

Surface Uniformization

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

August 29, 2023

Surface Uniformization

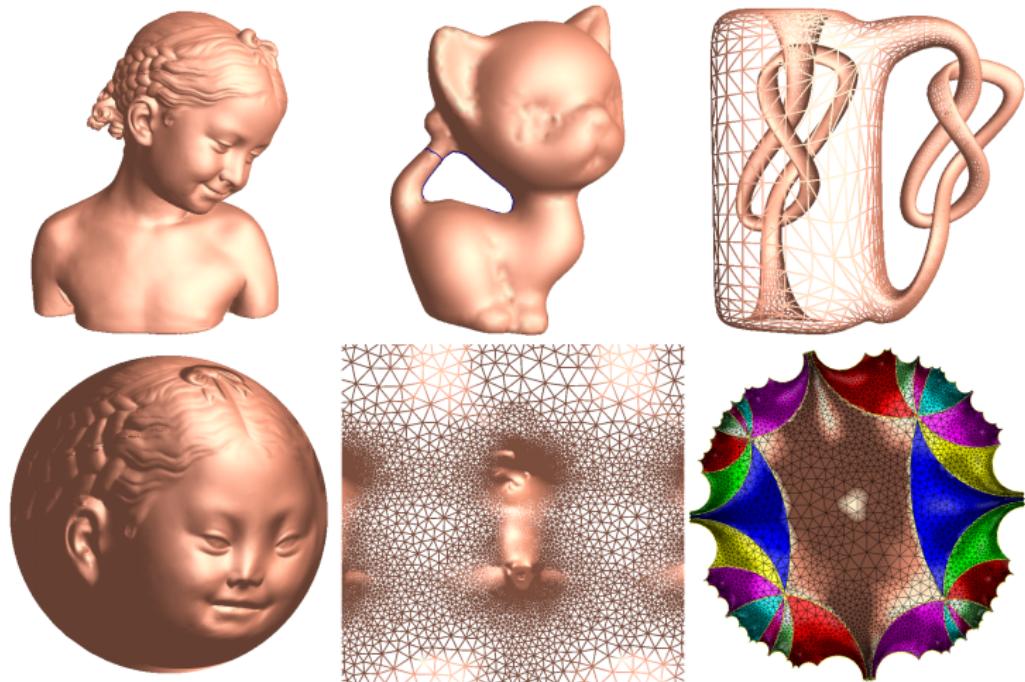


Figure: Closed surface uniformization.

Surface Uniformization

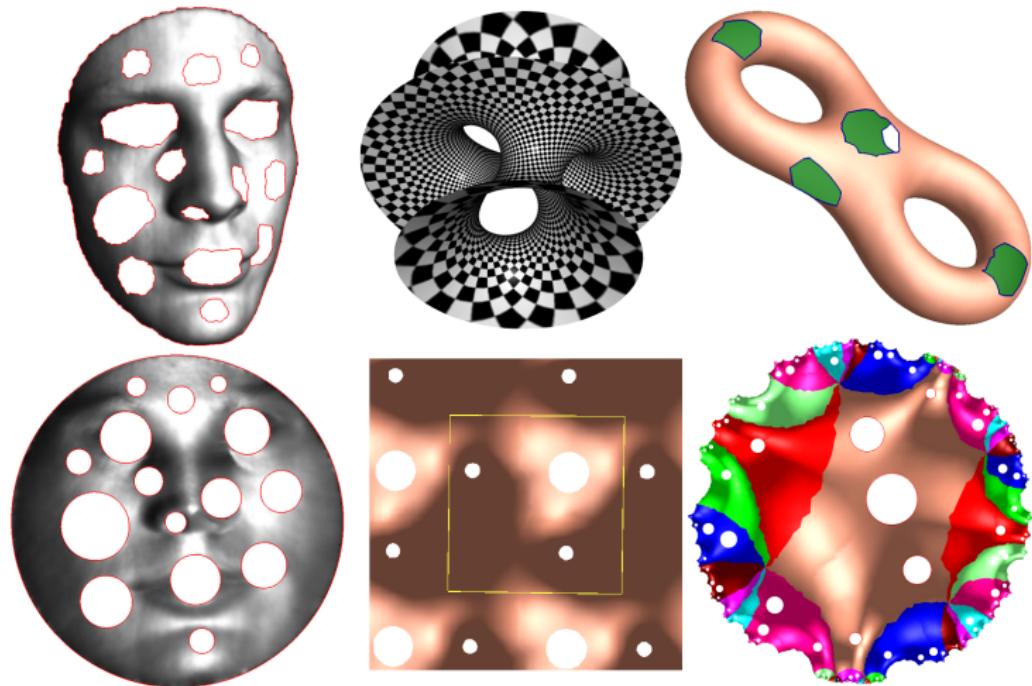


Figure: Open surface uniformization.

Conformal Mapping of Infinite Triangle Mesh

Problem

Suppose we have an infinite triangle mesh, \tilde{M} , such as the universal covering space of a closed mesh, fix a point $v_0 \in \tilde{M}$, choose a sequence of neighborhood $E_n \subset \tilde{M}$,

$$v_0 \in E_0 \subset E_1 \subset E_2 \cdots E_n \cdots$$

where each E_k is a topological disk, construct discrete conformal mapping $\varphi_n : E_n \rightarrow \mathbb{D}^n$, such that

$$\varphi_n(v_0) = 0, \quad \varphi'_n(v_0) > 0,$$

then what is the limit of the sequence $\{\varphi_n(v_0)'\}$?

Conformal Mapping of Infinite Triangle Mesh

Answer

- ① If \tilde{M} is the universal covering of a torus, then the limit is 0;
- ② If \tilde{M} is the universal covering space of a high genus mesh, then the limit is a positive number $\delta > 0$.

Conformal Mapping of Infinite Triangle Mesh

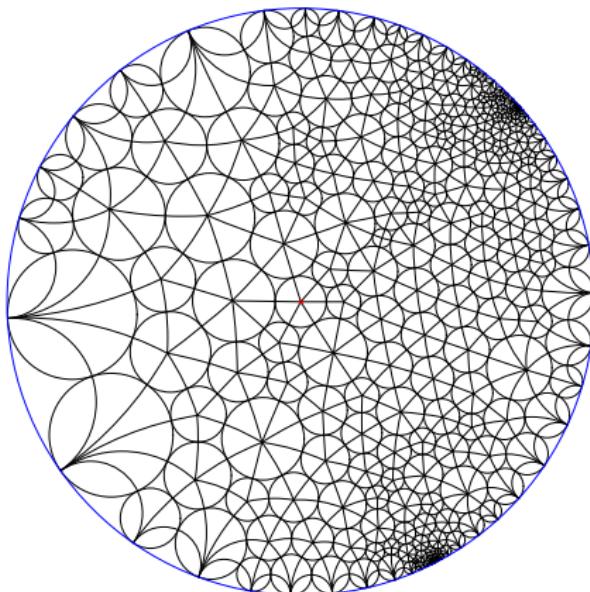


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

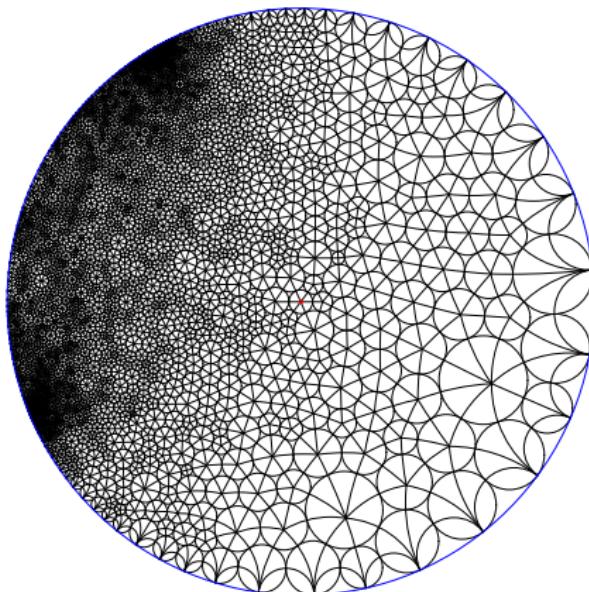


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

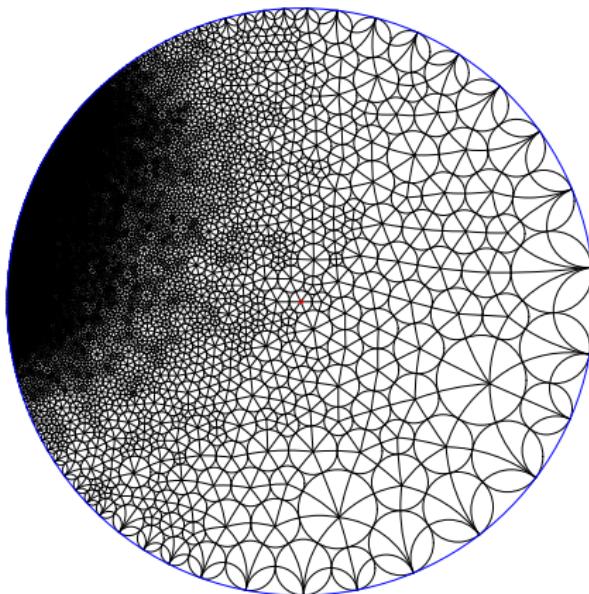


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

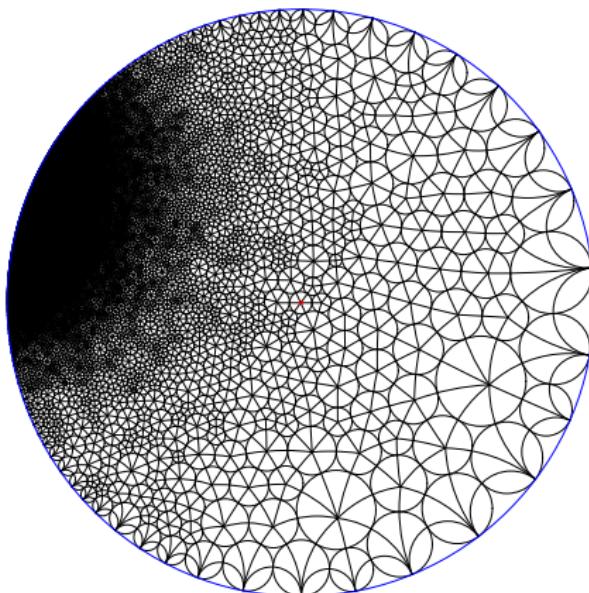


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

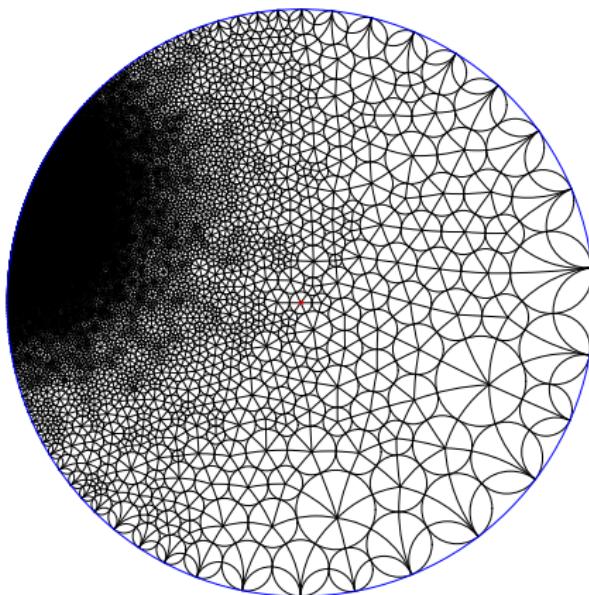


Figure: Discrete Riemann mapping of triangle mesh.

Conformal Mapping of Infinite Triangle Mesh

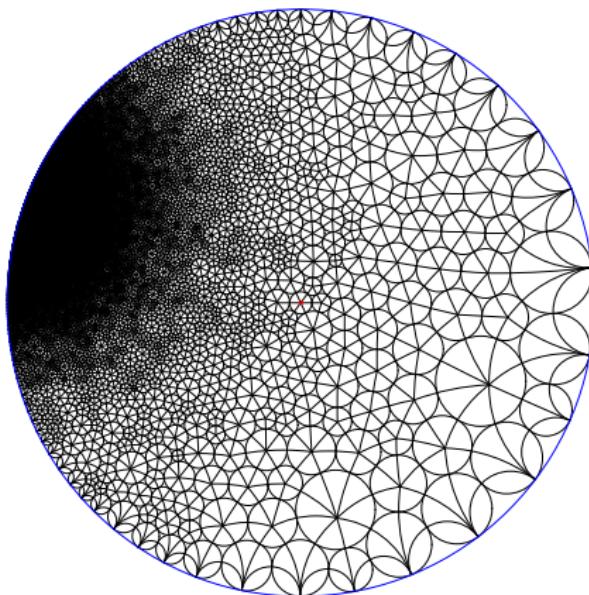


Figure: Discrete Riemann mapping of triangle mesh.

Liuville Theorem

Theorem (Liuville)

Suppose a holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ is bounded, $|f(z)| < C$, for all $z \in \mathbb{C}$, then $f(z) = \text{const.}$

Proof.

According to Cauchy's formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-a)^{n+1}} dz,$$

here Γ is a circle centered at a with radius r ,

$$|f'(a)| = \left| \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-a)^2} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{C}{r} d\theta = \frac{C}{r},$$

let $r \rightarrow \infty$, the derivative goes to 0. Hence the holomorphic function $f(z)$ is constant.



Liuville Theorem

The unit sphere \mathbb{S}^2 is conformal equivalent to the augmented complex plane $\hat{\mathbb{C}}$. Complex plane \mathbb{C} and the unit open disk \mathbb{D} are open sets, therefore they are not homeomorphic to the compact set \mathbb{S}^2 . Liuville theorem shows \mathbb{C} and \mathbb{D} are not conformally equivalent to each other.

Corollary

The complex plane \mathbb{C} and the unit disk \mathbb{D} are not conformally equivalent.

Proof.

Suppose they are equivalent, there is a biholomorphic function $f : \mathbb{C} \rightarrow \mathbb{D}$, according to Liuville, $f(z)$ is constant. Contradiction to biholomorphic function. □

Crescent and Full-Moon Theorem

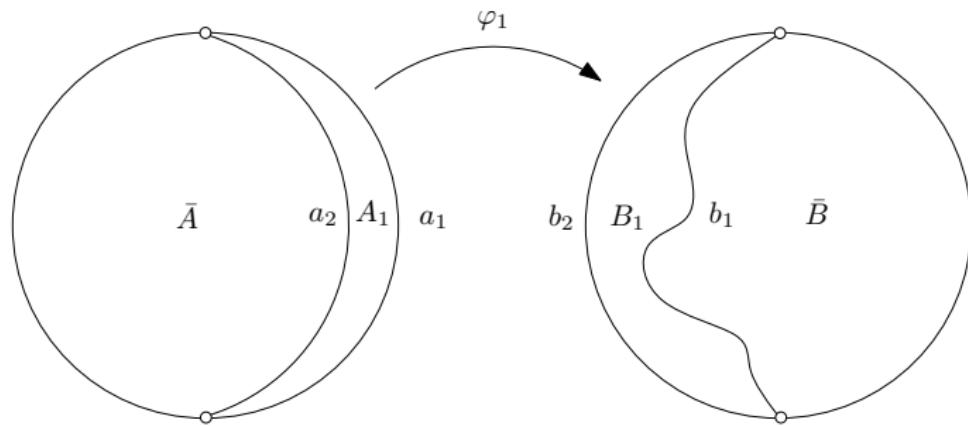


Figure: Initial Map.

Crescent and Full-Moon Theorem

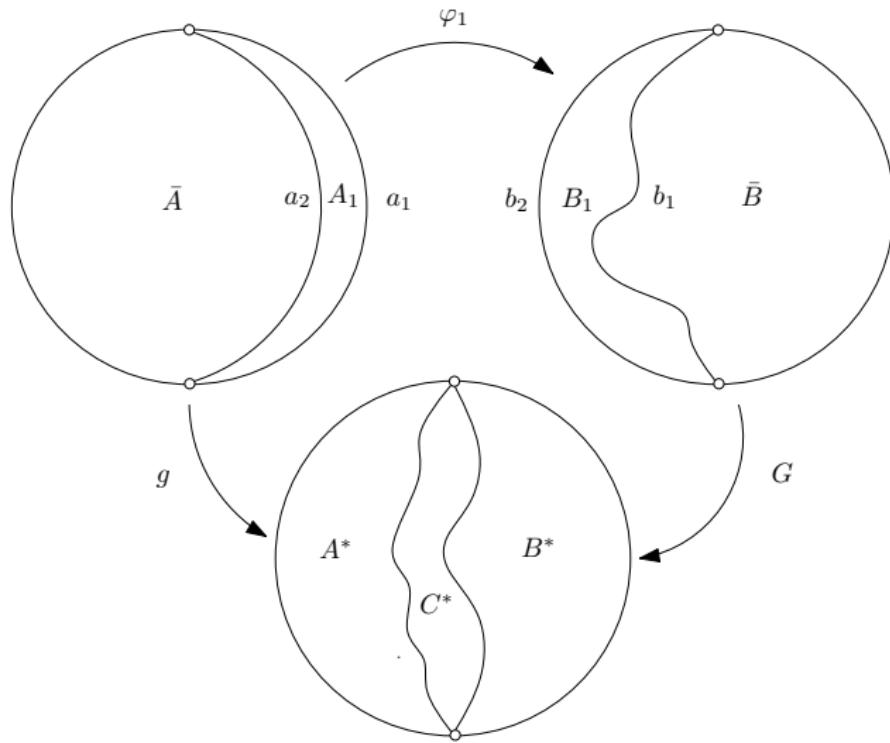


Figure: Analytic extension result.

Crescent and Full Moon

Lemma (Crescent and Full Moon)

As shown in Fig. 9, the boundaries of the crescent domain A_1 are circular arcs a_1 and a_2 , they have intersection angle $\pi/2^m$, $m \in \mathbb{Z}^+$. A conformal map $\varphi_1 : A_1 \rightarrow B_1$ is defined on the crescent A_1 , $\varphi_1(a_k) = b_k$, $k = 1, 2$, b_2 is a circular arc. Then there exist analytic functions, $g, G : \mathbb{D} \rightarrow \mathbb{D}$, as shown Fig. 10, satisfying

- ① $A^* = g(\bar{A})$, $C^* = g(A_1)$;
- ② $B^* = G(\bar{B})$, $C^* = G(B_1)$;
- ③ $g|_{A_1} = G \circ \varphi_1|_{A_1}$;

and the restriction on a_k 's and b_k 's, the mappings g and G are homeomorphisms.

Crescent and Full-Moon Theorem

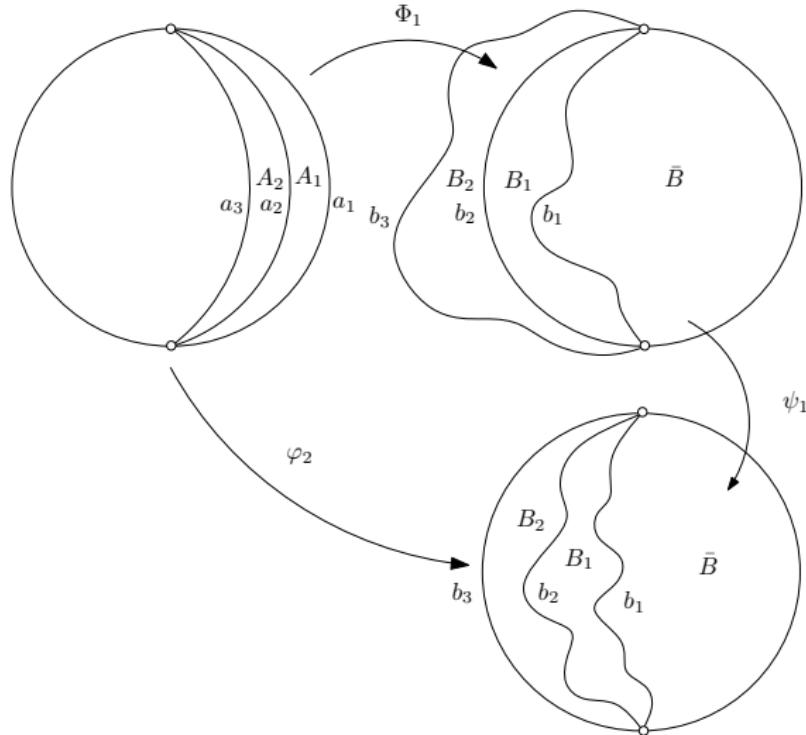


Figure: Analytic extension, step one.

Crescent and Full Moon

Proof.

As shown in Fig. (11), crescents A_1 and A_2 are symmetric about a_2 , by the Schwartz reflection principle, analytic function $\varphi_1 : A_1 \rightarrow B_1$ can be extended about the circular arc a_2 to

$$\Phi_1 : A_1 + A_2 \rightarrow B_1 + B_2,$$

using Riemann mapping

$$\psi_1 : B_1 + B_2 + \bar{B} \rightarrow \mathbb{D},$$

which maps the target to the unit disk. For convenience, we relabel $\psi_1(B_1)$, $\psi_1(B_2)$ as B_1 and B_2 , then the composition map is:

$$\varphi_2 = \psi_1 \circ \Phi_1 : A_1 + A_2 \rightarrow B_1 + B_2.$$

Crescent and Full-Moon Theorem

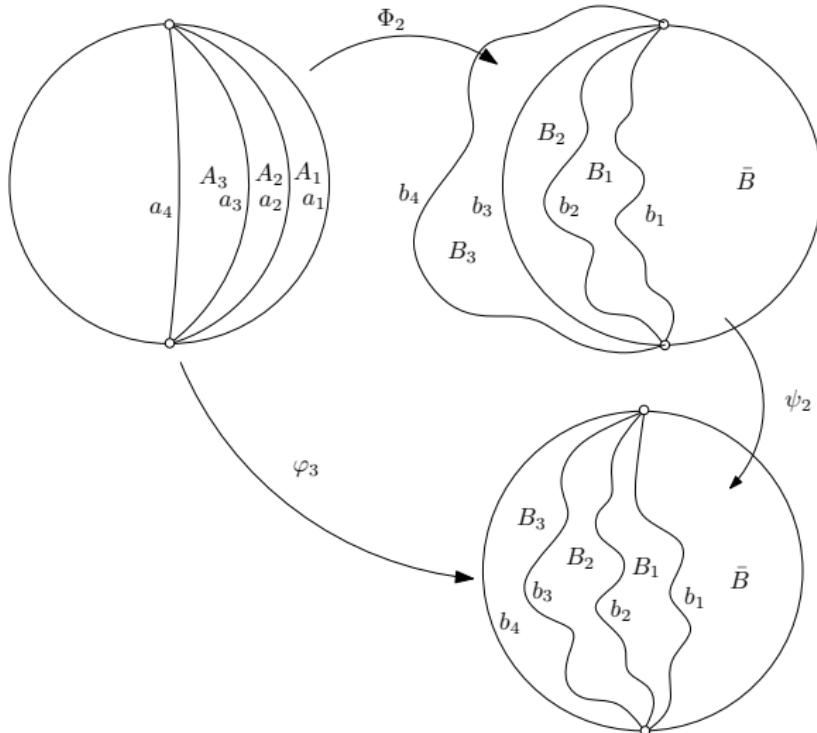


Figure: Analytic extension, step two.

Crescent and Full Moon

continued.

As shown in Fig. (12), we extend $\varphi_2 : A_1 + A_2 \rightarrow B_1 + B_2$ again, $A_1 + A_2$ is reflected about a_3 to a crescent A_3 , by Schwartz reflection principle,

$$\Phi_2 : (A_1 + A_2) + A_3 \rightarrow (B_1 + B_2) + B_3,$$

then composed with the Riemann mapping $\psi_2 : B_1 + B_2 + B_3 + \bar{B} \rightarrow \mathbb{D}$, we get the result for the second step extension,

$$\varphi_3 = \psi_2 \circ \Phi_2 : A_1 + A_2 + A_3 \rightarrow B_1 + B_2 + B_3.$$

Repeat this procedure, by analytic extension we get conformal mappings:

$$\varphi_k : \sum_{i=1}^k A_i \rightarrow \sum_{j=1}^k B_j,$$

Crescent and Full Moon

continued.

Consider the inner angle of the crescents, the angle of A_k is θ_k , we have recursive relations,

$$\begin{cases} \theta_1 = \pi/2^m \\ \theta_2 = \pi/2^m \\ \theta_k = \sum_{j=1}^{k-1} \theta_j, \quad k > 2 \end{cases}$$

therefore at the $m + 1$ step, all the crescents cover the whole disk. Hence, we obtain analytic function

$$G = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_2 \circ \psi_1,$$

and

$$g = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_2 \circ \phi_1.$$

Uniformization

We use a combinatorial representation to define a Riemann surface. Given a Riemann surface M , and a triangulation \mathcal{T} . If \mathcal{T} has finite number of faces, then M is a compact surface; if the surface has countable infinite number of faces, then M is an open surface. Van der Waerden proves the existence of a special type of triangulation.

Lemma (Van der Waerden)

Assume \tilde{M} is an open surface, then its triangulation can be sorted,

$$\mathcal{T} = \{\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_n, \dots\}$$

such that for any $n = 1, 2, \dots$,

$$\mathcal{T}_n := \bigcup_{k=1}^n \Delta_k$$

and Δ_{n+1} has only one intersection edge (and the third non-intersecting vertex), or two edges, namely \mathcal{T}_n is a topological disk.

Uniformization

Let \tilde{M} be the universal covering space of a Riemann surface, then \tilde{M} is a simply connected Riemann surface, its triangulation \mathcal{T} is sorted in Van der Waerden pattern. All the edges of \mathcal{T} are analytic arcs, and every face Δ_k is covered by a conformal local chart.

Lemma

For any $n > 0$, the interior of

$$E_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

is conformally mapped onto the open unit disk, $\varphi_n : E_n \rightarrow R_n$, R_n is an open unit disk, and the restriction on the boundary,

$$\varphi_n|_{\partial E_n} : \partial E_n \rightarrow \partial R_n$$

is topological homeomorphic.

Uniformization

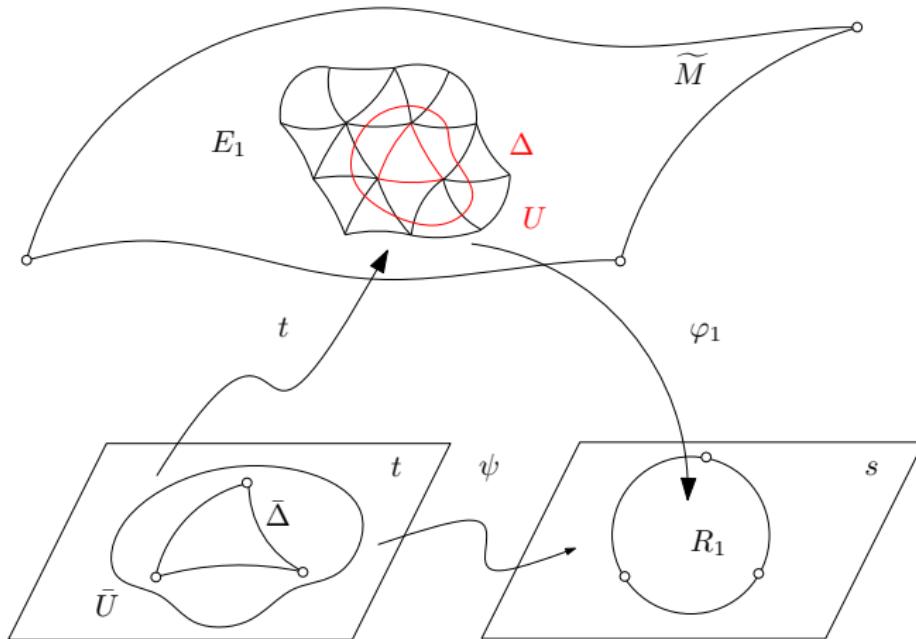


Figure: Initial induction step.

Uniformization

Proof.

Step one: when $n = 1$, as shown in Fig. (13), E_1 only includes one triangle Δ_1 , denote $\Delta = \Delta_1$. Δ is covered by a conformal coordinate system (U, t) , $\Delta \subset U$. Let $\bar{\Delta}$, \bar{U} are the pre-images of Δ , U on the t -plane,

$$t(\bar{\Delta}) = \Delta, \quad t(\bar{U}) = U.$$

$\bar{\Delta}$ is a simply connected domain, its boundary is piecewise analytic curves. According to Riemann mapping theorem, there is a holomorphic map $\psi : \bar{\Delta} \rightarrow R_1$, from $\bar{\Delta}$ to the unit disk R_1 on s -plane, and the restriction on the boundary is topological homeomorphic,

$$\psi|_{\partial\bar{\Delta}} : \partial\bar{\Delta} \rightarrow \partial R_1,$$

then construct a holomorphic map $\varphi_1 = \psi \circ t^{-1} : E_1 \rightarrow R_1$, its restriction on the boundary is a homeomorphism. □

Uniformization

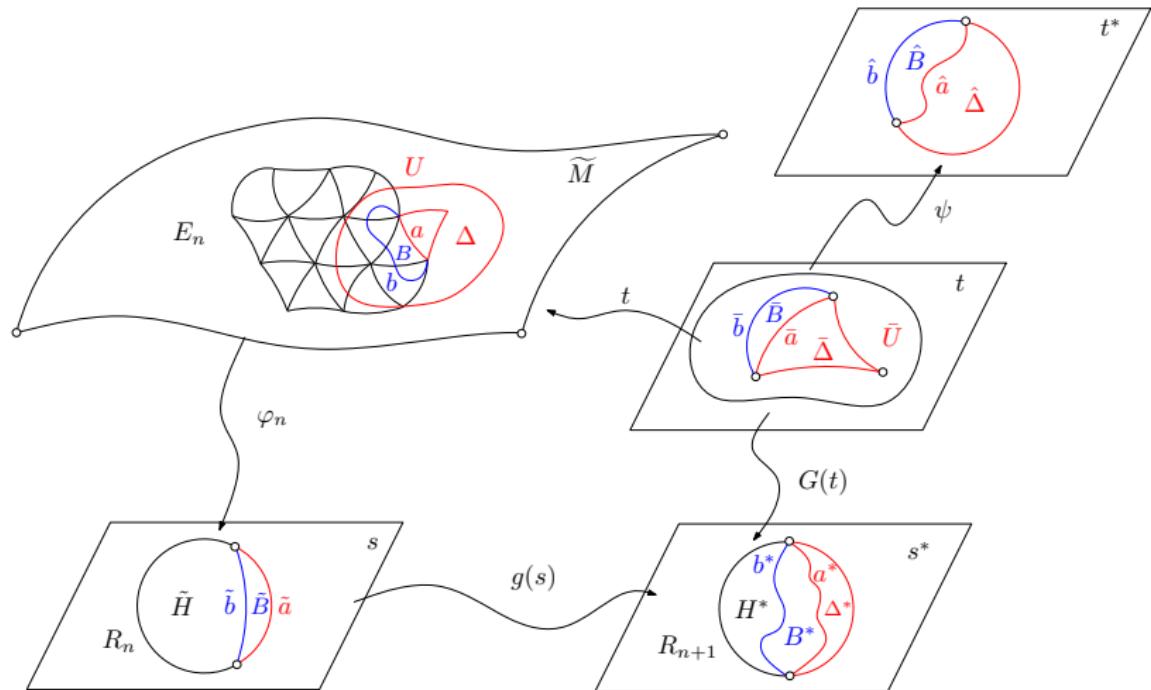


Figure: Induction step.

Uniformization

continued.

Step two: when $n > 1$, assume at the n -th step, E_n is conformally mapped onto the unit disk R_n on s -plane, $\varphi_n : E_n \rightarrow R_n$, the restriction on the boundary $\varphi_n|_{\partial E_n} : \partial E_n \rightarrow \partial R_n$ is homeomorphic.

As shown in Fig. (14), we consider $E_{n+1} = E_n + \Delta_{n+1}$. Let $\Delta = \Delta_{n+1}$, covered by a local conformal coordinates (U, t) , the preimages of U and Δ are \bar{U} and $\bar{\Delta}$ respectively in the local coordinate system,

$$t(\bar{U}) = U, \quad t(\bar{\Delta}) = \Delta.$$

E_n and Δ intersect at an analytic arc a , $\Delta \cap E_n = a$. The image of a under φ_n is \tilde{a} , $\varphi_n(a) = \tilde{a}$. The conformal local parametric representation of a is \bar{a} , $t(\bar{a}) = a$.

Uniformization

continued.

In the unit disk R_n on the s -plane, draw a circular arc \tilde{b} , two circular arcs \tilde{a} and \tilde{b} have the same ending points, and the intersection angles at the ending points equal to $\pi/2^k$, where k is a big positive integer. The circular arcs bound a crescent \tilde{B} , the pre-image of \tilde{B} on \tilde{M} is B ; the image of B on the t -image is \bar{B} , $\varphi_n(B) = \tilde{B}$, $t(\bar{B}) = B$. We want to show the existence of holomorphic maps $s^* = g(s)$ and $s^* = G(t)$, satisfying:

- ① $g(\tilde{B}) = B^*$, $g(\tilde{H}) = H^*$, where $\tilde{H} = R_n - \tilde{B}$;
- ② $G(\bar{B}) = B^*$, $G(\bar{\Delta}) = \Delta^*$;
- ③ on domain \bar{B} , $G(t) = g \circ \varphi_n \circ t$;
- ④ $R_{n+1} = B^* + H^* + \Delta^*$

The combination of $g(s)$ and $G(t)$ gives the conformal mapping from E_{n+1} to R_{n+1} .

Uniformization

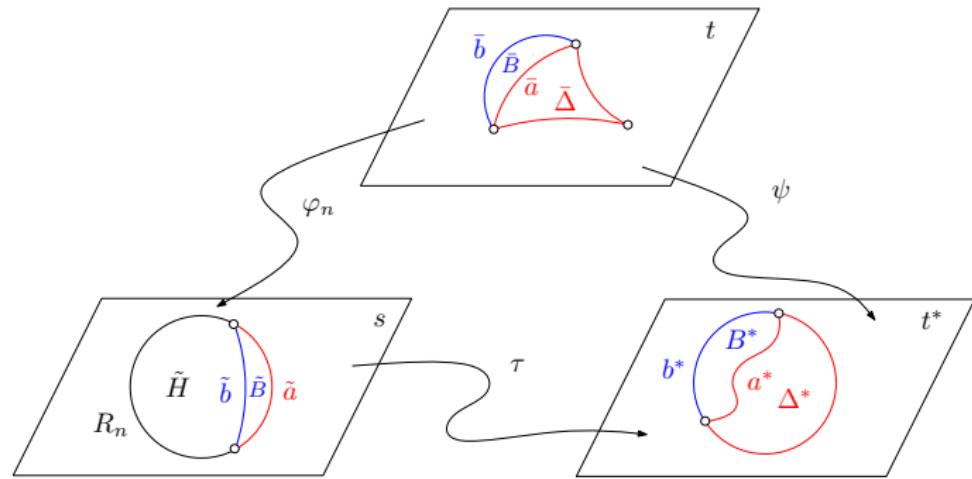


Figure: Combination of conformal mappings.

Uniformization

continued.

As shown in Fig. (15), by Riemann mapping, there is a mapping $t^* = \psi(t)$, mapping $\tilde{\Delta} + \tilde{B}$ to $\Delta^* + B^*$, the center of the disk is inside Δ^* . Then the composition

$$\tau = \psi \circ \varphi_n^{-1}, \quad t^* = \tau(s)$$

maps the crescent \tilde{B} to B^* . Note that $\tau : \tilde{B} \rightarrow B^*$ is defined on crescent \tilde{B} , not defined on \tilde{H} . By crescent-full moon lemma, there exist holomorphic functions g and G , this proves the existence of $\varphi_{n+1} : E_{n+1} \rightarrow R_{n+1}$. By induction, the lemma holds.

Uniformization

Theorem (Open Riemann Surface Uniformization)

Simply connected open Riemann surface is conformal equivalent to the whole complex plane \mathbb{C} or the unit open disk \mathbb{D} .

Proof.

Construct a sequence of holomorphic functions

$$\varphi_{1,n}(s) = \varphi_n \circ \varphi_1^{-1},$$

univalent on R_1 , and normalized at $s = 0$, $\varphi_{1,n}(0) = 0$, $\varphi'_{1,n}(0) = 1$. Then $\{\varphi_{1,n}\}$ is a normal family. We choose subsequence $\Gamma_1 \subset \{\varphi_{1,n}\}$, which converges to univalent function in the interior of R_1 , denoted as

$$\Gamma_1 : \varphi_1^1(p), \varphi_2^1(p), \varphi_3^1(p), \dots$$

converges to a univalent function $\varphi_0(p)$ in E_1 .



Construction of Normal Family

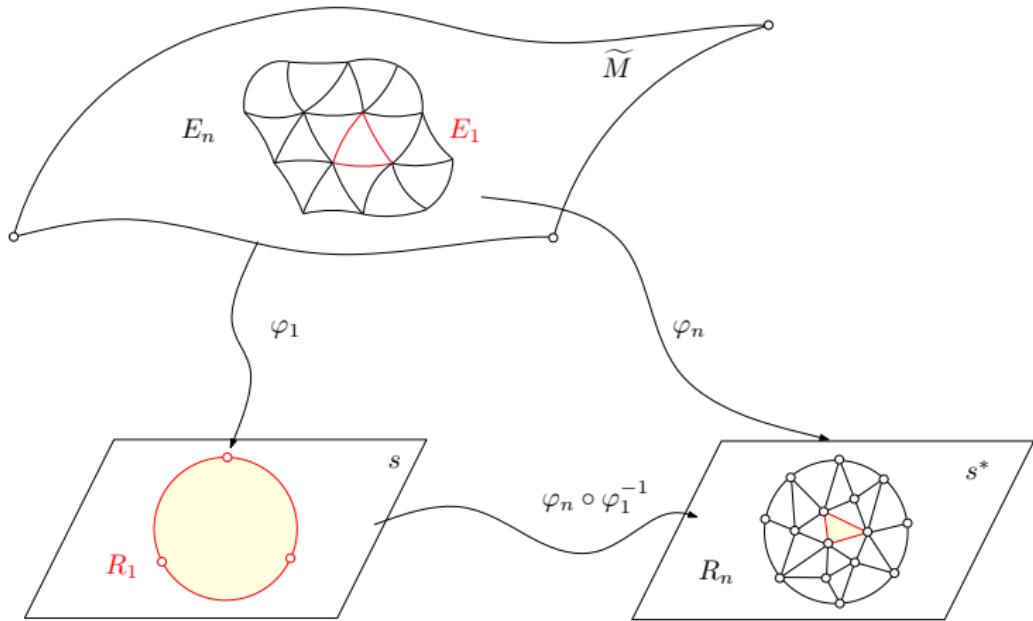


Figure: Construction of normal family $\{\varphi_n \circ \varphi_1^{-1}\}$.

Uniformization

continued.

Construct a sequence of holomorphic functions

$$\varphi_{2,n}(s) = \varphi_n^1 \circ \varphi_2^{-1}, \quad \varphi_n^1 \in \Gamma_1,$$

from $\{\varphi_{2,n}\}$ choose subsequence

$$\Gamma_2 : \varphi_1^2(p), \varphi_2^2(p), \dots$$

converges to a univalent holomorphic function on E_2 , and the restriction on E_1 equals to $\varphi_0(p)$, we still denote it as $\varphi_0(p)$.

Uniformization

continued.

Furthermore, construct a sequence of functions

$$\varphi_{3,n}(s) = \varphi_n^2 \circ \varphi_3^{-1}, \quad \varphi_n^1 \in \Gamma_2,$$

from $\{\varphi_{3,n}\}$ choose subsequence

$$\Gamma_3 : \varphi_1^3(p), \varphi_2^3(p), \dots$$

converges to a univalent holomorphic function on E_3 , and the restriction on E_2 equals to $\varphi_0(p)$, we still denote it as $\varphi_0(p)$. Repeat this step, apply diagonal principle, we obtain a function sequence

$$\varphi_1^1(p), \varphi_2^2(p), \varphi_3^3(p), \dots$$

where $\varphi_k^k(p)$ are well defined on E_n ($k \geq n$), and converge to $\varphi_0(p)$ on E_n .

continued.

Since E_n exhausts the whole open Riemann surface \tilde{M} , $\varphi_0(p)$ is univalent, and maps \tilde{M} to a simply connected domain R on s -plane.

Since \tilde{M} is open, R can't be the augmented complex plane. Hence, R is either the whole complex plane \mathbb{C} , or a domain on the complex plane. In the second situation, by Riemann mapping theorem, R can be conformally mapped to the unit disk \mathbb{D} .

Compact Surface Uniformization

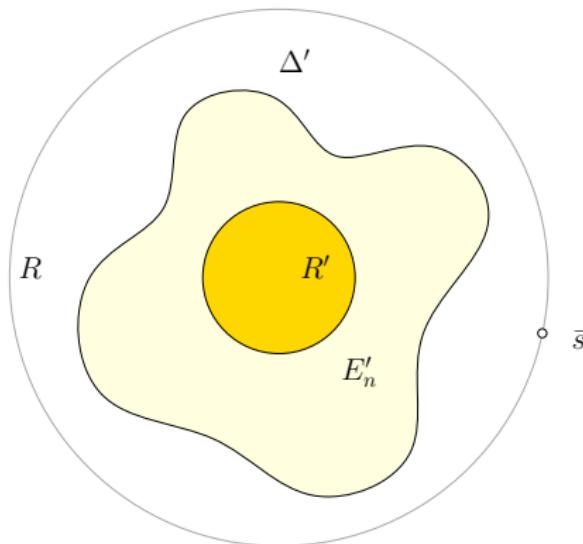


Figure: Compact surface case.

Uniformization

Theorem (Compact Riemann Surface Uniformization)

Compact simply connected Riemann surface is conformal equivalent to the unit sphere.

Proof.

Suppose \tilde{M} has a triangulation \mathcal{T} , which includes a finite number of faces,

$$\mathcal{T}_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

the last triangle Δ_n has three common edges with \mathcal{T}_{n-1} . Choose an interior point $q \in \Delta_n$, remove this point, we obtain an open Riemann surface,

$$\tilde{M}_0 = \tilde{M} \setminus \{q\},$$

according to open Riemann surface uniformization theorem, there is a conformal mapping, $\varphi : \tilde{M}_0 \rightarrow \mathbb{C}$, $s = \varphi(p)$, which maps the open Riemann surface either to a unit disk or the whole complex plane.



Uniformization

continued.

on s -complex plane, let $\varphi(\Delta_n \setminus \{q\}) = \Delta'$, $\varphi(E_{n-1}) = E'_n$, point $o \in E_{n-1}$, $\varphi(o) = 0$. Let $R' \subset E'_n$ be a disk centered at the origin, then Δ' is outside R' .

Function $w = 1/z$ maps Δ' to a bounded domain on w -plane. Consider a function $w = 1/\varphi(p)$ defined on $\tilde{M} \setminus \{q\}$, w is bounded in a neighborhood of q , hence q is a removable singularity of function w . Let the image of q in w -plane is $w(q)$.

Assume $R = \varphi(\tilde{M} \setminus \{q\})$ is not the whole complex plane, but the unit disk. Choose a point sequence s_1, s_2, \dots , its accumulation point is on the unit circle. The corresponding point sequence on the surface is p_1, p_2, \dots . Since \tilde{M} is compact, the accumulation point of the point sequence is on the surface. But the images of all points on $\tilde{M} \setminus \{q\}$ on s -plane are not on the unit circle, hence

$$q = \lim_{n \rightarrow \infty} p_n.$$

Uniformization

continued.

For any point on the unit circle, $\bar{s} \in \partial R$, there is a point sequence converging to \bar{s} , hence

$$1/\bar{s} = w(q),$$

but \bar{s} has infinite many value, hence $w(q)$ has infinite, contradiction. Hence the assumption is incorrect, $R = \varphi(\tilde{M} \setminus \{q\})$ is the whole complex plane, \tilde{M} is conformal equivalent to the augmented complex plane. \square