Algebraic Function Field on Riemann Surfaces

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Meromorphic Function Field

Lemma

Suppose $f \in \mathfrak{M}(M)$ is not constant, then $\mathbb{C}(f)$ is a pure transcendental extension of $\mathbb{C}$.

Proof.

Otherwise, $\exists P(x) \in \mathbb{C}[x]$, such that $P(x) \equiv 0$. Assume $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$,

$$P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n \equiv 0,$$

for any point $p \in M$, $f(p) \in \mathbb{C} \cup \{\infty\}$, $f(p)$ is a root of $P(x)$, therefore the range of f consists of finite number of points. This contradicts to the fact that $f : M \to \mathbb{S}^2$ is a branched covering with finite number of sheets. $\square$

Meromorphic Function Field

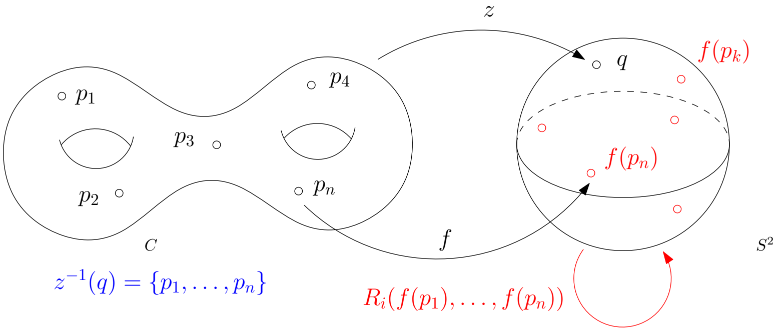

Figure: Construction of γ_i, holomorphic map $Q_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $Q_i = R_i \circ f \circ z^{-1}$, a rational polynomial.

$$Q_i : \hat{\mathbb{C}} \xrightarrow{z^{-1}} M^n \xrightarrow{f} \hat{\mathbb{C}}^n \xrightarrow{R_i} \hat{\mathbb{C}}$$

$$Q_i : q \xrightarrow{z^{-1}} (p_1, p_2, \dots, p_n) \xrightarrow{f} (f(p_1), \dots, f(p_n)) \xrightarrow{R_i} R_i(f(p_1), \dots, f(p_n))$$

Meromorphic Function Field

Theorem (Algebraic Function Field)

Suppose M is a compact Riemann surface, then $\mathfrak{M}(M)$ is an algebraic function field in one variable: if z is a meromorphic function with n $(n > 0)$ poles, then

$$[\mathfrak{M}(M) : \mathbb{C}(z)] = n.$$

Assume $(z) = (z)_0 - (z)_\infty$, where $(z)_0$ represent the zeros of z, $(z)_\infty$ the poles of z. Let $A = (dz)_0$, the zeros of dz (branch points of z), choose a point $a \in \mathbb{S}^2 - \{z(A), \infty\}$, then $z - a$ has simple zeros, $(z-a)^{-1}$ has simple poles. Since $\mathbb{C}(z) = \mathbb{C}(\frac{1}{z-a})$, if it is necessary we use $(z-a)^{-1}$ to replace z. Hence we can assume $(z)_\infty = p_1 + p_2 + \cdots + p_n$ consists of distinct points. Then we show

$$[\mathfrak{M}(M) : \mathbb{C}(z)] \geq n \quad \text{and} \quad [\mathfrak{M}(M) : \mathbb{C}(z)] \leq n.$$

Meromorphic Function Field

Lemma

$$[\mathfrak{M}(M) : \mathbb{C}(z)] \leq n.$$

Proof.

Claim: if $f_0 \in \mathfrak{M}(M)$, and $[\mathbb{C}(z)(f_0) : \mathbb{C}(z)]$ is maximized, then

$$\boxed{\mathfrak{M}(M) = \mathbb{C}(z)(f_0) = \mathbb{C}(z, f_0)}$$

Otherwise, there is $h \in \mathfrak{M}(M) - \mathbb{C}(z)(f_0)$, such that

$$[\mathbb{C}(z, f_0)(h) : \mathbb{C}(z, f_0)] = l > 1.$$

The classical primitive element theorem states: Every separable field extension of finite degree is simple. □

Meromorphic Function Field

continued.

The field $\mathbb{C}(z, f_0)(h) = \mathbb{C}(z)(f_0, h)$ must be a simple extension of $\mathbb{C}(z)$. Hence there is a $h_1 \in \mathfrak{M}(M)$, such that

$$\mathbb{C}(z)(f_0, h) = \mathbb{C}(z)(h_1).$$

$$\begin{aligned}
[\mathbb{C}(z)(h_1) : \mathbb{C}(z)] &= [\mathbb{C}(z, f_0)(h) : \mathbb{C}(z)] \\
&= [\mathbb{C}(z, f_0)(h) : \mathbb{C}(z, f_0)][\mathbb{C}(z, f_0) : \mathbb{C}] \\
&= l[\mathbb{C}(z, f_0) : \mathbb{C}(z)] \\
&> [\mathbb{C}(z, f_0) : \mathbb{C}(z)]
\end{aligned}$$

Contradiction to the assumption that $[\mathbb{C}(z)(f_0) : \mathbb{C}(z)]$ is maximized. Therefore $\mathbb{C}(z)(f_0) = \mathfrak{M}(M)$. $\square$

continued.

Claim,

$$\boxed{\forall f \in \mathfrak{M}(M) \quad [\mathbb{C}(z)(f) : \mathbb{C}(z)] \leq n}$$

Fix a $f \in \mathfrak{M}(M)$, assume there are $r_i(z) \in \mathbb{C}(z)$, $i = 1, 2, \dots, n$,

$$P(z, f) = f^n + r_1(z) f^{n-1} + r_2(z) f^{n-2} + \cdots + r_n(z) \equiv 0. \tag{1}$$

The problem boils down how to find $r_i(z)$'s. □

Meromorphic Function Field

continued.

Now z is an n-sheet branched covering from M to $\mathbb{S}^2$. Suppose $q \in \mathbb{S}^2$, such that $z^{-1}(q) = \{p_1, p_2, \dots, p_n\}$ are distinct. It is obvious that every point on $\mathbb{S}^2$ except a finite number of points has distinct pre-images. Let

$$\alpha_i = r_i(z(p_i)) = r_i(q).$$

If Eqn. (1) holds, then $f(p_1), f(p_2), \dots, f(p_n)$ are all the roots of the following polynomial:

$$W^n + \alpha_1 W^{n-1} + \cdots + \alpha_n,$$

and $\alpha_i = (-1)^i R_i(f(p_1), \dots, f(p_n))$, where R_i is the i-th elementary symmetric polynomial,

$$R_i(x_1, \cdots, x_n) = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} \cdots x_{j_i}, \quad 1 \le i \le n.$$

Meromorphic Function Field

continued.

Define map $Q_i : \mathbb{S}^2 \to \mathbb{S}^2$,

$$Q_i(q) = (-1)^i R_i(f(p_1), f(p_2), \cdots, f(p_n)).$$

It can be shown that Q_i is a holomorphic map from $\mathbb{S}^2$ to itself, hence Q_i is a rational function.

$$Q_i(z(p_j)) = Q_i(q) = (-1)^i R_i(f(p_1), \cdots, f(p_n)) = \alpha_i,$$

but $\alpha_i = r_i(z(p_j))$, hence we define

$$r_i(z) := Q_i(z)$$

which is a rational function of z. The construction of $Q_i(z)$ depends on f. $\square$

Meromorphic Function Field

continued.

Now reverse the whole process, $\forall f \in \mathfrak{M}(M)$, we can construct $r_i(z) \in \mathbb{C}(z)$, such that

$$P(z, f) \equiv f^n + r_1(z) f^{n-1} + \cdots + r_n(z) = 0$$

holds on M, this completes the proof of the lemma. $\square$

Meromorphic Function Field

Lemma

$$[\mathfrak{M}(M) : \mathbb{C}(z)] \geq n.$$

Proof.

Assume $(z)_\infty = p_1 + p_2 + \cdots + p_n$ consisting of distinct points, all poles are simple.
Claim: $\forall i = 1, 2, \cdots, n$, $\exists \omega_i \in \mathfrak{M}(M)$, such that

$$\boxed{\omega_i(p_1) = \omega_i(p_2) = \cdots = \omega_i(p_{i-1}) = 0, \omega_i(p_i) = 1}$$

□

Meromorphic Function Field

continued.

We show $\omega_1, w_2, \cdots, \omega_n$ are linearly independent with respect to $\mathbb{C}(z) = \mathbb{C}(\frac{1}{z})$. Otherwise, there are $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}(\frac{1}{z})$, such that

$$\sum_{i=1}^{n} \alpha_i \omega_i = 0.$$

Let β be the least common multiple of the denominators of $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let $\gamma_i = \beta\alpha_i$, thus

$$\sum_{i=1}^{n} \gamma_i \omega_i = 0, \quad \gamma_i \in \mathbb{C}[1/z], 1 \leq i \leq n.$$

Let d be the greatest common divisor of $\gamma_1, \gamma_2, \cdots, \gamma_n$, multiply the above equation by $1/d$, the coefficient γ_i/d is still denoted as α_i, then $\alpha_i \in \mathbb{C}(1/z)$ and α_i has no trivial common divisor. $\square$

Meromorphic Function Field

continued.

Hence at least one α_i has non-zero constant term, namely there is a $r \leq n$, such that $\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$ has zero constant terms, and α_r has non-zero constant term. At p_r, $1/z(p_r) = 0$, therefore

$$\alpha_1(p_r) = \alpha_2(p_r) = \cdots = \alpha_{r-1}(p_r) = 0.$$

By the choice of ω_i, we know

$$\omega_{r+1}(p_r) = \omega_{r+2}(p_r) = \cdots = \omega_n(p_r) = 0.$$

plug into $\sum_{i=1}^{n} \alpha_i \omega_i = 0$, we obtain

$$\alpha_r(p_r) \cdot \omega_r(p_r) = 0.$$

This contradicts to the fact that $\alpha_r(p_r) \neq 0$ and $\omega_r(p_r) \neq 0$. $\square$

Meromorphic Function Field

Lemma

Suppose M is a compact Riemann surface, D is a divisor and $deg(D) \geq 2g-1$, then

$$\boxed{diml(-D) = deg(D) + (1-g)} \tag{2}$$

Proof.

Since $\deg(D) \geq 2g-1$,
$\deg((\omega)-D) = \deg(\omega) - \deg(D) = 2g-2-\deg(D) < 0$, therefore

$$\Omega(D) \cong l(-((\omega)-D)) = \{0\}.$$

By Riemann-Roch

$$\dim l(-D) = \dim\Omega(D) + \deg(D) + (1-g) = \deg(D) + (1-g).$$

□

Meromorphic Function Field

continued.

Now we prove the claim. Select a point $q \in M - \{p_1, p_2, \cdots, p_n\}$ and $k \geq 2g - 1 + n$. Let $D = kq - (p_1 + p_2 + \cdots + p_{i-1})$, $\deg(D) > 2g - 1$, by lemma (Eqn. 2), $\dim l(-D) = \deg(D) + (1 - g)$, hence $\forall i = 1, \dots, n$

$$\dim l(-(kq - (p_1 + p_2 + \cdots + p_{i-1})) = 1 + \dim l(-(kq - (p_1 + p_2 + \cdots + p_i)).$$

Therefore, there exists

$$\omega_i \in l(-(kq - (p_1 + p_2 + \cdots + p_{i-1})) - l(-(kq - (p_1 + p_2 + \cdots + p_i)),$$

namely

$$\omega_i(p_1) = \omega_i(p_2) = \cdots = \omega_i(p_{i-1}) = 0, \omega_i(p_i) \neq 0.$$

This completes the proof for the claim. $\square$

Valuation

Definition (Valuation)

Let K be a field. A (discrete) valuation on K is a surjective map $v : K \to \mathbb{Z} \cup \{\infty\}$ with the following properties:

1. $v^{-1}(\infty) = \{0\}$.
2. $v(f+g) \geq \min(v(f), v(g))$ for all $f, g \in K$.
3. $v(fg) = v(f) + v(g)$ for all $f, g \in K$.

Valuation

Valuation $v : K^* \to \mathbb{Z}$ has the properties: we define $v(0) = +\infty$,

- $v(1) = 0, v(-1) = 0, v(f) = v(-f)$;
- when $v(f) \neq v(g)$, $v(f+g) = \min\{v(f), v(g)\}$.
- if $\mathbb{C} \subset K$, then $v(c) = 0$, $\forall c \in \mathbb{C}^*$. Since $v(c) = n \cdot v(c^{1/n})$, if $v(c) \neq 0$, then $v(c^{1/n}) \neq 0$, let $n \to \infty$, induce contradiction.

Valuation

Lemma (Valuation)

Suppose M is a compact Riemann surface, v is a valuation on $\mathfrak{M}(M)$, then there is a unique point $p \in M$, such that $v = \nu_p$.

Proof.

Take a meromorphic function h, such that $v(h) = 1$, obviously h is not constant (since $v(c) = 0$). If $a \in \mathbb{C}^*$, then $v(h - a) = \min\{v(h), v(-a)\} = 0$. Therefore, if r is a rational function, then

$$v(r(h)) = \nu_0(r).$$

Denote the zeros of h as $p_1, \cdots, p_n$. Choose an arbitrary meromorphic function f, then f satisfies the equation

$$f^n + r_1(h) f^{n-1} + \cdots + r_n(h) = 0,$$

r_i's are rational functions. $\square$

Valuation

continued.

$$
\begin{aligned}
nv(f) &\geq \min\{v(r_i(h)f^{n-i}), i = 1, \cdots, n\} \\
&= \min\{v(r_i(h)) + (n-i)v(f), i = 1, \cdots, n\} \\
&= \min\{\nu_0(r_i) + (n-i)v(f), i = 1, \cdots, n\}
\end{aligned}
$$

r_i's are rational functions. If $v(f) < 0$, then there is an i, such that $\nu_0(r_i) < 0$. Because $r_i(0)$ is the elementary symmetric function of the values of f at $p_1, p_2, \cdots, p_n$, hence f must have a pole at some p_j. By similar argument on $1/f$, we have if $v(f) > 0$, then f must have a zero at some p_k. □

Valuation

continued.

Now we label distinct points in $\{p_1, p_2, \cdots, p_n\}$ by $\{q_1, q_2, \cdots, q_m\}$. Choose a meromorphic function g, such that g is holomorphic at $\{q_i\}$, $g(q_i)$ are mutually different nonzero complex numbers, furthermore each $dg(q_i)$ is non-zero. By above argument, $g(q_i)$ is neither 0 nor ∞, $v(g)$ can not be positive or negative, hence $v(g) = 0$.
Consider function

$$h\Pi_{i=1}^{m}(g - g(q_i))^{-\nu_{q_i}(h)} \tag{3}$$

At q_i, the order of h is $\nu_{q_i}(h)$, the order of $g - g(q_i)$ is 1, hence the order of $h(g - g(q_i))^{-\nu_{q_i}(h)}$ is 0. Function (3) is holomorphic at $\{q_i\}$ with non-zero value, therefore its valuation is zero. Hence

$$1 = v(h) = \sum_{i=1}^{m} \nu_{q_i}(h) v(g - g(q_i)).$$

□

Valuation

continued.

Because $v(g - g(q_i)) \geq \min\{v(g), v(g(q_i))\} = 0$, there is a unique q_k, such that

$$v(g - g(q_k)) = 1 = \nu_{q_k}(h), \quad v(g - g(q_j)) = 0, \quad j \neq k.$$

For any meromorphic function f, consider function

$$f \prod_{i=1}^{m} (g - g(q_i))^{-\nu_{q_i}(f)}$$

It is holomorphic at $\{q_i\}$ with non-zero value, therefore its valuation is zero. Hence

$$v(f) = \sum_{i=1}^{m} \nu_{q_i}(f) v(g - g(q_i)) = \nu_{q_k}(f).$$

This completes the proof. □

Meromorphic Function Field

Theorem

Suppose M, N are compact Riemann surfaces, $\varphi : \mathfrak{M}(N) \to \mathfrak{M}(M)$ is a field isomorphism and its restriction on $\mathbb{C}$ is identity, then there exists a unique holomorphic map $h : M \to N$, such that

$$\varphi = h^*.$$

Meromorphic Function Field

Proof.

Suppose $p \in M$, $\forall f \in \mathfrak{M}(N)$, $\varphi(f) \in \mathfrak{M}(M)$, define valuation ν_p on $\mathfrak{M}(N)$ as

$$\nu_p(f) = \nu_p(\varphi(f)), \quad f \in \mathfrak{M}(N).$$

According to previous Valuation Lemma, there exists a unique point $h(p) \in N$, such that $\nu_p = \nu_{h(p)}$, namely

$$\nu_p(\varphi(f)) = \nu_{h(p)}(f), \quad f \in \mathfrak{M}(N).$$

In this way, we obtain the map $h : M \to N$. We claim

$$\varphi(f)(p) = f(h(p)), \quad f \in \mathfrak{M}(N), p \in M.$$

□

Meromorphic Function Field

continued.

Assume $c \in \mathbb{C}$, we have

$$\begin{aligned} f(h(p)) = c &\iff \nu_{h(p)}(f - c) > 0 \\ &\iff \nu_p(\varphi(f - c)) > 0 \\ &\iff \nu_p(\varphi(f) - c) > 0 \\ &\iff \varphi(f)(p) = c. \end{aligned}$$

therefore $f(h(p)) = \varphi(f)(p)$. $\square$

Meromorphic Function Field

continued.

Claim: h is continuous. Otherwise, there exist a point sequence $\{p_n\} \subset M$, such that $p_n \to p_0 \in M$, $h(p_n) \to q \in N$ and $q \neq h(p_0) = q_0$. On the other hand, for any meromorphic function $f \in \mathfrak{M}(N)$, we have

$$\begin{aligned} f(q) &= \lim_{n\to\infty} f(h(p_n)) \\ &= \lim_{n\to\infty} \varphi(f)(p_n) \\ &= \varphi(f)(p_0) = f(h(p_0)) = f(q_0), \end{aligned}$$

therefore f can't differentiate q and q_0, contradict to previous conclusion. $\square$

Meromorphic Function Field

continued.

Claim: h is holomorphic. Select an arbitrarily point $p \in M$, choose a meromorphic function $f \in \mathfrak{M}(N)$, such that f is biholomorphic in a local coordinate disk U of $h(p)$. $\varphi(f)$ is not constant (as an isomomorphism, φ is injective and identity on $\mathbb{C}$). Choose an open neighborhood $V \subset M$ of p, such that $h(V) \subset U$, and $\varphi(f)(V) \subset f(U)$. By $\varphi(f)(p) = f(h(p))$, we have

$$h(p') = f^{-1} \circ \varphi(f)(p'), \quad p' \in V.$$

Hence h is holomorphic. $\square$

Meromorphic Function Field

Corollary

If $\varphi : \mathfrak{M}(N) \to \mathfrak{M}(M)$ is a field isomorphism, then $h : M \to N$ is an isomorphism between Riemann surfaces.

The second proof doesn't use valuation.

Proof.

Suppose $f \in \mathfrak{M}(M)$, its minimal polynomial with respect to $\mathbb{C}(z)$ is

$$W^n + r_1(z)W^{n-1} + r_2(z)W^{n-2} + \cdots + r_n(z),$$

where $r_i(z) \in \mathbb{C}(z)$. After multiply the least common multiple, we obtain

$$G(W,z) \equiv S_0(z)W^n + S_1(z)W^{n-1} + S_2(z)W^{n-2} + \cdots + S_n(z),$$

where $S_i(z) \in \mathbb{C}[z]$, $G(f,z) \equiv 0$. □

Meromorphic Function Field

Proof.

$G(W,z) \in \mathbb{C}[W,z]$ is an algebraic function, and $G(W,z)$ is irreducible in $\mathbb{C}[W,z]$. $G(W,z) \equiv 0$ determines a n-valued holomorphic function in z on $\mathbb{C}$, namely for each z (with finite number of exceptions $z \in \mathbb{C}$), $W(z)$ has n values. By the fundamental theorem on algebraic functions, this locally determines n single valued holomorphic functions $W_1(z), \cdots, W_n(z)$, such that $G(W_i(z), z) \equiv 0$, $\forall 1 \leq i \leq n$; furthermore, every $W_i(z)$ is the analytic extension of each $W_j(z)$. $\{W_i(z)\}$ induces a Riemann surface M_0, such that $W(z)$ on M_0 is a single valued holomorphic function. M_0 is the so-called Riemann surface of the algebraic function $G(W,z)$.
By the construction of M_0, there is a canonical map $\varphi : M \to M_0$ and φ is biholomorphic. The construction of M_0 is solely determined by $\mathfrak{M}(M)$. $\square$

Meromorphic Function Field

Proof.

Similarly, we can construct N_0 from $\mathfrak{M}(N)$. If $\mathfrak{M}(M)$ is isomorphic to $\mathfrak{M}(N)$, then M_0 is isomorphic to N_0, so M and N are isomorphic. $\square$