$f: \mathbb{R}^n \to \mathbb{R}^m$

$B = \{v_1 \dots v_n\}$ base di $\mathbb{R}^n$

$C = \{w_1 \dots w_m\}$ base di $\mathbb{R}^m$

$\exists M^f_{BC} \in M(m,n) \qquad \forall v \in \mathbb{R}^n$

$$\underbrace{f(v)}_{\in M(m,1)} = \underbrace{(M^f_{B,C})}_{\in M(m,n)} \cdot \underbrace{v}_{\in M(n,1)}$$

$$M^f_{B,C} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$f(v_n) = a_{1n} w_1 + \dots a_{mn} w_n$

$f: \mathbb{R}^n \to \mathbb{R}^m \qquad g: \mathbb{R}^m \to \mathbb{R}^k$

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^k$

$g \circ f$

$B = \{v_1, \dots, v_n\}$ base di $\mathbb{R}^n$

$C = \{w_1 \dots w_n\}$ $\mathbb{R}^m$

$D = \{u_1 \dots u_k\}$ $\mathbb{R}^k$

$\Rightarrow M^{g \circ f}_{B,D} = M^g_{C,D} \cdot M^f_{B,C}$

se $g = f^{-1}$ ottengo che $M^{f^{-1}}_{CB} = (M^f_{BC})^{-1}$

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{f^{-1}} \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$B = (v_1 \dots v_n)$

$C = (w_1 \dots w_m)$ $\Big\} \to M^f_{B,C}$

$B' = \{\underline{v}'_1 \dots \underline{v}'_n\}$

$C' = \{\underline{w}'_1 \dots \underline{w}'_m\}$ $\Big\} \to M^f_{B',C'}$

$$\begin{array}{ccccccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{id} & \mathbb{R}^m \\ B' & & B & & C & & C' \end{array}$$

$$M^f_{B',C'} = M^{id}_{C,C'} \cdot M^f_{B,C} \cdot M^{id}_{B',B} = \underbrace{(M^{id}_{C,C'})^{-1}}_{Q^{-1}} \cdot M^f_{B,C} \cdot \underbrace{M^{id}_{B',B}}_{P}$$

queste matrici P e Q si chiamano di cambio di base

① $B = \{(1,0), (0,1)\} = \{\underline{v}_1, \underline{v}_2\}$

$C = \{(0,1), (1,0)\} = \{\underline{w}_1, \underline{w}_2\}$

$B' = \{(1,0), (1,1)\} = \{\underline{v}'_1, \underline{v}'_2\}$

$C' = \{(2,1), (0,2)\} = \{\underline{w}'_1, \underline{w}'_2\}$

basi di $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$(1,0) \to (1,2)$

$(0,1) \to (2,3)$

- calcolare $M^f_{B,C}$
- calcolare $M^{id}_{B',B}$ e $M^{id}_{C,C'}$
- calcolare $M^f_{B',C'}$

immagine con combinazione lineare base originale

$M^f_{BC} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $\quad f(v_1) = (1,2) = 2(0,1) + (1,0) = 2\underline{w}_1 + 1\underline{w}_2$

$f(v_2) = (2,3) = 3(0,1) + 2(1,0) = 3\underline{w}_1 + 2\underline{w}_2$

es. $f(\underline{v}_1) = \sum_{i=1}^{K} a_{j1}\underline{w}_1$

- calcolare $M^{id}_{B',B}$ e $M^{id}_{C,C'}$

$B = \{(1,0), (0,1)\} = \{v_1, v_2\}$

$B' = \{(1,0), (1,1)\} = \{v'_1, v'_2\}$

scrivo i vettori della base nuova come comb lin. di quelli della base originale

$id(v'_1) = id(1,0) = (1,0) = 1 \cdot v_1 + 0 \cdot v_2$

$id(v'_2) = id(1,1) = (1,1) \quad 1 \cdot v_1 + 1 v_2$

$M^{id}_{B',B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$C = \{(0,1), (1,0)\}$

$C' = \{(2,1), (0,2)\}$

scrivo la nuova base delle immagini come comb. lin. di quella vecchia

$M^{id}_{C,C'}$

$id(w_1) = id(0,1) = (0,1) = 0(2,1) + \frac{1}{2}(0,2)$

$id(w_2) = (1,0) = \frac{1}{2}(0,1) - \frac{1}{4}(0,2)$

$M^{id}_{C,C'} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & -1/4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$

ⓒ $M^{f}_{B',C'} = M^{id}_{C,C'} \cdot M^{f}_{BC} \cdot M^{id}_{B'B}$

$$= \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1/2 & 3/2 \\ 3/4 & 7/4 \end{pmatrix}$$

$f(v'_1) = \frac{1}{2} w'_1 + \frac{3}{4} w'_2$

$f(v'_2) = \frac{3}{4} w'_1 + \frac{7}{4} w'_2$

per controllare

$f(v'_1) = f((1,0)) = (1,2)$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \overset{?}{=} \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 1/2 + 3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$f(\underline{v}_2) = f(1,1) = (3,5) \overset{?}{=} 3/2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{7}{4}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$(1,0) \to (1,2)$

$(0,1) \to (2,3)$

$f(x,y) = f(x,0) + f(0,y) = x\, f(1,0) + y\, f(0,1) =$

$= x(1,2) + y(2,3) = (x-2y,\ 2x+3y)$

② consideriamo basi di $\mathbb{R}^3$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$B' = \{(1,0,-1), (0,1,1), (1,0,1)\}$

ⓐ scrivere $M^{id}_{B,B'}$ e $M^{id}_{B',B}$

ⓑ sia $f: \mathbb{R}^3 \to \mathbb{R}^3$ lineare tc: $M^f_{B,B} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$

calcolare $M^f_{B',B'}$

$id(1,0,0) = \alpha(1,0,-1) + \beta(0,1,1) + \gamma(1,0,1)$

$\begin{cases} \alpha + \gamma = 1 \\ \beta = 0 \\ -\alpha + \beta + \gamma = 0 \end{cases} \begin{cases} \alpha = 1/2 \\ \beta = 0 \\ \gamma = 1/2 \end{cases}$ 1° colonna

$id((0,1,0)) = (0,1,0) = \alpha(1,0,-1) + \beta(0,1,1) + \gamma(1,0,1)$

$\begin{cases} \alpha + \gamma = 0 \\ \beta = 1 \\ -\alpha + \beta + \gamma = 0 \end{cases} \begin{cases} \alpha = -1 \\ \beta = 1 \\ \gamma = -\frac{1}{2} \end{cases}$ 2° colonna

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha + \gamma \\ \beta \\ -\alpha + \beta + \gamma \end{pmatrix} \Rightarrow \begin{cases} \alpha = -1/2 \\ \beta = 0 \\ \gamma = 1/2 \end{cases}$ 3° colonna

$M^{id}_{B,B'} = \begin{pmatrix} 1/2 & 1 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}$

$M^{id}_{B',B}$ $\quad B' = \{(1,0,1), (0,1,1), (1,0,1)\}$

N.B B è base canonica quindi ez

$M^{id}_{B',B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

NB $M^{id}_{B,B'} = \left(M^{id}_{B',B}\right)^{-1}$ proviamo

$$M^{id}_{B',B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

calcolo det

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 & 1 \end{matrix}$$

$$1-(-1) = 2$$

scrivo trasposta

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

calcolo complementi algebrici

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix} \cdot \frac{1}{2}$$ ritorna il calcolo

(b) $$M^{f}_{B,B} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

$$M^{f}_{B',B'} = ? = P^{-1} \cdot M^{f}_{B,B} \cdot P$$
$$= M^{id}_{B,B'} \cdot M^{f}_{B,B} = M^{id}_{B',B}$$

$$\begin{pmatrix} 1/2 & 1 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3-1 & -1+2 & 3+1 \\ 0 & 2 & 0 \\ 1\cdot 3 & -1+3 & 1+3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ -2 & 2 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

é una matrice diagonale quindi f é diagonalizzabile e B' é una base di autovettori, gli autovalori di f sono 2, 2, 4

③ considero $\mathbb{R}_2[x]$ con le seguenti basi

$$B = \left\{ x^2+x+1,\ x^2+1,\ x-1 \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$B' = \left\{ 2x^2+3x+1,\ 2x^2+2x+1,\ -x^2-2 \right\} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \right\}$$

ⓐ calcolare le matrici di cambio di base

ⓑ polinomio con coordinate $(1, 2, 3)$ rispetto alla base B, scriverlo nella base B'

$p = 1(2x^2+3x+1) + 2(2x^2+2x+1) + 3(-x^2-2)$

scrivere coordinate di p rispetto B'

$R_2[x] \longrightarrow \mathbb{R}^3$

$x^2 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$x \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$1 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$