Probability recap

conditional probability: $P(x|y) = \frac{P(x,y)}{P(y)}$

Product rule: $P(x,y) = P(x|y)P(y)$

chain rule: $P(x_1, .. x_n) = P(x_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots = \prod_{i=1}^{n} P(X_i | X_1, .. X_{i-1})$

X, Y indidendent iff : $\forall x,y: P(x,y) = P(x)P(y)$

X and Y are conditionally indipendent given Z iff : $\forall x,y,z: P(x,y|z) = P(x|z)P(y|z)$
$(X \perp\!\!\!\perp Y | Z)$

Reasoning over time or space

often we want to reason about a sequence of observations
so we need to introduce time/space in our models

Markov's models

$X_1 \to X_2 \to X_3 \to \cdots \qquad P(X_1) \qquad P(X_t) = P(X_t \mid X_{t-1})$

- value of X at given time → state
- Parameters: called transition probabilities (or dynamics), specify how the state evolves over time (also initial state prob.s)
- stationary assumption: transition probabilities the same at all times

basic conditional indipendence:

- past and future indipendent given the present
- each time step only depends on the previous

} first order Markov property

the chain of events is just a (growable) BN
we can reason like a generic BN if we truncate at a fixed length

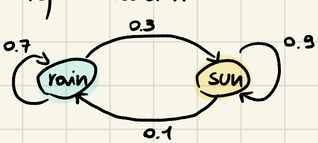
<u>example</u>: weather

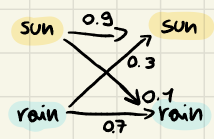
states : $X = \{rain, sun\}$
Initial distributions: 1.0 sun
CPT $P(X_t \mid X_{t-1})$:

conditional probability table

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

representation





initial distribution : 1.0 sun
prob. distribution after one step?

$P(X_2 = sun) = P(X_2 = sun \mid X_1 = sun)\, P(X_1 = sun) + P(X_2 = sun \mid X_1 = rain)\, P(X_1 = rain)$

$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$

Mini forward algorithm

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow \dots$$

what's $P(X)$ on some day t?

forward simulation

$P(x_1)$ = known $\qquad P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t) = \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$

<u>example</u>

from initial observation of sun

$0.9 \cdot 0.84 + 0.3 \cdot 0.16 = 0.804$

$$\underset{P(X_1)}{\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle} \Longrightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

$0.9 \cdot 0.9 + 0.3 \cdot 0.1 = 0.81 + 0.3 = 0.84$

same result

from initial observation of rain

$$\underset{P(X_1)}{\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle} \quad \left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle \Longrightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

in general

$$\underset{P(X_1)}{\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle} \quad \dots \quad \Longrightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

Stationary Distributions

- for most chains:
  - influence of the initial distributions gets less and less over time
  - The distribution we end up is indipendent from the initial one

stationary distribution P_∞ of the chain

it satisfies $P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X \mid x) P_\infty(x)$

<u>example</u>:

what's $P(x)$ at time $t = \infty$

$$X_1 \rightarrow X_2 \rightarrow X_3 \dashrightarrow$$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

$P_\infty(sun) = P(sun \mid sun) P_\infty(sun) + P(sun \mid rain) P_\infty(rain)$

$P_\infty(rain) = P(rain \mid sun) P_\infty(sun) + P(rain \mid rain) P_\infty(rain)$

$P_\infty(sun) = 0.9 P_\infty(sun) + 0.3 P_\infty(rain)$

$P_\infty(rain) = 0.1 P_\infty(sun) + 0.7 P_\infty(rain)$

$P_\infty(sun) = 3 P_\infty(rain)$ $\quad \xrightarrow{\text{also}} \quad P_\infty(sun) + P_\infty(rain) = 1 \Longrightarrow$ $P_\infty(sun) = 3/4$, $P_\infty(rain) = 1/4$

$P_\infty(rain) = 1/3 P_\infty(sun)$

Gibbs Sampling

each joint instatiation over all hidden and query variables is a state: $\{X_1, \dots X_n\} = H \cup Q$

trasitions:

with probabilitys $1/n$ resample X_j according to

$P(X_j \mid x_1, x_2 \dots x_{j-1}, x_{j+1}, \dots, e_1, \dots e_m)$

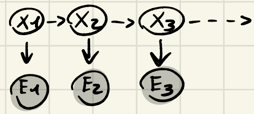
stationary distribution:

- conditional distribution $P(X_1, X_2, \dots X_n \mid e_1, \dots e_m)$

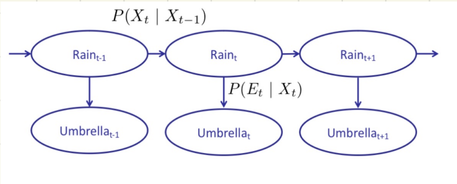
that means we are running Gibbs sampling long enough we get a sample from the desired distribution

Hidden Markov models

- Markov chains are not so usefull for most agents
  Need observations to update beliefs
- HMMs
  underlying Markov chain over states x
  observe outputs (effects) at each time step



example: weather HMM



R_{t-1}	R_t	$P(R_t \mid R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

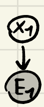
HMM defined by:
- Initial distribution: $P(X_1)$
- transition: $P(X_t \mid X_{t+1})$
- Emissions: $P(E_t \mid X_t)$

Properties
- Markov hidden process: future depends on past via the present
- Current observation indipendent of all else given current state

(evidence variables tend to be correlated by the hidden state)

Filtering/modelling
the task of tracking the distribution
$B_t(X) = P_t(X_t \mid e_1 \dots e_t)$ (the belief state) over time.
start with $B_1(X_1)$ usually uniform
As time passes (or we get obvservations) we update $B(X)$

Inference: base cases



$P(X_1 \mid e_1)$

$P(x_1 \mid e_1) = P(x_1, e_1)/P(e_1)$ eliminate

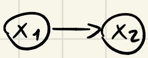
only variable is x_1

$\propto_{X_1} \; P(X_1, e_1)$

$= P(x_1)P(e_1 \mid x_1)$

proportional

X_1	$\alpha P(x_1 \mid e_1)$	
0	0.2	$0.2/0.3+0.2$
1	0.3	$0.3/0.2+0.3$



$P(X_2)$

$P(x_2) = \sum_{x_1} P(x_1 \mid x_2) = \sum_{x_1} P(x_1) P(x_2 \mid x_1)$

Passage of time

Assume we have current belief $P(X|\text{evidence to date})$ $\quad X_1 \to X_2$

$$B(X_t) = P(X_t | e_{1:t})$$

then after one step:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) =$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t})$$

more compact $\Longrightarrow$

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t) B(x_t)$$

Beliefs get "pushed" through transitions

observation

Assume we have current belief $P(X|\text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

then the evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t}) / P(e_{t+1}|e_{1:t})$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|e_{1:t}, X_{t+1}) P(X_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|X_{t+1}) P(X_{t+1}|e_{1:t})$$

compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1}) B'(X_{t+1})$$

basic idea: beliefs "reweighted" by likelihood of evidence

we have to renormalize

The forward algorithm

we are given evidence at each time and want to know

$$B_t(X) = P(X_t \mid e_{1:t})$$

we can derive the following updates

$$P(x_t \mid e_{1:t}) \propto P(x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t \mid x_{t-1}) P(e_t \mid x_t)$$

$$= P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

every time step, we start with current $P(X \mid \text{evidence})$

we update for time

$$P(x_t \mid e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1}) \cdot P(x_t \mid x_{t-1})$$

we update for evidence

$$P(x_t \mid e_{1:t}) \propto P(x_t \mid e_{1:t-1}) \cdot P(e_t \mid x_t)$$

the forward algo. does both at once and doesn't normalize

Filtering

elapse time : compute $P(X_t | e_{1:t-1})$

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

observe : compute $P(X_t | e_{1:t})$

$$P(x_t|e_{1:t}) \propto P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

~ · ~ · ~ · ~ · ~ · ~ · ~ · ~

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ E_1 & E_2 \end{tikzcd}$$

Belief : < P(rain), P(sun) >

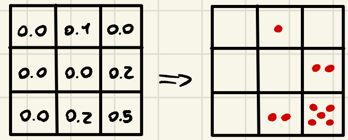
$P(X_1)$	<0.5, 0.5>	Prior on X_1
$P(X_1 \| E_1 = \text{umbrella})$	<0.82, 0.18>	observe
$P(X_2 \| E_1 = \text{umbrella})$	<0.63, 0.37>	elapse time
$P(X_2 \| E_1 = \text{umb}, E_2 = \text{umb})$	<0.88, 0.12>	observe

Particle filtering

Sometimes |X| is too big to use exact inference

Solution: approximate

- track samples (particles) of X, not all values
- time per step is linear in the number of samples



Our representation of P(x) is now a list of N particles (samples)

$N << |X|$

P(x) aproximated by number of particles with value x

many x may have $P(x)=0$

for now all particles have a weight of 1

elapse time

each particle is moved by sampling its next position from the transition model

$x' = \text{sample}(P(X'|x))$

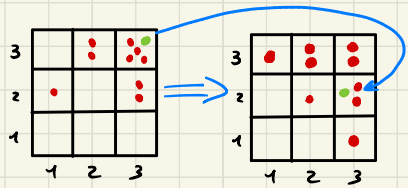
if enough samples, consistent

Some change, some remain the same

this represents the passage of time

particles

(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)



particles

(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

observe

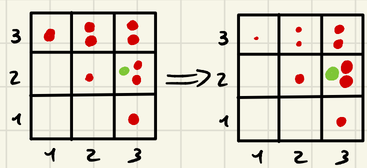
Fixes observation

downweight samples based on the evidence

$w(x) = P(e|x)$

$B(X) \propto P(e|X)B'(X)$

they don't sum to 1



particles	weight
(3,2)	.9
(2,3)	.2
(3,2)	.9
(3,1)	.4
(3,3)	.4
(3,2)	.9
(1,3)	.1
(2,3)	.2
(3,2)	.9
(2,2)	.4

Resample

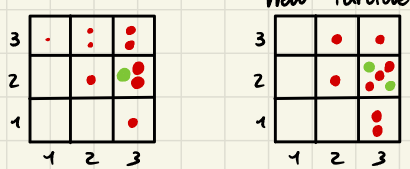
N times we choose from our weighted sample distribution

(draw with replacement)

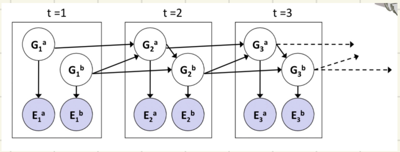
equivalent to normalize

repeat

new Particles



Dynamic Bayes nets



track multiple variables over time, multiple sources of evidence

idea: repeat a fixed BN structure at each time

variables at time t can condition on those from $t-1$

A generalization of HMMs

exact inference

we can apply variable elimination

"unroll" the network for T time steps, the eliminate variables until $P(X_t | e_{1:t})$ is computed

online belief updates: eliminate all variables from the previous time step; store factors for current time only

DBNs particle filters

a particle is a complete sample for a time step.

Init: generate prior samples for the $t=1$ BN

elapse time: sample a successor for each particle

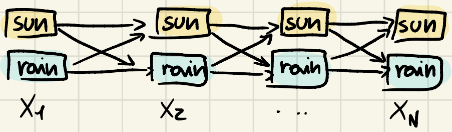
observe: weight each entire sample by the likelihood of the evidence conditioned on the sample

resample: select prior samples (tuples of values) in proportion to their likelihood

Most likely explanation

HMMs are defined by:

- states X
- Observations E
- initial distribution: $P(X_1)$
- transitions: $P(X | X_{-1})$
- emissions: $P(E|X)$



new query: most likely explanation: $\arg\max_{X_{1:t}} P(x_{1:t} | e_{1:t})$

each arch represent some transition $x_{t-1} \to x_t$

they have weight $P(x_1 | x_{t-1}) P(e_1 | x_t)$

each path is a sequence of states

Π_w of a path is that sequence's probability along with the evidence

forward algorithm (sum)

$f_1[x_t] = P(x_t, e_{1:t}) =$

$= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) f_{t-1}[x_{t-1}]$

Viterbi algo. (max)

$m_t[x_t] = \max_{x_{1:t-1}}$