Serre Duality

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Generalized Hodge Theorem

Hodge Star Operator

Definition (Hodge Star Operator)

$$\star : A^{p,q} \mapsto A^{1-p,1-q}, \quad \star 1 = \Omega, \star\Omega = 1, \star dz_\alpha = -idz_\alpha, \star d\bar{z}_\alpha = id\bar{z}_\alpha.$$

Definition (Inner Product)

The inner product of $A^p(M)$ can be written as

$$(\eta_1, \eta_2) := \int_M \eta_1 \wedge {}^\star\eta_2.$$

The Hodge star operator has the following properties:

1. $\star^2 = (-1)^{p+q}$
2. $(\star\eta_1, \star\eta_2) = (\eta_1, \eta_2)$

Operator δ and ϑ

Definition (δ and ϑ operators)

$$\delta = -\star d \star \quad \vartheta = -\star \partial \star$$

On a compact Riemann surface δ and ϑ are adjoint operators of d and ∂:

$$(d\eta_1, \eta_2) = (\eta_1, \delta\eta_2) \quad (\partial\eta_1, \eta_2) = (\eta_1, \vartheta\eta_2)$$

Operator Δ and $\Box$

Definition (Δ and $\Box$ operators)

$\Delta, \Box : A^{p,q}(M) \to A^{p,q}(M)$

$$\Delta = d\delta + \delta d \quad \Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}$$

Lemma

On a compact Riemann surface

$$\Delta\omega = 0 \iff d\omega = 0, \delta\omega = 0$$
$$\Box\omega = 0 \iff \bar{\partial}\omega = 0, \vartheta\omega = 0$$

and

$$\star\Delta = \Delta\star \quad \star\Box = \Box\star$$

Proved by direct computation.

Operator $\Box$

Lemma

$$\Box = \frac{1}{2}\Delta$$

Proof.

$d = \partial + \bar{\partial}$ and $\delta = \vartheta + \bar{\vartheta}$,

$$\begin{aligned}\Delta &= d\delta + \delta d = (\partial + \bar{\partial})(\vartheta + \bar{\vartheta}) + (\vartheta + \bar{\vartheta})(\partial + \bar{\partial})\\ &= \Box + \bar{\Box} + (\partial\vartheta + \vartheta\partial) + \overline{\partial\vartheta + \vartheta\partial},\end{aligned}$$

by direct computation,we have

$$\Box = \bar{\Box} \quad \partial\vartheta + \vartheta\partial = 0.$$

hence $\Delta = 2\Box$. $\Box$

Harmonic Form

Definition (Harmonic Form)

Suppose $\Box\omega = 0$, the ω is called a harmonic form.

If f is a smooth function defined on a compact Riemann surface, then $\int_M f\Omega = 0$ if and only if there is a function g, such that $f = \Box g$.

Generalized Hodge $*$ operator

Suppose M is a Riemann surface, L is a holomorphic line bundle on M. Choose Hermite metrics h for T_hM and g for L respectively. Generalize Hodge star operator $* : A^{p,q}(M) \to A^{1-p,1-q}(M)$ to L-valued (p,q) forms.

Definition (Hodge Star Operator)

The Hodge star operator $* : A^{p,q}(L) \to A^{1-p,1-q}(L)$ acting on a L-valued (p,q)-form $\sigma = \omega s$, where ω is a local (p,q)-form, s a local section, let

$$*\sigma = (*\omega)s.$$

It can be easily verified that

$$*^2 = (-1)^{p+q}.$$

Generalized Inner Product

Definition (Inner Product)

Suppose $\sigma_1 = \omega_1 s_1$ and $\sigma_2 = \omega_2 s_2$ are L-valued (p, q) differential forms, let

$$(\sigma_1, \sigma_2) = \int_M \langle s_1, s_2 \rangle \omega_1 \wedge *\bar{\omega}_2,$$

this gives an Hermite inner product on $A^{p,q}(L)$.

L-valued differential forms with different degrees are orthogonal. This defines an Hermite inner product on $A(L)$, Hodge $*$ operator preserves the inner product.

Generalized Connection Operator

The generalized connection operator

$$D : A^{p,q}(L) \to A^{p+1,q}(L) \oplus A^{p,q+1}(L),$$

D can be decomposed into $D = D' + \bar{\partial}$,

$$\begin{aligned} D' &: A^{p,q}(L) \to A^{p+1,q}(L) \\ \bar{\partial} &: A^{p,q}(L) \to A^{p,q+1}(L) \end{aligned}$$

Suppose $\sigma = \omega s_\alpha$, then

$$\begin{aligned} D'\sigma &= (\partial\omega + (-1)^{p+q}\omega \wedge \theta_\alpha)s_\alpha \\ \bar{\partial}\sigma &= (\bar{\partial}\omega)s_\alpha. \end{aligned}$$

Generalized Operator ϑ

Definition (ϑ operator)

The operator

$$\vartheta : A^{p,q}(L) \to A^{p,q-1}(L), \vartheta = - * D' *$$

Lemma

The operators $\bar{\partial}$ and ϑ are adjoint with respect to $(,)$,

$$(\bar{\partial}\sigma_1, \sigma_2) = (\sigma_1, \vartheta\sigma_2), \quad \forall \sigma_1 \in A^{p,q-1}(L), \sigma_2 \in A^{p,q}(L).$$

$\bar{\partial}$-Laplace operator

Definition ($\bar{\partial}$-Laplace operator)

The operator $\square : A^{p,q}(L) \to A^{p,q}(L)$,

$$\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$$

- $\square = \frac{1}{2}\Delta$
- $*\Delta = \Delta *$, $*\square = \square *$
- Self-adjoint: $\forall \sigma_1, \sigma_2 \in A^{p,q}(L)$,

$$(\square\sigma_1, \sigma_2) = (\sigma_1, \square\sigma_2).$$

- $\Delta\omega = 0 \iff d\omega = 0, \delta\omega = 0; \square\omega = 0 \iff \bar{\partial}\omega = 0, \vartheta\omega = 0.$

$\square$ Local representation

Suppose s_α is a local holomorphic section of L, nowhere zero, the Hermite metric of L has local representation $\{g_\alpha\}$, the Hermite metric of $T_h M$ has local representation

$$h = h_\alpha dz_\alpha \otimes d\bar{z}_\alpha.$$

the volume form of h is $\Omega = \frac{\sqrt{-1}}{2} h_\alpha dz_\alpha \wedge d\bar{z}_\alpha$, the curvature form of g is $\Theta = \bar{\partial}\partial \log g_\alpha$, the curvature K is:

$$\Theta = \frac{K}{\sqrt{-1}}\Omega, \quad K = -\frac{2}{h_\alpha}\frac{\partial^2 \log g_\alpha}{\partial z_\alpha \bar{\partial} z_\alpha}.$$

The local representation of the operator $\square_0$ is:

$$\square_0 = \frac{-2}{h_\alpha}\left(\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha} + \frac{\partial \log g_\alpha}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha}\right)$$

$\square$ Local representation

when $\sigma = \sigma_\alpha s_\alpha \in A^{0,0}(L)$,

$$\square\sigma = (\square_0\sigma_\alpha)s_\alpha;$$

when $\sigma = \sigma_\alpha dz_\alpha s_\alpha \in A^{1,0}(L)$,

$$\square\sigma = \left\{\left(\square_0 - 2\frac{\partial h_\alpha^{-1}}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha}\right)\sigma_\alpha\right\} dz_\alpha \otimes s_\alpha;$$

when $\sigma = \sigma_\alpha d\bar{z}_\alpha s_\alpha \in A^{0,1}(L)$,

$$\square\sigma = \left\{\left(\square_0 - 2\frac{\partial h_\alpha^{-1}}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha} + \left[K - 2\frac{\partial h_\alpha^{-1}}{\partial z_\alpha}\frac{\partial \log g_\alpha}{\partial \bar{z}_\alpha}\right]\right)\sigma_\alpha\right\} d\bar{z}_\alpha \otimes s_\alpha;$$

when $\sigma = \sigma_\alpha \Omega s_\alpha \in A^{1,1}(L)$,

$$\sigma = [(\square_0 + k)\sigma_\alpha]\Omega \otimes s_\alpha.$$

Hodge Theorem

Definition (Harmonic L-valued (p, q) form)

Denote $\mathcal{H}^{p,q}(L) = \{\sigma \in A^{p,q}(L) | \Box\sigma = 0\}$, the elements in $\mathcal{H}^{p,q}(L)$ are called the L-valued harmonic (p, q) form. The set of all harmonic forms is denoted as $\mathcal{H}(L) := \oplus_{p,q}\mathcal{H}^{p,q}(L)$.

Suppose $\sigma \in A^{p,q}(L)$ is a L-valued harmonic (p, q)-form, for any $\tau \in A^{p,q-1}(L)$,

$$\begin{aligned}
(\sigma + \bar{\partial}\tau, \sigma + \bar{\partial}\tau) &= (\sigma, \sigma) + (\sigma, \bar{\partial}\tau) + (\bar{\partial}\tau, \sigma) + (\bar{\partial}\tau, \bar{\partial}\tau) \\
&= (\sigma, \sigma) + (\vartheta\sigma, \tau) + (\tau, \vartheta\sigma) + (\bar{\partial}\tau, \bar{\partial}\tau) \\
&= (\sigma, \sigma) + (\bar{\partial}\tau, \bar{\partial}\tau) \geq (\sigma, \sigma)
\end{aligned}$$

Therefore, each Dolbeault cohomological class has at most one harmonic form.

Hodge Theorem

Theorem (Hodge)

Suppose L is a holomorphic line bundle on a compact Riemann surface M, there are Hermite metrics on $T_h M$ and L respectively, then

1. *$\mathcal{H}(L)$ is a finite dimensional vector space;*
2. *there is a compact operator $G : A(L) \to A(L)$, such that*

$$Ker\ G = \mathcal{H}(L), G(A^{p,q}(L)) \subset A^{p,q}(L), \quad G\bar{\partial} = \bar{\partial}G, \vartheta G = G\vartheta.$$

and

$$A(L) = \mathcal{H}(L) \oplus \Box GA(L) = \mathcal{H}(L) \oplus G\Box A(L).$$

Hodge Theorem

Definition (Projection)

The projection map $H : A(L) \to \mathcal{H}(L)$ is defined as

$$H(\sigma) := \sigma - G\Box\sigma \in \mathcal{H}(L).$$

Therefore, we have the unique decomposition

$$\sigma = H(\sigma) + G\Box\sigma, \quad \forall \sigma \in A(L).$$

If $\bar{\partial}\sigma = 0$, then

$$\begin{aligned}\sigma &= H(\sigma) + G\Box\sigma \\ &= H(\sigma) + G(\vartheta\bar{\partial} + \bar{\partial}\vartheta)\sigma \\ &= H(\sigma) + \bar{\partial}(G\vartheta\sigma).\end{aligned}$$

In the Dolbeault cohomological class $[\sigma]$, there is a unique harmonic representative $[H(\sigma)]$.

Hodge Theorem

Corollary

For any $p, q \geq 0$, there are isomorphisms

$$H^q(M; \Omega^p(L)) \cong H_{\bar{\partial}}^{p,q}(M) \cong \mathcal{H}^{p,q}(L).$$

Proof.

The linear map $H : H_{\bar{\partial}}^{p,q}(M) \to \mathcal{H}^{p,q}(L)$, $[\sigma] \mapsto [H(\sigma)]$ is well defined, and is injective and surjective, hence it is an isomorphism. □

Serre Duality

Suppose L is a holomorphic line bundle on a compact Riemann surface M, h is the Hermite metric of T_hM with local representation $\{h_\alpha\}$.

	L	$-L$
Hermite Metric	$\{g_\alpha\}$	$\{g_\alpha^{-1}\}$
Transition function	$\{f_{\beta\alpha}\}$	$\{f_{\beta\alpha}^{-1}\}$
Connection 1-form	θ_α	$\tilde{\theta}_\alpha = -\theta_\alpha$
local holomorphic section nowhere zero	$\{s_\alpha\} = \{\psi_\alpha^{-1}(\cdot, 1)\}$	$\{\tilde{s}_\alpha\} = \{\tilde{\psi}_\alpha^{-1}(\cdot, 1)\}$

Table: Duality

Serre Duality

Definition (Dual Operator $\sim$)

The operator $\sim$: $A^{p,q}(L) \to A^{q,p}(-L)$,

$$\sigma = \omega s_\alpha \mapsto \tilde{\sigma} = \bar{\omega} g_\alpha \tilde{s}_\alpha.$$

Suppose σ has another local representation $\sigma = \omega' s_\beta$, $\omega' = f_{\beta\alpha}\omega$, then

$$\bar{\omega}' g_\beta \tilde{s}_\beta = \bar{\omega} \bar{f}_{\beta\alpha} g_\beta f_{\beta\alpha} \tilde{s}_\alpha = \bar{\omega} g_\alpha \tilde{s}_\alpha,$$

namely $\sim$ is well defined. $\sim^2 = -1$.

Definition (Dual Operator $\tilde{*}$)

The operator $\tilde{*} : A^{p,q}(L) \to A^{1-p,1-q}(-L)$,

$$\tilde{*} = * \circ \sim = \sim \circ * .$$

It is easy to see

$$\tilde{*}(f\sigma) = \bar{f}\tilde{*}(\sigma), \quad \forall f \in A^{0,0}(M), \sigma \in A^{p,q}(L).$$

hence $\tilde{*}$ is conjugate isomorphic.

Serre Duality

Suppose the connection of $-L$ is $\tilde{D}$, let

$$\tilde{\vartheta} : A^{p,q}(-L) \to A^{p,q-1}(-L), \quad \tilde{\vartheta} = - * \tilde{D}' *,$$

and the $\bar{\partial}$-Laplace operator

$$\tilde{\Box} : A^{p,q}(-L) \to A^{p,q}(-L), \quad \tilde{\Box} = \tilde{\vartheta}\tilde{\partial} + \tilde{\partial}\tilde{\vartheta}.$$

Serre Duality

Lemma ($\tilde{*}$-operator)

The $\tilde{*}$-operator has the properties: for any $\sigma \in A^{p,q}(L)$,

1. $\tilde{*}\vartheta\sigma = (-1)^{p+q}\bar{\partial}\tilde{*}\sigma$;
2. $\tilde{*}\bar{\partial}\sigma = (-1)^{p+q+1}\tilde{\vartheta}\tilde{*}\sigma$;
3. $\tilde{\Box} \circ \tilde{*} = \tilde{*} \circ \tilde{\Box}$.

This lemma shows $\tilde{*}$ maps harmonic forms to harmonic forms.

Corollary

$$\tilde{*} : \mathcal{H}^{p,q}(L) \to \mathcal{H}^{1-p,1-q}(-L), \forall p, q \geq 0$$

is a conjugate isomorphism between two complex vector spaces.

Serre Duality

Theorem (Serre Duality)

Suppose L is a holomorphic line bundle on a compact Riemann surface, then

$$H^q(M;\Omega^p(L)) \cong H^{1-q}(M;\Omega^{1-p}(-L)), \quad p,q \geq 0.$$

Particularly, when $L=\lambda(D)$,

$$H^0(M;\Omega^1(\lambda(-D)) \cong H^1(M;\Omega^0(\lambda(D)).$$

Serre Duality

Lemma

For any divisor D, $l(D) \cong \Gamma_h(\lambda(D)) = H^0(M; \Omega^0(\lambda(D))$.

Lemma

For any holomorphic line bundle

$$\{s \in \mathfrak{M}(L) | (s) - D \geq 0\} \cong \Gamma_h(L - \lambda(D))$$

particularly,

$$H^0(M; \Omega^1(\lambda(-D)) \cong i(D) \cong \Gamma_h(T_h^* M - \lambda(D)) = H^0(M; \Omega^0(T_h^* M - \lambda(D))).$$

Serre Duality

Proof.

Let $\omega \in H^0(M; \Omega^1(\lambda(-D)))$ with local representations ω_α and ω_β, which are holomorphic 1-forms. we obtain $\omega_\beta = \omega_\alpha f_{\beta\alpha} = \omega_\alpha \frac{f_\beta}{f_\alpha}$, where $(f_\alpha) = -D \cap U_\alpha$ therefore

$$\omega' := \frac{\omega_\beta}{f_\beta} = \frac{\omega_\alpha}{f_\alpha}$$

is a globally defined meromorphic 1-form, $\omega' \in T_h^* M$,

$$((\omega') - D) \cap U_\alpha = (\omega_\alpha) \cap U_\alpha - (f_\alpha) - D \cap U_\alpha = (\omega_\alpha) + D \cap U_\alpha - D \cap U_\alpha = (\omega_\alpha) \geq 0$$

hence

$$\omega' \in \{s \in \mathfrak{M}(T_h^* M) | (s) - D \geq 0\} = \Gamma_h(T_h^* M - \lambda(D)).$$

$\square$

Riemann-Roch

Euler Characteristic Number

Definition (Euler Characteristic Number)

Suppose L is a holomorphic line bundle on a Riemann surface M, the Euler characteristic number of L is

$$\chi^p(L) := \sum_{i=0}^{\infty}(-1)^i \dim H^i(M, \Omega^p(L))$$

when $p = 0$, we use $\chi(L)$ to replace $\chi^0(L)$.

Suppose $L = \lambda(D)$,

$$\begin{aligned}
\chi(\lambda(D)) &= \dim H^0(M, \Omega(\lambda(D))) - \dim H^1(M, \Omega(\lambda(D))) \\
&= \dim H^0(M, \Omega(\lambda(D))) - \dim H^0(M, \Omega^1(\lambda(-D))) \quad (SerreDual) \\
&= \dim l(D) - \dim i(D).
\end{aligned}$$

$$\chi_0(M) := \chi(\lambda(0)) = \dim H^0(M, \Omega^0) - \dim H^0(M, \Omega^1) = 1 - g.$$

Euler Characteristic Number

Lemma

Suppose L is a holomorphic line bundle on a Riemann surface M, $\forall p \in M$, there exists an exact sequence:

$$0 \to \Omega(L - \lambda(p)) \to \Omega(L) \to \mathcal{S}_p \to 0. \tag{1}$$

where $\mathcal{S}_p$ is a skyscraper sheaf.

Proof.

$$\begin{aligned}
\Gamma(L-\lambda(D)) &\cong \{s \in \mathfrak{M}(L) : (s) - D \geq 0\} \\
\Gamma(L-\lambda(D))(W) &\cong \{s \in \mathfrak{M}(L|W) : (s) - D \cap W \geq 0\}, \quad \forall W \text{ open set} \\
\Omega(L-\lambda(D))(W) &\cong \{s \in \Gamma(L|W) : (s) - D \cap W \geq 0\}, \quad \text{when } D \geq 0 \\
\Omega(L-\lambda(p))(W) &\cong \{s \in \Omega(L)(W) : (s) - p \cap W \geq 0\}
\end{aligned}$$

□

Euler Characteristic Number

Lemma

Suppose

$$0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0 \tag{2}$$

is a sheaf exact sequence, if

$$dim H^j(M, \mathcal{F}_i) < \infty, \quad \forall i, j$$

and $H^j(M, \mathcal{F}_i) = 0$ *when* j *is sufficiently large, then*

$$\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3).$$

Proof.

The exact sheaf sequence Eqn. (2) induces the exact sequence of cohomology groups

$$H^{i-1}(F_3) \xrightarrow{\delta^*} H^i(\mathcal{F}_1) \xrightarrow{\alpha^*} H^i(\mathcal{F}_2) \xrightarrow{\beta^*} H^i(\mathcal{F}_3) \xrightarrow{\delta^*} H^{i+1}(\mathcal{F}_1)$$

we have the isomorphism

$$\beta^* H^i(\mathcal{F}_2) \cong H^i(\mathcal{F}_2)/\alpha^* H^i(\mathcal{F}_1).$$

hence

$$\dim H^i(\mathcal{F}_2) = \dim \alpha^* H^i(\mathcal{F}_1) + \dim \beta^* H^i(\mathcal{F}_2)$$

□

Euler Characteristic Number

continued.

Similarly

$$
\begin{aligned}
\dim H^i(\mathcal{F}_1) &= \dim \delta^* H^{i-1}(\mathcal{F}_3) + \dim \alpha^* H^i(\mathcal{F}_1)\\
\dim H^i(\mathcal{F}_3) &= \dim \delta^* H^i(\mathcal{F}_3) + \dim \beta^* H^i(\mathcal{F}_2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\chi(\mathcal{F}_2) - \chi(\mathcal{F}_1) - \chi(\mathcal{F}_3)\\
=&\sum_{i=0}^{\infty} (-1)^i \dim \alpha^* H^i(\mathcal{F}_1) + (-1)^i \dim \beta^* H^i(\mathcal{F}_2)\\
+&\sum_{i=0}^{\infty} (-1)^{i+1} \dim \delta^* H^{i-1}(\mathcal{F}_3) + (-1)^{i+1} \dim \alpha^* H^i(\mathcal{F}_1)\\
+&\sum_{i=0}^{\infty} (-1)^{i+1} \dim \delta^* H^i(\mathcal{F}_3) + (-1)^{i+1} \dim \beta^* H^i(\mathcal{F}_2)\\
=&0.
\end{aligned}
$$

□

Riemann-Roch

Theorem

Given a divisor D, the Euler characteristic number of $\lambda(D)$ is

$$\chi(\lambda(D)) = deg(D) + (1-g) = diml(D) - dimi(D).$$

Proof.

Suppose $D = D_1 - D_2$, $D_1 \geq 0$ and $D_2 \geq 0$, similar as the proof of Eqn. (1), we have the following exact sequence of sheaves

$$0 \to \Omega(L - \lambda(D_2)) \to \Omega(L) \to \mathcal{S}_{D_2} \to 0.$$

first let $L = \lambda(D_1)$, $\square$

Riemann-Roch

continued.

hence

$$\chi(\lambda(D_1)) = \chi(\lambda(D)) + \chi(S_{D_2}) = \chi(\lambda(D)) + \deg(D_2),$$

the skyscraper sheaf S_{D_2} is a fine sheaf, hence $\dim H^0(S_{D_2}) = \deg(D_2)$ and $H^q(S_{D_2}) = 0$, $\chi(S_{D_2}) = \deg(D_2)$.
second let $L = \lambda(D_1)$ and replace D_2 by D_1,

$$\chi(\lambda(D_1)) = \chi(\lambda(D_1) - \lambda(D_1)) + \chi(\mathcal{S}_{D_1}) = \chi_0(M) + \deg(D_1)$$

we obtain

$$\begin{aligned}\chi(\lambda(D)) &= \chi(\lambda(D_1)) - \deg(D_2)\\ &= \chi_0(M) + \deg(D_1) - \deg(D_2)\\ &= \chi_0(M) + \deg(D).\end{aligned}$$

Serre Duality

$$
\begin{aligned}
& dim\ l(D) - dim\ i(D) \\
=& dim\ \Gamma_h(\lambda(D)) - dim\ \Gamma_h(T_h^* M - \lambda(D)) \\
=& dim\ H^0(M; \Omega^0(\lambda(D))) - dim\ H^0(M; \Omega^0(T_h^* M - \lambda(D))) \\
=& dim\ H^0(M; \Omega^0(\lambda(D))) - dim\ H^0(M; \Omega^1(\lambda(-D))) \quad (Serre\ Dual) \\
=& dim\ H^0(M; \Omega^0(\lambda(D))) - dim\ H^1(M; \Omega^0(\lambda(D))) \\
=& \chi(\lambda(D)).
\end{aligned}
$$

We obtain Riemann-Roch,

$$
\boxed{\chi(\lambda(D)) = (1-g) + deg(D)}
$$