Geometric Interpretation of Abel-Jacobi Theorem

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Meromorphic Differential

Abel Differential of the Third Type

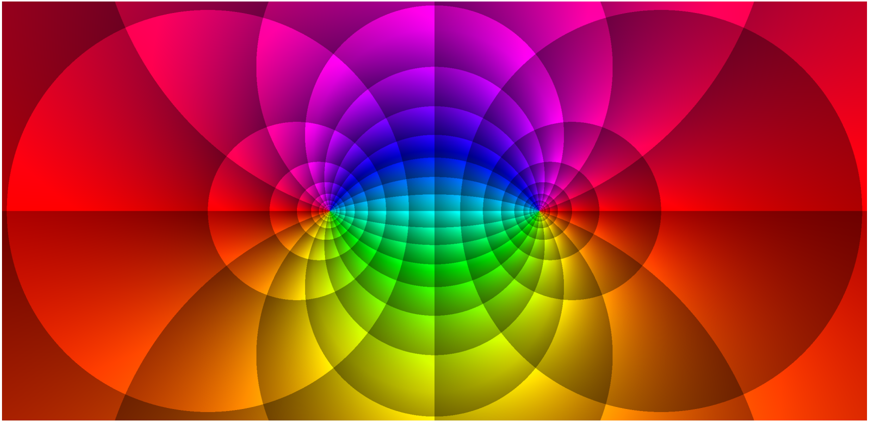

Figure: $f(z) = \log(z+1) - \log(z-1)$, $df(z) = \left(\frac{1}{z+1} - \frac{1}{z-1}\right) dz$

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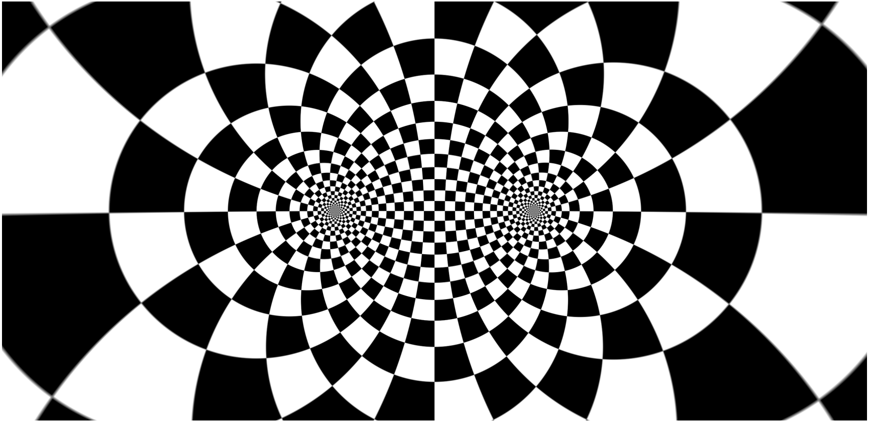

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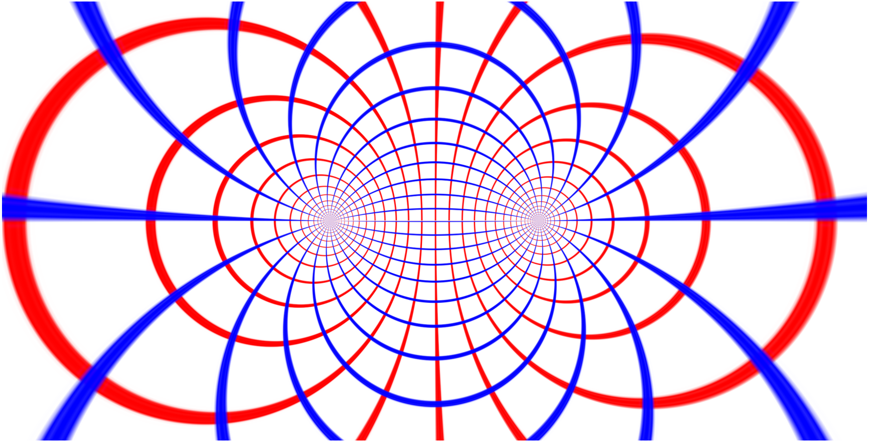

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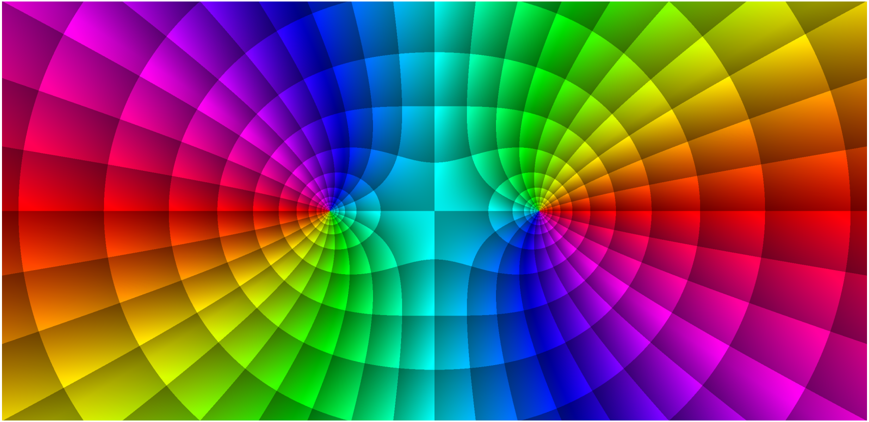

Figure: $f(z) = \log(z+1) + \log(z-1)$, $df(z) = \left(\frac{1}{z+1} + \frac{1}{z-1}\right) dz$

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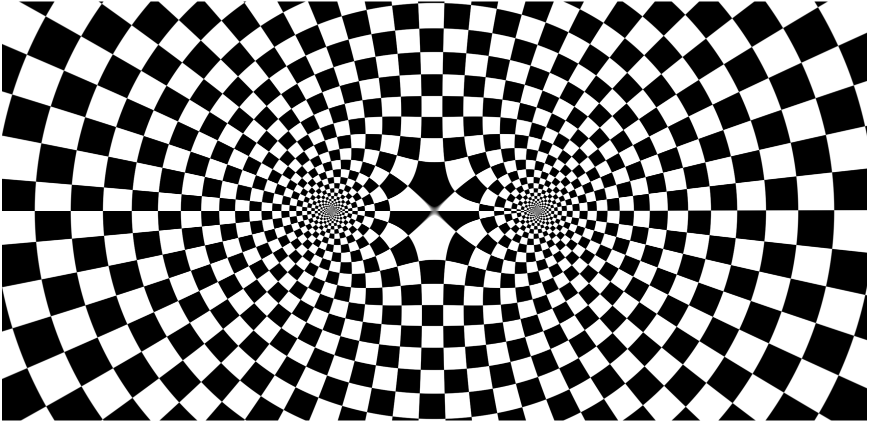

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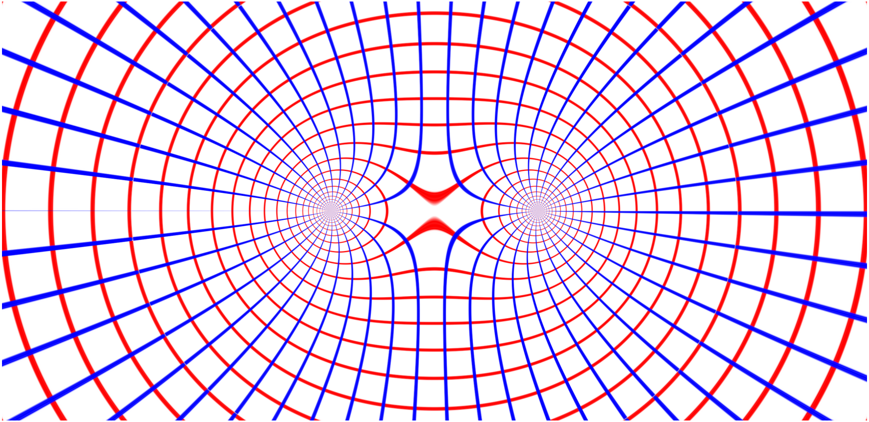

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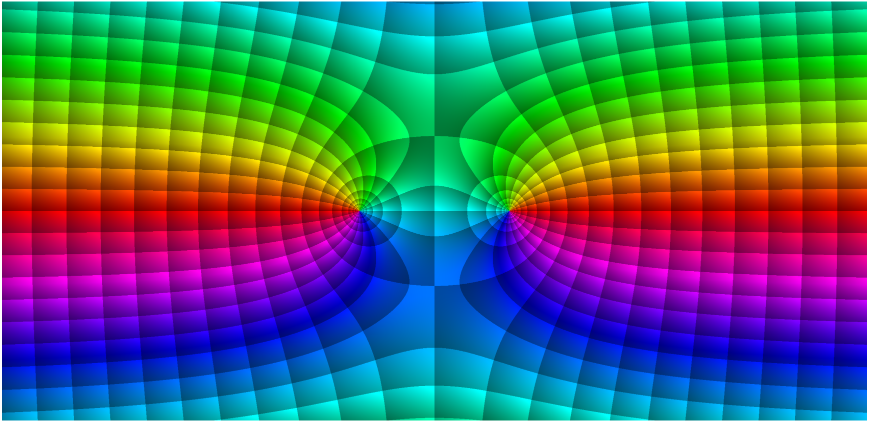

Figure: $f(z) = z + \log(z-1) - \log(z+1)$, $df(z) = \left(1 + \frac{1}{z-1} - \frac{1}{z+1}\right) dz$

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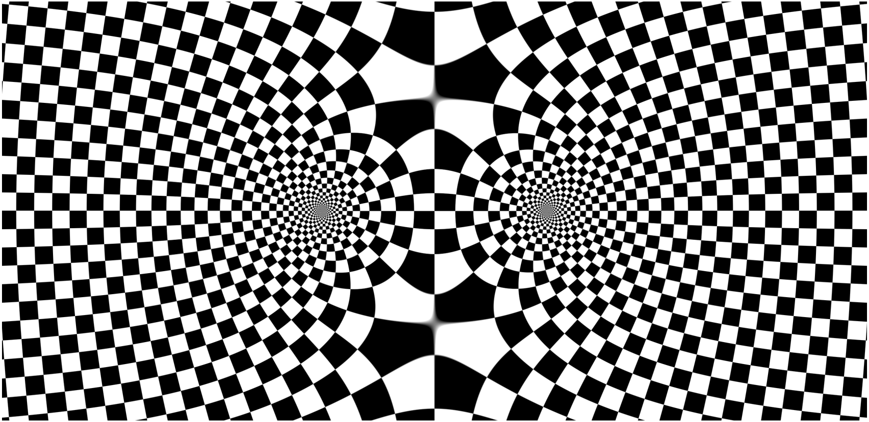

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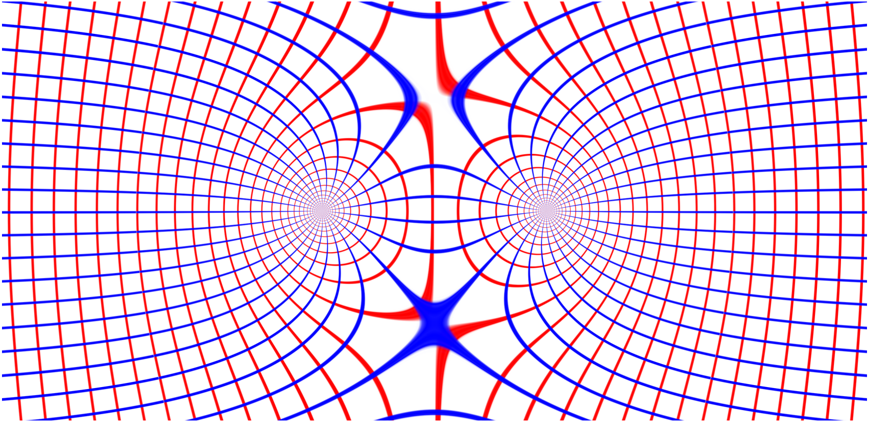

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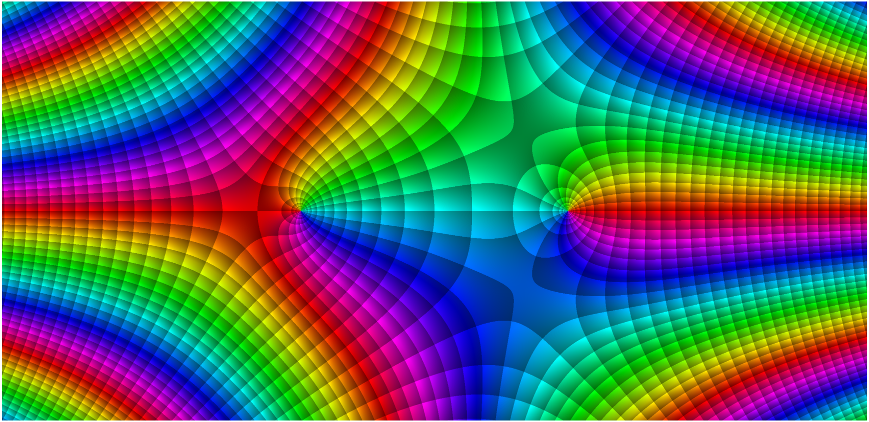

Figure: $f(z) = \frac{1}{2}(z^2 + \log(z-1) - \log(z+1))$, $df(z) = \frac{1}{2}\left(2z + \frac{1}{z-1} - \frac{1}{z+1}\right) dz$

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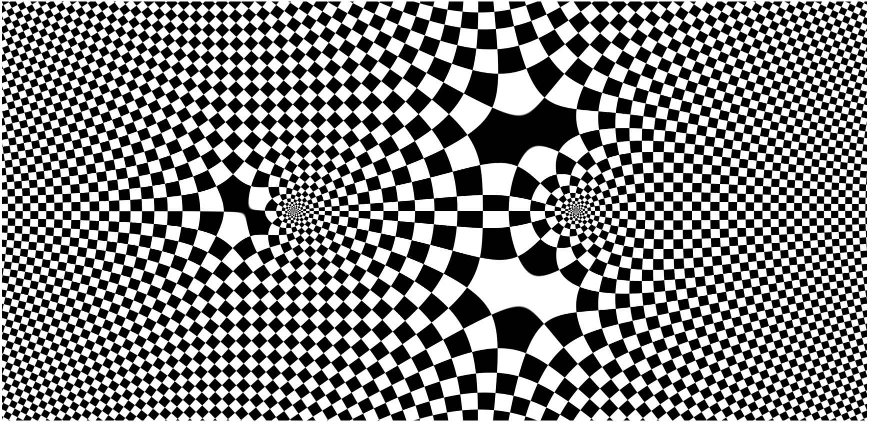

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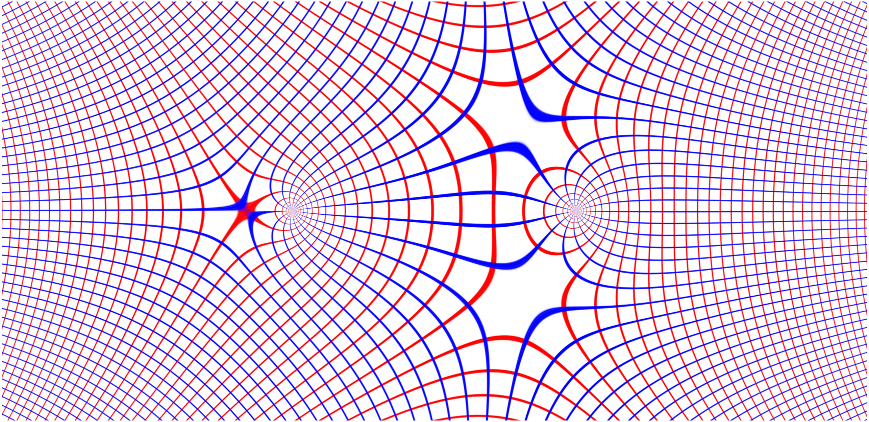

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Quadrilateral Mesh Generation Theory

Colorable Quad-Mesh

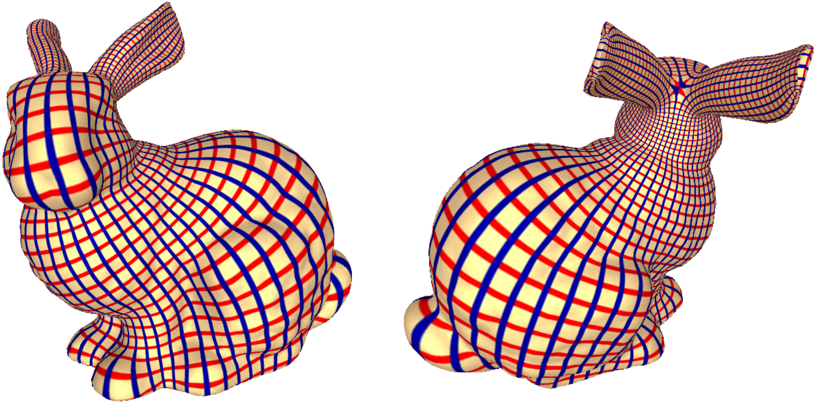


Figure: A red-blue (colorable) Quad-Mesh.

Colorable Quad-Mesh

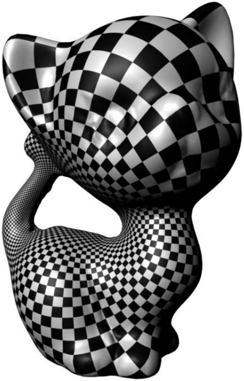
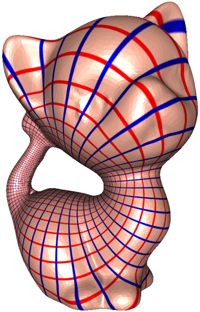


Figure: A quad-mesh induced by a holomorphic 1-form.

Singularities on a Topological Torus



Topological Torus

$$\chi = 2 - 2g = 0,$$

$$\sum K = 2\pi\chi = 0.$$

It is impossible to construct a quad mesh on a topological torus with one valence 3 singular point and one valence 5 singular point.

Otherwise, the valence 3 vertex p and the valence 5 vertex q become to the pole and the zero of a meromorphic function. By Abel condition, $\mu(p) = \mu(q)$, the pole and the zero coincide, contradiction.

Quad-Mesh

The number of singularities, and the layouts of separatrices are different.

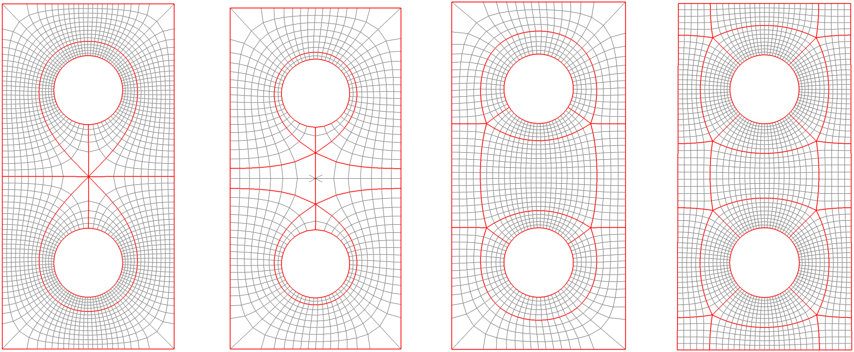


Figure: Quad-meshes with different number of singularities.

Quad-Meshes

Aim

Establish complete mathematical theory for structural mesh.

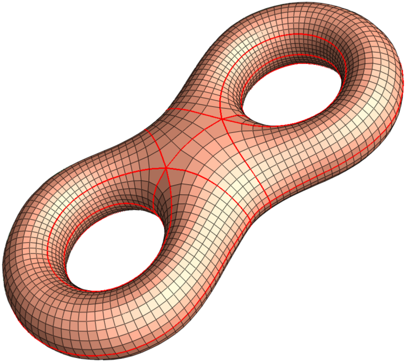


Figure: A quad-mesh of a genus two surface with different number of singularities.

Central Questions

Given a Riemannian surface $(S, \mathbf{g})$ two quad-meshes are equivalent if they differ by a finite step of subdivisions,

1. How many quad-mesh equivalent classes are there on S ?
   infinite
2. What is the dimension of the space of all the quad-mesh equivalent classes on S?
   Riemann-Roch theorem
3. What is the governing equation for the singularities ?
   Abel-Jacobi theorem

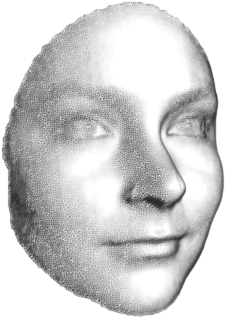
Mathematical View of Structural Quad Mesh

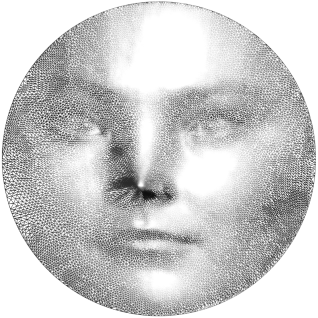
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ \ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

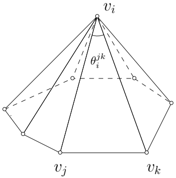
Definition (Discrete Curvature)

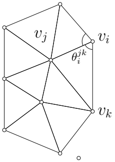
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v_i) = 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M; K(v_i) = \pi - \sum_{jk} \theta_{jk}, v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$

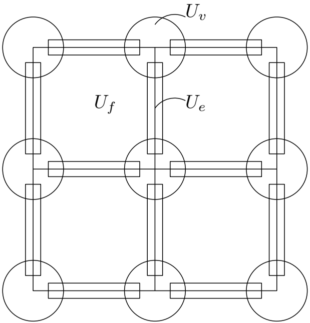


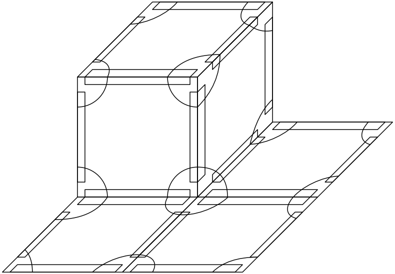


Quad-Mesh Metric

Definition (Quad-Mesh Metric)

Given a quad-mesh $\mathcal{Q}$, each face is treated as the unit planar square, this will define a Riemannian metric, the so-called quad-mesh metric $\mathbf{g}_{\mathcal{Q}}$, which is a flat metric with cone singularities.





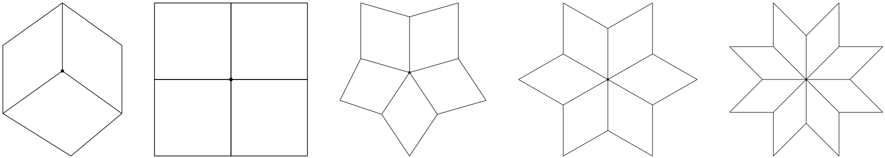
Discrete Gauss Curvature

Definition (Curvature)

Given a quad-mesh $\mathcal{Q}$, for each vertex v_i, the curvature is defined as

$$K(v) = \begin{cases} \frac{\pi}{2}(4 - k(v)) & v \notin \partial\mathcal{Q} \\ \frac{\pi}{2}(2 - k(v)) & v \in \partial\mathcal{Q} \end{cases}$$

where $k(v)$ is the topological valence of v, i.e. the number of faces adjacent to v.



$k = \pi/2$ $\quad$ $k = 0$ $\quad$ $k = -\pi/2$ $\quad$ $k = -\pi$ $\quad$ $k = -2\pi$

Quad-Mesh Metric Conditioins

Quad-Mesh Metric Conditions

Theorem (Quad-Mesh Metric Conditions)

Given a quad-mesh $\mathcal{Q}$, the induced quad-mesh metric is $\mathbf{g}_{\mathcal{Q}}$, which satisfies the following four conditions:

1. *Gauss-Bonnet condition;*
2. *Holonomy condition;*
3. *Boundary Alignment condition;*
4. *Finite geodesic lamination condition.*

1. Gauss-Bonnet Condition

Theorem (Gauss-Bonnet)

Given a quad-mesh $\mathcal{Q}$, the induced metric is $\mathbf{g}_{\mathcal{Q}}$, the total curvature satisfies

$$\sum_{v_i \in \partial \mathcal{Q}} K(v_i) + \sum_{v_i \notin \partial \mathcal{Q}} K(v_i) = 2\pi\chi(\mathcal{Q}).$$

Namely

$$\sum_{v_i \in \partial \mathcal{Q}} (2 - k(v_i)) + \sum_{v_i \notin \partial \mathcal{Q}} (4 - k(v_i)) = 4\chi(\mathcal{Q}).$$

2. Holonomy Condition

Theorem (Holonomy Condition)

Suppose $\mathcal{Q}$ is a closed quad-mesh, then the holonomy group induced by $\mathbf{g}_{\mathcal{Q}}$ is a subgroup of the rotation group $\{e^{i\frac{k}{2}\pi}, k \in \mathbb{Z}\}$.

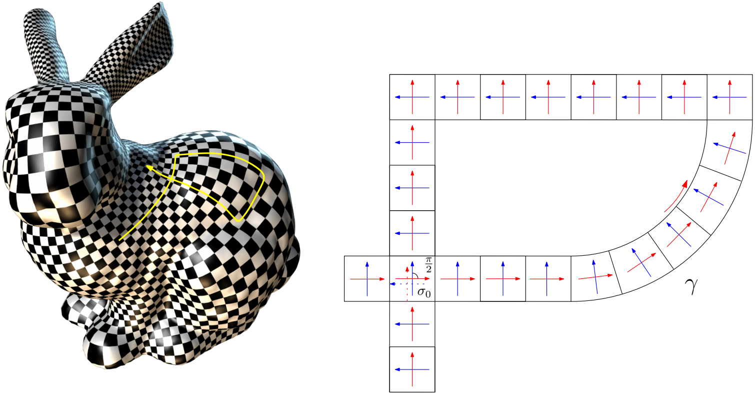


Figure: Parallel transportation along a face loop.

Definition (Boundary Alignment Condition)

Given a quad-mesh $\mathcal{Q}$, with induced metric $\mathbf{g}_{\mathcal{Q}}$, one can define a global cross field by parallel transportation, which is aligned with the boundaries.

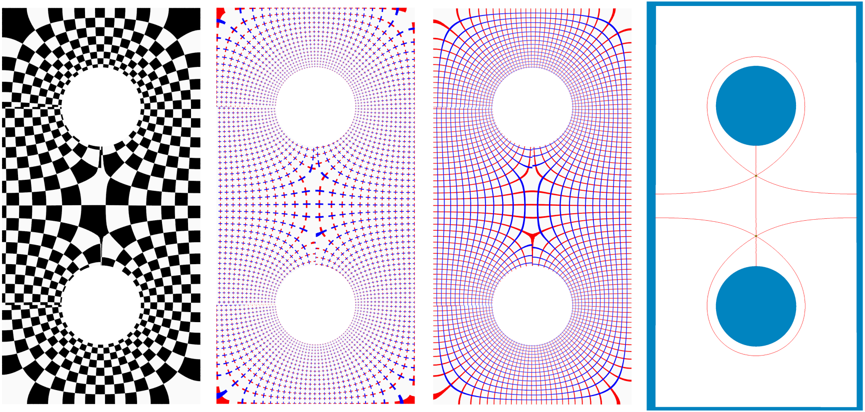


Figure: Aligned and mis-aligned with the inner boundaries

3. Boundary Alignment Condition

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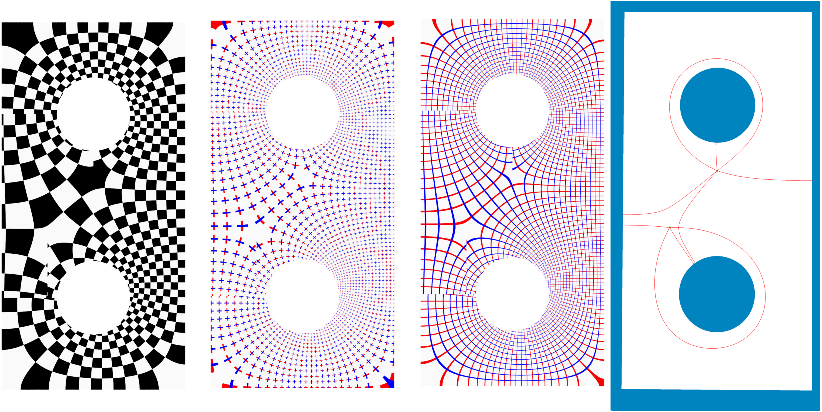
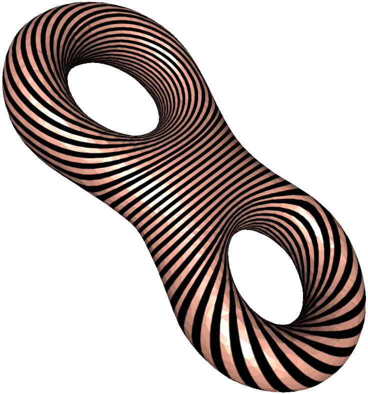


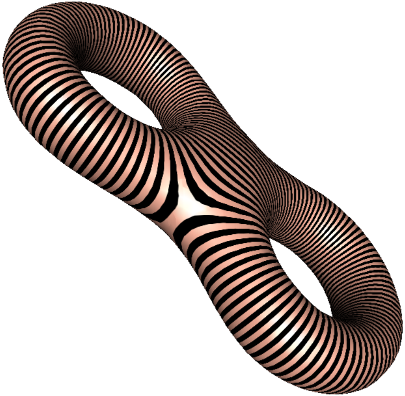
Figure: Aligned and mis-aligned with the inner boundaries.

4. Finite Geodesic Lamination Condition

Definition (Finite Geodesic Lamination Condition)

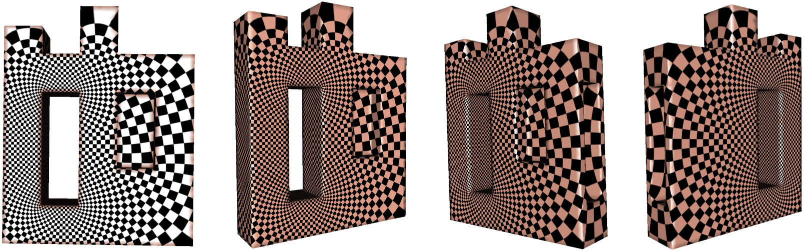
The stream lines parallel to the cross field are finite geodesic loops. This is the finite geodesic lamination condition.





Genus One Polycube Surface Example

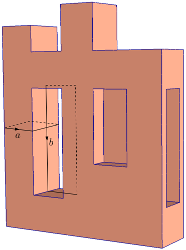
A genus one closed surface S, which is a polycube surface (union of canonical unit cubes). The holomorphic one form $\omega \in \Omega^1(S)$.

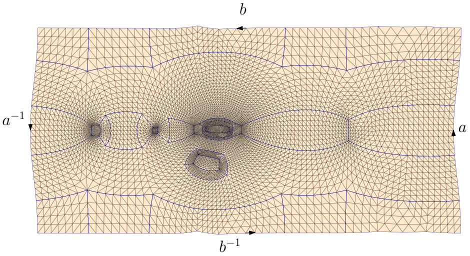


The homology basis is $\{a, b\}$, the surface is sliced along $\{a, b\}$ to get a fundamental domain D, $\partial D = abab^{-1}b^{-1}$. The conformal mapping $\mu : D \to \mathbb{C}$ is given by

$$\mu(q) = \int_p^q \omega,$$

where p is a base point and the integration path is arbitrarily chosen.

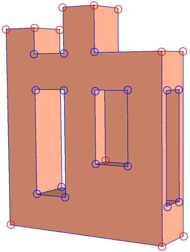


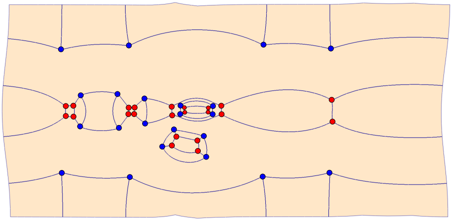


Genus One Polycube Surface Example

Suppose q_i's are poles (degree 3), p_j's are zeros (degree 5), then we have found that the number of poles equals to that of the zeros, furthermore,

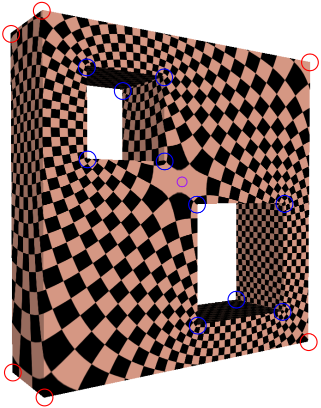
$$\sum_{j=1}^{22} \mu(p_j) - \sum_{i=1}^{22} \mu(q_i) = 0.$$



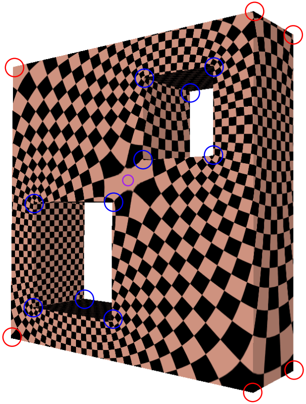


Genus Two Polycube Surface Example

Ia Suppose S is a genus two polycube surface, ω is a holomorphic one-form. The red circles show the poles (degree 3), the blue circles show the zeros (degree 5), the purple circles the zeros of ω.



(a). front view



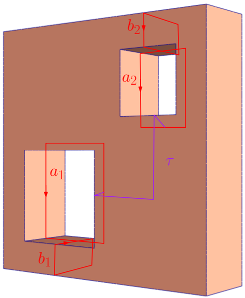
(b). back view

Genus Two Polycube Surface Example

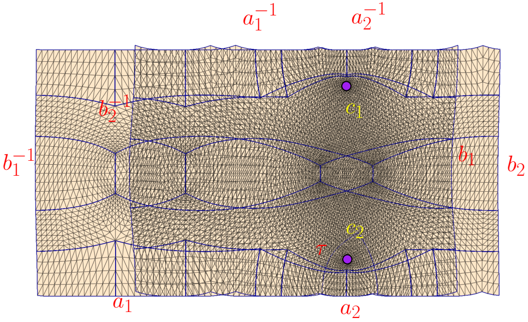
The surface is sliced along a_1, b_1, a_2, b_2, τ, and integrate ω to obtain $\mu : S \to \mathbb{C}$

$$\mu(q) = \int_p^q \omega,$$

it branch covers the plane, the branching points are zeros of ω, c_1, c_2.



(a). cuts

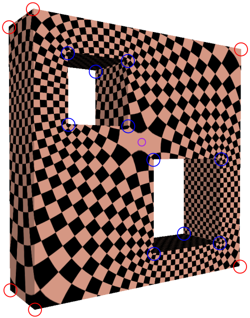


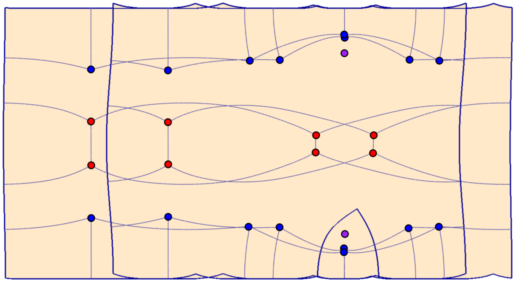
(b). conformal fattening

Genus Two Polycube Surface Example

Suppose p_i's are zeros (degree 5), q_j's are poles (degree 3), c_k's are branch points, then we have

$$\sum_{i=1}^{16} \mu(p_i) - \sum_{j=1}^{8} \mu(q_j) = 4 \sum_{k=1}^{2} \mu(c_k).$$





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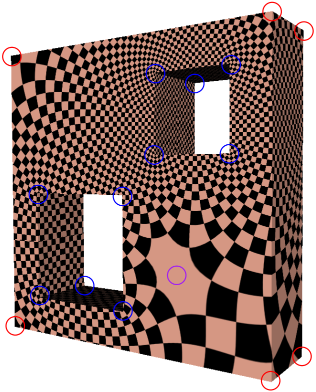

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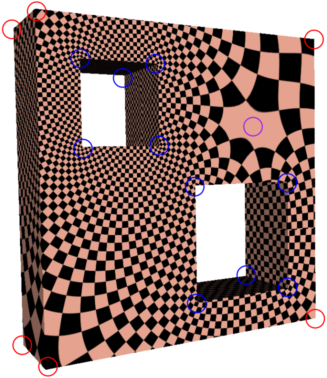

(b). back view

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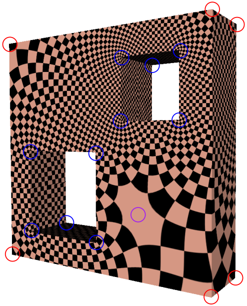

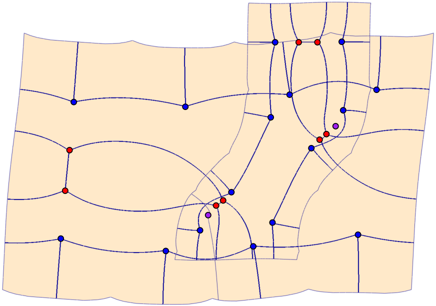

Algorithm Pipeline

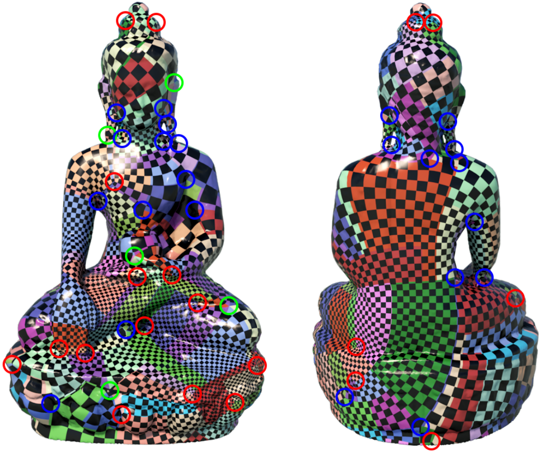

Figure: Step 1. Compute the singularities by optimizing Abel-Jacobi condition.

Algorithm Pipeline

Figure: Step 2. Compute the flat cone metric using surface Ricci flow, and compute the motorcycle graph.

Algorithm Pipeline

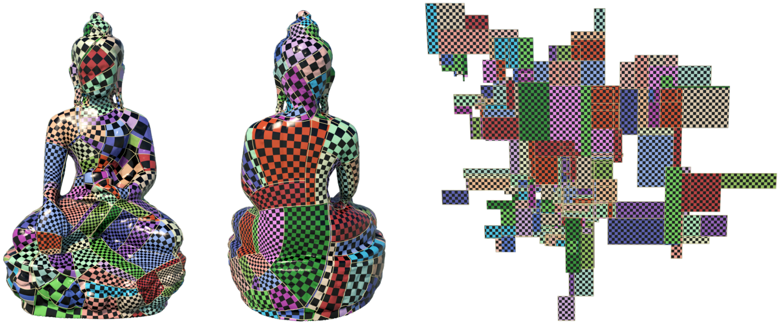

Figure: Step 3. Partition the surface into patches, each patch is conformally flattened onto a quadrilateral.

T-Meshes

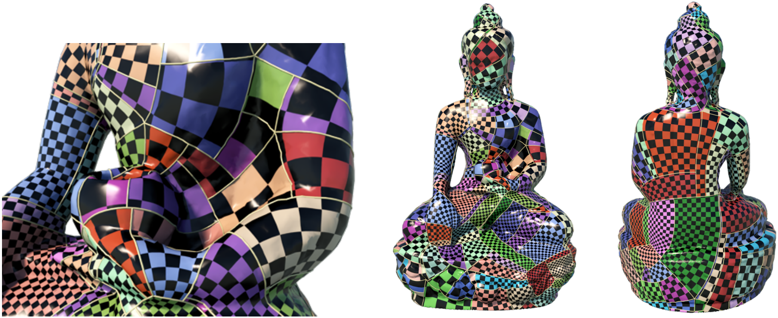

Figure: Step 4. Construct quad-meshes on each patch, with consistent boundary condition and adjust the width and the height of each quadrilateral.

Algorithm Pipeline

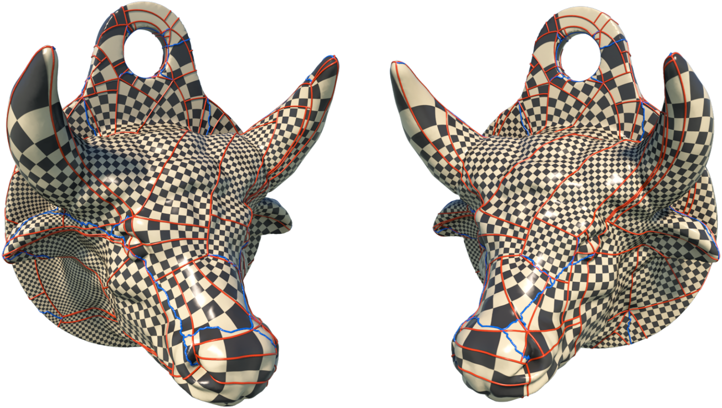

Figure: Step 2. Compute the flat cone metric using surface Ricci flow, and compute the motorcycle graph.

Algorithm Pipeline

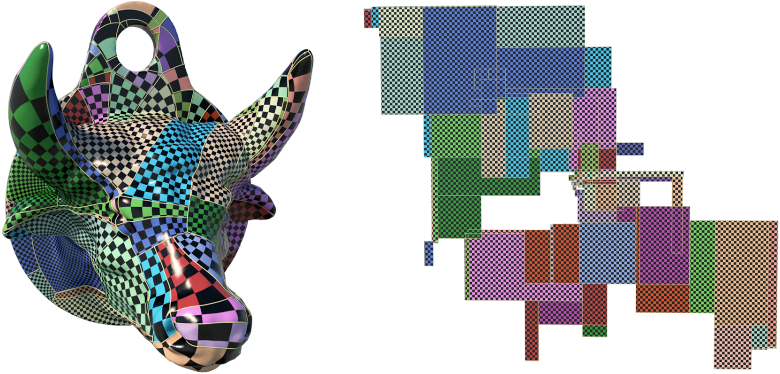

Figure: Step 3. Partition the surface into patches, each patch is conformally flattened onto a quadrilateral.

T-Meshes

Figure: Step 4. Construct quad-meshes on each patch, with consistent boundary condition and adjust the width and the height of each quadrilateral.