# Sheaf Cech Cohomology

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# Basic definitions of sheaf

# Sheaf Definition

## Definition (Structure Sheaf)

Suppose $M$ is a Riemann surface, $W$ is an open set on $M$. Let $\mathcal{O}(W)$ be the holomorphic function ring on $W$, define

$$\mathcal{O}(M) := \{\mathcal{O}(W) : \text{open set } W \subset M\}$$

$\mathcal{O}$ satisfies the following properties:

1. if $V \subset W$, let $\rho_{W,V} : \mathcal{O}(W) \to \mathcal{O}(V)$ be the restriction of homomorphism, namely $\rho_{W,V}(f) = f|_V$, then

$$U \subset V \subset W \implies \rho_{W,U} = \rho_{VU} \circ \rho_{W,V}$$

2. if $W = \bigcup_i W_i$, $W_i \subset M$ open, $\exists S_i \in \mathcal{O}(W_i)$, such that on $W_i \cap W_j$,

$$\rho_{W_i,W_i \cap W_j}(S_i) = \rho_{W_j,W_j \cap W_i}(S_j)$$

   then $\exists S \in \mathcal{O}(W)$m such that $\rho_{W,W_i}(S) = S_i$, $\forall i$.

3. if $W = \bigcup_i W_i$, $W_i \subset M$ open, if $S \in \mathcal{O}(W)$, such that $\forall i$, $\rho_{W,W_i}(S)$ is the zero element in $\mathcal{O}(W_i)$, then $S = 0$.

Then $\mathcal{O}$ is called the structure sheaf of $M$.

# Germs of holomorphic functions

## Definition (Germs of holomorphic functions)

For each point $x \in M$, classify $\bigcup\{\mathcal{O}(U) : x \in U\}$ by the equivalence relation: suppose $S_w \in \mathcal{O}(W)$ and $S_V \in \mathcal{O}(V)$,

$$S_w \sim S_v \iff \exists U \subset W \cap V, \rho_{W,U}(S_W) = \rho_{V,U}(S_V).$$

Each equivalence class of

$$\bigcup\{\mathcal{O}(U) : x \in U\}/\sim$$

is called a germ of holomorphic function at $x$. The set of all equivalence classes is denoted as $\mathcal{O}(x)$, and called the stalk at $x$.

# Sheaf of holomorphic functions

## Definition (Sheaf of holomorphic functions)

Define $\tilde{O}$ as the set of all germs of holomorphic functions,

$$\tilde{\mathcal{O}} := \{\mathcal{O}(x) : x \in M\}.$$

Introduce a topology on $\tilde{\mathcal{O}}$: for any open set $W \subset M$, $\forall f \in \mathcal{O}(W)$, define $\bigcup_{x \in W}[f]_x$ is an open set of $\tilde{\mathcal{O}}$, $[f]_x$ is the germ in the stalk $\mathcal{O}(x)$. Such kind of open sets define the topology base, such that $\tilde{O}$ becomes a topological space, the so-called sheaf of holomorphic functions.

# Sheaf projection

**Definition (Sheaf projection)**

Define the sheaf projection map as

$$\pi : \tilde{\mathcal{O}} \to M : \quad \pi(\mathcal{O}(x)) = x.$$

$\pi$ maps an open set of $\tilde{\mathcal{O}}$ to an open set of $M$ injectively, hence $\pi$ is a local homeomorphism. $\pi^{-1}(x) = \mathcal{O}(x)$ is the stalk at $x$.

# Sheaf section

## Definition (Sheaf section)

Suppose $W$ is an open set on $M$, if a continuous map $S : W \to \tilde{\mathcal{O}}$, satisfies

$$\pi \circ S = \mathrm{id}_W,$$

then $S$ is a section of $\tilde{\mathcal{O}}$ on $W$. The set of all sections of $\tilde{\mathcal{O}}$ on $W$ is denoted as $\Gamma(\tilde{\mathcal{O}}, W)$.

$\{\tilde{\mathcal{O}}, \pi\}$ is called the espace etalé of $\mathcal{O}$. In general, they are treated as the same.

## Lemma

$$\Gamma(\tilde{\mathcal{O}}, W) = \mathcal{O}(W).$$

# Sheaf

## Definition (Sheaf)

Suppose $M$ is a topological space, a sheaf $\mathcal{F}$ on $M$ is a family $\{\mathcal{F}(U) : U \text{ open set on } M\}$, where $\mathcal{F}(U)$ is an Abelian group, and $\forall U \subset W$, $\exists$ restriction homomorphism map

$$\rho_{W,U} : \mathcal{F}(W) \to \mathcal{F}(U),$$

which is a group homomorphism. $\mathcal{F}$ and $\rho_{W,U}$ satisfies the following properties:

1. if $V \subset W$, let $\rho_{W,V} : \mathcal{F}(W) \to \mathcal{F}(V)$ be the restriction of homomorphism, namely $\rho_{W,V}(f) = f|_V$, then

$$U \subset V \subset W \implies \rho_{W,U} = \rho_{VU} \circ \rho_{W,V}$$

2. if $W = \bigcup_i W_i$, $W_i \subset M$ open, $\exists S_i \in \mathcal{O}(W_i)$, such that on $W_i \cap W_j$,

$$\rho_{W_i, W_i \cap W_j}(S_i) = \rho_{W_j, W_j \cap W_i}(S_j)$$

   then $\exists S \in \mathcal{F}(W)$m such that $\rho_{W,W_i}(S) = S_i$, $\forall i$.

3. if $W = \bigcup_i W_i$, $W_i \subset M$ open, if $S \in \mathcal{F}(W)$, such that $\forall i$, $\rho_{W,W_i}(S)$ is the zero element in $\mathcal{F}(W_i)$, then $S = 0$.

# Germs

## Definition (Germs)

For each point $x \in M$, classify $\bigcup\{\mathcal{F}(U) : x \in U\}$ by the equivalence relation: suppose $f_w \in \mathcal{F}(W)$ and $f_V \in \mathcal{F}(V)$,

$$f_w \sim f_v \iff \exists U \subset W \cap V, \rho_{W,U}(f_W) = \rho_{V,U}(f_V).$$

Each equivalence class of

$$\bigcup\{\mathcal{F}(U) : x \in U\}/\sim$$

is called a germ at $x$. The set of all equivalence classes is denoted as $\mathcal{F}(x)$, and called the stalk at $x$.

# Sheaf of functions

## Definition (Sheaf of functions)

Define $\tilde{F}$ as the set of all germs,

$$\tilde{\mathcal{F}} := \{\mathcal{F}(x) : x \in M\}.$$

Introduce a topology on $\tilde{\mathcal{F}}$: for any open set $W \subset M$, $\forall f \in \mathcal{F}(W)$, define $\bigcup_{x\in W}[f]_x$ is an open set of $\tilde{\mathcal{F}}$, $[f]_x$ is the germ in the stalk $\mathcal{F}(x)$. Such kind of open sets define the topology base, such that $\tilde{F}$ becomes a topological space, the so-called sheaf of functions.

# Sheaf projection

## Definition (Sheaf projection)

Define the sheaf projection map as

$$\pi : \tilde{\mathcal{F}} \to M : \quad \pi(\mathcal{F}(x)) = x.$$

$\pi$ maps an open set of $\tilde{\mathcal{F}}$ to an open set of $M$ injectively, hence $\pi$ is a local homeomorphism. $\pi^{-1}(x) = \mathcal{F}(x)$ is the stalk at $x$.

# Sheaf section

## Definition (Sheaf section)

Suppose $W$ is an open set on $M$, if a continuous map $S : W \to \tilde{\mathcal{F}}$, satisfies

$$\pi \circ S = \mathrm{id}_W,$$

then $S$ is a section of $\tilde{\mathcal{F}}$ on $W$. The set of all sections of $\tilde{\mathcal{F}}$ on $W$ is denoted as $\Gamma(\tilde{\mathcal{F}}, W)$.

$\{\tilde{\mathcal{F}}, \pi\}$ is called the espace etalé of $\mathcal{F}$. In general, they are treated as the same.

## Lemma

$$\Gamma(\tilde{\mathcal{F}}, W) = \mathcal{F}(W).$$

# Sheaf Examples

Suppose $M$ is a Riemann surface, $U \subset M$ is an open set on $M$, the followings are common sheaves:

- $\mathcal{O}(U)$ - sheaf of holomorphic function germs, holomorhic functions on $U$;
- $\mathcal{A}^0(U)$ - sheaf of $C^\infty$ function germs, $C^\infty$ functions on $U$;
- $\mathcal{A}^p(U)$ - sheaf of $C^\infty$ $p$-form germs, $C^\infty$ $p$-forms on $U$, $p = 0, 1, 2$;
- $\mathcal{A}^{p,q}(U)$ - sheaf of $C^\infty$ $(p,q)$-form germs, $C^\infty$ $(p,q)$-forms on $U$, $0 \leq p, q \leq 1$;

# Sheaf Examples

Suppose $M$ is a Riemann surface, $U \subset M$ is an open set on $M$, $L$ is a holomorphic line bundle on $M$, the followings are common sheaves:

- $\Omega(L)(U)$ - sheaf of holomorphic sections of $L$, the group of all holomorphic sections $\{S : U \to L\}$ of $L$ on $U$, $\Omega(L)(M) = \Gamma(M)$; if $U$ is trivalization neighborhood, then

$$\Omega(L)(U) \cong \mathcal{O}(U),$$

  and then $S = \{S_\alpha : S_\alpha = f_{\alpha\beta} S_\beta\}$.
- $\mathcal{A}^0(L)(U)$ - sheaf of $C^\infty$ sections of $L$, the group of all $C^\infty$ sections $f : U \to L$ on $U$, $\pi \circ f = \mathrm{id}_U$;

# Sheaf Examples

$\mathcal{A}^p(L)(U)$ - sheaf of $L$-valued $C^\infty$ $p$-form germs, $C^\infty$ $p$-forms on $U$, $p = 0, 1, 2$;

$$
\begin{aligned}
\mathcal{A}^p(L)(U) &= \mathcal{A}^p(U) \otimes_{\mathcal{A}^0(U)} \mathcal{A}^0(L)(U) \\
&= \left\{ \sum_i S_i \otimes S_i : \omega_i \ C^\infty \ p - \text{form}, S_i \ C^\infty \text{ section of L on U} \right\}
\end{aligned}
$$

$\mathcal{A}^p(L)(U)$ is a module on the commutative ring $\mathcal{A}^0(U)$.
$\forall S, S' \in \mathcal{A}^0(L)(U)$, $\omega, \omega' \in \mathcal{A}^p(U)$

$$
\begin{aligned}
\omega(S + S') &= \omega S + \omega S' \\
(\omega + \omega')S &= \omega S + \omega' S \\
f(\omega S) = (f\omega)S &= \omega(fS), \forall f \in \mathcal{A}^0(U)
\end{aligned}
$$

# Sheaf Examples

$\mathcal{A}^{p,q}(U)$ - sheaf of $L$-valued $C^\infty$ $(p,q)$-form germs, $C^\infty$ $(p,q)$-forms on $U$, $0 \le p, q \le 1$;

$$\mathcal{A}^{p,q}(L)(U) = \mathcal{A}^{p,q}(U) \otimes_{\mathcal{A}^0(U)} \mathcal{A}^0(L)(U)$$
$$= \left\{ \sum_i S_i \otimes S_i : \omega_i \ C^\infty \ (p,q) - \text{form}, S_i \ C^\infty \text{ section of L on U} \right\}$$

# Sheaf Homomorphism

## Definition (Sheaf homomorphism)

Suppose $M$ is a topologial space, $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on $M$, assume $\mathcal{U} = \{U\}$ is the total set of all open sets on $M$, if there are a family of group (ring) homomorphisms $\{\varphi_U\}$ $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, such that the diagram

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\varphi_W"] \arrow[d, "\rho_{W,U}"'] & \mathcal{G}(W) \arrow[d, "\rho'_{W,U}"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U) \end{tikzcd}$$

commutes $\forall U, W \in \mathcal{U}$, $U \subset W$. We call $\varphi = \{\varphi_U\}$ as the sheaf homomorphism from $\mathcal{F}$ to $\mathcal{G}$. If each $_U$ is a group (ring) isomorphism, then $\varphi$ is a sheaf isomorphism.

For any sheaf homomorphism $\varphi = \{\varphi_U : \mathcal{F} \to \mathcal{G}\}$, it induces a map between their espace etalé's, $\tilde{\varphi} : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$, such that for each $x \in M$,

$$\tilde{\varphi}([f]_x) = [\varphi(f)]_x.$$

$\varphi$ is a sheaf isomorphism if and only if $\tilde{\varphi}$ restricted on each stalk is an isomorphism from $\mathcal{F}(x)$ to $\mathcal{G}(x)$.

# Sheaf Exact Sequence

**Definition (sheaf exact sequence)**

$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k, \cdots$ are sheaves on the topological space $M$, $i_k : \mathcal{A}_k \to \mathcal{A}_{k+1}$ is a sheaf homomorphism, $k = 0, 1, 2, \cdots$

$$\mathcal{A}_0 \xrightarrow{i_0} \mathcal{A}_1 \xrightarrow{i_1} \mathcal{A}_2 \xrightarrow{i_2} \mathcal{A}_3 \xrightarrow{i_3} \cdots$$

is called a sheaf exact sequence, if $\forall x \in M$,

$$\mathcal{A}_0(x) \xrightarrow{i_0} \mathcal{A}_1(x) \xrightarrow{i_1} \mathcal{A}_2(x) \xrightarrow{i_2} \mathcal{A}_3(x) \xrightarrow{i_3} \cdots$$

is a group (ring) exact sequence.

# Short Sheaf Exact Sequence

We use 0 to represent zero sheaf,

**Definition (Short sheaf exact sequence)**

If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are sheaves

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

is called a short sheaf exact sequence.

1. $i : \mathcal{A} \to \mathcal{B}$ is injective;
2. $j : \mathcal{B} \to \mathcal{C}$ is surjective;
3. $j : \mathcal{B}/\mathcal{A} \to \mathcal{C}$ is a sheaf isomorphism.

# Cech Cohomology

# Cech Cochain

## Definition (Cěch Cochain)

Suppose $\mathcal{S}$ is a sheaf on $M$, $\mathcal{U} = \{U_\alpha\}$ is an open covering of $M$. For $q \in \mathbb{Z}_{\geq 0}$, consider the mapping $f$, which maps any $q+1$ ordered open sets $U_0, U_1, \dots, U_q$ in $\mathcal{U}$ to a section of $\mathcal{S}$ on $U_0 \cap U_1 \cap \dots U_q$, $f(U_0, U_1, \dots, U_q)$, such that

1. when $U_0 \cap U_1 \cdots \cap U_q = \emptyset$, $f(U_0, U_1, \dots, U_q) = 0$;
2. When two open sets are exchanged,

$$f(\cdots, U_i, \cdots, U_j, \cdots) = -f(\cdots, U_j, \cdots, U_i, \cdots).$$

we call $f$ a q-cochain in the open covering $\mathcal{U}$, the set of all q-cochains is denoted as $C^q(\mathcal{U}, \mathcal{S})$, which is an Abel group under the addition, the so-called $q$ dimensional cochain group. When $q < 0$, $C^q(\mathcal{U}, \mathcal{S})$ is zero.

# Cech Cochain

## Definition (Coboundary $\delta$-operator)

Define homomorphism $\delta : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ as follows:

$$\delta f(U_0, U_1, \cdots, U_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(U_0, U_1, \cdots, \hat{U}_i, \cdots, U_{q+1}),$$

where $\hat{U}_i$ means the component $U_i$ is removed, and the summation is the summation of sections restricted on $U_0 \cap U_1 \cap \cdots \cap U_{q+1}$.

## Lemma

*$\delta$ has the property*

$$\boxed{\delta^2 = \delta \circ \delta = 0}$$

# Cech Cohomology

## Definition (Cech Cohomology Group)

If $\delta f = 0$, then $f$ is called a q-closed cochain, the subgroup of all q-closed chains is denoted as $Z^q(\mathcal{U};\mathcal{S})$; if $f = \delta g$, then $f$ is called a q-boundary cochain, the subgroup of all q-boundary cochains is denoted as $B(\mathcal{U};\mathcal{S})$. The quotient group

$$H^q(\mathcal{U};\mathcal{S}) = \frac{Z^q(\mathcal{U};\mathcal{S})}{B^q(\mathcal{U};\mathcal{S})}$$

is called the q degree Cech cohomology associated with the open cover $\mathcal{U}$ and the coefficient group of the sections of the sheaf $\mathcal{S}$.

# Cech Cohomology

## Lemma

$$H^0(\mathcal{U};\mathcal{S}) = \Gamma(\mathcal{S}).$$

## Proof.

By definition $H^0(\mathcal{U};\mathcal{S}) = Z^0(\mathcal{U};\mathcal{S})$. For $f \in Z^0(\mathcal{U};\mathcal{S})$,

$$\begin{aligned}
\delta f = 0 &\iff f(U_\beta) - f(U_\alpha) = \delta f(U_\alpha, U_\beta) = 0, \quad \forall U_\alpha, U_\beta \in \mathcal{U} \\
&\iff f(U_\beta)|_{U_\alpha \cap U_\beta} = f(U_\alpha)|_{U_\beta \cap U_\alpha} \\
&\iff \exists i(f) \in \Gamma(\mathcal{S}),\ s.t.\ i(f)|_{U_\alpha} = f(U_\alpha), \quad \forall U_\alpha \in \mathcal{U}.
\end{aligned}$$

Therefore $i : Z^0(\mathcal{U};\mathcal{S}) \to \Gamma(\mathcal{S})$ is isomorphic. $\square$

# Sheaf Homomorphism

Suppose $\varphi : \mathcal{S} \to \mathcal{T}$ is a sheaf homomorphism, it induces homomorphism between cochain groups $\varphi^* : C^q(\mathcal{U}; \mathcal{S}) \to C^q(\mathcal{U}; \mathcal{T})$:

$$(\varphi^* f)(U_0, U_1, \cdots, U_q) = \varphi \circ f(U_0, U_1, \cdots, U_q).$$

It is easy to show $\delta\varphi^* f = \varphi^* \delta f$, therefore $\varphi$ induces the homomorphism between cohomology groups

$$\varphi^* : H^q(\mathcal{U}; \mathcal{S}) \to H^q(\mathcal{U}; \mathcal{T}).$$

# Subdivision - Cech Cohomology

If $\mathcal{U}, \mathcal{V}$ are two open coverings, and $\mathcal{U}$ is a subdivision of $\mathcal{V}$, namely there is a subdivision mapping: $\tau : \mathcal{U} \to \mathcal{V}$, such that $U_\alpha \subset \tau(U_\alpha)$. Then $\tau$ induces a cochain group homomorphism: $\tau^* : C^q(\mathcal{V}; \mathcal{S}) \to C^q(\mathcal{U}; \mathcal{S})$

$$(\tau^* f)(U_0, U_1, \cdots, U_q) = f(\tau(U_0), \tau(U_1), \cdots, \tau(U_q)),$$

$\tau^*$ and $\delta$ are exchangable, hence induces the group homomorphism:

$$\tau^* : H^q(\mathcal{V}; \mathcal{S}) \to H^q(\mathcal{U}; \mathcal{S}).$$

# Open Cover Independent Cech Cohomology

## Definition ($q$-th degree Cech cohomology group)

Suppose $\mathcal{S}$ is a sheaf on $M$, consider all possible open covers of $M$, let

$$H^q(M;\mathcal{S}) = \coprod_{\mathcal{U}} H^q(\mathcal{U};\mathcal{S})/\sim,$$

where equivalence relation $\sim$ is defined as follows:
$[f] \in H^q(\mathcal{U};\mathcal{S}) \sim [g] \in H^q(\mathcal{V};\mathcal{S})$, if and only if there exists a common subdivision $\mathcal{W}$ of both $\mathcal{U}$ and $\mathcal{V}$, such that $\tau_1^*[f] = \tau_2^*[g]$, where $\tau_1 : \mathcal{W} \to \mathcal{U}$ and $\tau_2 : \mathcal{W} \to \mathcal{V}$ are the subdivision mappings. $H^q(M,\mathcal{S})$ is called the $q$-th degree Cech cohomology group with coefficients in the sheaf $\mathcal{S}$.

# Cech Cohomology Group

- $H^0(M;\mathcal{S}) = \Gamma(\mathcal{S})$;
- the quotient map $i : H^1(\mathcal{U};\mathcal{S}) \to H^1(M;\mathcal{S})$ is injective;
- If $H^1(M;\mathcal{R}) = 0$, then the short exact sequence

$$0 \to \mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{j} \mathcal{T} \to 0$$

induces the short exact sequence of the Abelian groups

$$0 \to \Gamma(\mathcal{R}) \xrightarrow{i^*} \Gamma(\mathcal{S}) \xrightarrow{j^*} \Gamma(\mathcal{T}) \to 0$$

# Sheaf Cohomology

## Problem

*Suppose $f$ is a harmonic function defined on a Riemann surface $M$, whether there is a holomorphic function $h$, such that $Re(h) = f$?*

Select an open cover $\mathcal{U} = \{U_\alpha\}$. On each open set $U_\alpha$, there is a holomorphic function $h_\alpha$, such that $Re(h_\alpha) = f|_{U_\alpha}$. On $U_\alpha \cap U_\beta$, $Re(h_\beta - h_\alpha) = 0$, therefore $h_\beta - h_\alpha$ is local constant functions, which is a 1-cochain of the constant sheaf $\mathbb{C}$, denoted as $i(f)$. Clearly, $i(f)$ is a closed cochain. If $H^1(M, \mathbb{C}) = 0$, then there is a 0-cochain $g$, $i(f) = \delta g$, on $U_\alpha \cap U_\beta$, we have

$$h_\beta - h_\alpha = g_\beta - g_\alpha \implies h_\beta - g_\beta = h_\alpha - g_\alpha,$$

therefore there is a holomorphic function $h$ defined on $M$, such that $h|_{U_\alpha} = h_\alpha - g_\alpha$. Therefore $Re(h) - f$ is a local constant harmonic function on $M$, after minusing constants, we can assume $Re(h) = f$.

# Mittag-Leffler Problem

## Problem (Mittag-Leffler)

*Given a set of discrete points $\{p_i\} \subset M$, a finite sum $f_i = \sum_{j \geq 1} a_j z^{-j}$ in the neighborhood of each $p_i$. Find a meromorphic function $f$, such that $f - f_i$ is holomorphic in the neighborhood of $p_i$.*

Select an open cover $\mathcal{U} = \{U_\alpha\}$, on $U_\alpha$ there is a meromorphic function $h_\alpha$, such that when $p_i \in U_\alpha$, $h_\alpha - f_i$ is holomorphic in the neighborhood of $p_i$. Therefore, on $U_\alpha \cap U_\beta$, $h_{\beta\alpha} = h_\beta - h_\alpha$ is holomorphic, $h_{\beta\alpha} \in \mathcal{O}(U_\alpha \cap U_\beta)$ is a closed 1-cochain. If $H^1(M, \mathcal{O}) = 0$, there are holomorphic functions $\{g_\alpha\}$, such that $h_{\alpha\beta} = g_\beta - g_\alpha$. Hence

$$h_{\alpha\beta} = h_\beta - h_\alpha = g_\beta - g_\alpha \implies h_\beta - g_\beta = h_\alpha - g_\alpha$$

there is a globally defined meromorphic function $h$, $h|_{U_\alpha} = h_\alpha - g_\alpha$.

# Trivial Line Bundle

## Problem (Trivial Line Bundle)

*When is a holomorphic line bundle isomorphic to a trivial bundle ?*

Suppose $L$ is a holomorphic line bundle over $M$, $\{U_\alpha\}$ is a local trivialization open cover, $f_{\beta\alpha}$ the transit functions. We treat $f_{\beta\alpha}$ as the local section on $U_\alpha \cap U_\beta$ of the sheaf $O^*$, satisfying the cocycle condition,

$$f_{\beta\alpha} \cdot f_{\alpha\gamma} \cdot f_{\gamma\beta} = 1.$$

So $\{f_{\beta\alpha}\}$ is a closed 1-cochain. If $H^1(M; \mathcal{O}^*) = 0$, then there is non-zero holomorphic functions $\{f_\alpha\}$ on $U_\alpha$, such that $f_{\beta\alpha} = f_\alpha / f_\beta$, therefore $L$ is trivial.

# Short-Long Exact Sequence

# short-long exact sequence

## Theorem (short-long exact sequence)

*Given a short exact sequence of sheaves* $0 \to \mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{j} \mathcal{T} \to 0$, *there is a connecting homomorphism* $\delta_q^* : H^q(M;\mathcal{T}) \to H^{q+1}(M;\mathcal{R})$, *such that the following is a long exact sequence of Abelian groups*

$$\begin{aligned}
&0 \to H^0(M;\mathcal{R}) \xrightarrow{i^*} H^0(M;\mathcal{S}) \xrightarrow{j^*} H^0(M;\mathcal{T}) \\
&\quad \xrightarrow{\delta_0^*} H^1(M;\mathcal{R}) \xrightarrow{i^*} H^1(M;\mathcal{S}) \xrightarrow{j^*} H^1(M;\mathcal{T}) \\
&\quad \xrightarrow{\delta_1^*} H^2(M;\mathcal{R}) \xrightarrow{i^*} H^2(M;\mathcal{S}) \xrightarrow{j^*} H^2(M;\mathcal{T}) \\
&\quad \cdots\cdots \\
&\quad \xrightarrow{\delta_{p-1}^*} H^p(M;\mathcal{R}) \xrightarrow{i^*} H^p(M;\mathcal{S}) \xrightarrow{j^*} H^p(M;\mathcal{T})
\end{aligned}$$

# short-long exact sequence

Namely

$$
\begin{aligned}
0 \to \Gamma(\mathcal{R}) &\xrightarrow{i^*} \Gamma(\mathcal{S}) \xrightarrow{j^*} \Gamma(\mathcal{T}) \\
&\xrightarrow{\delta_0^*} H^1(M;\mathcal{R}) \xrightarrow{i^*} H^1(M;\mathcal{S}) \xrightarrow{j^*} H^1(M;\mathcal{T}) \\
&\xrightarrow{\delta^*} H^2(M;\mathcal{R}) \xrightarrow{i^*} H^2(M;\mathcal{S}) \xrightarrow{j^*} H^2(M;\mathcal{T}) \\
&\cdots\cdots \\
&\xrightarrow{\delta_{p-1}^*} H^p(M;\mathcal{R}) \xrightarrow{i^*} H^p(M;\mathcal{S}) \xrightarrow{j^*} H^p(M;\mathcal{T})
\end{aligned}
$$

Most powerful theorem in homology algebra.

# Chase on the graph

**Lemma (Connecting Homomorphism)**

*Given a short exact sequence of sheaves* $0 \to \mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{j} \mathcal{T} \to 0$, *there is a connecting homomorphism* $\delta^* : H^p(M;\mathcal{T}) \to H^{p+1}(M;\mathcal{R})$.

The proof is based on the classical method in homological algebra chase on the graph.

# Chase on the graph

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p}(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^{p}(M,T) \arrow[r] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure: Step 0: Communative diagram, the columns are exact sequences.

# Chase on the graph

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^p(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^p(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^p(M,T) \arrow[r] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$s \in C^p(M,S)$

$t \in Z^p(M,T)$

exact

Figure: Step 1: Choose $t \in Z^p(M,T)$, since $j^* : C^p(M,S) \to C^p(M,T)$ is surjective, $\exists s \in C^p(M,S)$, such that $j^*(s) = t$.

# Chase on the graph

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p}(M,S) \arrow[r, "\delta"] \arrow[d, "j^*"'] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^{p}(M,T) \arrow[r] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

exact

$s$ $\delta(s)$ $t$ $0$

Figure: Step 2: $\delta \circ j^*(s) = \delta(t) = 0$. Because the diagram commutes, $j^* \circ \delta(s) = 0$.

# Chase on the graph

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p}(M,S) \arrow[r, "\delta"] \arrow[d, "j^*"'] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^{p}(M,T) \arrow[r, "\delta"'] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

exact

Figure: Step 3: Because $j^* \circ \delta(s) = 0$, $\delta(s) \in \mathrm{Ker} J^*$. Since each column is a short exact sequence, $\mathrm{Ker} j^* = \mathrm{Img} i^*$, therefore $\exists r \in C^{p+1}(M,R)$, such that $i^*(r) = \delta(s)$.

# Chase on the graph

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 \arrow[ddd, "\text{exact}"'] & \cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p}(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r, "\delta"] \arrow[d, "i^*"] & \cdots \\
 & \cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p}(M,S) \arrow[r, "\delta"] \arrow[d, "j^*"] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
 & \cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^{p}(M,T) \arrow[r, "\delta"'] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 {} & & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
r \arrow[r, "i^*"] \arrow[d] & 0 \\
\delta(s) \arrow[r, "\delta"] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r] \arrow[d] & 0 \\
t \arrow[r] & 0
\end{tikzcd}
$$

Figure: Step 4: Because $\delta \circ i^*(r) = \delta \circ \delta(s) = 0$, the diagram commutes, $\imath^*\delta(r) = 0$. Because the 4-th column is exact, $i^*$ is injective, hence $\delta(r) = 0$, $r \in Z^{p+1}(M, R)$.

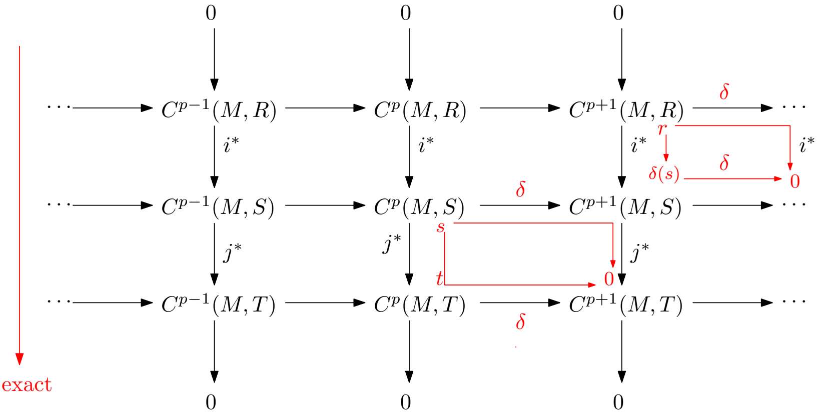


Figure: Finally, we obtain the connecting homomorphis $\delta^* : Z^p(M,T) \to Z^{p+1}(M,R)$, $\delta^*(t) = r$.

# Chase on the graph

**Lemma**

*The following sequences are exact:*

$$H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R) \xrightarrow{i^*} H^{p+1}(M,S)$$

$$H^p(M,S) \xrightarrow{j^*} H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R)$$

# Chase on the graph

We want to show $H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R) \xrightarrow{i^*} H^{p+1}(M,S)$ is exact, namely

1. $\text{Img}\delta^* \subset \text{Ker}i^*$
2. $\text{Ker}i^* \subset \text{Img}\delta^*$

Since $\delta^*(t) = r$, $i^*(r) = \delta(s)$, $\delta(s) = 0$ in $H^{p+1}(M,S)$, therefore $i^* \circ \delta^*(t) = 0$, $\text{Img}\delta^* \subset \text{Ker}i^*$.

# Chase on the graph

Show $H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R) \xrightarrow{i^*} H^{p+1}(M,S)$ is exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^p(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^p(M,S) \arrow[r] \arrow[d, "j^*"] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^p(M,T) \arrow[r] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$r \in Z^{p+1}(M,R)$

$\delta(s)$

$s$

exact

Figure: Step 1. choose $r \in \mathrm{Ker}\, i^*$, namely $r \in Z^{p+1}(M,R)$, $i^*(r) = 0$ in $H^{p+1}(M,S)$, hence $\exists s \in C^p(M,S)$, such that $i^*(r) = \delta(s)$.

Next we show $H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R) \xrightarrow{i^*} H^{p+1}(M,S)$ is exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^p(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^p(M,S) \arrow[r] \arrow[d, "j^*"'] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^p(M,T) \arrow[r] \arrow[d] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & r \in Z^{p+1}(M,R) \arrow[d] \\
s \arrow[r, "\delta"] \arrow[d] & \delta(s) \arrow[d] \\
t \arrow[r, "\delta"] & 0
\end{tikzcd}
$$

exact

Figure: Step 2. Each column is exact, hence $j^* \circ i^*(r) = j^*(\delta(s)) = 0$. Because the diagram commutes, $j^* \circ \delta(s) = 0$ implies $\delta \circ j^*(s) = 0$, hence $t = j^*(s) \in Z^p(M,T)$. By definition, $\delta^*(t) = r$, $r \in \text{Img } \delta^*$, this shows $\text{Ker} i^* \subset \text{Img}\delta^*$. Hence $\text{Ker} i^* = \text{Img}\delta^*$.

# Chase on the graph

We want to show $H^p(M,S) \xrightarrow{j^*} H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R)$ is exact, namely

1. $\text{Img} j^* \subset \text{Ker}\delta^*$
2. $\text{Ker}\delta^* \subset \text{Img} j^*$

Show $H^p(M,S) \xrightarrow{j^*} H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R)$ is exact,

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^p(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^p(M,S) \arrow[r, "\delta"] \arrow[d, "j^*"] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^p(M,T) \arrow[r, "\delta"] \arrow[d] \arrow[ruu, "\delta^*"] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$r \in Z^{p+1}(M,R)$

$\delta(s)$

$s$

$t$

$0$

exact

Figure: Step 1. Choose any $s \in Z^p(M,S)$, $t = j^*(s)$, $r = \delta^*(t)$,

$$\delta(s) = 0 \implies j^* \circ \delta(s) = 0 \implies \delta \circ j^*(s) = 0 \implies \delta(t) = 0 \implies t \in Z^p(M,T)$$

$$\delta(s) = 0 \implies i^*(r) = 0 \implies r = 0 \implies \delta^*(t) = 0 \implies t \in \mathrm{Ker}\delta^* \implies \mathrm{Img} j^* \subset \mathrm{Ker}\delta^*$$

# Chase on the graph

Show $H^p(M,S) \xrightarrow{j^*} H^p(M,T) \xrightarrow{\delta^*} H^{p+1}(M,R)$ is exact,

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^{p-1}(M,R) \arrow[r] \arrow[d, "i^*"] & C^p(M,R) \arrow[r] \arrow[d, "i^*"] & C^{p+1}(M,R) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,S) \arrow[r] \arrow[d, "j^*"] & C^p(M,S) \arrow[r, "\delta"] \arrow[d, "j^*"'] & C^{p+1}(M,S) \arrow[r] \arrow[d, "j^*"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(M,T) \arrow[r] \arrow[d] & C^p(M,T) \arrow[r] \arrow[d] \arrow[ruu, "\delta^*"] & C^{p+1}(M,T) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$r = 0 \to i^*(r) = 0$

$s \to t$, $s \to 0$, $t \xrightarrow{\delta} 0$

exact

Figure: Step 2. Choose $0 = r \in Z^{p+1}(M,R)$,

$$r = 0 \implies i^*(r) = 0 \implies \delta(s) = 0 \implies j^* \circ \delta(s) = 0 \implies \delta \circ j^*(s) = 0$$

$$\delta(s) = 0 \implies s \in Z^p(M,S), t = j^*(s) \in Z^p(M,T) \implies \text{Ker}\delta^* \subset \text{Img } j^*$$

# Connecting Homomorphism $\delta^*$

## Lemma

*Given a short exact sequence of sheaves* $0 \to \mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{j} \mathcal{T} \to 0$, *there is a connecting homomorphism* $\delta^* : H^0(M;\mathcal{T}) \to H^1(M;\mathcal{R})$.

## Proof.

$$
\begin{aligned}
f \in \Gamma(\mathcal{T}) &\implies \exists g_\alpha \in \Gamma(U_\alpha, \mathcal{S}), s.t.\ j^* g_\alpha = f \quad j^*\text{surjective}\\
&\implies j^*(g_\beta - g_\alpha) = f|_{U_\beta} - f|_{U_\alpha} = 0, \text{on } U_\alpha \cap U_\beta\\
(g_\beta - g_\alpha) \in \text{Ker } j^* &\implies g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathbb{R}), i^* g_{\alpha\beta} = (g_\beta - g_\alpha)\\
&\text{since Img } i^* = \text{Ker } j^*\\
i^* g_{\alpha\beta} + i^* g_{\beta\gamma} + i^* g_{\gamma\alpha} = 0 &\implies g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0 \quad i^* \text{ injective}\\
\{g_{\alpha\beta}\} \text{ closed} &\implies [\{g_{\alpha\beta}\}] \in H^1(M;\mathcal{R})
\end{aligned}
$$

$$
\delta^* : [f] \to [\{g_{\alpha\beta}\}], H^0(M;\mathcal{T}) \to H^1(M;\mathcal{R}).
$$