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Exact rotation-minimizing frames for spatial Pythagorean-hodograph curves

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Abstract

An exact specification of the rotation-minimizing frame on a spatial Pythagorean-hodograph (PH) curve can be derived by integration of a rational function. The result is an angular function $\theta(t)$ of the curve parameter, comprising in general both rational and logarithmic terms, that specifies the orientation of the rotation-minimizing frame relative to the Frenet frame. For PH cubics and quintics, the solution employs only arithmetic operations on the curve coefficients and some complex square and cube root extractions. Moreover, the generalization to PH curves of arbitrary order entails only standard polynomial algorithms (i.e., arithmetic, greatest common divisors, and resultants), solution of a linear system, and a minimal element of polynomial root-solving. Rotation-minimizing frames are employed in computer animation, the construction of swept surfaces, and in robotics applications where the axis of a tool or probe should remain tangential to a given spatial path while minimizing changes of orientation about this axis.

Keywords: Pythagorean-hodograph curves; Differential geometry; Frenet–Serret frame; Rotation-minimizing frame; Rational function integration

1. Introduction

At (almost) each point of a regular space curve $\mathbf{r}(t)$, the *Frenet frame* defines an orthonormal basis for vectors in $\mathbb{R}^3$ aligned with the local intrinsic curve geometry. The elements of this basis are the curve tangent $\mathbf{t}$, normal $\mathbf{n}$, and binormal $\mathbf{b}$, specified [24] by

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$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \quad \mathbf{n} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{1}$$

Note that, for a polynomial or rational curve $\mathbf{r}(t)$, the unit vectors (1) do not depend rationally on the curve parameter t (however, see [31] for discussion of a special class of curves that possess rational Frenet frames).

On a regular curve (i.e., $\mathbf{r}'(t) \neq \mathbf{0}$ for all t) the tangent is defined at every point, but the normal and binormal are undefined at *inflection* points, where $\mathbf{r}''(t)$ becomes parallel to $\mathbf{r}'(t)$ or vanishes. In fact, $\mathbf{n}$ and $\mathbf{b}$ as defined by (1) may experience sudden reversals upon passing through inflections.

At each point of the curve with $\mathbf{r}' \times \mathbf{r}'' \neq \mathbf{0}$, the *osculating*, *normal*, and *rectifying* planes are spanned by the pairs of vectors $(\mathbf{t}, \mathbf{n})$, $(\mathbf{n}, \mathbf{b})$, and $(\mathbf{b}, \mathbf{t})$, respectively. The variation of the Frenet frame with curve arc length s may be described [24] by the equations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{t}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{n}, \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{b}, \tag{2}$$

where the *Darboux vector*

$$\mathbf{d} = \kappa \mathbf{b} + \tau \mathbf{t} \tag{3}$$

is defined in terms of the curvature and torsion, given by

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \qquad \text{and} \qquad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}. \tag{4}$$

These quantities are invariant under any (regular) curve re-parameterization. Equations (2) characterize the instantaneous variation of the Frenet frame as a rotation about the vector $\mathbf{d}$, at a rate given by the "total curvature"

$$\omega = |\mathbf{d}| = \sqrt{\kappa^2 + \tau^2}. \tag{5}$$

In applications requiring control of the orientation of a rigid body, as its center of mass executes a given path, alignment of the body's principal axes with the Frenet frame at each point may appear to be the obvious solution. However, other useful orthonormal frames $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ may be defined along a space curve [2]. In most contexts it is natural to choose $\mathbf{e}_1 = \mathbf{t}$, and $(\mathbf{e}_2, \mathbf{e}_3)$ are then obtained from $(\mathbf{n}, \mathbf{b})$ by a rotation in the normal plane:

$$\begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{6}$$

This allows us to remedy the indeterminacy of the Frenet frame at inflections, and also provides additional flexibility to adapt the orthonormal frame to the requirements of specific applications. An example is the *rotation-minimizing frame* introduced by Klok [23] for the construction of swept surfaces, which are defined by the motion of a planar "profile" curve along a spatial "sweep" curve. The profile curve remains in the normal plane of the sweep curve, but the variation of its orientation in that plane must be specified.

For the purpose of orienting a profile curve along a given sweep curve, the rotation-minimizing frame is preferable to the Frenet frame in the following sense. By substituting (3) into (2), we obtain

$$\begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where dots indicate derivatives with respect to s. This reveals that $\mathbf{t}$ changes at instantaneous rate κ in the direction of $\mathbf{n}$. The instantaneous change of $\mathbf{n}$ has two components: rate $-\kappa$ in the direction of $\mathbf{t}$, and rate τ in the direction of $\mathbf{b}$. Finally, $\mathbf{b}$ changes at instantaneous rate $-\tau$ in the direction of $\mathbf{n}$. Now changes in the direction of $\mathbf{t}$ are unavoidable if we choose a basis with $\mathbf{e}_1 = \mathbf{t}$. The change of $\mathbf{n}$ in the direction of $\mathbf{b}$, and of $\mathbf{b}$ in the direction of $\mathbf{n}$, however, correspond to a rotation of these vectors in the normal plane.

By a suitable choice for the variation of the angle θ in (6), an orthonormal frame that eliminates this "unnecessary" rotation may be defined. Klok [23] showed that, with $\mathbf{e}_1 = \mathbf{t}$, the remaining basis vectors must satisfy

$$\mathbf{e}_k'(t) = -\frac{\mathbf{r}''(t) \cdot \mathbf{e}_k(t)}{|\mathbf{r}'(t)|^2}\mathbf{r}'(t), \quad k = 2, 3$$

in order to define such a rotation-minimizing frame. Substituting from (6), one can verify that this amounts to the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -|\mathbf{r}'|\tau = -|\mathbf{r}'|\frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} \tag{7}$$

for the angular function $\theta(t)$ used to obtain $(\mathbf{e}_2, \mathbf{e}_3)$ from $(\mathbf{n}, \mathbf{b})$. Hence, as noted by Guggenheimer [17], this function has the form [1]

$$\theta(t) = \theta_0 - \int_0^t \tau(u)|\mathbf{r}'(u)|\,\mathrm{d}u. \tag{8}$$

Unfortunately, the above integral does not admit a closed-form reduction for the polynomial and rational curves employed in computer graphics, computer-aided design, robotics, and similar applications. Consequently, a number of schemes have been proposed to approximate the rotation-minimizing frame of a given curve, or to approximate a given curve by "simpler' segments (e.g., cicular arcs) with known rotation-minimizing frames [20–22,32].

Approximation schemes always incur concerns over accuracy, robustness, and data volume. Our intent here is to avoid such concerns by deriving *exact* rotation-minimizing frames for a special class of curves—the *Pythagorean-hodograph* (PH) *curves*. PH curves incorporate special algebraic structures, that offer many computational advantages [1,7,8,12–14] in design and manufacturing applications. For example, their arc lengths can be computed *precisely*, they have *rational* offsets, and one

[1] An incorrect sign before the integral is given in [17].

can formulate real-time CNC interpolators that drive multi-axis machines along curved paths, at fixed or varying speeds, from their exact analytic descriptions [11,15,28].

For PH curves, the integrand in (8) is a *rational function* and thus admits closed-form integration. Now the integral of a rational function involves, in general, both rational and transcendental (logarithmic) terms. The rational term requires only arithmetic operations on polynomials, a greatest common divisor, and the solution of a linear system for its determination. Although the logarithmic terms cannot, in general, be determined without introducing new algebraic constants, methods have been proposed [26,27] that minimize the amount of polynomial root-solving required to determine these constants.

2. Rotation-minimizing frames on PH curves

For the hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ of a polynomial curve to satisfy the Pythagorean equation

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t), \tag{9}$$

where $\sigma(t)$ is a polynomial, it is sufficient and necessary that its components be expressible [6] in terms of polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ in the form

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ y'(t) &= 2[u(t)q(t) + v(t)p(t)], \\ z'(t) &= 2[v(t)q(t) - u(t)p(t)], \\ \sigma(t) &= u^2(t) + v^2(t) + p^2(t) + q^2(t). \end{aligned} \tag{10}$$

Choi et al. [4] give an elegant characterization of the hodograph (10) in terms of quaternions. Consider the quaternion polynomial

$$\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}, \tag{11}$$

where the non-commutative basis elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy [25] the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

and hence $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The hodograph (10) can be constructed as the quaternion product

$$\begin{aligned} \mathbf{r}'(t) &= \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t) \\ &= [u^2(t) + v^2(t) - p^2(t) - q^2(t)]\mathbf{i} \\ &\quad + 2[u(t)q(t) + v(t)p(t)]\mathbf{j} \\ &\quad + 2[v(t)q(t) - u(t)p(t)]\mathbf{k}, \end{aligned} \tag{12}$$

where $\mathcal{A}^*(t) = u(t) - v(t)\mathbf{i} - p(t)\mathbf{j} - q(t)\mathbf{k}$ is the *conjugate* of (11).

The form (12) is invariant [9] under arbitrary spatial rotations. A rotation by angle θ about an axis specified by the unit vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ is characterized by a unit quaternion $\mathcal{U} = \cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\mathbf{n}$ satisfying $\mathcal{U}\mathcal{U}^* = 1$—the rotated instance of the hodograph can be written as

$$\tilde{\mathbf{r}}'(t) = \mathcal{U}\mathbf{r}'(t)\mathcal{U}^* = \tilde{\mathcal{A}}(t)\mathbf{i}\tilde{\mathcal{A}}^*(t),$$

where we define the new quaternion polynomial $\tilde{\mathcal{A}}(t) = \mathcal{U}\mathcal{A}(t)$.

The simplest non-trivial spatial PH curves are cubics—they correspond to segments of non-circular helices (i.e., the ratio κ/τ of curvature to torsion is constant), and may be characterized by certain geometrical constraints on their Bézier control polygons [14]. To guarantee sufficient shape flexibility for typical applications, we must employ quintic PH curves. The construction of spatial PH quintics as first-order Hermite interpolants is described in [10].

In lieu of the Frenet frame and rotation-minimizing frame, Choi and Han [3] have proposed the *rational* "Euler–Rodrigues frame" defined by

$$\mathbf{e}_1 = \mathcal{U}(t)\mathbf{i}\,\mathcal{U}^*(t), \quad \mathbf{e}_2 = \mathcal{U}(t)\mathbf{j}\,\mathcal{U}^*(t), \quad \mathbf{e}_3 = \mathcal{U}(t)\mathbf{k}\,\mathcal{U}^*(t),$$

where $\mathcal{U}(t) = \mathcal{A}(t)/|\mathcal{A}(t)|$, and they characterize the angular velocity of this frame relative to the rotation-minimizing frame for PH cubics and quintics. Sufficient and necessary conditions for the Euler–Rodrigues frame to coincide with a rotation-minimizing frame on PH quintics are also given.

Our intent is to derive *exact* rotation-minimizing frames for PH curves. We begin by writing the relation (7) in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{p(t)}{q(t)}, \tag{13}$$

where

$$p(t) = |\mathbf{r}'(t)|[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t), \quad q(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)|^2.$$

Now if $\mathbf{r}(t)$ is a polynomial curve of degree n, we have

$$\deg(\mathbf{r}' \times \mathbf{r}'') = 2n - 4 \qquad \text{and} \qquad \deg((\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''') = 3n - 9$$

due to cancellation of highest-order terms, while $|\mathbf{r}'(t)|$ is the square root of a polynomial of degree $2n - 2$ in t. In general, the latter term precludes the possibility of a closed-form integration of equation (13).

For the PH curves, however, some striking simplifications arise. First, we have $|\mathbf{r}'(t)| = \sigma(t)$—a *polynomial* (of degree $n - 1$) in t, and the right-hand side of (13) is thus a rational function. Furthermore, a common factor may be cancelled from the numerator and denominator. Substituting (10) into

$$|\mathbf{r}' \times \mathbf{r}''|^2 = (y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2,$$

we deduce for PH curves the remarkable factorization

$$|\mathbf{r}' \times \mathbf{r}''|^2 = \sigma^2\rho,$$

where $\sigma = u^2 + v^2 + p^2 + q^2$ as in (10), and the polynomial ρ is defined by

$$\rho = 4[(up' - u'p)^2 + (uq' - u'q)^2 + (vp' - v'p)^2 + (vq' - v'q)^2 + 2(uv' - u'v)(pq' - p'q)],$$

with $\deg(\rho) = 2n - 6$. Cancelling the common factor $\sigma(t)$ from $p(t)$ and $q(t)$, we may write (13) in the case of a PH curve as

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\sigma(t)\,\rho(t)}. \tag{14}$$

Now for a degree-n PH curve, $\deg((\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''') = 3n - 9$, $\deg(\sigma) = n - 1$, and $\deg(\rho) = 2n - 6$. Hence, the right-hand side is a *proper* rational fraction whose numerator is degree 2 less than the denominator. Specifically, for PH cubics $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ and ρ are constants, while σ is quadratic. For PH quintics, $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ is of degree 6, while σ and ρ are both quartic in t.

For $n \geqslant 5$, the partial fraction expansion of (14) is defined by polynomials $a(t)$ and $b(t)$, with $\deg(a) \leqslant n - 2$ and $\deg(b) \leqslant 2n - 7$, such that

$$[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t) = a(t)\,\rho(t) + b(t)\,\sigma(t). \tag{15}$$

This is an identity among polynomials of degree $3n - 8$. Equating coefficients of like terms yields $3n - 7$ linear equations for the $(n - 1) + (2n - 6) = 3n - 7$ unknown coefficients of $a(t)$ and $b(t)$. Solving for these coefficients, we have

$$\frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\sigma(t)\,\rho(t)} = \frac{a(t)}{\sigma(t)} + \frac{b(t)}{\rho(t)}. \tag{16}$$

With $\theta = \theta_0$ when $t = 0$, integration of (13) then yields

$$\theta(t) = \theta_0 - \int_0^t \frac{a(\tau)}{\sigma(\tau)}\,\mathrm{d}\tau - \int_0^t \frac{b(\tau)}{\rho(\tau)}\,\mathrm{d}\tau. \tag{17}$$

3. Integration of rational functions

For PH cubics, the integration of (14) is a trivial task, since the numerator is a constant and the denominator is quadratic. For PH quintics, we use (17), where $a(t)$, $b(t)$ are at most cubic and $\sigma(t)$, $\rho(t)$ are quartics. Before treating these specific cases in detail, we review some general principles governing the integration of rational functions in as exact a manner as possible.

In general, the indefinite integral of a rational function

$$\int \frac{p(t)}{q(t)}\,\mathrm{d}t, \tag{18}$$

where $\gcd(p, q) = 1$ and $\deg(p) < \deg(q)$, yields a function with both rational and transcendental (logarithmic) terms. The naive approach is to attempt to completely factorize $q(t)$ into linear factors over $\mathbb{C}$, or into linear/quadratic factors over $\mathbb{R}$, and then perform a decomposition of the integrand into partial fractions. In general, however, such factorizations incur algebraic constants that can only be approximated in floating-point arithmetic, even though the final integral may not depend on all of them in an essential way.

The study of algorithmic integration of rational functions, with minimal introduction of algebraic constants, was motivated by the advent of computer algebra systems [5,29]. The first step involves extracting the rational part of (18) by the method of Horowitz [18,19]. Using Euclid's algorithm [30] to compute $\gcd(q(t), q'(t))$, we define

$$q_1(t) = \gcd(q(t), q'(t)) \quad \text{and} \quad q_2(t) = \frac{q(t)}{\gcd(q(t), q'(t))}$$

so that

$$q(t) = q_1(t) q_2(t). \tag{19}$$

We may assume, without loss of generality, that $q_2(t)$ is a monic polynomial (i.e., that its highest-order coefficient is 1). Note that the roots of $q_1(t)$ are the *multiple* roots of $q(t)$. Specifically, if z is a root of q of multiplicity $m \geqslant 2$, then it is a root of q_1 of multiplicity $m-1$. Moreover, each of the distinct (simple or multiple) roots of $q(t)$ is a *simple* root of $q_2(t)$.

We seek polynomials $p_1(t)$, $p_2(t)$ such that $p(t)/q(t)$ can be expressed as

$$\frac{p(t)}{q(t)} = \left(\frac{p_1(t)}{q_1(t)}\right)' + \frac{p_2(t)}{q_2(t)}. \tag{20}$$

Carrying out the differentiation and simplifying, we obtain the relation

$$p(t) = q_2(t) p_1'(t) - s(t) p_1(t) + q_1(t) p_2(t), \tag{21}$$

where we set

$$s(t) = \frac{q_1'(t) q_2(t)}{q_1(t)}.$$

Now by differentiating (19), we can re-write this as

$$s(t) = \frac{q'(t)}{\gcd(q(t), q'(t))} - q_2'(t),$$

and since $\gcd(q, q')$ divides q' without remainder, $s(t)$ must be a polynomial. Since the polynomials $p(t)$, $q_1(t)$, $q_2(t)$, $s(t)$ in (21) are known, comparing like terms in this equation yields a linear system of equations for the unknown coefficients of the polynomials $p_1(t)$, $p_2(t)$. Once these coefficients have been determined, we can express the integral of (20) as

$$\int \frac{p(t)}{q(t)}\, dt = \frac{p_1(t)}{q_1(t)} + \int \frac{p_2(t)}{q_2(t)}\, dt, \tag{22}$$

where $q_2(t)$ is "square-free" (i.e., has no multiple roots).

The integral on the right in (22) is the transcendental part. If $\deg(q_2) = N$, then $q_2(t)$ has distinct roots $z_1, \ldots, z_N$ and the integrand has the complete partial fraction decomposition

$$\frac{p_2(t)}{q_2(t)} = \sum_{k=1}^{N} \frac{c_k}{t - z_k}, \tag{23}$$

and hence we have

$$\int \frac{p_2(t)}{q_2(t)}\,\mathrm{d}t = \sum_{k=1}^{N} c_k \ln(t - z_k).$$

Since $p_2(t)$ and $q_2(t)$ are real, complex terms in this sum occur in conjugate pairs, and may be combined to yield explicitly real expressions.

In general, the roots $z_1, \ldots, z_N$ are algebraic numbers that do not admit exact, finite decimal representations. In floating-point arithmetic, they must be approximated. A "defect" of the complete partial-fraction decomposition (23) is that it employs all these roots, although the integral may ultimately be expressible in a form that does not require all of them.

The following approach, due to Rothstein [26] and Trager [27], evaluates such integrals with a minimal algebraic extension of the set of constants. Let $f(t)$, $g(t)$ be polynomials satisfying $\deg(f) < \deg(g)$, $\gcd(f, g) = 1$, with $g(t)$ monic and square-free. Then if $c_1, \ldots, c_h$ are the distinct roots of

$$h(c) = \mathrm{Resultant}_t(f(t) - cg'(t), g(t)) = 0 \tag{24}$$

we have

$$\int \frac{f(t)}{g(t)}\,\mathrm{d}t = \sum_{k=1}^{h} c_k \ln v_k(t),$$

where the polynomials $v_1(t), \ldots, v_h(t)$ are defined by

$$v_k(t) = \gcd(f(t) - c_k g'(t), g(t)).$$

Apart from the need for a numerical determination of the roots of (24), this method is essentially exact for rational functions of arbitrary order.

4. Frames for PH cubics and quintics

In principle, the procedure described in Section 3 allows rotation-minimizing frames to be computed for PH curves of arbitrary order. We now give more specific details for the PH cubics and quintics. The former admit a particularly simple closed-form reduction, but in general PH cubics do not offer sufficient shape flexibility for free-form design applications. The PH quintics are a little more involved, but provide much greater geometrical versatility.

PH cubics are constructed by inserting four linear polynomials, expressed in the Bernstein form $u(t) = u_0(1 - t) + u_1 t$ and similarly for $v(t)$, $p(t)$, $q(t)$, into (10), and integrating the hodograph. In this case, $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)$ and $\rho(t)$ are both constants, and their ratio is the quantity

$$k = 2(u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0).$$

The orientation of the rotation-minimizing frame relative to the Frenet frame is thus defined by the function

$$\theta(t) = \theta_0 - k\int_0^t \frac{\mathrm{d}\tau}{\sigma(\tau)},$$

where the parametric speed is the quadratic

$$\sigma(t) = \sigma_0(1-t)^2 + \sigma_1 2(1-t)t + \sigma_2 t^2$$

with Bernstein coefficients

$$\sigma_0 = u_0^2 + v_0^2 + p_0^2 + q_0^2,$$

$$\sigma_1 = u_0u_1 + v_0v_1 + p_0p_1 + q_0q_1,$$

$$\sigma_2 = u_1^2 + v_1^2 + p_1^2 + q_1^2.$$

Hence [16] we have

$$\theta(t) = \theta_0 + \frac{k}{\sqrt{\sigma_1^2 - \sigma_0\sigma_2}} \tanh^{-1} \frac{(\sigma_2 - 2\sigma_1 + \sigma_0)t + \sigma_1 - \sigma_0}{\sqrt{\sigma_1^2 - \sigma_0\sigma_2}}$$

or

$$\theta(t) = \theta_0 - \frac{k}{\sqrt{\sigma_0\sigma_2 - \sigma_1^2}} \tan^{-1} \frac{(\sigma_2 - 2\sigma_1 + \sigma_0)t + \sigma_1 - \sigma_0}{\sqrt{\sigma_0\sigma_2 - \sigma_1^2}}$$

according to whether $\sigma_1^2 - \sigma_0\sigma_2$ is positive or negative.

PH quintics are defined by inserting four quadratic polynomials, $u(t) = u_0(1-t)^2 + u_1 2(1-t)t + u_2 t^2$ and similarly for $v(t), p(t), q(t)$, into (10), and integrating. In this case, $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)$ is of degree 6, while $\rho(t)$ and $\sigma(t)$ are quartics. We use the form (17), where $a(t)$ and $b(t)$ are determined by solving the linear system defined by Eq. (15).

We begin by dividing the numerator and denominator of the integrands in (17) by the highest-order coefficient of the denominator, so we can assume that $\sigma(t)$ and $\rho(t)$ are monic. These two quartics can be explicitly factorized by invoking Ferrari's method [30] to compute their roots (see Appendix A). If we denote these roots by z_1, z_2, z_3, z_4 and w_1, w_2, w_3, w_4, respectively, the coefficients c_1, c_2, c_3, c_4 and d_1, d_2, d_3, d_4 in the partial fraction expansions

$$\frac{a(t)}{\sigma(t)} = \sum_{k=1}^{4} \frac{c_k}{t - z_k} \quad \text{and} \quad \frac{b(t)}{\rho(t)} = \sum_{k=1}^{4} \frac{d_k}{t - w_k}$$

can be found by clearing the denominators, and setting t equal to each of the roots in succession, to obtain

$$c_k = \frac{a(z_k)}{\prod_{j \neq k}(z_k - z_j)} \quad \text{and} \quad d_k = \frac{b(w_k)}{\prod_{j \neq k}(w_k - w_j)} \tag{25}$$

for $k = 1, \ldots, 4$. Integration then gives

$$\theta(t) = \theta_0 - \sum_{k=1}^{4} c_k \ln(t - z_k) + d_k \ln(t - w_k).$$

Now since $\sigma(t)$ and $\rho(t)$ are real polynomials, their complex roots must occur as conjugate pairs, and the corresponding partial-fraction coefficients are also complex conjugates. Logarithmic terms that correspond to such pairs can be combined to give explicitly real expressions: for example, if z, $\bar{z}$ and c, $\bar{c}$ are conjugate roots and coefficients, we have

$$c \ln(t - z) + \bar{c} \ln(t - \bar{z}) = 2[\mathrm{Re}(c) \ln|t - z| - \mathrm{Im}(c) \arg(t - z)].$$

Here $\arg(t - z)$ must be interpreted as a *continuous* function—i.e., it should not be reduced modulo 2π.

Example. Consider the PH quintic with $\mathbf{r}(0) = (0, 0, 0)$ and Bernstein coefficents $(u_0, u_1, u_2) = (2, 0, 2)$, $(v_0, v_1, v_2) = (1, 1, 0)$, $(p_0, p_1, p_2) = (0, -2, 0)$, $(q_0, q_1, q_2) = (1, 2, 1)$ for the quadratic polynomials in (10). In this case,

$$\begin{aligned}[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t) = {} & 16032t^6 - 49152t^5 + 74592t^4 - 66560t^3 \\ & + 32256t^2 - 6912t - 576\end{aligned}$$

and

$$\sigma(t) = 37t^4 - 72t^3 + 46t^2 - 12t + 6,$$

$$\rho(t) = 80t^4 - 544t^3 + 3376t^2 - 2976\,t + 720.$$

The partial fraction decomposition (16) is then defined by the polynomials

$$a(t) = 8t^2 - 8t \quad \text{and} \quad b(t) = 416t^2 - 384t - 96.$$

Ferrari's method (see Appendix A) then gives

$$z_1, \bar{z}_1 = 0.012018388019 \pm 0.394440354901\,\mathrm{i}$$
$$z_2, \bar{z}_2 = 0.960954584954 \pm 0.343344423853\,\mathrm{i}$$

and

$$w_1, \bar{w}_1 = 0.493692520439 \pm 0.069302855662\,\mathrm{i}$$
$$w_2, \bar{w}_2 = 2.906307479561 \pm 5.269302855662\,\mathrm{i}$$

for the roots of $\sigma(t)$ and $\rho(t)$. The corresponding partial fraction coefficients, obtained from (25), are

$$c_1, \bar{c}_1 = -0.096099539514 \mp 0.030185214163\,\mathrm{i}$$
$$c_2, \bar{c}_2 = 0.096099539514 \mp 0.014590334426\,\mathrm{i}$$

and

$$d_1, \bar{d}_1 = \pm 0.5\mathrm{i}, \quad d_2, \bar{d}_2 = \mp 0.5\mathrm{i}.$$

In terms of the above complex values, we now have

$$\theta(t) = \theta_0 - 2\sum_{k=1}^{2}[\mathrm{Re}(c_k)\ln|t - z_k|\mathrm{Im}(c_k)\arg(t - z_k)]$$
$$- 2\sum_{k=1}^{2}[\mathrm{Re}(d_k)\ln|t - w_k| - \mathrm{Im}(d_k)\arg(t - w_k)],$$

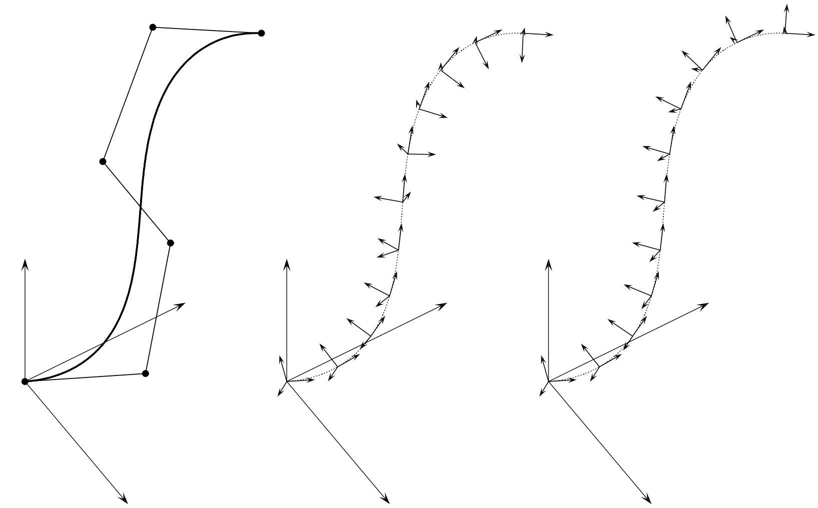

Fig. 1. A PH quintic space curve (left), with Bézier control polygon. Also shown are the Frenet frame (center) and rotation-minimizing frame (right).

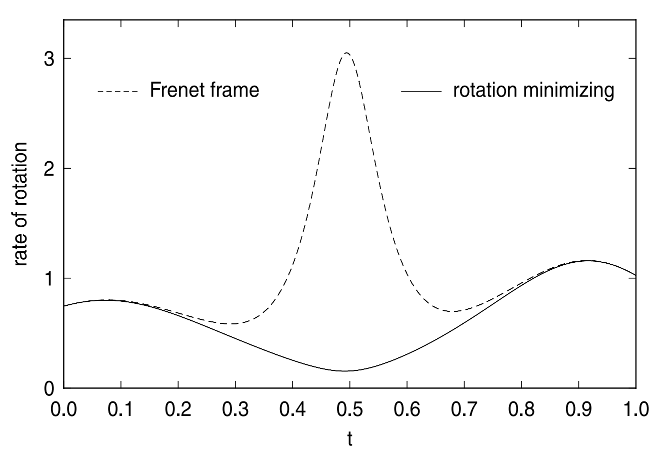

Fig. 2. Comparison of instantaneous rates of rotation for the Frenet frame and the rotation-minimizing frame along the PH quintic shown in Fig. 1.

where the integration constant θ_0 may be chosen such that $\theta(0) = 0$. Fig. 1 compares the variation of the Frenet and rotation-minimizing frames along the example PH curve. A quantitative comparison is presented in Fig. 2, which shows the instantaneous rates of rotation for both frames—namely, $\omega = \sqrt{\kappa^2 + \tau^2}$ for the Frenet frame, and κ for the rotation-minimizing frame. It is apparent that, compared to the rotation-minimizing frame, the motion of the Frenet frame incurs a great deal of "unnecessary" rotation.

5. Closure

For applications such as computer animation, robotics, and construction of swept surfaces, in which the computation of rotation-minimizing frames plays a key role, the spatial PH curves offer exact solutions that obviate concerns over the accuracy, efficiency, and data volume of approximation schemes. For PH cubics and quintics, in particular, integration of the torsion to obtain the relative orientation of the rotation-minimizing and Frenet frames involves a partial-fraction decomposition of rational functions with (at most) quartic denominators, which admit complete factorization through the determination of their roots by radicals. The resulting angular function $\theta(t)$ specifying the rotation-minimizing frame comprises, in general, a rational function plus a sum of logarithmic terms. The method can, in principle, be extended to PH curves of higher order with minimal introduction of new algebraic constants. There is also a quite straightforward extension to PH spline curves.

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Appendix A. Ferrari's method for quartics

The four roots of the quartic equation

$$t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0 \tag{A.1}$$

may be computed by Ferrari's method [30]. Namely, let z be a real root of the *resolvent* cubic equation

$$z^3 + c_2 z^2 + c_1 z + c_0 = 0 \tag{A.2}$$

with $c_2 = -a_2$, $c_1 = a_1 a_3 - 4a_0$, and $c_0 = 4a_2 a_0 - a_1^2 - a_3^2 a_0$. Then the roots of (A.1) are the same as the roots of the two quadratic equations

$$t^2 + \left(\tfrac{1}{2}a_3 \pm E\right) t + \left(\tfrac{1}{2}z \pm F\right) = 0, \tag{A.3}$$

where we define

$$E = \tfrac{1}{2}\sqrt{a_3^2 + 4(z - a_2)} \quad \text{and} \quad F = \frac{a_3 z - 2a_1}{4E}. \tag{A.4}$$

The roots of the cubic (A.2) may be obtained by Cardano's method [30]. Set

$$Q = \frac{3c_1 - c_2^2}{9}, \quad R = \frac{9c_1c_2 - 27c_0 - 2c_2^3}{54}, \quad \Delta = Q^3 + R^2,$$

and let S be any of the three complex values specified by

$$S = \left(R + \sqrt{\Delta}\right)^{1/3}. \tag{A.5}$$

Then, writing

$$A = S - \frac{Q}{S} \quad \text{and} \quad B = S + \frac{Q}{S}, \tag{A.6}$$

the roots of (A.2) are given by

$$z = \begin{cases} -\frac{1}{3}c_2 + A, \\ -\frac{1}{3}c_2 - \frac{1}{2}A + \frac{1}{2}\sqrt{3}\,\mathrm{i}B, \\ -\frac{1}{3}c_2 - \frac{1}{2}A - \frac{1}{2}\sqrt{3}\,\mathrm{i}B. \end{cases} \tag{A.7}$$

One of the roots (A.7) is real and the other two are complex conjugates when $\Delta > 0$; all three roots are real and distinct when $\Delta < 0$; and when $\Delta = 0$ there is a multiple root. Note that, even if all three roots are real, complex arithmetic is generally required to evaluate the quantities (A.5)–(A.7), and also to compute (A.4) and the roots of the quadratic equations (A.3).

References

[1] G. Albrecht, R.T. Farouki, Construction of C^2 Pythagorean hodograph interpolating splines by the homotopy method, Adv. Comput. Math. 5 (1996) 417–442.
[2] R.L. Bishop, There is more than one way to frame a curve, Am. Math. Monthly 82 (1975) 246–251.
[3] H.I. Choi, C.Y. Han, Euler–Rodrigues frames on spatial Pythagorean-hodograph curves, Comput. Aided Geom. Design 19 (2002) 603–620.
[4] H.I. Choi, D.S. Lee, H.P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, Adv. Comput. Math. 17 (2002) 5–48.
[5] J.H. Davenport, Y. Siret, E. Tournier, Computer Algebra, Academic Press, London, 1988.
[6] R. Dietz, J. Hoschek, B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, Comput. Aided Geom. Design 10 (1993) 211–229.
[7] R.T. Farouki, Pythagorean hodograph curves in practical use, in: R.E. Barnhill (Ed.), Geometry Processing for Design and Manufacturing, SIAM, Philadelphia, PA, 1992, pp. 3–33.
[8] R.T. Farouki, The conformal map $z \to z^2$ of the hodograph plane, Comput. Aided Geom. Design 11 (1994) 363–390.
[9] R.T. Farouki, M. al-Kandari, T. Sakkalis, Structural invariance of spatial Pythagorean hodographs, Comput. Aided Geom. Design 19 (2002) 395–407.
[10] R.T. Farouki, M. al-Kandari, T. Sakkalis, Hermite interpolation by rotation-invariant spatial Pythagorean-hodograph curves, Adv. Comput. Math. 17 (2002) 369–383.

[11] R.T. Farouki, J. Manjunathaiah, D. Nicholas, G.-F. Yuan, S. Jee, Variable feedrate CNC interpolators for constant material removal rates along Pythagorean hodograph curves, Comput. Aided Design 30 (1998) 631–640.
[12] R.T. Farouki, C.A. Neff, Hermite interpolation by Pythagorean hodograph quintics, Math. Comput. 64 (1995) 1589–1609.
[13] R.T. Farouki, T. Sakkalis, Pythagorean hodographs, IBM J. Res. Develop. 34 (1990) 736–752.
[14] R.T. Farouki, T. Sakkalis, Pythagorean hodograph space curves, Adv. Comput. Math. 2 (1994) 41–66.
[15] R.T. Farouki, S. Shah, Real-time CNC interpolators for Pythagorean hodograph curves, Comput. Aided Geom. Design 13 (1996) 583–600.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980.
[17] H. Guggenheimer, Computing frames along a trajectory, Comput. Aided Geom. Design 6 (1989) 77–78.
[18] E. Horowitz, Algorithm for symbolic integration of rational functions, Ph.D. Thesis, University of Wisconsin, 1969.
[19] E. Horowitz, Algorithms for partial fraction decomposition and rational function integration, in: Proceedings of Second ACM Symposium on Symbolic and Algebraic Manipulation, 1971, pp. 441–457.
[20] B. Jüttler, Generating rational frames of space curves via Hermite interpolation with Pythagorean hodograph cubic splines, in: Geometric Modeling and Processing '98, Bookplus Press, 1998, pp. 83–106.
[21] B. Jüttler, C. Mäurer, Cubic Pythagorean hodograph spline curves and applications to sweep surface modelling, Comput. Aided Design 31 (1999) 73–83.
[22] B. Jüttler, C. Mäurer, Rational approximation of rotation minimizing frames using Pythagorean-hodograph cubics, J. Geom. Graphics 3 (1999) 141–159.
[23] F. Klok, Two moving coordinate frames for sweeping along a 3D trajectory, Comput. Aided Geom. Design 3 (1986) 217–229.
[24] E. Kreyszig, Differential Geometry, University of Toronto Press, 1959.
[25] J. Roe, Elementary Geometry, Oxford University Press, 1993.
[26] M. Rothstein, Aspects of symbolic integration and simplification of exponential and primitive functions, Ph.D. Thesis, University of Wisconsin, 1976.
[27] B.M. Trager, Algebraic factoring and rational function integration, in: Proceedings of 1976 ACM Symposium on Symbolic and Algebraic Computation, 1976, pp. 219–226.
[28] Y.-F. Tsai, R.T. Farouki, B. Feldman, Performance analysis of CNC interpolators for time-dependent feedrates along PH curves, Comput. Aided Geom. Design 18 (2001) 245–265.
[29] J. von zur Gathen, J. Gerhard, Modern Computer Algebra, Cambridge University Press, Cambridge, MA, 1999.
[30] J.V. Uspensky, Theory of Equations, McGraw-Hill, New York, 1948.
[31] M. Wagner, B. Ravani, Curves with rational Frenet-Serret motion, Comput. Aided Geom. Design 15 (1997) 79–101.
[32] W. Wang, B. Joe, Robust computation of the rotation minimizing frame for sweep surface modelling, Comput. Aided Design 29 (1997) 379–391.