic to a subalgebra of $\mathrm{M}(2, \mathbb{C})$, which is isomorphic to the quaternion algebra $\mathbb{H}$. This isomorphism is achieved by use of the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

that satisfy the following basic multiplication relations

$$\begin{aligned} \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0, \qquad & \sigma_1\sigma_2 = \mathrm{i}\sigma_3 = -\sigma_2\sigma_1, \\ \sigma_2\sigma_3 = \mathrm{i}\sigma_1 = -\sigma_3\sigma_2, \qquad & \sigma_3\sigma_1 = \mathrm{i}\sigma_2 = -\sigma_1\sigma_3. \end{aligned}$$

In particular, we can select an algebra embedding $\phi : \mathcal{C}\ell^+(3) \hookrightarrow \mathrm{M}(2,\mathbb{C})$ such that $\phi(1) = \sigma_0$, $\phi(\mathbf{e}_{23}) = \mathrm{i}\sigma_1$, $\phi(\mathbf{e}_{31}) = -\mathrm{i}\sigma_2$ and $\phi(\mathbf{e}_{12}) = \mathrm{i}\sigma_3$. Let $\mathbf{p} = p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12} \in \mathcal{C}\ell^+(3)$. Check that its corresponding matrix representation $\phi(\mathbf{p})$ is given by

$$\phi(\mathbf{p}) = \begin{pmatrix} p_0 + \mathrm{i}p_3 & -p_2 + \mathrm{i}p_1 \\ p_2 + \mathrm{i}p_1 & p_0 - \mathrm{i}p_3 \end{pmatrix} = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix},$$

where $z_1 = p_0 + \mathrm{i}p_3$ and $z_2 = p_2 + \mathrm{i}p_1$. This suggests the identification of $\mathbf{p} = p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12}$ with the pair of complex numbers (z_1, z_2) by the formula:

$$\begin{aligned} \mathbf{p} &= p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12} \\ &= (p_0 + p_3\mathbf{e}_{12}) + \mathbf{e}_{31}(p_2 + p_1\mathbf{e}_{12}) = z_1 + \mathbf{e}_{31}z_2, \end{aligned}$$

where $z_1 = p_0 + p_3\mathbf{e}_{12}$ and $z_2 = p_2 + p_1\mathbf{e}_{12}$. From now on, we sometimes abuse notation by dropping ϕ. Thus, it is to be understood from the context when we say things like

$$\mathbf{p} = z_1 + \mathbf{e}_{31}z_2 = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}.$$

Note that

$$\mathrm{Spin}(3) = \{\mathbf{p} \in \mathcal{C}\ell^+(3) \colon N(\mathbf{p}) = 1\}.$$

The above identification (algebra isomorphism) says that the matrix form of Spin(3) is

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} : |z_1|^2 + |z_2|^2 = 1 \right\}.$$

We now write the PH representation map $\mathbf{x} \mapsto T_{\mathbf{e}_3}(\mathbf{x}) = \mathbf{x}\mathbf{e}_3\overline{\mathbf{x}}$ in this SU(2) setting. First, check that

$$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \overline{z_1} & \overline{z_2} \\ -z_2 & z_1 \end{pmatrix} = \begin{pmatrix} \mathrm{i}\big(|z_1|^2 - |z_2|^2\big) & \mathrm{i}\big(2z_1\overline{z_2}\big) \\ \mathrm{i}\big(2\overline{z_1}z_2\big) & -\mathrm{i}\big(|z_1|^2 - |z_2|^2\big) \end{pmatrix}. \tag{13}$$

According to our algebra isomorphism ϕ, the matrix on the right-hand side of (13) is identified with

$$\begin{aligned} &\mathrm{i}\big(|z_1|^2 - |z_2|^2\big) + \mathbf{e}_{31}\big(\mathrm{i}(2\overline{z_1}z_2)\big) \\ &\quad = \big(|z_1|^2 - |z_2|^2\big)\mathbf{e}_{12} + \mathbf{e}_{31}\big(2\,\mathrm{Im}\,(z_1\overline{z_2}) + 2\,\mathrm{Re}\,(z_1\overline{z_2})\mathbf{e}_{12}\big) \\ &\quad = \big(|z_1|^2 - |z_2|^2\big)\mathbf{e}_{12} + 2\,\mathrm{Re}\,(z_1\overline{z_2})\mathbf{e}_{23} + 2\,\mathrm{Im}\,(z_1\overline{z_2})\mathbf{e}_{31}. \end{aligned}$$

This shows that the matrix formulation of (13) coincides with the standard PH representation map given in (11). A word concerning the choice of the matrix

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

in (13) is in order: the Hodge $*$ operator maps the vector $\mathbf{e}_3$ to $\mathbf{e}_{12}$ that is identified with

$$\mathrm{i}\sigma_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},$$

which motivates our choice. A more precise and mathematically correct justification can be given if we delve deeper into the matrix algebra realization of the full Clifford algebra $\mathcal{C}\ell(3)$. But this would unnecessarily complicate our current presentation.

Let us now study the PH representation map from the group-theoretic (homogeneous space) point of view. Let

$$M = \left\{ \begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix} \in \mathrm{SU}(2) \colon \mathrm{Re}(w_1) = 0 \right\}.$$

Thus, M is diffeomorphic to the standard unit 2-sphere S^2. Note that (13) says that SU(2) acts on M via conjugation. First, it is easy to check that the isotropy subgroup of SU(2) at $\mathrm{i}\sigma_3$ is

$$\mathrm{U}(1) = \left\{ \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

We claim that SU(2) acts on M transitively, i.e., for a given

$$\begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix} \in M,$$

there exists

$$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \in \mathrm{SU}(2)$$

such that

$$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \overline{z_1} & \overline{z_2} \\ -z_2 & z_1 \end{pmatrix} = \begin{pmatrix} w_1 & -\overline{w_2} \\ w_2 & \overline{w_1} \end{pmatrix}.$$

Upon direct computation, the matrix

$$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{1+\mathrm{Im}(w_1)}}{\sqrt{2}}\mathrm{i} & \dfrac{-\overline{w_2}}{\sqrt{2}\sqrt{1+\mathrm{Im}(w_1)}} \\ \dfrac{w_2}{\sqrt{2}\sqrt{1+\mathrm{Im}(w_1)}} & -\dfrac{\sqrt{1+\mathrm{Im}(w_1)}}{\sqrt{2}}\mathrm{i} \end{pmatrix} \tag{14}$$

is seen to satisfy the requirement. In fact, since the isotropy subgroup is U(1), any such solution should be of the form

$$\begin{pmatrix} \frac{\sqrt{1+\mathrm{Im}(w_1)}}{\sqrt{2}}\,\mathrm{i} & \frac{-\overline{w_2}}{\sqrt{2}\sqrt{1+\mathrm{Im}(w_1)}} \\ \frac{w_2}{\sqrt{2}\sqrt{1+\mathrm{Im}(w_1)}} & -\frac{\sqrt{1+\mathrm{Im}(w_1)}}{\sqrt{2}}\,\mathrm{i} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$$

for some $\theta \in \mathbb{R}$.

Since SU(2) acts on M transitively with the isotropy subgroup U(1), the standard theory of homogeneous space gives the following theorem.

Theorem 4.1 (Hopf fibration). By the PH representation map T_{e_3} in $\mathcal{C}\ell^+(3)$, we get the following isomorphism:

$$\mathrm{SU}(2)/\mathrm{U}(1) \cong M \cong S^2. \tag{15}$$

4.3. Singular foliation of Hopf fibrations

The group-theoretic (homogeneous space) picture of the previous section has another geometric counterpart: namely, the algebro-geometric description of the Hopf map. Although the two viewpoints, group-theoretic and algebro-geometric, are equivalent, we choose to pursue an algebro-geometric approach in this section because it is more intuitive and familiar. However, the same analysis can be done entirely in the homogeneous space setting. (In chapter 5, we in fact do just that, because the homogeneous space setting is straightforward to handle, whereas the algebro-geometric setting is harder to come by in the Minkowski space setting.)

Note that the PH representation map sends (z_1, z_2) and $(z_1', z_2') \in S^3$ to the same point if and only if they lie in the same coset space, i.e.,

$$\begin{pmatrix} z_1' & -\overline{z_2'} \\ z_2' & \overline{z_1'} \end{pmatrix} = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. This means that $(z_1', z_2') = \mathrm{e}^{\mathrm{i}\theta}(z_1, z_2)$, i.e.,

$$[z_1', z_2'] = [z_1, z_2]$$

in the homogeneous coordinates. Thus the Hopf fibration is simply a map $\pi : S^3 \subset \mathbb{C}^2 \to \mathbb{C}\mathrm{P}^1$ given by $\pi(z_1, z_2) = [z_1, z_2]$ for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$. When $\mathbb{C}\mathrm{P}^1$ is identified with S^2, the Hopf map can be defined as a map $S^3 \to S^2$. This description can be found in the differential geometry or topology literature – for instance, one may consult [1]. The Hopf map can be decomposed into two maps: one from $S^3 \subset \mathbb{C}^2$ onto $\mathbb{C}\mathrm{P}^1$ by taking the homogeneous coordinates; and the other from $\mathbb{C}\mathrm{P}^1$ to $S^2 \subset \mathbb{R}^3$ via the inverse of the stereographic projection. Since we have to deal with spheres of all radii, we present the Hopf map in a slightly general fashion. The Hopf map is decomposed as in the following diagram.

$$\begin{array}{ccccccc}
\mathbb{C}^2 & & & & \mathbb{R}^3 & & \mathbb{R}^3 \\
\cup & & & & \cup & & \cup \\
S^3(\sqrt{r}) & \xrightarrow{\pi} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{C} = \mathbb{R}^2 & \xrightarrow{p^{-1}} & S^2(r) \\
(z_1, z_2) & \longmapsto & [z_1, z_2] = \left[r\dfrac{z_1}{z_2}, r \right] & \longmapsto & w = r\dfrac{z_1}{z_2} & \longmapsto & (x_1, x_2, x_3),
\end{array}$$

where p^{-1} is the inverse of the stereographic projection from $(0, 0, r)$ of $S^2(r)$ into the xy-plane given by

$$x_1 = r^2 \frac{w + \overline{w}}{r^2 + |w|^2}, \qquad x_2 = r^2 \frac{w - \overline{w}}{\mathrm{i}(r^2 + |w|^2)}, \qquad x_3 = r\frac{|w|^2 - r^2}{r^2 + |w|^2},$$

and A is the usual map taking the affine plane $\mathbb{C}$ in $\mathbb{CP}^1$. For later purposes, note that p itself maps $(x_1, x_2, x_3) \in S^2(r) \subset \mathbb{R}^3$ to

$$w = r\frac{x_1 + \mathrm{i}x_2}{r - x_3}. \tag{16}$$

Therefore, the image (x_1, x_2, x_3) of (z_1, z_2) under the Hopf map $H = p^{-1} \circ A \circ \pi$ is

$$x_1 = 2\,\mathrm{Re}\,\big(z_1\overline{z_2}\big), \qquad x_2 = 2\,\mathrm{Im}\,\big(z_1\overline{z_2}\big), \qquad x_3 = |z_1|^2 - |z_2|^2,$$

which gives

$$H(z_1, z_2) = \big(2z_1\overline{z_2},\ |z_1|^2 - |z_2|^2\big)$$

as a well defined smooth map from $\mathbb{C}^2$ into $\mathbb{R}^3$. Note that H maps $S^3(\sqrt{r})$ to $S^2(r)$. However, this Hopf map degenerates into a point map when $r = 0$. This phenomenon is reminiscent of the *branched* double covering map $z \mapsto z^2$ in the 2-dimensional case; hence the name "singular" foliation of Hopf fibrations.

The Hopf fibration gives the fiber bundle structure on S^3. For $m, m' \in S^3$, we have $H(m) = H(m')$ if and only if $m' = \lambda m$ for some unit complex number λ. Therefore, the three sphere S^3 can be considered as a circle bundle over the base space S^2 (this fact can be also seen from the homogeneous setup).

4.4. *Geometric proof of Dietz–Hoschek–Jüttler theorem*

The Hopf map description of the PH representation map gives a wealth of information and geometric insight. As an illustration, we give an independent geometric proof, based on our Hopf fibration framework, of Dietz et al. [12] theorem.

Let $\gamma(t) = (x(t), y(t), z(t))$ be a PH curve given by equation (12). The essence of the Dietz–Hoschek–Jüttler theorem is the converse. To properly describe their theorem, we need to discuss the notion of a *primitive* curve.

Note that if $\sigma(t) = \langle \gamma'(t), \gamma'(t)\rangle_{\mathbb{R}^3}$ vanishes at some point, say at t_0, then so do $x'(t),\ y'(t),\ z'(t)$. Thus, they all have a common factor $t - t_0$. One can then extract those common factors and assume that $x'(t)$, $y'(t)$, $z'(t)$ have no common factor, in which case $\gamma(t)$ is called a *primitive* PH curve. Under this assumption, we can prove the following theorem:

Theorem 4.2 (Solution of PH characterization problem for $\mathbb{R}^3$). Let $\gamma(t) = (x(t), y(t), z(t))$ be a primitive curve whose hodograph satisfies

$$\langle \gamma'(t), \gamma'(t) \rangle_{\mathbb{R}^3} = x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$. Then $\gamma(t)$ is a PH curve in $\mathbb{R}^3$, i.e., there exist four polynomials $\mathbf{x}(t) = p_0(t)+p_1(t)\mathbf{e}_{23}+p_2(t)\mathbf{e}_{31}+p_3(t)\mathbf{e}_{12} \in \mathcal{C}\ell^+(3)[t]$ such that $\gamma'(t) = T_{\mathbf{e}_3}(\mathbf{x}(t))$.

Proof. In view of the Hopf map description of the PH representation map, the proof is reduced to a lifting problem via the Hopf map. Assume $\gamma(t)$ is a curve such that $x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$. Since γ is primitive, we may assume without loss of generality that $\sigma(t)$ is positive for each t. Then for each t, $\gamma'(t)$ lies on the sphere $S^2(\sigma(t))$. In fact, for different t, the radii $\sigma(t)$ differ, but as far as the lifting goes, we can work with each fixed t. Then, the stereographic projection of $\gamma'(t)$ gives

$$\sigma(t)\frac{x'(t) + \mathrm{i}y'(t)}{\sigma(t) - z'(t)}$$

(see (16)). And it corresponds $[\sigma(t)Z(t), \sigma(t)]$ in the homogeneous coordinates of $\mathbb{C}\mathrm{P}^1$, where

$$Z(t) = \frac{x'(t) + \mathrm{i}y'(t)}{\sigma(t) - z'(t)}.$$

Since the inverse image of $S^2(\sigma(t))$ by the Hopf map is $S^3(\sqrt{\sigma(t)})$, we rewrite $[\sigma(t)Z(t), \sigma(t)]$ as

$$\left[\sqrt{\sigma(t)}\frac{Z(t)}{\sqrt{|Z(t)|^2+1}},\ \sqrt{\sigma(t)}\frac{1}{\sqrt{|Z(t)|^2+1}}\right].$$

The only ambiguity in this expression is now the fiber coordinate $\lambda(t)$ whose modulus is identically equal to 1.

So the lifting in $S^3(\sqrt{r})$ of $\gamma'(t)$ is

$$\left(\sqrt{\sigma(t)}\frac{Z(t)}{\sqrt{|Z(t)|^2+1}}\lambda(t),\ \sqrt{\sigma(t)}\frac{1}{\sqrt{|Z(t)|^2+1}}\lambda(t)\right),$$

where the fiber coordinate $\lambda(t)$ has to be decided. Since $|Z(t)|^2 + 1 = 2\sigma(t)/(\sigma(t) - z'(t))$, the lifted point can be written as

$$\left(\frac{x'(t) + \mathrm{i}y'(t)}{\sqrt{2}\sqrt{\sigma(t) - z'(t)}}\lambda(t),\ \frac{\sqrt{\sigma(t) - z'(t)}}{\sqrt{2}}\lambda(t)\right) \in S^3\bigl(\sqrt{\sigma(t)}\bigr). \tag{17}$$

Since $\sigma(t)$ is positive and $\sigma(t) \geqslant z'(t)$, $\sigma(t) - z'(t)$ is also a real positive polynomial. So $\sigma(t) - z'(t)$ is factored into

$$\prod_{i=1}^{l}(t-\alpha_i)\big(t-\overline{\alpha_i}\big) \cdot \prod_{j=1}^{m}(t-a_j)^2,$$

where $\alpha_i \in \mathbb{C}\backslash\mathbb{R}$ and $a_j \in \mathbb{R}$. Then from the relation

$$\big(x'(t)+\mathrm{i}y'(t)\big)\big(x'(t)-\mathrm{i}y'(t)\big) = \big(\sigma(t)+z'(t)\big)\big(\sigma(t)-z'(t)\big),$$

we know that

$$\prod_{i=1}^{l}(t-\alpha_i)\big(t-\overline{\alpha_i}\big) \cdot \prod_{j=1}^{m}(t-a_j)^2$$

divides $(x'(t)+\mathrm{i}y'(t))(x'(t)-\mathrm{i}y'(t)) = x'(t)^2 + y'(t)^2$. First, note that if α_i is a root of $x'(t)+\mathrm{i}y'(t)$, then $\overline{\alpha_i}$ is a root of $x'(t)-\mathrm{i}y'(t)$; second, note also that if a_j is a root of $x'(t)+\mathrm{i}y'(t)$ then a_j is also a root of $x'(t)-\mathrm{i}y'(t)$. Thus, after rearranging complex roots, we may assume that $x'(t)+\mathrm{i}y'(t)$ is divisible by $\prod_{i=1}^{l}(t-\alpha_i)\prod_{j=1}^{m}(t-a_j)$.

We now choose the fiber coordinate $\lambda(t)$ to be $\sqrt{\prod_{i=1}^{l}(t-\overline{\alpha_i})/(t-\alpha_i)}$. Since the argument of the square root is a complex number, it may look like one has to worry about the choice of the branch of the logarithm. However, one can rewrite $\lambda(t)$ as

$$\lambda(t) = \prod_{i=1}^{l}\frac{t-\overline{\alpha_i}}{\sqrt{(t-\alpha_i)(t-\overline{\alpha_i})}},$$

and in this form the formula inside the square root sign is positive. Then both coordinates in (17) become complex valued polynomials. If we set $z_1(t)$ as

$$\frac{x'(t)+\mathrm{i}y'(t)}{\sqrt{2}\sqrt{\sigma(t)-z'(t)}}\lambda(t)$$

and $z_2(t)$ as

$$\frac{\sqrt{\sigma(t)-z'(t)}}{\sqrt{2}}\lambda(t),$$

then these polynomials construct the Pythagorean hodograph $\gamma'(t)$ through the Hopf fibration map. □

Remark 4.3. As alluded to in section 4.3, the algebro-geometric description of the Hopf map used above has a counterpart in the group-theoretic framework. One can use this homogeneous space setting to give an alternative geometric proof of the Dietz–Hoschek–Jüttler theorem. In this setting, the essential argument is the same except that formula (17) has to be replaced with the scaled version of (14). This approach is in fact taken in the next section, from which the interested reader can easily construct the argument.

5. Three-dimensional Minkowski Pythagorean hodographs

In this section, we investigate the algebraic structure of *Minkowski* Pythagorean hodographs in $\mathbb{R}^{2,1}$ with the corresponding Clifford algebra $\mathcal{C}\ell(2,1)$. According to the third author [22], a polynomial curve $\gamma(t) = (x(t), y(t), z(t))$ is called a *Minkowski Pythagorean hodograph* (MPH) curve if there exists a polynomial $\sigma(t)$ such that $x'(t)^2 + y'(t)^2 - z'(t)^2 = \sigma(t)^2$. He gave a necessary and sufficient condition for a polynomial curve to be an MPH curve. Namely, $\gamma(t)$ is an MPH curve if and only if

$$\begin{aligned} x'(t) &= -2p_0(t)p_3(t) - 2p_1(t)p_2(t), \\ y'(t) &= p_0(t)^2 + p_1(t)^2 - p_2(t)^2 - p_3(t)^2, \\ z'(t) &= 2p_0(t)p_1(t) + 2p_2(t)p_3(t), \\ \sigma(t) &= p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2, \end{aligned} \tag{18}$$

for some polynomials $p_0(t),\ p_1(t),\ p_2(t),\ p_3(t)$. The *primitivity* condition is not needed in this case. This version of the Pythagorean hodograph arises from the study of the medial axis transform of a planar region in which $(x(t), y(t))$ represents the medial axis (spine) curve, and $z(t)$ is the radius of the maximal inscribed circle centered at $(x(t), y(t))$. The formulae in (18) differ from those given in [22]. However, after permutation of variables, (18) can be transformed to the form in [22].

5.1. *Clifford algebra formalism*

The algebraic structure in (18) resembles that in (9), but with signs mixed up. In this section, we show that this algebraic structure can be easily captured using the PH representation map in an appropriate Clifford algebra setting. Since the underlying vector space is the Minkowski space $\mathbb{R}^{2,1}$, the natural Clifford algebra should be $\mathcal{C}\ell(2,1)$.

We now construct the Minkowski Pythagorean hodographs from the even Clifford algebra. Recall that the map $T : \mathcal{C}\ell^+(2,1) \to \mathrm{End}(\mathcal{C}\ell(2,1))$ is defined by

$$T(\mathbf{x})(\mathbf{y}) = \mathbf{x}\mathbf{y}\overline{\mathbf{x}}$$

for $\mathbf{x} \in \mathcal{C}\ell^+(2,1)$ and $\mathbf{y} \in \mathcal{C}\ell(2,1)$. One can easily show that for any $\mathbf{x} \in \mathcal{C}\ell^+(2,1)$, $T(\mathbf{x})$ acts on $\mathbb{R}^{2,1}$ as an endomorphism, which means $\Lambda(2,1) = \mathcal{C}\ell^+(2,1)$. Let us now choose the base vector a in the definition of the PH representation map to be $\mathbf{e}_2$. Thus we work with $T_{\mathbf{e}_2} : \mathcal{C}\ell^+(2,1) \to \mathbb{R}^{2,1}$ given by $T_{\mathbf{e}_2}(\mathbf{x}) = T(\mathbf{x})(\mathbf{e}_2)$. Suppose $\mathbf{p}(t) = p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12}$ is a polynomial in $\mathcal{C}\ell^+(2,1)$. Then $T_{\mathbf{e}_2}$ maps $\mathbf{p}(t)$ to a curve in $\mathbb{R}^{2,1}$ as follows:

$$\begin{aligned} T_{\mathbf{e}_2}\big(\mathbf{p}(t)\big) &= \big(p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12}\big)\mathbf{e}_2 \\ &\quad \times \big(p_0(t) - p_1(t)\mathbf{e}_{23} - p_2(t)\mathbf{e}_{31} - p_3(t)\mathbf{e}_{12}\big) \end{aligned}$$

$$
\begin{aligned}
&= -\big(2p_0(t)p_3(t) + 2p_1(t)p_2(t)\big)\mathbf{e}_1 \\
&\quad + \big(p_0(t)^2 + p_1(t)^2 - p_2(t)^2 - p_3(t)^2\big)\mathbf{e}_2 \\
&\quad + \big(2p_0(t)p_1(t) + 2p_2(t)p_3(t)\big)\mathbf{e}_3.
\end{aligned}
$$

And the norm $N(\mathbf{p}(t))$ of $\mathbf{p}(t)$ is

$$
\begin{aligned}
N\big(\mathbf{p}(t)\big) &= \big(p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12}\big) \\
&\quad \times \big(p_0(t) - p_1(t)\mathbf{e}_{23} - p_2(t)\mathbf{e}_{31} - p_3(t)\mathbf{e}_{12}\big) \\
&= p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2.
\end{aligned}
$$

Therefore, the image of $T_{\mathbf{e}_2}$ satisfies the Minkowski Pythagorean condition (18), and the magnitude of $T_{\mathbf{e}_2}(\mathbf{p}(t))$ is given by the norm $N(\mathbf{p}(t))$.

5.2. *Homogeneous space framework*

We now investigate a matrix algebra version. For reasons that become clear as we proceed, we need to investigate the matrix algebra corresponding to the full Clifford algebra $\mathcal{C}\ell(2,1)$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of $\mathbb{R}^{2,1}$ such that $\mathbf{e}_1$ and $\mathbf{e}_2$ are spacelike and $\mathbf{e}_3$ is timelike. Thus $\mathbf{e}_1^2 = \mathbf{e}_2^2 = -1$ and $\mathbf{e}_3^2 = 1$. Similarly, $\mathbf{e}_{12}^2 = -1$, $\mathbf{e}_{23}^2 = \mathbf{e}_{31}^2 = 1$.

We represent $\mathcal{C}\ell(2,1)$ as a matrix algebra using the Pauli matrices. The isomorphism ψ we look for from $\mathcal{C}\ell(2,1)$ to the 2×2 complex matrix algebra $\mathrm{M}(2,\mathbb{C})$ is given by the algebra extension of the relations

$$
\psi(\mathbf{e}_1) = \mathrm{i}\sigma_1, \qquad \psi(\mathbf{e}_2) = -\mathrm{i}\sigma_2, \qquad \psi(\mathbf{e}_3) = \sigma_3.
$$

Then, for other basis of $\mathcal{C}\ell(2,1)$, the isomorphism ψ is given by

$$
\psi(1) = \sigma_0, \qquad \psi(\mathbf{e}_{23}) = \sigma_1, \qquad \psi(\mathbf{e}_{31}) = -\sigma_2, \qquad \psi(\mathbf{e}_{12}) = \mathrm{i}\sigma_3, \qquad \psi(\mathbf{e}_{123}) = \mathrm{i}\sigma_0.
$$

These eight matrices $\mathrm{i}\sigma_1$, $-\mathrm{i}\sigma_2$, σ_3, σ_1, $-\sigma_2$, $\mathrm{i}\sigma_3$, σ_0, and $\mathrm{i}\sigma_0$ are independent over the *real* field $\mathbb{R}$. Therefore, they form a *real* basis of $\mathrm{M}(2,\mathbb{C})$, thereby making $\mathrm{M}(2,\mathbb{C})$ algebra isomorphic to $\mathcal{C}\ell(2,1)$ as *real* algebras. Using these, we can write an even Clifford element $\mathbf{p} = p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12}$ as

$$
\begin{aligned}
\psi(\mathbf{p}) &= p_0\sigma_0 + p_1\sigma_1 + p_2(-\sigma_2) + p_3(\mathrm{i}\sigma_3) \\
&= \begin{pmatrix} p_0 + \mathrm{i}p_3 & p_1 + \mathrm{i}p_2 \\ p_1 - \mathrm{i}p_2 & p_0 - \mathrm{i}p_3 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix},
\end{aligned} \tag{19}
$$

where $z_1 = p_0 + \mathrm{i}p_3$ and $z_2 = p_1 + \mathrm{i}p_2$. This suggests a way of writing $\mathbf{p} = p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12}$ as a pair (z_1, z_2) of complex numbers. First note that $\mathbf{e}_{12}^2 = -1$, which suggests the identification of $a + b\mathbf{e}_{12}$ with $a + b\mathrm{i}$. Then we write

$$
\begin{aligned}
\mathbf{p} &= p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12} \\
&= (p_0 + p_3\mathbf{e}_{12}) + (p_1 + p_2\mathbf{e}_{12})\mathbf{e}_{23} = z_1 + z_2\mathbf{e}_{23},
\end{aligned}
$$

where $z_1 = p_0 + p_3\mathbf{e}_{12}$ and $z_2 = p_1 + p_2\mathbf{e}_{12}$. In what follows, we sometimes abuse notation by dropping ψ. Thus, it is to be understood from the context when we say things like

$$\mathbf{p} = z_1 + z_2\mathbf{e}_{23} = \begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}.$$

A useful property of the isomorphism ψ is that it transforms the norm into the matrix determinant, i.e.,

$$N(\mathbf{p}) = \det\bigl(\psi(\mathbf{p})\bigr).$$

For the subsequent discussion, we need the matrix representation of vectors in $\mathbb{R}^{2,1}$. For a vector $v = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, its matrix representation $\psi(v)$ is given by

$$\begin{aligned}\psi(v) &= \psi(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathrm{i}\sigma_1 + x_2(-\mathrm{i}\sigma_2) + x_3\sigma_3 \\ &= \mathrm{i}\begin{pmatrix} -\mathrm{i}x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & \mathrm{i}x_3 \end{pmatrix} = \mathrm{i}\begin{pmatrix} \overline{w_2} & w_1 \\ \overline{w_1} & w_2 \end{pmatrix},\end{aligned} \tag{20}$$

where $w_1 = x_1 + \mathrm{i}x_2$, and $w_2 = \mathrm{i}x_3$.

The PH representation map is then given by

$$\begin{aligned}T_{\mathbf{e}_2}(\mathbf{p}) &= (p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12})\mathbf{e}_2(p_0 - p_1\mathbf{e}_{23} - p_2\mathbf{e}_{31} - p_3\mathbf{e}_{12}) \\ &= (z_1 + z_2\mathbf{e}_{23})\mathbf{e}_2\bigl(\overline{z_1} - z_2\mathbf{e}_{23}\bigr) \\ &= -\operatorname{Im}\bigl(z_1^2 + z_2^2\bigr)\mathbf{e}_1 + \operatorname{Re}\bigl(z_1^2 + z_2^2\bigr)\mathbf{e}_2 + 2\operatorname{Re}\bigl(z_1\overline{z_2}\bigr)\mathbf{e}_3.\end{aligned} \tag{21}$$

This computation can also be performed in the following matrix form:

$$\begin{aligned}\psi\bigl(\mathbf{p}\mathbf{e}_2\overline{\mathbf{p}}\bigr) &= \mathrm{i}\begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \overline{z_1} & -z_2 \\ -\overline{z_2} & z_1 \end{pmatrix} \\ &= \mathrm{i}\begin{pmatrix} -\mathrm{i}2\operatorname{Re}\bigl(z_1\overline{z_2}\bigr) & \mathrm{i}\bigl(z_1^2 + z_2^2\bigr) \\ -\mathrm{i}\bigl(\overline{z_1}^2 + \overline{z_2}^2\bigr) & -\mathrm{i}2\operatorname{Re}\bigl(z_1\overline{z_2}\bigr) \end{pmatrix}.\end{aligned} \tag{22}$$

The above two results in (21) and (22) coincide under the matrix representation of vectors given in (20).

Note that the matrix $\psi(\overline{\mathbf{p}})$ in (22) is not the usual matrix conjugation of $\psi(\mathbf{p})$. The matrix $\psi(\overline{\mathbf{p}})$ is the adjoint matrix of $\psi(\mathbf{p})$ under the Hermitian inner product of $\mathbb{C}^{1,1}$. The Hermitian bilinear form B on $\mathbb{C}^{1,1}$ is defined by

$$B\bigl((\alpha_1, \alpha_2)^{\mathrm{T}}, (\beta_1, \beta_2)^{\mathrm{T}}\bigr) = \alpha_1\overline{\beta_1} - \alpha_2\overline{\beta_2},$$

for any 2-dimensional column vectors $(\alpha_1, \alpha_2)^{\mathrm{T}}$ and $(\beta_1, \beta_2)^{\mathrm{T}}$. Straightforward computation then shows the adjoint relation between $\psi(\mathbf{p})$ and $\psi(\overline{\mathbf{p}})$ in the following manner:

$$B\bigl(\psi(\mathbf{p})(\alpha_1, \alpha_2)^{\mathrm{T}}, (\beta_1, \beta_2)^{\mathrm{T}}\bigr) = B\bigl((\alpha_1, \alpha_2)^{\mathrm{T}}, \psi(\overline{\mathbf{p}})(\beta_1, \beta_2)^{\mathrm{T}}\bigr).$$

This Hermitian bilinear form B reflects also the norm N of the Clifford algebra $\mathcal{C}\ell(2,1)$, i.e.,

$$N(z_1 + z_2\mathbf{e}_{23}) = B\big((z_1, z_2)^{\mathrm{T}}, (z_1, z_2)^{\mathrm{T}}\big).$$

Along this observation, we establish the identification of the spin group $\mathrm{Spin}(2,1)$ and the unitary group $\mathrm{U}(1,1)$ of $\mathbb{C}^{1,1}$. Since the spin group $\mathrm{Spin}(2,1)$ is characterized by

$$\mathrm{Spin}(2,1) = \big\{\mathbf{p} = z_1 + z_2\mathbf{e}_{23} \in \mathcal{C}\ell^{2,1}\colon\ N(\mathbf{p}) = |z_1|^2 - |z_2|^2 = \pm 1\big\},$$

it has two connected components depending on their norms. Among them, the connected component of the identity is denoted by

$$\mathrm{Spin}^0(2,1) = \big\{\mathbf{p} = z_1 + z_2\mathbf{e}_{23} \in \mathcal{C}\ell^{2,1}\colon\ N(\mathbf{p}) = |z_1|^2 - |z_2|^2 = 1\big\},$$

and the isomorphism ψ maps $\mathrm{Spin}^0(2,1)$ onto the special unitary group

$$\mathrm{SU}(1,1) = \left\{\begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}\colon\ |z_1|^2 - |z_2|^2 = 1\right\}.$$

In the 3-dimensional Minkowski space, the group $\mathrm{SU}(1,1)$ plays the role of $\mathrm{SU}(2)$ in the 3-dimensional Euclidean space of the previous section.

There is another possible representation of $\mathrm{Spin}^0(2,1)$ using the special linear group $\mathrm{SL}(2,\mathbb{R})$ instead of the special unitary group $\mathrm{SU}(1,1)$. This representation can be obtained by choosing a different isomorphism such as the algebra isomorphism defined by

$$\psi(\mathbf{e}_1) = \mathrm{i}\sigma_3, \qquad \psi(\mathbf{e}_2) = \mathrm{i}\sigma_1, \qquad \psi(\mathbf{e}_3) = -\sigma_2.$$

We now formulate the homogeneous space framework that is analogous to that of the Hopf fibration. The set of all space-like unit vectors in $\mathbb{R}^{2,1}$ is identified with

$$M = \left\{\mathrm{i}\begin{pmatrix} \overline{w_2} & w_1 \\ \overline{w_1} & w_2 \end{pmatrix}\colon\ |w_1|^2 - |w_2|^2 = 1,\ \mathrm{Re}(w_2) = 0\right\}.$$

And the matrix action given in (22) can be generalized to the $\mathrm{SU}(1,1)$-action on M. Then, it is easy to check the isotropy subgroup of this action at $-\mathrm{i}\sigma_2$ is

$$I = \left\{\begin{pmatrix} a & \mathrm{i}b \\ -\mathrm{i}b & a \end{pmatrix}\colon\ a, b \in \mathbb{R},\ a^2 - b^2 = 1\right\}.$$

Moreover, this action is also transitive, i.e., for a given

$$\mathrm{i}\begin{pmatrix} \overline{w_2} & w_1 \\ \overline{w_1} & w_2 \end{pmatrix} \in M,$$

there exists

$$\begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \in \mathrm{SU}(1,1)$$

such that

$$\mathrm{i}\begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \overline{z_1} & -z_2 \\ -\overline{z_2} & z_1 \end{pmatrix} = \mathrm{i}\begin{pmatrix} \overline{w_2} & w_1 \\ \overline{w_1} & w_2 \end{pmatrix}. \tag{23}$$

Since the given matrix

$$\mathrm{i}\begin{pmatrix} \overline{w_2} & w_1 \\ \overline{w_1} & w_2 \end{pmatrix}$$

corresponds to a unit vector (x_1, x_2, x_3) in $\mathbb{R}^{2,1}$ satisfying $w_1 = x_1 + \mathrm{i}x_2$ and $w_2 = \mathrm{i}x_3$, we can rewrite the matrix in the form

$$\mathrm{i}\begin{pmatrix} -\mathrm{i}\sinh\phi & (-\sin\theta + \mathrm{i}\cos\theta)\cosh\phi \\ (-\sin\theta - \mathrm{i}\cos\theta)\cosh\phi & \mathrm{i}\sinh\phi \end{pmatrix}$$

for some real numbers θ and ϕ. We here measure the angle θ from the x_2-axis since we use $\mathbf{e}_2$ as the base of PH representation map instead of $\mathbf{e}_1$. Upon a straightforward calculation, one can easily check that the matrix

$$\begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2}\cosh\dfrac{\phi}{2} & \mathrm{e}^{\mathrm{i}\theta/2}\sinh\dfrac{\phi}{2} \\ \mathrm{e}^{-\mathrm{i}\theta/2}\sinh\dfrac{\phi}{2} & \mathrm{e}^{-\mathrm{i}\theta/2}\cosh\dfrac{\phi}{2} \end{pmatrix} \tag{24}$$

satisfies equation (23). In fact, since the isotropy subgroup is I, any solution of equation (23) should be of the form

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2}\cosh\dfrac{\phi}{2} & \mathrm{e}^{\mathrm{i}\theta/2}\sinh\dfrac{\phi}{2} \\ \mathrm{e}^{-\mathrm{i}\theta/2}\sinh\dfrac{\phi}{2} & \mathrm{e}^{-\mathrm{i}\theta/2}\cosh\dfrac{\phi}{2} \end{pmatrix}\begin{pmatrix} a & \mathrm{i}b \\ -\mathrm{i}b & a \end{pmatrix}$$

for some real numbers a and b with $a^2 - b^2 = 1$.

By the standard theory of homogeneous space, we get the following isomorphism. In contrast with the Euclidean version of the Hopf fibration (15), we can call (25) the "Minkowski–Hopf fibration".

Theorem 5.1 (Minkowski–Hopf fibration). By the Clifford representation map $T_{\mathbf{e}_2}$, we get the following isomorphism.

$$\mathrm{SU}(1,1)/\mathrm{I} \cong H, \tag{25}$$

where H is the unit hyperboloid of one sheet in $\mathbb{R}^{2,1}$ given by

$$H = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1}\colon\ x_1^2 + x_2^2 - x_3^2 = 1\}.$$

Therefore SU(1, 1) is the hyperbola (the isotropy subgroup I) bundle over the hyperboloid H. Comparing with the Hopf fibration (15) in the Euclidean space, we name the fibration (25) the "Minkowski–Hopf fibration".

5.3. Singular foliation of Minkowski–Hopf fibrations

In this section, we describe the foliation of the Minkowski–Hopf fibrations that is analogous to the foliation of Hopf fibrations in the Euclidean space given in section 4.3.

At first, let $H(r)$ denote the set of space-like vectors of length r in the Minkowski space $\mathbb{R}^{2,1}$, i.e.,

$$H(r) = \{v = (x_1, x_2, x_3) \in \mathbb{R}^{2,1}\colon\ x_1^2 + x_2^2 - x_3^2 = r^2\}.$$

Then the set of all space-like vectors in $\mathbb{R}^{2,1}$ is foliated by $H(r)$ in the following way:

$$\{v = (x_1, x_2, x_3)\colon\ x_1^2 + x_2^2 - x_3^2 > 0\} = \bigcup_{r>0} H(r) = \mathbb{R}^+ \times H(r).$$

On the other hand, if we define $S(r)$ by

$$S(r) = \{\mathbf{p} \in \mathcal{C}\ell^+(2,1)\colon\ N(\mathbf{p}) = r\},$$

then the even Clifford algebra $\mathcal{C}\ell^+(2,1)$ is also foliated by $S(r)$ in the form

$$\mathcal{C}\ell^+(2,1) = \bigcup_{r\in\mathbb{R}} S(r) = \mathbb{R} \times S(1) = \mathbb{R} \times \mathrm{Spin}^0(2,1).$$

In fact, the PH representation map $T_{\mathbf{e}_2}$ in (21) is a map between these two foliation structures. If $T_{\mathbf{e}_2}$ is restricted to $S(r)$ for $r > 0$, we have

$$T_{\mathbf{e}_2} : S(r) \to H(r),$$

and this map has the same fiber structure with the Minkowski–Hopf fibration for $r = 1$. Let $v = (x_1, x_2, x_3)$ is a space-like vector in $H(r)$, then we can rewrite v as

$$v = (-r \sin\theta \cosh\phi, r\cos\theta\cosh\phi, r\sinh\phi).$$

for appropriate θ and ϕ. Applying scaling methods to (24), one can find a pre-image of v, which is

$$\sqrt{r}\mathrm{e}^{\mathbf{i}\theta/2}\cosh\frac{\phi}{2} + \sqrt{r}\mathrm{e}^{\mathbf{i}\theta/2}\sinh\frac{\phi}{2}\mathbf{e}_{23}.$$

And all pre-images of v are obtained by multiplying the isotropy subgroup. Thus, any pre-image $\mathbf{p} = z_1 + z_2\mathbf{e}_{23} \in S(\sqrt{r})$ of v is given by

$$z_1 = \sqrt{r}\left(a\mathrm{e}^{\mathrm{i}\theta/2}\cosh\frac{\phi}{2} - \mathrm{i}b\mathrm{e}^{\mathrm{i}\theta/2}\sinh\frac{\phi}{2}\right),$$
$$z_2 = \sqrt{r}\left(\mathrm{i}b\mathrm{e}^{\mathrm{i}\theta/2}\cosh\frac{\phi}{2} + a\mathrm{e}^{\mathrm{i}\theta/2}\sinh\frac{\phi}{2}\right),$$

for some real numbers a and b satisfying $a^2 - b^2 = 1$.

5.4. *Geometric proof of MPH representation theorem*

The group-theoretic description of the Minkowski–Hopf map reveals the rich geometric and algebraic structures underlying the MPH curve formalism, which should be useful for future study. In this section, we shall give a geometric proof of the MPH representation formula due to the third author [22]. Although the algebraic version discovered by the third author does not distinguish cases, we need to assume certain geometric conditions to make our geometric picture conform to the actual geometric intuition! Suppose $\gamma(t) = (x(t), y(t), z(t))$ is a polynomial curve satisfying

$$\left|\gamma'(t)\right|^2 = x'(t)^2 + y'(t)^2 - z'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$. If $\sigma(t)$ vanishes at some point t_0, it means that $\gamma(t)$ becomes light-like at that point. On the other hand, if $\gamma(t)$ represents the medial axis transform of some planar domain, $\gamma(t)$ has to be space-like except perhaps at the end point of its interval of definition where it may become light-like. Being light-like corresponds to having inscribed osculating circle at the terminal node of the medial axis transform [5]. This kind of light-like case can be handled by a limiting argument. Thus we may assume $\gamma(t)$ is space-like, which means that $\sigma(t)$ never vanishes. By reversing sign if necessary, we may assume that $\sigma(t) > 0$ for all t.

Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a PH curve in the Minkowski space $\mathbb{R}^{2,1}$ given by $T_{\mathbf{e}_2}(\mathbf{x}(t))$ with $\mathbf{x}(t) = p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12} \in \mathcal{C}\ell^+(3,1)[t]$, i.e.,

$$\begin{aligned} x_1'(t) &= -2p_0(t)p_3(t) - 2p_1(t)p_2(t), \\ x_2'(t) &= p_0(t)^2 + p_1(t)^2 - p_2(t)^2 - p_3(t)^2, \\ x_3'(t) &= 2p_0(t)p_1(t) + 2p_2(t)p_3(t). \end{aligned} \tag{26}$$

Then it is easy to check that $\gamma(t)$ is an PH curve such that

$$\sigma(t) = p_0(t)^2 - p_1(t)^2 - p_2(t)^2 + p_3(t)^2. \tag{27}$$

The following theorem is the converse.

Theorem 5.2 (Solution of PH characterization problem for $\mathbb{R}^{2,1}$). Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a polynomial curve in $\mathbb{R}^{2,1}$ satisfying

$$\left|\gamma'(t)\right|^2 = x'(t)^2 + y'(t)^2 - z'(t)^2 = \sigma(t)^2 \tag{28}$$

$\sigma(t) > 0$ for all t. Then $\gamma(t)$ is a PH curve in $\mathbb{R}^{3,1}$, i.e., there exist $\mathbf{x}(t) = p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12} \in \mathcal{C}\ell^+(3,1)[t]$ so that $\gamma'(t) = T_{\mathbf{e}_2}(\mathbf{x}(t))$.

Proof. Let $\gamma(t)$ be a polynomial curve in $\mathbb{R}^{2,1}$ satisfying (28). Since what we are interested in is the hodograph of $\gamma(t)$, to simplify notations, we put $\gamma'(t) = (x_1(t), x_2(t), x_3(t))$ which satisfies $x_1(t)^2 + x_2(t)^2 - x_3(t)^2 = \sigma(t)^2$. By applying polar decomposition, we can rewrite $\gamma'(t)$ as

$$\begin{aligned} x_1(t) &= -\sigma(t)\sin\theta(t)\cosh\phi(t), \\ x_2(t) &= \sigma(t)\cos\theta(t)\cosh\phi(t), \\ x_3(t) &= \sigma(t)\sinh\phi(t). \end{aligned} \tag{29}$$

Then, by (26), we know that the pre-image $\mathbf{p}(t) = z_1(t) + z_2(t)\mathbf{e}_{23} \in \mathcal{C}\ell^+(2,1)$ of $\gamma'(t)$ under the PH representation map is given by

$$\begin{aligned} z_1(t) &= \sqrt{\sigma(t)}\left(a(t)\mathrm{e}^{\mathrm{i}\theta(t)/2}\cosh\frac{\phi(t)}{2} - \mathrm{i}b(t)\mathrm{e}^{\mathrm{i}\theta(t)/2}\sinh\frac{\phi(t)}{2}\right), \\ z_2(t) &= \sqrt{\sigma(t)}\left(\mathrm{i}b(t)\mathrm{e}^{\mathrm{i}\theta(t)/2}\cosh\frac{\phi(t)}{2} + a(t)\mathrm{e}^{\mathrm{i}\theta(t)/2}\sinh\frac{\phi(t)}{2}\right). \end{aligned} \tag{30}$$

So the proof is reduced to finding the fiber coordinates function $a(t)$ and $b(t)$, which make $z_1(t)$ and $z_2(t)$ into complex polynomials, while satisfying $a(t)^2 - b(t)^2 = 1$.

Form now on, we drop the parameter t to simplify notations. Dependency on the curve parameter t should be understood from the context. First, we define two polynomials p_0 and p_2 by the following equations:

$$p_0 - p_2 = \gcd(\sigma + x_2, x_1 + x_3), \qquad p_0 + p_2 = \frac{\sigma + x_2}{2(p_0 - p_2)}.$$

Then, by the MPH condition $(x_1 + x_3)(x_1 - x_3) = (\sigma + x_2)(\sigma - x_2)$, the polynomial $(x_1 + x_3)/(p_0 - p_2)$ is a factor of $\sigma - x_2$. So we can define two polynomials p_1 and p_3 by the relations:

$$p_1 - p_3 = \frac{x_1 + x_3}{2(p_0 - p_2)}, \qquad p_1 + p_3 = -\frac{\sigma - x_2}{p_1 - p_3}.$$

By straightforward computation, one can show that the above four polynomials p_0, p_1, p_2 and p_3 satisfy equations (26) and (27).

We now decide the fiber coordinates a and b which make z_1 and z_2 in (30) into

$$z_1 = p_0 + \mathrm{i}p_3, \qquad z_2 = p_1 + \mathrm{i}p_2.$$

By direct solving equation (30) with the above conditions, we get

$$\begin{aligned} a &= \frac{1}{\sqrt{\sigma}}\left\{\mathrm{e}^{-\mathrm{i}\theta/2}\cosh\frac{\phi}{2}(p_0 + \mathrm{i}p_3) - \mathrm{e}^{\mathrm{i}\theta/2}\sinh\frac{\phi}{2}(p_1 - \mathrm{i}p_2)\right\}, \\ b &= \frac{1}{\sqrt{\sigma}}\left\{\mathrm{e}^{-\mathrm{i}\theta/2}\cosh\frac{\phi}{2}(p_1 + \mathrm{i}p_2) - \mathrm{e}^{\mathrm{i}\theta/2}\sinh\frac{\phi}{2}(p_0 - \mathrm{i}p_3)\right\}. \end{aligned}$$

We have to make sure that a and b belong to the isotropy subgroup I, that is, they are real-valued functions with $a^2 - b^2 = 1$. Although the above expressions involve some complex functions, both a and b are actually real-valued. In fact, the imaginary part of a is given by

$$\mathrm{Im}(a) = \cosh\frac{\phi}{2}\left(p_3\cos\frac{\theta}{2} - p_0\sin\frac{\theta}{2}\right) - \sinh\frac{\phi}{2}\left(p_1\sin\frac{\theta}{2} - p_2\cos\frac{\theta}{2}\right).$$

By taking square and using relations (29), it can be shown that $\mathrm{Im}(a)^2 = 0$. And $\mathrm{Im}(b) = 0$ can also be proved in a similar way.

Therefore, the fiber coordinate a and b are actually *real*-valued functions given by

$$a = \cosh\frac{\phi}{2}\left(p_0\cos\frac{\theta}{2} + p_3\sin\frac{\theta}{2}\right) - \sinh\frac{\phi}{2}\left(p_1\cos\frac{\theta}{2} + p_2\sin\frac{\theta}{2}\right),$$
$$b = \cosh\frac{\phi}{2}\left(p_2\cos\frac{\theta}{2} - p_1\sin\frac{\theta}{2}\right) - \sinh\frac{\phi}{2}\left(p_0\sin\frac{\theta}{2} - p_3\cos\frac{\theta}{2}\right).$$

(Note that a and b do not need to be polynomials.) The fact that $a^2 - b^2 = 1$ can also be checked in a similar way. □

6. Four-dimensional Minkowski Pythagorean hodographs

In preceding sections, we have shown that the Clifford algebra is well-suited to explain the algebraic structure of Pythagorean and Minkowski Pythagorean hodographs. We now generalize MPH curves to the 4-dimensional Minkowski space through the corresponding PH representation map.

Suppose $\gamma(t) = (x(t), y(t), z(t), r(t))$ is a polynomial curve in $\mathbb{R}^{3,1}$. As we have done in the 3-dimensional Minkowski space, we can regard a point (x, y, z, r) in $\mathbb{R}^{3,1}$ as a sphere in $\mathbb{R}^3$ centered at (x, y, z) of radius r. Thus, the curve $\gamma(t)$ expresses a one parameter family of spheres in $\mathbb{R}^3$, whose envelope is a canal surface.

In [23], Peternell and Pottmann show that any canal surface with polynomial spine curve and polynomial radius function has a rational parameterization if it is a real canal surface, i.e., $x'(t)^2 + y'(t)^2 + z'(t)^2 - r'(t)^2 \geqslant 0$. They rely on the fact that any nonnegative polynomial can be decomposed into the sum of two squared polynomials. Thus, if we have $x'(t)^2 + y'(t)^2 + z'(t)^2 - r'(t)^2 = f(t) \geqslant 0$ for all t, then there exist two polynomials $f_1(t)$ and $f_2(t)$ such that $f(t) = f_1(t)^2 + f_2(t)^2$. They provide an algorithm to construct the rational parameterization of the canal surface with $f_1(t)$ and $f_2(t)$.

The major difficulty in applying their algorithm to a given canal surface is the step to decompose $f(t)$ into $f_1(t)^2 + f_2(t)^2$. In order to do this, one has to find all complex zeros of $f(t)$. It requires numerical computation together with algebraic algorithm. They also mention that the rational parameterization of a given canal surface is not so amenable. The parametric grid curves generally make lots of rotation around the spine curve.

In this chapter, we show that every canal surface with rational spine curve and rational radius function can be represented by the 4-dimensional MPH. Invoking the result of Pottmann and Peternell [23], our result gives an alternative approach to proving that every canal surface with rational spine curve and rational radius function can be rationally parameterized.

6.1. Clifford algebra formalism

To generalize MPH curves to $\mathbb{R}^{3,1}$, we first discuss the structure of the Clifford algebra $\mathcal{C}\ell(3,1)$. The Clifford algebra $\mathcal{C}\ell(3,1)$ is of degree 16 and has basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}, \mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}, \mathbf{e}_{1234}\}$. These elements are distinguished according to degree: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are called *vectors*; $\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$ are *bivectors*; $\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}$ are *tri-vectors*; and $\mathbf{e}_{1234}$ is a *pseudo-scalar*. An element $\mathbf{x}$ in the even Clifford algebra $\mathcal{C}\ell^+(\mathbb{R}^{3,1})$ is of the form $x_0 + x_1\mathbf{e}_{12} + x_2\mathbf{e}_{13} + x_3\mathbf{e}_{14} + x_4\mathbf{e}_{23} + x_5\mathbf{e}_{24} + x_6\mathbf{e}_{34} + x_7\mathbf{e}_{1234}$.

According to the classification results on the Clifford algebra, $\mathcal{C}\ell(3,1)$ is isomorphic to the 2×2 quaternion matrix algebra $\mathbb{H}(2)$. This is obtained from the well-known isomorphism $\mathcal{C}\ell(r+1, s+1) \cong \mathcal{C}\ell(r,s)\otimes\mathcal{C}\ell(1,1)$. Thus, $\mathcal{C}\ell(3,1) \cong \mathcal{C}\ell(2)\otimes\mathcal{C}\ell(1,1)$. We already mentioned that $\mathcal{C}\ell(2) \cong \mathbb{H}$. And one can easily show that $\mathcal{C}\ell(1,1) \cong \mathrm{M}(2,\mathbb{R})$. Moreover, $\mathcal{C}\ell^+(3,1)$ is isomorphic to $\mathcal{C}\ell(2,1) \cong \mathrm{M}(2,\mathbb{C})$. This is a general fact for arbitrary n, i.e., $\mathcal{C}\ell^+(n+1,m) \cong \mathcal{C}\ell(n,m)$.

Here we are going to use the Minkowski version of the biquaternion notation. The even Clifford algebra $\mathcal{C}\ell^+(3,1)$ has the basis

$$\{1, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{14}, \mathbf{e}_{24}, \mathbf{e}_{34}, \mathbf{e}_{1234}\}.$$

We write $\mathbf{e}_{23} = \mathbf{i},\ \mathbf{e}_{31} = \mathbf{j},\ \mathbf{e}_{12} = \mathbf{k}$ and $\mathbf{e}_{1234} = \omega$. The elements $1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}$ are basis of the quaternion algebra $\mathbb{H}$. Thus, an element $\mathbf{x}$ in $\mathcal{C}\ell^+(3,1)$ can be written as

$$\mathbf{x} = \mathbf{p} + \omega\mathbf{q} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} + \omega(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}).$$

Or, by complex notation, we have $\mathbf{x} = p + wq = z_1 + z_2\mathbf{j} + \omega(w_1 + w_2\mathbf{j})$, where $z_1 = p_0 + p_1\mathrm{i},\ z_2 = p_2 + p_3\mathrm{i},\ w_1 = q_0 + q_1\mathrm{i}$, and $w_2 = q_2 + q_3\mathrm{i}$.

We now consider the map $T : \mathcal{C}\ell^+(3,1) \to \mathrm{End}(\mathcal{C}\ell(3,1))$ defined in the same way as before,

$$T(\mathbf{x})(\mathbf{y}) = \mathbf{x}\mathbf{y}\overline{\mathbf{x}}$$

for $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$ and $\mathbf{y} \in \mathcal{C}\ell(3,1)$.

The PH curve in $\mathbb{R}^{3,1}$ with base vector $\mathbf{e}_1$ is given by

$$\begin{aligned} T_{\mathbf{e}_1}(\mathbf{x}) &= \mathbf{x}\mathbf{e}_1\overline{\mathbf{x}} = (p + \omega q)\mathbf{e}_1\big(\overline{p} + \omega\overline{q}\big) \\ &= \big(p_0^2 + p_1^2 - p_2^2 - p_3^2 + q_0^2 + q_1^2 - q_2^2 - q_3^2\big)\mathbf{e}_1 \\ &\quad + 2(p_1p_2 + p_0p_3 + q_1q_2 + q_0q_3)\mathbf{e}_2 \\ &\quad + 2(p_1p_3 - p_0p_2 + q_1q_3 - q_0q_2)\mathbf{e}_3 \\ &\quad + 2(p_1q_0 - p_0q_1 + p_2q_3 - p_3q_2)\mathbf{e}_4. \end{aligned} \tag{31}$$

And by using the complex notation, we have a more compactified form of $T_{\mathbf{e}_1}(\mathbf{x})$ as follows:

$$\begin{aligned} T_{\mathbf{e}_1}(\mathbf{x}) &= \mathbf{x}\mathbf{e}_1\overline{\mathbf{x}} = (p + \omega q)\mathbf{e}_1\big(\overline{p} + \omega\overline{q}\big) \\ &= \big(|z_1|^2 - |z_2|^2 + |w_1|^2 - |w_2|^2\big)\mathbf{e}_1 + 2\,\mathrm{Im}(z_1z_2 + w_1w_2)\mathbf{e}_2 \\ &\quad - 2\,\mathrm{Re}(z_1z_2 + w_1w_2)\mathbf{e}_3 + 2\,\mathrm{Im}\big(z_1\overline{w_1} - z_2\overline{w_2}\big)\mathbf{e}_4. \end{aligned}$$

The pseudoscalar ω has some nice properties. Since $\omega\mathbf{e}_k = -\mathbf{e}_k\omega$ for $k = 1, \dots, 4$, it follows that ω commutes with every element of the even Clifford algebra. And $\omega^2 = -1$. Then we have the following result.

Proposition 6.1. For any $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$ and $v \in \mathbb{R}^{3,1}$, $T(\mathbf{x})(v) \in \mathbb{R}^{3,1}$. So $\Lambda(3,1) = \mathcal{C}\ell^+(3,1)$.

Proof. Let us write $\mathbf{x}$ as $p + \omega q$ for some quaternions p and q. Since $\overline{\omega} = \omega$ and ω commutes with quaternions, $\bar{\mathbf{x}}$ equals $\overline{p} + \omega\overline{q}$. Suppose v is chosen to be $\mathbf{e}_4$. Since $\mathbf{e}_4$ commutes with quaternions, $T(\mathbf{x})(\mathbf{e}_4)$ is given by

$$\begin{aligned} T(\mathbf{x})(\mathbf{e}_4) &= (p + \omega q)\mathbf{e}_4\bigl(\overline{p} + \omega\overline{q}\bigr) = \mathbf{e}_4(p - \omega q)\bigl(\overline{p} + \omega\overline{q}\bigr) \\ &= \bigl(|p|^2 + |q|^2\bigr)\mathbf{e}_4 - \mathbf{e}_{123}\bigl(p\overline{q} - q\overline{p}\bigr). \end{aligned}$$

Then $T(\mathbf{x})(\mathbf{e}_4)$ is also a vector, because $p\overline{q} - q\overline{p}$ is a pure quaternion.

If v is $\mathbf{e}_1$, then the commutation relation is a little bit different. Since $\mathbf{e}_1\mathbf{e}_{23} = \mathbf{e}_{23}\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_{31} = -\mathbf{e}_{31}\mathbf{e}_1$ and $\mathbf{e}_1\mathbf{e}_{12} = -\mathbf{e}_{12}\mathbf{e}_1$, we have $\mathbf{e}_1(p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) = (p_0 + p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k})\mathbf{e}_1$. Let $\tilde{p}$ denote $p_0 + p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k}$. Then,

$$\begin{aligned} T(\mathbf{x})(\mathbf{e}_1) &= (p + \omega q)\mathbf{e}_1\bigl(\overline{p} + \omega\overline{q}\bigr) = \mathbf{e}_1\bigl(\tilde{p} - \omega\tilde{q}\bigr)\bigl(\overline{p} + \omega\overline{q}\bigr) \\ &= \mathbf{e}_1\bigl(\tilde{p}\overline{p} + \tilde{q}\overline{q} + \omega\bigl(\tilde{p}\overline{q} - \tilde{q}\overline{p}\bigr)\bigr). \end{aligned}$$

By straightforward calculation, we can show that the quaternions $\tilde{p}\overline{p} + \tilde{q}\overline{q}$ do not have an $\mathbf{i} = \mathbf{e}_{23}$ term and $\tilde{p}\overline{q} - \tilde{q}\overline{p}$ has only an $\mathbf{i} = \mathbf{e}_{23}$ term. Therefore, $T(\mathbf{x})(\mathbf{e}_1)$ is contained in $\mathbb{R}^{3,1}$.

For $\mathbf{e}_2$ and $\mathbf{e}_3$, we can show that $T(\mathbf{x})(\mathbf{e}_k) \in \mathbb{R}^{3,1}$ by similar computation. Therefore, by the linearity of $T(\mathbf{x})$, we have shown that $T(\mathbf{x})(v) \in \mathbb{R}^{3,1}$ for all $v \in \mathbb{R}^{3,1}$. $\square$

We now consider the norm of $\mathbf{x}$. The norm $N(\mathbf{x})$ is given by

$$N(\mathbf{x}) = (p + \omega q)\bigl(\overline{p} + \omega\overline{q}\bigr) = \bigl(|p|^2 - |q|^2\bigr) + \omega\bigl(p\overline{q} + q\overline{p}\bigr).$$

Thus, the norm of $\mathbf{x}$ has not only a scalar term, but also a pseudoscalar term. Let $f_1(\mathbf{x})$ be the scalar part of $N(\mathbf{x})$, and let $f_2(\mathbf{x})$ be the coefficient of the pseudoscalar part.

Proposition 6.2. For any $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$, $T(\mathbf{x})$ acts on $\mathbb{R}^{3,1}$ as a linear transform in $\mathbb{R}^+ \times \mathrm{SO}(3,1)$. That is, $T(\mathbf{x})$ consists of a special orthogonal transform and a magnification. And the magnification factor is $\sqrt{f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2}$.

Proof. What we need to show is that

$$\bigl\langle T(\mathbf{x})(v), T(\mathbf{x})(w)\bigr\rangle_{\mathbb{R}^{3,1}} = \bigl(f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2\bigr)\langle v, w\rangle_{\mathbb{R}^{3,1}},$$

for any $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$ and $v, w \in \mathbb{R}^{3,1}$. But we only need show $Q(T(\mathbf{x})(\mathbf{e}_k)) = (f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2)Q(\mathbf{e}_k)$, since the bilinear form B is defined by the polarization of the quadratic form Q. The actual computation is given below.

$$\begin{aligned} Q\big(T(\mathbf{x})(\mathbf{e}_k)\big) &= Q\big(\mathbf{x}\mathbf{e}_k\overline{\mathbf{x}}\big) = -\mathbf{x}\mathbf{e}_k\overline{\mathbf{x}}\mathbf{x}\mathbf{e}_k\overline{\mathbf{x}} = -\mathbf{x}\mathbf{e}_k\big(f_1(\mathbf{x}) + \omega f_2(\mathbf{x})\big)\mathbf{e}_k\overline{\mathbf{x}} \\ &= \mathbf{x}\big(f_1(\mathbf{x}) - \omega f_2(\mathbf{x})\big)Q(\mathbf{e}_k)\overline{\mathbf{x}} = Q(\mathbf{e}_k)\big(f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2\big). \qquad \square \end{aligned}$$

In order to understand the meaning of the pseudoscalar, we need the following lemma.

Lemma 6.3. For some $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$, if $T(\mathbf{x})$ is the identity in $\mathrm{SO}_+(3,1)$ and $N(\mathbf{x})$ is a nonzero scalar, then $\mathbf{x}$ equals ± 1 or $\pm\omega$.

Proof. Let $\mathbf{x}$ be $p + \omega q$ as before. And let f denote $N(\mathbf{x})$, i.e., $f = |p|^2 - |q|^2 \neq 0$. Since $T(\mathbf{x})(\mathbf{e}_4) = \mathbf{e}_4$, by multiplying $\mathbf{x}$ on the right of both sides, we have $\mathbf{x}\mathbf{e}_4\overline{\mathbf{x}}\mathbf{x} = \mathbf{e}_4\mathbf{x}$. Thus, $(p - \omega q)f = p + \omega q$. So we get the following two equations:

$$fp = p \quad \text{and} \quad -fq = q.$$

For the first case, if $p \neq 0$, then f becomes 1. And q becomes 0. Thus, $\mathbf{x} = p$ with $|p| = 1$. Since the quaternion p is in Spin(3) and the mappings T and χ are the same on the spin group, $T(\mathbf{x}) = Id$ implies $\mathbf{x} = \pm 1$. For the other case, suppose p is 0. Then q should be nonzero. And f becomes -1. So q is a unit quaternion. Thus $\mathbf{x} = \omega q$. Since $Id = T(\mathbf{x}) = T(\omega q) = T(\omega)T(q)$ and $T(\omega) = Id$, we get $q = \pm 1$. Therefore $\mathbf{x}$ equals $\pm\omega$. $\square$

Since we defined T in a different way from χ, the corresponding group Λ has different structure from the Clifford group Γ. The Clifford group $\Gamma(3,1)$ is isomorphic to $\mathbb{R}^+ \times \mathrm{Spin}(3,1)$. And proposition 6.2 shows that $\Lambda(3,1)$ equals $\mathcal{C}\ell^+(3,1)$. Here, we discuss the relation between the Clifford group $\Gamma(3,1)$ and the norm.

Theorem 6.4. For any $\mathbf{x} \in \Lambda(3,1)$, assume $N(\mathbf{x}) \neq 0$. Then $\mathbf{x} \in \Gamma(3,1)$ if and only if $N(\mathbf{x})$ is a scalar.

Proof. Suppose $\mathbf{x}$ is in $\Gamma(3,1)$, then $\mathbf{x}$ can be written as products of even number of vectors (we call such $\mathbf{x}$ is *decomposable*). Thus, it is obvious that $N(\mathbf{x}) \in \mathbb{R}$.

For sufficiency, we again employ the biquaternion notation. Let $\mathbf{x}$ be $p + \omega q$. Then $N(\mathbf{x})$ is given by $f_1(\mathbf{x}) + \omega f_2(\mathbf{x})$, where $f_1(\mathbf{x}) = |p|^2 - |q|^2$ and $f_2(\mathbf{x}) = p\overline{q} + q\overline{p}$. We now assume $f_2(\mathbf{x}) = 0$. By proposition (6.2), $T(\mathbf{x})$ is expressed as $f_1(\mathbf{x})^2 A$ for some linear transform A in O(3, 1). Since Spin(3, 1) is a double cover by the covering map χ and T is the same as χ on Spin(3, 1), there exists $\mathbf{y} \in \mathrm{Spin}(3,1)$ such that $T(\mathbf{y}) = A^{-1}$. Let $\mathbf{z}$ be $\mathbf{y}/f_1(\mathbf{x})$, then $T(\mathbf{z}) = T(\mathbf{x})^{-1}$. So $T(\mathbf{z}\mathbf{x}) = T(\mathbf{z})T(\mathbf{x}) = Id$. And $N(\mathbf{z}\mathbf{x}) = \mathbf{z}\mathbf{x}\overline{\mathbf{z}}\,\overline{\mathbf{x}} = \mathbf{z}f_1(\mathbf{x})^2\overline{\mathbf{z}} = 1$. Therefore, by lemma (6.3), we get $\mathbf{z}\mathbf{x} = \pm 1$ or $\pm\omega$. Since $\mathbf{z}$ is decomposable, $\mathbf{x} = \pm\mathbf{z}^{-1}$ or $\pm\omega\mathbf{z}^{-1}$. Hence, $\mathbf{x}$ is also decomposable. $\square$

By the equation (31), we have the following expression for the speed of Minkowski Pythagorean hodograph curves. This result is a straightforward consequence of proposition 6.2.

Proposition 6.5. If $\gamma(t) = (x(t), y(t), z(t), r(t))$ is an MPH curve defined by polynomials $p_k(t),\ q_k(t)$ for $k = 0, \ldots, 3$, then the squared speed of $\gamma(t)$ measured under the Minkowski metric is expressed as

$$\left|\gamma'(t)\right|^2 = x'(t)^2 + y'(t)^2 + z'(t)^2 - r'(t)^2 = f_1(t)^2 + f_2(t)^2,$$

where

$$\begin{aligned} f_1(t) &= p_0(t)^2 + p_1(t)^2 + p_2(t)^2 + p_3(t)^2 - \big(q_0(t)^2 + q_1(t)^2 + q_2(t)^2 + q_3(t)^2\big), \\ f_2(t) &= 2\big(p_0(t)q_0(t) + p_1(t)q_1(t) + p_2(t)q_2(t) + p_3(t)q_3(t)\big). \end{aligned}$$

Therefore, if the spine curve $s(t)$ together with the radius function $r(t)$ is given by PH curve in $\mathbb{R}^{3,1}$, then the canal surface generated by $s(t)$ and $r(t)$ has a rational parameterization in a natural way. One can directly apply the parameterization method proposed in [23] without the decomposition step.

In fact, the converse of proposition 6.5 is true which can be proven in theorem 6.9.

6.2. *Dirac matrix representation*

We now investigate a matrix representation of $\mathcal{C}\ell(3,1)$. For reasons that will become clear as we proceed, we need to write down the matrix algebra version of the full Clifford algebra $\mathcal{C}\ell(3,1)$. Let $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_4$ be an orthonormal basis of $\mathbb{R}^{3,1}$ such that $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ are space-like and $\mathbf{e}_4$ is time-like (this is opposite to the usual convention used in physics). Thus we have:

$$\begin{aligned} &\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1, \qquad && \mathbf{e}_4^2 = 1, \\ &\mathbf{e}_{12}^2 = \mathbf{e}_{23}^2 = \mathbf{e}_{31}^2 = -1, && \mathbf{e}_{41}^2 = \mathbf{e}_{42}^2 = \mathbf{e}_{43}^2 = 1. \end{aligned}$$

It is well known that $\mathcal{C}\ell(3,1)$ over the complex field $\mathbb{C}$ is an algebra that is isomorphic to the 4×4 matrix algebra $\mathrm{M}(4,\mathbb{C})$. And $\mathcal{C}\ell^+(3,1)$ is isomorphic to a subalgebra of $\mathrm{M}(4,\mathbb{C})$, which is isomorphic to the biquaternion algebra $\mathbb{H} \times \mathbb{H}$. This isomorphism is achieved by use of the Pauli spin matrices introduced in section 4.2.

Let $F_{\mathbb{C}}$ be the complex algebra generated in $\mathrm{End}(\mathbb{C})$ by the four matrices γ_α, $\alpha = 0, 1, 2, 3$,

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}.$$

These are the Dirac matrices commonly used in physics but, as described above, the norm convention is different. Then the following lemma is easily proved (see [10]):

Lemma 6.6. The algebra generated by the four Dirac matrices over $\mathbb{C}$ is $\mathrm{M}(4, \mathbb{C})$.

The isomorphism ψ we look for from $\mathcal{C}\ell(3, 1)$ to the 4×4 complex matrix algebra $\mathrm{M}(4, \mathbb{C})$ is given by the algebra extension of the map ψ such that

$$\psi(\mathbf{e}_1) = \gamma_1, \qquad \psi(\mathbf{e}_2) = \gamma_2, \qquad \psi(\mathbf{e}_3) = \gamma_3, \qquad \psi(\mathbf{e}_4) = \gamma_4.$$

Then, by the above isomorphism and the Pauli relations, the elements of $\mathcal{C}\ell^+(3, 1)$ are explicitly represented as:

$$\psi(\mathbf{e}_{12}) = \gamma_1\gamma_2 = \begin{pmatrix} \mathrm{i}\sigma_1 & 0 \\ 0 & \mathrm{i}\sigma_1 \end{pmatrix}, \qquad \psi(\mathbf{e}_{31}) = \gamma_3\gamma_1 = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix},$$
$$\psi(\mathbf{e}_{23}) = \gamma_2\gamma_3 = \begin{pmatrix} \mathrm{i}\sigma_3 & 0 \\ 0 & \mathrm{i}\sigma_3 \end{pmatrix}, \qquad \psi(\mathbf{e}_{1234}) = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & -\mathrm{i}\sigma_0 \\ -\mathrm{i}\sigma_0 & 0 \end{pmatrix}.$$

Here we adopt a somewhat different matrix representation of quaternions from that in section 4.2. We interchange the role of $\mathbf{e}_{12}$ with that of $\mathbf{e}_{23}$, compared with section 4.2, for the convenience of numbering. For a quarternion $p = p_0 + p_1\mathbf{e}_{23} + p_2\mathbf{e}_{31} + p_3\mathbf{e}_{12} = z_1 + z_2\mathbf{e}_{31}$ with $z_1 = p_0 + p_1\mathrm{i}$, $z_2 = p_2 + p_3\mathrm{i}$, the matrix isomorphism $p \to \mathrm{M}_p$ is represented by

$$\mathrm{M}_p = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}.$$

Note that $\mathrm{M}_p\mathrm{M}_q = \mathrm{M}_{pq}$ and $\mathrm{M}_{\overline{p}} = \mathrm{M}_p^*$ where M^* is the conjugate transpose of the matrix M.

Then for an element $\mathbf{x} = p + wq = z_1 + z_2\mathbf{e}_{31} + \omega(z_3 + z_4\mathbf{e}_{31}) \in \mathcal{C}\ell^+(3, 1)$, where $z_1 = p_0 + p_1\mathrm{i}$, $z_2 = p_2 + p_3\mathrm{i}$, $z_3 = q_0 + q_1\mathrm{i}$, and $z_4 = q_2 + q_3\mathrm{i}$, the corresponding 4×4 matrix $\psi(\mathbf{x})$ is given by the following:

$$\psi(\mathbf{x}) = \psi\big(z_1 + z_2\mathbf{e}_{31} + \omega(z_3 + z_4\mathbf{e}_{31})\big) = \begin{pmatrix} \mathrm{M}_p & -\mathrm{i}\mathrm{M}_q \\ -\mathrm{i}\mathrm{M}_q & \mathrm{M}_p \end{pmatrix}.$$

And the corresponding matrix for the image $T_{\mathbf{e}_1}(x)$ is represented by the following:

$$\begin{aligned} \psi\big(\mathbf{x}\mathbf{e}_1\overline{\mathbf{x}}\big) &= \psi(\mathbf{x})\psi(\mathbf{e}_1)\psi\big(\overline{\mathbf{x}}\big) \\ &= \begin{pmatrix} \mathrm{M}_p & -\mathrm{i}\mathrm{M}_q \\ -\mathrm{i}\mathrm{M}_q & \mathrm{M}_p \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{M}_{\overline{p}} & -\mathrm{i}\mathrm{M}_{\overline{q}} \\ -\mathrm{i}\mathrm{M}_{\overline{q}} & \mathrm{M}_{\overline{p}} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}} + \overline{\mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}}} & -\mathrm{i}\big[\mathrm{M}_p(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}} + \mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{q}}\big] \\ -\mathrm{i}\big[\mathrm{M}_p(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}} + \mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{q}}\big] & \mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}} + \overline{\mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}}} \end{pmatrix}. \end{aligned}$$

Here we note that the forms $\mathrm{M}_p(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{p}}$ and $\mathrm{M}_q(\mathrm{i}\sigma_3)\mathrm{M}_{\overline{q}}$ above correspond to the images p and q by the PH representation map of $\mathcal{C}\ell^+(3)$ in section 4.

6.3. *PH characterization theorem in* $\mathbb{R}^{3,1}$

In this section, we shall investigate the PH representation map with a basis $\mathbf{e}_1$, $T_{\mathbf{e}_1}: \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$, more carefully. The main goal is the *PH characterization theorem* in $\mathbb{R}^{3,1}$ (theorem 6.9), which is the converse of proposition 6.5. To achieve this goal, we compute the isotropy subgroup of $T_{\mathbf{e}_1}$ in lemma 6.7 and compute the preimages of a given real polynomial curve $\gamma(t) = (x(t), y(t), z(t), r(t)) \in \mathbb{R}^{3,1}$ by $T_{\mathbf{e}_1}$ action.

We mainly use complex notations, which make the computations easier. We use the notation z_i to represent complex numbers. One should distinguish them from $z(t)$, which is a real polynomial as a coordinate function of $\gamma(t)$.

Let us rewrite $T_{\mathbf{e}_1}(\mathbf{x})$ using the complex notation. Let $\mathbf{x} = p + wq = z_1 + z_2\mathbf{e}_{31} + \omega(z_3 + z_4\mathbf{e}_{31})$, where $z_1 = p_0 + p_1\mathrm{i}$, $z_2 = p_2 + p_3\mathrm{i}$, $z_3 = q_0 + q_1\mathrm{i}$, and $z_4 = q_2 + q_3\mathrm{i}$. Then $T_{\mathbf{e}_1}(\mathbf{x})$ is represented by the following:

$$\begin{aligned} T_{\mathbf{e}_1}(\mathbf{x}) &= \mathbf{x}\mathbf{e}_1\overline{\mathbf{x}} = (p + \omega q)\mathbf{e}_1\left(\overline{p} + \omega\overline{q}\right) \\ &= \left(|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2\right)\mathbf{e}_1 + 2\,\mathrm{Im}(z_1z_2 + z_3z_4)\mathbf{e}_2 \\ &\quad - 2\,\mathrm{Re}(z_1z_2 + z_3z_4)\mathbf{e}_3 + 2\,\mathrm{Im}\left(z_1\overline{z_3} - z_2\overline{z_4}\right)\mathbf{e}_4. \end{aligned} \tag{32}$$

The following lemma describes the isotropy subgroup I of $T_{\mathbf{e}_1}(\mathbf{x})$.

Lemma 6.7. The isotropy subgroup I of $T_{\mathbf{e}_1}(\mathbf{x})$ is given by the following:

$$\begin{aligned} \mathrm{I} = \Bigg\{ &\mathbf{x} = p + \omega q \in \mathcal{C}\ell^+(3,1)\colon\ p = \tau w_1 - \mu w_2\mathbf{e}_{31},\ q = \mu w_1 + \tau w_2\mathbf{e}_{31}, \\ &|w_1|^2 - |w_2|^2 = \frac{1}{\tau^2 + \mu^2},\ w_1, w_2 \in \mathbb{C},\ \tau, \mu \in \mathbb{R},\ (\tau, \mu) \neq (0,0) \Bigg\}. \end{aligned}$$

Proof. By the equation (32), $T_{\mathbf{e}_1}(x) = \mathbf{e}_1$ implies the following system of equations:

$$\begin{aligned} |z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2 &= 1, \\ 2(z_1z_2 + z_3z_4) &= 0, \\ 2\,\mathrm{Im}\left(z_1\overline{z_3} - z_2\overline{z_4}\right) &= 0. \end{aligned} \tag{33}$$

By the second equation, we have $z_1z_2 = -z_3z_4$.

If $z_1z_2 \neq 0$, there exists a $\tau \in \mathbb{C} - \{0\}$ such that $z_1 = \tau z_3$ and $z_2 = -z_4/\tau$. By applying this to the third equation of (33), we find that τ is a real number. And we get $(\tau^2 + 1)(|z_3|^2 - |z_4|^2/|\tau|^2) = 1$ from the first equation of (33). Replacing z_4/τ to z_4 simplifies the representation of the isotropic elements in I to $\mathbf{x} = \tau z_3 - z_4\mathbf{e}_{31} + \omega(z_3 + \tau z_4\mathbf{e}_{31})$, where $\tau \in \mathbb{R}$, $z_3 \in \mathbb{C}$, and $z_4 \in \mathbb{C}$ satisfy $(\tau^2 + 1)(|z_3|^2 - |z_4|^2) = 1$.

If $z_1 = 0$, then we get two possibilities, $z_3 = 0$ or $z_4 = 0$, from the second equation of (33). The case $z_3 = 0$ is impossible by the first equation of (33). For the case $z_4 = 0$, we get $-|z_2|^2 + |z_3|^2 = 1$. This implies that the isotropic elements are given by $\mathbf{x} = z_3 + \omega(z_4\mathbf{e}_{31})$, where $z_3 \in \mathbb{C}$, $z_4 \in \mathbb{C}$, and $|z_3|^2 - |z_4|^2 = 1$.

If we reverse the role of z_3 and z_4 in the above calculation, we can get another result. All these varieties can be unified by the formula in the statement of lemma. □

Note that the algebraic dimension of I is not 5, but 4. In fact, we can derive a more simplified formula for elements in I by dropping μ or τ when $\mu \neq 0$ or $\tau \neq 0$. For example, when $\mu \neq 0$, by the transformation $\tau \to \tau/\mu$, $w_1 \to \mu w_1$ and $w_2 \to \mu w_2$, we get the simpler form

$$\mathbf{x} = \tau w_1 - w_2\mathbf{e}_{31} + \omega(w_1 + \tau w_2\mathbf{e}_{31}), \tag{34}$$

where $|w_1|^2 - |w_2|^2 = 1/(\tau^2+1)$, and $w_1, w_2 \in \mathbb{C}$. And if $\mu = 0$ or $\tau = 0$, theorem 6.4 implies the following corollary.

Corollary 6.8. If $\mu = 0$ or $\tau = 0$, that the corresponding elements in I are contained in Spin(3, 1).

Now we claim that this action is transitive: for a given

$$\gamma'(t) = \big(x'(t), y'(t), z'(t), r'(t)\big) \in \mathbb{R}^{3,1},$$

we can find $\mathbf{x} \in \mathcal{C}\ell^+(3,1)$ satisfying $T_{\mathbf{e}_1}(\mathbf{x}) = x'(t)\mathbf{e}_1 + y'(t)\mathbf{e}_2 + z'(t)\mathbf{e}_3 + r'(t)\mathbf{e}_4$. One can easily check that

$$\mathbf{x} = \frac{1}{2} + \big(-z' + y'\mathrm{i}\big)\mathbf{j} + \omega\big[s - r'i\big],$$

where $s = \sqrt{(y')^2 + (z')^2 - (r')^2 + x' - \frac{1}{4}}$, satisfies the relation $T_{\mathbf{e}_1}(\mathbf{x}) = \gamma'$. Note that above candidate is valid only when s is a real number, i.e., for the case that $(y')^2 + (z')^2 - (r')^2 + x' - \frac{1}{4} \leqslant 0$. When $(y')^2 + (z')^2 - (r')^2 + x' - \frac{1}{4} \geqslant 0$, we can find a somewhat different candidate directly. But here we note that this sign problem is not significant, and the above candidate is just a transient formal solution which finally leads us to find rational or polynomial candidate as in theorem 6.9, where we do not worry about the problem of sign and logarithmic branch.

So another pre-image of $\gamma'(t)$ is given by the product of the chosen candidate with an element of the isotropy subgroup I in lemma 6.7: any element in the pre-image of γ' by $T_{\mathbf{e}_1}$ looks like $[z_1 + z_2\mathbf{e}_{31}] + \omega[z_3 + z_4\mathbf{e}_{31}]$, where

$$\begin{aligned}
z_1 &= \left(\frac{\tau}{2} - s + r'\mathrm{i}\right)w_1 + \big(-z' + y'\mathrm{i}\big)\overline{w_2},\\
z_2 &= \big(-z' + y'\mathrm{i}\big)\tau\overline{w_1} - \left(\frac{1}{2} + \big(s - r'\mathrm{i}\big)\tau\right)w_2,\\
z_3 &= \left(\frac{1}{2} + \big(s - r'\mathrm{i}\big)\tau\right)w_1 - \big(-z' + y'\mathrm{i}\big)\tau\overline{w_2},\\
z_4 &= \big(-z' + y'\mathrm{i}\big)\overline{w_1} + \left(\frac{\tau}{2} - s + r'\mathrm{i}\right)w_2.
\end{aligned} \tag{35}$$

The next theorem is the converse of proposition 6.5, and is called the *PH characterization theorem in* $\mathbb{R}^{3,1}$.

Theorem 6.9 (Solution of PH characterization problem for $\mathbb{R}^{3,1}$). Let $\gamma(t) = (x(t), y(t), z(t), r(t))$ be a space-like polynomial curve in the 4-dimensional Minkowski space $\mathbb{R}^{3,1}$. Then it is a Pythagorean hodograph curve in $\mathbb{R}^{3,1}$, that is, there exist polynomial curves $p_i(t)$ and $q_i(t)$ for $i = 1, \ldots, 3$ such that $\mathbf{x}(t) = p_0(t) + p_1(t)\mathbf{e}_{23} + p_2(t)\mathbf{e}_{31} + p_3(t)\mathbf{e}_{12} + \omega(q_0(t) + q_1(t)\mathbf{e}_{23} + q_2(t)\mathbf{e}_{31} + q_3(t)\mathbf{e}_{12}) \in \mathcal{C}\ell^+(3,1)[t]$ so that $\gamma'(t)$ is given by (31).

Proof. Since γ is assumed to be space-like,

$$\left|\gamma'(t)\right|^2 = x'(t)^2 + y'(t)^2 + z'(t)^2 - r'(t)^2$$

is a positive real polynomial. Thus by the result of Pottmann and Peternell [23], there exist two polynomials $f_1(t)$ and $f_2(t)$ such that

$$\left|\gamma'(t)\right|^2 = f_1(t)^2 + f_2(t)^2.$$

To find the polynomials $p_k(t), q_k(t)$ for $k = 0, \ldots, 3$ for which $T_{\mathbf{e}_1}(p(t) + wq(t)) = x'(t)\mathbf{e}_1 + y'(t)\mathbf{e}_2 + z'(t)\mathbf{e}_3 + r'(t)\mathbf{e}_4$, we must find the τ, w_1, and w_2 satisfying $|w_1|^2 - |w_2|^2 = 1/(\tau^2+1)$ which make z_1, z_2, z_3, z_4 in (35) polynomials. To simplify the notation, we drop primes ($'$) and use x, y, z, and r instead of x', y', z', and r'. The four equations in (35) can be joined to result in the following two matrix equations:

$$\begin{pmatrix} \dfrac{\tau}{2} - s + r\mathrm{i} & -z + y\mathrm{i} \\ -z - y\mathrm{i} & \dfrac{\tau}{2} - s - r\mathrm{i} \end{pmatrix} \begin{pmatrix} w_1 \\ \overline{w_2} \end{pmatrix} = \begin{pmatrix} z_1 \\ \overline{z_4} \end{pmatrix}, \tag{36}$$

$$\begin{pmatrix} \tau(z + y\mathrm{i}) & \dfrac{1}{2} + (s + r\mathrm{i})\tau \\ \dfrac{1}{2} + (s - r\mathrm{i})\tau & \tau(z - y\mathrm{i}) \end{pmatrix} \begin{pmatrix} w_1 \\ \overline{w_2} \end{pmatrix} = \begin{pmatrix} -\overline{z_2} \\ z_3 \end{pmatrix}. \tag{37}$$

Note that the left-hand matrices in (36) and (37) are in SU(1, 1), with some multiplication factors.

From (36) with the equation $|w_1|^2 - |w_2|^2 = 1/(\tau^2+1)$, we get

$$|z_1|^2 - |z_4|^2 = \frac{r^2 + (\tau/2 - s)^2 - y^2 - z^2}{\tau^2 + 1},$$

which should be a polynomial. Let us denote this by k. Here we note that the determinant of the matrix in (36) is $k(\tau^2+1)$, which cannot be zero if $k \neq 0$. So the inverse of the matrix can exist if $k \neq 0$. We use the formal inverse of the matrix, which can be guaranteed by proper choice of k below.

Let us consider the relation between τ and k. When k is given, τ is formally given by

$$\tau = \frac{-2s \pm \sqrt{4s^2 - (4k-1)(4k+1-4x)}}{4k-1}. \tag{38}$$

And in the reverse order, this τ makes $|z_1|^2 - |z_4|^2 = k$.

Applying this τ to (37), we obtain another polynomial equation. By direct computation, we have the following equation:

$$|z_2|^2 - |z_3|^2 = k - x.$$

The next step is that of relating (z_1, z_4) in (36) to (z_2, z_3) in (37).

By taking the formal inverse of (36), we get $(w_1, \overline{w_2})$, and by applying these $(w_1, \overline{w_2})$ to (37), we get a very complicated matrix equation:

$$\begin{pmatrix} \tau(z+y\mathrm{i}) & \frac{1}{2} + (s+r\mathrm{i})\tau \\ \frac{1}{2} + (s-r\mathrm{i})\tau & \tau(z-y\mathrm{i}) \end{pmatrix} \begin{pmatrix} \frac{\tau}{2} - s + r\mathrm{i} & -z + y\mathrm{i} \\ -z - y\mathrm{i} & \frac{\tau}{2} - s - r\mathrm{i} \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ \overline{z_4} \end{pmatrix} = \begin{pmatrix} -\overline{z_2} \\ z_3 \end{pmatrix}.$$

After some calculation, we get

$$\frac{1}{E} \begin{pmatrix} (z+y\mathrm{i})\frac{1+\tau^2}{2} & \frac{\tau}{2} - \tau x + \frac{(\tau^2-1)s}{2} + \frac{(\tau^2+1)r\mathrm{i}}{2} \\ \frac{\tau}{2} - \tau x + \frac{(\tau^2-1)s}{2} - \frac{(\tau^2+1)r\mathrm{i}}{2} & (z-y\mathrm{i})\frac{\tau^2+1}{2} \end{pmatrix} \begin{pmatrix} z_1 \\ \overline{z_4} \end{pmatrix} = \begin{pmatrix} -\overline{z_2} \\ z_3 \end{pmatrix}$$

where $E = x + \tau^2/4 - \tau s - 1/4$. And by applying τ to (38), we get

$$\begin{pmatrix} \frac{z+y\mathrm{i}}{2k} & m_{12} \\ \overline{m_{12}} & \frac{z-y\mathrm{i}}{2k} \end{pmatrix} \begin{pmatrix} z_1 \\ \overline{z_4} \end{pmatrix} = \begin{pmatrix} -\overline{z_2} \\ z_3 \end{pmatrix},$$

where m_{12} is given by the following

$$\begin{aligned} &\Big(s\big(2y^2 + 2z^2 - 2r^2 + 8kx - 8k^2\big) + r\mathrm{i}\big(x + 2y^2 + 2z^2 - 2r^2 - 2k + 4kx\big) \\ &\quad - \big(x + 2y^2 + 2z^2 - 2r^2 - 2k + 4kx + 2sr\mathrm{i}\big)\sqrt{4s^2 - (4k-1)(4k+1-4x)}\Big) \\ &\quad \Big/ \Big(2k\big(-2r^2 + x + k(-2+4x) + 2y^2 + 2z^2 - 2s\sqrt{4s^2 - (4k-1)(4k+1-4x)}\big)\Big), \end{aligned}$$

and $\overline{m_{12}}$ is a complex conjugate of m_{12}.

By this matrix relation, polynomials z_1 and z_4 should generate polynomials z_2 and z_3. This can be achieved by proper choices of k. By varying the polynomial k, one can ascertain that a polynomial matrix can be obtained only when $k = (x \pm f_2)/2$ or $k = (x \pm f_1)/2$. Here we note that all these four choices of k cannot be zero at the same time at each t. From now on, let us assume that $(x + f_1)/2$ is not zero.

If we adopt $k = (x + f_1)/2$ and apply this to the matrix mentioned above, we get the following simple matrix relation of $z_1,\ z_2,\ z_3$, and z_4:

$$\frac{1}{x + f_1}\begin{pmatrix} z + y\mathrm{i} & -f_2 + r\mathrm{i} \\ -f_2 - r\mathrm{i} & z - y\mathrm{i} \end{pmatrix}\begin{pmatrix} z_1 \\ \overline{z_4} \end{pmatrix} = \begin{pmatrix} -\overline{z_2} \\ z_3 \end{pmatrix}, \tag{39}$$

where $|z_1|^2 - |z_4|^2 = (x + f_1)/2$. Note that there are many pairs of polynomials (z_1, z_4) which satisfy $|z_1|^2 - |z_4|^2 = (x + f_1)/2$.

Here we have two versions: one is a rational version where z_1 and z_4 are polynomials and z_3 and z_4 are rationals. The other is a polynomial version where all $z_1,\ z_2,\ z_3$, and z_4 are polynomials.

If we want rational pre-images of γ by T_{e_1}, then we are already done. The procedure is as follows: we can choose z_1 and z_2 from many polynomial candidates satisfying the relation $|z_1|^2 - |z_4|^2 = (x + f_1)/2$, and apply this to (39) to get two rational functions z_2 and z_3. Finally, we get the rational pre-images of the form $z_1 + z_2\mathbf{e}_{31} + \omega(z_3 + z_4\mathbf{e}_{31})$.

For the polynomial version, we need the well-known *Euclidean Algorithm*. Letting $z_1 = a + b\mathrm{i}$ and $\overline{z_4} = c + d\mathrm{i}$, we obtain the following four equations for $\mathrm{Re}(-\overline{z_2}),\ \mathrm{Im}(-\overline{z_2}),\ \mathrm{Re}(z_3)$, and $\mathrm{Im}(z_4)$ from (39), which should be polynomials:

$$\frac{za - yb - f_2c - rd}{x + f_1}, \qquad \frac{ya + zb + rc - f_2d}{x + f_1},$$
$$\frac{-f_2a + rb + zc + yd}{x + f_1}, \qquad \frac{-ra - f_2b - yc + zd}{x + f_1},$$

where $a^2 + b^2 - c^2 - d^2 = (x + f_1)/2$ and $y^2 + z^2 - r^2 - f_2^2 = f_1^2 - x^2$.

Adding and extracting each of two equations and rearranging the above four equations, we get the following four equations which should be polynomials and be divided by $(x + f_1)$.

$$\begin{aligned} &(z - f_2)(a + c) - (y - r)(b - d), \\ &(z + f_2)(a - c) - (y + r)(b + d), \\ &(y - r)(a - c) + (z - f_2)(b + d), \\ &(y + r)(a + c) + (z + f_2)(b - d). \end{aligned} \tag{40}$$

First, if $\gcd(z - f_2, y - r) = 1$ where $\gcd(\cdot,\cdot)$ is the greatest common divisor of the two elements, we can find polynomials A and B such that $A\cdot(z - f_2) + B\cdot(y - r) = 1$ by the Euclidean Algorithm. And these A and B are applied to the following four equations to get $a,\ b,\ c$, and d:

$$a + c = \frac{A(x + f_1)}{2}, \qquad b - d = -\frac{B(x + f_1)}{2},$$
$$a - c = z - f_2, \qquad b + d = -y + r.$$

Then these choices of a, b, c, and d satisfy the required conditions (40).

Second, if $\gcd(z-f_2, y-r) = g$, then there exist polynomials A' and B' satisfying $A'(z-f_2) + B'(y-r) = g$ by the Euclidean Algorithm. And to find a, b, c, and d, these A' and B' are applied to the following equations:

$$a+c = \frac{A'(x+f_1)}{2}, \qquad b-d = -\frac{B'(x+f_1)}{2},$$
$$a-c = \frac{z-f_2}{g}, \qquad b+d = \frac{-y+r}{g}.$$

All these are polynomials satisfying $a^2+b^2-c^2-d^2 = (x+f_1)/2$. And these choices of $a,\ b,\ c$, and d satisfy the required conditions of (40). $\square$

We have also a rational version of theorem 6.9, as we can see in the proof of the above theorem, which is much easier and more convenient when we begin with a rational curve $\gamma(t) = (x(t), y(t), z(t), r(t))$.

Corollary 6.10. Let $\gamma(t) = (x(t), y(t), z(t), r(t))$ be a space-like rational polynomial curve in the four dimensional Minkowski space $\mathbb{R}^{3,1}$. Then it is a Pythagorean hodograph curve in $\mathbb{R}^{3,1}$, that is, there exist rational curves $p_i(t)$ and $q_i(t)$ for $i = 1, \ldots, 3$ such that $\mathbf{x}(t) = p_0(t)+p_1(t)\mathbf{e}_{23}+p_2(t)\mathbf{e}_{31}+p_3(t)\mathbf{e}_{12}+\omega(q_0(t)+q_1(t)\mathbf{e}_{23}+q_2(t)\mathbf{e}_{31}+q_3(t)\mathbf{e}_{12}) \in \mathcal{C}\ell^+(3,1)(t)$ so that $\gamma'(t)$ is given by (31).

Remark 6.11 (Rational version of PH characterization problem). We can apply the approach like that of corollary 6.10 to all results of the PH characterization problems in this paper, i.e., we can also have rational versions for theorems 3.1, 4.2 and 5.2. These rational versions are much easier and more convenient when we begin with a rational curve.

Remark 6.12 (PH characterization problem for general dimensions). It is not easy to deal with the PH characterization problem in general dimensions, because of some obstacles which do not appear in the dimensions less than 4. Here we suggest the method of the general approach.

We first look at the feature of the PH curve $\gamma(t) \in V[t]$ whose hodograph is written by $\gamma'(t) = T_a((\mathbf{x}(t)))$ for some $\mathbf{x}(t) \in \Lambda(V,Q)[t]$. The subalgebra $\Lambda(V,Q)$ is not simple as in the case of $\dim(V) \leqslant 4$ where we have just simple results where $\Lambda(V,Q) = \mathcal{C}\ell^+(V,Q)$. For example, we have $\Lambda(V,Q) \subsetneq \mathcal{C}\ell^+(V,Q)$ for $V = \mathbb{R}^5$ as we show in example 2.2.

For the next step, we should find some squaring conditions which are related to the norm of this hodograph:

$$\begin{aligned}\langle\gamma'(t),\gamma'(t)\rangle &= Q(\gamma'(t)) = |\gamma(t)|_Q^2 = -(T_a(\mathbf{x}))\overline{(T_a(\mathbf{x}))}\\ &= -\mathbf{x}(t)a\overline{\mathbf{x}(t)}x(t)a\overline{\mathbf{x}(t)}.\end{aligned}$$

In $\dim(V) \leqslant 4$, it is just $N(\mathbf{x}(t))^2$, where the norm is the square of a single polynomial or the sum of two polynomials. What does this look like? In this paper we suggest this general approach and defer the solutions for another paper.

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