ChatGPT

Known recurrence sequences for $1+\sqrt m$

Classical Pell-type recurrences yield special cases of the form $a(n)=2a(n-1)+(m-1)a(n-2)$ whose dominant root is $1+\sqrt m$. For example, the Pell numbers $(A000129)$ satisfy $a(n)=2a(n-1)+a(n-2)$ (so $m=2$), and it is well known that

$$\lim_{n\to\infty}\frac{a(n)}{a(n-1)}=\sqrt{2}+1\approx 2.414\ldots$$

[1]. Likewise, other specific recurrences appear in OEIS. For $m=8$, one finds

$$a(n)=2a(n-1)+7a(n-2)\ (a(0)=1,a(1)=1)$$

with characteristic roots $1\pm\sqrt8$, hence $\lim a(n)/a(n-1)=1+\sqrt8=1+2\sqrt2$ [2]. This recurrence (due to Paul Barry, 2003) indeed approximates $1+2\sqrt2$. In general, one can construct such sequences for any fixed m by choosing appropriate initial values, but these were not packaged as a unified "m–parameter" family in the literature prior to 2019.

Similarly, recurrences of the second form $a(n)=2a(n-1)+(k^2m-1)a(n-2)$ yield limit $1+k\sqrt m$. The OEIS contains instances for small (k,m). For example, Kyle Smith's entry A330390 (Dec 2019) gives $a(n)=2a(n-1)+17a(n-2)$ with $\lim a(n)/a(n-1)=1+3\sqrt2$ [3], and A328606 (Oct 2019) has $a(n)=2a(n-1)+11a(n-2)$ with $\lim a(n)/a(n-1)=1+2\sqrt3$ [4]. In each case the formula matches our pattern ($17=3^2\cdot2-1$, $11=2^2\cdot3-1$) and the limit is $1+k\sqrt m$ as claimed. For $m=10,k=1$, A328604 (Kyle Smith, 2019) gives $a(n)=2a(n-1)+9a(n-2)$ with limit $1+\sqrt{10}$ [5]. Other ad hoc sequences (e.g. A297189 for $m=7$, which uses a 4th-order recurrence, also yields $\lim a(n)/a(n-1)=1+\sqrt7$ [6]).

Prior literature or OEIS generalizations

Our search did not uncover any prior reference that **explicitly states** the general recurrence $a(n)=2a(n-1)+(m-1)a(n-2)$ or $a(n)=2a(n-1)+(k^2m-1)a(n-2)$ for arbitrary m,k. The known references tend to focus on particular cases: Pell numbers for $m=2$ [1], or isolated examples like $m=8$ [2]. We found no journal or textbook discussing the family of "one-sequence" integer approximations to $1+\sqrt m$ for general m. In the OEIS, index entries exist only for each concrete recurrence (e.g. [A084058] for $(2,7)$ signature [2]), not a metatheorem.

In particular, prior to 2019 there is no OEIS entry that parametrically describes "for any integer $m>0$, the sequence satisfying $a(n)=2a(n-1)+(m-1)a(n-2)$ has limit $1+\sqrt m$." The comment by Ross La Haye on A000129 (Pell numbers) notes $\lim a(n)/a(n-1)=1+\sqrt2$ [1], but it does not generalize to other m. Aside from scattered examples, the structure $2a(n-1)+(m-1)a(n-2)$ was not presented as a general indexed scheme. Similarly, the form $2a(n-1)+(k^2m-1)a(n-2)$ was not seen in earlier OEIS or literature apart from specific (k,m) cases.

By contrast, Kyle MacLean Smith's OEIS contributions in late 2019 systematically introduced many of these cases (e.g. A328604, A328606, A330390, etc.) with exactly the claimed property $\lim a(n)/a(n-1)=1+k\sqrt m$ [3] [4] [5]. We did not find any prior source – OEIS or published – that frames the recurrences as a unified general approximation method beyond the classical Pell examples.

Conclusion

In summary, while Pell recurrences and a few specific OEIS sequences for fixed m do yield limits $1+\sqrt m$ (e.g. [1] [2]), **no earlier reference explicitly gives the general family** of recurrences $a(n)=2a(n-1)+(m-1)a(n-2)$ or its k-weighted variant for arbitrary m,k. Kyle MacLean Smith's 2019 OEIS entries appear to be the first explicit formulation of this integer-only approximation scheme in indexed generality [3] [4] [5].

Sources: OEIS entries for Pell and related sequences [1] [2] and for Kyle Smith's sequences [3] [4] [5] provide the evidence summarized above.

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[1] A000129 - OEIS
https://oeis.org/A000129/internal

[2] A084058 - OEIS
https://oeis.org/A084058

[3] A330390 - OEIS
https://oeis.org/A330390

[4] A328606 - OEIS
https://oeis.org/A328606

[5] A328604 - OEIS
https://oeis.org/A328604

[6] A297189 - OEIS
https://oeis.org/A297189