

# Simplicial Homology and Cohomology Groups

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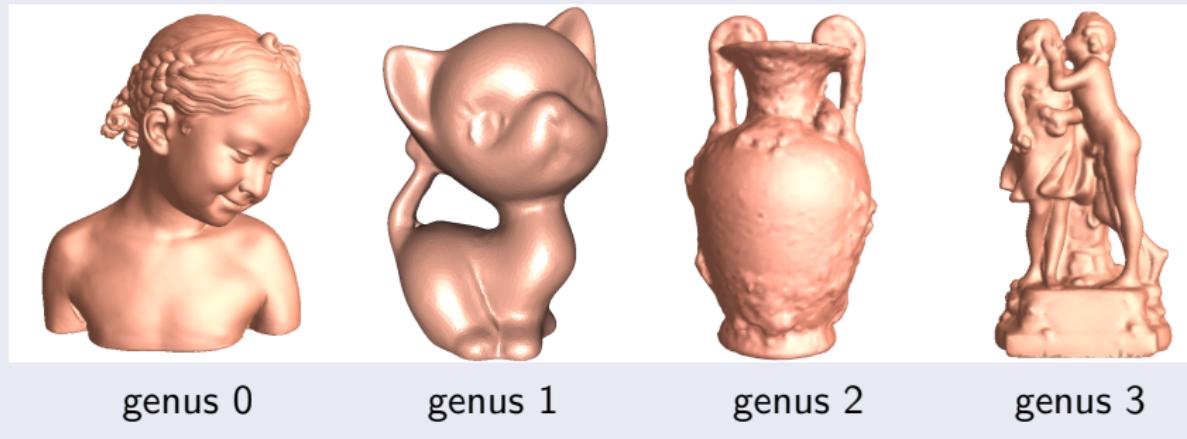
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August 11, 2023

# Homology and Cohomology Group

# Topology of Surfaces - Closed Surfaces



genus 0

genus 1

genus 2

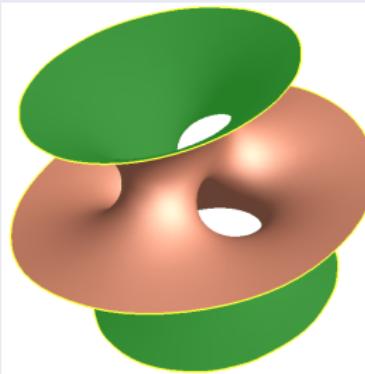
genus 3

**Figure:** Surface topological classification

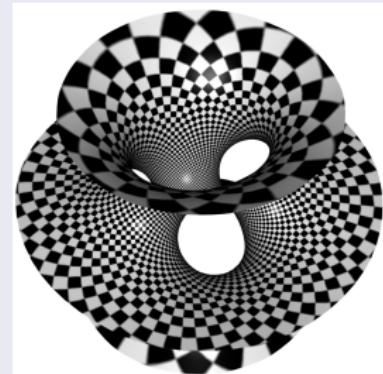
# Topology of Surfaces - Surfaces with boundaries



(0,1)



(1,3)



(1,3)

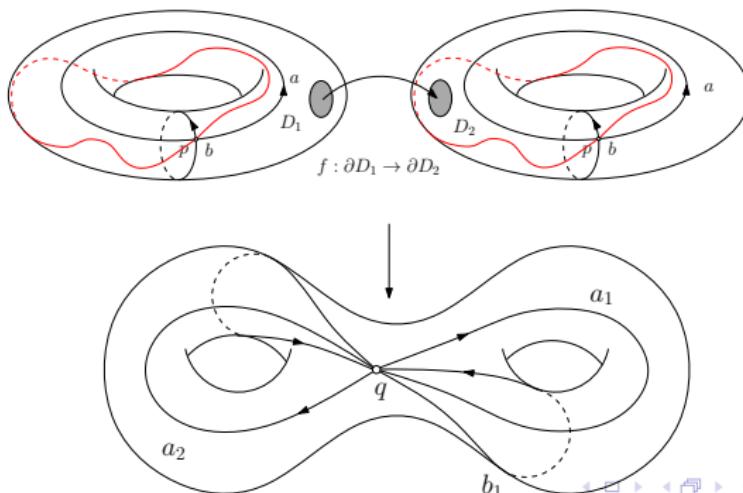
**Figure:** Topological classification for surfaces with boundaries  $(g, b)$ .

# Connected Sum

## Definition (connected Sum)

The connected sum  $S_1 \oplus S_2$  is formed by deleting the interior of disks  $D_i$  and attaching the resulting punctured surfaces  $S_i - D_i$  to each other by a homeomorphism  $h : \partial D_1 \rightarrow \partial D_2$ , so

$$S_1 \oplus S_2 = (S_1 - D_2) \cup_h (S_2 - D_2).$$



# Connected Sum



A Genus eight Surface, constructed by connected sum.

# Orientability

M.c. Escher



Möbius band.

# Projective Plane

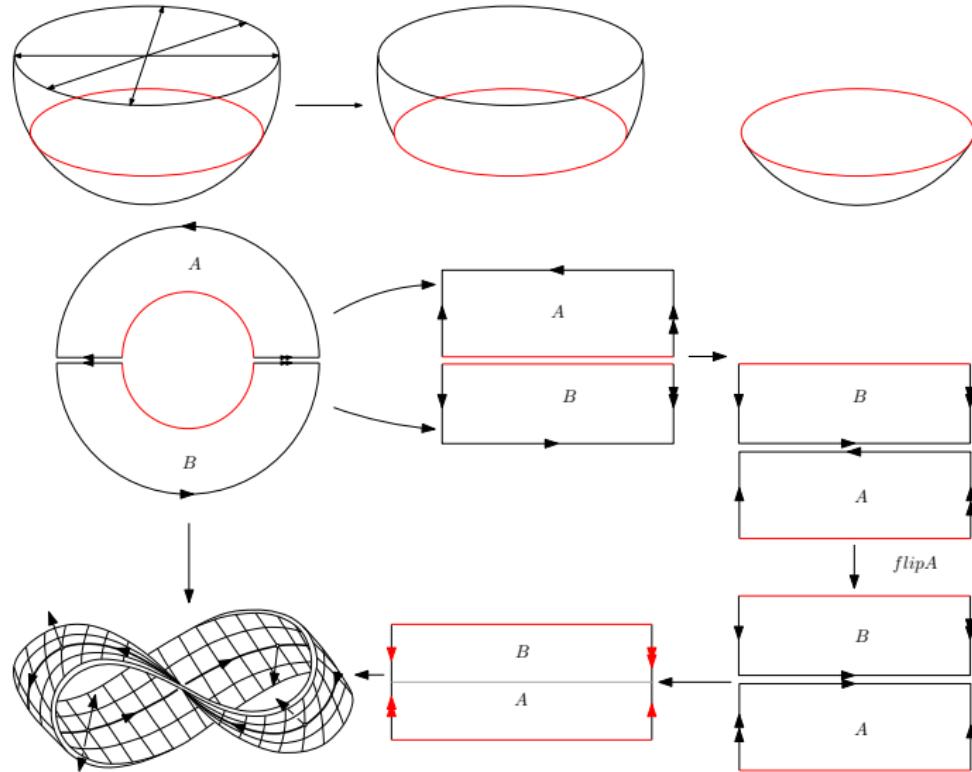
## Definition (Projective Plane)

All straight lines through the origin in  $\mathbb{R}^3$  form a two dimensional manifold, which is called the projective plane  $RP^2$ .

A projective plane can be obtained by identifying two antipodal points on the unit sphere. A projective plane with a hole is called a crosscap.

$$\pi_1(RP^2) = \{\gamma, e\}.$$

# Projective Plane



# Surface Topology

## Theorem (surface Topology)

*Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a sphere with a finite number of disjoint discs removed and with cross caps glued in their places. The sphere and connected sums of tori are orientable surfaces, whereas surfaces with crosscaps are unorientable.*

Any closed surface is the connected sum

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_g,$$

if  $S$  is orientable, then  $S_i$  is a torus. If  $S$  is non-orientable, then  $S_i$  is a projective plane.

# Triangular mesh

## Definition (triangular mesh)

A triangular mesh is a surface  $\Sigma$  with a triangulation  $T$ ,

- ① Each face is counter clockwise oriented with respect to the normal of the surface.
- ② Each edge has two opposite half-edges.



# Simplicial Complex

## Definition (Simplicial Complex)

Suppose  $k + 1$  points in the general positions in  $\mathbb{R}^n$ ,  $v_0, v_1, \dots, v_k$ , the standard simplex  $[v_0, v_1, \dots, v_k]$  is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \dots, v_k] = \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^k \lambda_i v_i, \sum_{i=0}^k \lambda_i = 1, \lambda_i \geq 0\},$$

we call  $v_0, v_1, \dots, v_k$  as the vertices of the simplex  $\sigma$ .

Suppose  $\tau \subset \sigma$  is also a simplex, then we say  $\tau$  is a facet of  $\sigma$ .

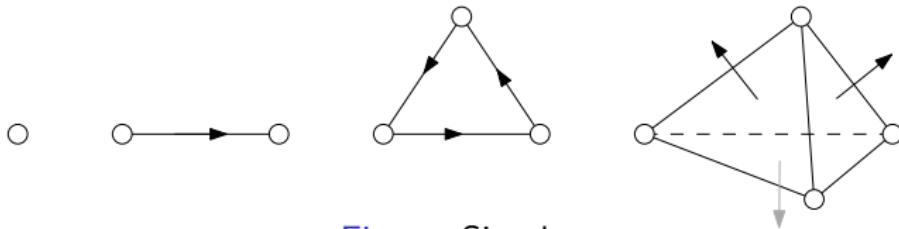


Figure: Simplex

# Simplicial Complex

## Definition (Simplicial complex)

A simplicial complex  $\Sigma$  is a union of simplices, such that

- ① If a simplex  $\sigma$  belongs to  $\Sigma$ , then all its facets also belongs to  $\Sigma$ .
- ② If  $\sigma_1, \sigma_2 \subset \Sigma$ ,  $\sigma_1 \cap \sigma_2 \neq \emptyset$ , then their intersection is also a common facet.

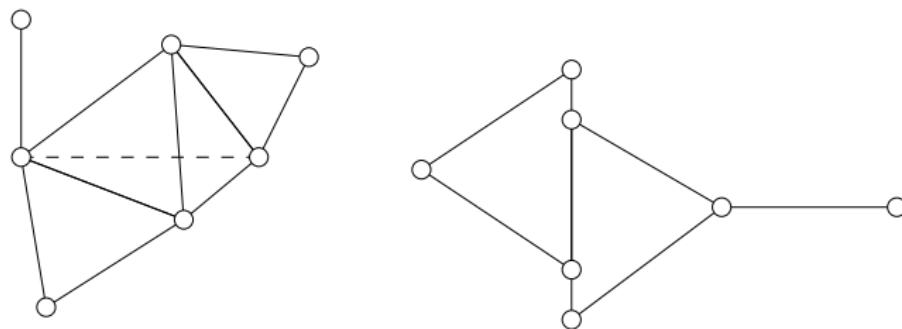


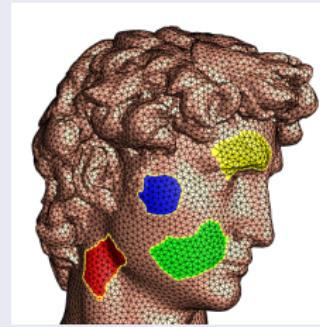
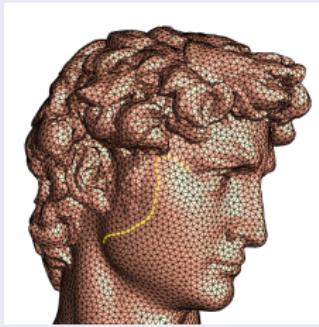
Figure: Simplicial complex.

# Chain Space

## Definition (Chain Space)

A  $k$  chain is a linear combination of all  $k$ -simplices in  $\Sigma$ ,  
 $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}$ . The  $k$  dimensional chain space is the linear space formed by all  $k$ -chains, denoted as  $C_k(\Sigma, \mathbb{Z})$ .

A curve on the mesh is a 1-chain, a surface patch is a 2-chain.



# Boundary Operator

## Definition (Boundary Operator)

The  $n$ -th dimensional boundary operator  $\partial_n : C_n \rightarrow C_{n-1}$  is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \dots, v_n] = \sum_i (-1)^i [v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n].$$

Boundary operator extracts the boundary of a chain.



# Boundary Operator

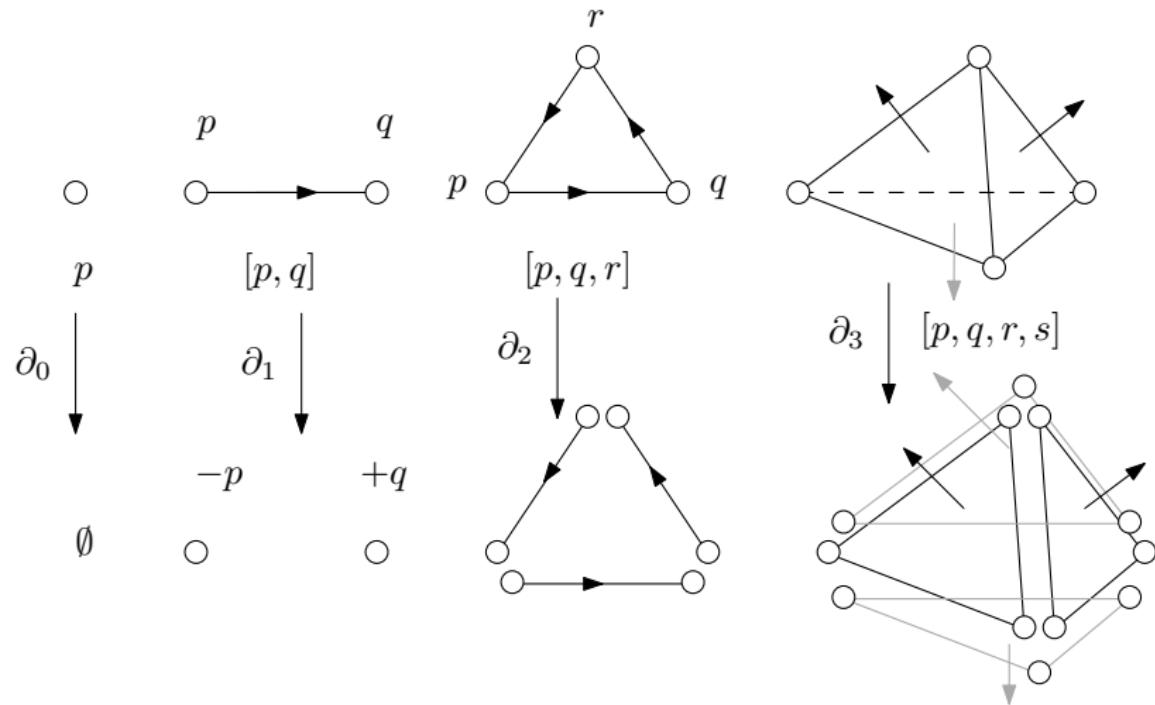


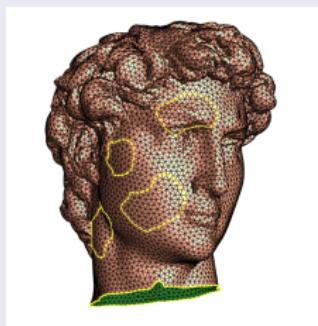
Figure: Boundary operator.

# Closed Chains

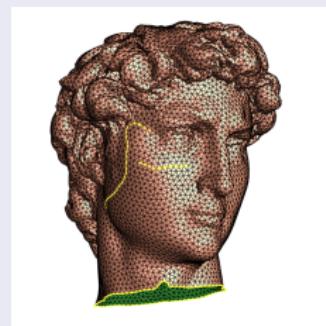
## Definition (closed chain)

A  $k$ -chain  $\gamma \in C_k(\sigma)$  is called a closed  $k$ -chain, if  $\partial_k \gamma = 0$ .

A closed 1-chain is a loop. A non-closed 1-chain has boundary vertices.



closed 1-chain

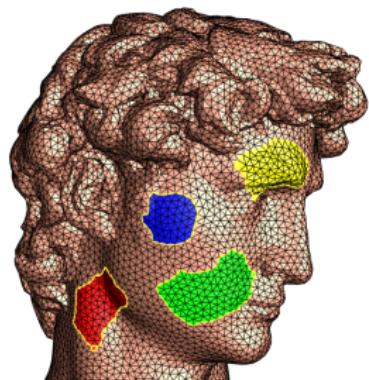


open 1-chain

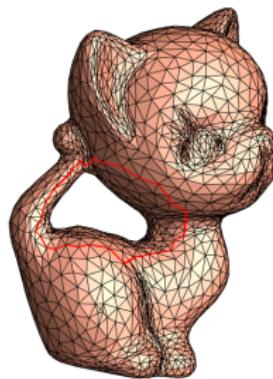
# Exact Chains

## Definition (Exact Chain)

A  $k$ -chain  $\gamma \in C_k(\sigma)$  is called an exact  $k$ -chain, if there exists a  $(k + 1)$  chain  $\sigma$ , such that  $\partial_{k+1}\sigma = \gamma$ .



exact 1-chain



closed, non-exact 1-chain

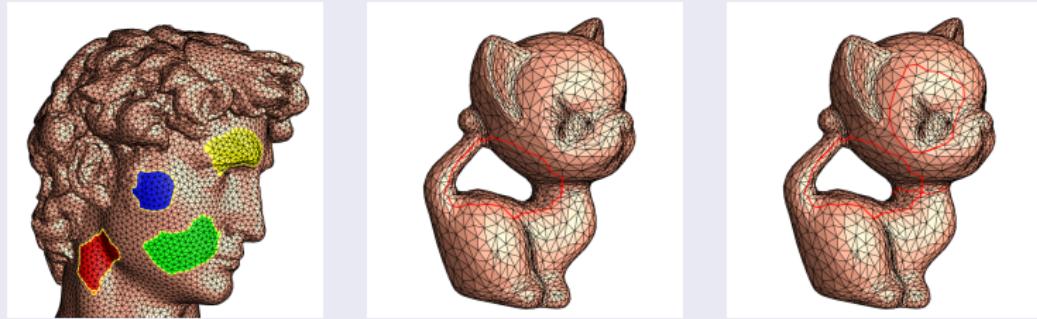
# Boundary of Boundary

## Theorem (Boundary of Boundary)

*The boundary of a boundary is empty*

$$\partial_k \circ \partial_{k+1} \equiv \emptyset.$$

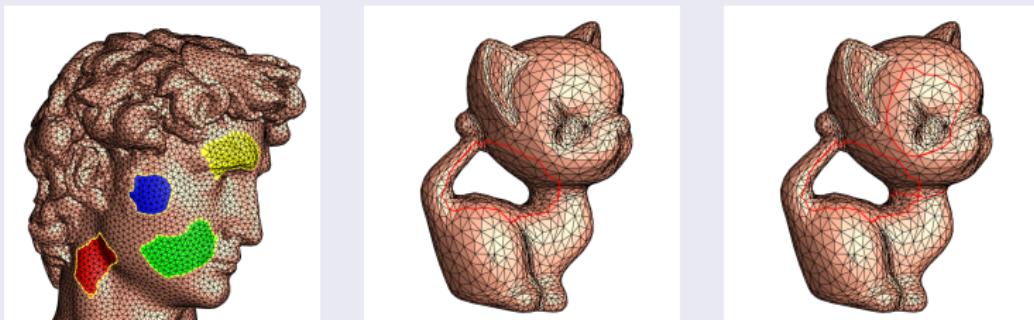
namely, exact chains are closed. But the reverse is not true.



# Homology

The difference between the closed chains and the exact chains indicates the topology of the surfaces.

- ① Any closed 1-chain on genus zero surface is exact.
- ② On tori, some closed 1-chains are not exact.



# Homology Group

Closed  $k$ -chains form the kernel space of the boundary operator  $\partial_k$ . Exact  $k$ -chains form the image space of  $\partial_{k+1}$ .

## Definition (Homology Group)

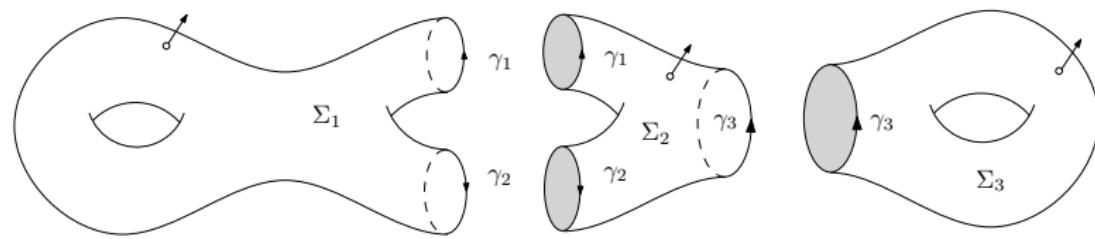
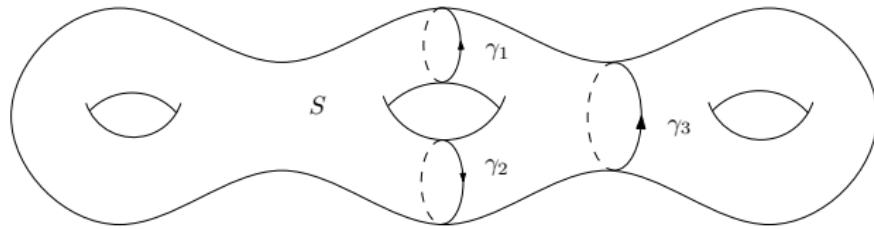
The  $k$  dimensional homology group  $H_k(\Sigma, \mathbb{Z})$  is the quotient space of  $\ker \partial_k$  and the image space of  $\text{img } \partial_{k+1}$ .

$$H_k(\Sigma, \mathbb{Z}) = \frac{\ker \partial_k}{\text{img } \partial_{k+1}}.$$

Two  $k$ -chains  $\gamma_1, \gamma_2$  are homologous, if they boundary a  $(k+1)$ -chain  $\sigma$ ,

$$\gamma_1 - \gamma_2 = \partial_{k+1} \sigma.$$

# Homological Classes



$$\partial\Sigma_1 = \gamma_1 - \gamma_2, \quad \partial\Sigma_2 = \gamma_3 - \gamma_1 + \gamma_2, \quad \partial\Sigma_3 = -\gamma_3.$$

$\gamma_1$  and  $\gamma_2$  are not homotopic but homological;  $\gamma_3$  is not homotopic to  $e$ , but homological to 0;  $\gamma_3$  is homological to  $\gamma_1 - \gamma_2$ .

# Homology vs. Homotopy

## Abelianization

The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

$$H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)].$$

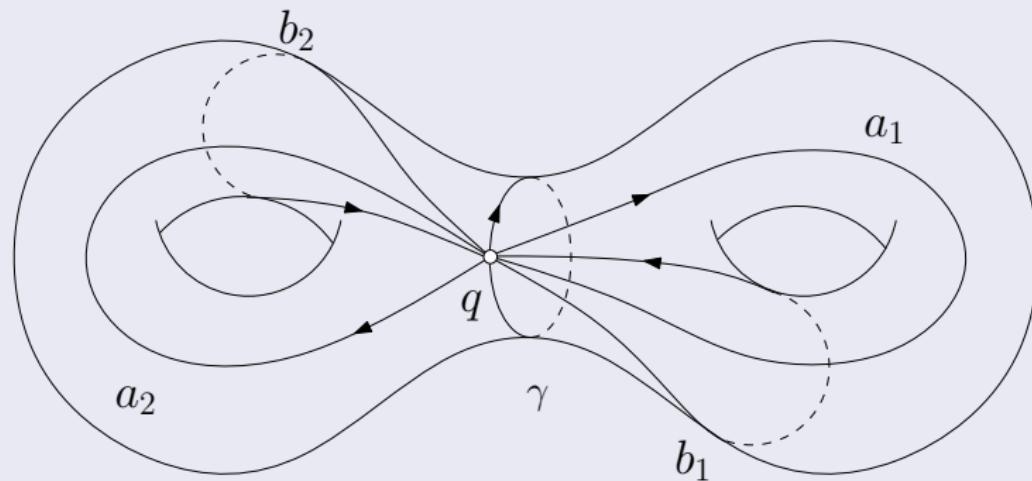
where  $[\pi_1(\Sigma), \pi_1(\Sigma)]$  is the commutator of  $\pi_1$ ,

$$[\gamma_1, \gamma_2] = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}.$$

Fundamental group encodes more information than homology group, but more difficult to compute.

# Homology vs. Homotopy

Homotopy group is non-abelian, which encodes more information than homology group.



- in homotopy group  $\pi_1(S, q)$ ,  $\gamma \sim [a, b]$ ,
- in homology group  $H_1(S, \mathbb{Z})$ ,  $\gamma \sim 0$ .

# Poincaré Duality

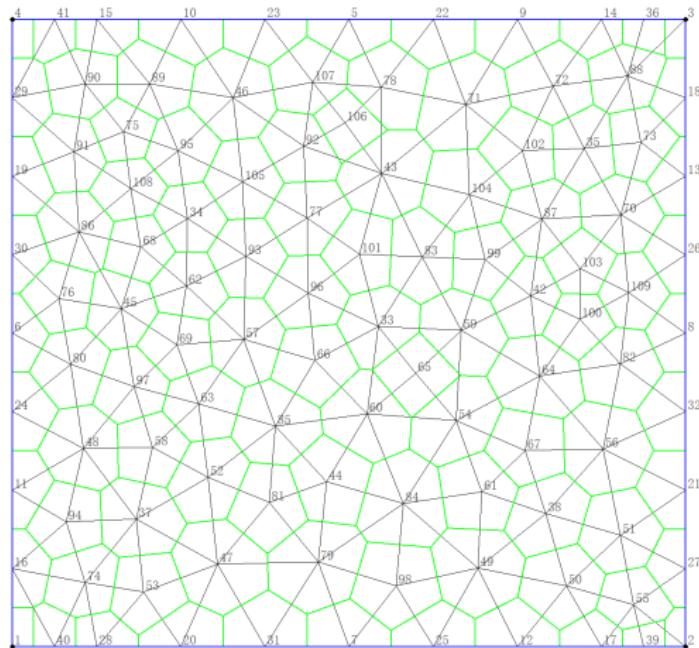
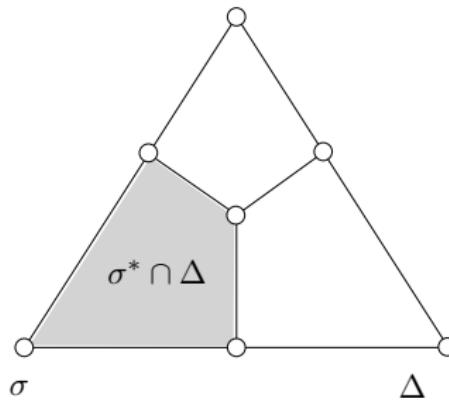


Figure: Poincaré Duality.

# Poincaré Duality

Given a triangulated manifold  $T$ , there is a corresponding dual polyhedral decomposition  $T^*$ , which is a cell decomposition of the manifold such that the  $k$ -cells of  $T^*$  are in bijective correspondence with the  $(n - k)$ -cells of  $T$ .

Let  $\sigma$  be a simplex of  $T$ . Let  $\Delta$  be a top-dimensional simplex of  $T$  containing  $\sigma$ , so we can think of  $\sigma$  as a subset of the vertices of  $\Delta$ . Define the dual cell  $\sigma^*$  corresponding to  $\sigma$  so that  $\Delta \cap \sigma^*$  is the convex hull in  $\Delta$  of the barycentres of all subsets of the vertices of  $\Delta$  that contain  $\sigma$ .



# Homology Group

## Theorem

Suppose  $M$  is a  $n$  dimensional closed manifold, then  
 $H_k(M, \mathbb{Z}) \cong H_{n-k}(M, \mathbb{Z})$ .

## Proof.

The intersection map  $C_k(T) \times C_{n-k}(T) \rightarrow \mathbb{Z}$  gives an isomorphism  
 $C_k(T) \rightarrow C^{n-k}(T^*)$ . □

## Theorem

Suppose  $M$  is a genus  $g$  closed surface, then  $H_0(M, \mathbb{Z}) \cong \mathbb{Z}$ ,  
 $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ ,  $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$ .

If  $H_0(M, \mathbb{Z}) = \mathbb{Z}^k$ , then  $M$  has  $k$  connected components.

# Computation for Homology Basis

Each boundary operator:  $\partial_k : C_k \rightarrow C_{k-1}$  is a linear map between linear spaces  $C_k$  and  $C_{k-1}$ , therefore it can be represented as a integer matrix. Suppose there are  $n_k$   $k$ -simplexes of  $\Sigma$ ,  $\{\sigma_1^k, \sigma_2^k, \dots, \sigma_{n_k}^k\}$ .

$$C_k = \left\{ \sum_{i=1}^{n_k} \lambda_i \sigma_i^k \right\}.$$

## Boundary Matrix

The boundary matrix is defined as:  $\partial_k = ([\sigma_i^{k-1}, \sigma_j^k])$ , where

$$[\sigma_i^{k-1}, \sigma_j^k] = \begin{cases} +1 & +\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ -1 & -\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ 0 & \sigma_i^{k-1} \notin \partial_k \sigma_j^k \end{cases}$$

# Computation for Homology Basis

## Combinatorial Laplace Operator

Construct linear operator  $\Delta_k : C_k \rightarrow C_k$ ,

$$\Delta_k := \partial_k^T \partial_k + \partial_{k+1} \partial_{k+1}^T,$$

the eigen vectors of zero eigen values of  $\Delta_k$  form the basis of  $H_k(M, \mathbb{Z})$ .

## Smith Norm

The eigen vectors can be found using Smith norm of integer matrix. The computational cost is very high.

# Simplicial Cohomology Group

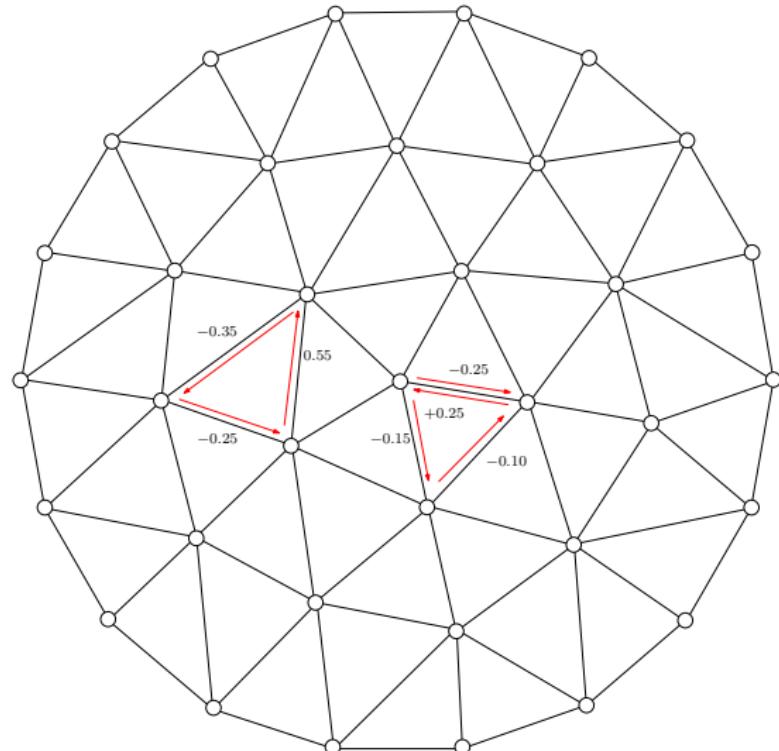


Figure: 1-Cochain.

# Simplicial Cohomology Group

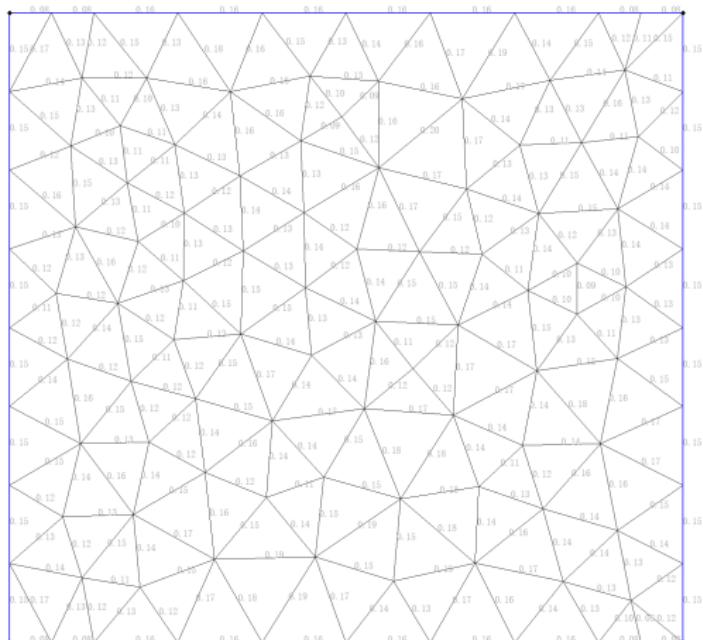


Figure: 1-Cochain.

# Simplicial Cohomology Group

## Definition (Cochain Space)

A  $k$ -cochain is a linear function

$$\omega : C_k \rightarrow \mathbb{Z}.$$

The  $k$  cochain space  $C^k(\Sigma, \mathbb{Z})$  is a linear space formed by all the linear functionals defined on  $C_k(\Sigma, \mathbb{Z})$ . A  $k$ -cochain is also called a  $k$ -form.

## Definition (Coboundary)

The coboundary operator  $\delta_k : C^k(\Sigma, \mathbb{Z}) \rightarrow C^{k+1}(\Sigma, \mathbb{Z})$  is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \quad \omega \in C^k(\Sigma, \mathbb{Z}).$$

# Simplicial Cohomology Group

## Example

$M$  is a 2 dimensional simplicial complex,  $\omega$  is a 1-form, then  $\delta_1\omega$  is a 2-form, such that

$$\begin{aligned}\delta_1\omega([v_0, v_1, v_2]) &= \omega(\partial_2[v_0, v_1, v_2]) \\ &= \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0])\end{aligned}$$

# Cohomology

Coboundary operator is similar to differential operator.  $\delta_0$  is the gradient operator,  $\delta_1$  is the curl operator.

## Definition (closed forms)

A  $k$ -form is closed, if  $\delta_k \omega = 0$ .

## Definition (Exact forms)

A  $k$ -form is exact, if there exists a  $k - 1$  form  $\sigma$ , such that

$$\omega = \delta_{k-1} \sigma$$

# Cohomology

suppose  $\omega \in C^k(\Sigma)$ ,  $\sigma \in C_k(\Sigma)$ , we denote the pair

$$\langle \omega, \sigma \rangle := \omega(\sigma).$$

## Theorem (Stokes)

$$\langle d\omega, \sigma \rangle = \langle \omega, \partial\sigma \rangle.$$

## Theorem

$$\delta^k \circ \delta^{k-1} \equiv 0.$$

All exact forms are closed. The curl of gradient is zero.

# Cohomology

The difference between exact forms and closed forms indicates the topology of the manifold.

## Definition (Cohomology Group)

The  $k$ -dimensional cohomology group of  $\Sigma$  is defined as

$$H^n(\Sigma, \mathbb{Z}) = \frac{\ker \delta^n}{\text{img } \delta^{n-1}}.$$

Two 1-forms  $\omega_1, \omega_2$  are cohomologous, if they differ by a gradient of a 0-form  $f$ ,

$$\omega_1 - \omega_2 = \delta_0 f.$$

# Homology vs. Cohomology

## Duality

$H_1(\Sigma)$  and  $H^1(\Sigma)$  are dual to each other. suppose  $\omega$  is a closed 1-form,  $\sigma$  is a closed 1-chain, then the pair  $\langle \omega, \sigma \rangle$  is a bilinear operator.

## Definition (dual cohomology basis)

suppose a homology basis of  $H_1(\Sigma)$  is  $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ , the dual cohomology basis is  $\{\omega_1, \omega_2, \dots, \omega_n\}$ , if and only if

$$\langle \omega_i, \gamma_j \rangle = \delta_i^j.$$

Cohomology was introduced by H. Whitney in order to represent stiefel whitney class characteristic class. Prof. Chern learned it from Whitney.

# Simplicial Mapping

## Definition (simplicial mapping)

Suppose  $M$  and  $N$  are simplicial complexes,  $f : M \rightarrow N$  is a continuous map,  $\forall \sigma \in M$ ,  $\sigma$  is a simplex,  $f(\sigma)$  is a simplex.

For each simplex, we can add its gravity center, and subdivide the simplex to multiple ones. The resulting complex is called the gravity center subdivision.

## Theorem

Suppose  $M$  and  $N$  are simplicial complexes embedded in  $\mathbb{R}^n$ ,  $f : M \rightarrow N$  is a continuous mapping. Then for any  $\epsilon > 0$ , there exists gravity subdivisions  $\tilde{M}$  and  $\tilde{N}$ , and a simplicial mapping  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$ , such that

$$\forall p \in |M|, |f(p) - \tilde{f}(p)| < \epsilon.$$

# Simplicial Approximation

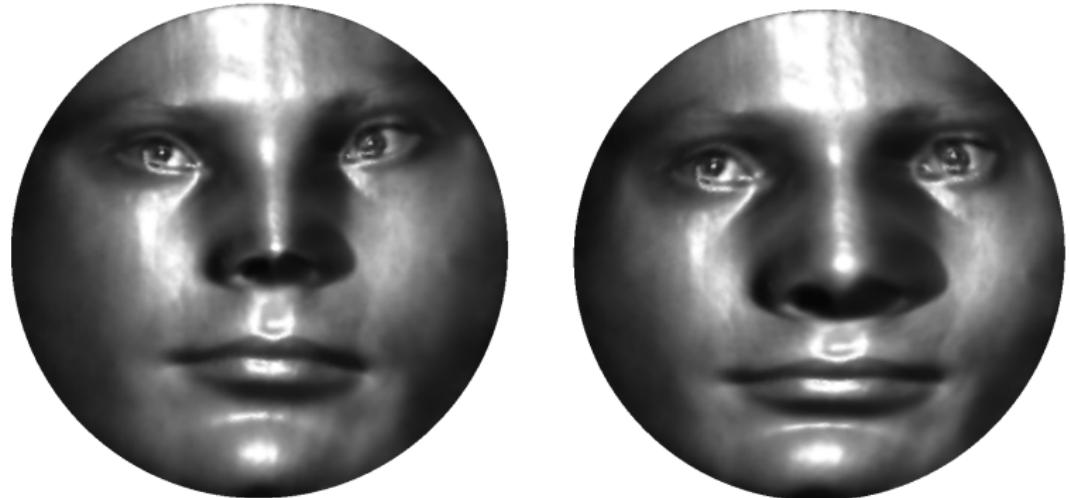


Figure: A planar map.

# Simplicial Approximation

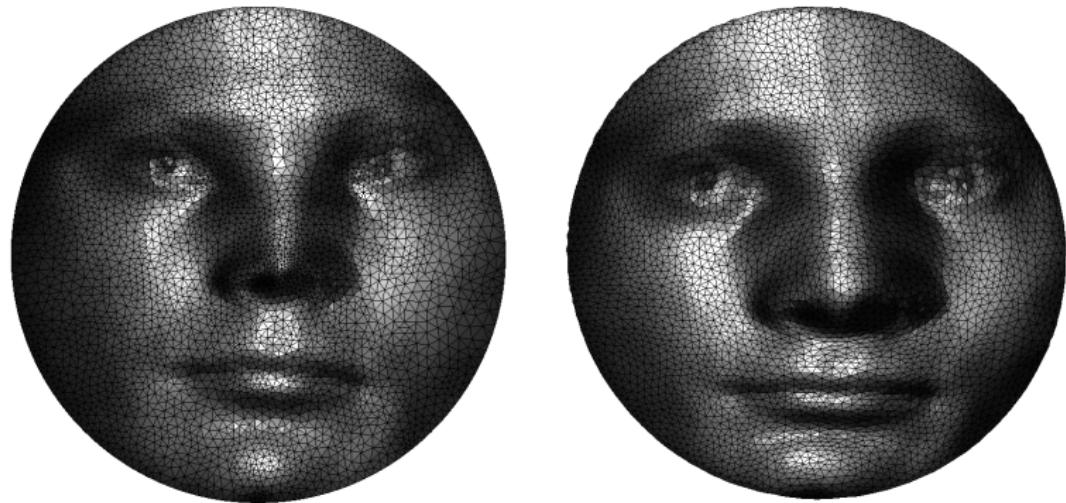


Figure: A planar map.

# Simplicial Mapping

## Definition (Pull-Back Map)

If  $f : M \rightarrow N$  is a continuous map, then  $f$  induces a homomorphism  $f_* : H_1(M) \rightarrow H_1(N)$ , which push forward the chains of  $M$  to the chains in  $N$ . Similarly,  $f$  induces a pull back map  $f^* : H^k(N) \rightarrow H^k(M)$ . Suppose  $\sigma \in C_1(M)$ ,  $\omega \in C^1(N)$ ,

$$f^* \omega(\sigma) = \omega(f_* \sigma) = \omega(f(\sigma)).$$

# Degree of a mapping

Suppose  $M$  and  $N$  are two closed surfaces.  $H_2(M, \mathbb{Z}) = \mathbb{Z}$ ,  $H_2(N, \mathbb{Z}) = \mathbb{Z}$ , suppose  $[M]$  is the generator of  $H_2(M)$ , which is the union of all faces. similarly,  $[n]$  is the generator of  $H_2(N)$ .  $f : M \rightarrow N$  is a continuous map. Then

$$f_* : \mathbb{Z} \rightarrow \mathbb{Z},$$

must has the form  $f_*(z) = cz, c \in \mathbb{Z}$ .

## Definition (Mapping Degree)

$f_*([M]) = c[N]$ , then the integer  $c$  is the degree of the map.

map degree is the algebraic number of pre-images  $f^{-1}(q)$  for arbitrary point  $q \in N$ , which is independent of the choice the point  $q$ .

# Degree of a mapping

## Example (Gauss-Bonnet)

$G : S \rightarrow \mathbb{S}^2$  is the Gauss map, which maps the point  $p$  to its normal  $\mathbf{n}(p)$ , then  $\deg(G) = 1 - g$ . The total area of the image is  $4\pi\deg(G) = 2\pi\chi(S)$ .

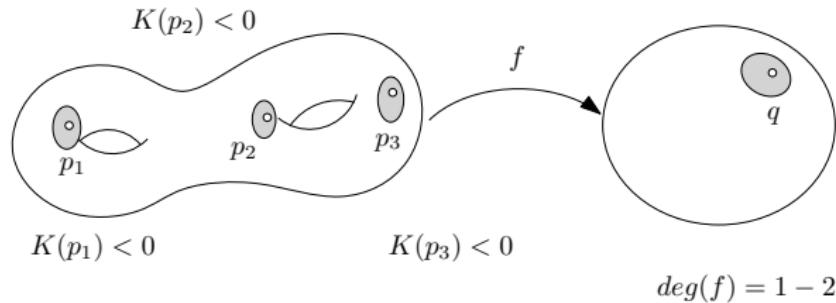


Figure: Map degree and Gauss-Bonnet theorem proof.

High dimensional Gauss-Bonnet theorem was first proved by Allendoerfer and Weil, Prof. Chern used different method to reprove it.

# Algorithm for Cohomology Group

## Algorithm for $H^1(M, \mathbb{R})$

Input: A genus  $g$  closed triangle mesh  $M$ ;

Output: A set of basis of  $H^1(M, \mathbb{R})$

- ① Compute a set of basis of  $H_1(M, \mathbb{Z})$ , denoted as

$$\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\},$$

- ② for each  $\gamma_i$ , slice  $M$  along  $\gamma_i$ , to obtain a mesh with two boundaries  $M_i, \partial M_i = \gamma_i^+ - \gamma_i^-$ ;
- ③ set a 0-form  $\tau_i$  on  $M_i$ , such that  $\tau_i(v) = 1$  for all  $v \in \gamma_i^+$  and  $\tau_i(w) = 0$ , for all  $w \in \gamma_i^-$ ; set  $\omega_i = d\tau_i$ ;
- ④ All  $\{\omega_1, \omega_2, \dots, \omega_{2g}\}$  form a basis of  $H^1(M, \mathbb{R})$ .