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Introduction

In this project, we study and describe category theory from the purview of the Modern Algebra course. In our examples, and in the motivation for the material, we keep in mind a student of Modern Algebra. As such, we use familiar algebraic structures in our examples: sets, groups, rings, and fields. At times, we introduce new structures not difficult to understand for a Modern Algebra student.

Samuel Eilenberg and Saunders Mac Lane introduced category theory from algebraic topology necessities, however, the language of categories is beneficial in other areas, even beyond mathematics[1]. Indeed, categories do not require a knowledge of higher mathematics, but experience with a variety of algebraic structures is beneficial for understanding the motivation. There is an ongoing debate about the appropriateness of introducing categorical language early on in a study of abstract algebra, and one of our sources, Paulo Aluffi's book [1], attempts this objective. Still, we think it is still important to have exposure to examples of algebraic structures, which makes category theory an interesting subject to explore for a Modern Algebra student.

One of the main ideas passing through the material is that everything is determined by morphisms. This shifts the set-theoretic perspective where sets are the first importance and everything is composed of sets. We introduce categories, morphisms, functors and natural transformations shortly. Objects, as we later show, are fully determined by their morphisms, making categories determined by a collection of morphisms. Functors are essentially morphisms between categories, while natural transformations are morphisms between functors. In fact, if one wants to strip away the definitions of category theory, all that is left are morphisms. However, the language of categories makes it easier to communicate ideas about abstract algebraic structures.

We conclude our project with one of the most important results in category theory: Yoneda lemma. Behind its succinct statement, it shows that objects are completely determined by their relationships to other objects, where relationships can be viewed as, again, morphisms. This proves the main idea of the material — objects, which are representable as collections of morphisms, are unique representations of relationships between other objects.

[1]Category theoretic notions have been especially fruitful in Computer Science, and particularly in the development of functional programming languages such as Haskell and Lisp.

Categories

2.1 Categories

Definition 1. A **category** $\mathcal{C}$ consists of a collection of **objects** $\text{ob}(\mathcal{C})$ and a collection of **morphisms** $\text{mor}(\mathcal{C})$ such that:

- Each morphism $f \in \text{mor}(\mathcal{C})$ has a specified **domain** object x and **codomain** object y where $x, y \in \text{ob}(\mathcal{C})$. We denote this by $f : x \to y$. The collection of morphism between x and y is denoted $C(x, y), \text{Hom}_{\mathcal{C}}(x, y)$, or simply $\text{Hom}(x, y)$ if the context is clear.
- For each object $x \in \text{ob}(\mathcal{C})$ there exists an **identity morphism** $1_x : x \to x \in \text{mor}(\mathcal{C})$.
- For any two morphisms $f : x \to y$ and $g : y \to z$, there exists a **composite morphism** $gf : x \to z$.
- (Associative Law) For any morphisms $f : w \to x, g : x \to y, h : y \to z$, the composite morphisms $h(gf) : w \to z$ and $(hg)f : w \to z$ are equal.
- (Identity Law) For any morphism $f : x \to y$, we have $1_y f = f$ and $f 1_x = f$.

Example 1. (a) The category of sets, denoted **Set**, consists of sets as objects and functions between sets as morphisms. To verify that **Set** is a category, note that each function between sets has a specified domain and codomain, there exists an identity function for each set, and ordinary function composition satisfies the required properties of composition for morphisms.

(b) The category of groups, denoted **Group**, consists of groups as objects, and group homomorphisms between groups as morphisms. Since the composition of group homomorphisms is a group homomorphism, we see that **Group** is a category. Similarly, the category of rings, denoted **Ring**, consists of rings as objects, and ring homomorphisms as morphisms. Likewise, we can define a category of fields **Field**, and a category of abelian groups **Ab**.

(c) The category of posets (partially ordered sets), denoted **Poset** consists of partially ordered sets as objects, and monotone functions, that is functions $f : X \to Y$ such that $a \leq_X b \implies f(a) \leq_Y f(b)$, as morphisms.

Recalling Russell's paradox, note that $\text{ob}(\textbf{Set})$, the collection of all sets, and $\text{mor}(\textbf{Set})$, the collection of all functions between all sets, are too "large" to be sets. Indeed, in our definition of a category, the word "collection" is used rather than "set". While the objects of **Set** do not form a set, given two sets A and B, $\text{Hom}(A, B) = B^A$, the collection of functions from A to B, is a set. These notions are made precise in the following definitions.

Definition 2. A category $\mathcal{C}$ is **small** if $\text{mor}(\mathcal{C})$ is a set. Since there is a one-to-one correspondence between the identity morphisms and the objects of a category, this also implies that $\text{ob}(\mathcal{C})$ is a set.

Definition 3. A category $\mathcal{C}$ is **locally small** if for every $x, y \in \text{ob}(\mathcal{C})$, $\text{Hom}(x, y)$ is a set. For the purposes of this text, we will be working with categories of this type.

2.2 Morphisms

Remark 1. From previous experience in mathematics, one might be tempted to ascribe too much importance to the objects of a category: to think of the category in terms of its objects. In the examples we have discussed thus far, all morphisms have been functions between sets. This function-set intuition fails us in some regard, carrying the assumption that morphisms are *determined* by objects. For instance, in case of **Group**, the objects—groups, seem to point toward a natural choice for structure preserving morphisms: group homomorphisms. Likewise in the case of **Ring**. Categories of this form are called **concrete** categories.

To break from this intuition, we will consider a category defined by a group and a poset.

Example 2. (a) Given a group G, we can consider the category BG with $\mathrm{ob}(\mathrm{B}G) = \{*\}$, $\mathrm{Hom}(*, *) = G$, with composition as group multiplication. Associativity is inherited from the group multiplication, and the identity element $1 \in G$ acts as the identity morphism. Note that the object in this case is entirely arbitrary, and does not determine the morphisms.

(b) A poset $(P, \leq)$ can also be viewed as a category $\mathcal{P}$. The objects of $\mathcal{P}$ are the elements of P, and for $p, q \in \mathrm{ob}(P)$, $\mathrm{Hom}(p, q)$ is a singleton if $p \leq q$, and $\mathrm{Hom}(p, q) = \emptyset$ otherwise. Note that reflexivity implies the existence of an identity morphism, and transitivity implies that composite morphisms exist. Similar to the case of a group, it is the morphisms that which determine the structure, in this case an ordering.

Right from the definition of categories, which devotes much to morphisms and less to objects, there is a vague sense that morphisms matter more than objects. This should not be unusual, in Linear Algebra for instance, we are not too concerned with finite dimensional vector spaces over a field F, which quickly prove as isomorphic to F^n; rather, we focus on the linear transformations between them. Throughout the rest of the text, especially as we develop the Yoneda Lemma, this notion will become clearer. ■

In Algebra, isomorphisms often take the form of a "bijective morphisms" as in case of group, ring and field isomorphisms. In Category theory the notion of an isomorphism is more abstract, in the sense that it does not make use of additional information about the objects and morphisms.

Definition 4. A morphism $f : x \to y$ in a category $\mathcal{C}$ is an **isomorphism** if there exists a morphism $g : y \to x$ such that $gf = 1_x$ and $fg = 1_y$. We say that the objects x and y of $\mathcal{C}$ are **isomorphic** if there exists an isomorphism between x and y, and write $x \cong y$. The element g is called the inverse of f.

Definition 5. A morphism whose domain equals it codomain is called an **endomorphism**. A morphism is called **automorphism** if it is an endomorphism and an isomorphism.

Example 3. (a) The isomorphisms of **Set** are bijections between sets.

(b) The isomorphisms of **Group**, **Ring**, **Field**, are group, ring, and field isomorphisms respectively (as in encountered in Algebra).

(c) To see where the category theoretic notion of an isomorphism differs to that of a "bijective morphism", we can consider isomorphisms in **Poset**. Consider the poset $X = \{a, b\}$ with $a \leq_X b$ along with the reflexive conditions, and the poset $Y = \{a, b\}$ only with reflexive conditions; in particular, a and b cannot be compared in Y. The map $i : Y \to X$ defined by $i(x) = x$ is clearly monotone, and therefore a morphism. But it is not an isomorphism since its inverse $i^{-1} : Y \to X$ is not a morphism, as $a \leq_Y b$ but $f(a) = a \not\leq_X f(b) = b$. We see that a "bijective morphism" fails to act as an order isomorphism.

2.3 Duality

Considering a category as a mathematical object in and of itself, we can begin to start asking natural questions about its properties. One such question is what happens when we reverse the direction of morphisms in a category? This leads us to the following definition.

Definition 6. Given a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined as follows:

- $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$
- For all $x, y \in \mathrm{ob}(\mathcal{C})$, $\mathcal{C}^{\mathrm{op}}(x, y) = \mathcal{C}(y, x)$. That is, the morphisms of $\mathcal{C}^{\mathrm{op}}$ are the morphisms of $\mathcal{C}$, where the domain and codomain are reversed. For clarity, we write $f^{\mathrm{op}} : x \to y$ for the morphism of $\mathcal{C}^{\mathrm{op}}$ corresponding to the morphism $f : y \to x$ of $\mathcal{C}$.
- Note that the pair of morphisms $f^{\mathrm{op}}, g^{\mathrm{op}}$ of $\mathcal{C}^{\mathrm{op}}$ are composable precisely when g, f are composable in $\mathcal{C}$. So we define $g^{\mathrm{op}} f^{\mathrm{op}} = (fg)^{\mathrm{op}}$.

$\mathcal{C}^{\mathrm{op}}$ defines a category. For each object $x \in \mathcal{C}^{\mathrm{op}}$, the morphism 1_X^{op} acts as the identity morphism, satisftying the identity law, and for morphisms $f^{\mathrm{op}} : w \to x, g^{\mathrm{op}} : x \to y, h^{\mathrm{op}} : y \to z$ we have

$$h^{\mathrm{op}}(g^{\mathrm{op}} f^{\mathrm{op}}) = h^{\mathrm{op}}(fg)^{\mathrm{op}} = ((fg)h)^{\mathrm{op}} = (f(gh))^{\mathrm{op}} = (gh)^{\mathrm{op}} f^{\mathrm{op}} = (h^{\mathrm{op}} g^{\mathrm{op}}) f^{\mathrm{op}},$$

so the associative law is satisfied.

Note that any theorem that quantifies over "all categories $\mathcal{C}$" applies to the opposites of these categories. We can then consider the **dual theorem**, by reversing the direction of morphisms, and replacing a composite fg by gf in the statement of the original theorem. The dual theorem is proved by the dual of the original proof. That is, "any proof in category theory simultaneously proves two theorems, the original statement and its dual" [5]. We will illustrate by proving the following:

Lemma 1. Let $\mathcal{C}$ be a category and $f : x \to y$ be a morphism of $\mathcal{C}$. The following are equivalent:

(i) f is an isomorphism

(ii) For all objects $c \in \mathrm{ob}(\mathcal{C})$ post-composition with f is a bijection. That is, the mapping

$$f_* : \mathcal{C}(c, x) \to \mathcal{C}(c, y), \qquad h \mapsto fh$$

is a bijection.

(iii) For all objects $c \in \mathrm{ob}(\mathcal{C})$ pre-composition with f is a bijection. That is, the mapping

$$f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c), \qquad h \mapsto hf$$

is a bijection.

Note that we have included the case of non-locally small categories. This is because in set theoretical foundations that allow for functions between collections larger than a set, the proof given applies.

Proof. We will first prove the equivalence (i) $\iff$ (ii). Suppose $f : x \to y$ is an isomorphism, with an inverse $g : y \to x$. We define

$$g_* : \mathcal{C}(c, y) \to \mathcal{C}(c, x), \qquad h \mapsto gh.$$

Given a morphism $h : c \to x \in \mathcal{C}(c, x)$, by the associative and identity laws, we have

$$(g_* \circ f_*)(h) = g_*(fh) = g(fh) = (gf)h = 1_x h = h,$$

Similarly, given a morphism $k : c \to y \in \mathcal{C}(c, y)$, we have

$$(f_* \circ g_*)(k) = f_*(gk) = f(gk) = (fg)k = 1_y k = k.$$

Therefore g_* is an inverse for f_*, so f_* is a bijection.

Conversely, suppose f_* is a bijection for all $c \in \mathrm{ob}(\mathcal{C})$. Taking $c = y$, we have that $f_* : \mathcal{C}(y, x) \to \mathcal{C}(y, y)$ is a bijection. In particular, f_* is surjective, so there exists a morphism $g : y \to x \in \mathcal{C}(y, x)$, such that $f_*(g) = 1_y \implies fg = 1_y$. Taking $c = x$, we have that $f_* : \mathcal{C}(x, x) \to \mathcal{C}(x, y)$ is a bijection. Consider the composite gf and identity 1_x in $\mathcal{C}(x, x)$. By the associative and identity laws we have

$$f_*(gf) = f(gf) = (fg)f = 1_x f = f,$$

and

$$f_*(1_x) = f 1_x = f.$$

Since f_* is injective and $f_*(gf) = f = f_*(1_x)$ we have that $gf = 1_x$. Therefore g is an inverse for f, so f is an isomorphism. Since we have proved the equivalence (i) $\iff$ (ii) for any locally small category. Given a category C, we also have proved it for its opposite category C^{op}. So we have that a morphism $f^{\mathrm{op}} : y \to x$ is an isomorphism if and only if

$$f^{\mathrm{op}}_* : \mathcal{C}^{\mathrm{op}}(c, y) \to \mathcal{C}^{\mathrm{op}}(c, x) \text{ is a bijection for all } c \in \mathrm{ob}(\mathcal{C}^{\mathrm{op}}).$$

However note that f^{op} is an isomorphism in $\mathcal{C}^{\mathrm{op}}$ if and only if f is an isomorphism in $\mathcal{C}$, $\mathcal{C}^{\mathrm{op}}(c, y) = \mathcal{C}(y, x)$, and $\mathcal{C}^{\mathrm{op}}(c, x) = \mathcal{C}(x, c)$. So the statement above expresses the same as the following: $f : x \to y$ in $\mathcal{C}$ is an isomorphism if and only if

$$f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c) \text{ is a bijection.}$$

That is, the equivalence (i) $\iff$ (ii) in $\mathcal{C}^{\mathrm{op}}$ gives us the equivalence (i) $\iff$ (iii) in $\mathcal{C}$ $\square$

Functors

3.1 Functors

A Modern Algebra student may notice that given any ring we can derive an abelian additive group from it by "forgetting" its multiplicative properties. Or we can derive a set from an abelian group by "forgetting" its associated binary operation. Those are trivial and not so useful derivations, but they capture the notion of transforming algebraic objects to algebraic objects of other kinds. As a more useful example, as it was shown in Galois theory in Algebra, we can derive a Galois groups of E over F, denoted $Gal(E/F)$, from field extensions $F \subseteq E$. This is the essential motivation for *functors*: capturing transformations from one category into another, and preserving its structure.

Definition 7. A **functor** $F : \mathcal{C} \to \mathcal{D}$, between categories $\mathcal{C}$ and $\mathcal{D}$, maps every object $c \in \mathrm{ob}(\mathcal{C})$ to an object $d \in ob(\mathcal{D})$, and maps every morphism $f : c \to c' \in \mathrm{mor}(\mathcal{C})$ to a morphism $F(f) : F(c) \to F(c') \in \mathrm{mor}(\mathcal{D})$, satisfying the following properties:

- For any composable pair of morphisms $f, g \in \mathrm{mor}(\mathcal{C})$, $F(g) \circ F(f) = F(gf)$.
- For each object $c \in ob(\mathcal{C})$, $F(1_c) = 1_{F_c}$

In other words, a functor maps objects and morphisms from one category to another and preserves its categorical structure, where identity morphisms and composite morphism are in agreement between the categories. In particular, note that the direction of composition is preserved.

Example 4. As mentioned above, a simple example of a functor is $F : \mathbf{Ring} \to \mathbf{Ab}$, from a category of rings to a category abelian groups, sends every ring $(R, +, \times) = r \in \mathrm{ob}(\mathbf{Ring})$ to $(R, +) = r' \in \mathrm{ob}(\mathbf{Ab})$; and it sends every ring homomorphism $\phi : R_1 \to R_2 \in \mathrm{mor}(\mathbf{Ring})$ to group homomorphisms $\psi : R_1 \to R_2 \in \mathrm{mor}(\mathbf{Ab})$. We can verify the identity and composite morphism are preserved. This type of functors that just remove some structure are called **forgetful functors**, which include $\mathbf{Ab} \to \mathbf{Group}$, $\mathbf{Group} \to \mathbf{Set}$, $\mathbf{Ring} \to \mathbf{Set}$ and so on. Those are the simplest functors. They are useful for expressing ideas such as "all rings are groups" in terms of categories.

A functor in Definition 7 is also called **covariant functor** to distinguish it from the following type of functor.

Definition 8. A **contravariant functor** F, between categories $\mathcal{C}$ and $\mathcal{D}$, is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. So, F maps every object $c \in \mathrm{ob}(\mathcal{C})$ to an object $d \in ob(\mathcal{D})$, and maps every morphism $f : c \to c' \in \mathrm{mor}(\mathcal{C})$ to a morphism $F(f) : F(c') \to F(c) \in \mathrm{mor}(\mathcal{D})$, satisfying the following properties:

- For any composable pair of morphisms $f, g \in \mathrm{mor}(\mathcal{C})$, $F(f) \circ F(g) = F(gf)$.
- For each object $c \in ob(\mathcal{C})$, $F(1_c) = 1_{F_c}$

As opposed to a covariant functor, contravariant functor reverses the direction of composition. We demonstrate this in the next example.

Example 5. Consider the category BG from Example 2. Then $inv : g \mapsto g^{-1}$ is a contravariant functor B$G \to$ BG. We have $inv(*) = *$ and $inv(g : * \to *) = g^{-1} : * \to *$. But because for $g, h \in mor(\mathrm{B}G)$, $(gh)^{-1} = h^{-1}g^{-1}$, it reverses the direction of composition, hence it is a contravariant functor.

Example 6. (The fundamental theorem of Galois theory) Let $F \subseteq E$ be finite, normal, separable extensions, and G be a Galois group of E/F, $G := Gal(E/F)$, which contains all automorphisms of E that fix F and $|Gal(E/F)| = [E : F]$. Let $\mathbf{Field}_F^E$ be a subcategory of **Field** which objects are subfields of E and field extensions of F. Let O_G be the category whose objects are all subgroups $H \subset G$. Then, by the fundamental theorem of Galois theory, we can define a contravariant functor $F : O_G^{op} \to \mathbf{Field}_F^E$ from intermediate fields $F \subseteq K \subseteq E$ to subgroups of $Gal(E/F)$. Since this functor is bijective, then $O_G^{op} \cong \mathbf{Field}_F^E$.

Lemma 2. Functors preserve isomorphisms.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and $f : x \to y$ a isomorphism in C with inverse $g : y \to x$. We have

$$F(g)F(f) = F(gf) = F(1_x) = 1_{F_x} \text{ and } F(f)F(g) = F(fg) = F(1_y) = 1_{F_y}$$

Therefore $F(f) : F(x) \to F(y)$ is an isomorphism in $\mathcal{D}$ with inverse $F(g) : F(y) \to F(x)$. □

Example 7. Consider the category BG for a group G. A functor $X : \mathrm{B}G \to \mathcal{C}$ for some category C, specifies the unique object of BG to an object $X \in \mathrm{ob}\,\mathcal{C}$, as well as an endomorphism $g_* : X \to X$ for each $g \in G$, satisfying the following: for each $g, h \in G$ we have $h_*g_* = (hg)_*$, and $1_* = 1_X$ for the identity element $1 \in G$. The functor B$G \to \mathcal{C}$ defines an **left action** (or simply **action**) of G on $X \in \mathrm{ob}(\mathcal{C})$. Similarly, the functor $\mathrm{B}G^{\mathrm{op}} \to \mathcal{C}$ defines an **right action** of G on $X \in \mathrm{ob}(\mathcal{C})$, for which each $g \in G$ determines an endomorphism $g^* : X \to X$, but instead we have $h^*g^* = (gh)^*$. When $\mathcal{C} = \mathbf{Set}$, the object X is called a G-**set**, a concept familiar to the reader. Since morphisms of BG are isomorphisms by 3.1, their images under the functor must also be isomorphisms.

Definition 9. Let $\mathcal{C}$ be a locally small category. For any object $c \in \mathrm{ob}(\mathcal{C})$ we can then define a covariant functor $\mathcal{C}(c, -)$ and a contravariant functor $\mathcal{C}(-, c)$ as follows. $\mathcal{C}(c, -)$ carries an object x to $\mathcal{C}(c, x)$ and a morphism $f : x \to y$ to the post-composition function $f_* : \mathcal{C}(c, x) \to \mathcal{C}(c, y)$. Similarly, $\mathcal{C}(-, c)$ carries an object x to $C(x, c)$ and a morphism $f : x \to y$ to the pre-composition function $f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c)$. We denote f_* and f^* by $\mathcal{C}(c, f)$ and $\mathcal{C}(f, c)$ respectively. We say that $\mathcal{C}(c, -)$ and $\mathcal{C}(-, c)$ are **represented by** c.

3.2 Natural Transformations

Raising the level of abstraction further, we consider a structure preserving map between functors.

Definition 10. Let F and G be functors both from category C to D. A **natural transformation** $\alpha : F \Rightarrow G$ consists a collection of morphisms $\alpha_c : F(c) \to G(c)$ for each object $c \in ob(C)$, such that for each morphism $f : c \to c' \in \mathrm{mor}(C)$ the following diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"'] & G(c') \end{tikzcd},$$

that is, if $\alpha_{c'} \circ F(f) = G(f) \circ \alpha_c$.

Example 8. Consider a natural transformation $\eta : 1_{Set} \to P$ from the identity to the powerset functor whose components $\eta_A : A \to \mathcal{P}(A)$ are functions that carry $a \in A$ to the singleton subset $\{a\} \in \mathcal{P}(A)$.

In fact, we can define a category $\mathrm{Fun}(C, D)$ whose objects are functors $C \to D$ and morphisms are natural transformations $F \Rightarrow G$.

Definition 11. A **natural isomorphism** is a natural transformation $\alpha : F \Rightarrow G$ such that every component α_c is an isomorphism. We can write a natural isomorphism as $\alpha : F \cong G$.

Example 9. Recall from Example 7 that a functor $X : \mathrm{B}G \to \mathcal{C}$ corresponds to a left action of G on X. We consider the natural transformation between $X, Y : \mathrm{B}G \rightrightarrows \mathcal{C}$. Since $\mathrm{B}G$ has one object a natural transformation $\alpha : X \to Y$ consists of a single morphism $\alpha : X \to Y$ in $\mathcal{C}$ that is G-**equivariant**; that is, the following diagram commutes for each $g \in G$:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "g_*"'] & Y \arrow[d, "g_*"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Definition 12. Let $\mathcal{C}$ be a locally small category.

(i) A covariant (or contravariant functor) F from $\mathcal{C}$ to **Set** is **representable** if there is an object $c \in \mathrm{ob}(\mathcal{C})$ and natural isomorphism to functors $\mathcal{C}(c, -)$ (or $\mathcal{C}(-, c)$). In this case, we say that functor F is **represented by** c.

(ii) A **representation** for a functor F is a choice of an object $c \in \mathrm{ob}(\mathcal{C})$ together with a specified natural isomorphism $\mathcal{C}(c, -) \cong F$ if F is covariant, or $\mathcal{C}(-, c) \cong F$ if F is contravariant.

The Yoneda Lemma

Since the category BG has a single object $*$, there is a unique covariant represented functor. As the elements of G define the set of automorphisms of the single object of BG, the covariant represented functor is the G-set G, with its action given by left multiplication. Dually, the unique contravariant represented functor is the G-Set G, with its action given by right multiplication. Now consider a G-set $X : \mathrm{B}G \to \mathbf{Set}$. A natural transformation $\phi : G \Rightarrow X$ is exactly its single component: the G-equivariant map $\phi : G \to X$ (Example 9). For $g \in G$, we must have $\phi(g \cdot h) = g \cdot \phi(h)$. Taking $h = 1$, the identity element, we have $\phi(g) = g \cdot \phi(1)$; that is, the choice of $\phi(1)$ determines ϕ.

This is not something that is unique to BG. Natural transformations with a domain that is a represented functor are determined by the choice of a single element, which is in the image of the evaluation of the representing element in the codomain functor. Further, this element is the image of the identity morphism at the representing object. We will show this in our proof of the Yoneda Lemma, which we give after the following lemma.

Lemma 3. Suppose $x, y \in \mathrm{ob}(\mathcal{C})$ and $f : x \to y$. We have a natural transformation

$$\mathcal{C}(-, f) : \mathcal{C}(-, x) \Rightarrow \mathcal{C}(-, y),$$

with components $C(z, f) : C(z, x) \to C(z, y)$ for $z \in \mathrm{ob}(\mathcal{C})$. Dually,

$$\mathcal{C}(f, -) : \mathcal{C}(y, -) \Rightarrow \mathcal{C}(x, -),$$

is a natural transformation with components $C(z, f) : C(x, z) \to C(y, z)$ for $z \in \mathrm{ob}(\mathcal{C})$.

Proof. Let $z, z' \in \mathcal{C}$ and $g : z \to z'$: for $k : z' \to x$

$$\begin{aligned}
[\mathcal{C}(g, y) \circ \mathcal{C}(z', f)](k) &= \mathcal{C}(g, y)(\mathcal{C}(z', f)(k)) \\
&= \mathcal{C}(g, y)(f \circ k) \\
&= (f \circ k) \circ g \\
&= f \circ (k \circ g) \\
&= \mathcal{C}(z, f)(k \circ g) \\
&= \mathcal{C}(z, f)(\mathcal{C}(g, x)(k)) \\
&= [\mathcal{C}(z, f) \circ \mathcal{C}(g, x)](k);
\end{aligned}$$

that is, $[\mathcal{C}(g, y) \circ \mathcal{C}(z', f)](k) = [\mathcal{C}(z, f) \circ \mathcal{C}(g, x)](k)$. Hence

$$\mathcal{C}(g, y) \circ \mathcal{C}(z', f) = \mathcal{C}(z, f) \circ \mathcal{C}(g, x).$$

□

Theorem 1. (The Yoneda lemma) Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to \mathbf{Set}$ a functor and $c \in \mathrm{ob}(\mathcal{C})$. There exists a bijective correspondence

$$\theta_{F,c} : \mathrm{Nat}\,(\mathcal{C}(c,-), F) \to F(c)$$

between the natural transformations from $\mathcal{C}(c,-)$ to F and the elements of $F(c)$; in particular, these natural transformations constitute a set. The collection of morphisms

$$(\theta_{F,c} : \mathrm{Nat}\,(\mathcal{C}(c,-), F) \to F(c))_{c \in \mathrm{ob}(\mathcal{C})}$$

constitutes a natural transformation. The collection of morphisms

$$(\theta_{F,c} : \mathrm{Nat}\,(\mathcal{C}(c,-), F) \to F(c))_{F \in \mathrm{Fun}(\mathcal{C},\mathbf{Set})}$$

constitutes a natural transformation.

Proof. For a natural transformation $\alpha : \mathcal{C}(c,-) \Rightarrow F$ from the functor $\mathcal{C}(c,-)$ to the functor F, let

$$\theta_{F,c}(\alpha) := \alpha_c(1_c).$$

Given $x \in F(c)$, we define a natural transformation $\tau_x : \mathcal{C}(c,-) \Rightarrow F$ as follows. For every $y \in \mathcal{C}$, let

$$\tau_{x,y} : \mathcal{C}(c,y) \to F(y)$$

be the mapping defined by $\tau_{x,y}(f) = [F(f)](x)$ for $f : c \to y$. We will first show that τ_x is a natural transformation from the functor $\mathcal{C}(c,-)$ to the functor F. Let $y, z \in \mathrm{ob}(\mathcal{C})$ and $f : y \to z$: then for $g : c \to y$, we have

$$\begin{aligned}
[\tau_{x,z} \circ C(c,f)](g) &= \tau_{x,z}(fg) \\
&= [F(fg)](x) \\
&= [F(f)F(g)](x) \\
&= F(f)(F(g)(x)) \\
&= F(f)(\tau_{x,y}(g)) \\
&= [F(f) \circ \tau_{x,y}](g);
\end{aligned}$$

that is, $[\tau_{x,z} \circ C(c,f))](g) = [F(f) \circ \tau_{x,y}](g)$. Hence $\tau_{x,z} \circ C(c,f) = F(f) \circ \tau_{x,y}$. This shows that $(\tau_{x,y} : \mathcal{C}(c,y) \to F(y))_{y \in \mathrm{ob}(\mathcal{C})}$ is a natural transformation. Hence τ is a mapping from $F(c)$ to $Nat(\mathcal{C}(c,-), F)$.

We will show that $\theta_{F,c}$ and τ are inverse to each other. Let $x \in F(c)$: then

$$\theta_{F,c}(\tau_x) = \tau_{x,c}(1_c) = F(1_c)(x) = 1_{F(c)}(x) = x;$$

so that $\theta_{F,c}(\tau_x) = x$. Thus $\theta_{F,c} \circ \tau = 1_{F(c)}$. On the other hand, let $\alpha : \mathcal{C}(c,-) \Rightarrow F$: for $y \in \mathrm{ob}(\mathcal{C})$ and $f : c \to y$

$$\begin{aligned}
\tau_{\theta_{F,c}(\alpha),y}(f) &= \tau_{\alpha_c(1_c),Y}(f) \\
&= F(f)(\alpha_c(1_c)) \\
&= (F(f) \circ \alpha_c)(1_c) \\
&= (\alpha_y \circ C(c,f))(1_c) \\
&= \alpha_y(f \circ 1_c) \\
&= \alpha_y(f);
\end{aligned}$$

that is, $\tau_{\theta_{F,c}(\alpha),y}(f) = \alpha_y(f)$. Thus $\tau_{\theta_{F,c}(\alpha),y} = \alpha_y$. Hence $(\tau \circ \theta_{F,c})(\alpha) = \alpha$. Therefore $\theta_{F,c}$ is a bijection.

We will now prove the naturality of the bijections. Consider the functor $N : \mathcal{C} \to \mathbf{Set}$ defined by

$$N(x) := \mathrm{Nat}(\mathcal{C}(x, -), F)$$

for $x \in \mathrm{ob}(\mathcal{C})$ and

$$N(f) : \mathrm{Nat}(\mathcal{C}(x, -), F) \to \mathrm{Nat}(\mathcal{C}(y, -), F), \qquad \alpha \mapsto \alpha \circ C(f, -)$$

for $x, y \in (\mathcal{C})$, $f : x \to y$ and $C(f, -) : \mathcal{C}(y, -) \Rightarrow \mathcal{C}(x, -)$ is the natural transformation such that

$$C(f, z)(g) = gf$$

for $z \in \mathrm{ob}(\mathcal{C})$ and $g : y \to z$. We are claiming the existence of a natural transformation $\eta : N \Rightarrow F$ defined by $\eta_x = \theta_{F,x}$. Let $x, y \in \mathrm{ob}(\mathcal{C})$, $\alpha : \mathcal{C}(x, -) \Rightarrow F$ and $f : x \to y$: then

$$\begin{aligned}
[\eta_y \circ N(f)](\alpha) &= \eta_y(\alpha \circ C(f, -)) \\
&= \theta_{F,y}(\alpha \circ C(f, -)) \\
&= [\alpha \circ C(f, -)](y)(1_y) \\
&= [\alpha_y \circ C(f, y)](1_Y) \\
&= \alpha_y(1_y \circ f) \\
&= \alpha_y(f) \\
&= \alpha_y(f \circ 1_x) \\
&= \alpha_y(C(x, f)(1_x)) \\
&= (\alpha_y C(x, f))(1_x) \\
&= (F(f) \circ \alpha_x)(1_x) \\
&= [F(f) \circ \theta_{F,x}](\alpha);
\end{aligned}$$

that is $[\eta_y \circ N(f)](\alpha) = [F(f) \circ \theta_{F,x}](\alpha)$. Hence $\eta_Y \circ N(f) = F(f) \circ \theta_{F,x}$. Therefore $\eta : N \Rightarrow F$ is a natural transformation.

Consider the category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ of functors from $\mathcal{C}$ to $\mathbf{Set}$ and natural transformations between them. For $x \in \mathrm{ob}(\mathcal{C})$ consider the functor $M : \mathrm{Fun}(\mathcal{C}, \mathbf{Set}) \to \mathbf{Set}$ defined by

$$M(F) := \mathrm{Nat}(\mathcal{C}(x, -), F)$$

for $F : \mathcal{C} \to \mathbf{Set}$. For $F, G : \mathcal{C} \rightrightarrows \mathbf{Set}$ and $\gamma : F \Rightarrow G$

$$M(\gamma) : \mathrm{Nat}(\mathcal{C}(x, -), F) \to \mathrm{Nat}(\mathcal{C}(c, -), F)$$

is defined by $M(\gamma)(\alpha) := \gamma \circ \alpha$ for each $\alpha : \mathcal{C}(x, -) \Rightarrow F$.

On the other hand, consider the functor **evaluation in** x, $\mathrm{ev}_x : \mathrm{Fun}(\mathcal{C}, \mathbf{Set}) \to \mathbf{Set}$, defined by

$$\mathrm{ev}_x(F) = F(x) \quad \& \quad \mathrm{ev}_x(\gamma) = \gamma_x$$

for $F : \mathcal{C} \to \mathbf{Set}$, $G : \mathcal{C} \to \mathbf{Set}$ and $\gamma : F \Rightarrow G$. We claim that $\mu : M \Rightarrow \mathrm{ev}_x$ defined by $\mu_F = \theta_{F,x}$ is a natural transformation. Let $F, G :\rightrightarrows \mathbf{Set}$, $\gamma : F \Rightarrow G$ and $\alpha : \mathcal{C}(x, -) \Rightarrow F$: then

$$\begin{aligned}
[\mu_G \circ M(\gamma)](\alpha) &= \theta_{G,x}(\gamma \circ \alpha) \\
&= (\gamma_x \circ \alpha_x)(1_x) \\
&= \gamma_x(\theta_{F,x}(\alpha)) \\
&= \mathrm{ev}_x(\gamma)(\theta_{F,x}(\alpha)) \\
&= [\mathrm{ev}_x(\gamma) \circ \theta_{F,x}](\alpha);
\end{aligned}$$

that is, $[\mu_G \circ M(\gamma)](\alpha) = [\mathrm{ev}_x(\gamma) \circ \theta_{F,x}](\alpha)$. Hence $\mu_G \circ M(\gamma) = \mathrm{ev}_x(\gamma) \circ \theta_{F,x}$. $\square$

Definition 13. A functor $F : \mathcal{C} \to \mathcal{D}$ is

- **full** if for each $x, y \in \mathrm{ob}(\mathcal{C})$, the map $\mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is surjective
- **faithful** if for each $x, y \in \mathrm{ob}(\mathcal{C})$, the map $\mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is injective.

Definition 14. The **covariant Yoneda embedding** is the functor $Y_* : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ defined by

$$\begin{aligned} Y_*(x) &= \mathcal{C}(-, x), \\ Y_*(f) &= \mathcal{C}(-, f) : \mathcal{C}(-, x) \Rightarrow \mathcal{C}(-, y), \end{aligned}$$

for $x, y \in \mathrm{ob}(\mathcal{C})$ and $f : x \to y$ in C. Dually, the **contravariant Yoneda embedding** is the functor $Y^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(C, \mathbf{Set})$ defined by

$$\begin{aligned} Y^*(x) &= \mathcal{C}(x, -), \\ Y^*(f) &= \mathcal{C}(f, -) : \mathcal{C}(x, -) \Rightarrow \mathcal{C}(y, -), \end{aligned}$$

for $x, y \in \mathrm{ob}(\mathcal{C})$ and $f : x \to y$ in C.

Corollary 1. (Yoneda embedding) The covariant and contravariant Yoneda embeddings are full and faithful.

Proof. We will prove it for the contravariant embedding

$$Y^* : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}).$$

Let $x, y \in \mathcal{C}$. Applying Yoneda lemma to the functor $\mathcal{C}(x, -) : \mathcal{C} \to \mathbf{Set}$ and object $y \in \mathcal{C}$, we have that the mapping

$$\theta_{\mathcal{C}(x,-),y} : \mathrm{Nat}(\mathcal{C}(y, -), \mathcal{C}(x, -)) \to \mathcal{C}(x, y)$$

is bijective. Note that

$$\begin{aligned} \mathrm{Nat}(\mathcal{C}(y, -), \mathcal{C}(x, -)) &= \mathrm{Fun}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(y, -), \mathcal{C}(x, -)) \\ &= \mathrm{Fun}(\mathcal{C}, \mathbf{Set})(Y^*(y), Y^*(x)). \end{aligned}$$

Therefore the mapping

$$\mathcal{C}(x, y) \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set})(Y^*(Y), Y^*(X))$$

is bijective. □

We conclude by applying the Yoneda Lemma to prove a familiar result in Algebra: Cayley's Theorem.

Theorem 2. (Cayley's Theorem) Any group is isomorphic to a subgroup of a permutation group.

Proof. We consider a group G as a category one-object $\mathrm{B}G$. The image of the unique object in the covariant Yoneda embedding $\mathrm{B}G \to \mathrm{Fun}(\mathrm{B}G^{\mathrm{op}}, \mathbf{Set})$ is the right G-set G, where G acts on itself by right multiplication. By the bijection in Corollary 1, the only G-equivariant endomorphisms of the right G-set G are maps defined by left multplication for a fixed element $g \in G$, in particular they are automorphisms. So, the Yoneda embedding defines an isomorphism between G and the automorphism group of the right G-set G. Finally, by composing with faithful forgetful functor $\mathrm{Fun}(\mathrm{B}G^{\mathrm{op}}, \mathbf{Set}) \to \mathbf{Set}$, we have an isomorphism between G and a subgroup of its automorphism group $\mathrm{Aut}(G)$. □

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