6.1) esprimere omomorfismo dell' es. 5 in coordinate

$f.\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definito da

$$\underbrace{f\begin{pmatrix}2\\0\\1\\0\end{pmatrix}}_{b_1}=\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{b'_1},\quad \underbrace{f\begin{pmatrix}-2\\1\\0\\2\end{pmatrix}}_{b_2}=\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{b'_2},\quad \underbrace{f\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{b_3}=\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{b'_3},\quad \underbrace{f\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{b_4}=\overbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}^{b'_4}$$

li ho già

1° modo

$$f\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = x\, f\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + y\, f\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + z\, f\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + w\, f\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Per calcolare $f\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ dobbiamo scrivere $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ come comb. lin. dei vettori della mia base.

$$\bullet \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \alpha\begin{pmatrix}2\\0\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-2\\1\\0\\2\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{cases} 2\alpha-2\beta=1 & 2\alpha=1 \quad \alpha=1/2 \\ \beta=0 \\ \alpha+\gamma=0 & \gamma=-\alpha \quad \gamma=-1/2 \\ 2\beta=0 \end{cases}$$

$$f\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{2} f\begin{pmatrix}2\\0\\1\\0\end{pmatrix} - \frac{1}{2} f\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1/2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \alpha\begin{pmatrix}2\\0\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-2\\1\\0\\2\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{cases} 2\alpha-2\beta=0 & \alpha=\beta=1 \\ \beta=1 \\ \alpha+\gamma=0 & \gamma=-1 \\ 2\beta+\lambda=0 & \lambda=-2 \end{cases}$$

$$f\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = f\begin{pmatrix}2\\0\\1\\0\end{pmatrix} + f\begin{pmatrix}-2\\1\\0\\2\end{pmatrix} - f\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - 2f\begin{pmatrix}0\\0\\0\\1\end{pmatrix} =$$
$$= \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$f\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = x\begin{pmatrix}0\\0\\1/2\end{pmatrix} + y\begin{pmatrix}0\\1\\2\end{pmatrix} + z\begin{pmatrix}1\\0\\0\end{pmatrix} w\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}z\\y\\\frac{x}{2}+2y\end{pmatrix}$$

← in coordinate

2° modo usiamo il teorema di cambiamento di base

$A_f(\underline{b}_j, \underline{b}_i) = Q^{-1} A_f(\underline{v}_j, \underline{v}_i) \cdot P$ con:

↑ vettori della base canonica

$$Q^{-1} = \begin{pmatrix}1&0&1\\0&1&0\\1&1&0\end{pmatrix}$$

come colonne vettori della base dell'immagine

$$P = (P^{-1})^{-1} = \begin{pmatrix}2&-2&0&0\\0&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}^{-1} = \begin{pmatrix}1/2&1&0&0\\0&1&0&0\\-1/2&-1&1&0\\0&-1&0&1\end{pmatrix}$$

↑ come colonne vettori della base

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{trasposta}} \begin{pmatrix} 2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \det = 2$$

$$a_{ij} = (-1)^{i+j} \frac{\det(A_{ij})}{\det(A)}$$

$$\Big\downarrow \text{ inversa}$$

$$\begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

siccome $f(b_1) = b'_1$ $\quad f(b_3) = b'_3$ $\quad$ e (b'_1, b'_2, b'_3) é la base $(\underline{w}_i)$ di $\mathbb{R}^3$

$f(b_2) = b'_2$ $\quad f(b_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$A_f\big((\underline{v}_j), (\underline{w}_i)\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

quindi
$$A_f\big((\underline{b}_j), (\underline{b}_i)\big) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 2 & 0 & 0 \end{pmatrix}$$

da cui
$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} z \\ y \\ \frac{x}{2} + 2y \end{pmatrix}$$
coincide con la soluzione trovata con l'altro metodo

é più facile calcolare l'immagine di un vettore con la funzione espressa in coordinate rispetto a usare la linearitá

ad es.
$$f\left(\begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \end{array}\begin{array}{l} x \\ y \\ z \\ w \end{array}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{array}{l} z \\ y \\ \frac{x}{2} + 2y \end{array}$$