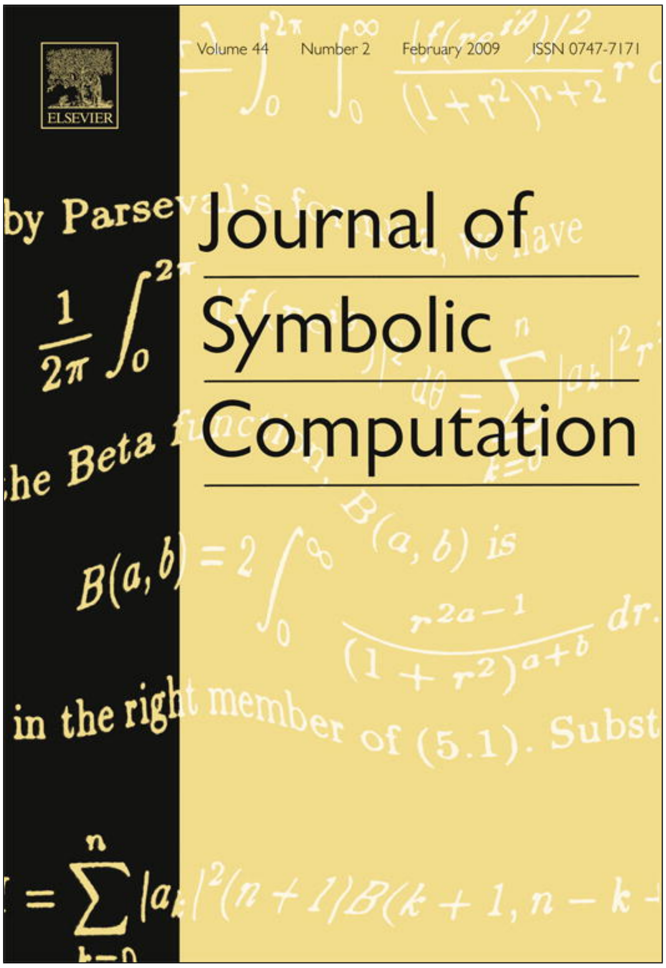
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Helical polynomial curves and double Pythagorean hodographs I. Quaternion and Hopf map representations

Rida T. Farouki [a,1], Carlotta Giannelli [b], Alessandra Sestini [c]

[a] Department of Mechanical and Aeronautical Engineering, University of California, Davis, CA 95616, USA
[b] Dipartimento di Sistemi e Informatica, Università degli Studi di Firenze, Viale Morgagni 65, 50134 Firenze, Italy
[c] Dipartimento di Matematica "Ulisse Dini", Università degli Studi di Firenze, Viale Morgagni 67a, 50134 Firenze, Italy

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ABSTRACT

For regular polynomial curves $\mathbf{r}(t)$ in $\mathbb{R}^3$, relations between the helicity condition, existence of rational Frenet frames, and a certain "double" Pythagorean-hodograph (PH) structure are elucidated in terms of the quaternion and Hopf map representations of spatial PH curves. After reviewing the definitions and properties of these representations, and conversions between them, linear and planar PH curves are identified as degenerate spatial PH curves by certain linear dependencies among the coefficients. Linear and planar curves are trivially helical, and all proper helical polynomial curves are PH curves. All spatial PH cubics are helical, but not all PH quintics. The two possible types of helical PH quintic (monotone and general) are identified as subsets of the PH quintics by constraints on their quaternion coefficients. The existence of a rational Frenet frame and curvature on polynomial space curves is equivalent to a certain "double" PH form, first identified by Beltran and Monterde, in which $|\mathbf{r}'(t)|$ and $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ are *both* polynomials in t. All helical PH curves are double PH curves, which encompass all PH cubics and all helical PH quintics, although non-helical double PH curves of higher order exist. The "double" PH condition is thoroughly analyzed in terms of the quaternion and Hopf map forms, and their connections. A companion paper presents a complete characterization of all helical and non-helical double PH curves up to degree 7.

E-mail addresses: farouki@ucdavis.edu (R.T. Farouki), giannelli@dsi.unifi.it (C. Giannelli), sestini@math.unifi.it (A. Sestini).
[1] Tel.: +1 530 752 1779; fax: +1 530 752 4153.

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1. Introduction

At each point of a space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ satisfying $\mathbf{r}'(t) \neq \mathbf{0}$ and $\mathbf{r}'(t) \times \mathbf{r}''(t) \neq \mathbf{0}$ for all t, the Frenet frame $(\mathbf{t}, \mathbf{p}, \mathbf{b})$ specified (Kreyszig, 1959) by

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \qquad \mathbf{p} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}, \qquad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \tag{1}$$

defines an orthonormal basis for $\mathbb{R}^3$ in terms of the local intrinsic geometry of the curve. The *tangent* $\mathbf{t}$ indicates the instantaneous direction of motion along the curve; the *principal normal* $\mathbf{p}$ points toward the center of curvature; and the *binormal* $\mathbf{b} = \mathbf{t} \times \mathbf{p}$ completes the right-handed frame.

The variation of the Frenet frame on $\mathbf{r}(t)$ is described by the *Frenet–Serret equations* (Kreyszig, 1959) in which the *curvature* and *torsion* functions,

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \qquad \text{and} \qquad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}, \tag{2}$$

appear. For general polynomial or rational curves $\mathbf{r}(t)$, however, the Frenet frame and the curvature do not exhibit a rational dependence on the curve parameter (although the torsion is rational)—a desirable property for use in computer-aided geometric design, robot motion planning, computer graphics and animation, and many other application contexts.

The spatial *Pythagorean-hodograph* (PH) *curves* (Farouki et al., 2002a,b; Farouki and Sakkalis, 1994) are polynomial curves characterized by the fact that their "parametric speed" $\sigma(t) = |\mathbf{r}'(t)|$ is a *polynomial* in the parameter t. This property is achieved by the *a priori* incorporation of a Pythagorean structure in the components of the hodograph (derivative) $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$. Thus, the tangent vector $\mathbf{t}$ of a spatial PH curve has a rational dependence on t, but the normal vectors $\mathbf{p}$ and $\mathbf{b}$ do not, since they depend upon the quantity $|\mathbf{r}' \times \mathbf{r}''|$. Also, the torsion τ is generically a rational function of t, but the curvature κ is not.

In order to secure a rational dependence of $(\mathbf{t}, \mathbf{p}, \mathbf{b})$ and κ, τ on the curve parameter, it is necessary to incorporate Pythagorean structures into both the first hodograph $\mathbf{r}'(t)$ *and* the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ of the first and second hodographs. The significance of this "double" PH structure was first noted by Beltran and Monterde (2007), in a study of helical polynomial curves. A *helix* is characterized (Kreyszig, 1959) by the fact that its tangent $\mathbf{t}$ maintains a constant inclination ψ (the *pitch angle*) with respect to a fixed unit vector $\mathbf{a}$ (the *axis*) in $\mathbb{R}^3$, so that

$$\mathbf{a} \cdot \mathbf{t} = \cos\psi. \tag{3}$$

Equivalently, the curvature and torsion have (Kreyszig, 1959) the constant ratio[2]

$$\frac{\kappa}{\tau} = \tan\psi. \tag{4}$$

As noted in Farouki et al. (2004), the satisfaction of (3) implies that every helical polynomial curve must be a PH curve. It was also implicitly noted in Farouki et al. (2004) – and more explicitly emphasized in Beltran and Monterde (2007) – that for a helical polynomial curve the quantity $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ must be a *polynomial* in t. Thus, all helical polynomial curves must be *double PH curves*, not just PH curves. However, the sets of helical polynomial curves and of double PH curves are not coincident—an example of a *non-helical* double PH curve of degree 7 is given in Beltran and Monterde (2007). Hence, the helical polynomial curves comprise a proper subset of the double PH curves.

Subsequently, (Monterde, in press) proposed an elegant general construction for helical polynomial curves of arbitrary degree, based on the Hopf map model for spatial PH curves and the property that helical curves exhibit a circular tangent indicatrix on the unit sphere, as observed in

[2] Since κ is by definition non-negative, but τ is a signed quantity, the constant in (4) may change sign at special curve points where $\kappa = \tau = 0$.

Farouki et al. (2004). The Hopf map and quaternion models[3] for spatial PH curves were introduced simultaneously by Choi et al. (2002). Although the quaternion model has since enjoyed greater use (Farouki et al. (2002b, 2008) and Šír and Jüttler (2005, 2007)) the study of double PH curves has relied more (Beltran and Monterde, 2007; Monterde, in press) on the Hopf map form, which appears somewhat better suited to this context. However, since the two forms are essentially equivalent, results obtained using one of them should be unambiguously transferrable to the other.

This two-part paper presents a more comprehensive treatment of helical polynomial curves and double PH curves, with emphasis on the relationship between the quaternion and Hopf map representations, and the enumeration of all curve types up to degree 7. This first paper begins with a review of the quaternion and Hopf models in Sections 2 and 3, while conversions between these forms are treated in Section 4. Conditions that incur *linear* and *planar* degenerations of spatial PH curves are then identified in Section 5, using the quaternion representation, since such curves are trivially helical.

The double PH condition is then discussed in Section 6, and analyzed in the context of the quaternion and Hopf map models of spatial PH curves in Sections 7 and 8, respectively. Section 9 extends the methods of Section 5 to proper helical PH quintics. Finally, Section 10 summarizes the main results of this paper. A companion paper (Farouki et al., in press) gives a complete enumeration of all the double PH curve types (helical and non-helical) up to degree 7, and includes methods for their construction and computed examples.

2. Quaternion form of spatial PH curves

The components of a Pythagorean hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ in $\mathbb{R}^3$ can be expressed (Choi et al., 2002; Dietz et al., 1993; Farouki et al., 2002a) in terms of four polynomials $u(t), v(t), p(t), q(t)$ as

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ y'(t) &= 2[u(t)q(t) + v(t)p(t)], \\ z'(t) &= 2[v(t)q(t) - u(t)p(t)], \end{aligned} \tag{5}$$

with corresponding polynomial parametric speed

$$\sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t), \tag{6}$$

such that

$$x'^2(t) + y'^2(t) + z'^2(t) \equiv \sigma^2(t).$$

If the polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ are of degree m at most, the PH curve $\mathbf{r}(t)$ obtained by integrating the hodograph $\mathbf{r}'(t)$ is of *odd* degree, $n = 2m + 1$.

The hodograph (5) admits (Choi et al., 2002; Farouki et al., 2002a) a compact description using the algebra of quaternions (see Chapters 5 and 22 of Farouki (2008) for a review). Namely, we write

$$\mathbf{r}'(t) = \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t), \tag{7}$$

where[4]

$$\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k} = \sum_{l=0}^{m} \mathcal{A}_l \binom{m}{l} (1-t)^{m-l} t^l \tag{8}$$

denotes a quaternion polynomial, of degree $m = \frac{1}{2}(n-1)$ for a PH curve of degree n, specified by the Bernstein coefficients

$$\mathcal{A}_l = u_l + v_l\mathbf{i} + p_l\mathbf{j} + q_l\mathbf{k}, \quad l = 0, \ldots, m. \tag{9}$$

[3] Yet another formulation of the spatial PH curves is based on the *geometric product* of Clifford algebra—see Perwass et al. (2007) for complete details.

[4] Quaternions are denoted by calligraphic letters, complex numbers by bold letters, and real numbers by italic letters. Bold letters are also employed to denote vectors in $\mathbb{R}^3$—the meaning should be clear from the context.

$\mathcal{A}^*(t) = u(t) - v(t)\mathbf{i} - p(t)\mathbf{j} - q(t)\mathbf{k}$ in (7) is the conjugate of $\mathcal{A}(t)$. In terms of the component polynomials $u(t), v(t), p(t), q(t)$ of $\mathcal{A}(t)$, we have

$$\begin{aligned}\mathbf{r}'(t) &= [u^2(t) + v^2(t) - p^2(t) - q^2(t)]\mathbf{i} \\ &\quad +2[u(t)q(t) + v(t)p(t)]\mathbf{j} + 2[v(t)q(t) - u(t)p(t)]\mathbf{k}.\end{aligned} \tag{10}$$

Note that a given Pythagorean hodograph $\mathbf{r}'(t)$ is generated through (7) by a *one-parameter family* of quaternion polynomials (Farouki et al., 2002a)—exactly the same hodograph is obtained by replacing any chosen $\mathcal{A}(t)$ with $\mathcal{A}(t)\mathcal{Q}(\phi)$, where $\mathcal{Q}(\phi) = \cos\phi + \sin\phi\,\mathbf{i}$ satisfies $\mathcal{Q}(\phi)\mathbf{i}\mathcal{Q}^*(\phi) = \mathbf{i}$ for $0 \le \phi \le 2\pi$. Moreover, the angular variable may be specified as a function $\phi(t)$ of the curve parameter t without any change in the hodograph (7).

Remark 1. A *primitive* hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ is characterized by the fact that $\gcd(x'(t), y'(t), z'(t)) =$ constant. Primitive hodographs are preferred in practice, since a common real root of $x'(t)$, $y'(t)$, $z'(t)$ generally incurs a *cusp* (sudden tangent reversal) on the curve $\mathbf{r}(t)$. However, choosing relatively prime polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ as components of $\mathcal{A}(t)$ does not guarantee a primitive $\mathbf{r}'(t)$. It can be shown (Farouki et al., 2004) that for relatively prime real polynomials u, v, p, q the common factor (if any) of x', y', z' is given by

$$\gcd(x', y', z') = |\gcd(u + \mathrm{i}v, p - \mathrm{i}q)|^2. \tag{11}$$

This defines a *real* even-degree polynomial $h(t)$, with no real roots. A non-primitive spatial Pythagorean hodograph can thus be written in the form

$$h(t)\,\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t) \tag{12}$$

for a suitable quaternion polynomial $\mathcal{B}(t)$, of degree $m - r$ when $\deg(h) = 2r$. Of course, if $u(t), v(t), p(t), q(t)$ are not relatively prime, $\gcd(u, v, p, q)$ will also contribute to $h(t)$ in (12).

The unit tangent $\mathbf{t}$ to a polynomial curve has a rational dependence on the parameter if and only if the curve hodograph is Pythagorean. It is defined in terms of the polynomials $u(t), v(t), p(t), q(t)$ and $\sigma(t)$ by

$$\mathbf{t} = \frac{\mathbf{r}'}{|\,\mathbf{r}'\,|} = \frac{(u^2 + v^2 - p^2 - q^2, 2(uq + vp), 2(vq - up))}{\sigma}. \tag{13}$$

Hence a *helical* polynomial curve, that satisfies (3), must be a PH curve (Farouki et al., 2004).

However, the principal normal $\mathbf{p}$ and binormal $\mathbf{b}$ defined by (1) are not, in general, rational unit vectors since the quantity $|\,\mathbf{r}' \times \mathbf{r}''\,|$ generically incurs the square root of a polynomial. Likewise, the curvature κ given by (2) does not, in general, have a rational dependence on t (although the torsion τ does). To secure a rational dependence of $(\mathbf{t}, \mathbf{p}, \mathbf{b})$ and κ, τ on t, we must consider the *double* PH curves—see Section 6.

3. Hopf map form of spatial PH curves

As an alternative to the quaternion representation, Choi et al. (2002) observed that the spatial Pythagorean hodograph (5) can be generated from a pair of complex polynomials through the *Hopf map*, $\mathbb{C} \times \mathbb{C} \to \mathbb{R}^3$. This map can be regarded as associating points $\mathbf{p} = (x, y, z) \in \mathbb{R}^3$ with complex number pairs $\boldsymbol{\alpha} = u + \mathrm{i}v$ and $\boldsymbol{\beta} = q + \mathrm{i}p$ according to

$$\mathbf{p} = H(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (|\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}), 2\,\mathrm{Im}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}})). \tag{14}$$

When we restrict (14) to complex numbers satisfying $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2 = 1$, it can be interpreted as a map between the "3-sphere" $S^3 : u^2 + v^2 + p^2 + q^2 = 1$ in the space $\mathbb{R}^4$ spanned by coordinates (u, v, p, q), and the familiar "2-sphere" $S^2 : x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ with (x, y, z) as coordinates. Thus, for example, the *great circles* of S^3 are mapped to *points* of S^2 by (14).

One can easily verify that the hodograph $\mathbf{r}'(t)$ defined by (5) is generated from the complex polynomials $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}v(t)$ and $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}p(t)$ as

$$\mathbf{r}'(t) = H(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)). \tag{15}$$

As with the quaternion form, the relationship between $\mathbf{r}'(t)$ and $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ is not one-to-one: we generate exactly the same hodograph on replacing the latter by $\boldsymbol{\alpha}(t)(\cos\phi + \mathrm{i}\sin\phi)$ and $\boldsymbol{\beta}(t)(\cos\phi + \mathrm{i}\sin\phi)$ for $0 \le \phi \le 2\pi$.

Remark 2. The hodograph (15) is primitive if and only if $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) = \text{constant}$. When $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ have a non-constant common factor $\mathbf{w}(t)$, we may write $\mathbf{r}'(t) = |\mathbf{w}(t)|^2 H(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t))$ where $\boldsymbol{\alpha}(t) = \mathbf{w}(t)\tilde{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\beta}(t) = \mathbf{w}(t)\tilde{\boldsymbol{\beta}}(t)$. The common factor $\mathbf{w}(t)$ influences only the *magnitude* of the hodograph—$\tilde{\boldsymbol{\alpha}}(t)$ and $\tilde{\boldsymbol{\beta}}(t)$ alone determine its direction.

The magnitude of the hodograph (15) is simply $|\mathbf{r}'(t)| = |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2$ but the orientational dependence of $\mathbf{r}(t)$ on $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ has a less-intuitive interpretation than the quaternion model. Writing $\mathcal{A}(t) = |\mathcal{A}(t)|(\cos\frac{1}{2}\theta(t) + \sin\frac{1}{2}\theta(t)\mathbf{n}(t))$ in the latter context, we may identify $|\mathcal{A}(t)|^2$ as the magnitude of the hodograph $\mathbf{r}'(t)$, while its orientation is obtained by rotating the vector $\mathbf{i}$ through angle $\theta(t)$ about the unit vector $\mathbf{n}(t)$.

As observed by Monterde (in press), however, the Hopf map can often be more convenient for the study of the *double* PH curves—and especially the *helical* polynomial curves that form a proper subset of them. The *tangent indicatrix* of the curve specified by (15) is the locus on the unit sphere defined by

$$\frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{(|\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}), 2\,\mathrm{Im}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}))}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{(|\mathbf{z}|^2 - 1, 2\,\mathrm{Re}(\mathbf{z}), 2\,\mathrm{Im}(\mathbf{z}))}{|\mathbf{z}|^2 + 1},$$

where we set $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$. Hence, the tangent indicatrix of a spatial PH curve evidently does not depend on the complex polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ *individually*, only on their *ratio*. Monterde notes (Monterde, in press) that the last expression above specifies the inverse of the *stereographic projection*, mapping points $\mathbf{z}$ of the (extended) complex plane to points $\mathbf{r}'/|\mathbf{r}'|$ on the unit sphere. Since helical curves have circular tangent indicatrices on the unit sphere (Farouki et al., 2004), with pre-images under the inverse stereographic projection that are lines or circles (Needham, 1997; Schwerdtfeger, 1979), the Hopf map model offers a very natural and intuitive approach to the analysis and construction of helical polynomial curves.

4. Conversion between representations

By identifying the imaginary unit i with the quaternion basis element $\mathbf{i}$, the polynomial $\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$ in the quaternion form (7) can be expressed in terms of the complex polynomials $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}v(t)$ and $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}p(t)$ in the Hopf map form (15) as

$$\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\boldsymbol{\beta}(t). \tag{16}$$

Conversely, we can obtain $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ from $\mathcal{A}(t)$ through the expressions

$$\boldsymbol{\alpha}(t) = \tfrac{1}{2}[\mathcal{A}(t) - \mathbf{i}\mathcal{A}(t)\mathbf{i}], \qquad \boldsymbol{\beta}(t) = -\tfrac{1}{2}\mathbf{k}[\mathcal{A}(t) + \mathbf{i}\mathcal{A}(t)\mathbf{i}]. \tag{17}$$

Of course, expressions (16) and (17) are actually specific instances among the one-parameter family of quaternion polynomials $\mathcal{A}(t)$ or complex polynomial pairs $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ that define a given hodograph $\mathbf{r}'(t)$ through (7) or (15).

Given a Pythagorean hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ the quaternion pre-image $\mathcal{A}(t)$ under the map $\mathbb{H} \to \mathbb{R}^3$ defined by (7) can be expressed as

$$\begin{aligned}\mathcal{A}(t) &= \sqrt{\tfrac{1}{2}(\sigma(t) + x'(t))}\,\bigg[-\sin\phi + \cos\phi\,\mathbf{i} \\ &\quad + \frac{y'(t)\cos\phi + z'(t)\sin\phi}{\sigma(t) + x'(t)}\mathbf{j} + \frac{z'(t)\cos\phi - y'(t)\sin\phi}{\sigma(t) + x'(t)}\mathbf{k}\bigg],\end{aligned}$$

where $\sigma(t) = |\mathbf{r}'(t)|$, and ϕ is a free angular parameter. For each t, the above relation identifies the pre-image of a given point $\mathbf{r}'(t)$ in $\mathbb{R}^3$ as a circle in the quaternion space $\mathbb{H}$, traced by increasing ϕ from

0 to 2π. For the Hopf map $\mathbb{C}\times\mathbb{C}\to\mathbb{R}^3$, the one-parameter family of pre-image complex polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ is given in terms of the free angular parameter ϕ by

$$\boldsymbol{\alpha}(t) = \sqrt{\tfrac{1}{2}(\sigma(t)+x'(t))}\,(-\sin\phi + \mathrm{i}\,\cos\phi),$$

$$\boldsymbol{\beta}(t) = \frac{[z'(t)\cos\phi - y'(t)\sin\phi] + \mathrm{i}\,[y'(t)\cos\phi + z'(t)\sin\phi]}{\sqrt{2(\sigma(t)+x'(t))}}.$$

5. Degenerate spatial PH curves

Integrating (5) may yield *linear* or *planar* PH curves as special cases, which are trivially double PH curves. In the linear case, $\mathbf{r}'$ and $\mathbf{r}''$ are always parallel, so $|\,\mathbf{r}'\times\mathbf{r}''\,|\equiv 0$. In the planar case, there exist coordinates such that $z'\equiv 0$ and $z''\equiv 0$, so $|\,\mathbf{r}'\times\mathbf{r}''\,|^2$ becomes the perfect square $(x'y''-x''y')^2$. Since we are interested in generic double PH curves, we need criteria to identify such degenerate cases. For this purpose we use primarily the quaternion form (7), but also express the results in terms of the Hopf map form (15).

If $u(t), v(t), p(t), q(t)$ are all constants, the hodograph is a single point, specifying a (uniformly-parameterized) straight line. Straight lines with non-uniform parameterizations also arise when $x'(t)$, $y'(t), z'(t)$ are non-constant, but exhibit constant ratios. This corresponds to vanishing of the curvature $\kappa=\sigma^{-3}|\,\mathbf{r}'\times\mathbf{r}''\,|$, and in Farouki et al. (2004) it was shown that all spatial PH curves satisfy

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2 = \sigma^2(t)\rho(t), \tag{18}$$

where the polynomial ρ may be specified (Farouki et al., 2004) in terms of u, v, p, q as

$$\begin{aligned}\rho\ &= 4[(up'-u'p)^2+(uq'-u'q)^2+(vp'-v'p)^2+(vq'-v'q)^2\\ &\quad + 2(uv'-u'v)(pq'-p'q)].\end{aligned} \tag{19}$$

The condition for degeneration to a straight line is thus equivalent to $\rho(t)\equiv 0$. For a degree-n PH curve, $\rho(t)$ is of degree $2n-6$ and is therefore a constant for PH cubics, and a quartic for PH quintics.

Planar degenerations of spatial PH curves correspond to vanishing of the torsion, so $(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t)$, which can be written in terms of $u(t), v(t), p(t), q(t)$ and their first and second derivatives, must be identically zero. For a degree-n PH curve, this polynomial is of degree $3n-9$. For the PH cubics, it reduces to a constant, while for PH quintics it is of degree 6.

Propositions 1 and 2 below state precise conditions for linear and planar degeneration of spatial PH curves in terms of the quaternion model. For the proofs, we refer the reader to Section 22.2 of Farouki (2008). In Remarks 3 and 4, these conditions are translated into the Hopf map model.

Proposition 1. *Let $\mathcal{A}_1$ be expressed in terms of $\mathcal{A}_0\ (\neq 0)$ as*

$$\mathcal{A}_1 = \mathcal{A}_0\,(\alpha+\beta\mathbf{i}+\gamma\mathbf{j}+\delta\mathbf{k}). \tag{20}$$

Then the spatial PH cubic defined by substituting $\mathcal{A}(t)=\mathcal{A}_0(1-t)+\mathcal{A}_1 t$ into (7) and integrating is a straight line if and only if $\gamma=\delta=0$, and a planar curve other than a straight line if and only if $\beta=0$ and $(\gamma,\delta)\neq(0,0)$.

Viewing quaternions as vectors in $\mathbb{R}^4$, a spatial PH cubic degenerates to a straight line if and only if $\mathcal{A}_1$ lies in the two-dimensional subspace spanned by $\mathcal{A}_0$, $\mathcal{A}_0\mathbf{i}$, and to a planar curve (other than a line) if and only if $\mathcal{A}_1$ lies in the three-dimensional subspace spanned by $\mathcal{A}_0$, $\mathcal{A}_0\mathbf{j}$, $\mathcal{A}_0\mathbf{k}$.

Proposition 2. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be expressed in terms of $\mathcal{A}_0(\neq 0)$ as*

$$\mathcal{A}_1 = \mathcal{A}_0(\alpha_1+\beta_1\mathbf{i}+\gamma_1\mathbf{j}+\delta_1\mathbf{k}),\quad \mathcal{A}_2 = \mathcal{A}_0(\alpha_2+\beta_2\mathbf{i}+\gamma_2\mathbf{j}+\delta_2\mathbf{k}). \tag{21}$$

Then the spatial PH quintic specified by $\mathcal{A}(t)=\mathcal{A}_0(1-t)^2+\mathcal{A}_1 2(1-t)t+\mathcal{A}_2t^2$ and (7) is a straight line if and only if $\gamma_1=\gamma_2=\delta_1=\delta_2=0$, and a plane curve other than a straight line if and only if $\beta_1=\beta_2=\gamma_1\delta_2-\gamma_2\delta_1=0$ with $\gamma_1,\gamma_2,\delta_1,\delta_2$ not all zero, provided that $\gcd(x',y',z')=$ constant in (7).

When $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$, the hodograph (7) reduces to $\mathbf{r}'(t) = \sigma(t)\mathbf{t}_0$, where $\sigma(t) = |\mathbf{r}'(t)|$ is the parametric speed and the unit vector

$$\mathbf{t}_0 = \frac{(u_0^2 + v_0^2 - p_0^2 - q_0^2,\, 2(u_0q_0 + v_0p_0),\, 2(v_0q_0 - u_0p_0))}{u_0^2 + v_0^2 + p_0^2 + q_0^2}$$

defines the fixed tangent direction. Since $\mathbf{r}'(t)$ has fixed direction, the locus is a (non-uniformly parameterized) straight line. If $\mathbf{r}(t)$ degenerates to a plane curve, the normal $\mathbf{n} = (n_x, n_y, n_z)$ to the plane in which it resides is given by

$$\begin{aligned} n_x : n_y : n_z = {} & 2\gamma_2(u_0q_0 - v_0p_0) - 2\delta_2(u_0p_0 + v_0q_0) \\ & : \gamma_2(v_0^2 + q_0^2 - u_0^2 - p_0^2) + 2\delta_2(u_0v_0 - p_0q_0) \\ & : \delta_2(v_0^2 + p_0^2 - u_0^2 - q_0^2) - 2\gamma_2(u_0v_0 + p_0q_0). \end{aligned}$$

Thus, PH quintics degenerate to straight lines if $\mathcal{A}_1, \mathcal{A}_2$ lie in the subspace of $\mathbb{R}^4$ spanned by $\mathcal{A}_0$, $\mathcal{A}_0\mathbf{i}$ (as with the PH cubics). For planar PH quintics $\mathcal{A}_1, \mathcal{A}_2$ must lie in the subspace spanned by $\mathcal{A}_0$, $\mathcal{A}_0\mathbf{j}$, $\mathcal{A}_0\mathbf{k}$, as with the cubics, but we also require $\gamma_1\delta_2 - \gamma_2\delta_1 = 0$ with $\gamma_1, \gamma_2, \delta_1, \delta_2$ not all zero. This implies that $\gamma_1 : \gamma_2 = \delta_1 : \delta_2$—so $\mathcal{A}_1, \mathcal{A}_2$ have, for some real number h, the form

$$\mathcal{A}_1 = \alpha_1\mathcal{A}_0 + \gamma_1\mathcal{A}_0\,\mathbf{j} + \delta_1\mathcal{A}_0\,\mathbf{k}, \quad \mathcal{A}_2 = \alpha_2\mathcal{A}_0 + h\,(\gamma_1\mathcal{A}_0\,\mathbf{j} + \delta_1\mathcal{A}_0\,\mathbf{k})$$

i.e. the components of $\mathcal{A}_1, \mathcal{A}_2$ in the subspace of $\mathbb{R}^4$ spanned by $\mathcal{A}_0\mathbf{j}$, $\mathcal{A}_0\mathbf{k}$ must be proportional. Hence, $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are not linearly independent—the combination $\lambda\,\mathcal{A}_0 + \mu\mathcal{A}_1 + \nu\mathcal{A}_2$ vanishes when $\lambda : \mu : \nu = \alpha_2 - h\alpha_1 : h : -1$.

Remark 3. In the Hopf map model, one can easily verify that the conditions of Proposition 1 are equivalent to requiring the Bernstein coefficients of the complex polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ to satisfy $\boldsymbol{\alpha}_1 : \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_0 : \boldsymbol{\beta}_0$ for degeneration to a straight line, and $\boldsymbol{\alpha}_1 = \lambda\,\boldsymbol{\alpha}_0 - \mathbf{z}\,\overline{\boldsymbol{\beta}}_0$ and $\boldsymbol{\beta}_1 = \lambda\,\boldsymbol{\beta}_0 + \mathbf{z}\,\overline{\boldsymbol{\alpha}}_0$ with λ real and $\mathbf{z}$ complex for degeneration to a planar curve other than a straight line.

Remark 4. The conditions of Proposition 2 for degeneration to a straight line are equivalent to requiring the Bernstein coefficients of the polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ in the Hopf map model to satisfy $\boldsymbol{\alpha}_2 : \boldsymbol{\beta}_2 = \boldsymbol{\alpha}_1 : \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_0 : \boldsymbol{\beta}_0$. For a plane curve other than a straight line, the conditions of Proposition 2 translate to $\boldsymbol{\alpha}_1 = \lambda_1\boldsymbol{\alpha}_0 - \mu_1\mathbf{z}\,\overline{\boldsymbol{\beta}}_0$, $\boldsymbol{\alpha}_2 = \lambda_2\boldsymbol{\alpha}_0 - \mu_2\mathbf{z}\,\overline{\boldsymbol{\beta}}_0$ and $\boldsymbol{\beta}_1 = \lambda_1\boldsymbol{\beta}_0 + \mu_1\mathbf{z}\,\overline{\boldsymbol{\alpha}}_0$, $\boldsymbol{\beta}_2 = \lambda_2\boldsymbol{\beta}_0 + \mu_2\mathbf{z}\,\overline{\boldsymbol{\alpha}}_0$ with $\lambda_1, \lambda_2, \mu_1, \mu_2$ real and $\mathbf{z}$ complex.

6. Characterization of "double" PH curves

As noted in Section 1, the tangent $\mathbf{t}$ and torsion τ of spatial PH curves have a rational dependence on the curve parameter, but the normal vectors $\mathbf{p}$ and $\mathbf{b}$ and curvature κ do not, because the quantity $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|$ generically incurs the square root of a polynomial. To investigate the possibility of constructing curves for which $(\mathbf{t}, \mathbf{p}, \mathbf{b})$ and κ, τ are all rational in the curve parameter, we must study the structure of $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|$ in greater detail.

By substituting from (5) into

$$|\,\mathbf{r}' \times \mathbf{r}''\,|^2 = (y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2, \tag{22}$$

one may directly verify Eq. (18) with $\rho(t)$ given by (19). The polynomial (19) may also be interpreted as

$$\rho(t) = |\mathbf{r}''(t)|^2 - \sigma'^2(t)$$

or as $|\mathbf{r}''(t)|^2 \sin^2\phi(t)$, where $\phi(t)$ is the angle between $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. In fact, ρ can be written in terms of the polynomials u, v, p, q and their derivatives u', v', p', q' in several different ways. For example, it can be written (Farouki, 2008) as

$$\begin{aligned} \rho = {} & 4\,[(uv' - u'v + pq' - p'q)^2 + (up' - u'p - vq' + v'q)^2 \\ & + (uq' - u'q + vp' - v'p)^2 - (uv' - u'v - pq' + p'q)^2], \end{aligned}$$

or as a sum of just two squares,

$$\rho = 4\,[(up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2]. \tag{23}$$

The form (23) was derived by Beltran and Monterde (2007) using the Hopf map model, and was independently discovered empirically by the present authors. Its importance is that it allows us to characterize the conditions under which $\rho(t)$ is a perfect square, and thus the Frenet frame $(\mathbf{t}, \mathbf{p}, \mathbf{b})$, the curvature κ and torsion τ all have a *rational* dependence on the curve parameter t.

A polynomial space curve $\mathbf{r}(t)$ is said to be a "double PH curve" if $|\,\mathbf{r}'(t)\,|$ and $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|$ are both *polynomial* functions of t, i.e. if the conditions

$$|\,\mathbf{r}'\,|^2 = x'^2 + y'^2 + z'^2 \;\equiv\; \sigma^2, \tag{24}$$

$$|\,\mathbf{r}' \times \mathbf{r}''\,|^2 = (y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2 \;\equiv\; (\sigma\omega)^2 \tag{25}$$

are simultaneously satisfied for some polynomials $\sigma(t)$, $\omega(t)$. In other words, in addition to the usual PH condition (24), for a double PH (or DPH) curve we require the polynomial ρ in the relation (18), satisfied by all PH curves, to be a perfect square: $\rho = \omega^2$ for some polynomial $\omega(t)$. Beltran and Monterde called such curves the "PH curves of second class" or "2-PH curves" (Beltran and Monterde, 2007), and they determined that the cubic and quintic double PH curves are exactly the *helical* PH curves (Farouki et al., 2004) of equal degree—but double PH curves of degree 7 exist that are not helical, i.e., they do not satisfy (3) and (4).

For a double PH curve with $\rho(t) = \omega^2(t)$, the Frenet frame vectors and the curvature and torsion functions are given by the rational expressions

$$\mathbf{t} = \frac{\mathbf{r}'}{\sigma}, \quad \mathbf{p} = \frac{\sigma\mathbf{r}'' - \sigma'\mathbf{r}'}{\sigma\omega}, \quad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{\sigma\omega}, \quad \kappa = \frac{\omega}{\sigma^2}, \quad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\sigma^2\omega^2}.$$

Hence, the DPH curves may be regarded as the complete set of polynomial curves that have rational Frenet frames (Wagner and Ravani, 1997).

As observed in Farouki et al. (2004), all helical space curves are PH curves, and for a pitch angle ψ they satisfy the relation

$$\rho^{3/2} = \tan\psi\;(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''',$$

implying that ρ is a perfect square (since the right-hand side is a polynomial). Hence, every *helical* PH curve must be a DPH curve. Beltran and Monterde (2007) showed that for, cubics and quintics, there is an exact coincidence of helical curves and DPH curves, but quoted an example of a DPH curve of degree 7 that is non-helical. For DPH curves, the curvature/torsion ratio becomes

$$\frac{\kappa(t)}{\tau(t)} = \frac{\omega^3(t)}{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}, \tag{26}$$

and hence we have the following observation.

Remark 5. If a polynomial space curve $\mathbf{r}(t)$ is helical, $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)$ must be proportional to the cube of a polynomial $\omega(t)$.

For PH cubics, the ratio (26) is always constant since the numerator and denominator are individually constant.[5] For the DPH quintics, they are both polynomials of degree 6, and $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ must be a multiple of ω^3 since all double PH quintics are helical. For higher-order DPH curves, satisfaction of the condition $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = \omega^3 \tan\psi$ for some constant $\tan\psi$ can be used to distinguish the helical DPH curves from those that are non-helical.

Now for a PH curve of degree n, $\deg(\rho) = 2n - 6$. In the present context, the expression (23) for $\rho(t)$ as a sum of squares is the most interesting, since it implies (Beltran and Monterde, 2007) that

[5] The fact that all PH cubics are helical curves is one of the first known properties (Farouki and Sakkalis, 1994) of the spatial PH curves.

to satisfy the second Pythagorean condition (25), the three polynomials $2(up' - u'p + vq' - v'q)$, $2(uq' - u'q - vp' + v'p)$, ω must comprise a Pythagorean triple, satisfying

$$4\,[(up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2] \equiv \omega^2. \tag{27}$$

The solutions of this equation must be (Kubota, 1972) of the form

$$\begin{aligned} up' - u'p + vq' - v'q &= h(a^2 - b^2), \\ uq' - u'q - vp' + v'p &= 2hab, \\ \omega &= 2h(a^2 + b^2), \end{aligned} \tag{28}$$

for polynomials $h(t)$, $a(t)$, $b(t)$ with $\gcd(a(t), b(t)) =$ constant. For instances with $\gcd(up' - u'p + vq' - v'q, uq' - u'q - vp' + v'p) =$ constant, we may take $h(t) = 1$, and we then have a *primitive* Pythagorean triple.

As observed in Farouki et al. (2004), the helical PH quintics comprise a proper subset of all spatial PH quintics. For PH quintics, $\rho(t)$ is not merely a constant, and for a double PH curve it must be the perfect square of a quadratic. The set of double PH quintics coincides precisely with the set of helical PH quintics, but this coincidence does not extend to higher-degree PH curves (Beltran and Monterde, 2007).

7. Quaternion form of double PH curves

Consider a spatial PH curve with parametric speed and first two derivatives specified in terms of a quaternion polynomial $\mathcal{A}(t)$ by

$$\sigma(t) = |\mathcal{A}(t)|^2, \quad \mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \quad \mathbf{r}''(t) = \mathcal{A}'(t)\,\mathbf{i}\,\mathcal{A}^*(t) + \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t).$$

Regarding $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ as pure vector quaternions, the quantity $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is the vector part of their quaternion product, and it can be expressed as one half this product minus its conjugate. Thus, writing

$$2\,\mathbf{r}' \times \mathbf{r}'' = (\mathcal{A}\,\mathbf{i}\,\mathcal{A}^*)\,(\mathcal{A}'\,\mathbf{i}\,\mathcal{A}^* + \mathcal{A}\,\mathbf{i}\,\mathcal{A}'^*) - (\mathcal{A}'\,\mathbf{i}\,\mathcal{A}^* + \mathcal{A}\,\mathbf{i}\,\mathcal{A}'^*)^*\,(\mathcal{A}\,\mathbf{i}\,\mathcal{A}^*)^*$$

and simplifying, we obtain

$$2\,\mathbf{r}' \times \mathbf{r}'' = \mathcal{A}\,\mathbf{i}\,(\mathcal{A}^*\mathcal{A}' - \mathcal{A}'^*\mathcal{A})\,\mathbf{i}\,\mathcal{A}^* + \sigma\,(\mathcal{A}'\mathcal{A}^* - \mathcal{A}\mathcal{A}'^*).$$

Now since $\sigma(t) = \mathcal{A}(t)\mathcal{A}^*(t) = \mathcal{A}^*(t)\mathcal{A}(t)$, we have

$$\sigma' = \mathcal{A}'\mathcal{A}^* + \mathcal{A}\,\mathcal{A}'^* = \mathcal{A}'^*\mathcal{A} + \mathcal{A}^*\mathcal{A}',$$

and by invoking these relations we deduce that

$$\mathbf{r}' \times \mathbf{r}'' = \mathcal{A}\,\mathbf{i}\,\mathcal{A}^*\mathcal{A}'\,\mathbf{i}\,\mathcal{A}^* + \sigma\,\mathcal{A}'\mathcal{A}^*.$$

Using $\sigma(t) = \mathcal{A}(t)\mathcal{A}^*(t)$, we re-write this as

$$\mathbf{r}' \times \mathbf{r}'' = \mathcal{A}\,(\,\mathbf{i}\,\mathcal{A}^*\mathcal{A}'\,\mathbf{i} + \mathcal{A}^*\mathcal{A}'\,)\,\mathcal{A}^*.$$

Now for $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$, the products $\mathcal{A}^*\mathcal{A}'$ and $\mathbf{i}\,\mathcal{A}^*\mathcal{A}'\,\mathbf{i}$ are given by

$$\begin{aligned} \mathcal{A}^*\mathcal{A}' &= (uu' + vv' + pp' + qq') + (uv' - u'v - pq' + p'q)\,\mathbf{i} \\ &\quad + (up' - u'p + vq' - v'q)\,\mathbf{j} + (uq' - u'q - vp' + v'p)\,\mathbf{k}, \\ \mathbf{i}\,\mathcal{A}^*\mathcal{A}'\,\mathbf{i} &= -\,(uu' + vv' + pp' + qq') - (uv' - u'v - pq' + p'q)\,\mathbf{i} \\ &\quad + (up' - u'p + vq' - v'q)\,\mathbf{j} + (uq' - u'q - vp' + v'p)\,\mathbf{k}, \end{aligned}$$

so $\mathbf{i}\,\mathcal{A}^*(t)\mathcal{A}'(t)\,\mathbf{i} + \mathcal{A}^*(t)\mathcal{A}'(t)$ is just twice the $(\mathbf{j}, \mathbf{k})$ part of $\mathcal{A}^*(t)\mathcal{A}'(t)$. Thus, in terms of the polynomials

$$\begin{aligned} f(t) &= u(t)p'(t) - u'(t)p(t) + v(t)q'(t) - v'(t)q(t), \\ g(t) &= u(t)q'(t) - u'(t)q(t) - v(t)p'(t) + v'(t)p(t), \end{aligned} \tag{29}$$

appearing in (23), we have

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathcal{A}(t)\,[f(t)\,\mathbf{j} + g(t)\,\mathbf{k}]\,\mathcal{A}^*(t). \tag{30}$$

Expanding this product in terms of components gives

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= 2\,[2(vp-uq)f + 2(up+vq)g]\,\mathbf{i} + 2\,[(u^2-v^2+p^2-q^2)f + 2(pq-uv)g]\,\mathbf{j} \\ &\quad + 2\,[2(uv+pq)f + (u^2-v^2-p^2+q^2)g]\,\mathbf{k},\end{aligned}$$

and the squared modulus of this vector is

$$|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|^2 = 4\,\sigma^2(t)\,[f^2(t) + g^2(t)],$$

where $\sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t)$. Hence, $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|$ is a polynomial in t if and only if the two polynomials $f(t), g(t)$ are elements of a Pythagorean triple, and are thus (Kubota, 1972) of the form

$$f(t) = h(t)\,[a^2(t) - b^2(t)], \qquad g(t) = 2\,h(t)a(t)b(t), \tag{31}$$

for polynomials $a(t), b(t), h(t)$ with $\gcd(a(t), b(t)) =$ constant.

Note that, since the components of $\mathbf{r}'(t) \times \mathbf{r}''(t)$ satisfy the Pythagorean condition (25) if $\mathbf{r}(t)$ is a double PH curve, it must be expressible in the form[6] (12) in terms of a real polynomial $h(t)$ and a quaternion polynomial $\mathcal{B}(t)$. If $\mathbf{r}(t)$ is of degree n, we must have $\deg(\mathbf{r}' \times \mathbf{r}'') = 2n - 4 = \deg(h) + 2\deg(\mathcal{B})$.

Proposition 3. *For a DPH curve $\mathbf{r}(t)$ specified by a quaternion polynomial (8) satisfying the conditions (27) and (28), the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ can be expressed in the quaternion Pythagorean form (12) with $\mathcal{B}(t)$ given by*

$$\mathcal{B}(t) = \mathcal{A}(t)\,\mathcal{C}(t), \quad \text{where} \;\; \mathcal{C}(t) = -\,b(t) + a(t)\,\mathbf{i} + a(t)\,\mathbf{j} + b(t)\,\mathbf{k}. \tag{32}$$

Proof. Invoking the form (30), and multiplying both sides of

$$\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t) = 2\,\mathcal{A}(t)\,[(a^2(t) - b^2(t))\,\mathbf{j} + 2\,a(t)b(t)\,\mathbf{k}]\,\mathcal{A}^*(t) \tag{33}$$

on the left by $\mathcal{A}^*(t)$ and the right by $\mathcal{A}(t)$, we obtain

$$\mathcal{Q}(t)\,\mathbf{i}\,\mathcal{Q}^*(t) = 2\,\sigma^2(t)\,[(a^2(t) - b^2(t))\,\mathbf{j} + 2\,a(t)b(t)\,\mathbf{k}],$$

where we set $\mathcal{Q}(t) = \mathcal{A}^*(t)\,\mathcal{B}(t)$. One may then deduce (Farouki et al., 2008) that the general solution to this equation has the form

$$\mathcal{Q} = \sigma\,\frac{((a^2+b^2)\,\mathbf{i} + (a^2-b^2)\,\mathbf{j} + 2ab\,\mathbf{k})}{\sqrt{a^2+b^2}}\,(\cos\phi + \sin\phi\,\mathbf{i}),$$

where ϕ is a free angular parameter. Hence we obtain

$$\mathcal{B} = \mathcal{A}\,\frac{((a^2+b^2)\,\mathbf{i} + (a^2-b^2)\,\mathbf{j} + 2ab\,\mathbf{k})}{\sqrt{a^2+b^2}}\,(\cos\phi + \sin\phi\,\mathbf{i}),$$

and in order to ensure that $\mathcal{B}(t)$ is a polynomial, we choose the dependence of ϕ on t defined by

$$\sin\phi(t) = \frac{b(t)}{\sqrt{a^2(t)+b^2(t)}}, \qquad \cos\phi(t) = \frac{a(t)}{\sqrt{a^2(t)+b^2(t)}}.$$

Substituting and simplifying, this gives the solution (32) for $\mathcal{B}(t)$. ∎

[6] We assume here the generic form, appropriate to the case where the components of $\mathbf{r}'(t) \times \mathbf{r}''(t)$ are not necessarily relatively prime.

8. Hopf map form of double PH curves

The Hopf map form (14) constructs spatial Pythagorean hodographs from two complex polynomials $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\, v(t)$ and $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}\, p(t)$. Forming the combination

$$\boldsymbol{\alpha}\, \boldsymbol{\beta}' - \boldsymbol{\alpha}' \boldsymbol{\beta} = (uq' - u'q - vp' + v'p) + \mathrm{i}\,(up' - u'p + vq' - v'q) \tag{34}$$

of these polynomials, we observe that

$$\rho(t) = 4\,|\, \boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\,|^2. \tag{35}$$

Thus, the DPH curves are spatial PH curves for which $|\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)|^2$ is the perfect square of a real polynomial. Due to its importance in the theory of double PH curves, we call $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ the *proportionality polynomial* of $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$. It vanishes identically if and only if $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ are (complex) constant multiples of each other. In the Hopf map representation, the curvature of spatial PH curves is given by

$$\kappa(t) = 2\, \frac{|\, \boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\,|}{(|\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2)^2},$$

so vanishing of (34) identifies degeneration to a straight line. When (34) does not vanish identically, its real roots (if any) identify *inflections* of a PH space curve, at which the normal vectors $\mathbf{p}$, $\mathbf{b}$ may suffer sudden reversals.

Now in the Hopf map representation, the conditions (28) for a spatial PH curve to be a *double* PH curve can be expressed as

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = h(t)\, \mathbf{w}^2(t) \tag{36}$$

for some real polynomial $h(t)$ and complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}\, b(t)$ with $\gcd(a(t), b(t)) =$ constant, such that

$$\deg(h(t)) + 2\, \deg(\mathbf{w}(t)) = 2\, \deg(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) - 2. \tag{37}$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, the complex polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ may be regarded as defining plane curves, and from the complex representation of planar PH curves (Farouki, 1994; Farouki et al., 2001; Farouki and Neff, 1995) the expression on the right in (36) is seen to define a planar Pythagorean hodograph. These observations reveal the following connection between double (spatial) PH curves and planar PH curves.

Proposition 4. *A spatial* PH *curve specified through the Hopf map* (15) *by two complex polynomials* $\boldsymbol{\alpha}(t)$ *and* $\boldsymbol{\beta}(t)$ *is a double PH curve if and only if their proportionality polynomial* (34) *defines a planar Pythagorean hodograph.*

One may deduce the Hopf map form of the polynomial (35) in the relation (18) directly, as follows. From (15) the components of $\mathbf{r}'(t)$ are written as

$$x'(t) = \boldsymbol{\alpha}(t)\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\beta}(t)\overline{\boldsymbol{\beta}}(t), \quad y'(t) + \mathrm{i}\, z'(t) = 2\,\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t), \tag{38}$$

and differentiating then gives

$$x''(t) = \boldsymbol{\alpha}'(t)\overline{\boldsymbol{\alpha}}(t) + \boldsymbol{\alpha}(t)\overline{\boldsymbol{\alpha}}'(t) - \boldsymbol{\beta}'(t)\overline{\boldsymbol{\beta}}(t) - \boldsymbol{\beta}(t)\overline{\boldsymbol{\beta}}'(t), \tag{39}$$

$$y''(t) + \mathrm{i}\, z''(t) = 2\,[\boldsymbol{\alpha}'(t)\overline{\boldsymbol{\beta}}(t) + \boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}'(t)]. \tag{40}$$

Substituting from (38) and (39)–(40) into

$$y'z'' - y''z' = -\tfrac{1}{2}\,\mathrm{i}\,[(y' - \mathrm{i}\, z')(y'' + \mathrm{i}\, z'') - (y' + \mathrm{i}\, z')(y'' - \mathrm{i}\, z'')],$$

$$(x'y'' - x''y') + \mathrm{i}\,(z'x'' - z''x') = x'(y'' - \mathrm{i}\, z'') - x''(y' - \mathrm{i}\, z'),$$

and writing

$$\boldsymbol{\eta}(t) = \boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t), \tag{41}$$

after some manipulation one obtains

$$\begin{aligned} &y'(t)z''(t) - y''(t)z'(t) = 2\,\mathrm{i}\,[\overline{\boldsymbol{\alpha}}(t)\overline{\boldsymbol{\beta}}(t)\boldsymbol{\eta}(t) - \boldsymbol{\alpha}(t)\boldsymbol{\beta}(t)\overline{\boldsymbol{\eta}}(t)], \\ &[x'(t)y''(t) - x''(t)y'(t)] + \mathrm{i}\,[z'(t)x''(t) - z''(t)x'(t)] = 2\,[\overline{\boldsymbol{\alpha}}^2(t)\boldsymbol{\eta}(t) + \boldsymbol{\beta}^2(t)\overline{\boldsymbol{\eta}}(t)]. \end{aligned}$$

By direct substitution and simplification, one can then deduce that

$$\begin{aligned} |\,\mathbf{r}' \times \mathbf{r}''\,|^2 &= (y'z'' - y''z'')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2 \\ &= 4\,|\overline{\boldsymbol{\alpha}}\,\overline{\boldsymbol{\beta}}\,\boldsymbol{\eta} - \boldsymbol{\alpha}\,\boldsymbol{\beta}\,\overline{\boldsymbol{\eta}}|^2 + 4\,|\overline{\boldsymbol{\alpha}}^2\boldsymbol{\eta} + \boldsymbol{\beta}^2\overline{\boldsymbol{\eta}}|^2 = 4\,(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2)^2\,|\boldsymbol{\eta}|^2. \end{aligned}$$

Since $\sigma(t) = |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2$ in the Hopf map representation, we deduce that $\rho(t) = 4\,|\boldsymbol{\eta}(t)|^2$ where $\boldsymbol{\eta}(t)$ is defined by (41).

Assuming $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ and $h(t)$, $\mathbf{w}(t)$ are specified in the Bernstein form as

$$\boldsymbol{\alpha}(t) = \sum_{l=0}^{m} \boldsymbol{\alpha}_l \binom{m}{l}(1-t)^{m-l}t^l, \quad \boldsymbol{\beta}(t) = \sum_{l=0}^{m} \boldsymbol{\beta}_l \binom{m}{l}(1-t)^{m-l}t^l,$$

$$h(t) = \sum_{l=0}^{d} h_l \binom{d}{l}(1-t)^{d-l}t^l, \quad \mathbf{w}(t) = \sum_{l=0}^{e} \mathbf{w}_l \binom{e}{l}(1-t)^{e-l}t^l,$$

where from (37) we must have $d + 2e = 2m - 2$, we now elucidate certain connections between the Hopf map and quaternion representations.

Remark 6. From the Hopf map form (35) of $\rho(t)$ and the condition (36) for a double PH curve we may infer that, for DPH curves, $\rho(t) = 4\,h^2(t)\,|\,\mathbf{w}(t)\,|^4$. Thus, the polynomial $\omega(t)$ defined for DPH curves by $\rho(t) = \omega^2(t)$ is simply $\omega(t) = 2\,h(t)\,|\,\mathbf{w}(t)\,|^2$. For *helical* curves, we deduce (see Remark 5) that the triple product $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)$ is proportional to $(2\,h(t)\,|\,\mathbf{w}(t)\,|^2)^3$.

Remark 7. If $h(t)$ is a non-constant polynomial, we must have $h(t) \geq 0$ for all t in order to write $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,| = 2\,\sigma(t)\,h(t)\,|\,\mathbf{w}(t)\,|^2$. Otherwise, if $h(t)$ is not non-negative for all t, we must replace $h(t)$ by $|h(t)|$ in the expression for $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|$, which is then the *absolute value* of a polynomial in t. In practice, the choice $h(t) =$ constant may be preferable—as with the case of *primitive* planar Pythagorean hodographs (Farouki and Sakkalis, 1990).

We conclude with two observations connecting the quaternion and Hopf map formulations of the double PH curves.

Remark 8 (*Quaternion Form of Proportionality Polynomial*). Identifying the imaginary unit i with the quaternion basis element $\mathbf{i}$, let the coefficients of $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ be

$$\boldsymbol{\alpha}_l = \alpha_l + a_l\,\mathbf{i} \qquad \text{and} \qquad \boldsymbol{\beta}_l = \beta_l + b_l\,\mathbf{i}, \qquad l = 0, \ldots, m.$$

Then the Bernstein coefficients of the corresponding quaternion polynomial (8) defined by (16) are

$$\mathcal{A}_l = \boldsymbol{\alpha}_l + \mathbf{k}\,\boldsymbol{\beta}_l = (\alpha_l + a_l\,\mathbf{i}) + \mathbf{k}\,(\beta_l + b_l\,\mathbf{i}) = \alpha_l + a_l\,\mathbf{i} + b_l\,\mathbf{j} + \beta_l\,\mathbf{k}$$

for $l = 0, \ldots, m$. One can then verify that

$$\mathbf{j}\,(\boldsymbol{\alpha}_k\boldsymbol{\beta}_l - \boldsymbol{\alpha}_l\boldsymbol{\beta}_k) = \tfrac{1}{2}(\mathcal{A}_k^*\mathcal{A}_l - \mathcal{A}_l^*\mathcal{A}_k)\times\mathbf{i}, \tag{42}$$

and hence the proportionality polynomial $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ in the Hopf map model is related to the quaternion polynomial $\mathcal{A}(t)$ by

$$\mathbf{j}\,[\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)] = \tfrac{1}{2}\,[\mathcal{A}^*(t)\mathcal{A}'(t) - \mathcal{A}'^*(t)\mathcal{A}(t)]\times\mathbf{i}. \tag{43}$$

Note here that $\frac{1}{2}\,[\mathcal{A}^*(t)\mathcal{A}'(t) - \mathcal{A}'^*(t)\mathcal{A}(t)] = \mathrm{vect}(\mathcal{A}^*(t)\mathcal{A}'(t))$ is a pure vector quaternion, and Eq. (43) amounts to identifying the real and imaginary parts of $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ with the $\mathbf{k}$ and $\mathbf{j}$ components of $\mathcal{A}^*(t)\mathcal{A}'(t)$.

Remark 9 (*Quaternion Form of Double PH Condition*). Identifying again the imaginary unit i with the quaternion basis element $\mathbf{i}$, the complex polynomial $\mathbf{w}(t) = a(t) + b(t)\,\mathbf{i}$ and the quaternion polynomial $\mathcal{C}(t)$ introduced in (32) satisfy

$$\tfrac{1}{2}\,\mathcal{C}(t)\,\mathbf{i}\,\mathcal{C}^*(t) = (a^2(t) - b^2(t))\,\mathbf{j} + 2\,a(t)b(t)\,\mathbf{k} = \mathbf{w}^2(t)\,\mathbf{j}.$$

Hence, using (43), the DPH condition (36) in the Hopf map model can be written in the quaternion form as

$$\begin{aligned}\left(\tfrac{1}{2}\,[\mathcal{A}^*(t)\mathcal{A}'(t) - \mathcal{A}'^*(t)\mathcal{A}(t)]\times\mathbf{i}\right)\mathbf{j} &= \mathbf{j}\left(\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\right)\mathbf{j}\\ &= h(t)\,\mathbf{j}\,\mathbf{w}^2(t)\,\mathbf{j} = \tfrac{1}{2}\,h(t)\,\mathbf{j}\,\mathcal{C}(t)\,\mathbf{i}\,\mathcal{C}^*(t).\end{aligned}$$

This relation can be more conveniently expressed in the form

$$[\mathcal{A}^*(t)\mathcal{A}'(t) - \mathcal{A}'^*(t)\mathcal{A}(t)]\times\mathbf{i} = h(t)\,\mathcal{D}(t)\,\mathbf{i}\,\mathcal{D}^*(t), \tag{44}$$

where we define

$$\mathcal{D}(t) = \mathbf{j}\,\mathcal{C}(t) = -\,a(t) + b(t)\,\mathbf{i} - b(t)\,\mathbf{j} - a(t)\,\mathbf{k}. \tag{45}$$

Hence, a spatial PH curve specified through (7) by a quaternion polynomial $\mathcal{A}(t)$ is a DPH curve if and only if the relation (44) holds for some quaternion polynomial $\mathcal{D}(t)$ of the special form (45).

9. Classification of helical PH quintics

We now characterize the general helical and monotone-helical PH quintics by relations among their quaternion coefficients $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, analogous to those of Section 5 for linear and planar degenerations of PH cubics and quintics. In Farouki et al. (2004) these helical PH quintics were distinguished using algebraic arguments, based on whether the tangent indicatrix is specified by a rational quartic or (by cancellation of common factors) a rational quadratic expression. Beltran and Monterde (2007) gave another algebraic characterization, based on whether the PH curve satisfies the Pythagorean equations (28) with $\deg(h) = 2$ and $a, b =$ constant, or with $h =$ constant and $\deg(a, b) = 1$.

These algebraic criteria are not very convenient for constructing helical PH curves through, say, the interpolation of Hermite data (Farouki et al., 2004). For example, knowing that the polynomials u, v, p, q satisfy (28) for given degrees of the polynomials h, a, b is not of much help in actually determining u, v, p, q. The criteria derived below, expressed in terms of constraints on the quaternion coefficients $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, are more easily imposed in the construction process.

Remark 10. When the axis of a helical space curve coincides with the z-axis, it has a parameterization of the form $\mathbf{r}(t) = (x(t), y(t), s(t)\cos\psi)$, where $s(t)$ is the arc-length function (Struik, 1961). For $\mathbf{r}(t)$ to be a polynomial curve, it must be a PH curve, since only PH curves have a polynomial for $s(t)$. If $\sigma(t) = \mathrm{d}s/\mathrm{d}t$ is the parametric speed, the projection $\tilde{\mathbf{r}}(t) = (x(t), y(t))$ onto the (x, y)-plane defines a *planar* PH curve, satisfying

$$x'^2(t) + y'^2(t) \;\equiv\; \sigma^2(t)\sin^2\psi.$$

Hence, in these special coordinates, helical polynomial curves can be obtained from *planar* PH curves through spatial hodographs of the form

$$\mathbf{r}'(t) = (u^2(t) - v^2(t), 2u(t)v(t), (u^2(t) + v^2(t))\cot\psi),$$

for relatively prime polynomials $u(t), v(t)$. The disadvantage of this approach is that, unlike the quaternion and Hopf map forms used here and in Farouki et al. (in press), the above description is not invariant under general rotations in $\mathbb{R}^3$. Moreover, it is not very useful in addressing the problem of determining whether a given polynomial curve is helical and, if so, identifying its axis.

We now consider the characterization of helical PH quintics, in terms of relationships among their quaternion coefficients $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$.

Proposition 5. *When $\mathcal{A}_1, \mathcal{A}_2$ are given in terms of $\mathcal{A}_0$ ($\neq 0$) as in (21), the spatial PH quintic specified by (7) and $\mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2$ is a helix satisfying (28) with $\deg(h) = 2$ and $a, b =$ constant if and only if*

$$\gamma_1{:}\gamma_2 = \delta_1{:}\delta_2 \quad \text{and} \quad \beta_1{:}\beta_2 = (\gamma_1^2 + \delta_1^2){:}(\gamma_1\gamma_2 + \delta_1\delta_2). \tag{46}$$

Proof. In the case $\deg(h) = 2$ and $a, b =$ constant, we set $h(t) = h_0(1-t)^2 + h_1 2(1-t)t + h_2 t^2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be specified as in (21), with the components

$$\begin{aligned} u_r &= \alpha_r u_0 - \beta_r v_0 - \gamma_r p_0 - \delta_r q_0, \quad & v_r &= \alpha_r v_0 + \beta_r u_0 - \gamma_r q_0 + \delta_r p_0, \\ p_r &= \alpha_r p_0 + \beta_r q_0 + \gamma_r u_0 - \delta_r v_0, \quad & q_r &= \alpha_r q_0 - \beta_r p_0 + \gamma_r v_0 + \delta_r u_0, \end{aligned} \tag{47}$$

for $r = 1, 2$. Then the Bernstein forms of the quadratic polynomials (29) are

$$\begin{aligned} f(t) &= 2(u_0p_1 - u_1p_0 + v_0q_1 - v_1q_0)\,(1-t)^2 + (u_0p_2 - u_2p_0 + v_0q_2 - v_2q_0)\,2(1-t)t \\ &\quad + 2(u_1p_2 - u_2p_1 + v_1q_2 - v_2q_1)\,t^2, \\ g(t) &= 2(u_0q_1 - u_1q_0 - v_0p_1 + v_1p_0)\,(1-t)^2 + (u_0q_2 - u_2q_0 - v_0p_2 + v_2p_0)\,2(1-t)t \\ &\quad + 2(u_1q_2 - u_2q_1 - v_1p_2 + v_2p_1)\,t^2. \end{aligned} \tag{48}$$

Substituting for u_1, v_1, p_1, q_1 and u_2, v_2, p_2, q_2 gives

$$\begin{aligned} f(t) &= 2\,|\mathcal{A}_0|^2\,[f_0(1-t)^2 + f_1\,2(1-t)t + f_2t^2], \\ g(t) &= 2\,|\mathcal{A}_0|^2\,[g_0(1-t)^2 + g_1\,2(1-t)t + g_2t^2], \end{aligned}$$

where $|\mathcal{A}_0|^2 = u_0^2 + v_0^2 + p_0^2 + q_0^2$ and

$$\begin{aligned} f_0 &= \gamma_1, \quad f_1 = \tfrac{1}{2}\gamma_2, \quad f_2 = \alpha_1\gamma_2 - \alpha_2\gamma_1 + \beta_1\delta_2 - \beta_2\delta_1, \\ g_0 &= \delta_1, \quad g_1 = \tfrac{1}{2}\delta_2, \quad g_2 = \alpha_1\delta_2 - \alpha_2\delta_1 - \beta_1\gamma_2 + \beta_2\gamma_1. \end{aligned} \tag{49}$$

Combining $f(t)$ and $g(t)$ into the complex polynomial $f(t) + \mathrm{i}\,g(t)$ and, a and b into the complex number $a + \mathrm{i}\,b$, Eqs. (28) are equivalent to

$$\begin{aligned} &(f_0 + \mathrm{i}\,g_0)(1-t)^2 + (f_1 + \mathrm{i}\,g_1)\,2(1-t)t + (f_2 + \mathrm{i}\,g_2)t^2 \\ &\qquad = (a^2 - b^2 + \mathrm{i}\,2ab)\,[h_0(1-t)^2 + h_1 2(1-t)t + h_2t^2]. \end{aligned} \tag{50}$$

Clearly, this is satisfied for suitable choices of h_0, h_1, h_2 and a, b if and only if

$$f_0{:}g_0 = f_1{:}g_1 = f_2{:}g_2,$$

i.e., if and only if

$$f_0g_1 - f_1g_0 = f_0g_2 - f_2g_0 = 0.$$

Substituting from (49) into these equations and simplifying, we obtain

$$\gamma_1\delta_2 - \gamma_2\delta_1 = (\gamma_1^2 + \delta_1^2)\beta_2 - (\gamma_1\gamma_2 + \delta_1\delta_2)\beta_1 = 0,$$

and the solutions may be characterized by the equality of ratios in (46). ∎

We may consider any four of $\beta_1, \gamma_1, \delta_1, \beta_2, \gamma_2, \delta_2$ as free parameters, and the relations (46) will then determine the other two. Note that the conditions for linear and planar PH quintics (see Proposition 2) are subsumed by (46). In the linear case, we have $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$, and conditions (46) are trivially satisfied, since $\gamma_1{:}\gamma_2 = \delta_1{:}\delta_2 = (\gamma_1^2 + \delta_1^2){:}(\gamma_1\gamma_2 + \delta_1\delta_2) = 0{:}0$. For the planar case, we have $\beta_1 = \beta_2 = \gamma_1\delta_2 - \gamma_2\delta_1 = 0$, so the first condition in (46) holds, and the second is trivially satisfied since $\beta_1{:}\beta_2 = 0{:}0$. One can verify that, for *non-zero* $\beta_1, \gamma_1, \delta_1, \beta_2, \gamma_2, \delta_2$ values, the conditions (46) are equivalent to

$$\beta_1{:}\beta_2 = \gamma_1{:}\gamma_2 = \delta_1{:}\delta_2. \tag{51}$$

Remark 11. In Farouki et al. (2004) we identified linear dependence of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, expressed by the relation

$$\mathcal{A}_1 = c_0\mathcal{A}_0 + c_2\mathcal{A}_2 \tag{52}$$

for real coefficients c_0, c_2, as a sufficient condition for a helical PH quintic. We can now ascertain that, for the generic case of non-zero values in (51), it is also necessary. Namely, (52) holds with $c_0 = \alpha_1 - \alpha_2/\ell$ and $c_2 = 1/\ell$ for a proportionality constant[7] ℓ in (51), with $\mathcal{A}_1$, $\mathcal{A}_2$ written in the form (21).

Corollary 1. *The spatial PH quintics with quaternion coefficients* (21) *that satisfy* (46), *so that* $\deg(h) = 2$ *and* $a, b = $ *constant in* (28), *correspond to general helical PH quintics.*

Proof. To avoid technical diversions concerning singular cases, we employ the form (51) of the sufficient-and-necessary conditions—valid for non-zero $\beta_1, \gamma_1, \delta_1, \beta_2, \gamma_2, \delta_2$. In this case, for some proportionality constant ℓ we have

$$\mathcal{A}_1 = \mathcal{A}_0(\alpha_1 + \beta_1\mathbf{i} + \gamma_1\mathbf{j} + \delta_1\mathbf{k}), \quad \mathcal{A}_2 = \mathcal{A}_0(\alpha_2 + \ell\beta_1\mathbf{i} + \ell\gamma_1\mathbf{j} + \ell\delta_1\mathbf{k}).$$

The components of the quaternions $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, i.e., the Bernstein coefficients of the polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$, are then given by

$$\begin{aligned}
&u_0, \quad u_1 = \alpha_1 u_0 - \beta_1 v_0 - \gamma_1 p_0 - \delta_1 q_0, \quad u_2 = \alpha_2 u_0 - \ell\beta_1 v_0 - \ell\gamma_1 p_0 - \ell\delta_1 q_0,\\
&v_0, \quad v_1 = \alpha_1 v_0 + \beta_1 u_0 - \gamma_1 q_0 + \delta_1 p_0, \quad v_2 = \alpha_2 v_0 + \ell\beta_1 u_0 - \ell\gamma_1 q_0 + \ell\delta_1 p_0,\\
&p_0, \quad p_1 = \alpha_1 p_0 + \beta_1 q_0 + \gamma_1 u_0 - \delta_1 v_0, \quad p_2 = \alpha_2 p_0 + \ell\beta_1 q_0 + \ell\gamma_1 u_0 - \ell\delta_1 v_0,\\
&q_0, \quad q_1 = \alpha_1 q_0 - \beta_1 p_0 + \gamma_1 v_0 + \delta_1 u_0, \quad q_2 = \alpha_2 q_0 - \ell\beta_1 p_0 + \ell\gamma_1 v_0 + \ell\delta_1 u_0.
\end{aligned}$$

In Farouki et al. (2004) we saw that, if the hodograph components x', y', z' have non-constant common factors, they are given by

$$\gcd(x', y', z') = \gcd(u + \mathrm{i}\,v, p - \mathrm{i}\,q) \cdot \gcd(u - \mathrm{i}\,v, p + \mathrm{i}\,q).$$

Using the above coefficients, we form the complex polynomials $u(t) \pm \mathrm{i}\,v(t)$ and $p(t) \mp \mathrm{i}\,q(t)$ in `MAPLE`, and take their resultant with respect to t, to obtain

$$\mathrm{Resultant}_t(u(t) \pm \mathrm{i}\,v(t), p(t) \mp \mathrm{i}\,q(t)) = |\mathcal{A}_0|^4\,(\ell^2 - 4\alpha_1\ell + 4\alpha_2)(\gamma_1 \mp \mathrm{i}\,\delta_1)^2.$$

Now $|\mathcal{A}_0| \neq 0$ by assumption, and in the generic case with non-zero γ_1, δ_1 we have $\gamma_1 \mp \mathrm{i}\,\delta_1 \neq 0$ since γ_1, δ_1 are real. Thus, except for the particular choice

$$\alpha_2 = \ell\left(\alpha_1 - \frac{\ell}{4}\right), \tag{53}$$

we see that $\gcd(u \pm \mathrm{i}\,v, p \mp \mathrm{i}\,q) = $ constant, so the PH quintic has no common hodograph factors, and it is thus a general (non-monotone) helix. It can be verified (see Remark 12) that the singular choice (53) for α_2 in terms of ℓ and α_1 identifies a monotone-helical PH quintic. ■

Example 1. The polynomials $u(t) = -19\,t^2+12\,t+5$, $v(t) = -22\,t^2+18\,t+1$, $p(t) = 15\,t^2-12\,t-1$, $q(t) = -31\,t^2 + 24\,t + 3$ yield the Pythagorean hodograph

$$\begin{aligned}
x'(t) &= -\,341\,t^4 + 600\,t^3 - 270\,t^2 - 12\,t + 16,\\
y'(t) &= 518\,t^4 - 588\,t^3 - 206\,t^2 + 252\,t + 28,\\
z'(t) &= 1934\,t^4 - 2988\,t^3 + 770\,t^2 + 300\,t + 16,
\end{aligned}$$

which defines a general helical PH quintic. In this case $\mathcal{A}_1$, $\mathcal{A}_2$ are specified in terms of $\mathcal{A}_0$ by the values

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = \frac{(39, 11, -13, 13)}{12}, \quad (\alpha_2, \beta_2, \gamma_2, \delta_2) = \frac{(-27, -11, 13, -13)}{36}$$

in (21), and one can easily verify that these values satisfy the conditions (51). Fig. 1 shows another example.

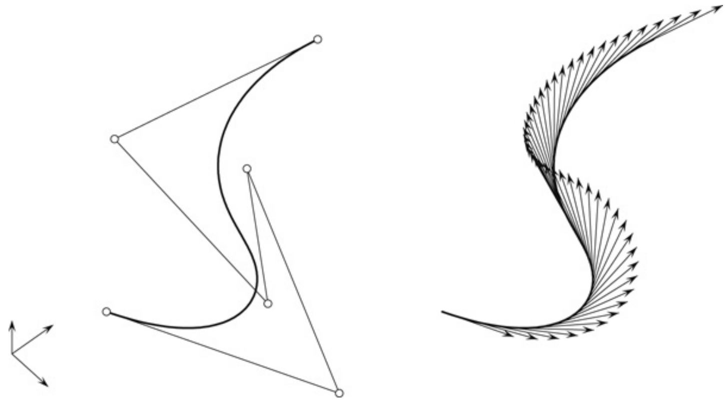

Fig. 1. A general helical PH quintic, constructed using Proposition 5 with $\mathcal{A}_0 = 1.6 - 0.4\,\mathbf{i} + 0.6\,\mathbf{j} + 1.0\,\mathbf{k}$, $\alpha_1 = -0.6$, $\beta_1 = -0.2$, $\gamma_1 = -0.4$, $\delta_1 = 0.4$, $\alpha_2 = 0.9$, and $\beta_2, \gamma_2, \delta_2$ fixed by the proportionality constant $\ell = 0.5$ in (51). Left: the PH curve and its Bézier control polygon. Right: the curve tangents exhibit a reversal in the sense of their rotation about the axis of the helix.

We now consider the case in which (28) is satisfied with $h =$ constant and $\deg(a, b) = 1$. We shall see that this identifies monotone-helical PH quintics.

Proposition 6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be given in terms of $\mathcal{A}_0$ ($\neq 0$) as in (21). Then the spatial PH quintic defined by (7) and $\mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2$ is a helix satisfying (28) with $h =$ constant and $\deg(a, b) = 1$ if and only if α_2, β_2 can be expressed in terms of $\alpha_1, \beta_1, \gamma_1, \delta_1$ and γ_2, δ_2 in the form*

$$\alpha_2 = \frac{\lambda\alpha_1 + \mu\beta_1}{\gamma_1^2 + \delta_1^2} + \frac{\mu^2 - \lambda^2}{4(\gamma_1^2 + \delta_1^2)^2}, \quad \beta_2 = \frac{\lambda\beta_1 - \mu\alpha_1}{\gamma_1^2 + \delta_1^2} + \frac{2\lambda\mu}{4(\gamma_1^2 + \delta_1^2)^2} \tag{54}$$

where $\lambda = \gamma_1\gamma_2 + \delta_1\delta_2$ and $\mu = \gamma_1\delta_2 - \gamma_2\delta_1$.

Proof. In the case $h =$ constant and $\deg(a, b) = 1$, without loss of generality we can take $h = 1$, $a(t) = a_0(1-t) + a_1 t$, $b(t) = b_0(1-t) + b_1 t$. Let $\mathcal{A}_1, \mathcal{A}_2$ be given in terms of $\mathcal{A}_0$ by (21), with the components (47) for $r = 1, 2$. Then the Bernstein forms of the quadratic polynomials (29) are given by (48). We combine f, g and a, b into the complex polynomials $\mathbf{s}(t) = f(t) + \mathrm{i}\,g(t)$ and $\mathbf{c}(t) = a(t) + \mathrm{i}\,b(t)$. Eqs. (28) are then equivalent to $\mathbf{s}(t) = \mathbf{c}^2(t)$, where

$$\begin{aligned} \mathbf{s}(t) &= (f_0 + \mathrm{i}\,g_0)(1-t)^2 + (f_1 + \mathrm{i}\,g_1)\,2(1-t)t + (f_2 + \mathrm{i}\,g_2)t^2, \\ \mathbf{c}^2(t) &= (a_0 + \mathrm{i}\,b_0)^2(1-t)^2 + (a_0 + \mathrm{i}\,b_0)(a_1 + \mathrm{i}\,b_1)\,2(1-t)t + (a_1 + \mathrm{i}\,b_1)^2 t^2. \end{aligned}$$

Clearly, $\mathbf{s}(t)$ can coincide with $\mathbf{c}^2(t)$ for suitable choices of a_0, a_1, b_0, b_1 if and only if its coefficients satisfy

$$(f_1 + \mathrm{i}\,g_1)^2 = (f_0 + \mathrm{i}\,f_0)(f_2 + \mathrm{i}\,g_2)$$

or, equating real and imaginary parts, if and only if

$$f_1^2 - g_1^2 = f_0 f_2 - g_0 g_2 \qquad \text{and} \qquad 2f_1 g_1 = f_0 g_2 + f_2 g_0. \tag{55}$$

Substituting from (49) for the Bernstein coefficients[8] of $f(t), g(t)$ into (55) yields two equations in the eight variables $\alpha_1, \beta_1, \delta_1, \gamma_1$ and $\alpha_2, \beta_2, \delta_2, \gamma_2$ which may be written as

$$\begin{aligned} &4(\gamma_1\gamma_2 - \delta_1\delta_2)\alpha_1 + 4(\gamma_1\delta_2 + \gamma_2\delta_1)\beta_1 - 4(\gamma_1^2 - \delta_1^2)\alpha_2 - 8\gamma_1\delta_1\beta_2 = \gamma_2^2 - \delta_2^2, \\ &4(\gamma_1\delta_2 + \gamma_2\delta_1)\alpha_1 - 4(\gamma_1\gamma_2 - \delta_1\delta_2)\beta_1 - 8\gamma_1\delta_1\alpha_2 + 4(\gamma_1^2 - \delta_1^2)\beta_2 = 2\gamma_2\delta_2. \end{aligned}$$

Now if $\gamma_1 = \delta_1 = 0$, all the terms on the left vanish, and we must also have $\gamma_2 = \delta_2 = 0$ if these equations are to be satisfied. This circumstance identifies the degenerate case of a straight line (Proposition 2). Discounting this case, we can be sure that $\gamma_1^2 + \delta_1^2 \neq 0$, and the real solutions of these

[7] Note that homogeneous parameters can be used to accommodate the case $\ell = 0$.

[8] Since equations (55) are homogeneous, we omit the common factor $4\,|\mathcal{A}_0|^2$.

equations for α_2, β_2 are then given in terms of $\alpha_1, \beta_1, \gamma_1, \delta_1, \gamma_2, \delta_2$ by expressions (54). Note that one could also solve for α_1, β_1 in terms of $\gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$. ■

Corollary 2. *The spatial PH quintics with quaternion coefficients* (21) *that satisfy* (54), *so that* h = *constant and* $\deg(a, b) = 1$ *in* (28), *correspond to the monotone-helical PH quintics, and the quadratic common factor of their hodograph components is*

$$\gcd(x'(t), y'(t), z'(t)) = \omega(t),$$

where $\omega(t)$ *is the polynomial defined in* (28) *by*

$$\omega^2 = 4\,[(up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2] = (a^2 + b^2)^2.$$

Proof. Substituting from (54) into expressions (47) for $r = 1, 2$ we form the polynomials $u(t), v(t), p(t), q(t)$. Using MAPLE, we then obtain the quadratic

$$\begin{aligned}&\gcd(u + \mathrm{i}\,v, p - \mathrm{i}\,q) \cdot \gcd(u - \mathrm{i}\,v, p + \mathrm{i}\,q)\\ &\quad = 4(\gamma_1^2 + \delta_1^2)(1 - t)^2 + 2(\gamma_1\gamma_2 + \delta_1\delta_2)2(1 - t)t + (\gamma_2^2 + \delta_2^2)t^2\end{aligned}$$

as the common factor of the three hodograph components. Hence, the curve is a monotone-helical PH quintic. Forming the polynomials (29), we observe that the sum of their squares is proportional to the square of this quadratic. Thus, for monotone-helical PH quintics, the common factor of the hodograph components x', y', z' is the polynomial ω defined by the helicity condition $\rho = 4\,[(up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2] = \omega^2$. ■

Remark 12. For the generic case of helical PH quintics, defined by equality of the ratios (51) with non-zero values, we have $\lambda = \ell(\gamma_1^2 + \delta_1^2)$ and $\mu = 0$ in Proposition 6 for a proportionality constant ℓ. Hence, substituting into (54), the conditions for a monotone-helical PH quintic become

$$\alpha_2 = \ell\left(\alpha_1 - \tfrac{1}{4}\ell\right), \quad \beta_2 = \ell\,\beta_1.$$

This coincides with the singular case (53) of general helical PH quintics, for which the curve becomes monotone-helical, and the hodograph components possess a non-constant common factor. This proves to be a special quadratic polynomial—it is the perfect square of a linear polynomial, namely

$$\gcd(x'(t), y'(t), z'(t)) = [2(1 - t) + \ell\,t]^2.$$

In Proposition 2, we excluded curves with $\gcd(x', y', z') \neq$ constant from the quoted sufficient-and-necessary conditions for degeneration of spatial PH quintics into planar curves (other than straight lines). These cases correspond to the planar degenerations of monotone-helical PH quintics. Using MAPLE, we find that they correspond to solutions of either

$$\gamma_1 = 4\,\beta_1\delta_1 - \gamma_2 = 4\,\delta_1^2\beta_2 - 4\,\alpha_1\delta_1\gamma_2 + \gamma_2\delta_2 = 4\,\delta_1^2\alpha_2 - 4\,\alpha_1\delta_1\delta_2 + \delta_2^2 = 0$$

or

$$\begin{aligned}&\beta_1\gamma_2 + \beta_2\gamma_1 - 4\,\alpha_1\beta_1\gamma_1 - 4\,\beta_1^2\delta_1 = \gamma_1\delta_2 - \gamma_2\delta_1 + 4\,\beta_1\gamma_1^2 + 4\,\beta_1\delta_1^2\\ &\quad = 4\,\gamma_1^2\alpha_2 - 4\,\alpha_1\gamma_1\gamma_2 - 8\,\beta_1\delta_1\gamma_2 + 16\,\alpha_1\beta_1\gamma_1\delta_1 + 16\,\beta_1^2\delta_1^2 + \gamma_2^2 = 0.\end{aligned}$$

One may verify that such solutions are compatible with the forms (54) given in Proposition 6: they define plane curves whose hodograph components have a common quadratic factor, that are not covered by the conditions for planar degeneration of spatial PH quintics in Proposition 2.

Example 2. The quaternion coefficients

$$\mathcal{A}_0 = 10\,\mathbf{i} + 5\,\mathbf{j} + 10\,\mathbf{k}, \quad \mathcal{A}_1 = \frac{-3 + 15\,\mathbf{i} + 13\,\mathbf{j} + 11\,\mathbf{k}}{2}, \quad \mathcal{A}_2 = -2 + 6\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k}$$

define the hodograph

$$\begin{aligned}x'(t) &= -\,3t^4 + 14t^3 - 36t^2 + 50t - 25,\\ y'(t) &= -\,2t^4 + 2t^3 - 14t^2 - 50t + 100,\\ z'(t) &= 6t^4 - 46t^3 + 1386t^2 - 250t + 200,\end{aligned}$$

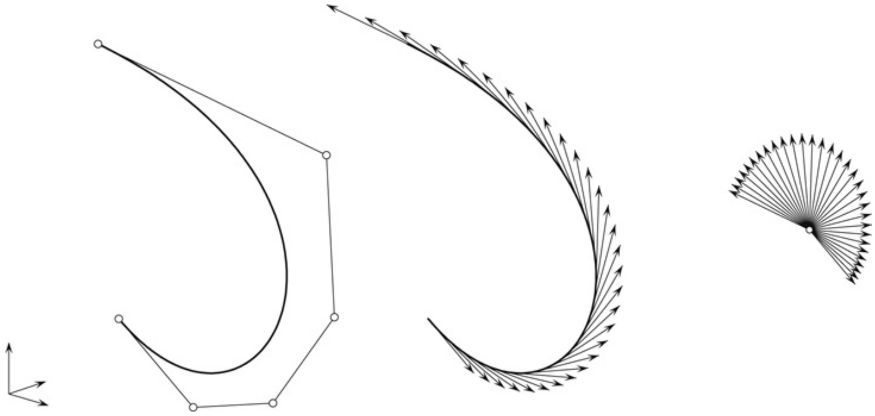

Fig. 2. A monotone-helical PH quintic, constructed using Proposition 6 with $\mathcal{A}_0 = 1.0 - 0.2\,\mathbf{i} + 0.6\,\mathbf{j} + 0.8\,\mathbf{k}$, $\alpha_1 = 0.6$, $\beta_1 = 0.2$, $\gamma_1 = -0.4$, $\delta_1 = 0.2$, $\gamma_2 = -0.8$, $\delta_2 = 1.0$, and α_2, β_2 determined by (54). Left: the PH curve with Bézier control polygon. Center: the curve tangents exhibit a fixed sense of rotation about the helical axis. Right: when regarded as emanating from a common origin, the tangents all lie on the surface of a cone about the axis.

whose components have the common quadratic factor $t^2 - 2t + 5$. Expressing $\mathcal{A}_1, \mathcal{A}_2$ in terms of $\mathcal{A}_0$ in the form (21) gives

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = \frac{(13, 7, -1, -1)}{18} \quad \text{and} \quad (\alpha_2, \beta_2, \gamma_2, \delta_2) = \frac{(22, 14, -2, -2)}{45}.$$

One can easily verify that these coefficients satisfy the conditions (54) for a monotone-helical PH quintic. Another monotone-helical PH quintic curve, constructed using Proposition 6, is shown in Fig. 2.

Remark 13. Helical PH quintics are an excellent class of curves for practical free-form design, through solutions of the first-order Hermite interpolation problem. The details of these solutions may be found in Farouki et al. (2004). Furthermore, it is shown in Farouki et al. (2008) that, among the two-parameter family of spatial PH quintics interpolating prescribed first-order Hermite data, the solutions of extremal arc length always correspond to *helical* PH quintics.

10. Closure

A comprehensive treatment of the theory of "double" PH curves and of helical polynomial curves, in terms of the complementary quaternion and Hopf map representations, has been presented. Although – as emphasized by Beltran and Monterde (2007) – the Hopf map model often provides a simpler perspective on double PH curves (which encompass all helical polynomial curves), the quaternion form has enjoyed more widespread use in practical algorithms for the construction and analysis of spatial PH curves. A detailed analysis of the relationships between these two alternative PH curve representations, and of conversions between them, was therefore needed.

The focus of the present paper has been on the basic principles underlying the two representations, without consideration of basic curve morphologies or constructive methods for generating example curves. The companion paper (Farouki et al., in press) gives a classification of double PH curve types up to degree 7, and describes their properties and algebraic algorithms for their construction.

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