

Introduction to Computational Conformal Geometry

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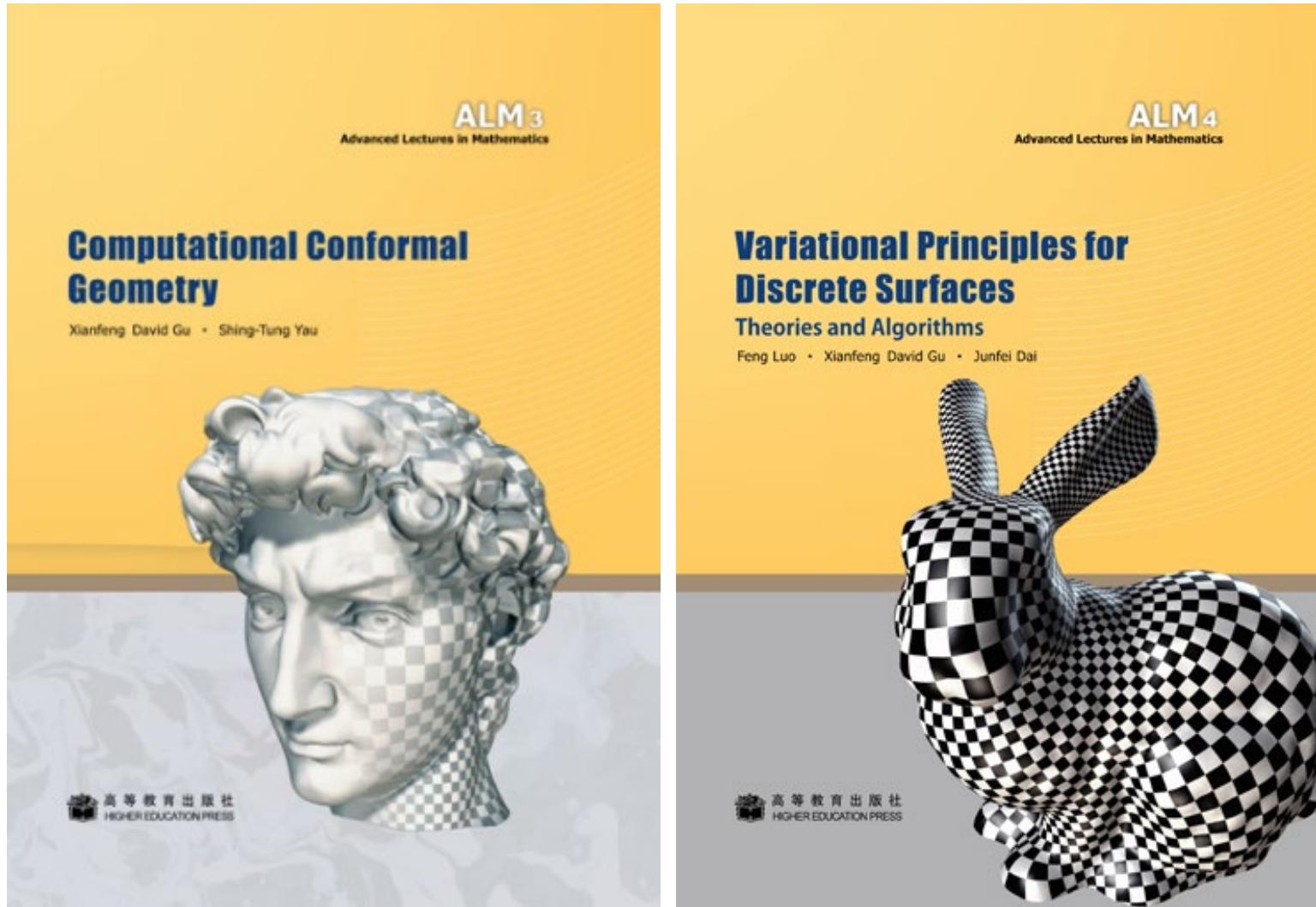
Textbook



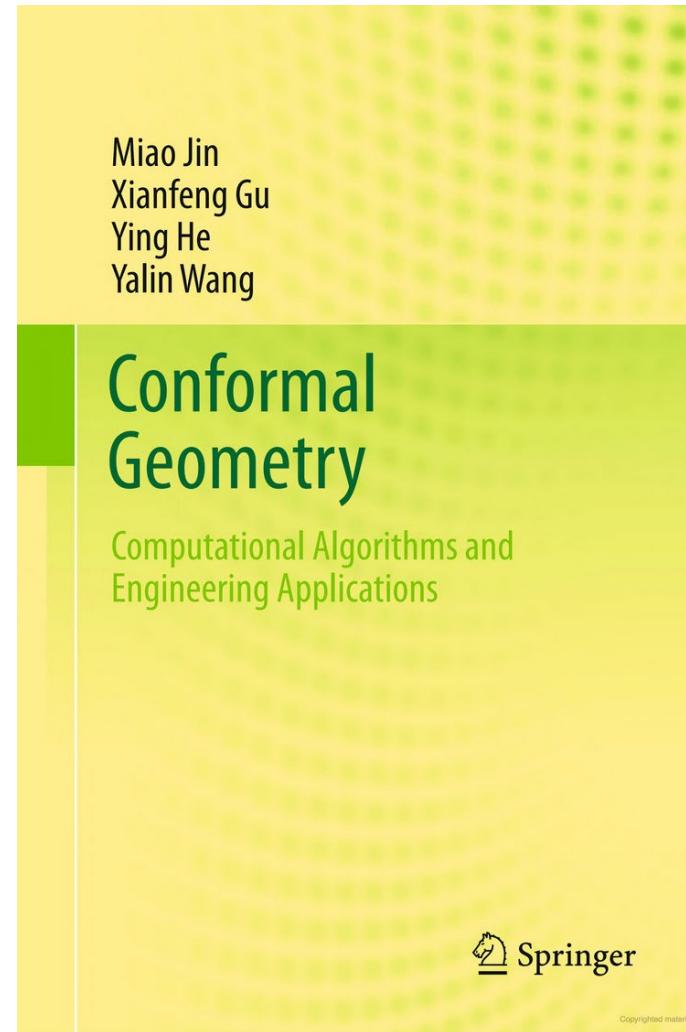
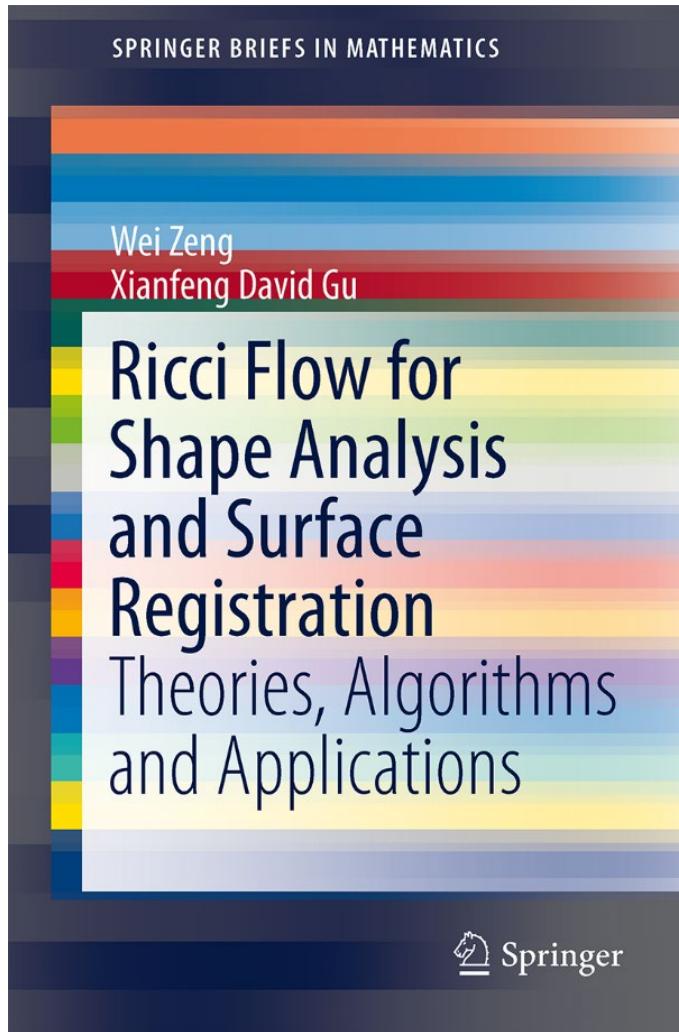
Textbook



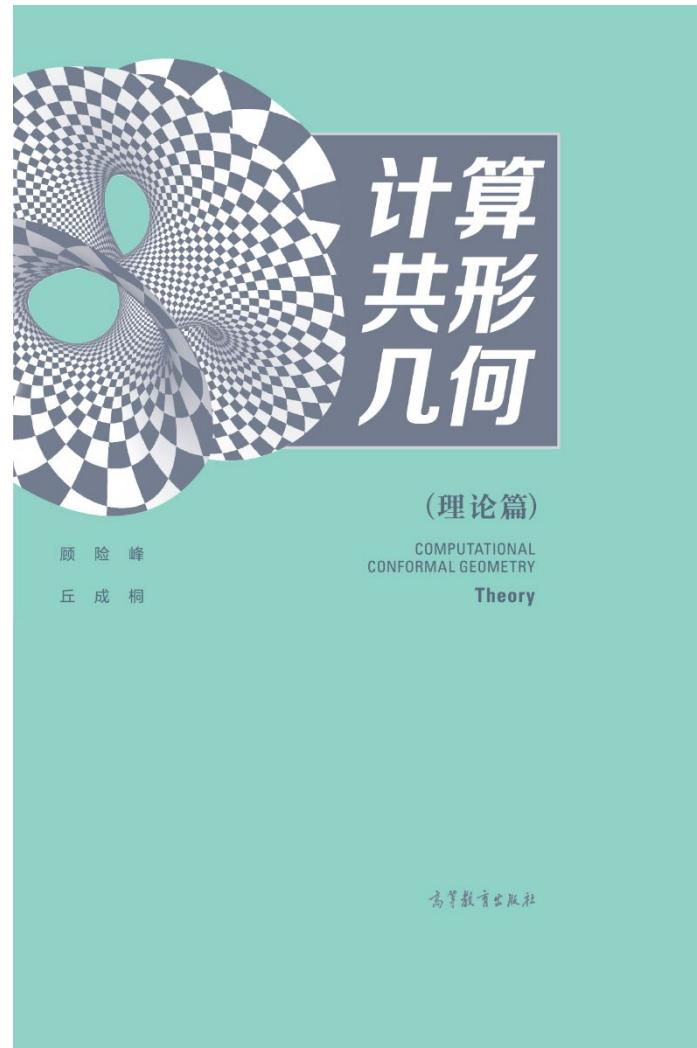
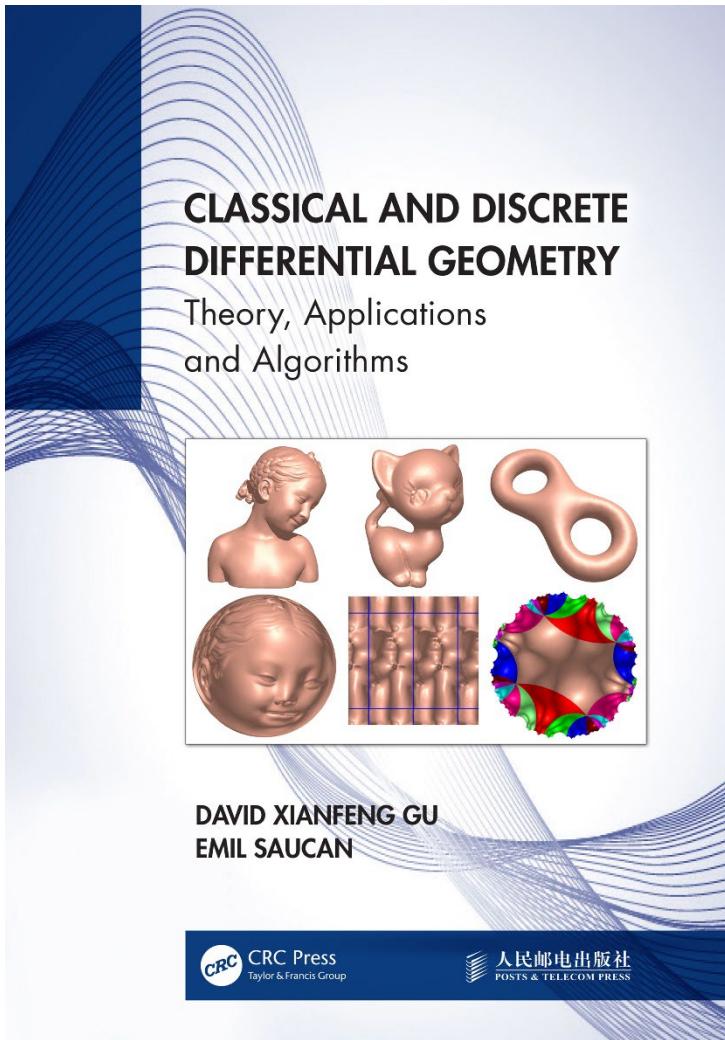
Textbook



Reference Books



Reference Books



Theoretical Approaches

- Complex Manifold : analysis, topology and differential geometry. Curvature, characteristic class, residue, elliptic differential operators;
- Algebraic Geometry: algebraic curve, ideals in polynomial ring, Galois theory, Grobner basis;
- Algebraic number theory: meromorphic function field with transandence degree one, valuation theory;

Computational Approaches

- Topology: Combinatorial and Algebraic methods
- Analysis: finite element method on surfaces
 - Dynamic combinatorial structure
 - Convex energy optimization using Newton method
 - Linearize nonlinear differential operators
 - Numerical Stability: Delaunay triangulation, Multiple Precision, Monitor Geometric distortion

Contents

- Concepts and Theorems : algebraic topology, surface differential geometry, Riemann surface theory and geometric partial differential equations;
- Computational Methods: surface fundamental group , homology group, harmonic maps, meromorphic differentials, foliation, conformal mapping, quasi-conformal mapping and Ricci flow.
- Applications : Computer Graphics, Computer Vision, Visualization, Geometric Modeling, Networking, Medical Imaging

3D Data Aquisition

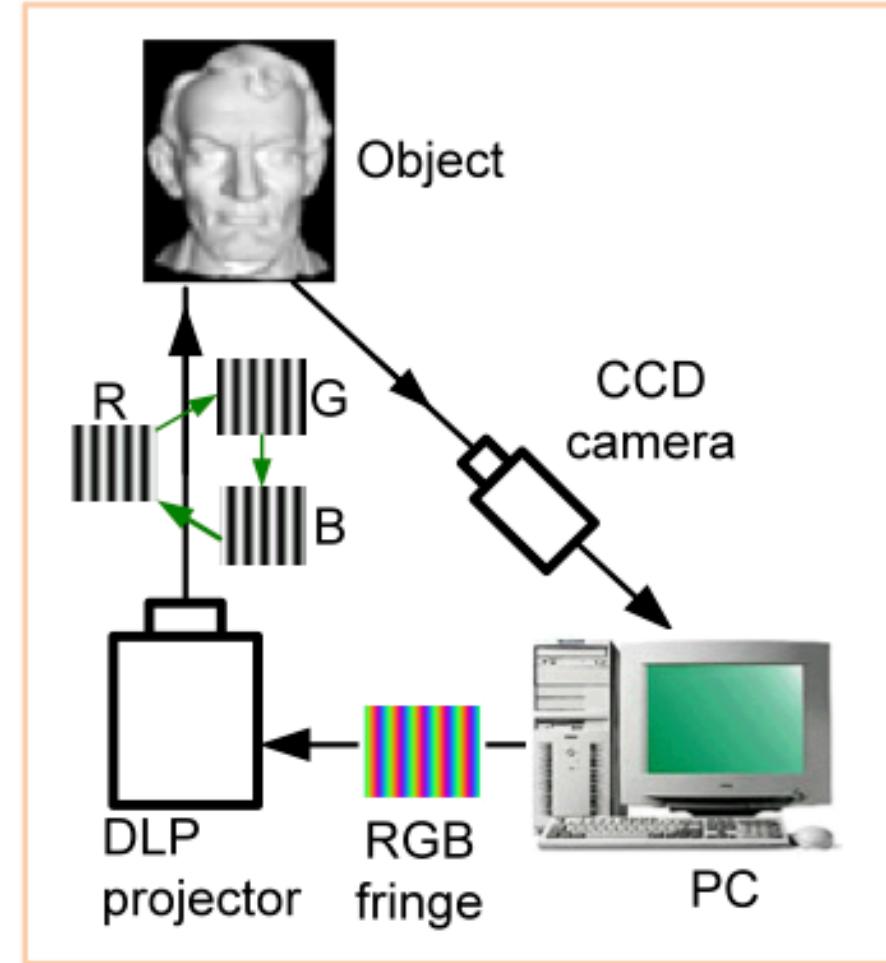
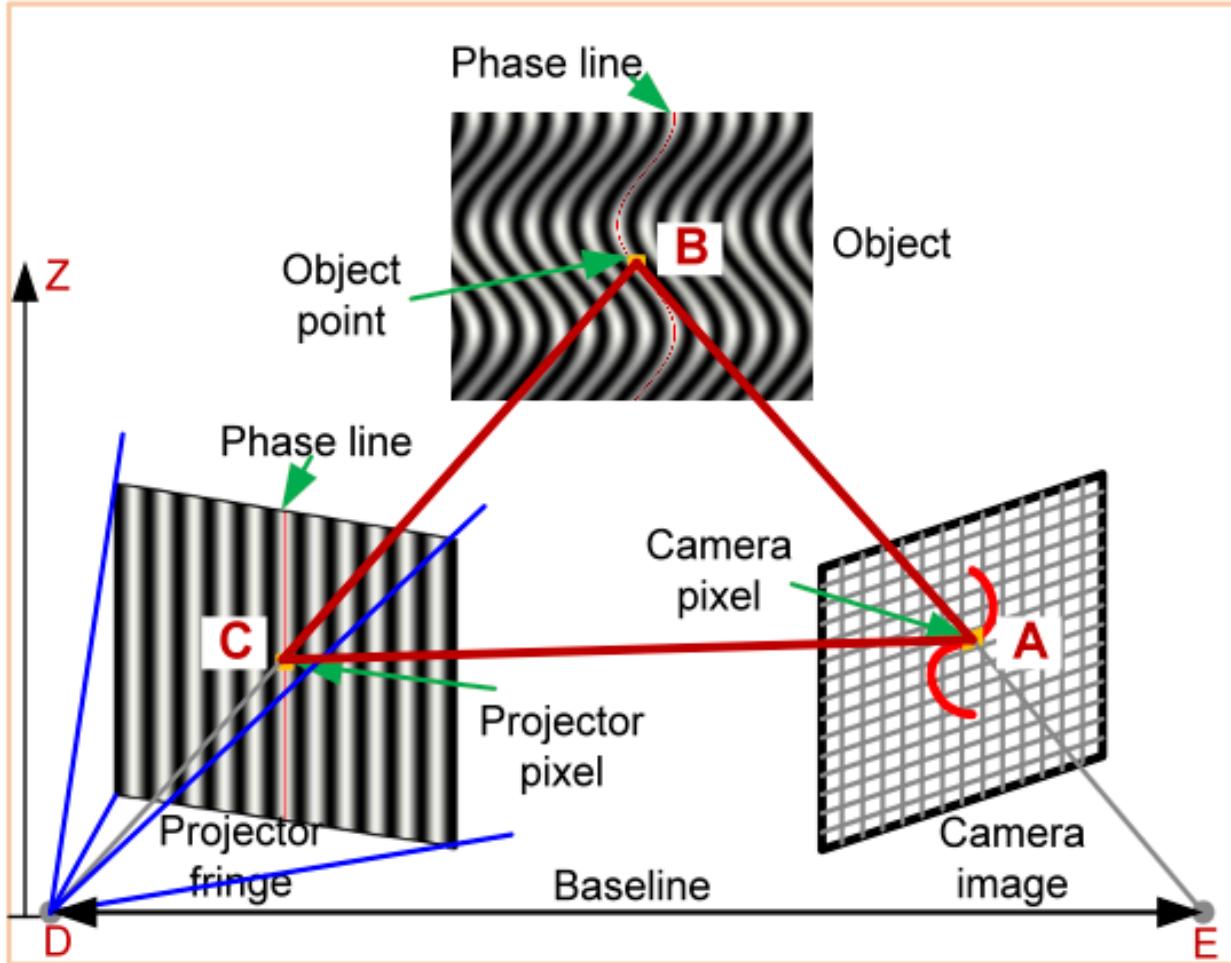
Dynamic 3D Scanning System

Problem: How to capture dynamic 3D data in real time with high quality?

Answer: 3D scanning system based on phase shifting structure light !

- Hand made 3D scanning system
- High speed, high depth accuracy, high resolution 3D camera
- Based on multi-wavelength phase shifting structured light principle

System Layout



Dynamic 3D Scanner (first generation)



Dynamic 3D Scanner (2nd, 3rd generations)

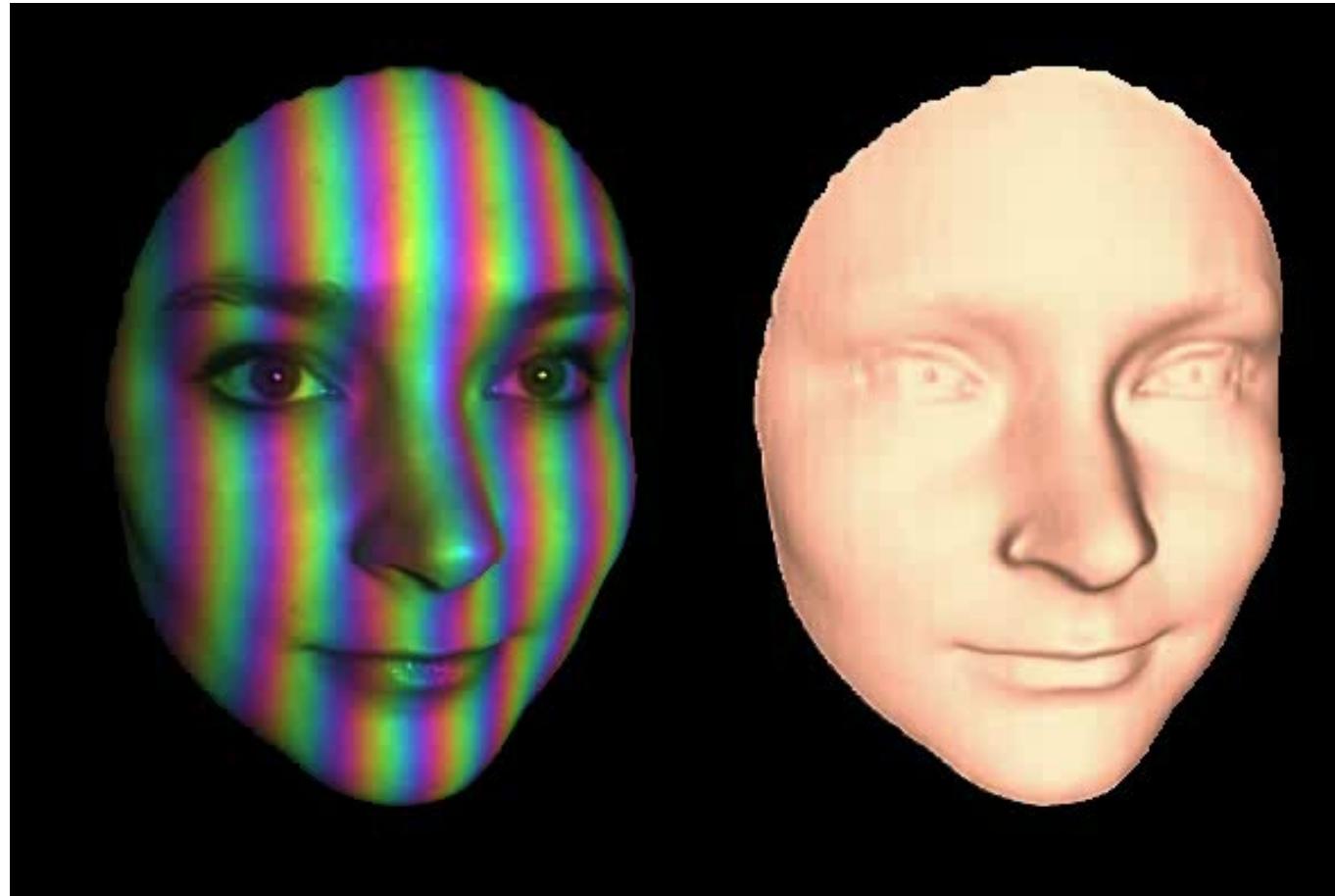


3D Data Acquisition

- Capture 3D human face with high speed and high accuracy.



Multi-wavelength Phase Shifting Structured Light

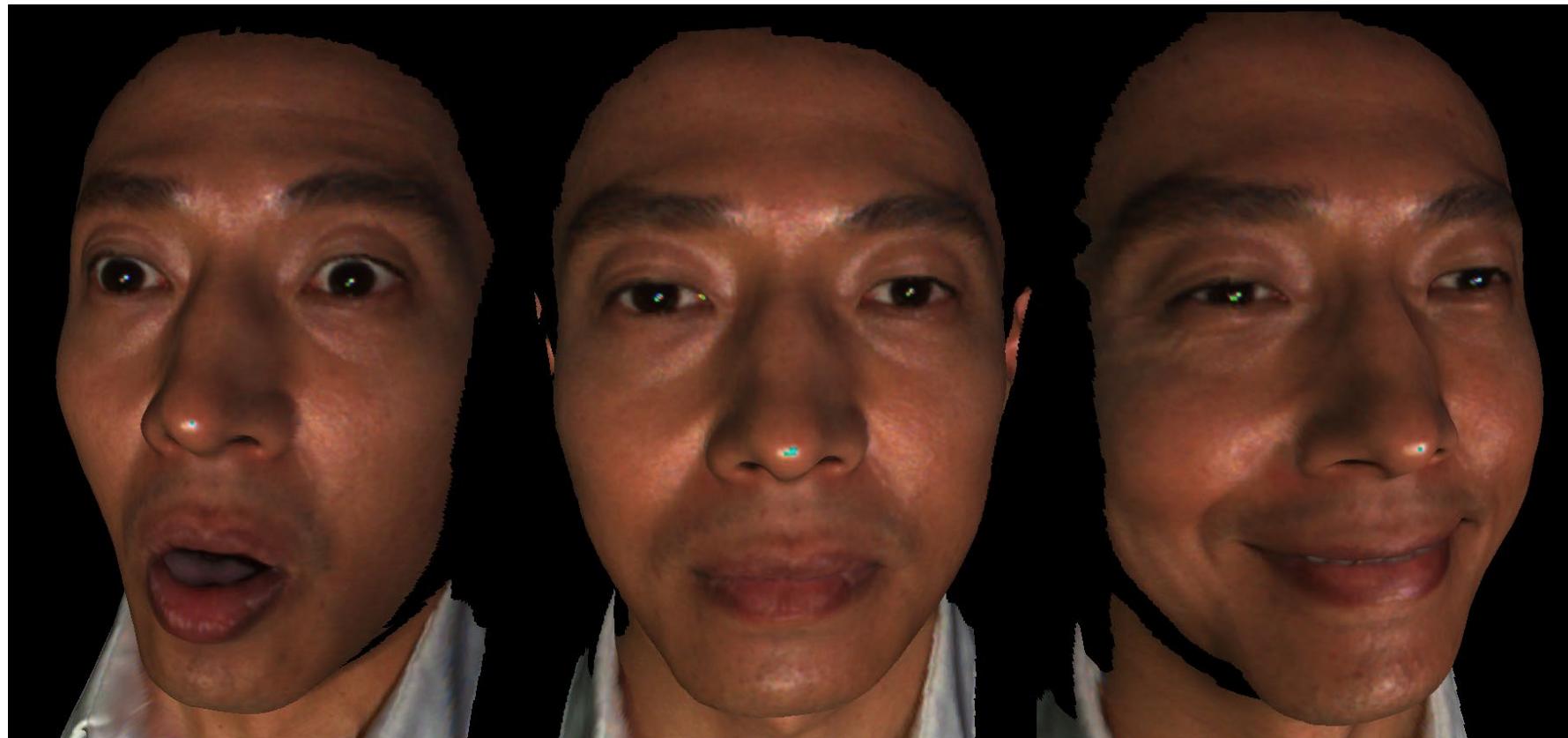


Capture 3D facial geometry with Dynamic Expression

- High speed (60fps)
- High resolution
- High accuracy (0.2mm depth)



Capture High Resolution Skin Color & Texture



Capture High Resolution Skin Color & Texture

- Texture, albedo



Conformal Geometry

Conformal Geometry

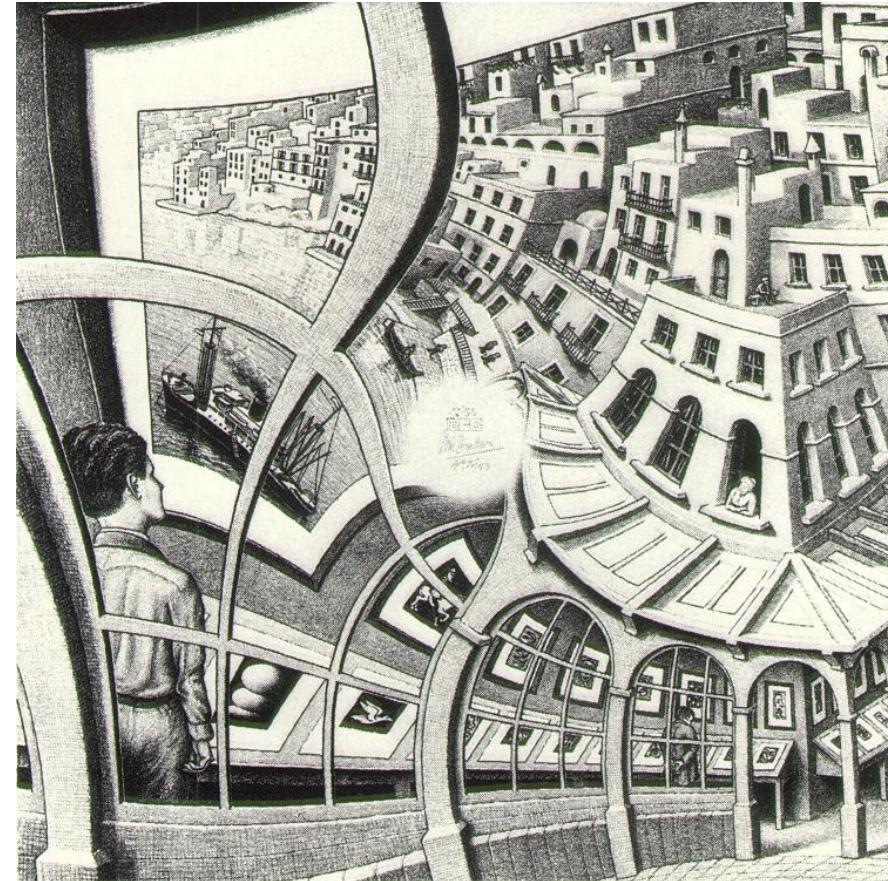
Problem: How to handle huge amount of dynamic 3D geometric data with large deformations ?

Answer: Computational conformal geometry !

- Combine differential geometry, algebraic topology, partial differential equations with computational geometry, numerical PDEs
- Good at shape classification, surface registration, dynamic geometric tracking, shape analysis and so on
- (Collaborated with Prof. Shing-Tung Yau, Prof. Feng Luo et al)

M.C. Escher Gallery 1956

- The virtual world and the real word are mixed together.



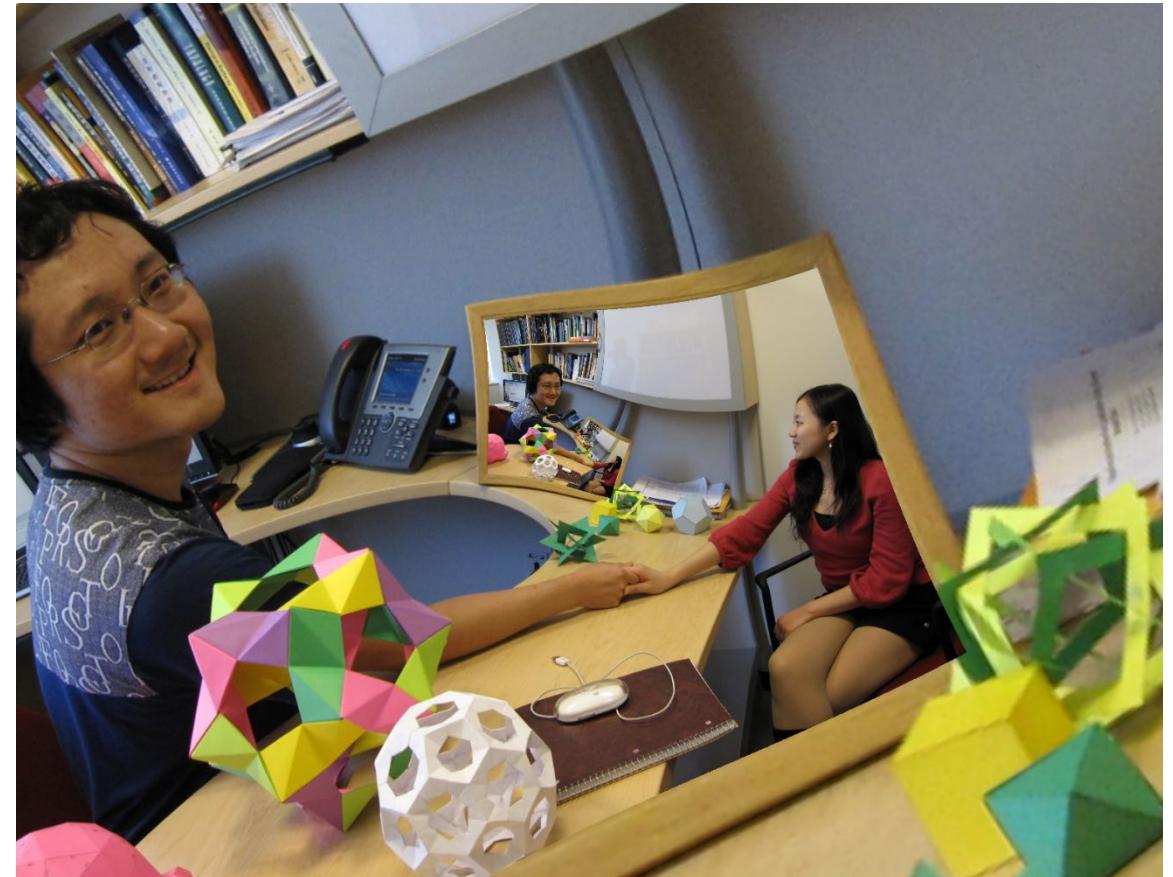
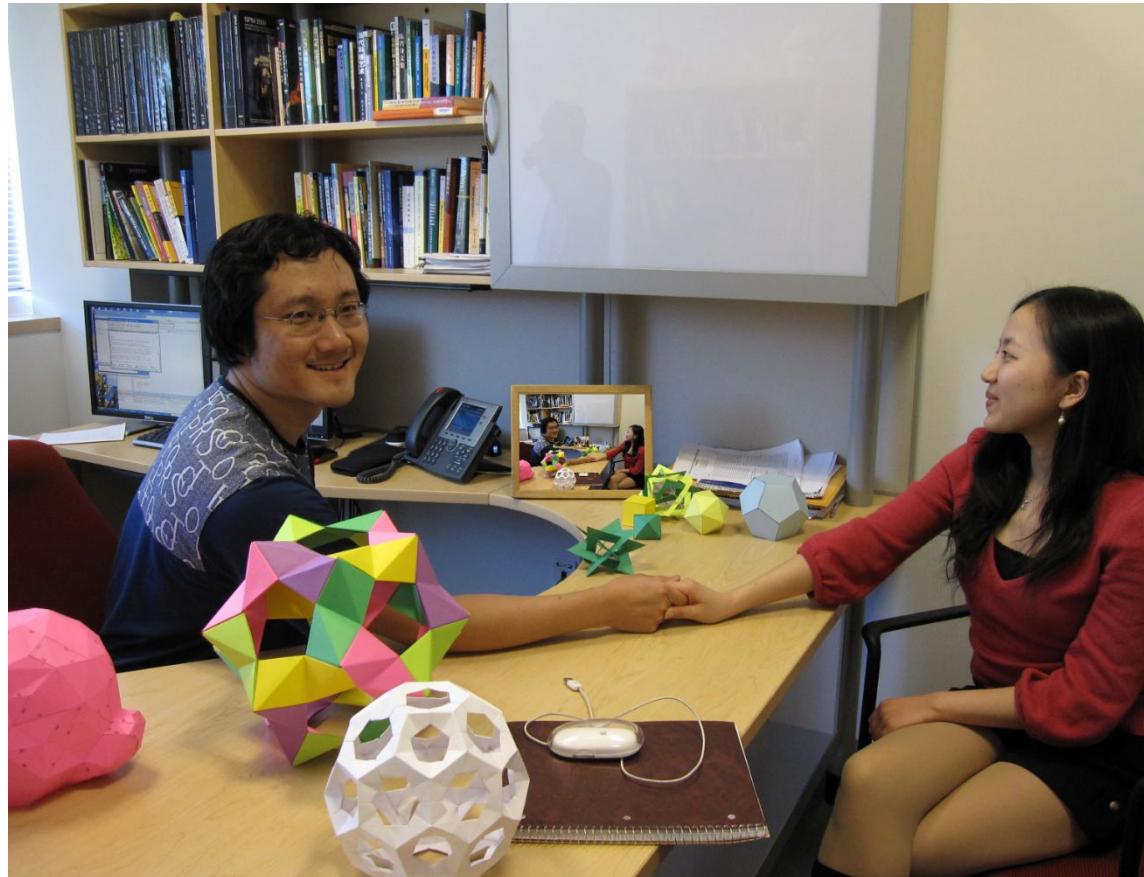
Planar Conformal Mapping

- Preserves local shapes, changes the global topology.



Planar Conformal Mapping

- Preserves local shapes, changes the global topology.



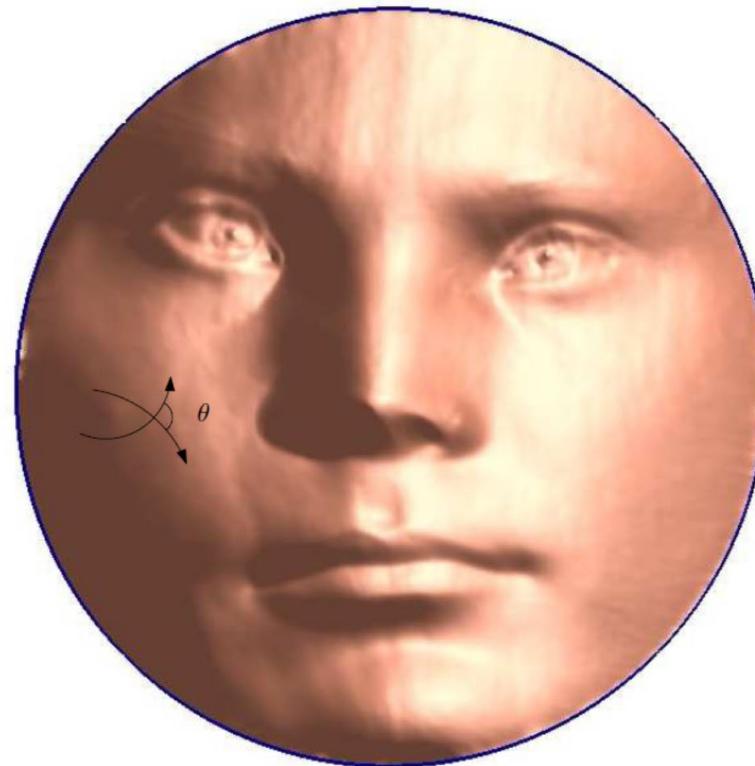
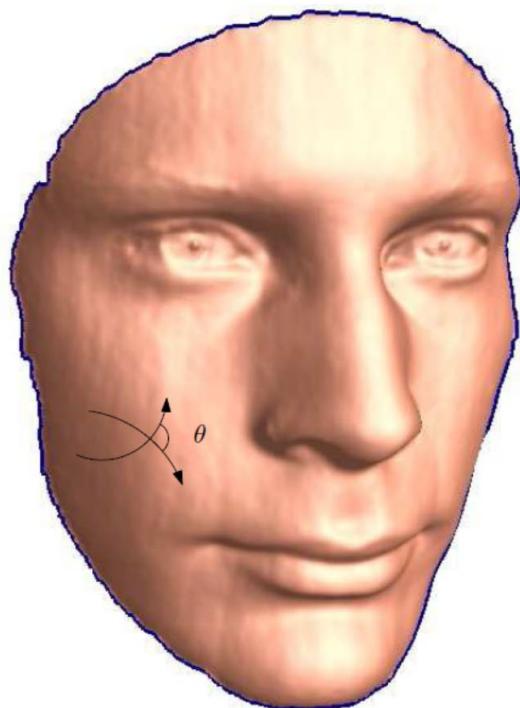
Surface Conformal Mapping

- Convert 3D geometry to 2D image, curved surfaces to flat plane.



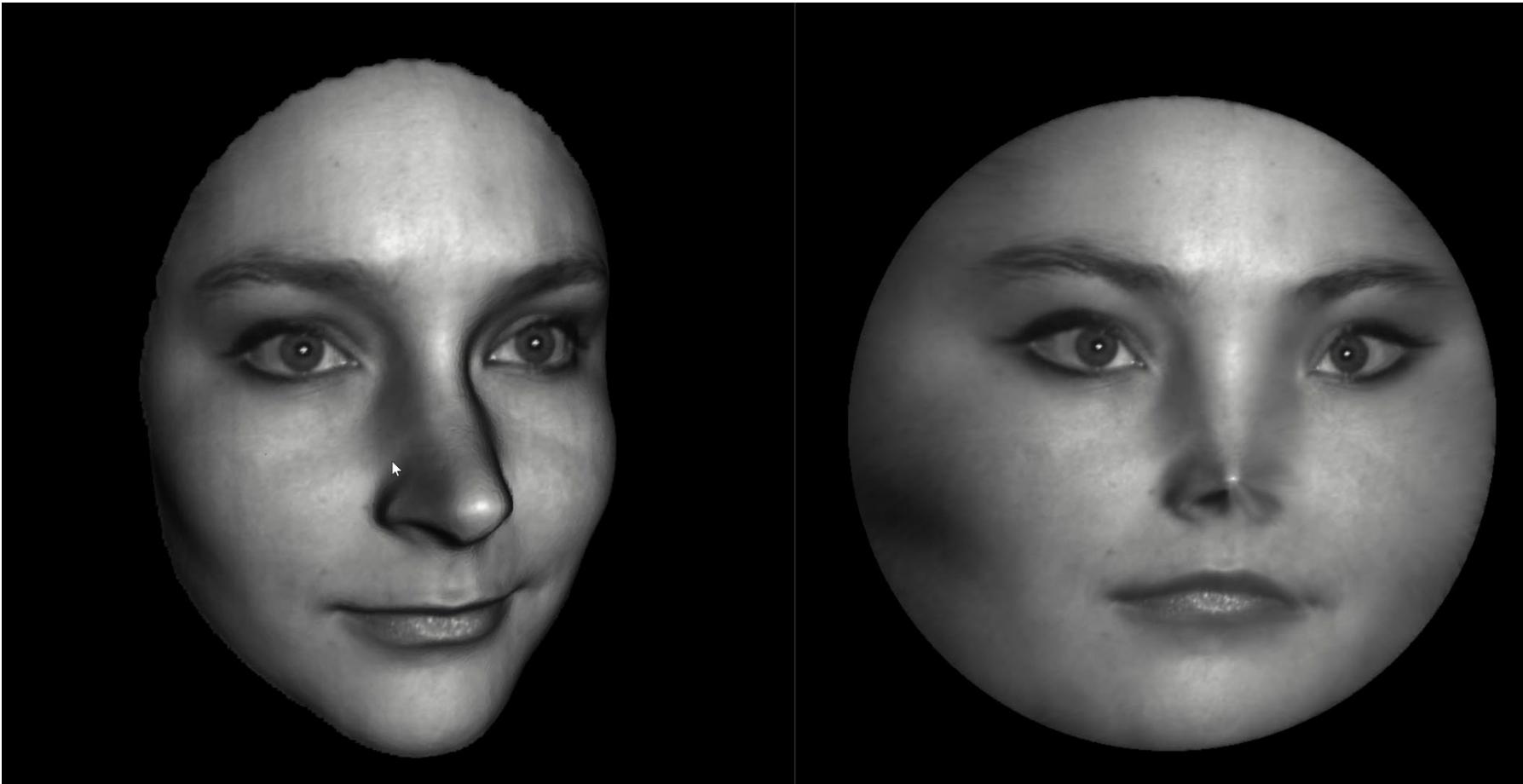
Conformal Mapping

- Angle preserving



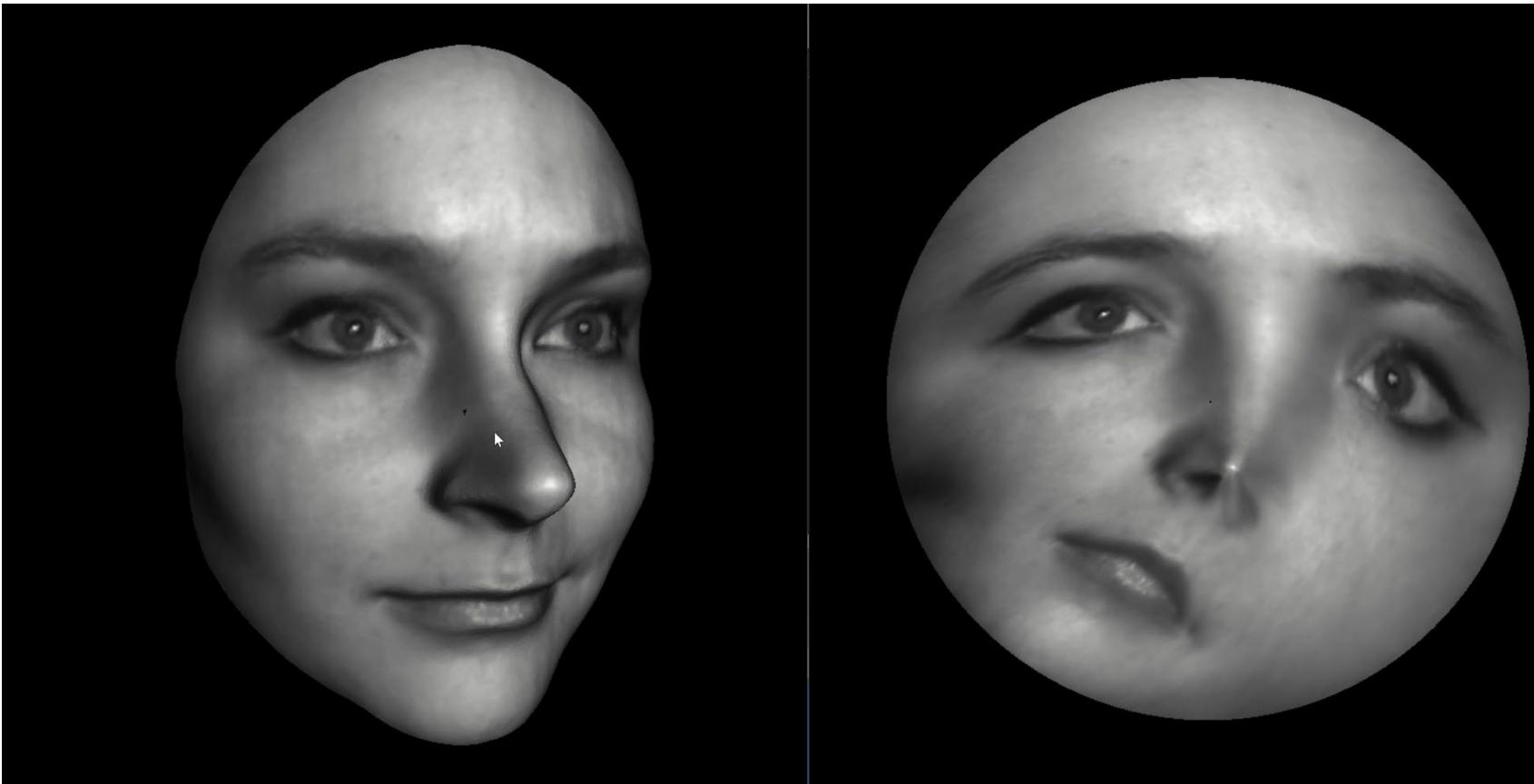
Conformal Mapping

- Preserve local shapes, conformal mapping.



Quasi-Conformal Mapping

- Maps infinitesimal ellipses to infinitesimal ellipses



Discrete Surface Ricci Flow

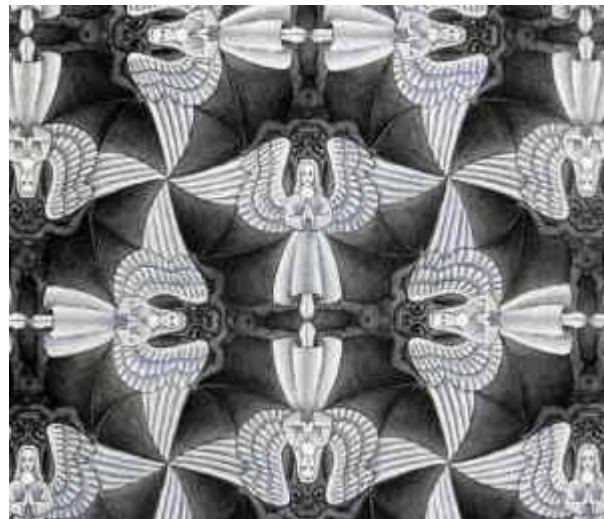
Problem: How to compute conformal mappings ?

Answer: Harmonic Map, Holomorphic Differential, Ricci flow

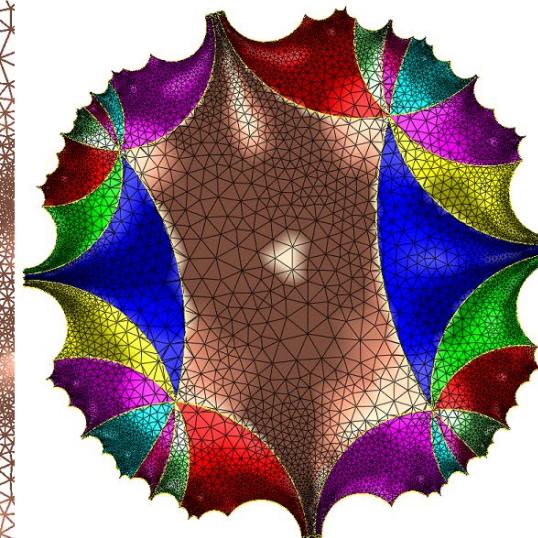
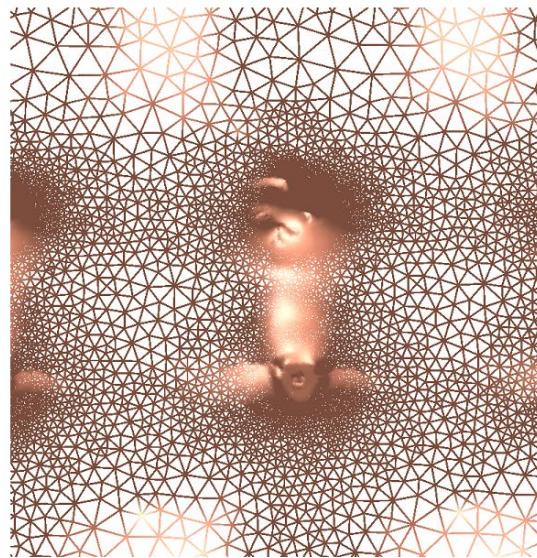
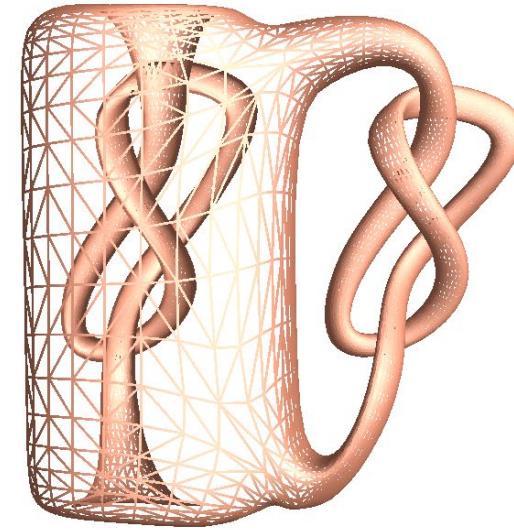
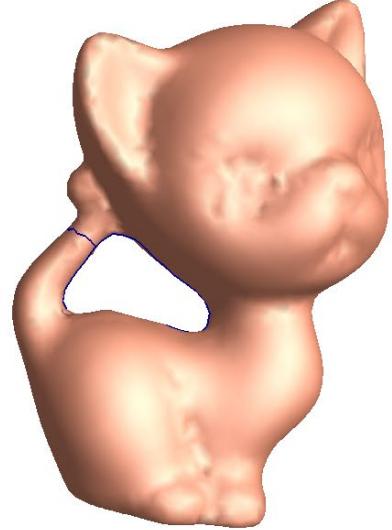
- Harmonic map minimizes the elastic deformation energy;
- Holomorphic differentials is based on Hodge theory, it finds harmonic vector fields on surfaces;
- Ricci flow is invented by Hamilton, used by Perelman et al. to prove Poincare conjecture. It can design Riemannian metrics from prescribed curvatures;

Escher Angels and Devils

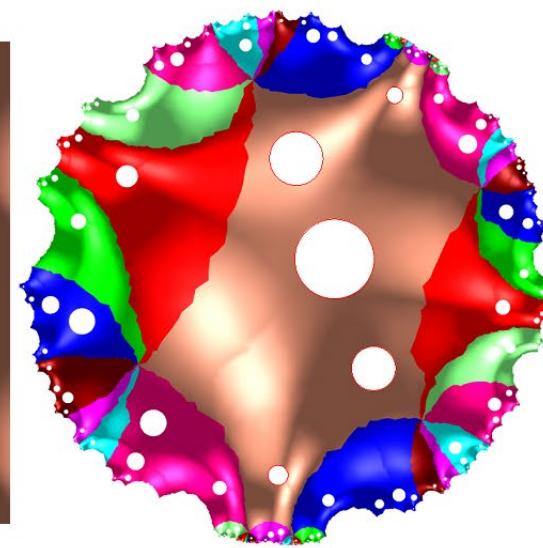
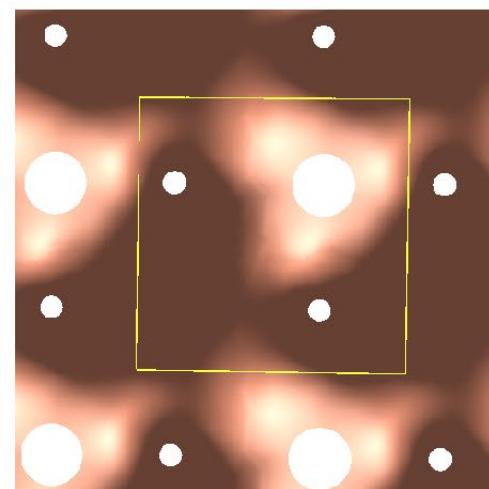
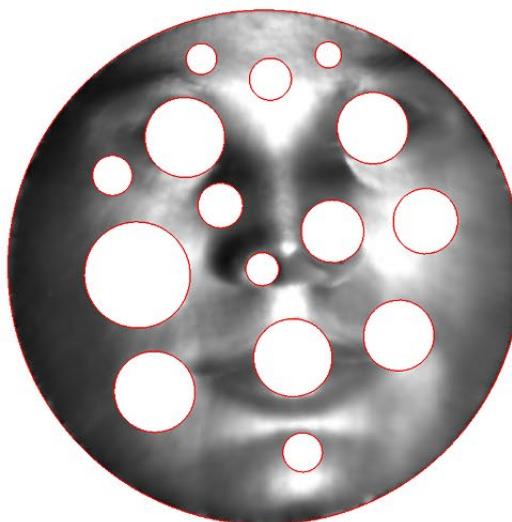
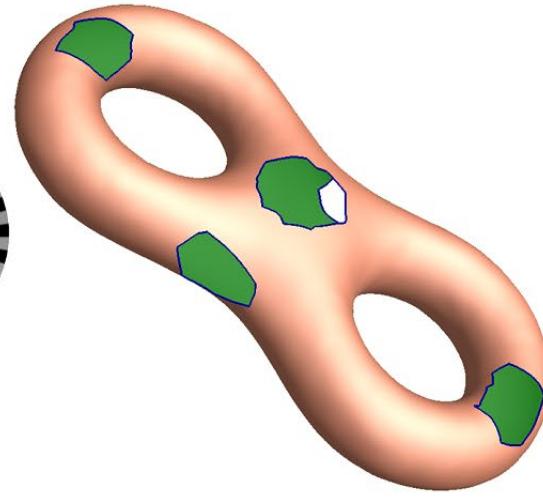
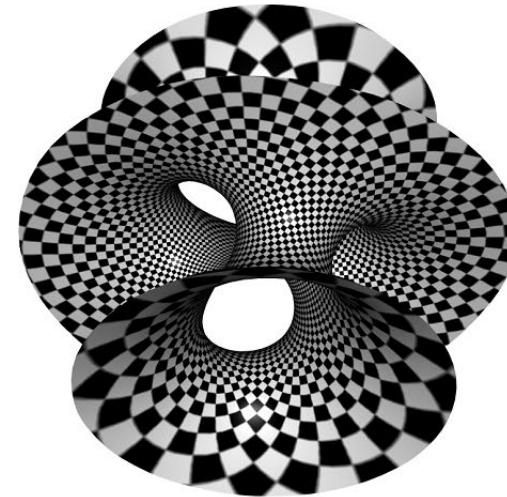
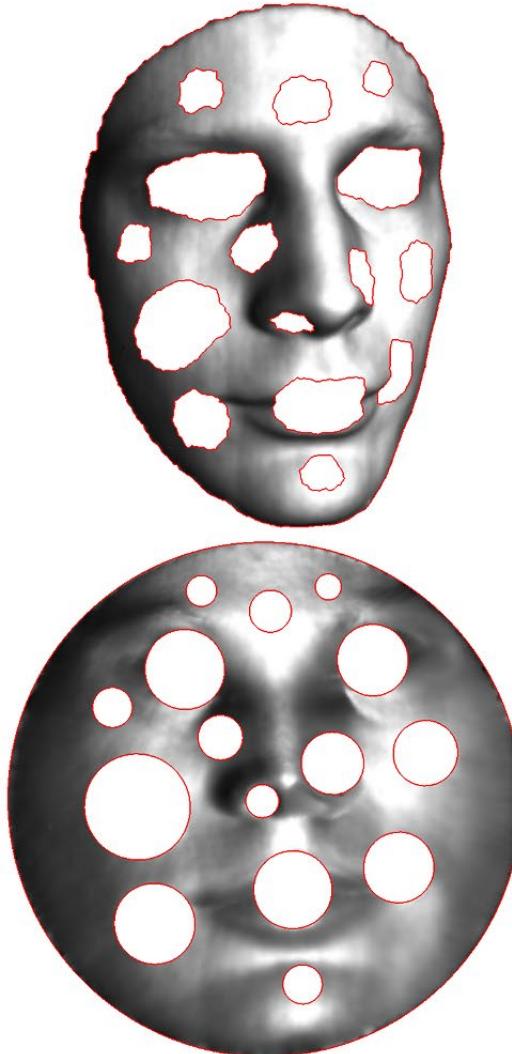
- Spherical, Euclidean and Hyperbolic Geometry



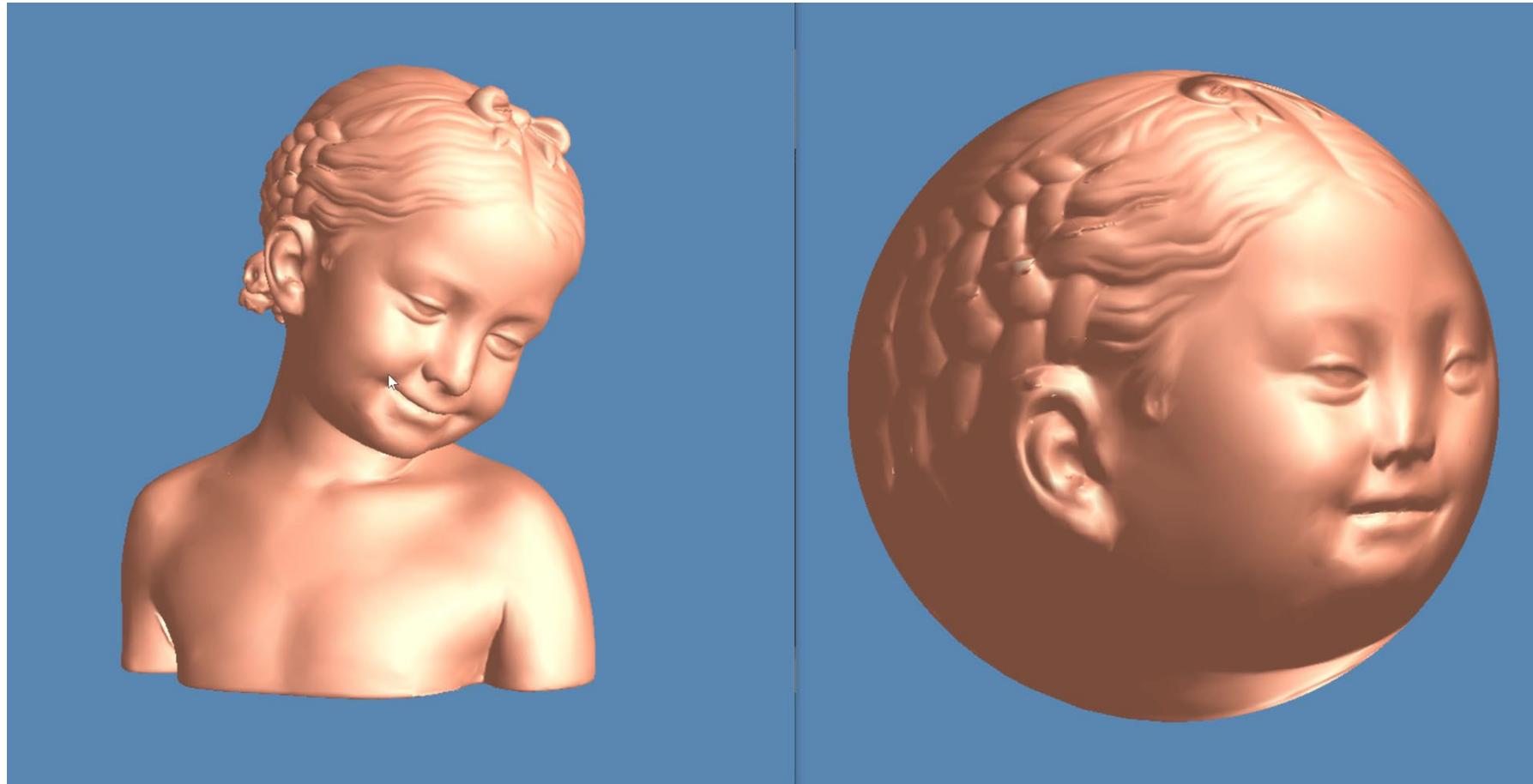
Surface Uniformization Theorem



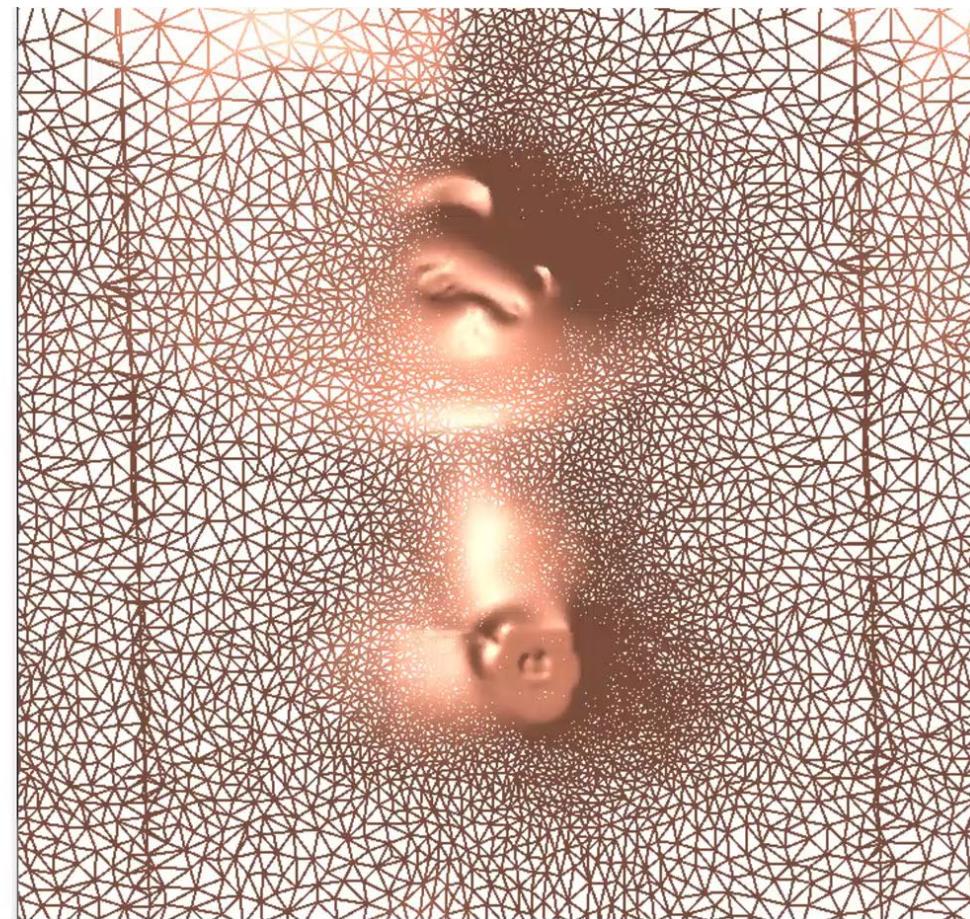
Surface Uniformization Theorem



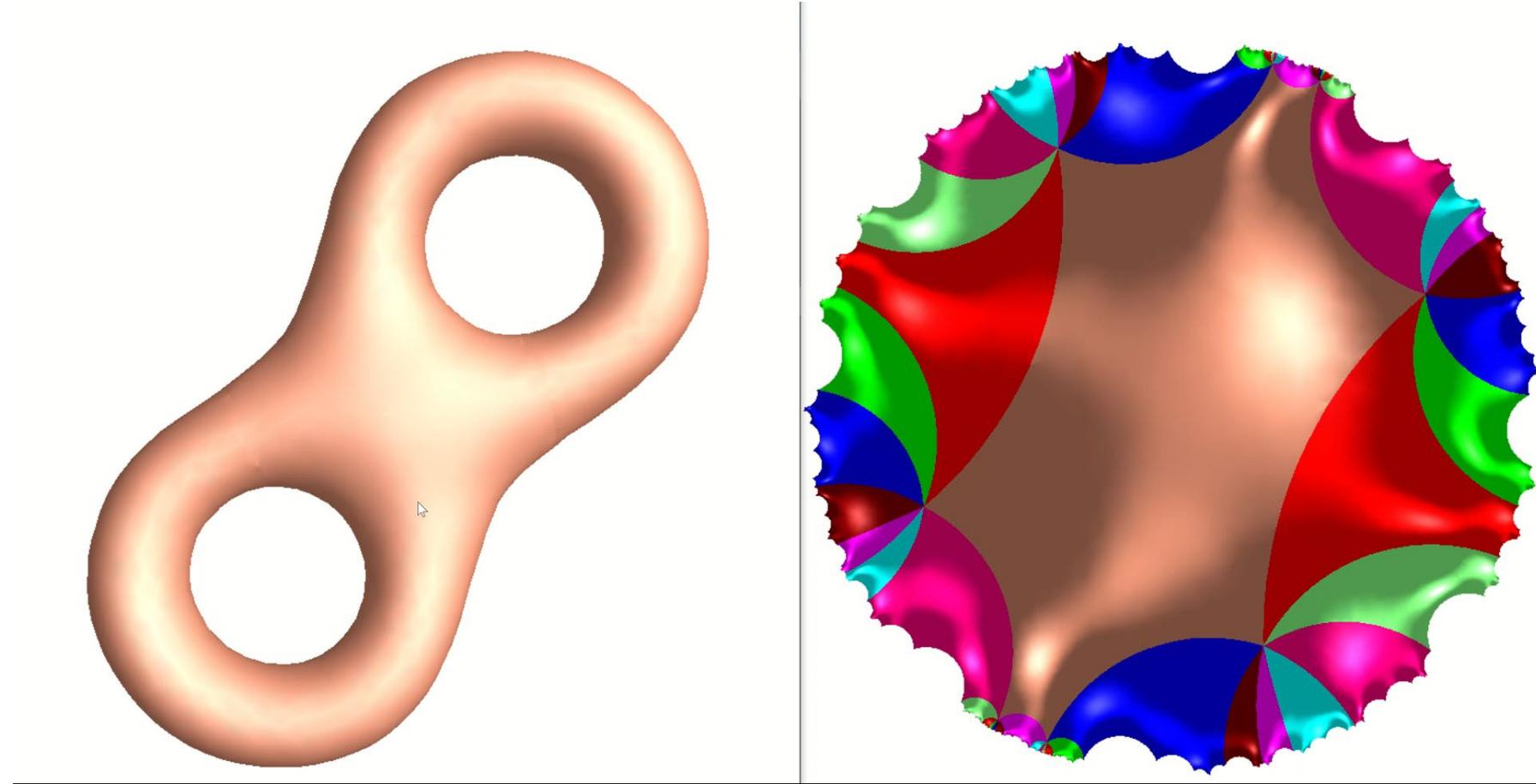
Genus Zero Surface: Harmonic map method



Genus One: holomorphic differential method



High Genus: Ricci flow method



Computer Graphics

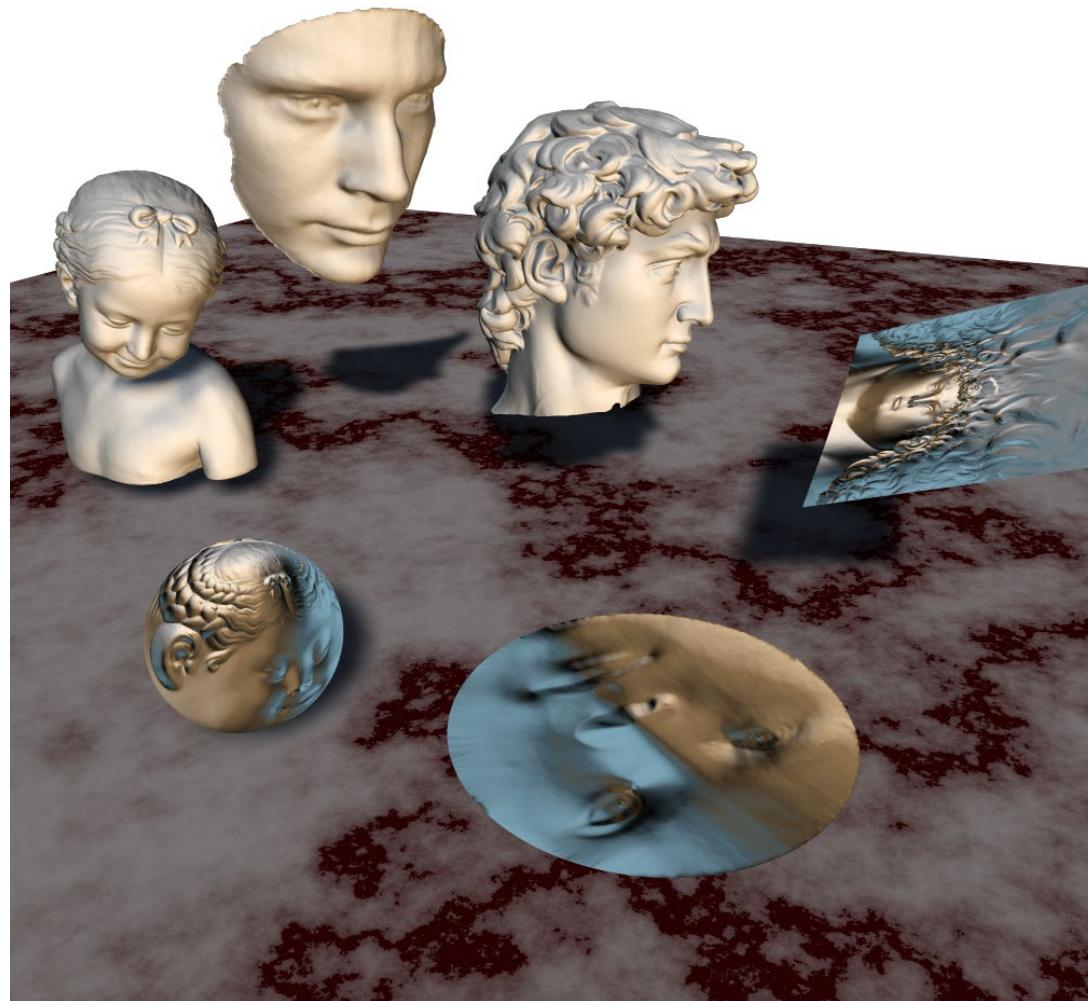
Global Surface Parameterization

Problem: How to parameterize general surfaces without partitioning and with minimal distortions ?

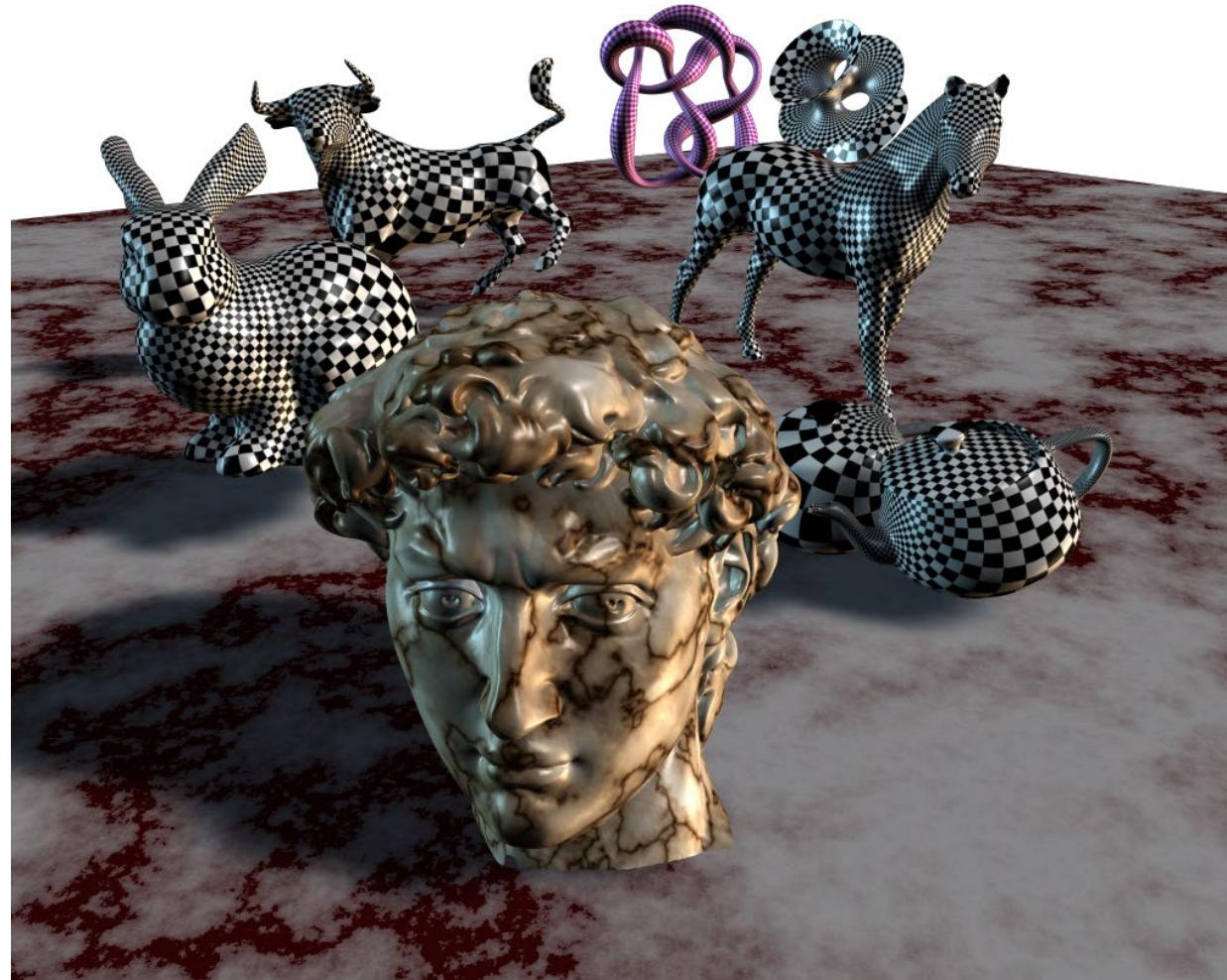
Answer: Global surface parameterization, angle preserving, area preserving

- Conventional methods are local parameterization, one has to decompose the surface into patches. Our method is global parameterization without decomposition
- Minimize the distortions induced by a parameterization, conformal parameterization preserves angle, optimal transportation maps preserve area-element.

Surface Texture Mapping - Parameterization



Surface Texture Mapping - Parameterization

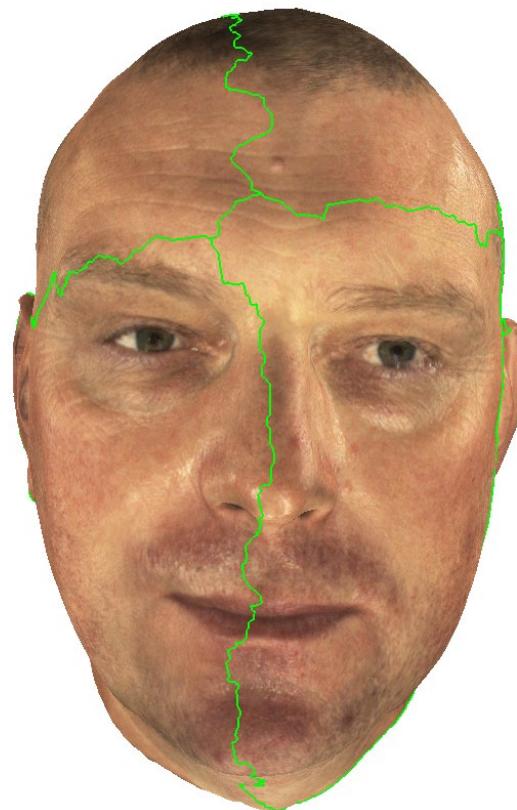
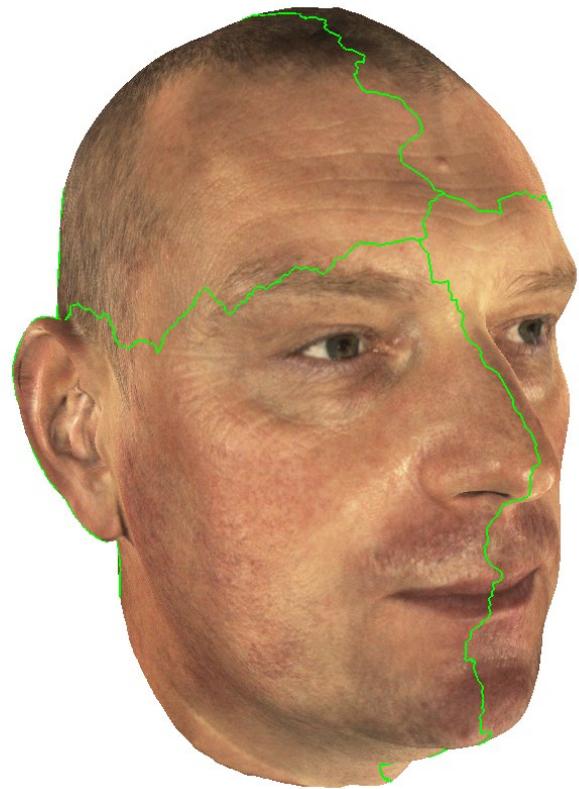


Virtual Makeup

- Dynamic change makeup.



Traditional Local Parameterization



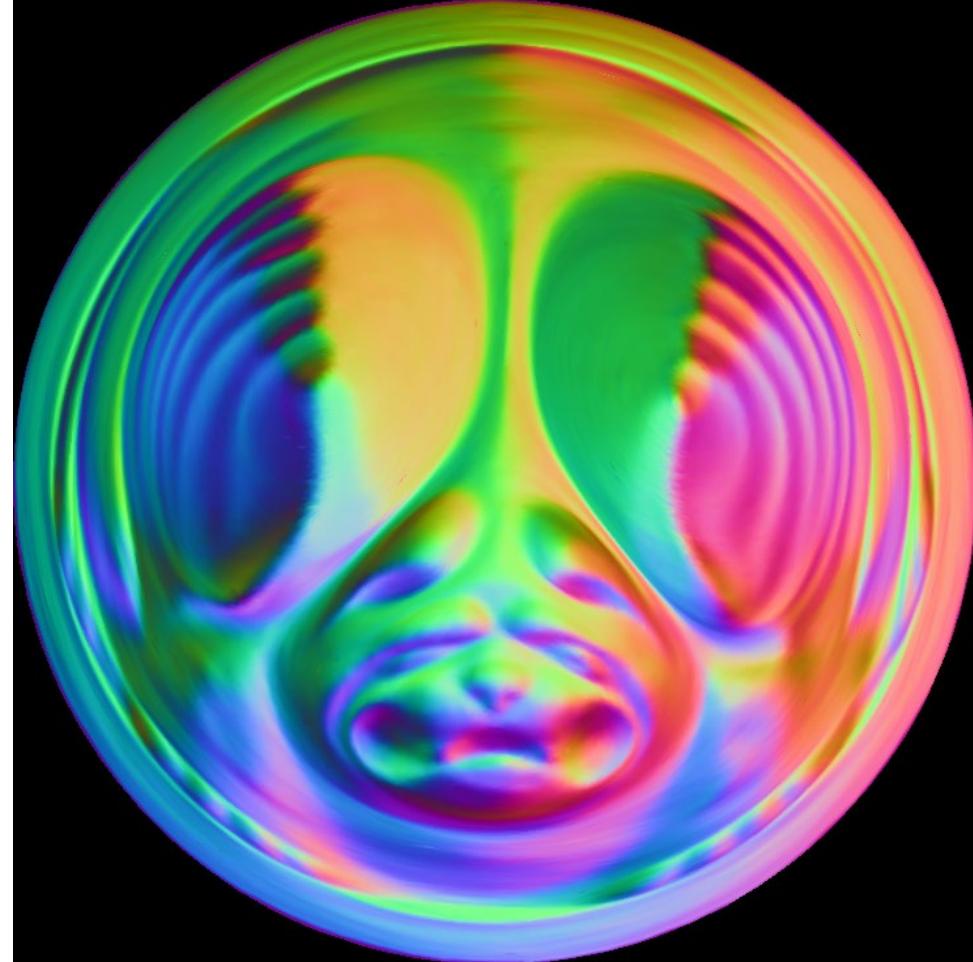
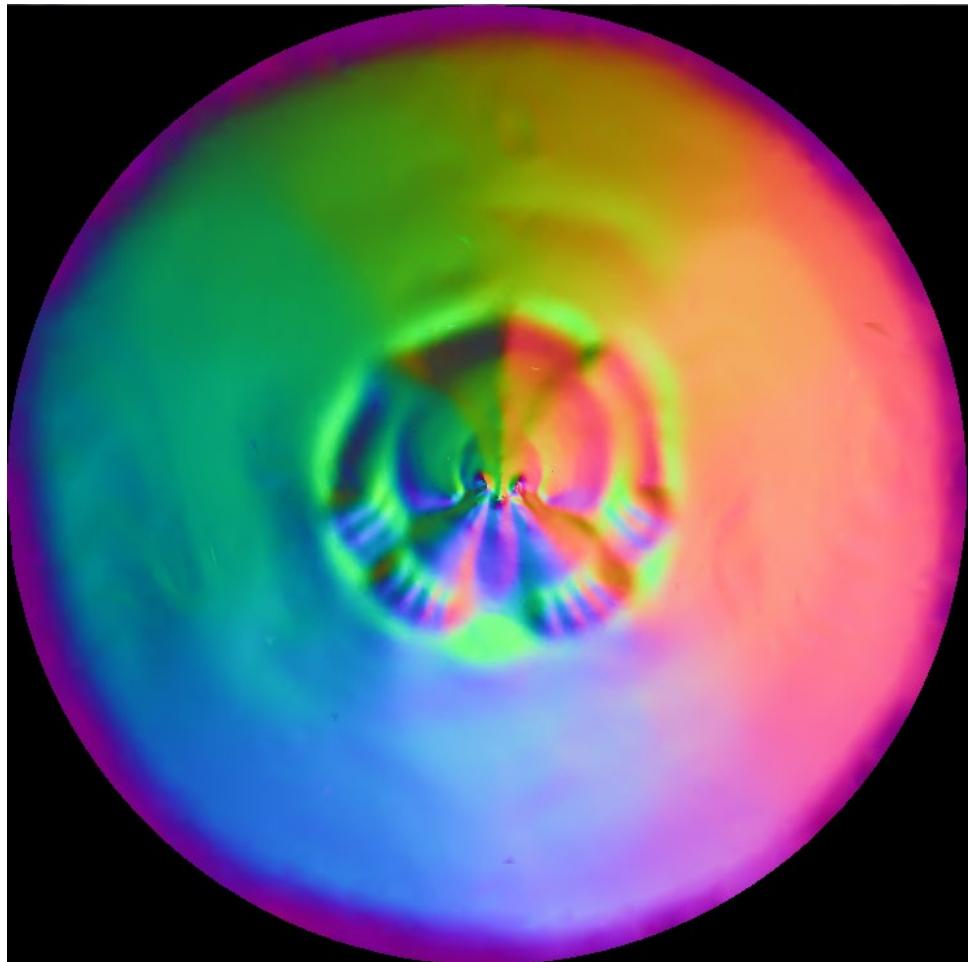
Gobal Surface Parameterization



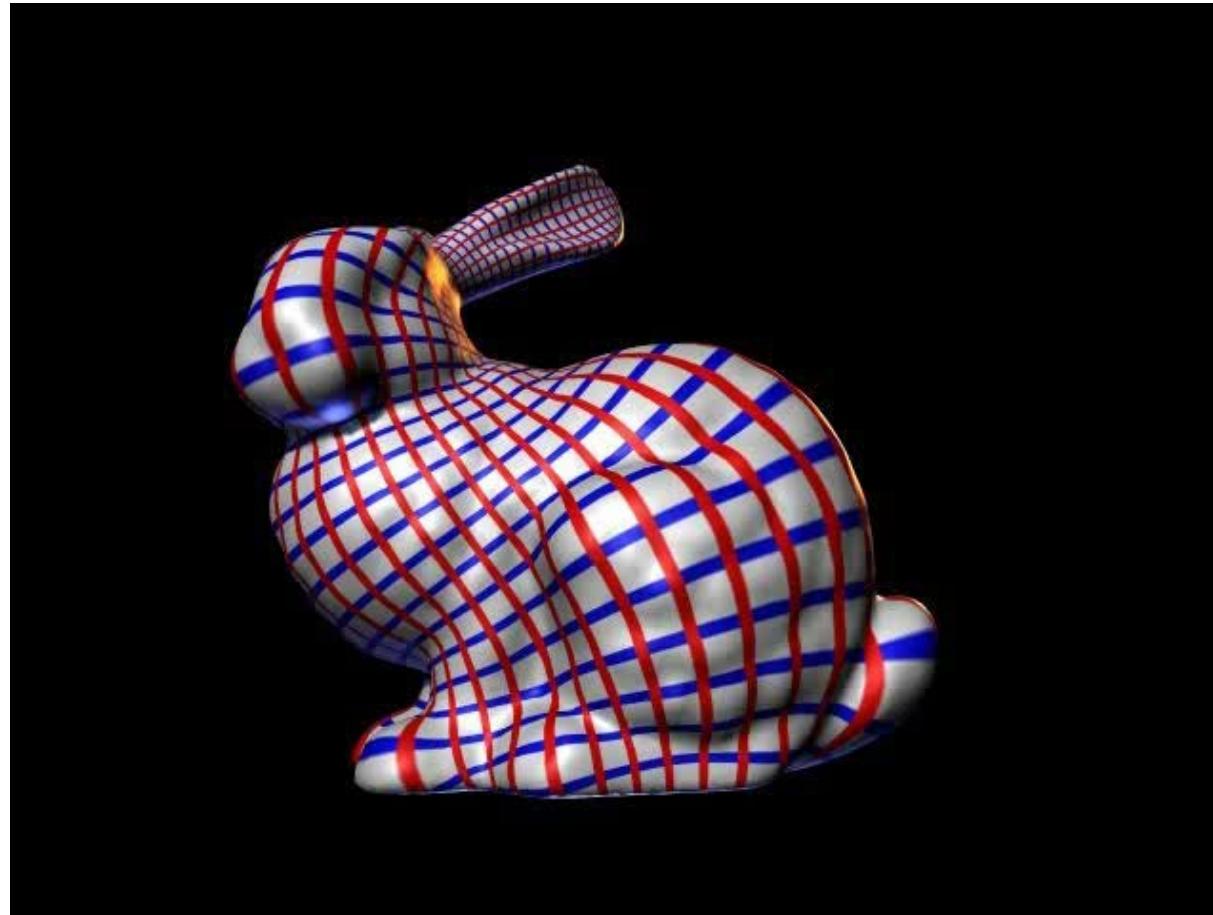
Surface Normal Mapping



Angle-preserving vs. Area-preserving

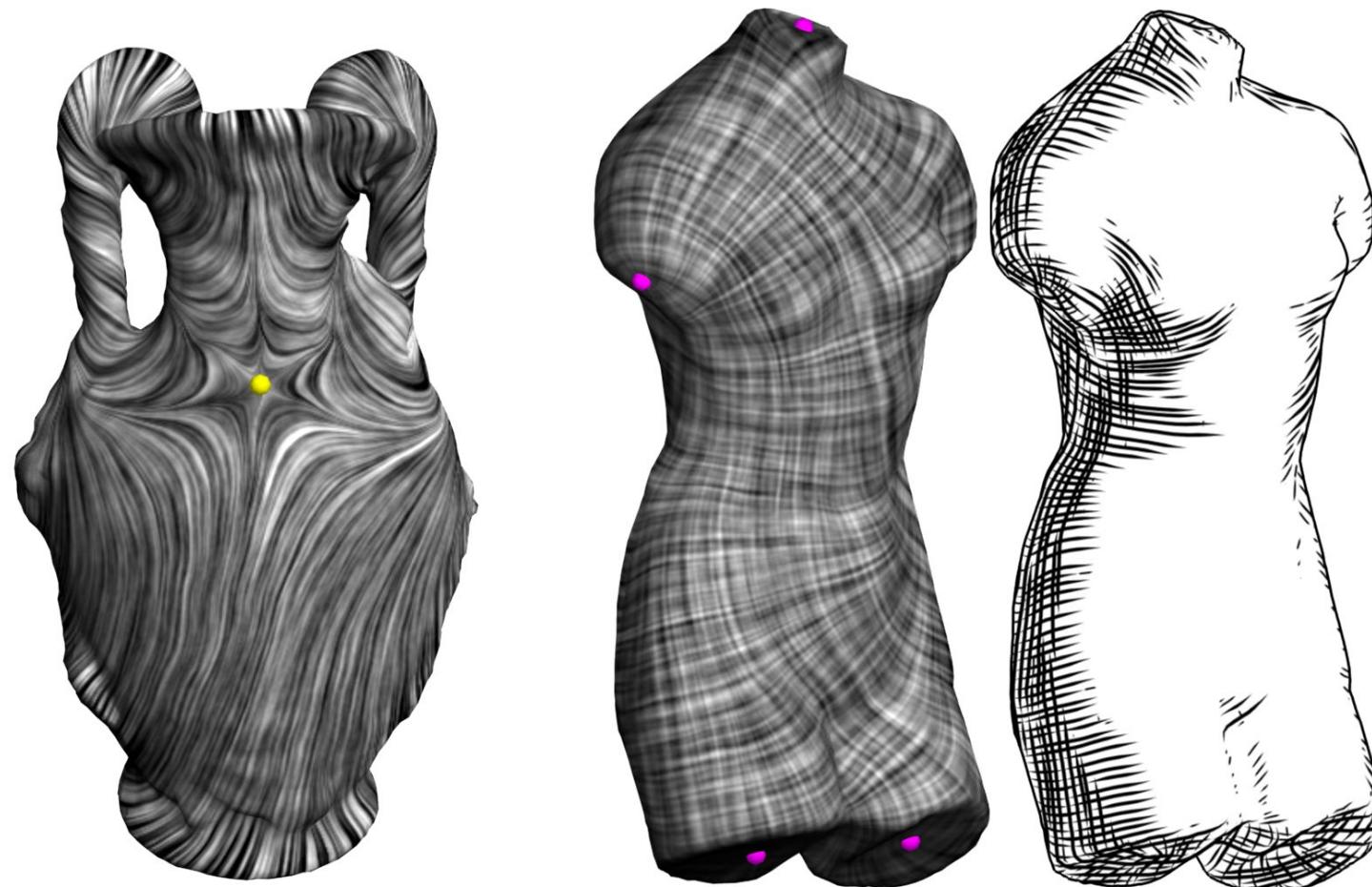


Global conformal surface parametrization



Smooth vector field design

- by prescribed singularities, with positions and indices.



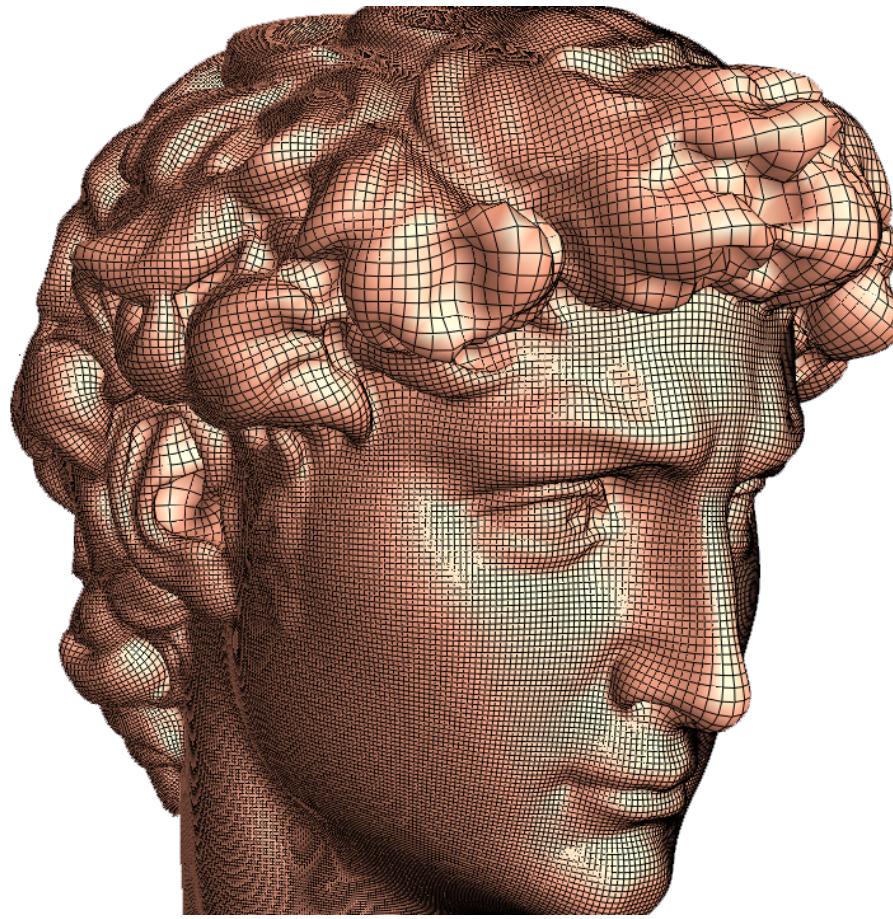
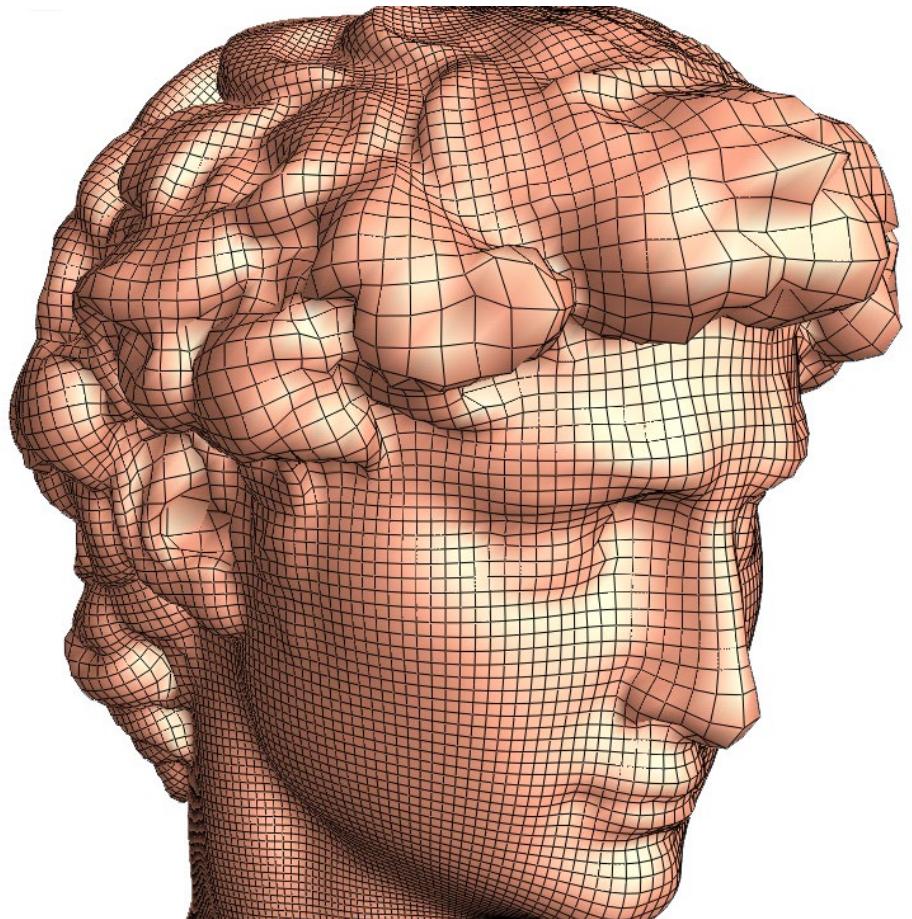
Geometry Image



Geometry Image



Geometry Image



Unreal 5 Nanite virtual micropolygon geometry



Computer Vision

Surface registration and tracking

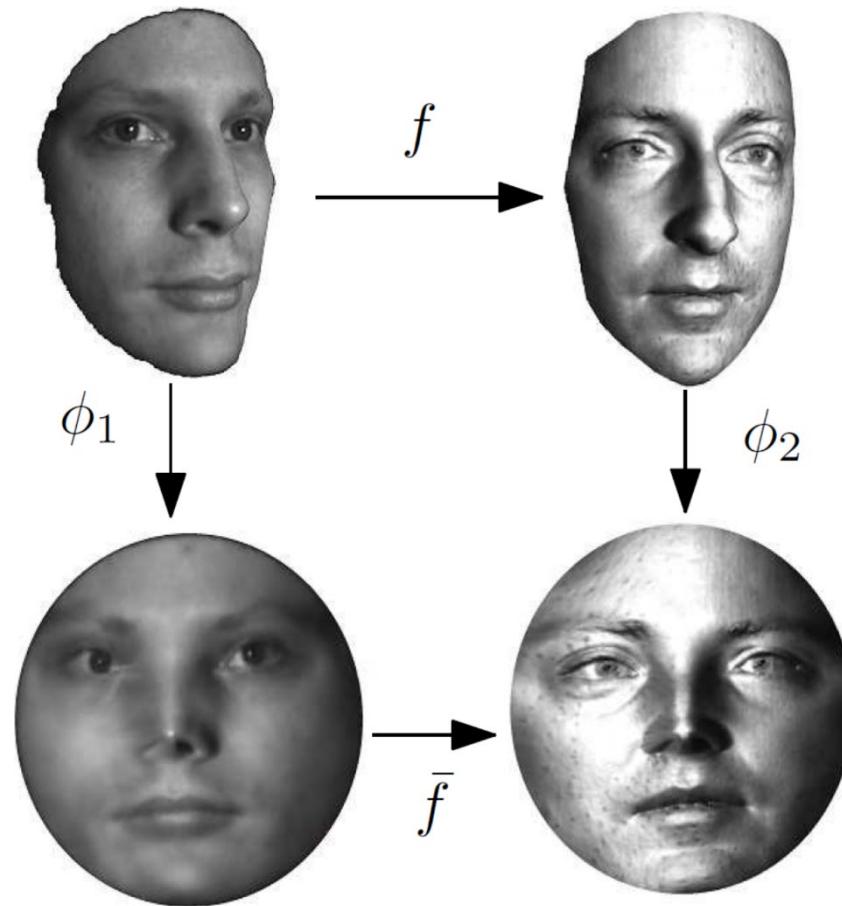
Problem: How to register 3D surfaces with large deformations, how to track dynamic surfaces with feature constraints and minimal distortions.

Answer: Conformal mapping, Teichmuller map based on Quasi-conformal geometry.

- Conformal mapping converts 3D surface registrations to 2D image registrations
- Teichmuller map minimizes local shape distortions

3D Surface Registration

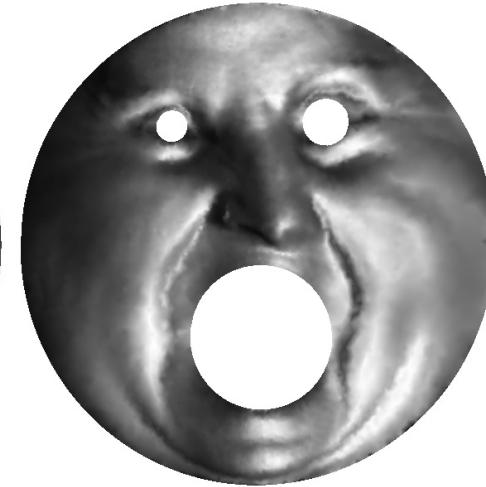
- 3D Matching is carried out by 2D matching.



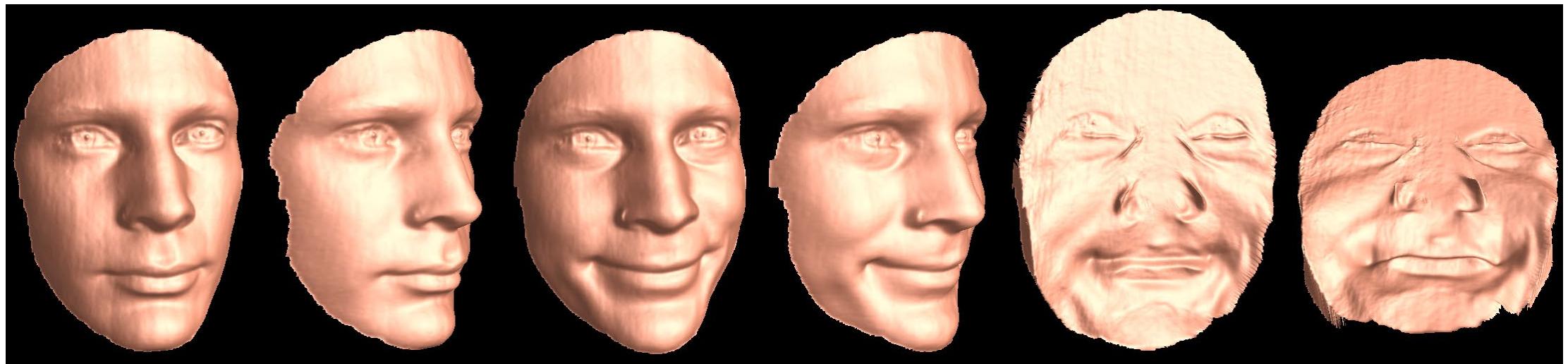
3D Facial Geometry Matching & Comparision



3D Facial Geometry Matching & Comparision

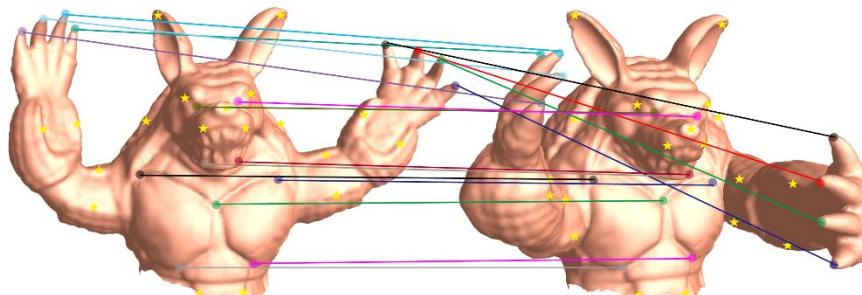


3D Facial Geometry Matching & Comparision



Surface Registration

- Non-rigid motion
- Complicated features
- Global optimality



(a) Armadillo #1

(b) Armadillo #2



(c) APP map #1



(d) APP map #2



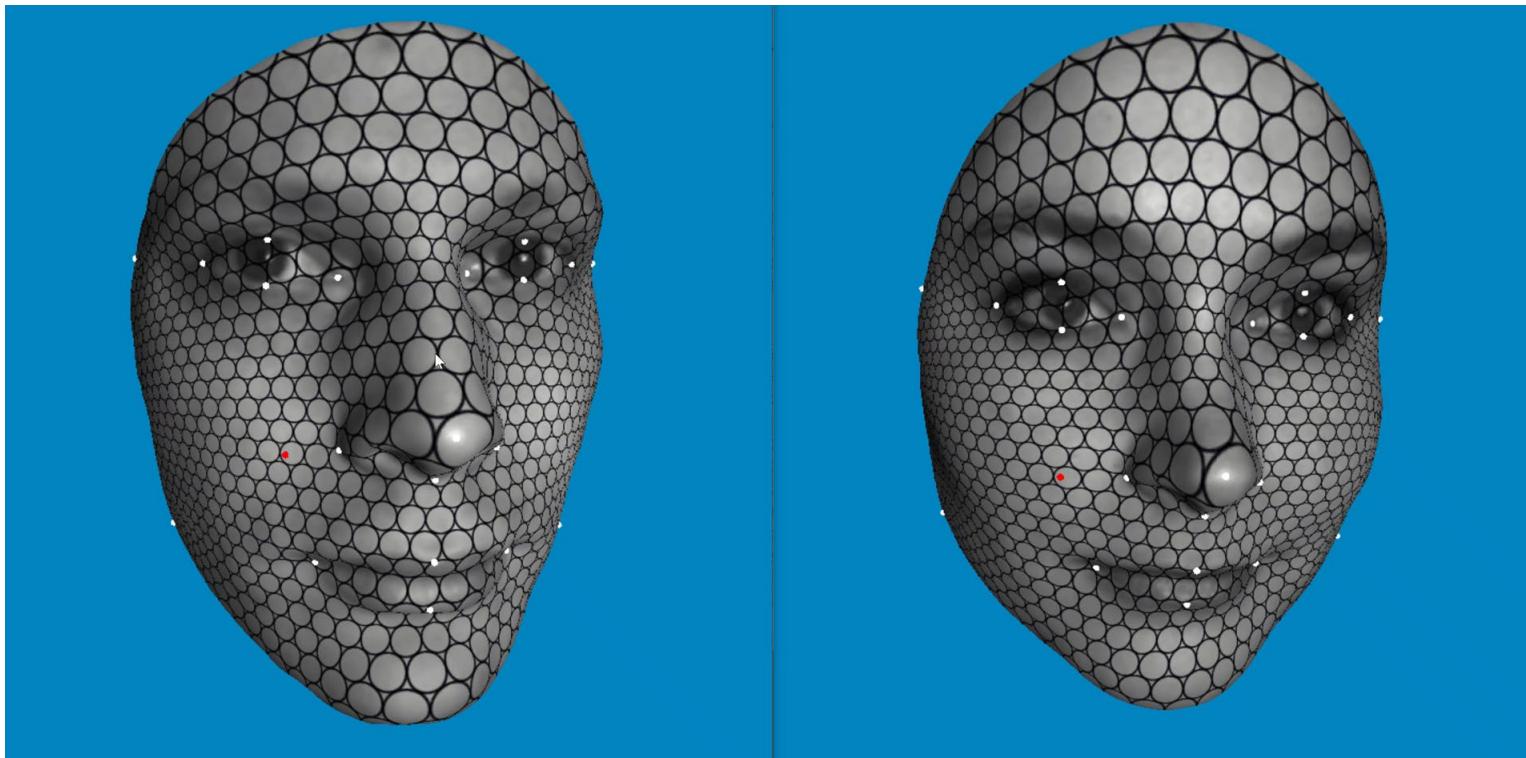
(e) Conformal map #1



(f) Conformal map #2

Teichmuller Map

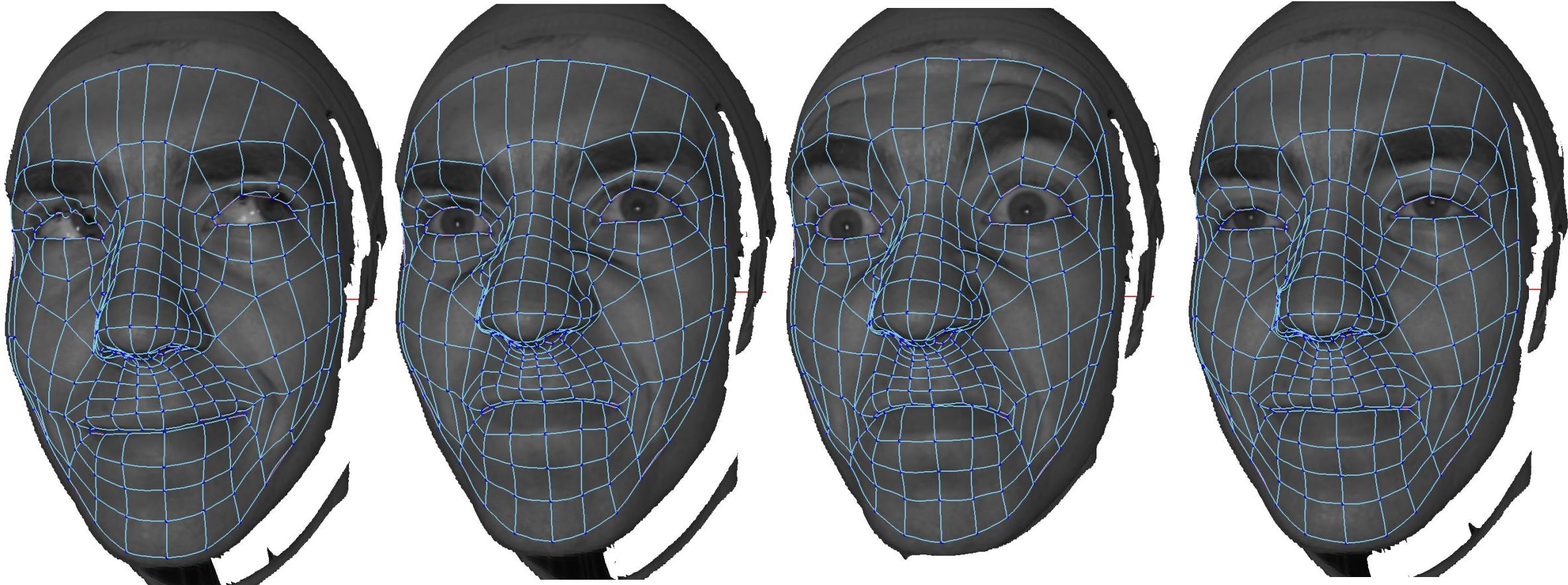
- Feature constraints; minimal angle distortion



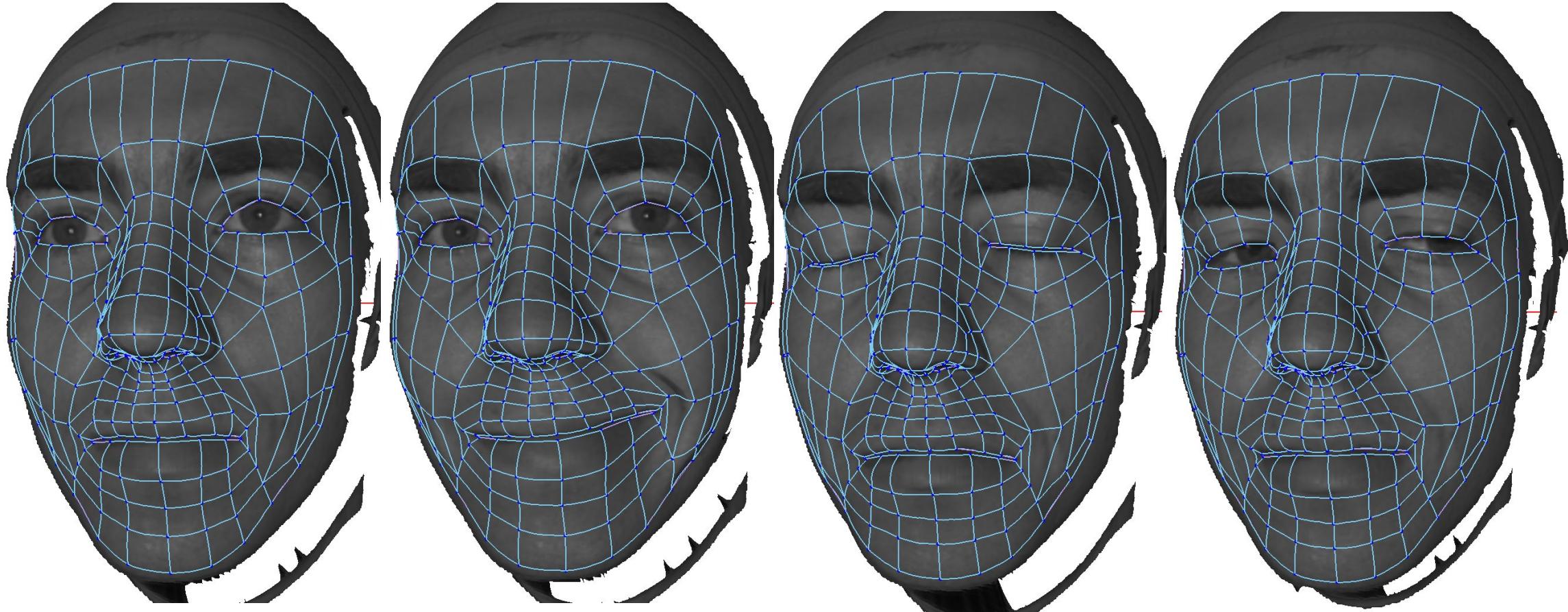
Expression Capture and Analysis



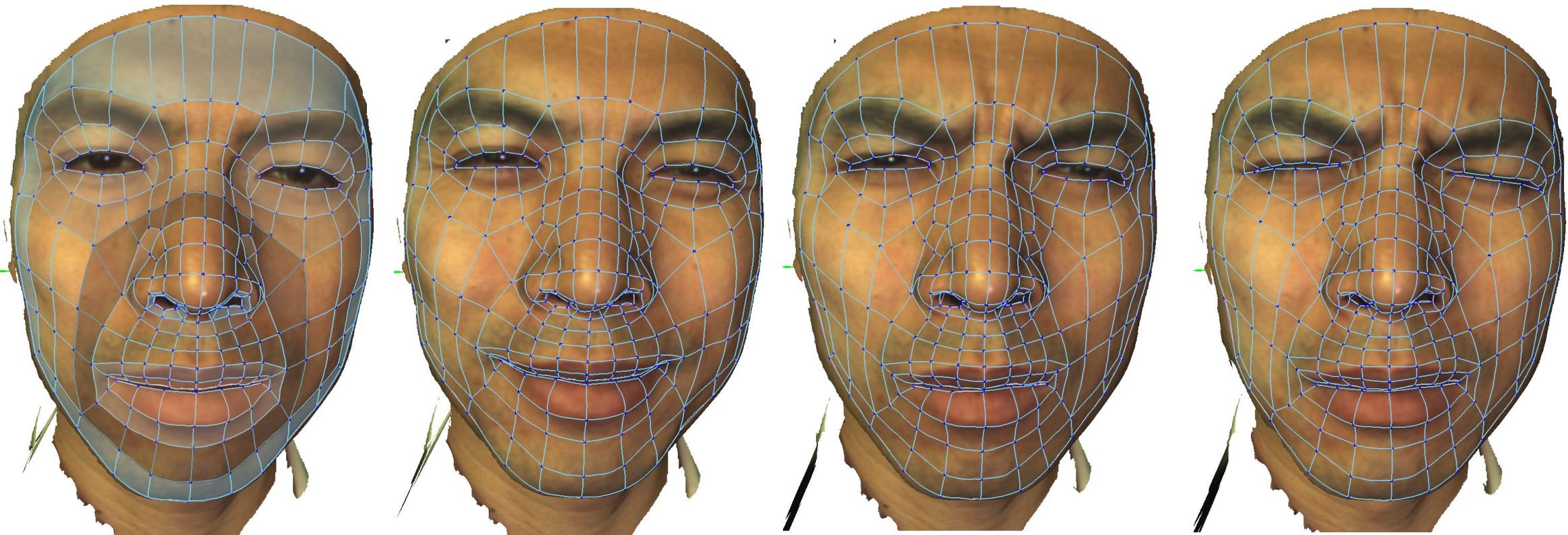
Expression Capture & Analysis



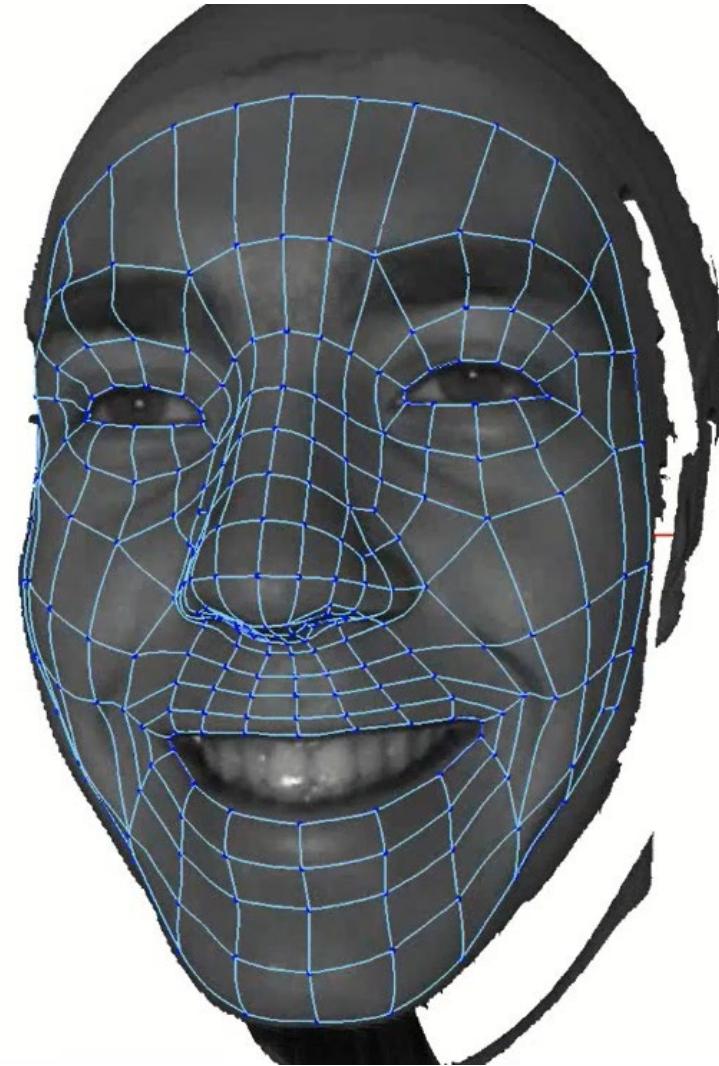
Expression Capture & Analysis



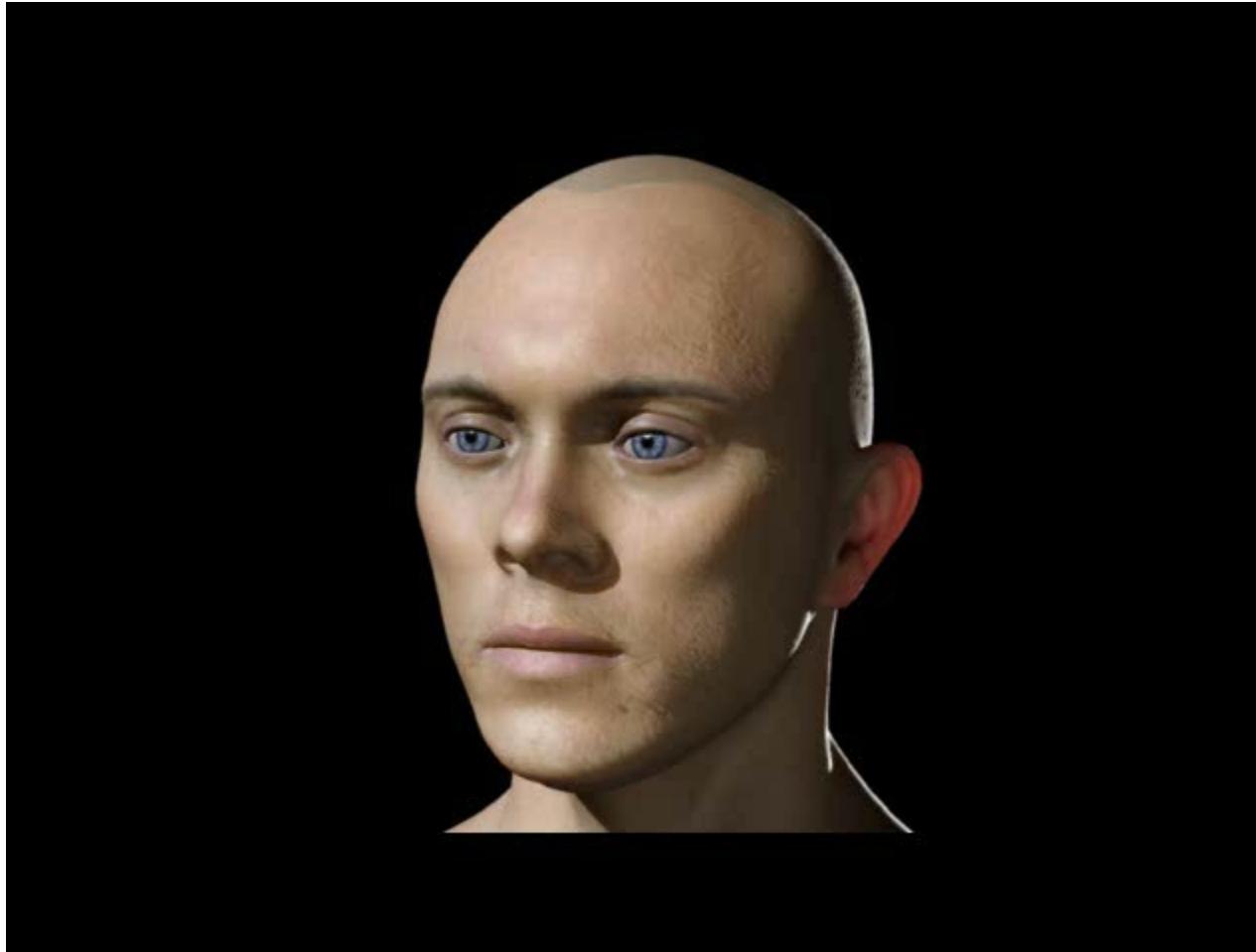
Facial Expression Tracking



Facial Expression Tracking



Virtual Actor



Virtual Actor

Property of VC Entertainment.

Not for Distribution.

Digital Geometry Processing

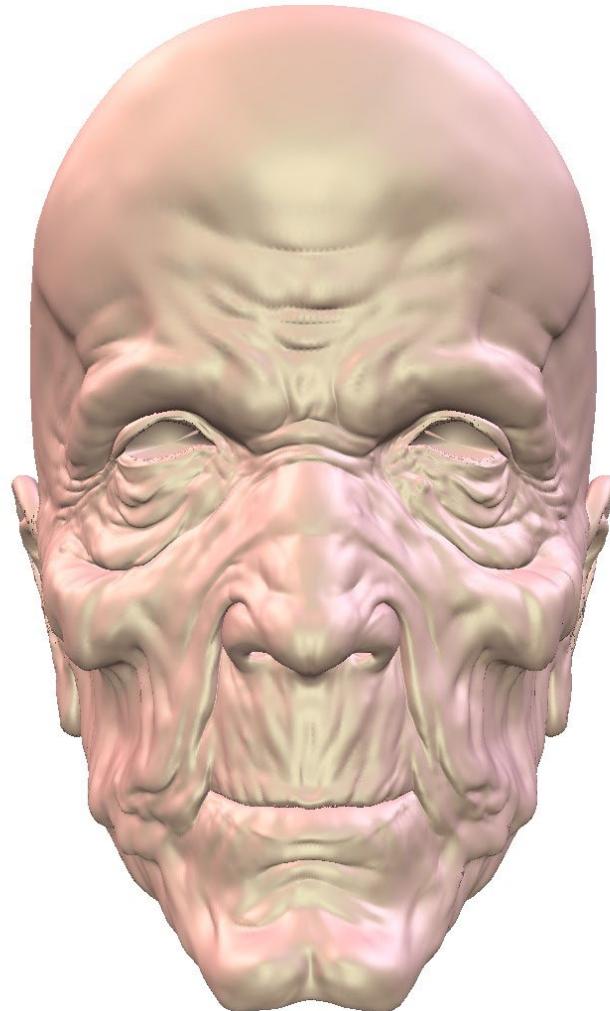
Geometric Approximation

Problem: How to use triangle mesh to approximate smooth surfaces ?

Answer: To approximate the normal cycle of the surface, not the surface itself using conformal mapping and optimal transportation map.

- Sampling on the image of the optimal transportation map
- Triangulation on the image of conformal mapping
- Guarantees the convergence of Hausdorff distance, normal field, Riemannian metric, Laplace-Beltrami operator, curvature measure

Curvature Sensitive Geometric Compression



Curvature Sensitive Geometric Compression



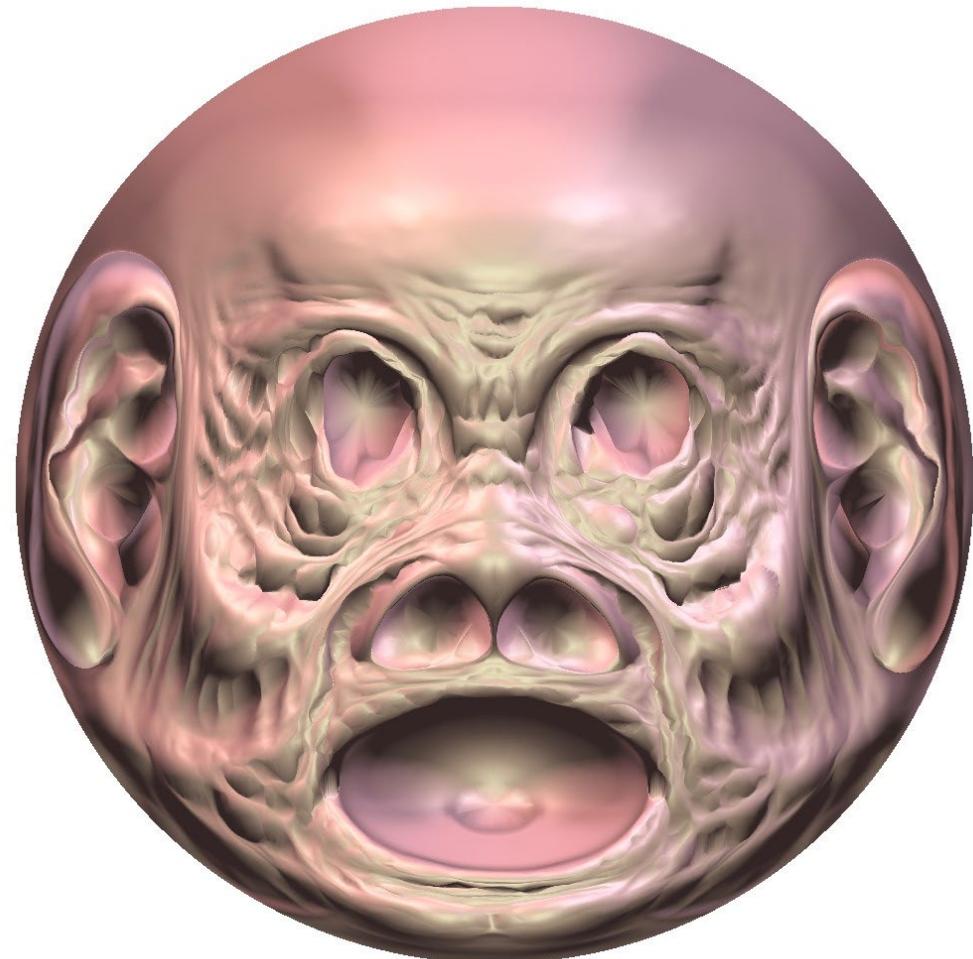
Curvature Sensitive Geometric Compression



Curvature Sensitive Geometric Compression



Curvature Sensitive Geometric Compression



Curvature Sensitive Geometric Compression



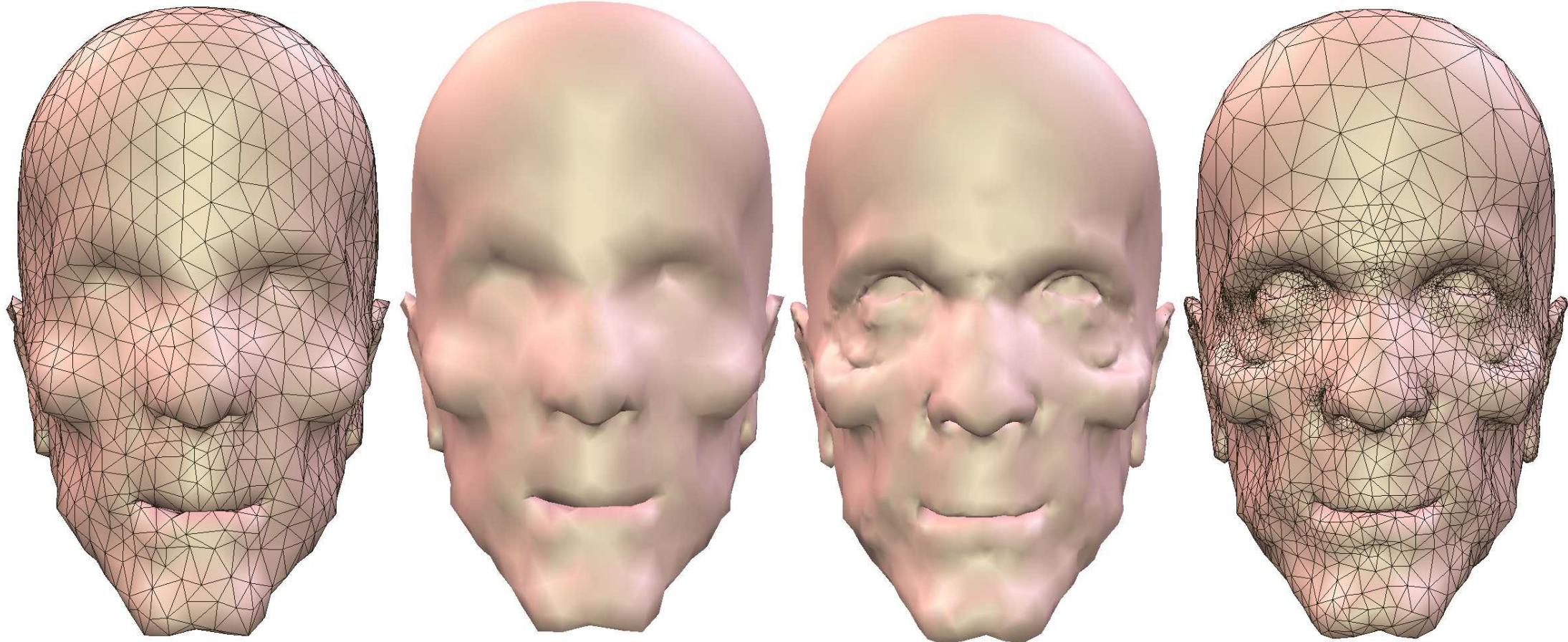
Curvature Sensitive Geometric Compression



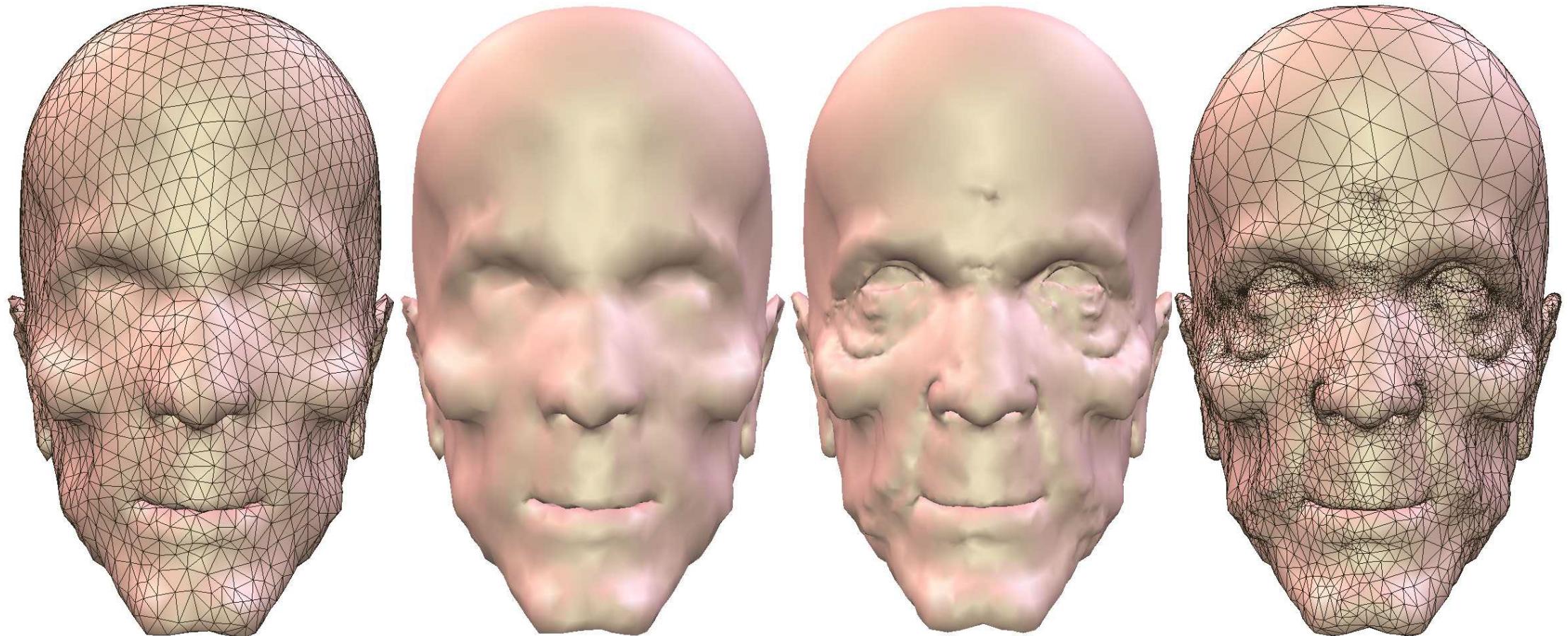
Curvature Sensitive Geometric Compression



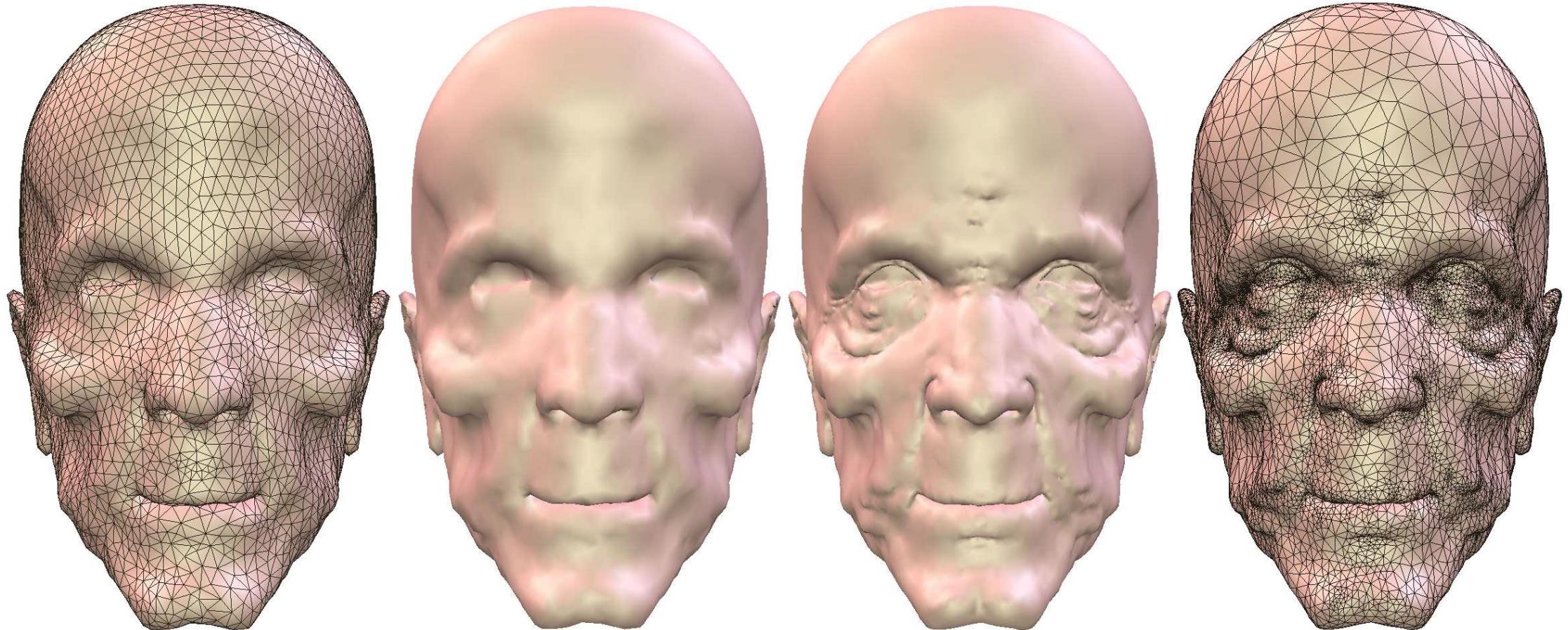
Geometric Compression – 1k vertices



Geometric Compression – 2k vertices



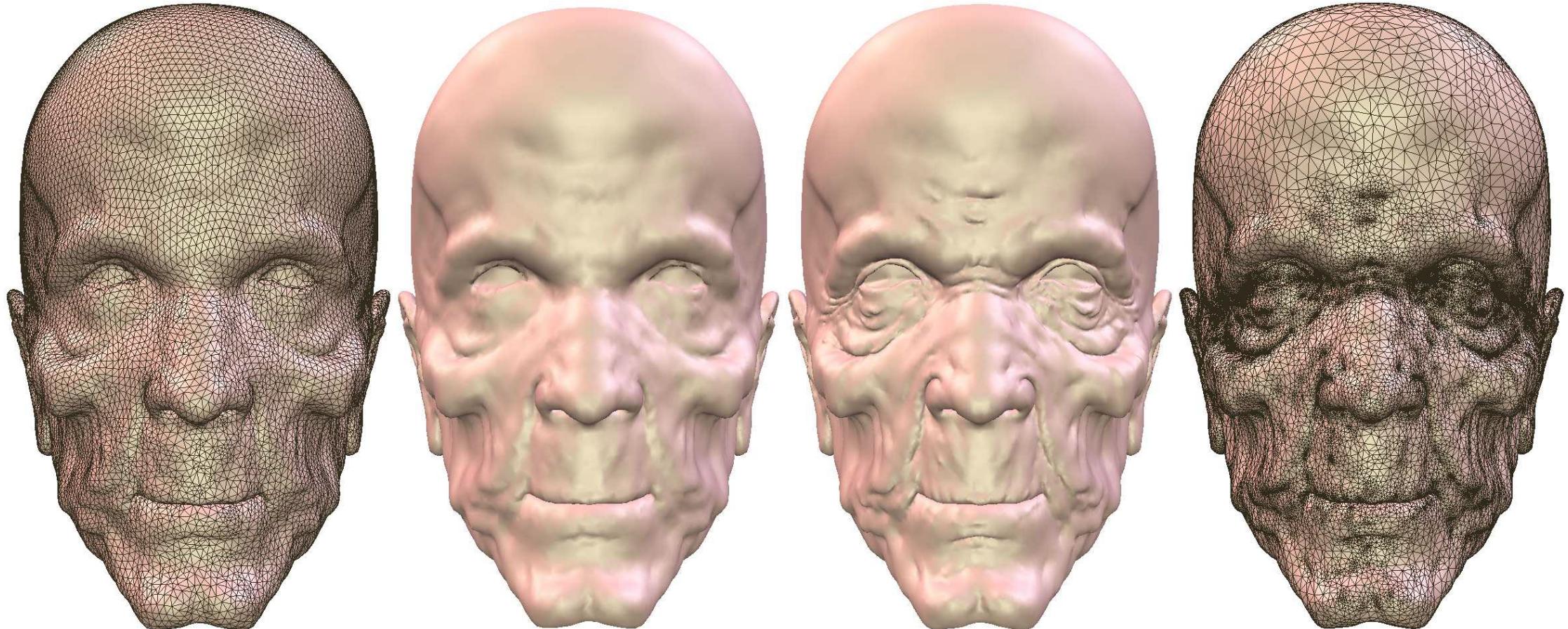
Geometric Compression – 4k vertices



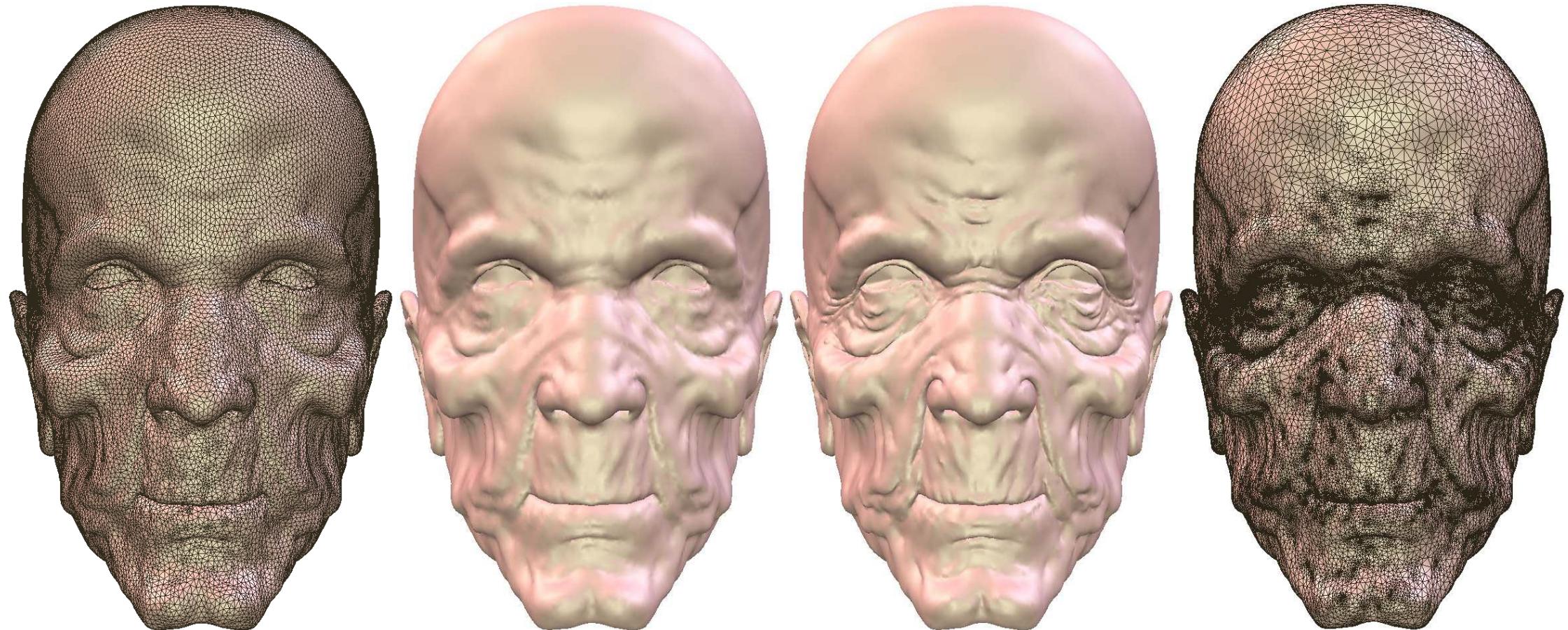
Geometric Compression – 8k vertices



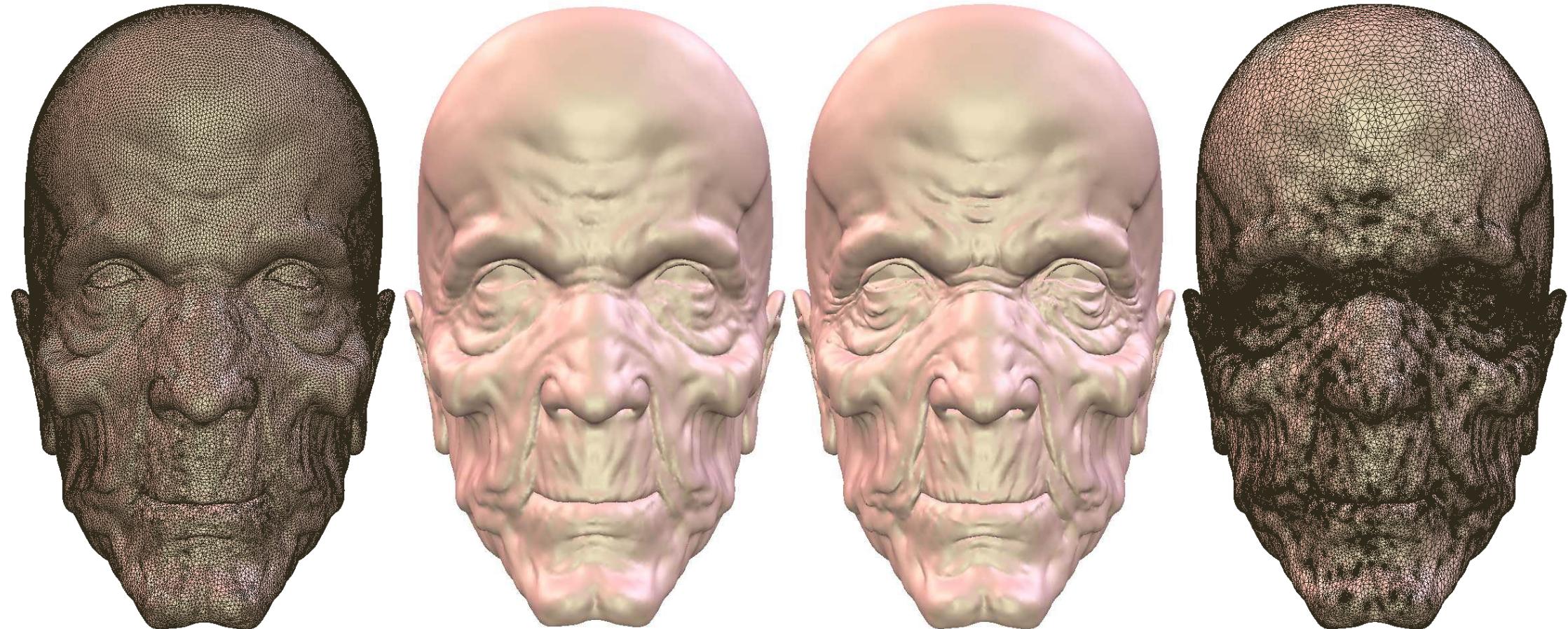
Geometric Compression – 16k vertices



Geometric Compression – 32k vertices



Geometric Compression – 64k vertices



Wireless Sensor Network

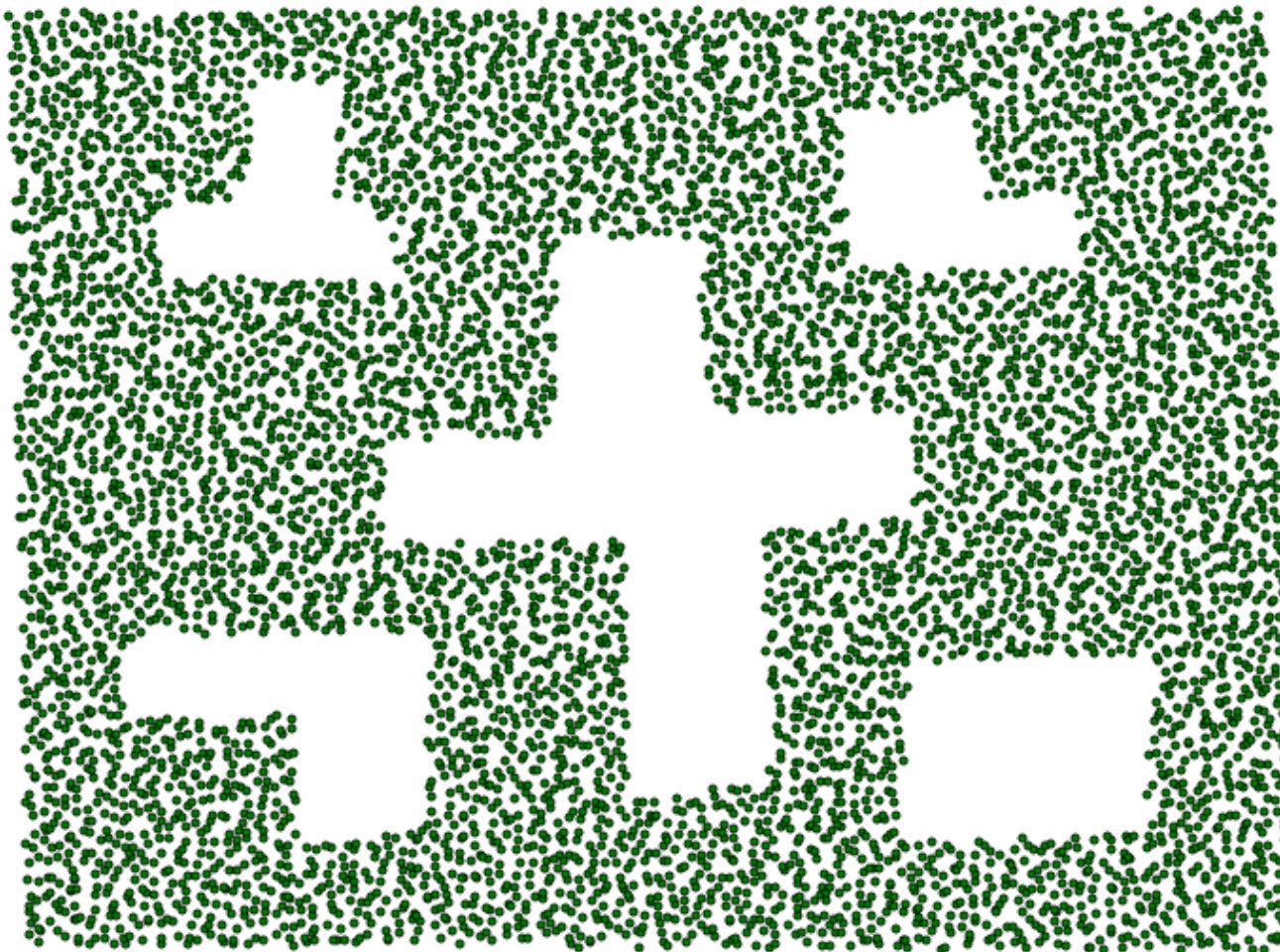
Geometric Routing

Problem: How to design a delivery guaranteed routing scheme? How to obtain load balancing?

Answer: Use canonical conformal mapping and topological covering map

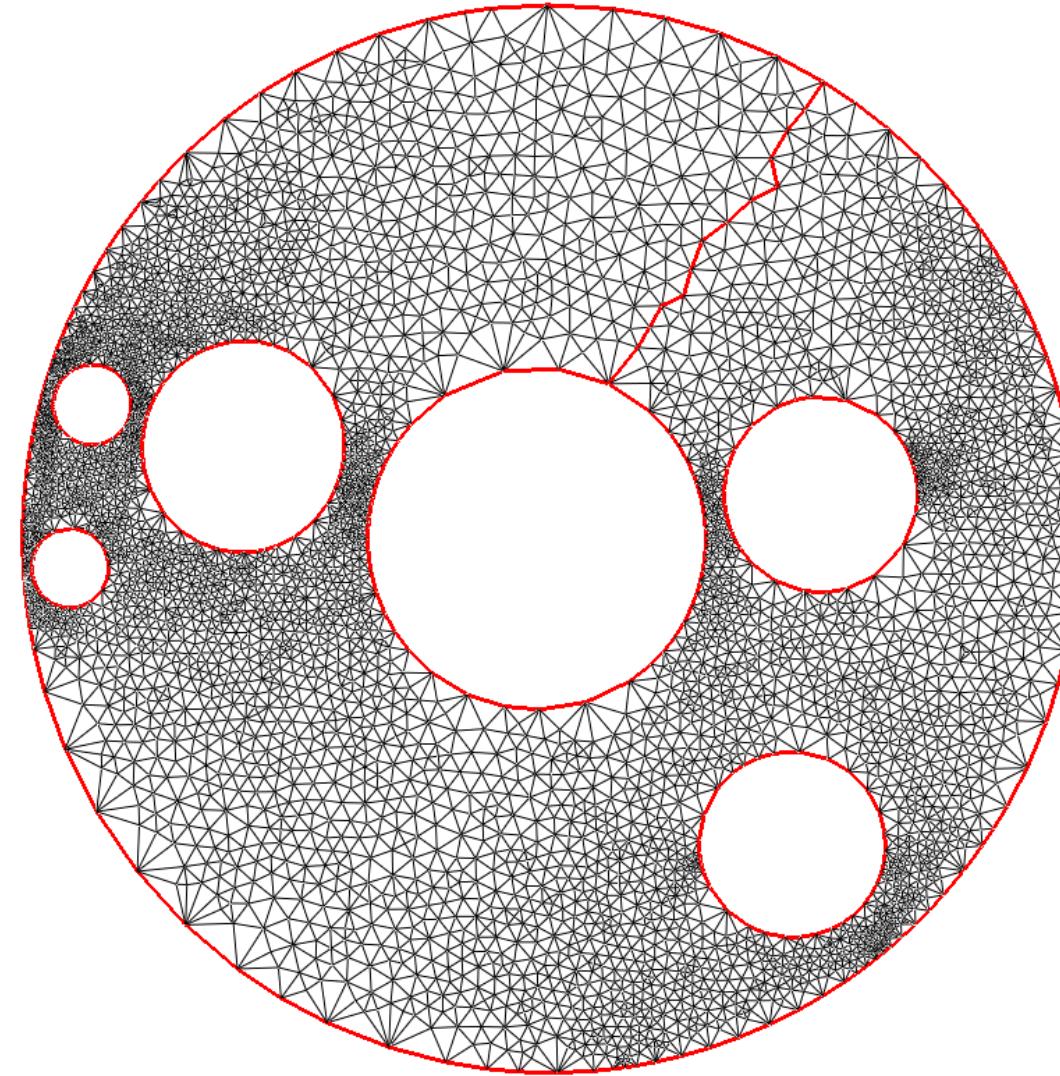
- Conformal mapping transforms all holes to circular holes, guarantees delivery
- Covering map fills all the holes, achieve load balancing

Delivery guaranteed greedy routing



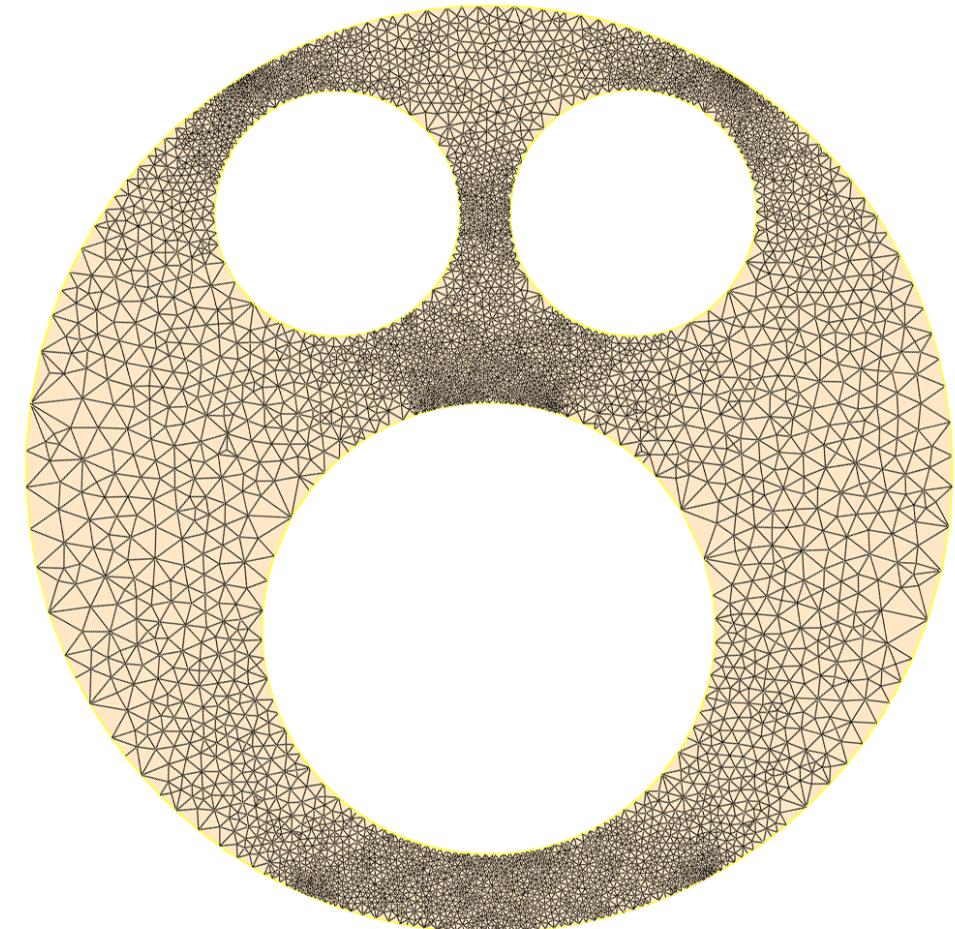
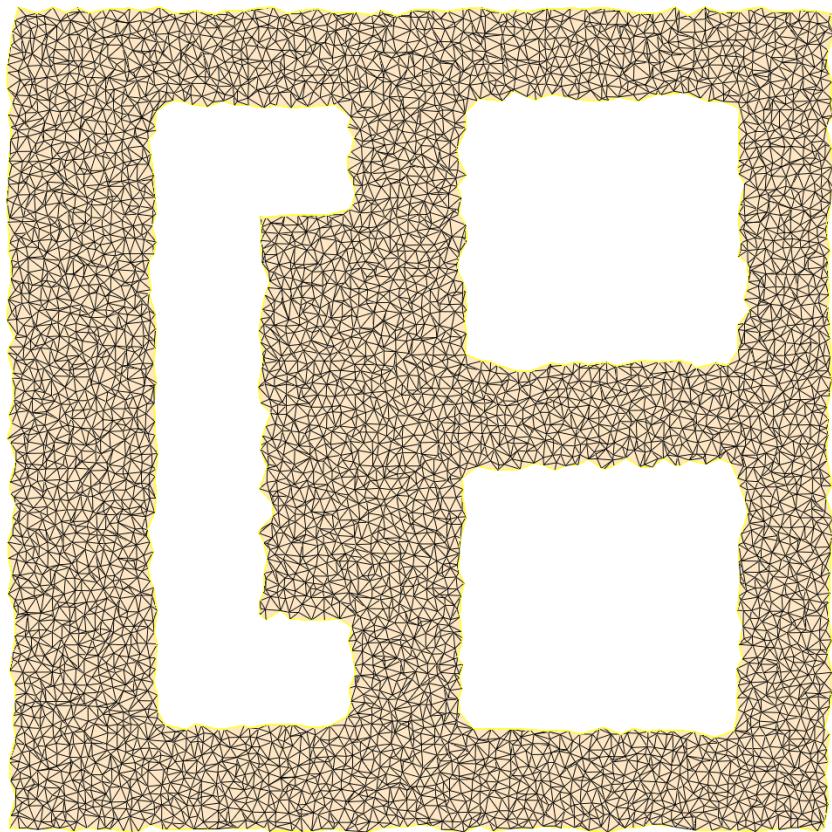
Delivery guaranteed greedy routing

- virtual coordinates



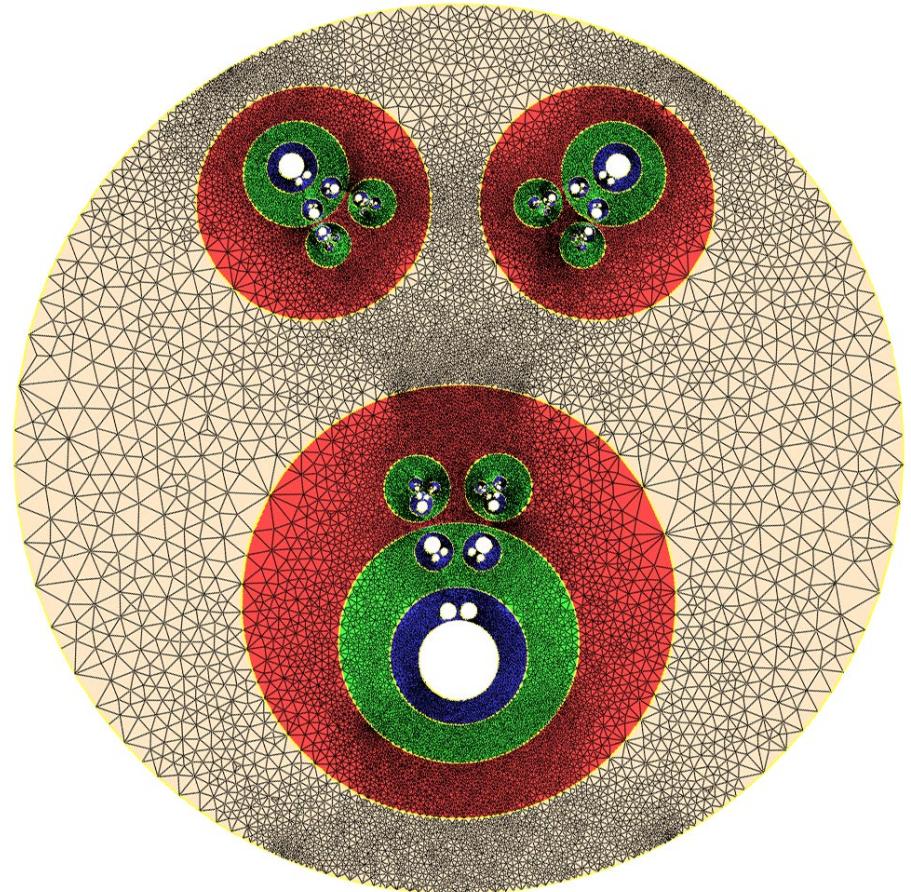
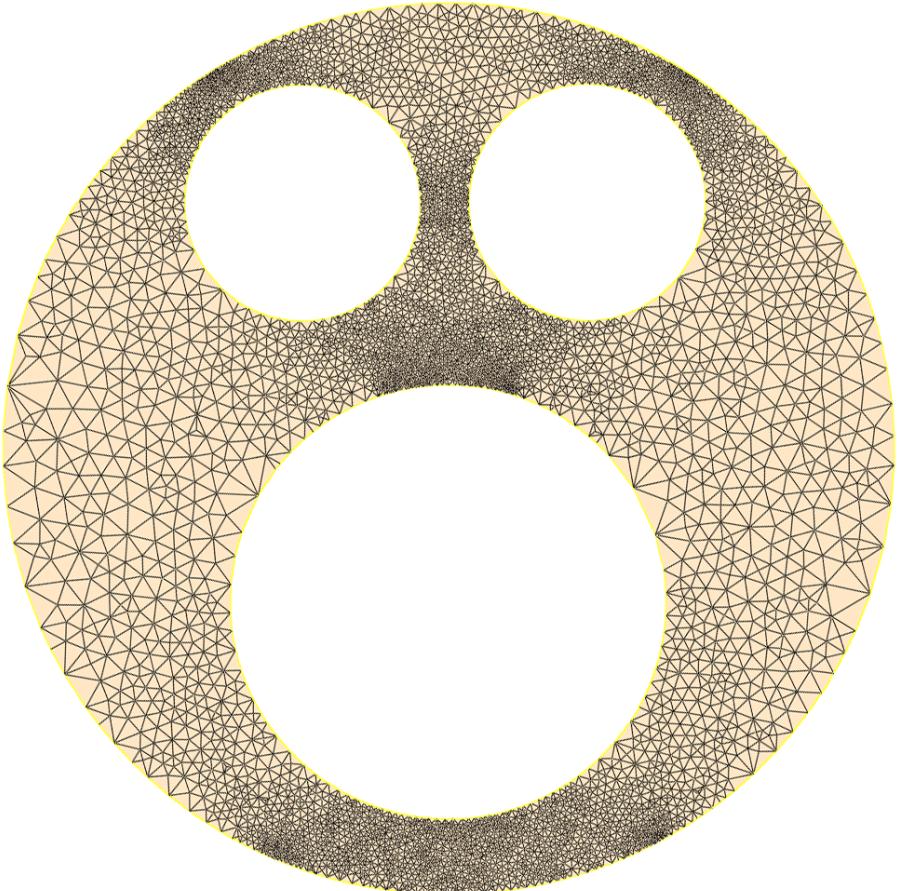
Load balancing

- Geodesics are along boundaries, sensors along the boundaries are overloaded.



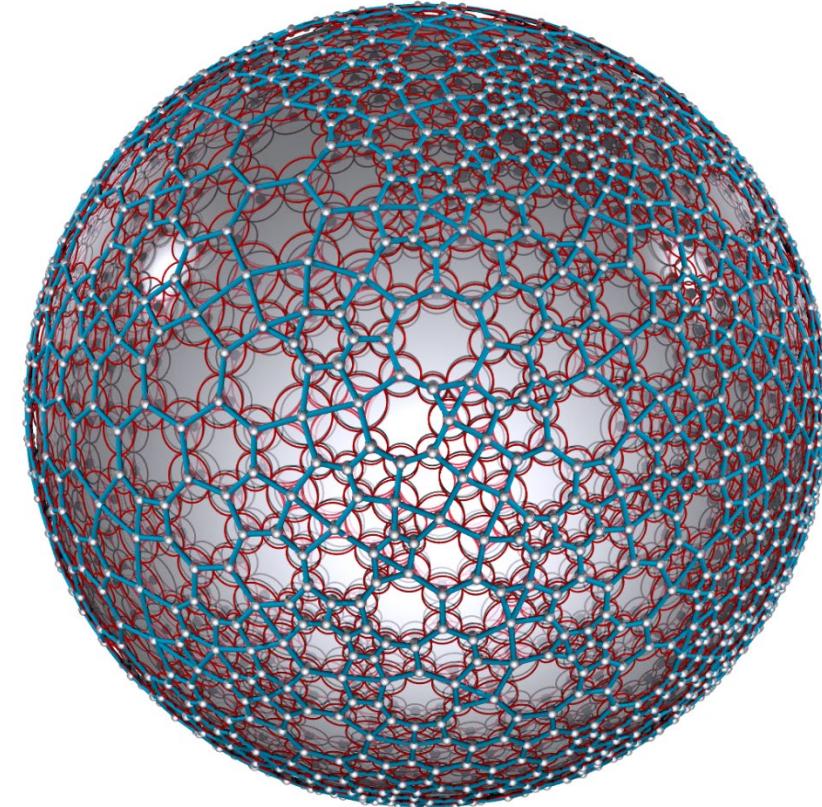
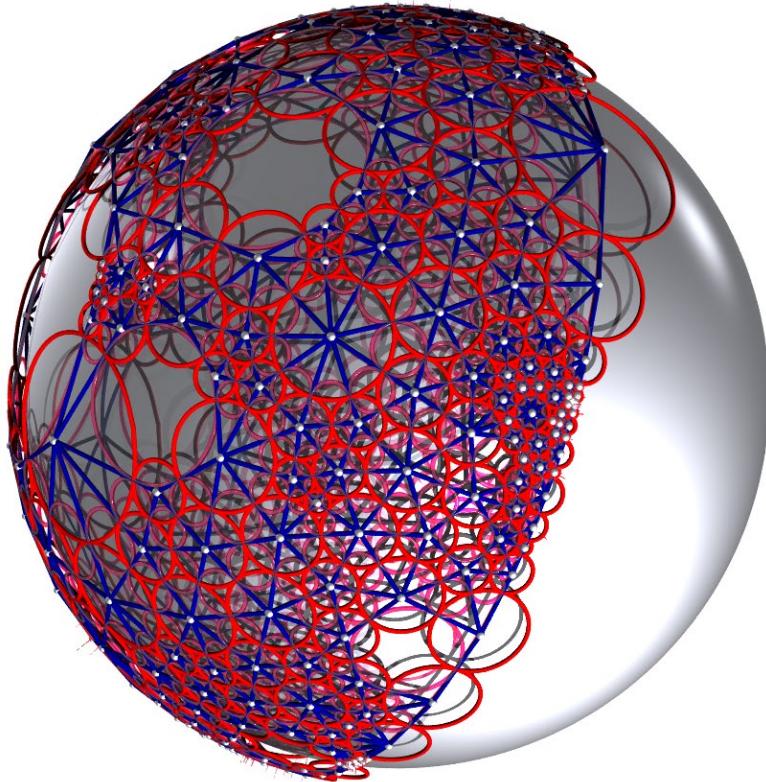
Load balancing

- Covering



Wireless sensor network underwater

- Graph embedding



Medical Imaging

Medical Imaging

Problem: How to compare/analyze the shapes of human organs precisely?

Answer: Surface registration based on conformal geometry.

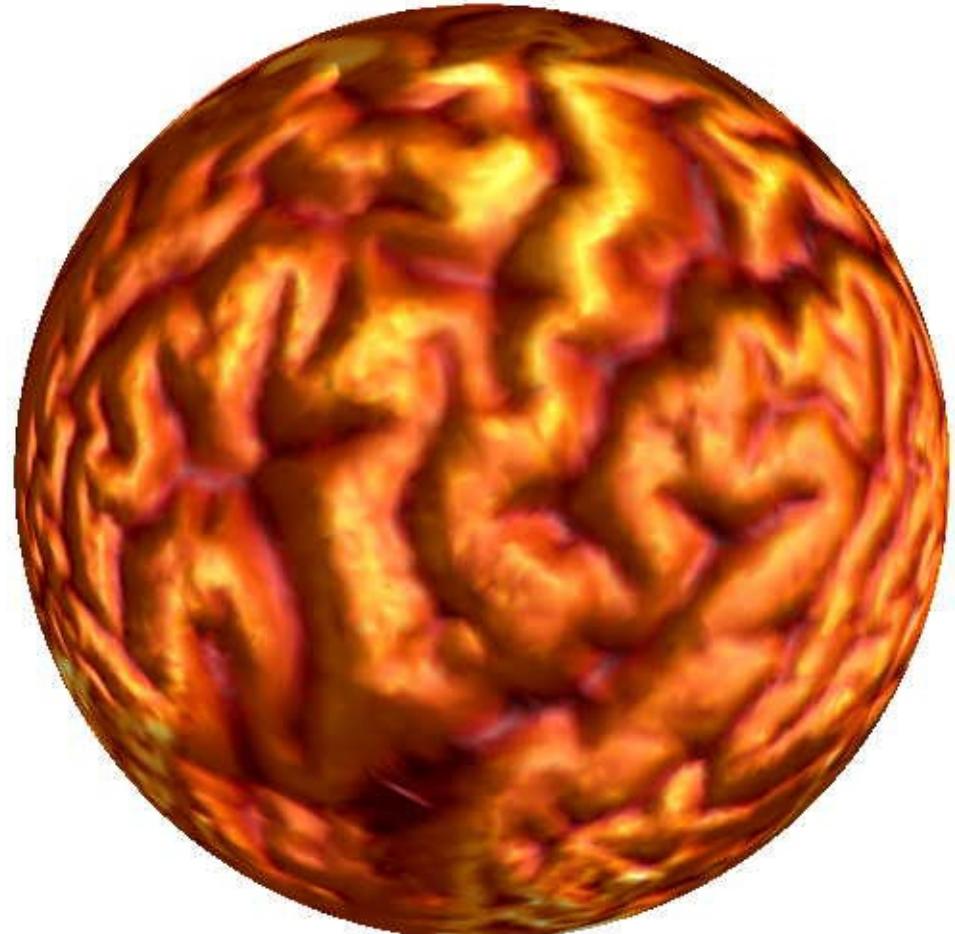
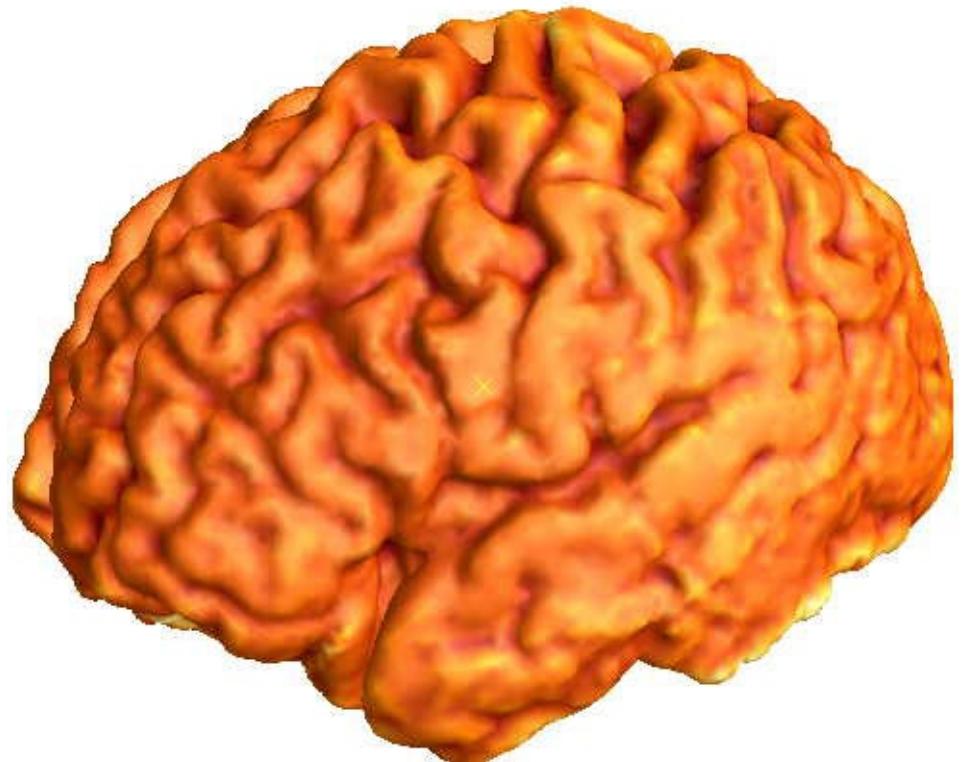
- Plastic surgery
- Conformal brain mapping
- Virtual colonoscopy

Plastic Surgery Analysis

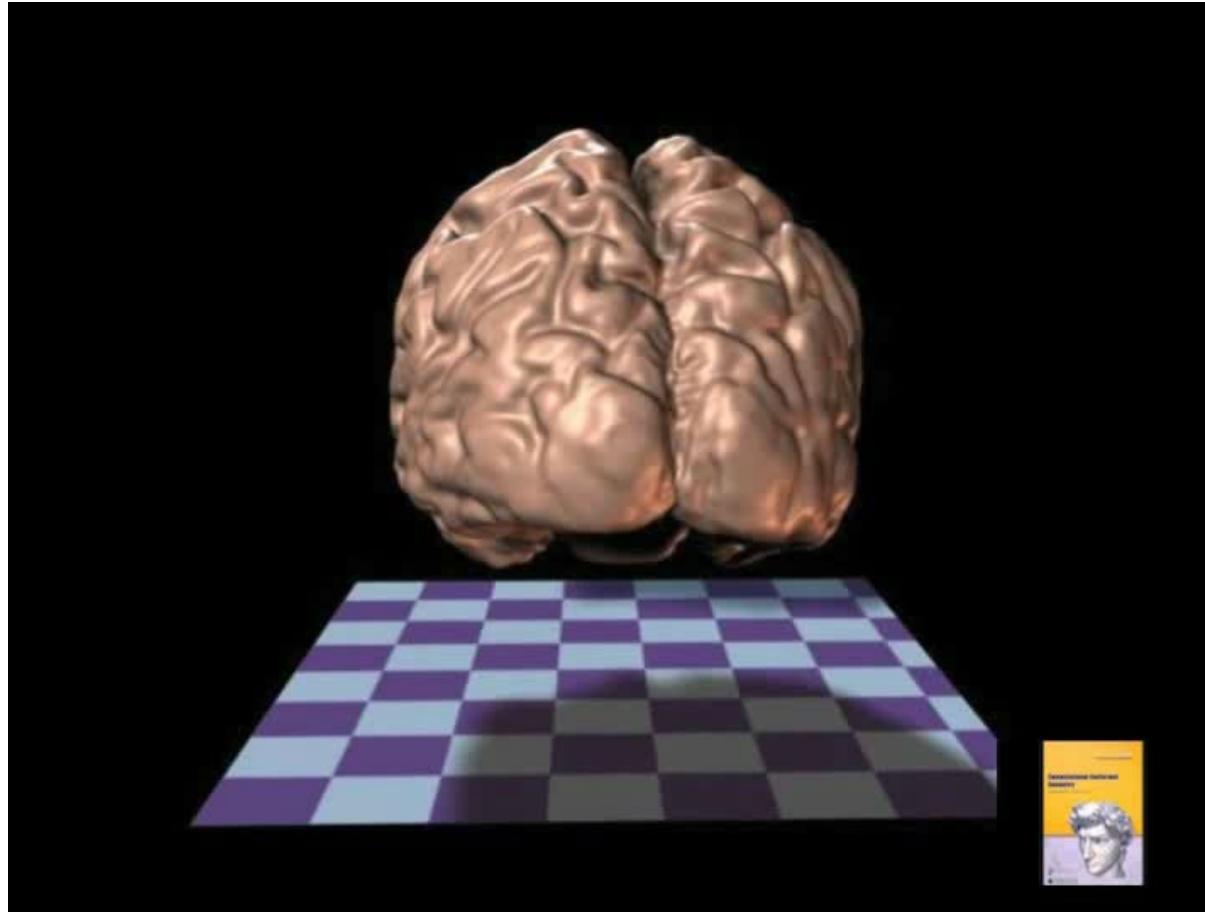


Brain mapping

- Map brain to the unit sphere, registration and comparison.

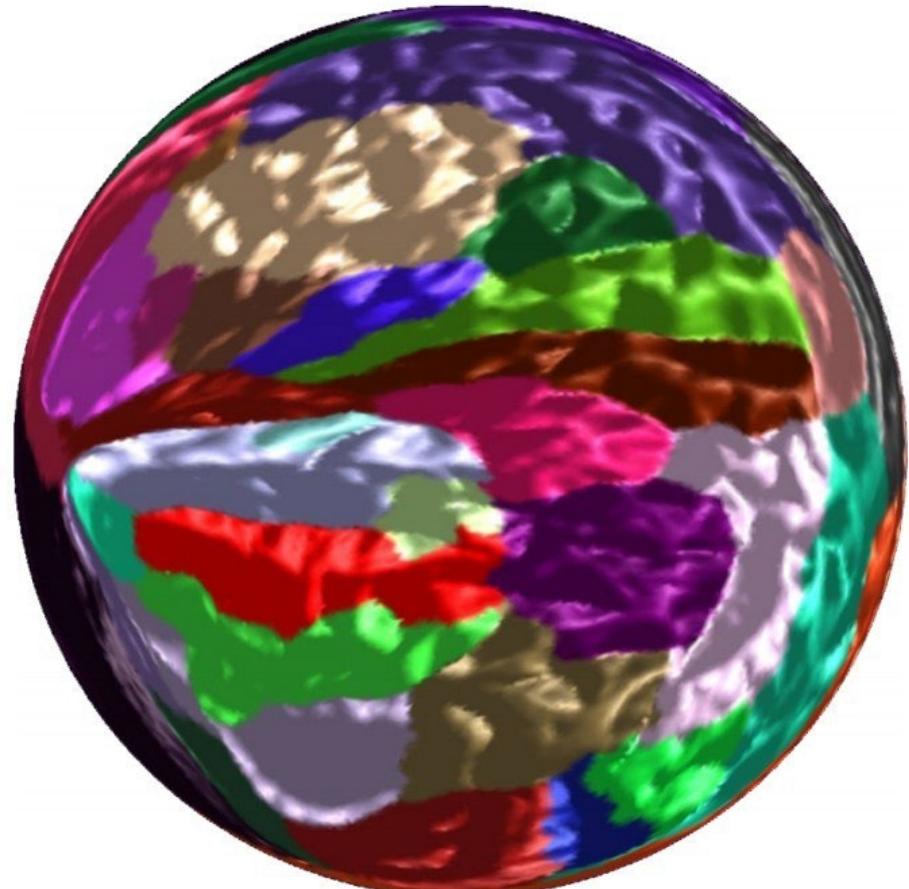
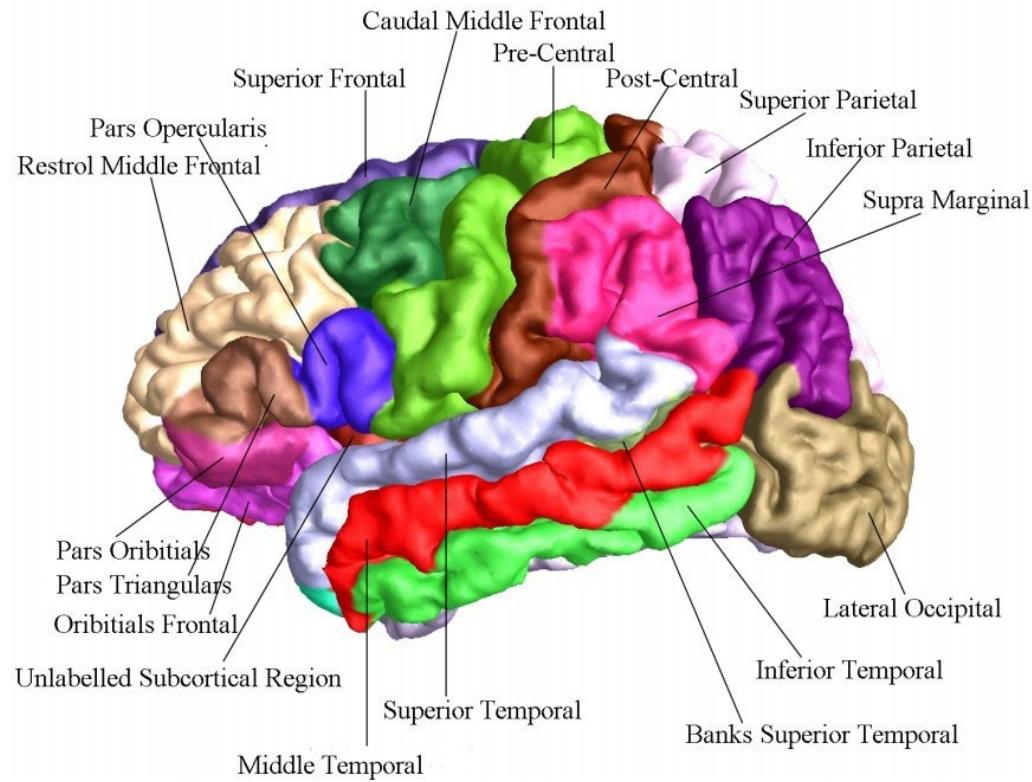


Conformal Brain Mapping



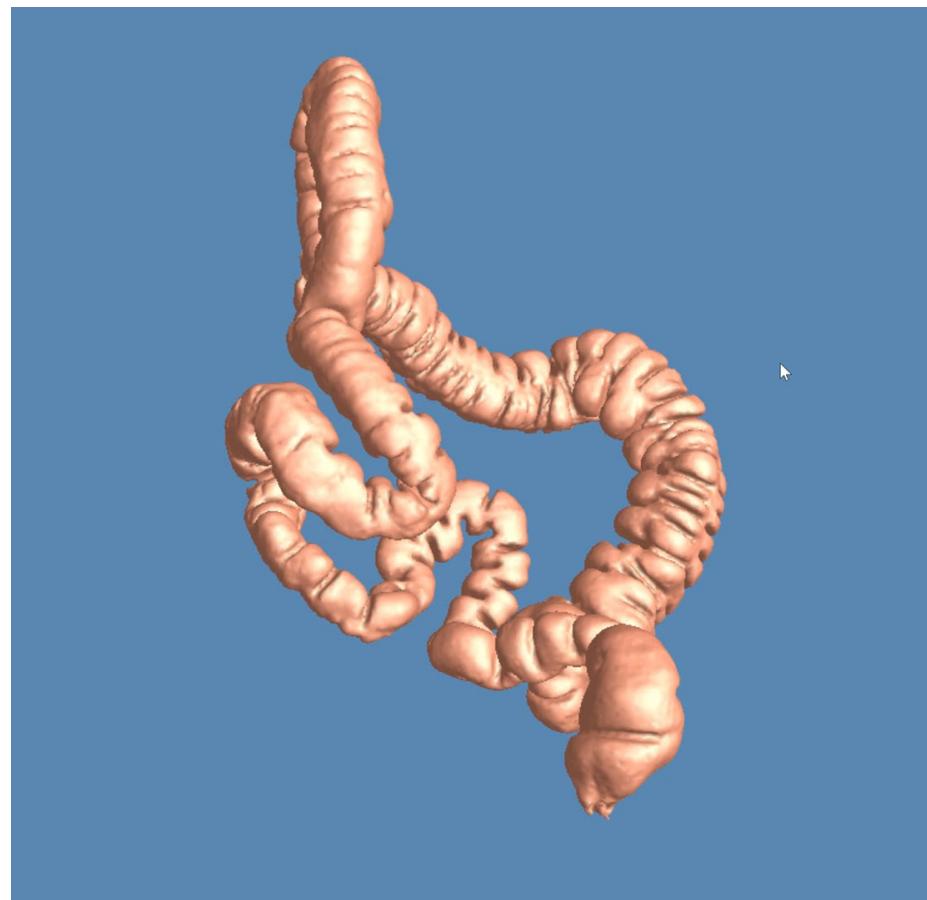
Brain Mapping

- Locate the illness, compare the deformation.

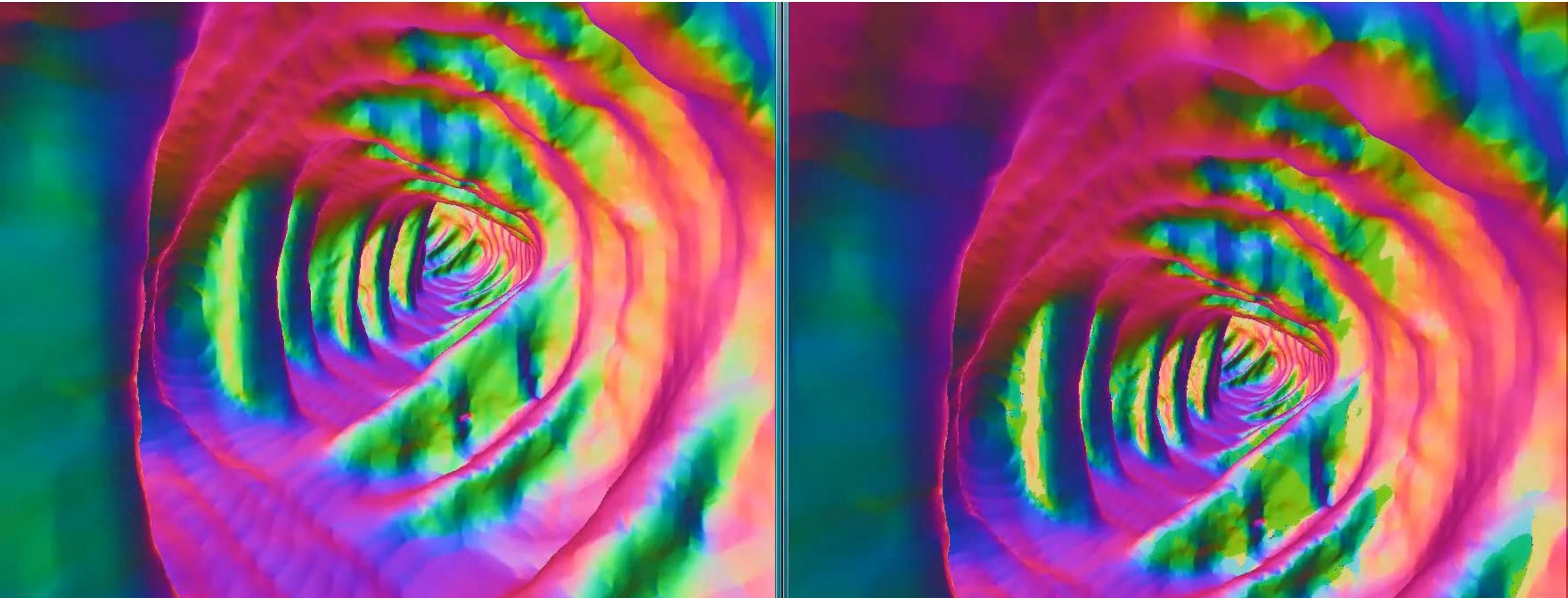


Virtual colonoscopy

- Reconstruct colon wall surface from CT images.

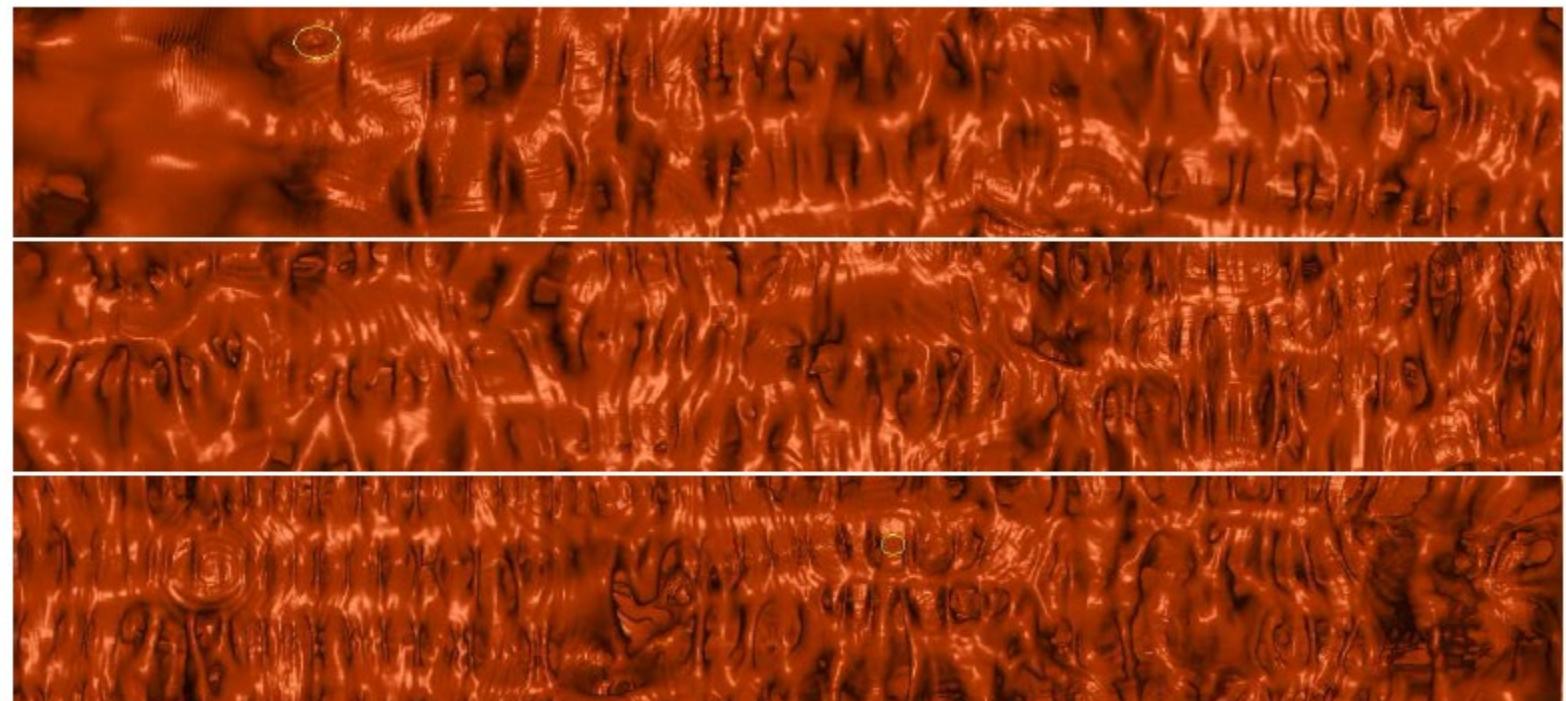
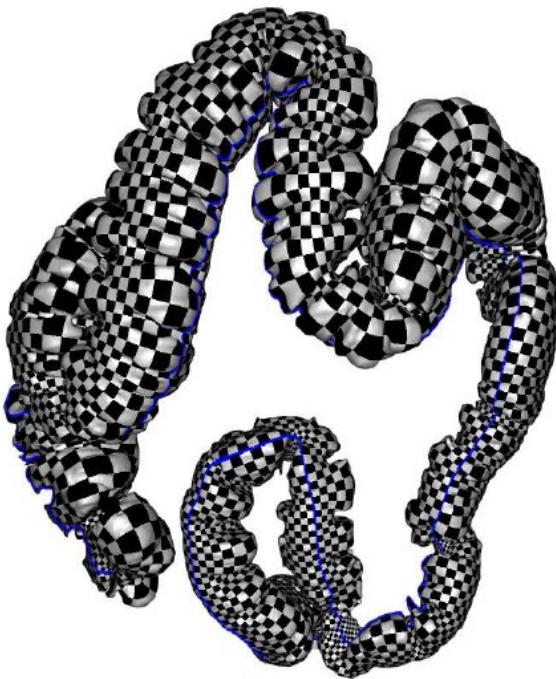


Virtual colonoscopy

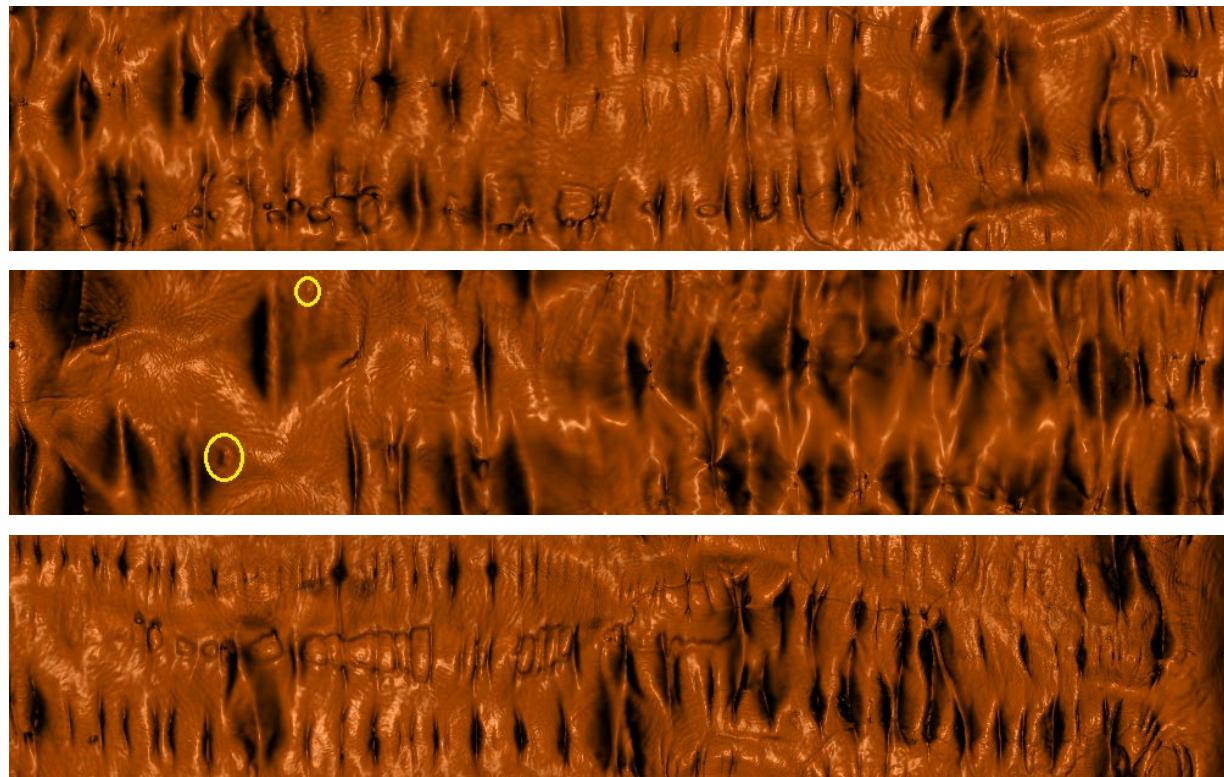
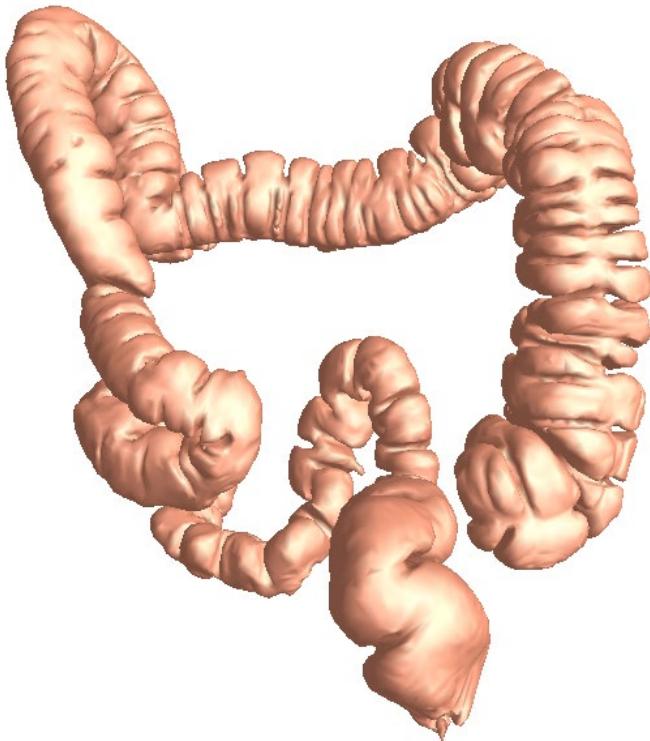


Virtual colonoscopy

- Unfold all the foldings on the colon surface, preserving local shapes.



Virtual Colonoscopy



Computer Geometric Design

Geometric Modeling

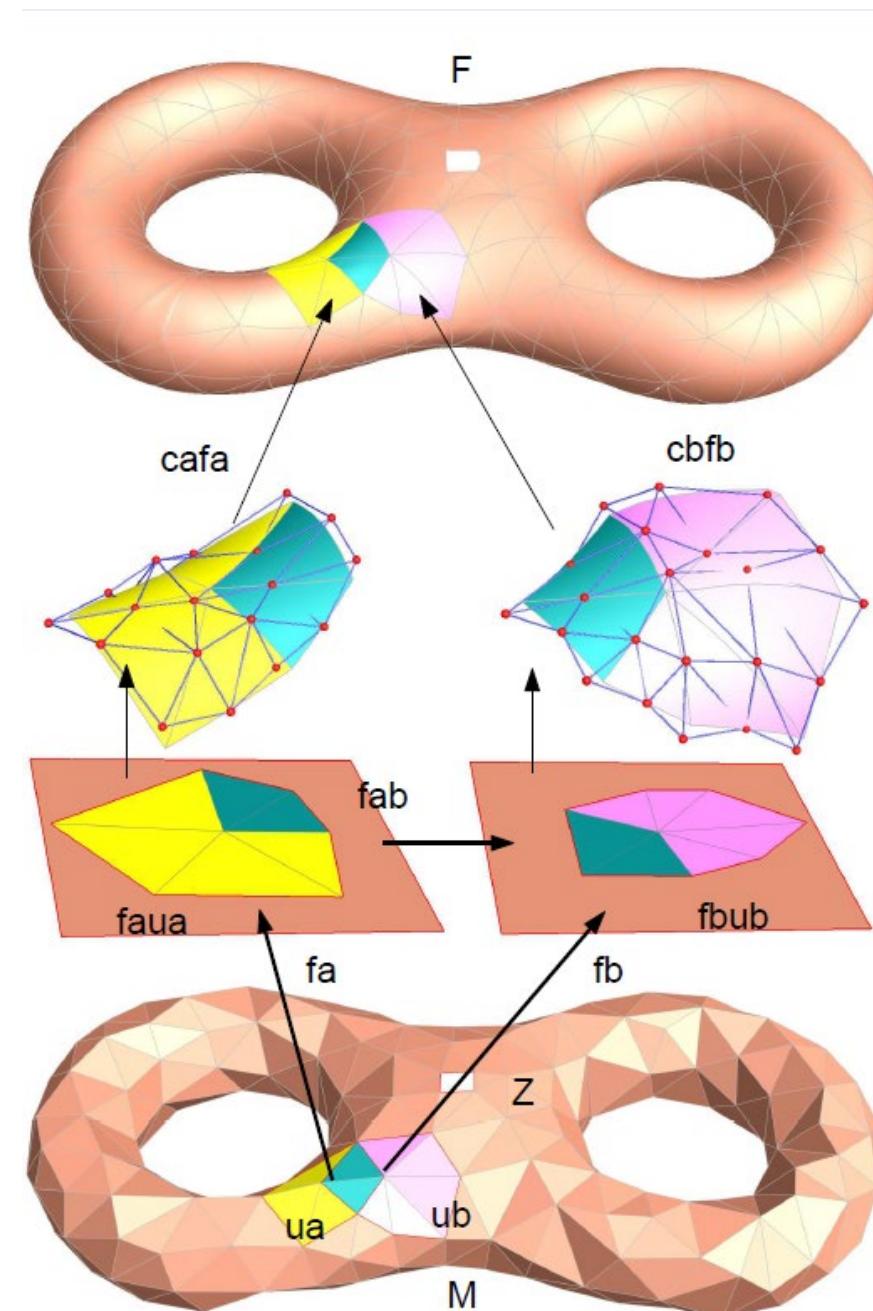
Problem: How to generalize Splines from planar domains to surface domains ? Can we avoid extraordinary points ? How to control extraordinary points?

Answer: It depends on the affine structure of the surface. Due to the topological obstruction, there is no affine structure in general.

- Compute flat metric with cone singularities
- Flat metric gives an affine structure
- Cone singularities give the extraordinary points

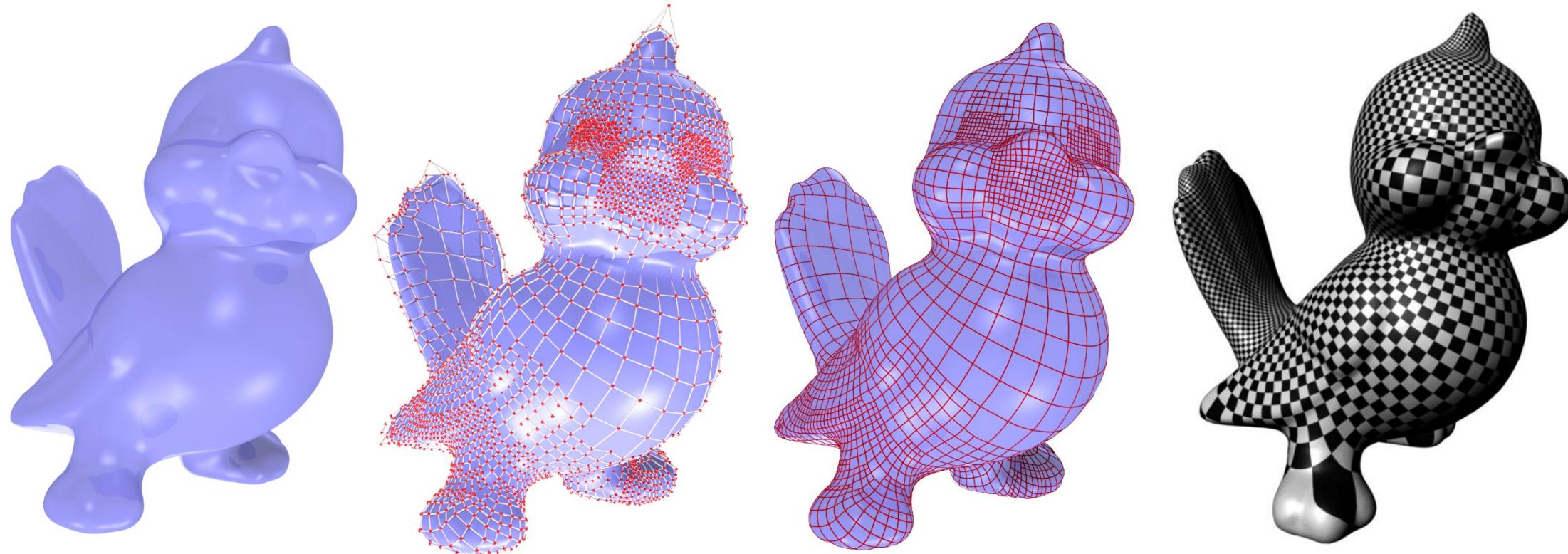
Manifold Splines

- Splines is based on Affine geometry
- Requires an Affine structure, an atlas with linear transitions
- Construct splines on each chart globally glued together
- Topological obstruction

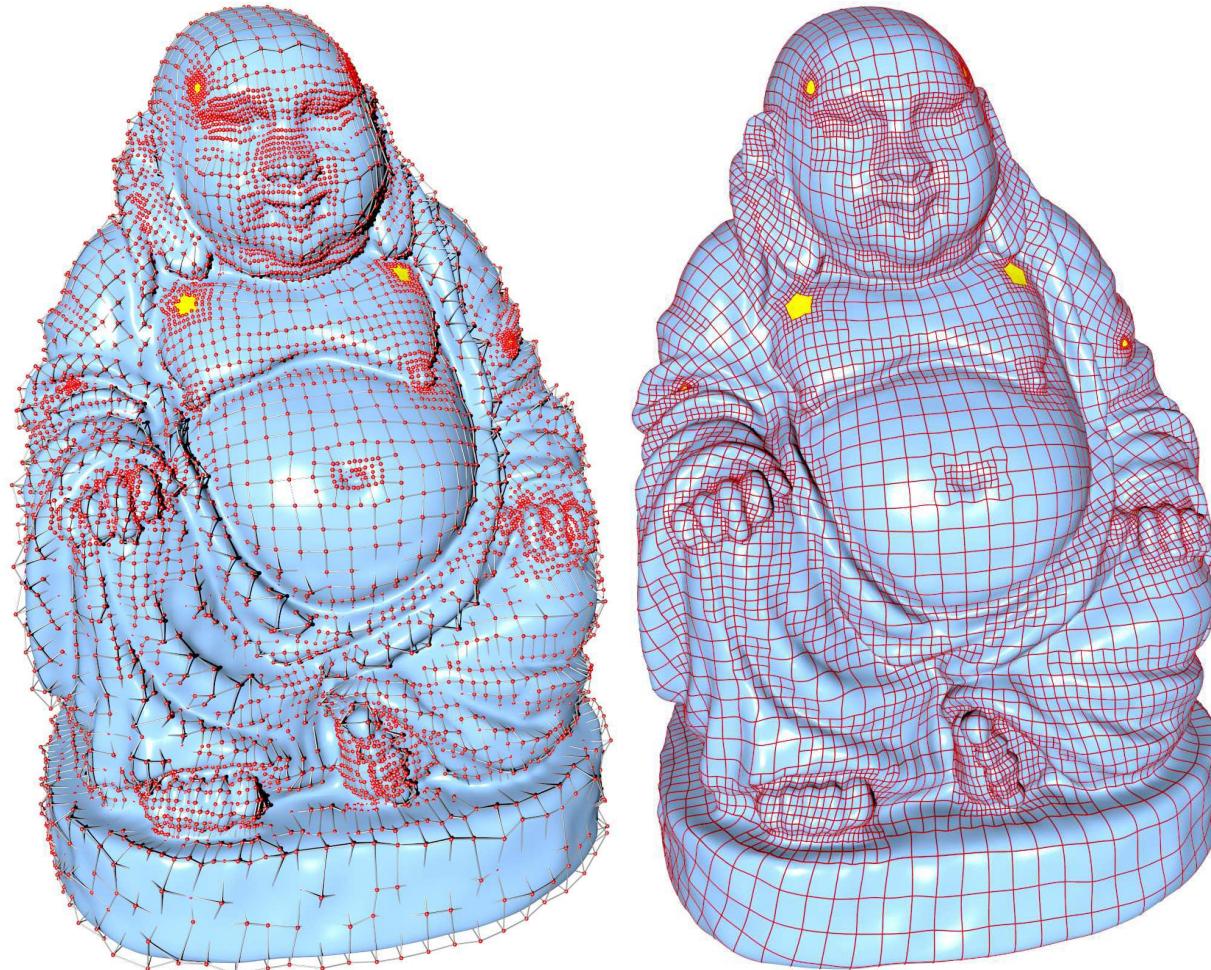


Manifold Splines

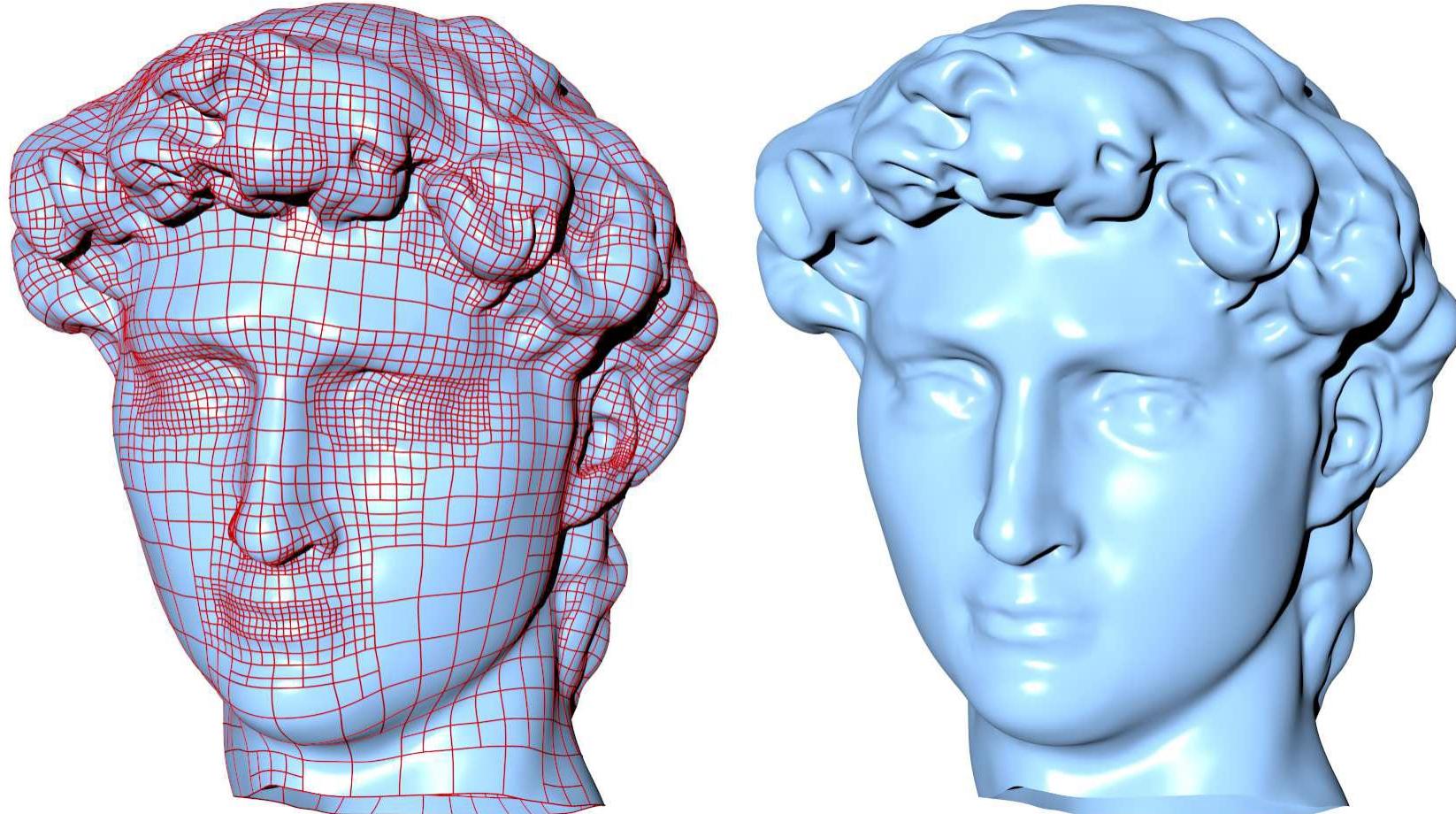
- Converting triangle meshes to manifold splines



Manifold Splines – control singularities



Manifold Splines – control approximation accuracy



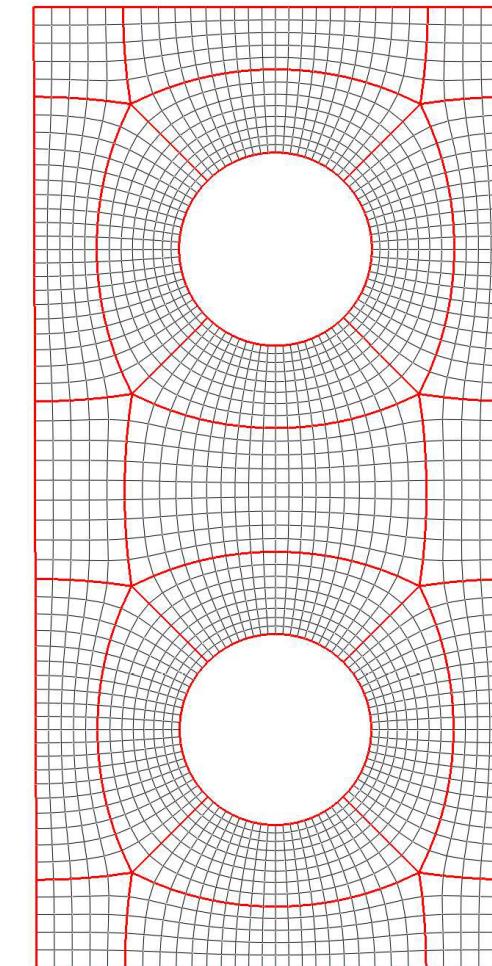
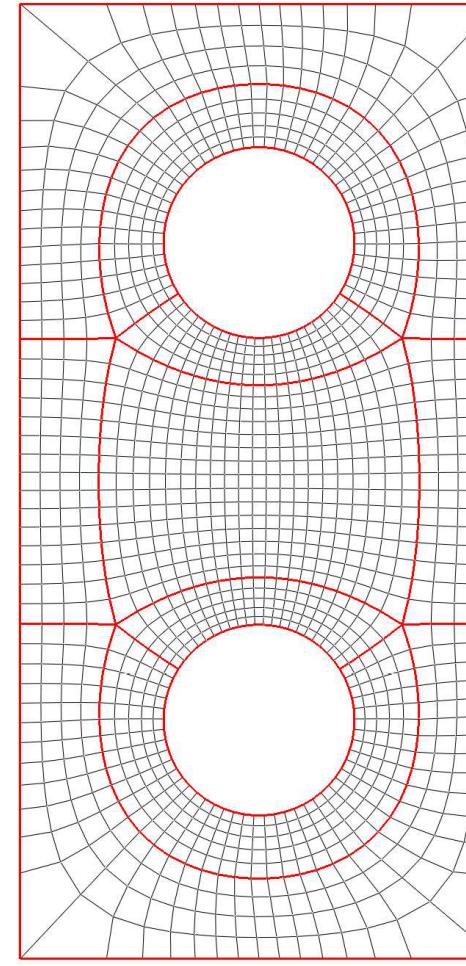
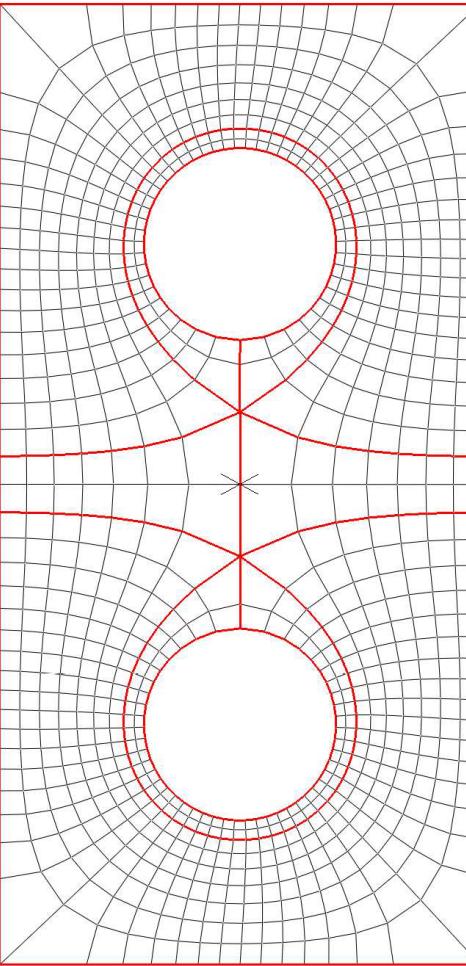
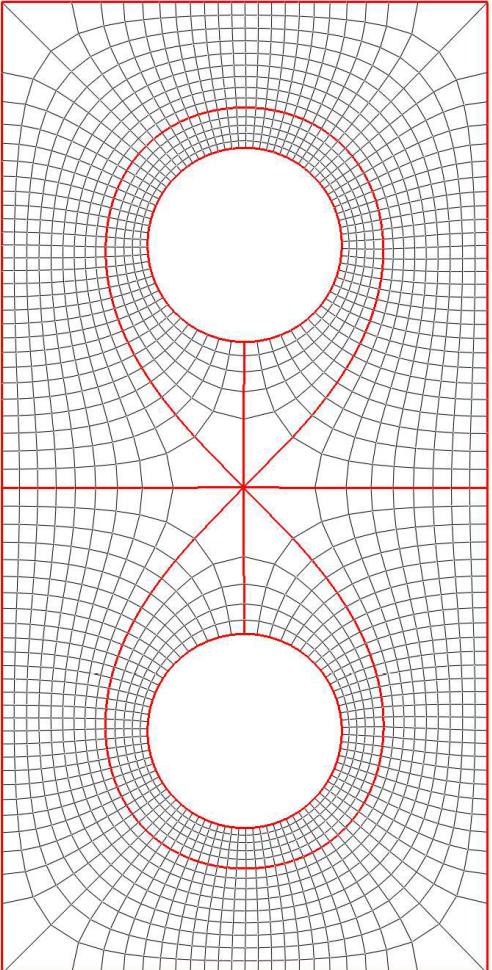
Surface quadrilateral meshing

Problem: How to construct global regular quad-meshes ?

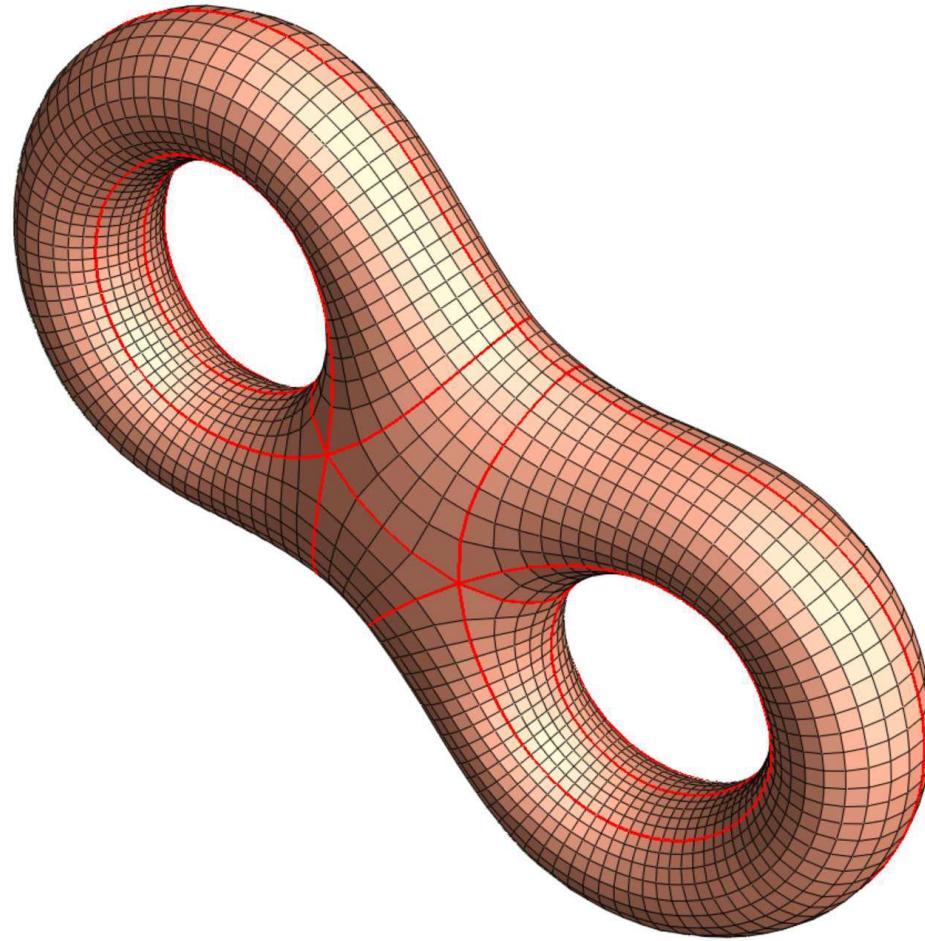
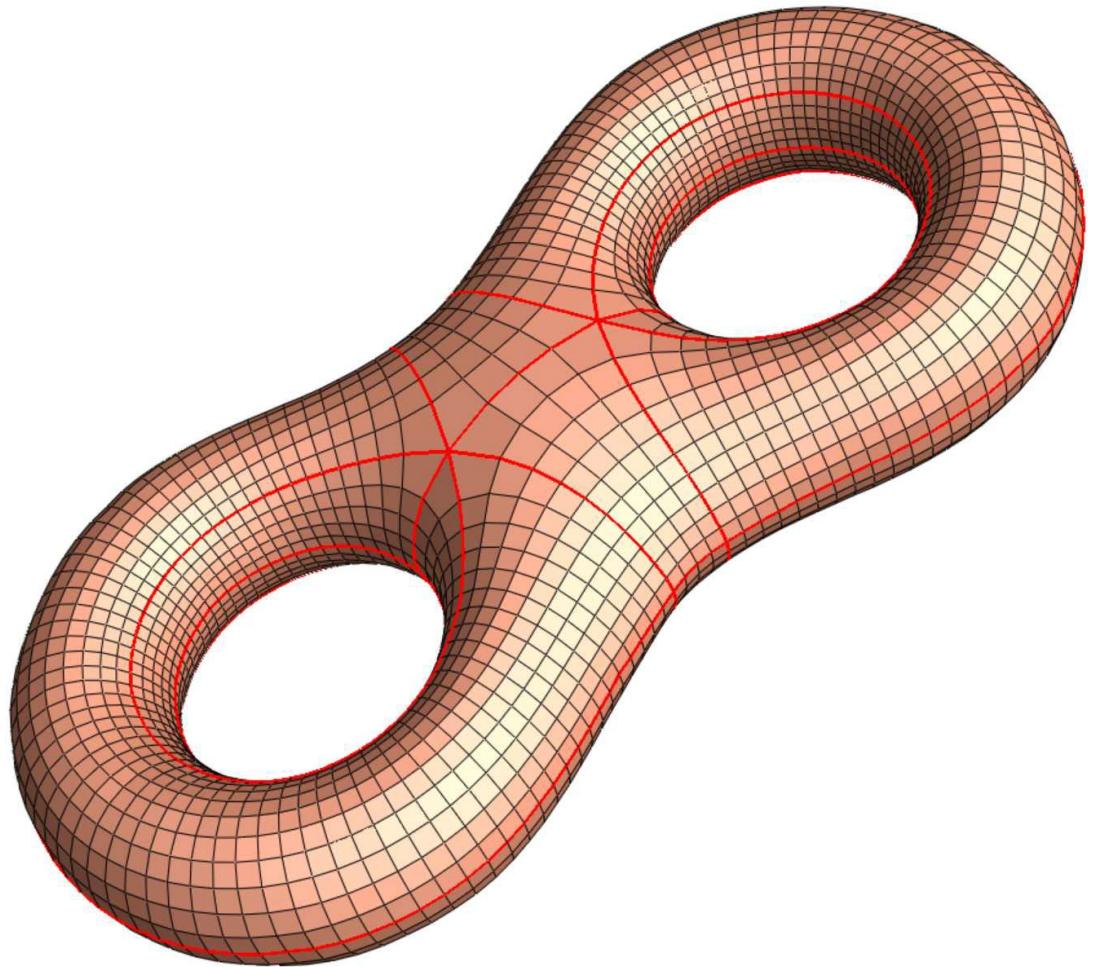
Answer: Use Abel-Jacobi theorem in Riemann surface.

- The singularities satisfy the Abel-Jacobi condition
- Compute the flat cone metric using Ricci flow
- Trace the geodesics to generate the quad-mesh

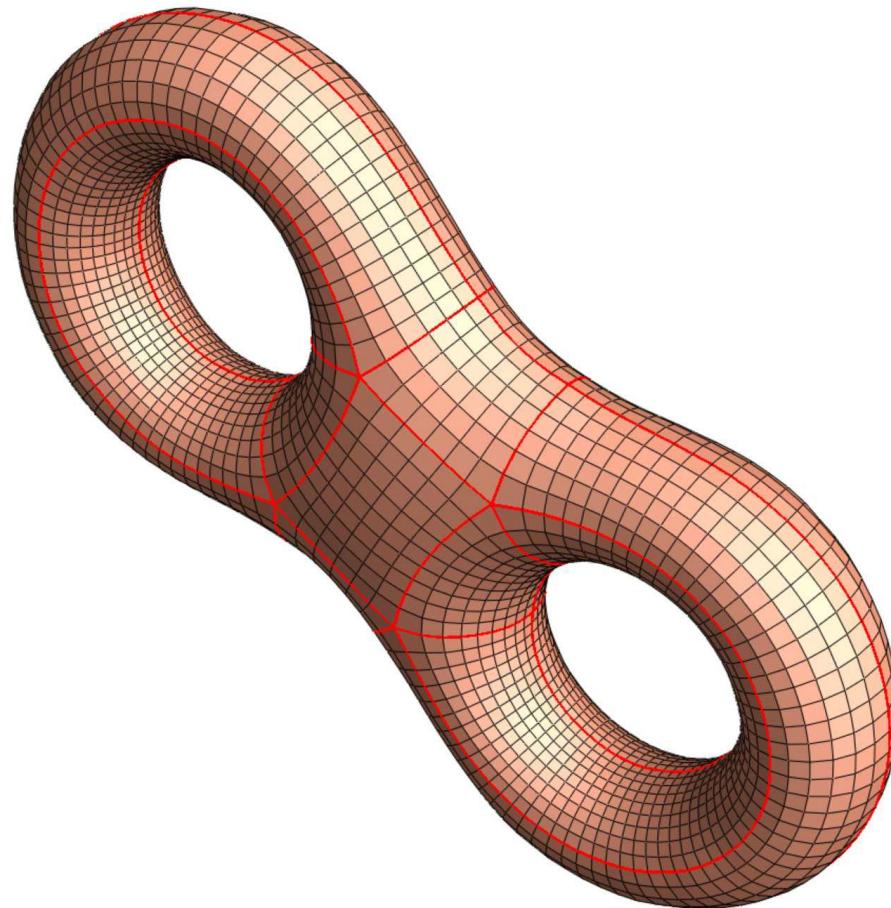
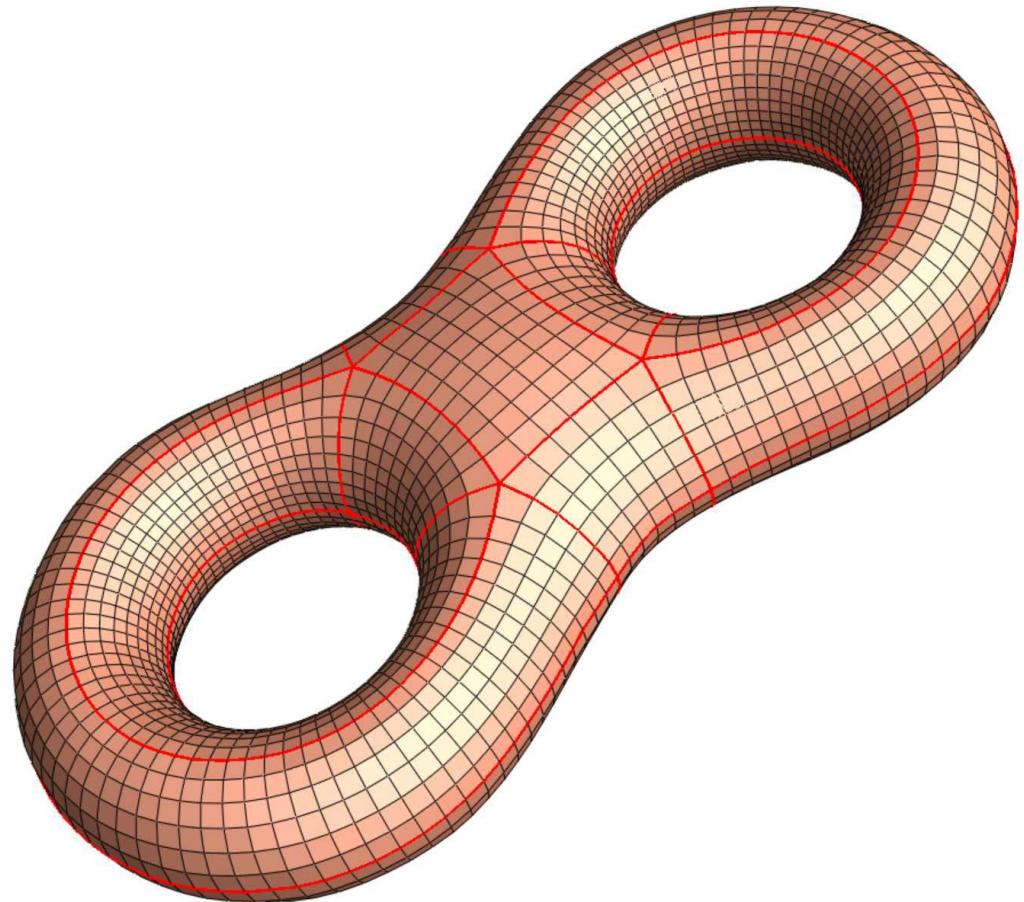
Quad-Mesh Theory



Quad-Mesh Theory



Quad-Mesh Theory



Holy Grid

Problem: How to construct global regular hex-meshes ?

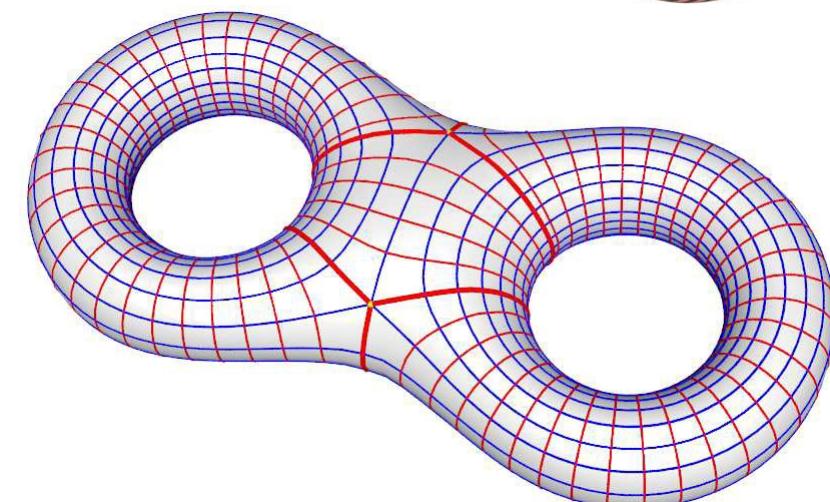
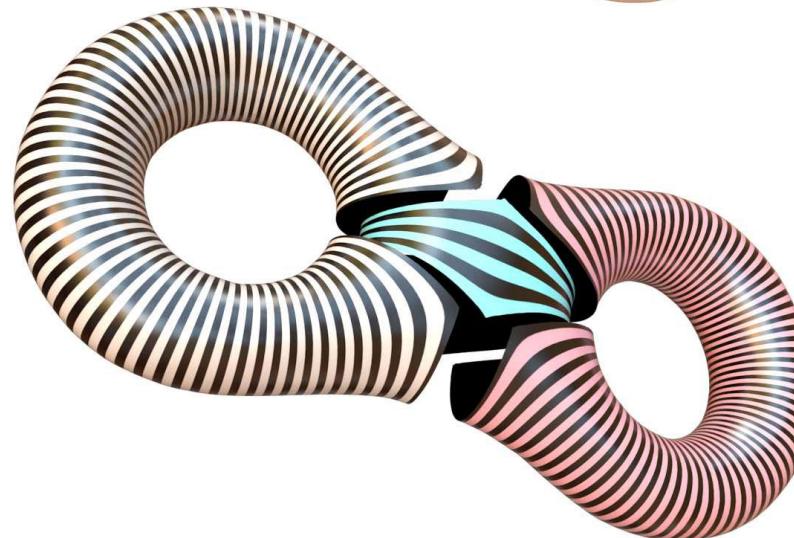
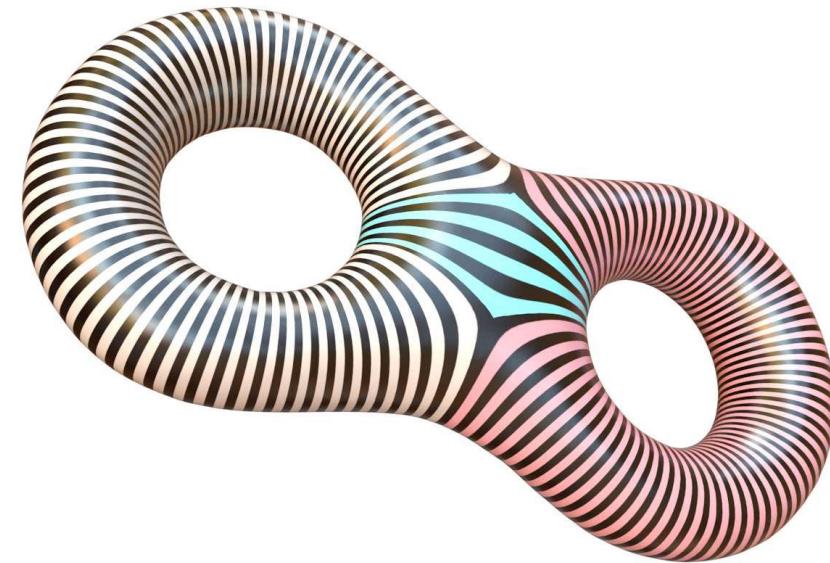
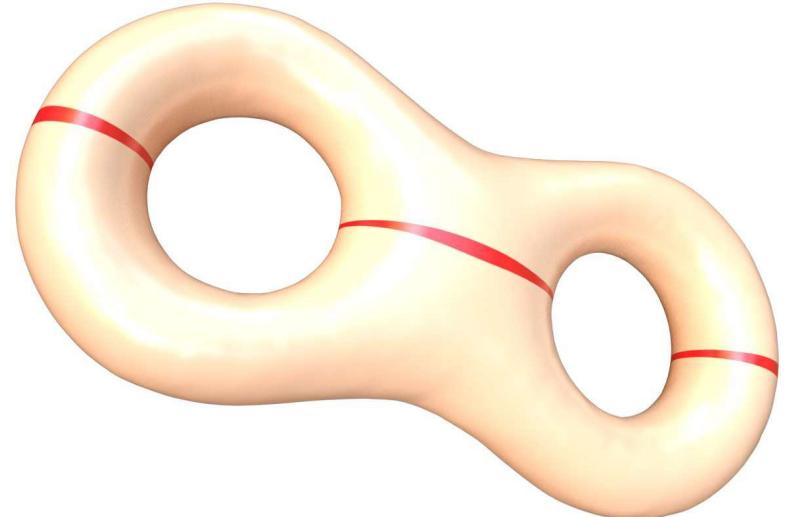
Answer: Use Strebel differential

- The Strebel differential induces a quad-mesh on the boundary surface
- The quad-mesh can be extended into the interior to form a hex-mesh

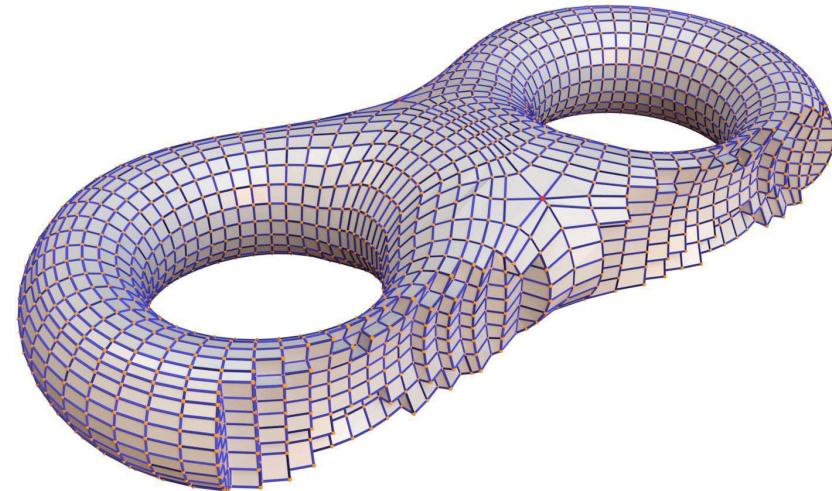
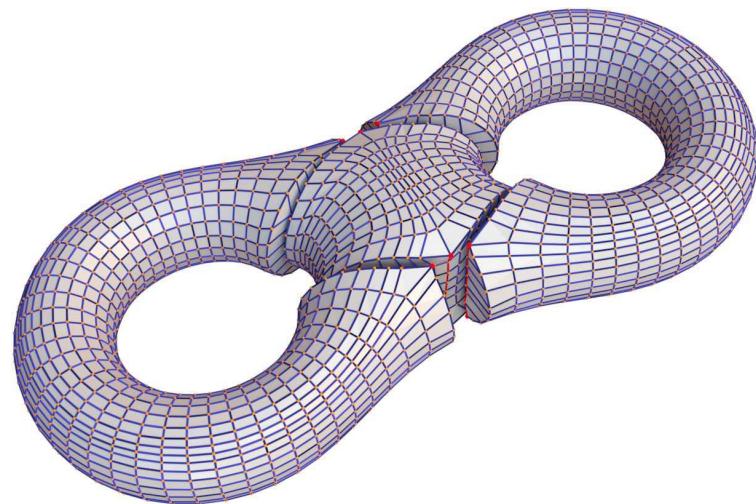
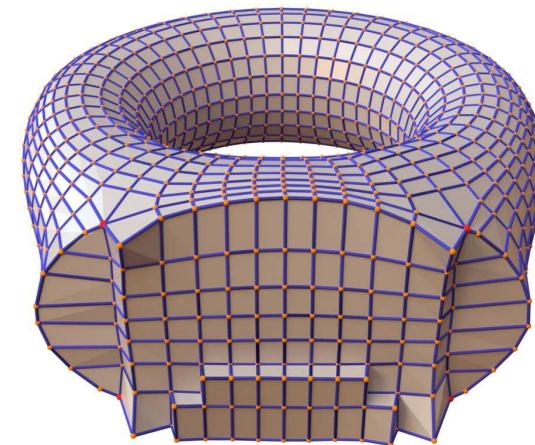
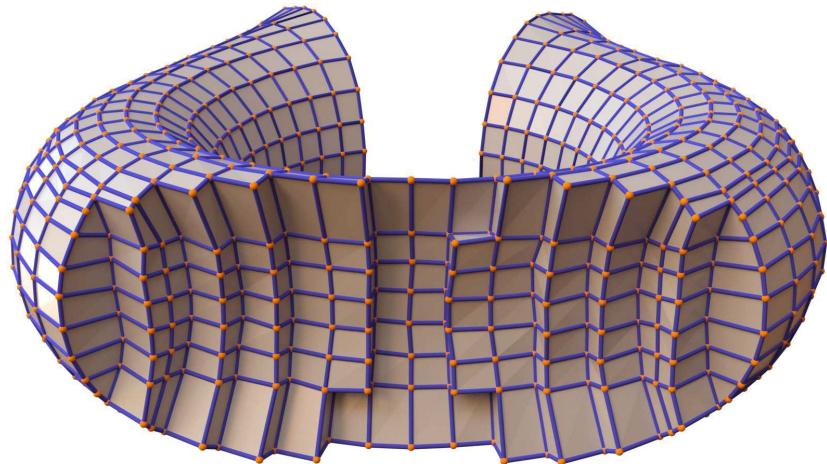
Foliation - Holomorphic Quadratic Differential



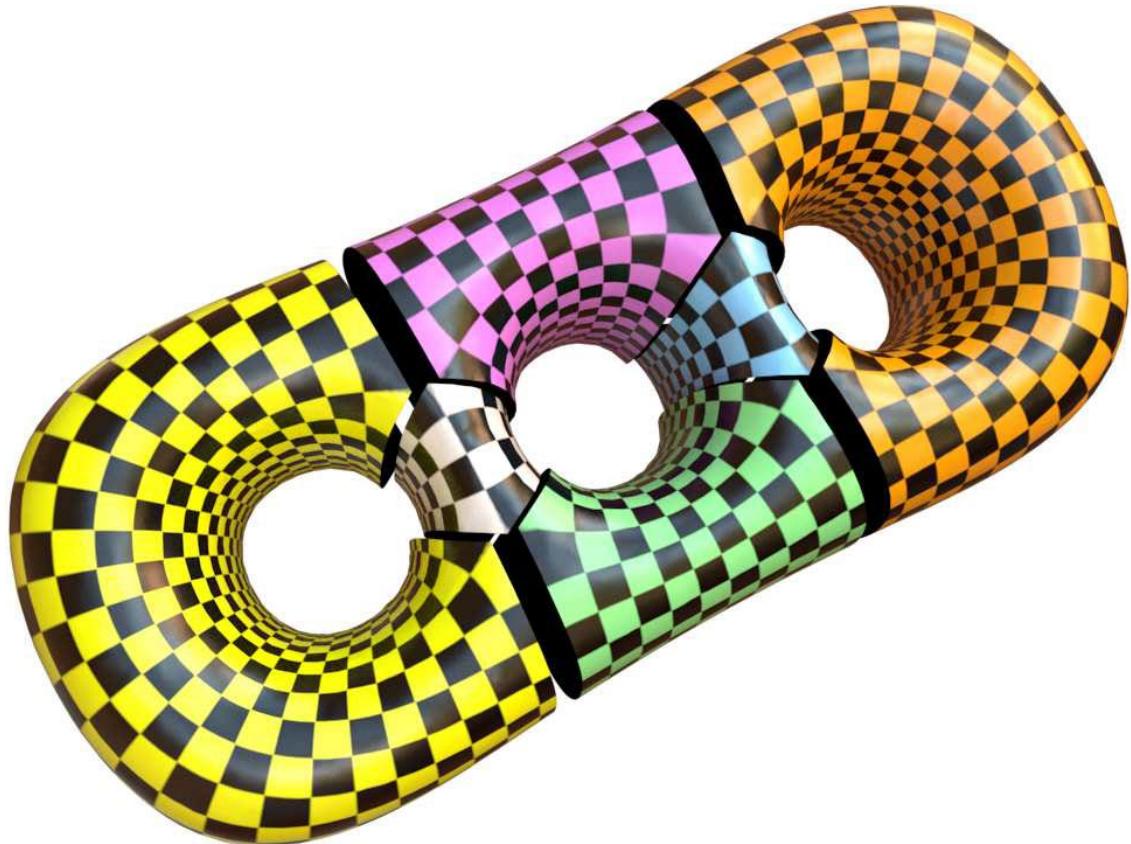
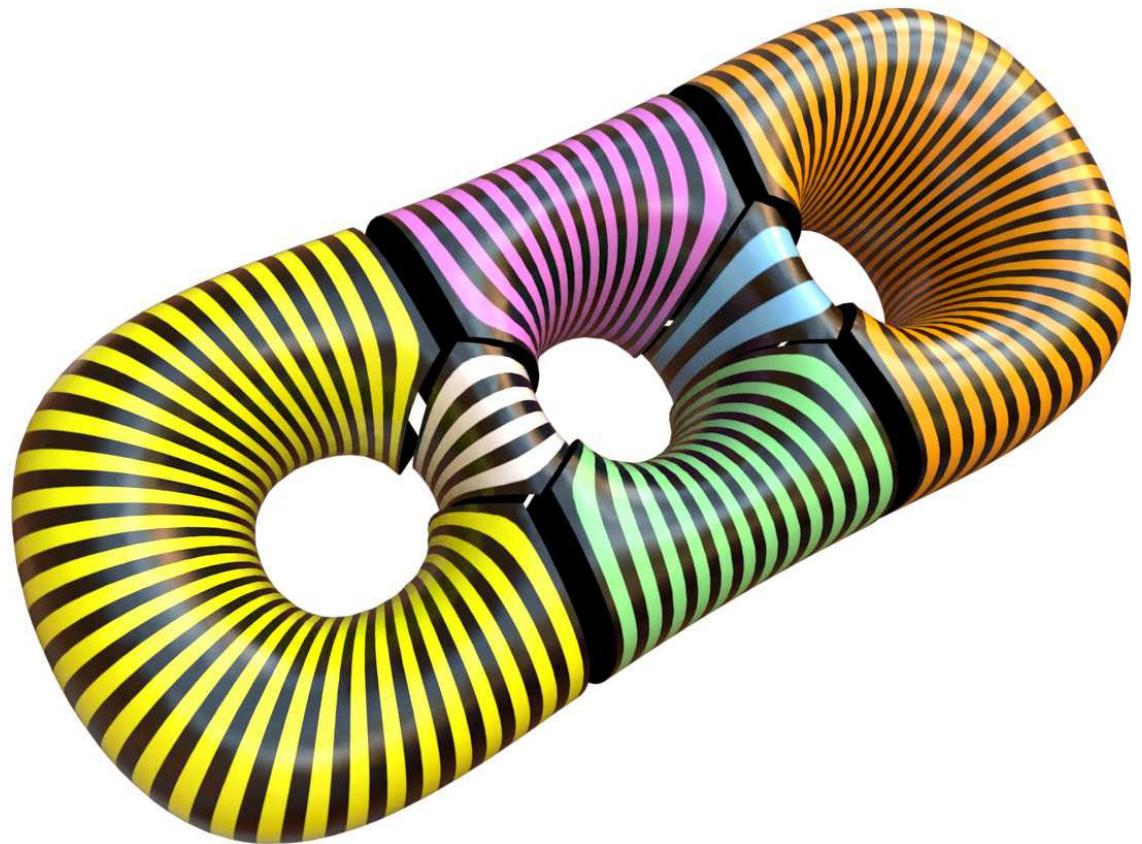
Holomorphic Quadratic Differential



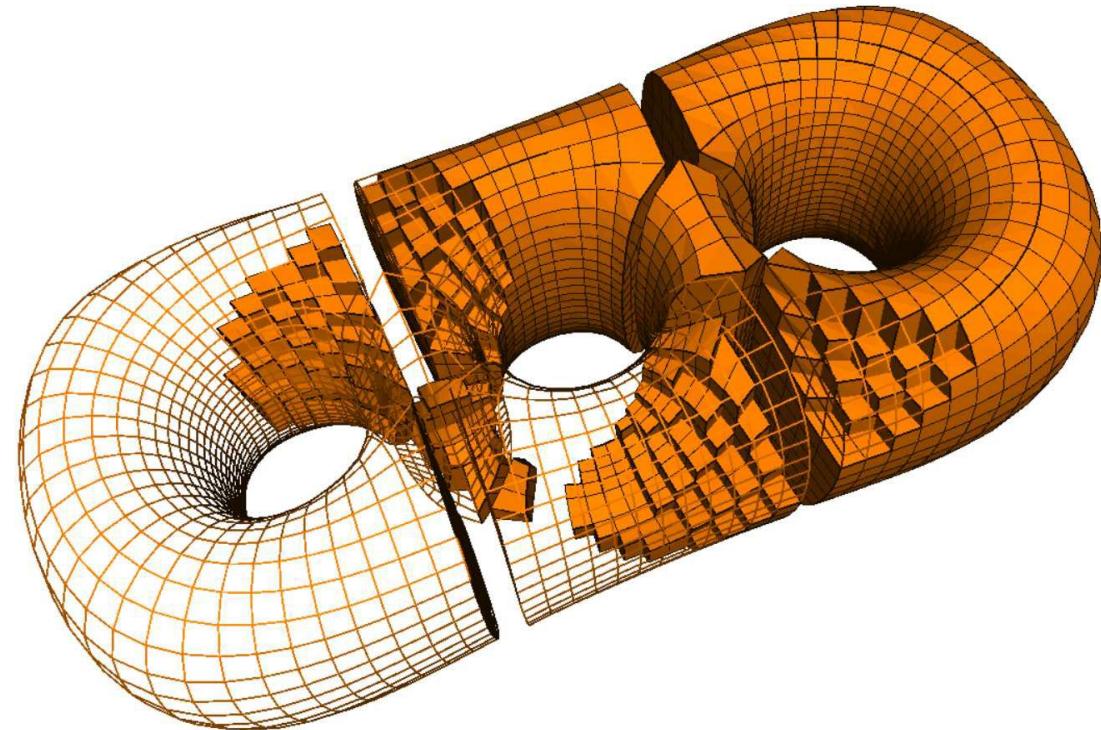
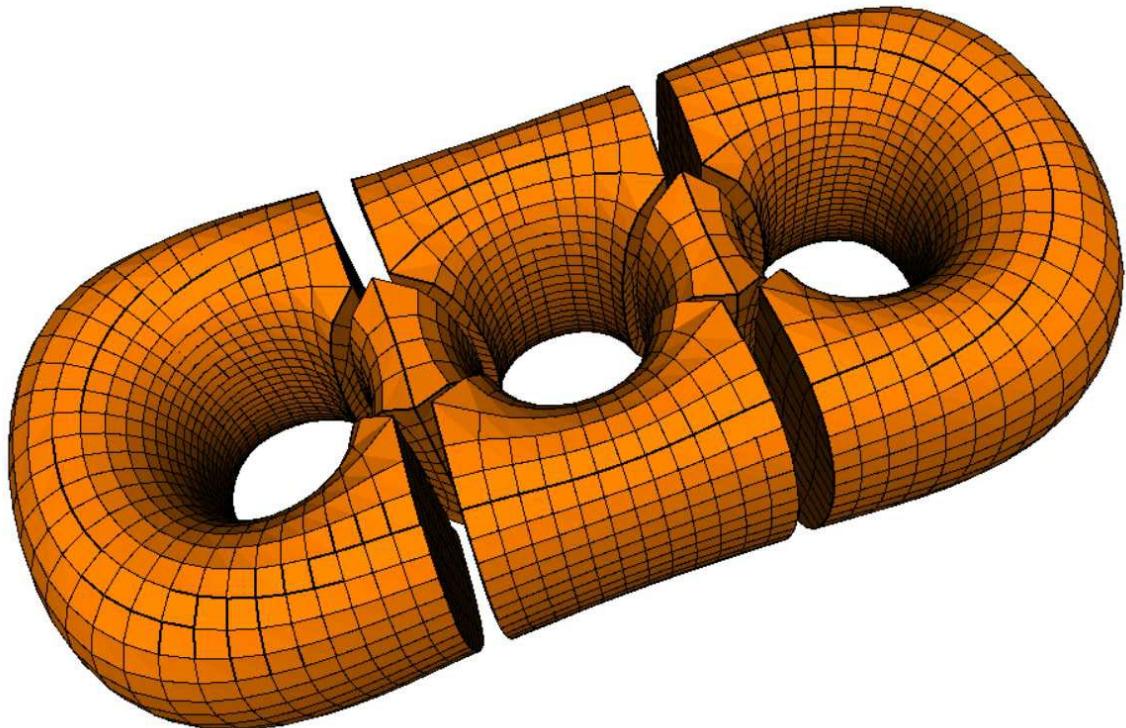
Hexahedral mesh generation



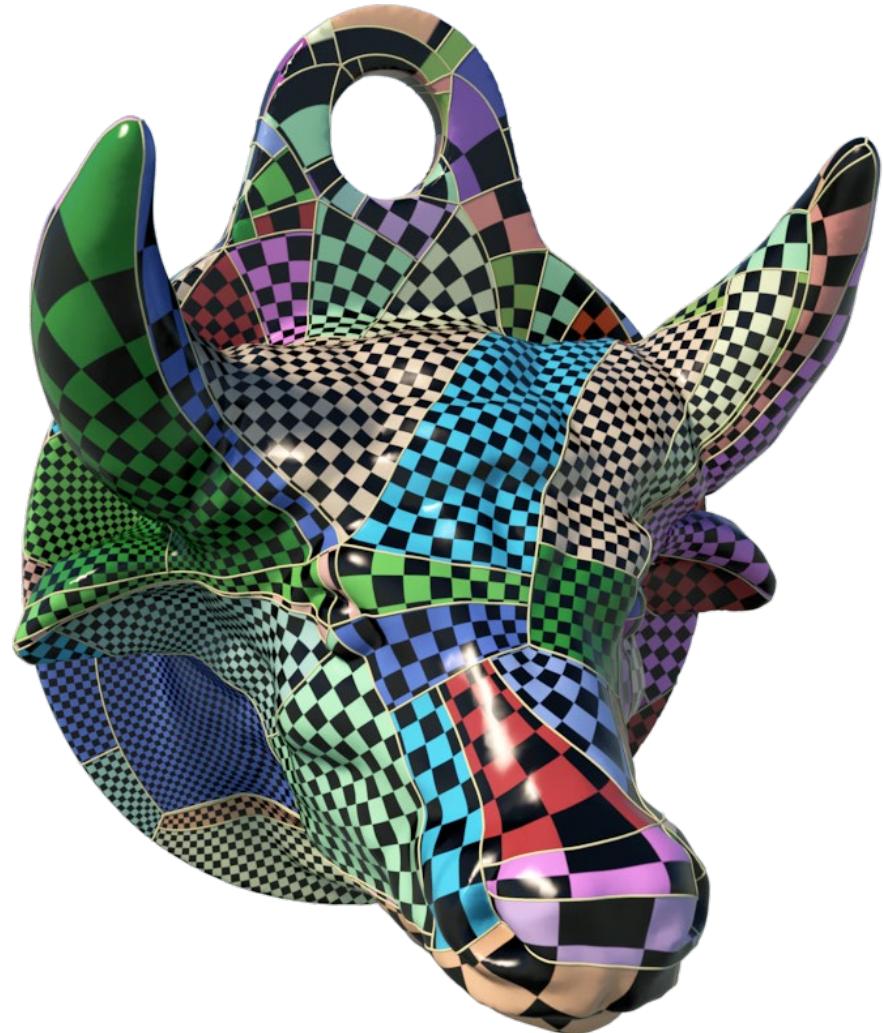
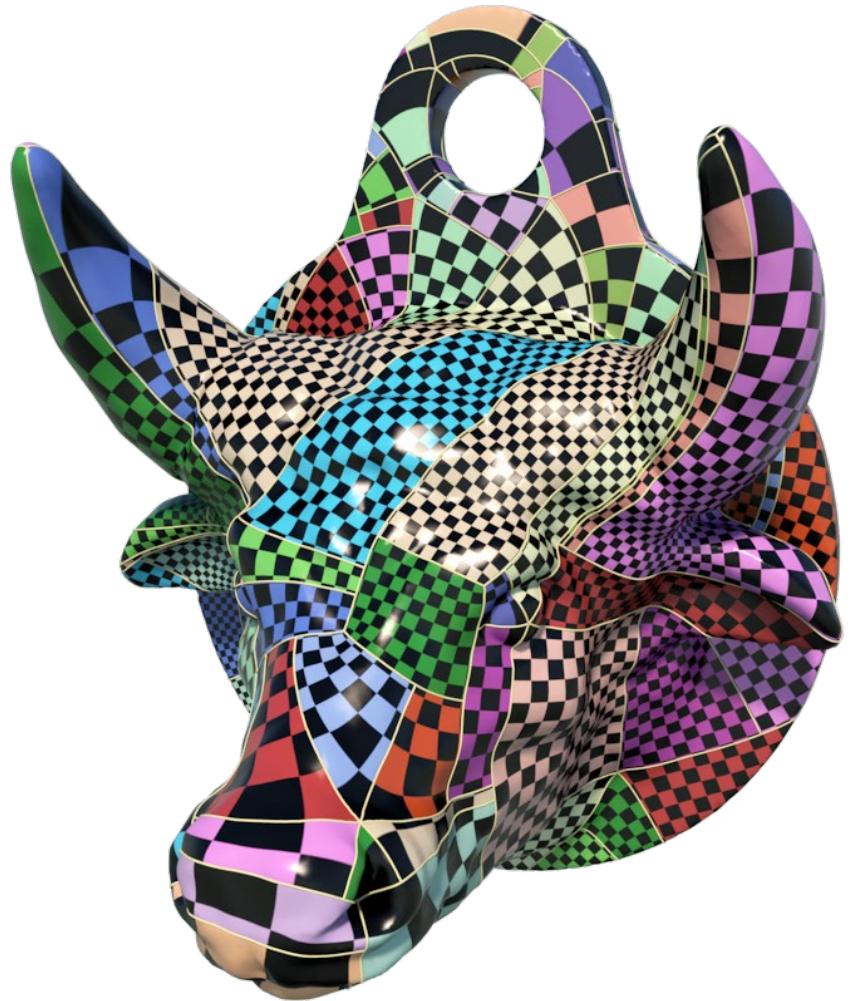
Quad-Mesh Theory



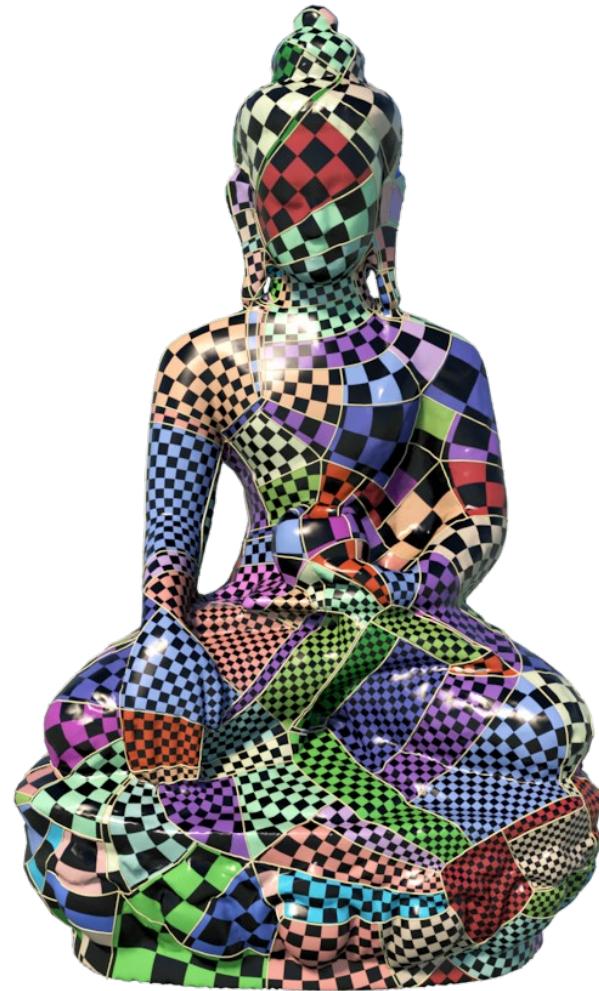
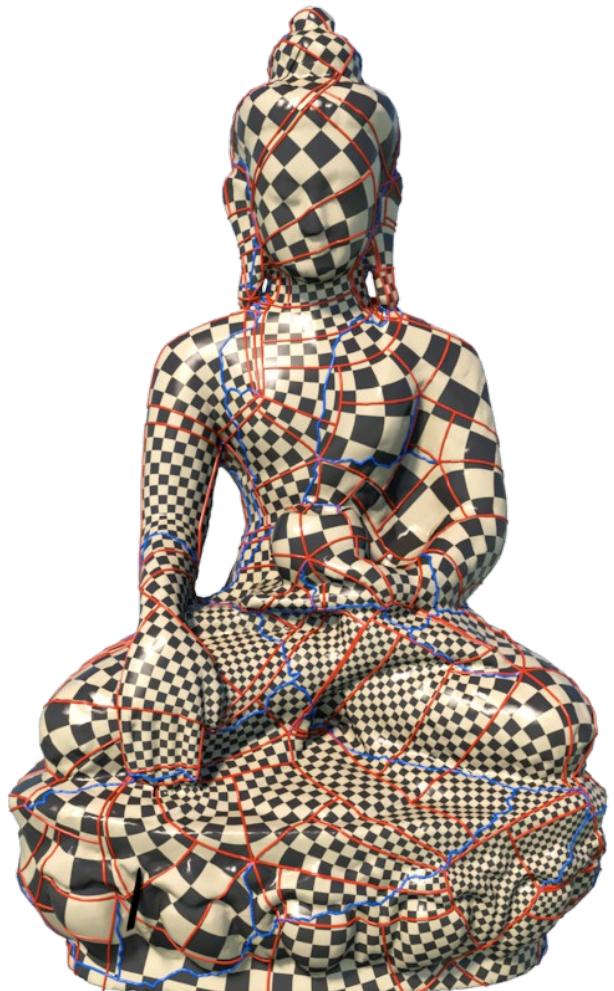
Quad-Mesh Theory



Quad-Mesh Theory



Mesh T-Spline



Mesh T-Spline



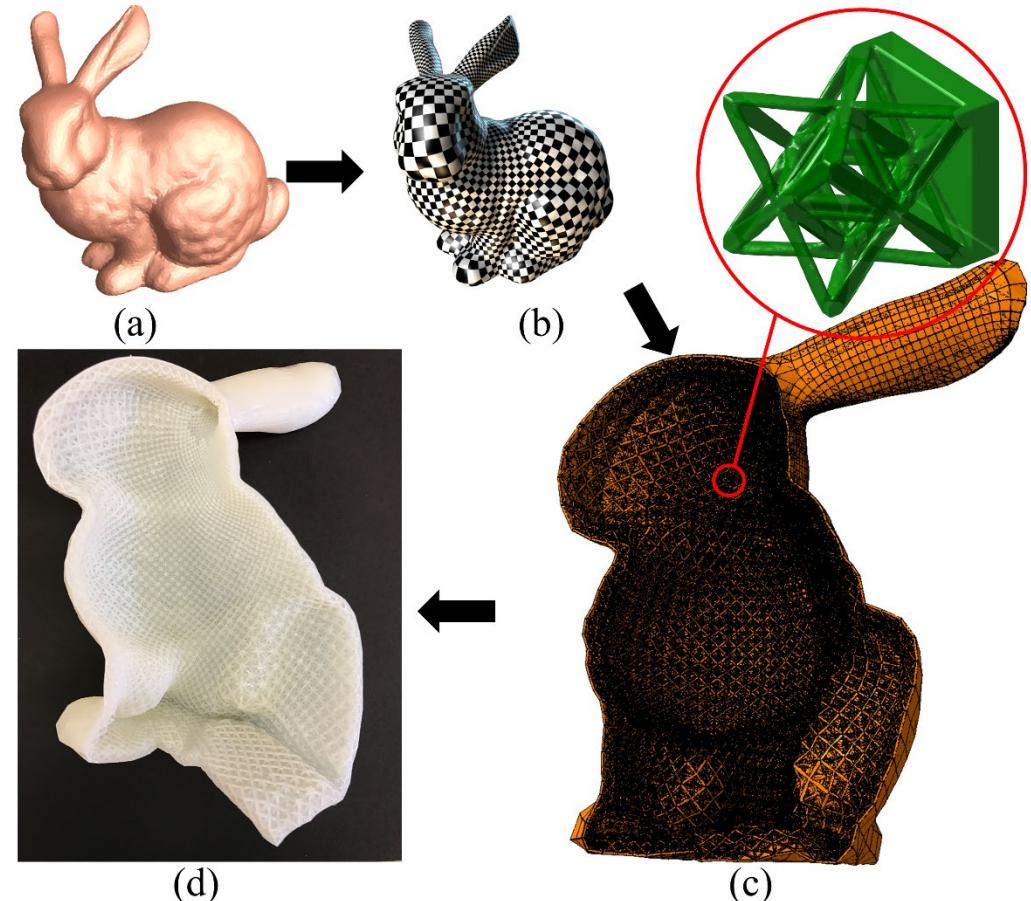
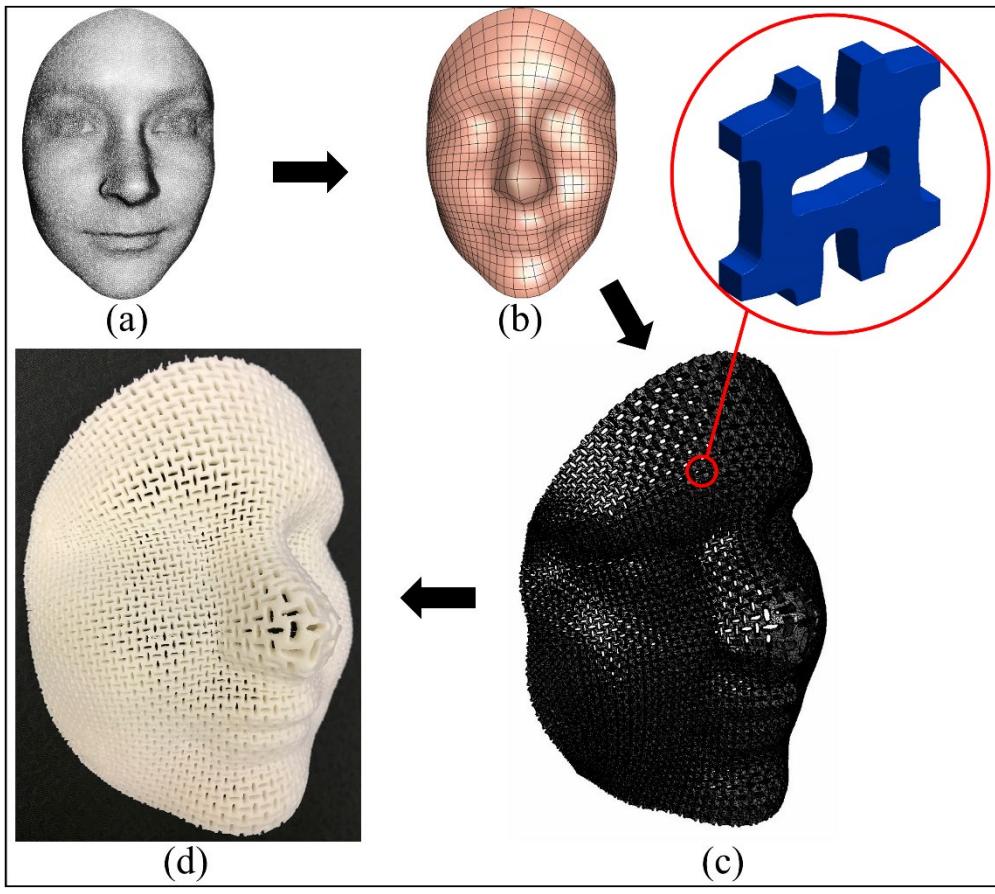
Digital Manufacture

Problem: How to design meta-materials on surfaces?

Answer: Using conformal mapping to generalize planar design to surface domain, cell structures.

3D Metamaterial Design

- 3D print bullet proof mask, ultra light and ultra stiff material.



Fabrication



Fabrication

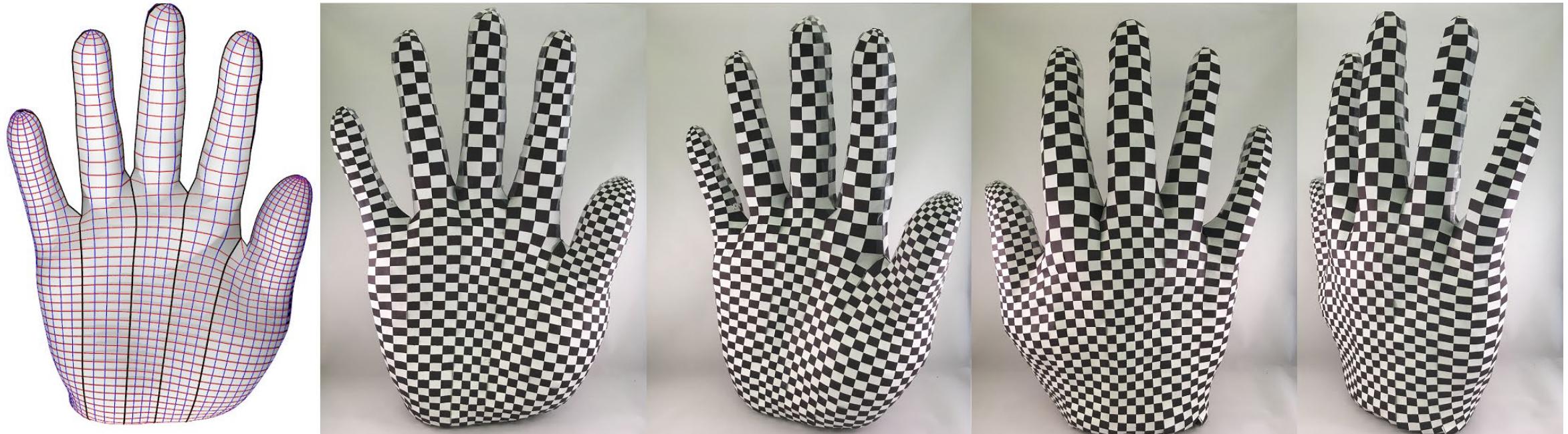


Fabrication by weaving

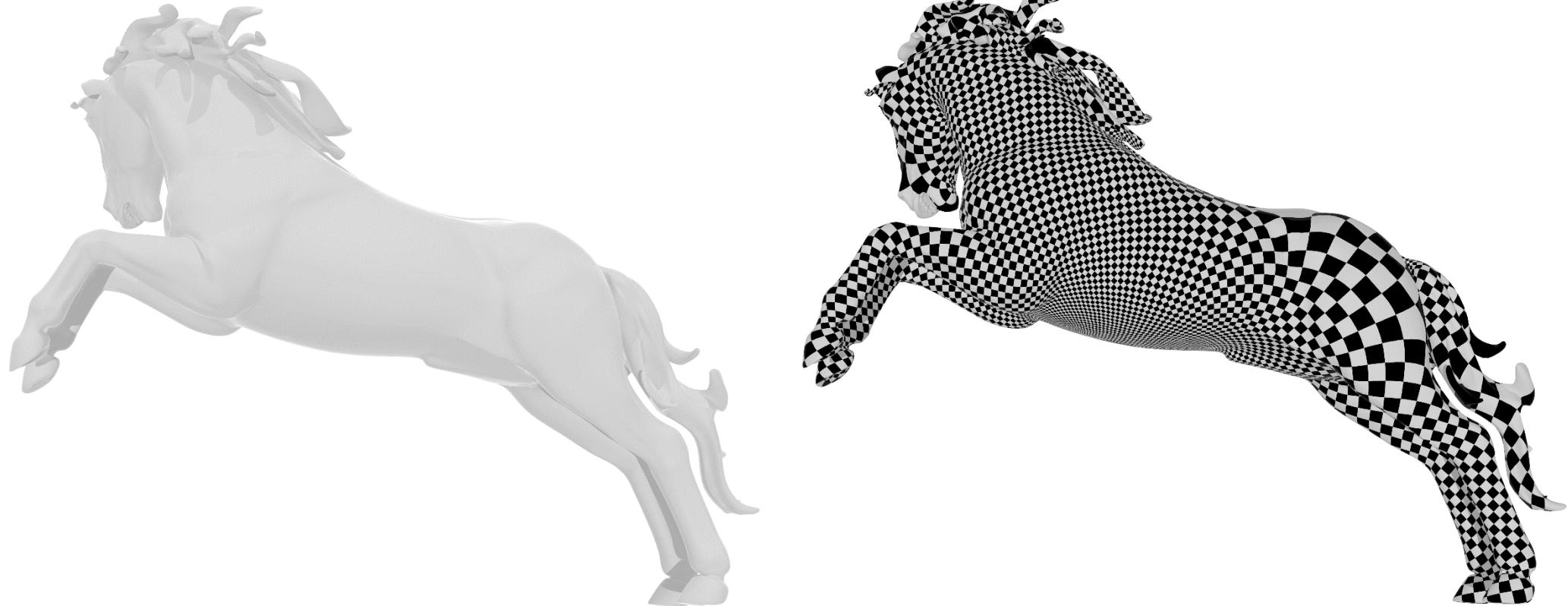
**This video demonstrates
how the paper hand model
is woven from orthogonal
strips generated by our
algorithm.**

Fabrication

- Carbon fiber



Sculpture Design



Sculpture



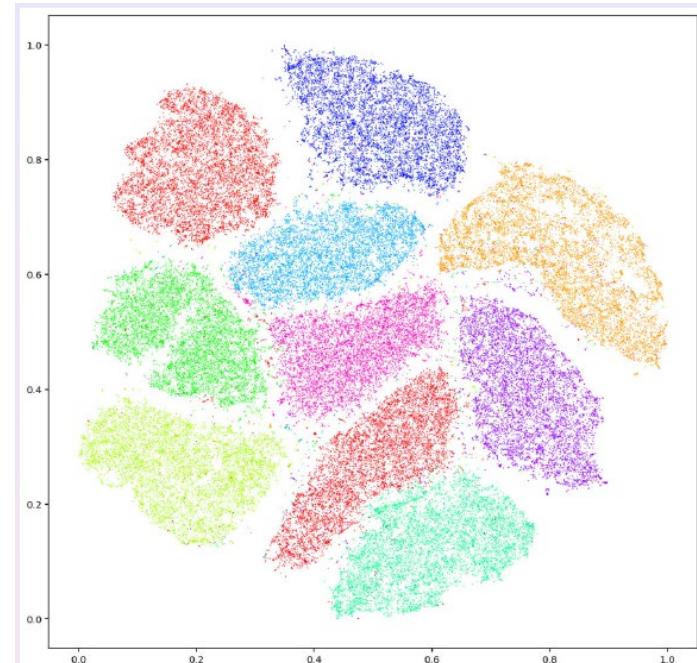
Geometric View to Deep Learning

A Geometric Framework for Deep Learning

Our physical world is geometric in nature.

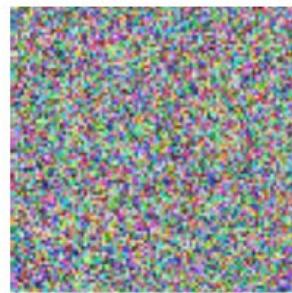
Manifold hypothesis:

Real data concentrates near low dimensional manifolds, different classes correspond to disjoint manifolds separated by low density areas.



Deep Generative Models

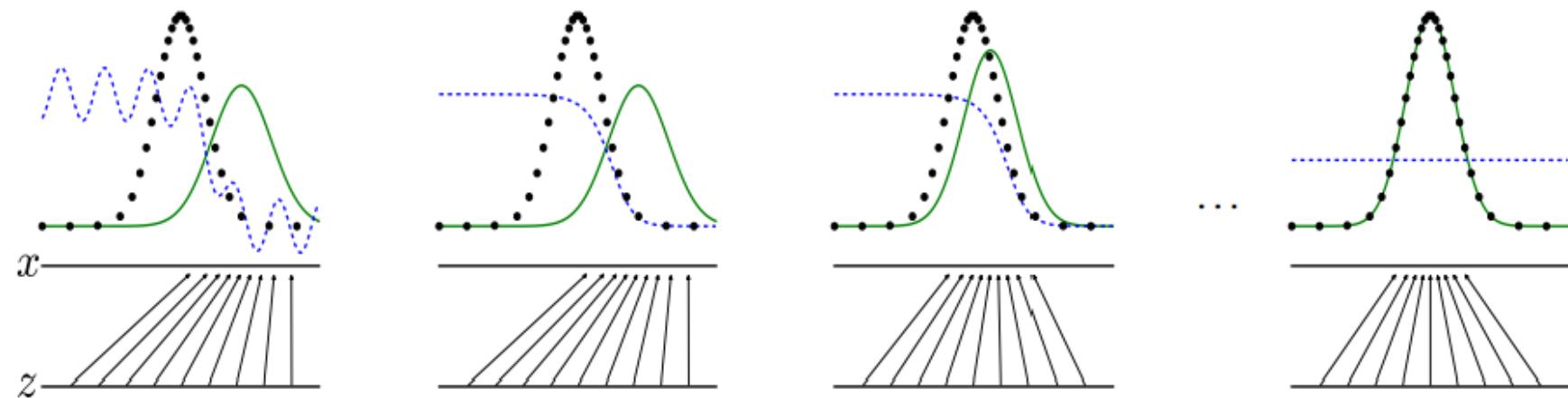
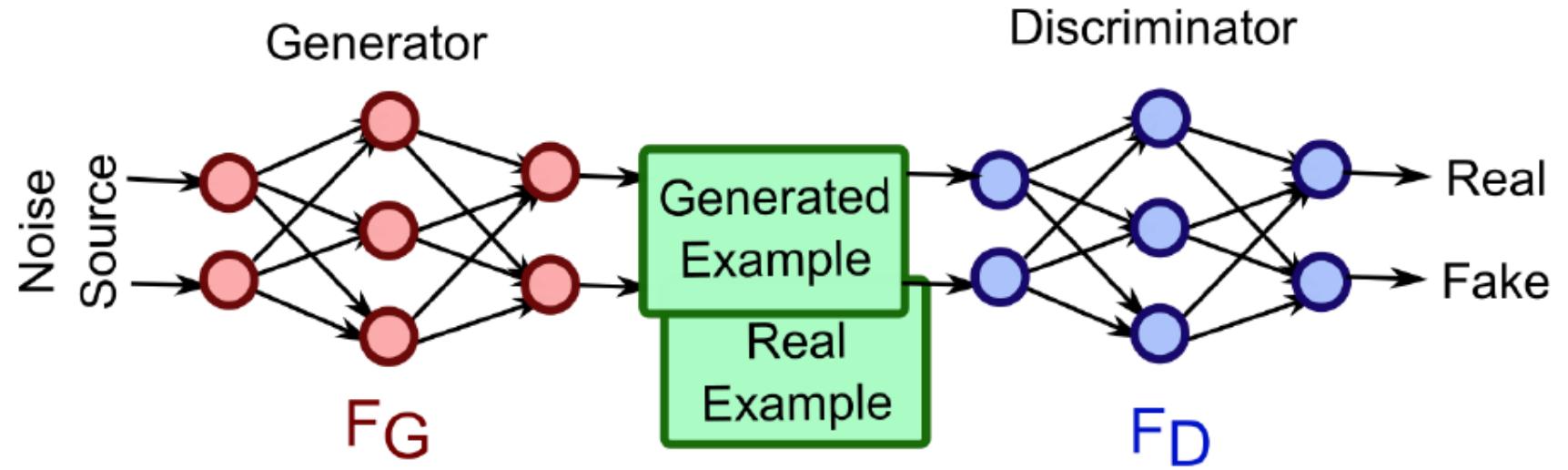
Noise $\sim N(0,1)$



Generative
Model

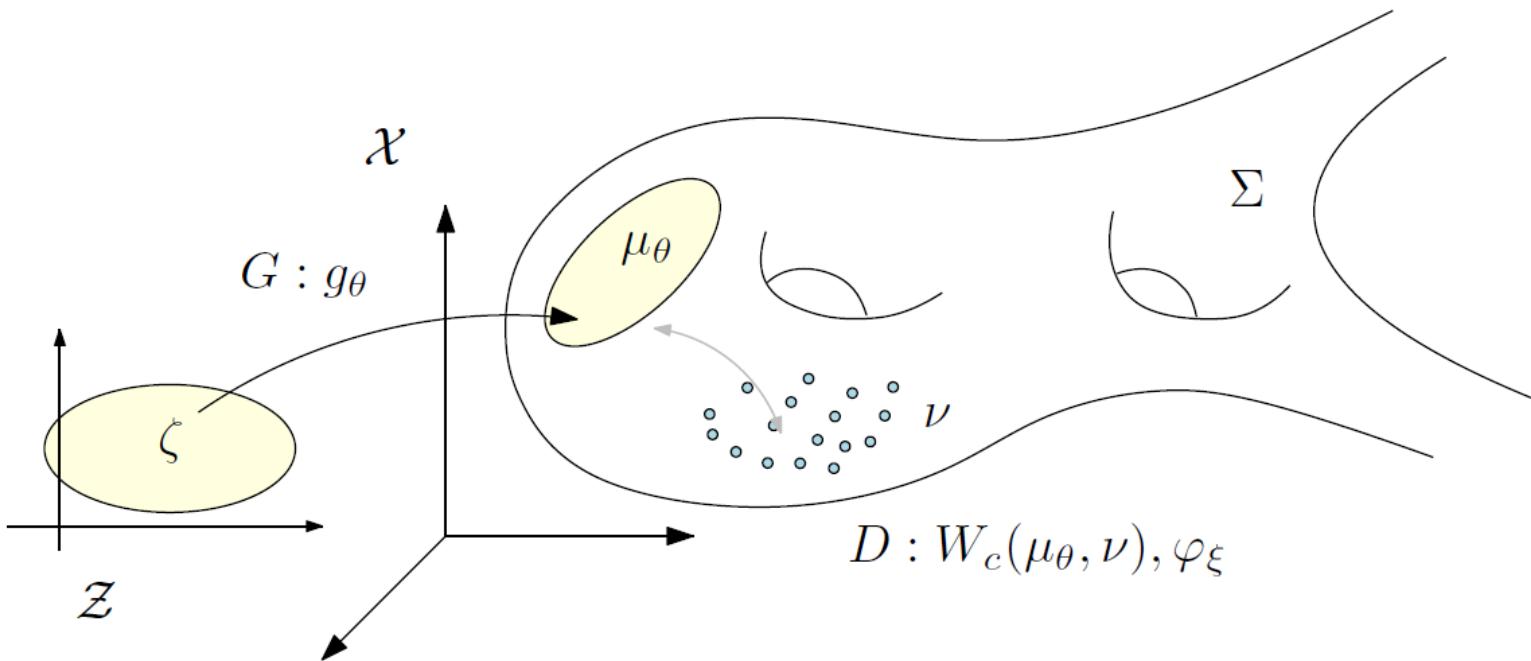


Generative Adversarial Networks



Generative Adversarial Networks

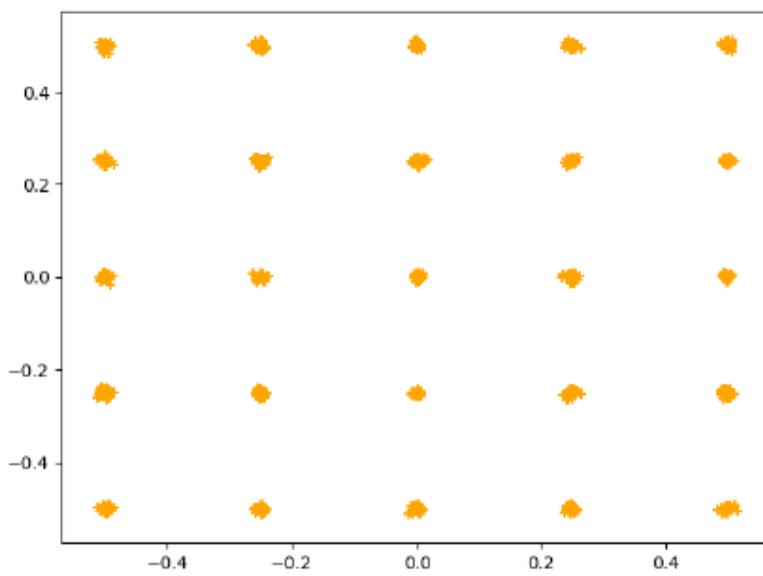
\mathcal{X} -image space; Σ -supporting manifold; \mathcal{Z} -latent space;



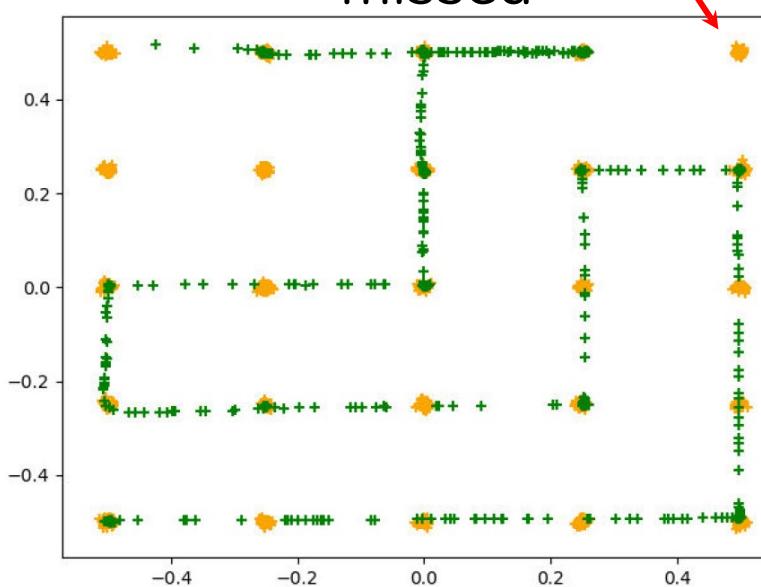
ν -training data distribution; ζ -uniform distribution;

$\mu_\theta = g_\theta \# \zeta$ -generated distribution; G - generator computes g_θ ; D - discriminator, measures the distance between ν and μ_θ , $W_c(\mu_\theta, \nu)$.

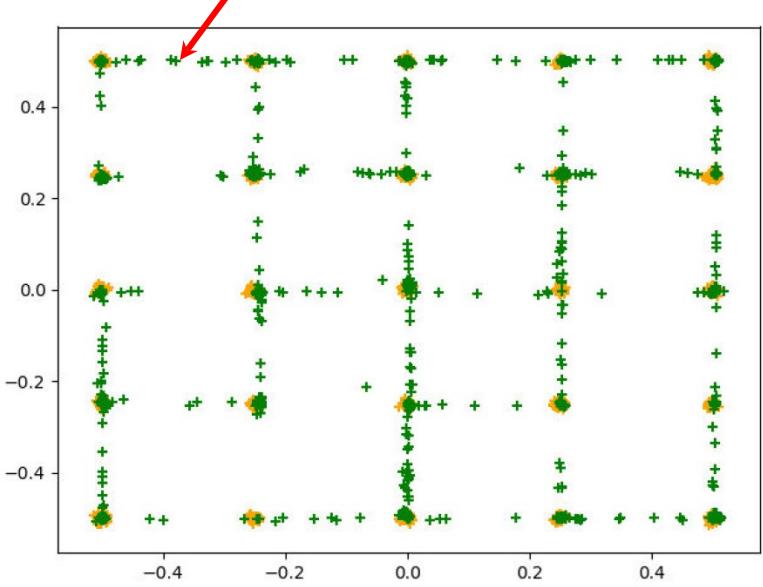
Mode Collapse in Generative Adversarial Networks



Training Data



GAN



PacGAN

Generated MNIST Samples



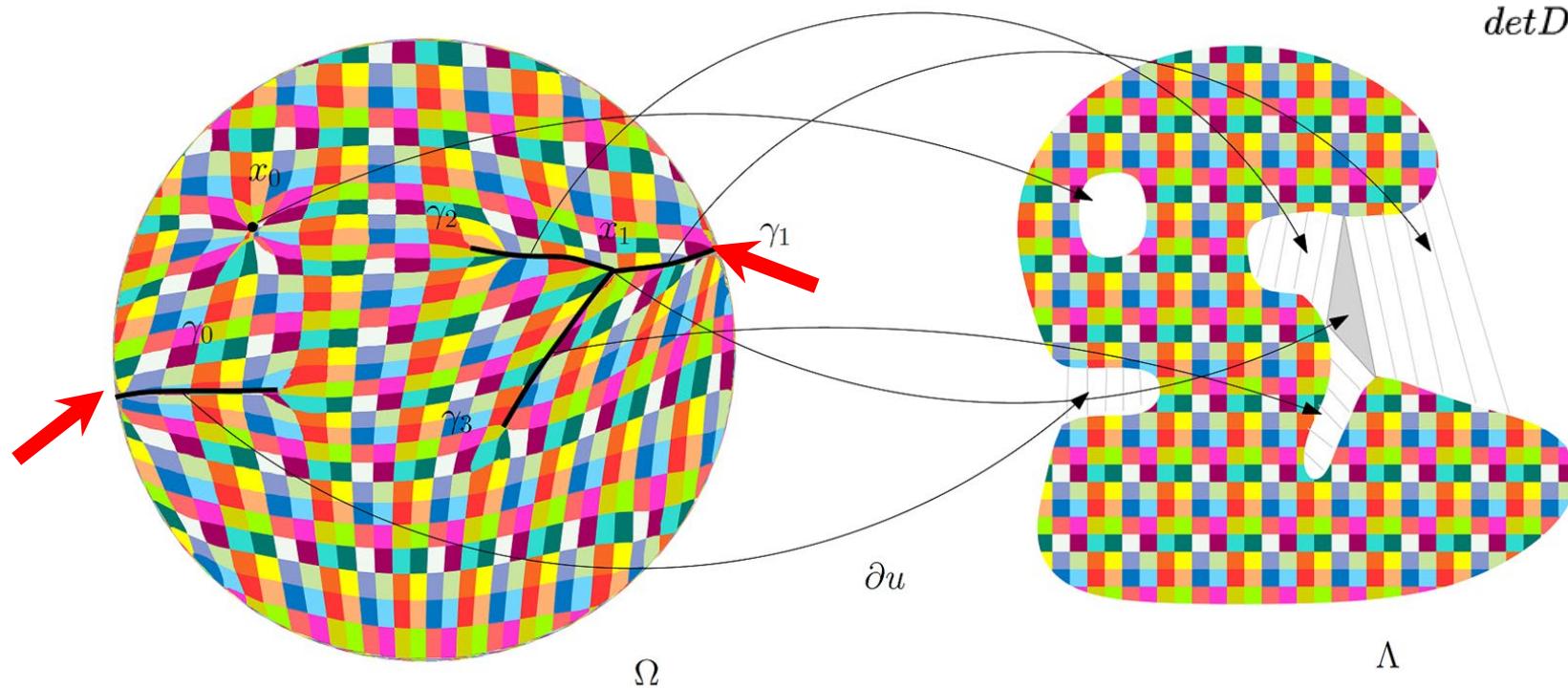
VAE



GAN

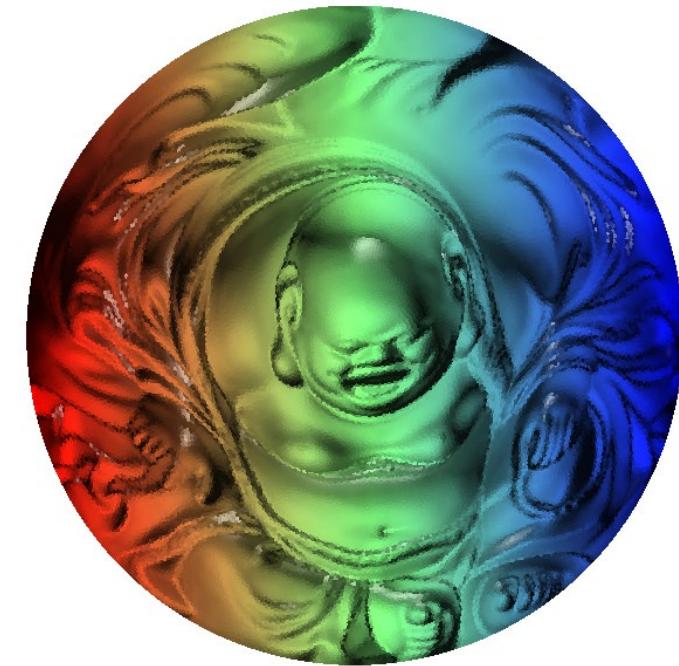
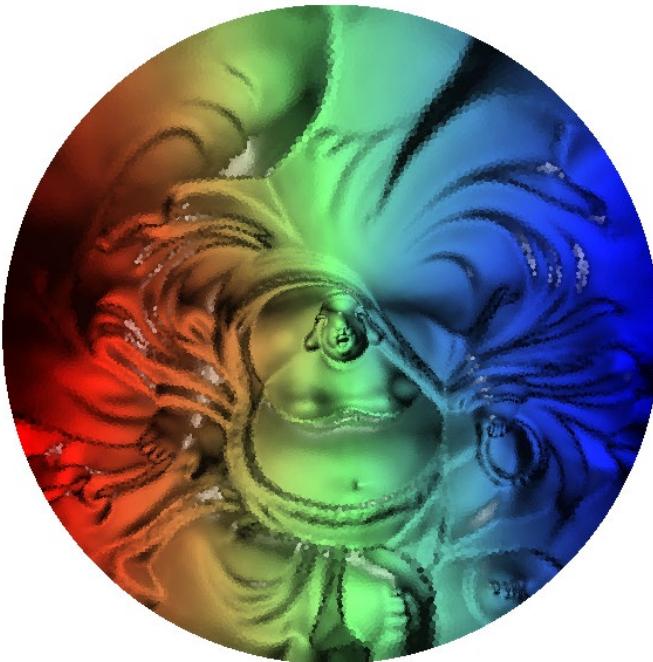
Regularity Theory

- Alessio Figalli's theorem: If the target support is non-convex, the **optimal transport map** is discontinuous at some singularity sets.

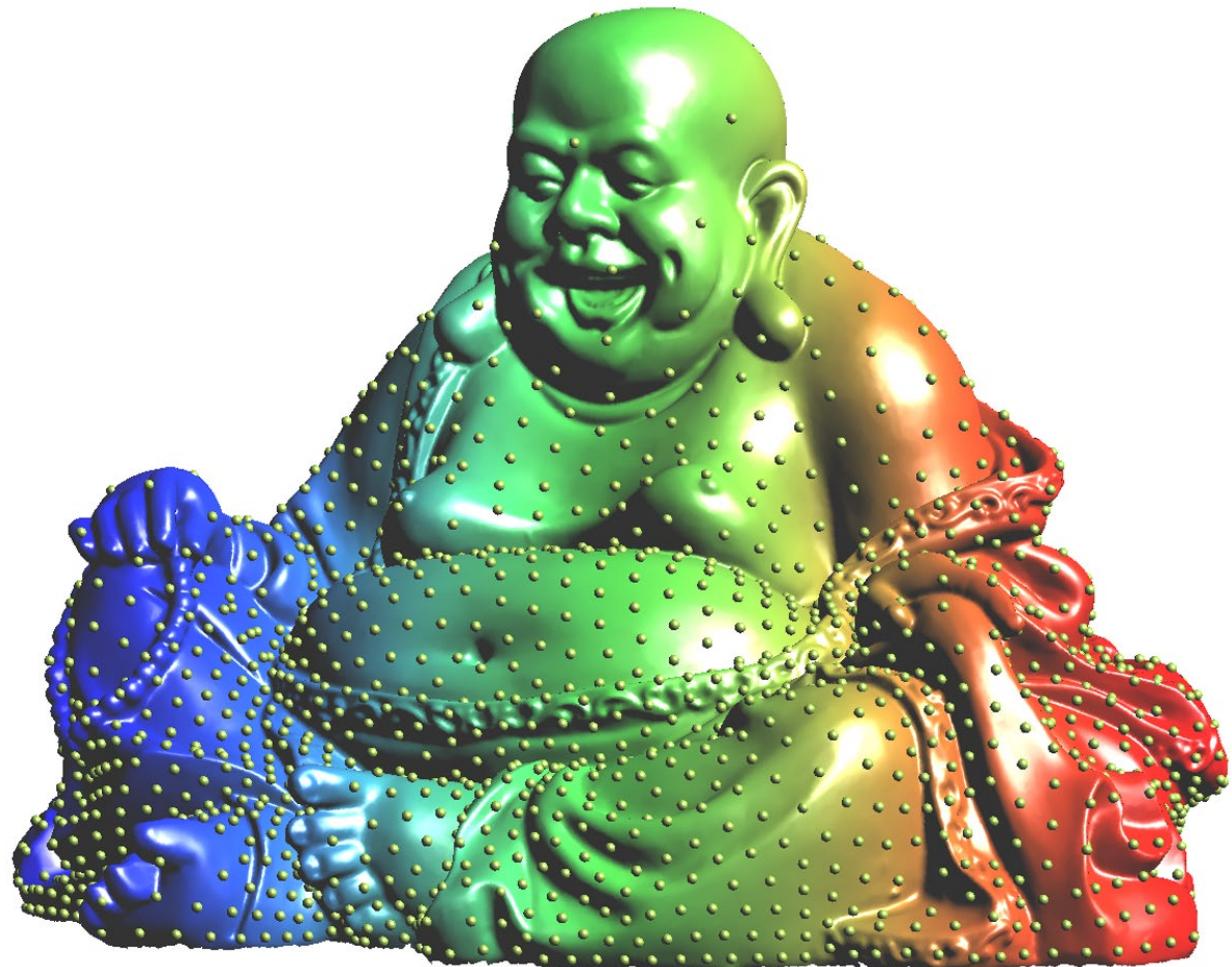


$$\det D^2 u(x) = \frac{f(x)}{g \circ \nabla u(x)}$$

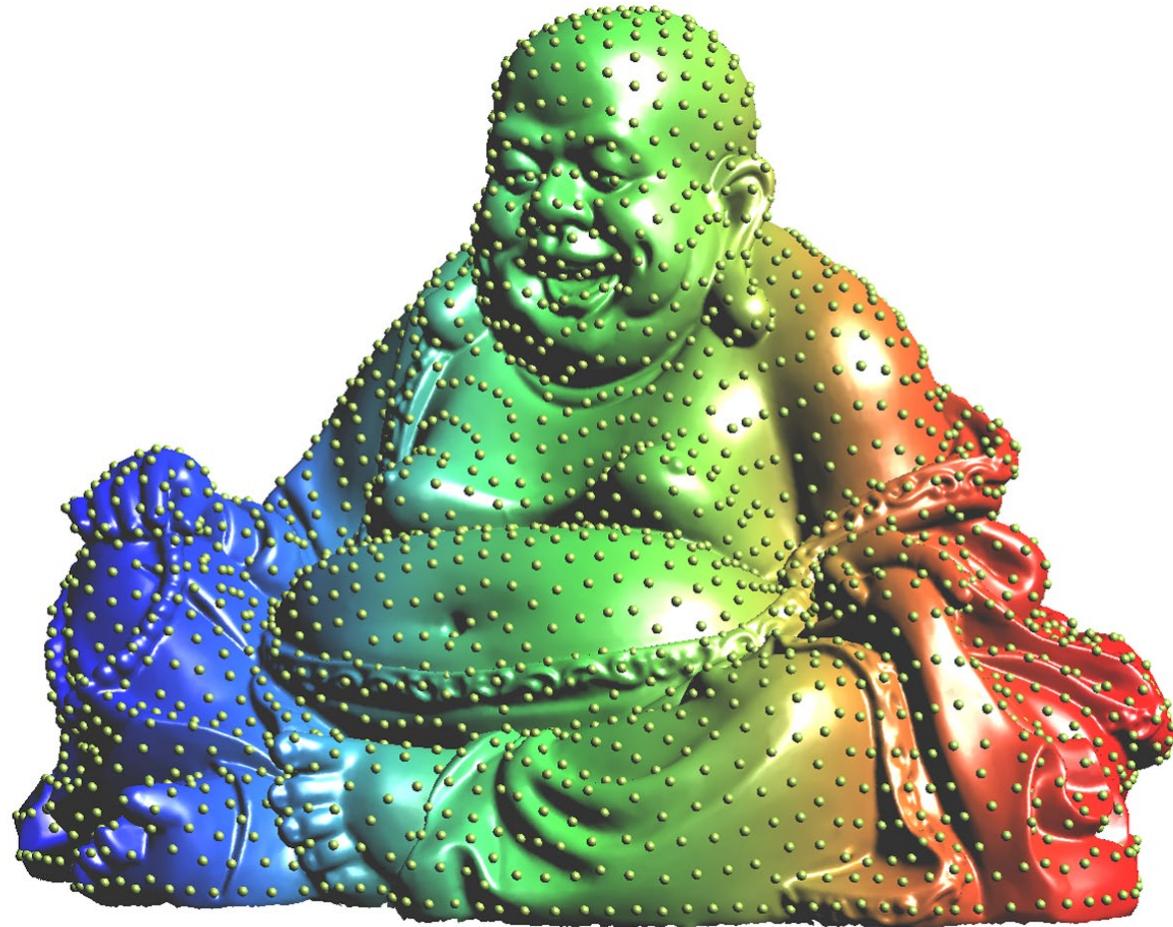
Optimal Transportation Map



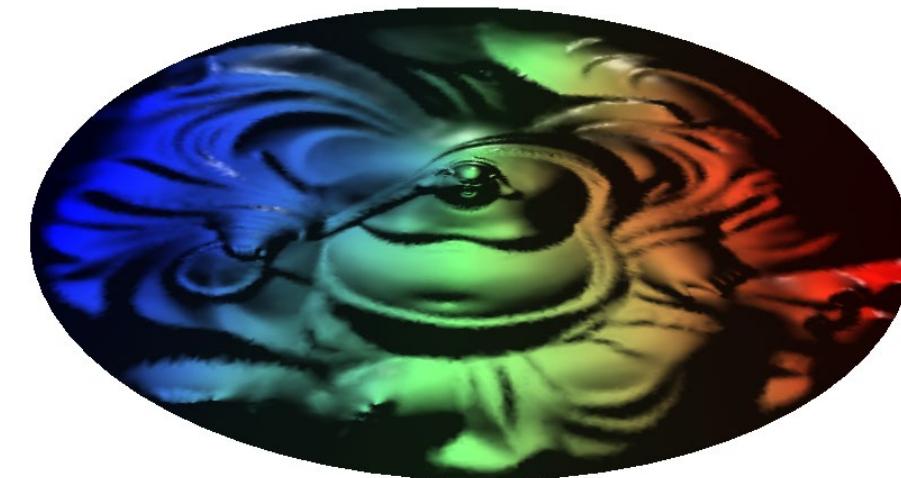
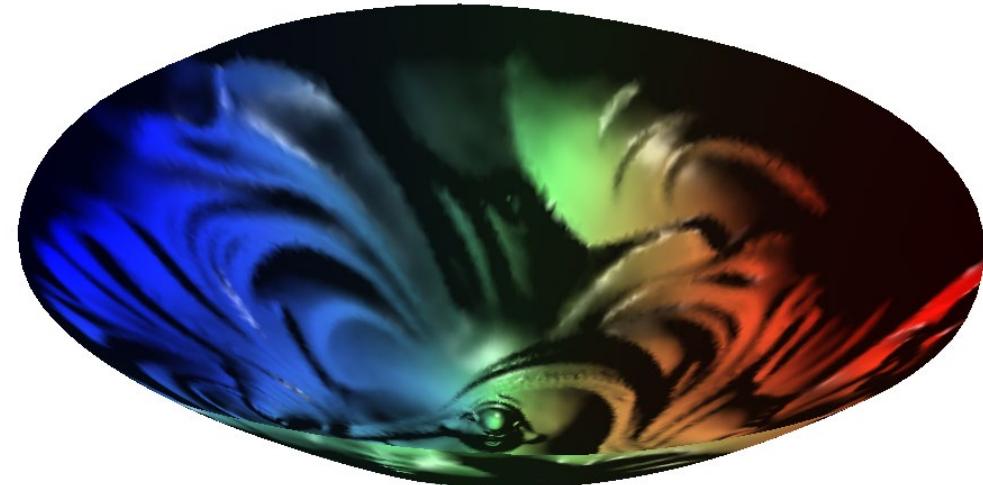
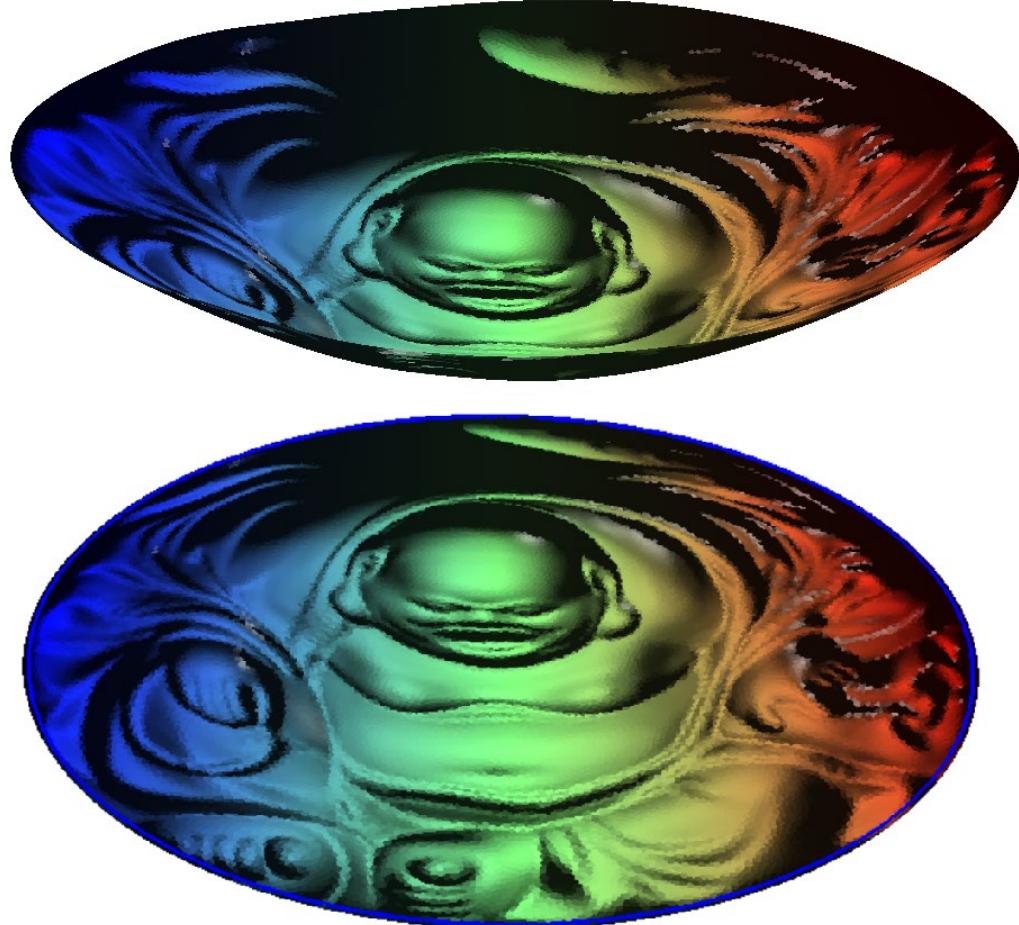
Optimal Transportation Map



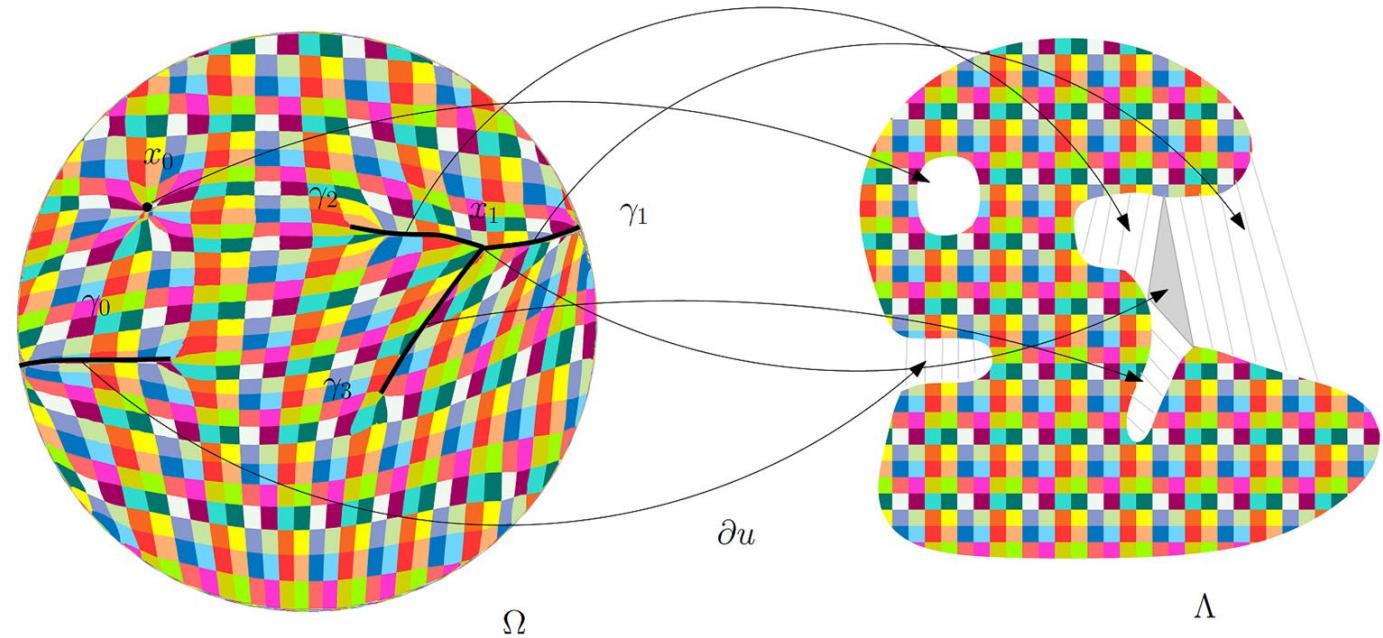
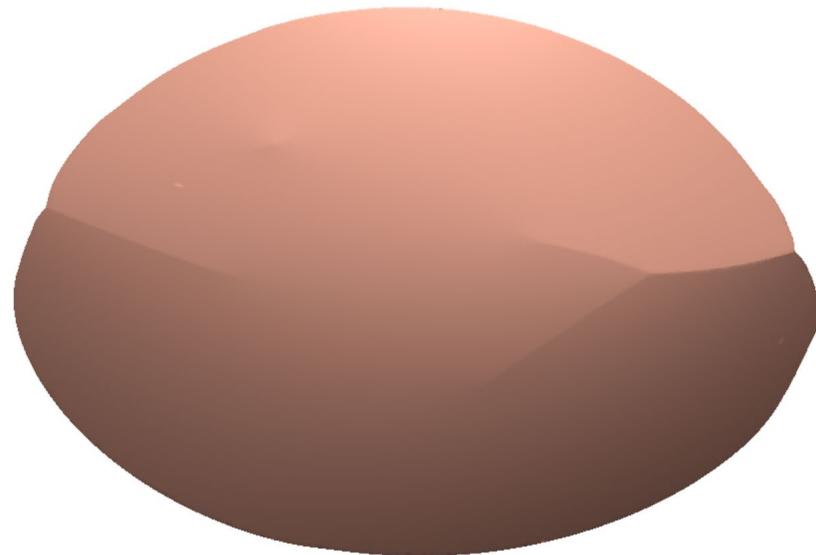
Optimal Transportation Map



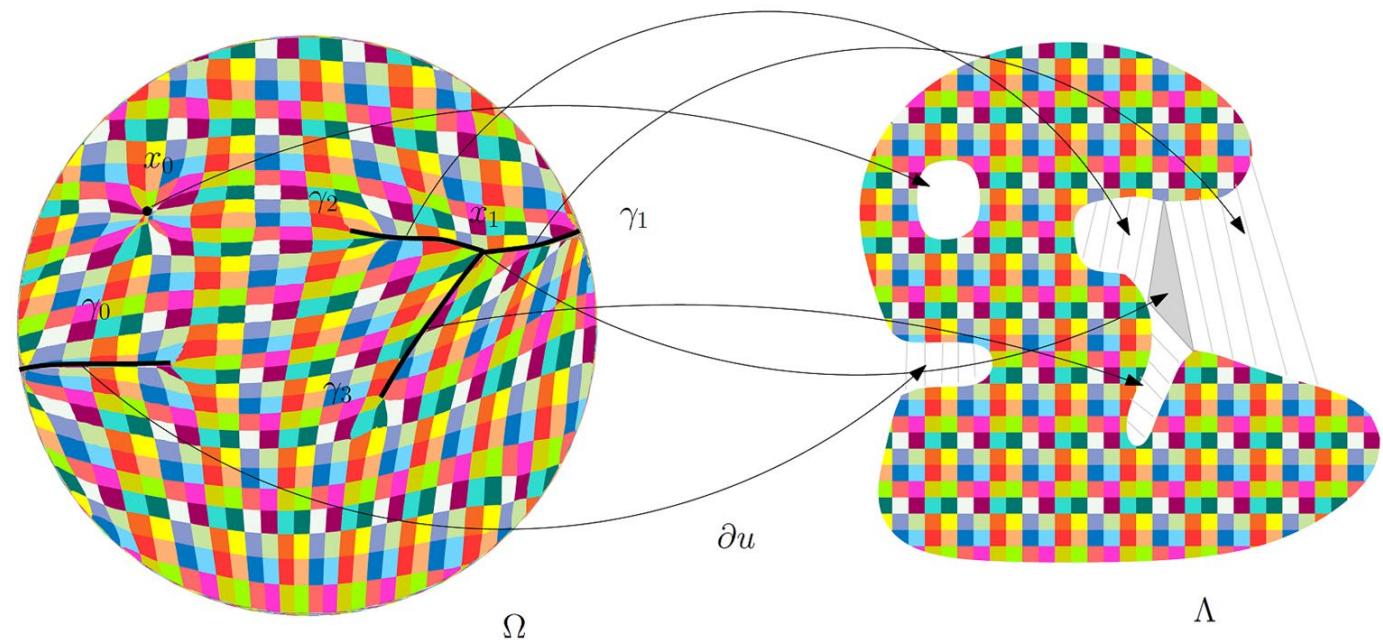
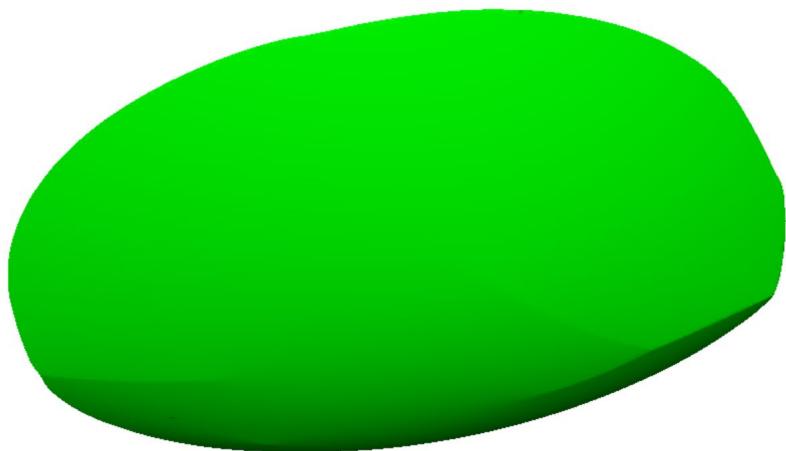
Optimal Transportation Map



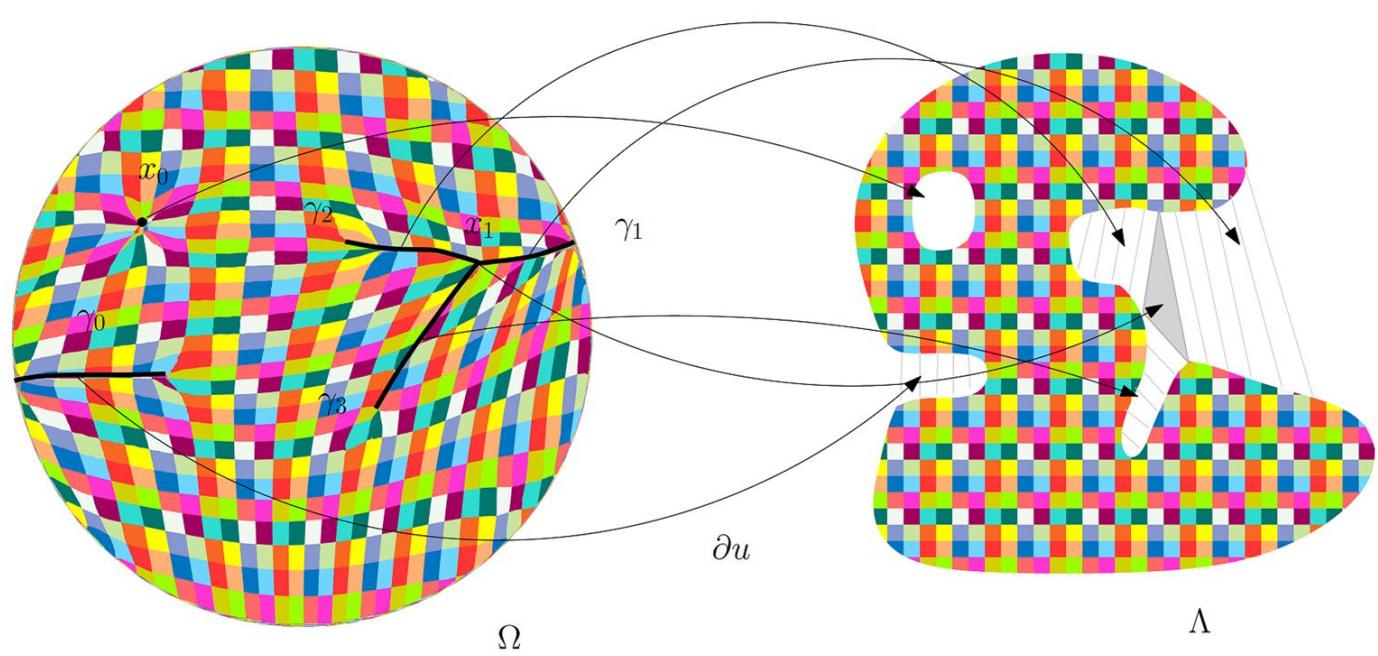
Brenier Potential Singularity



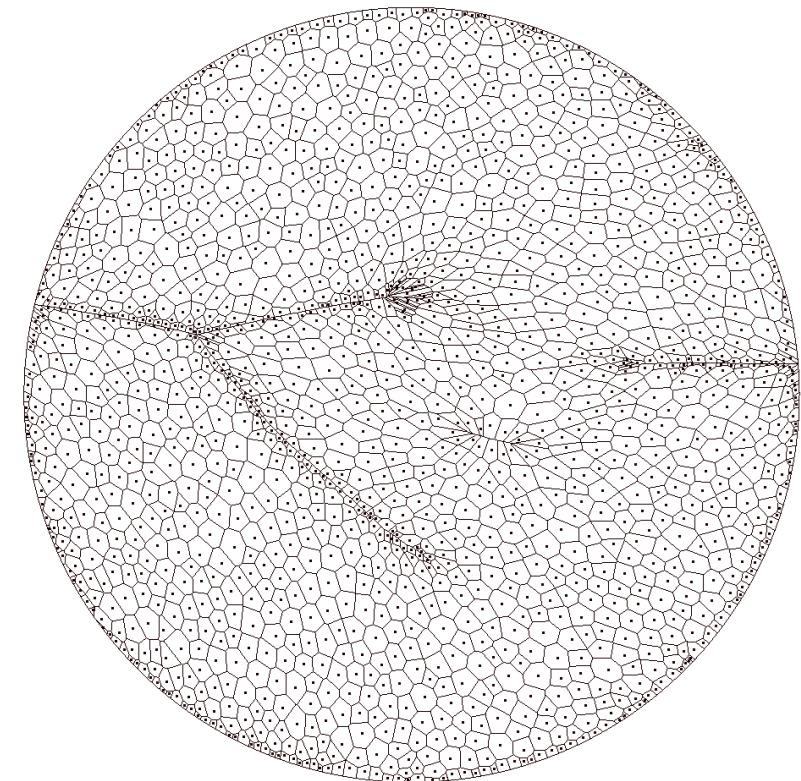
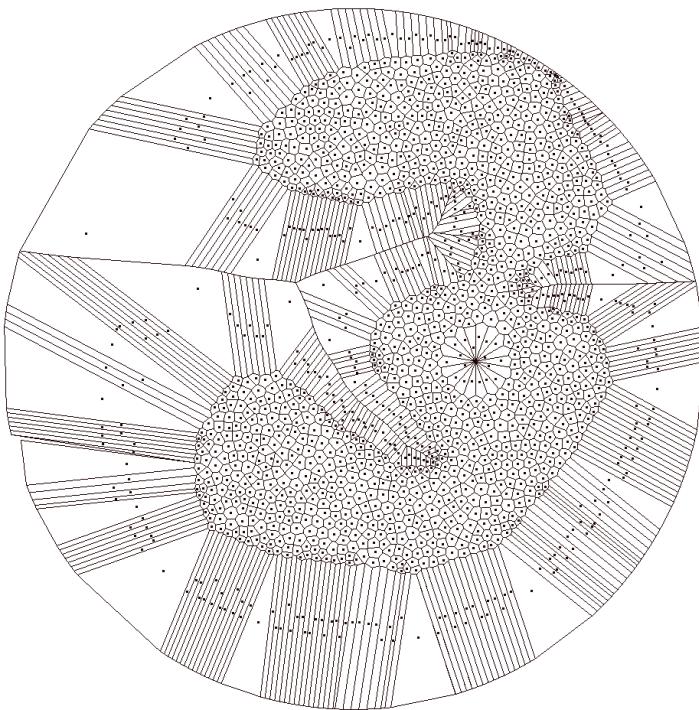
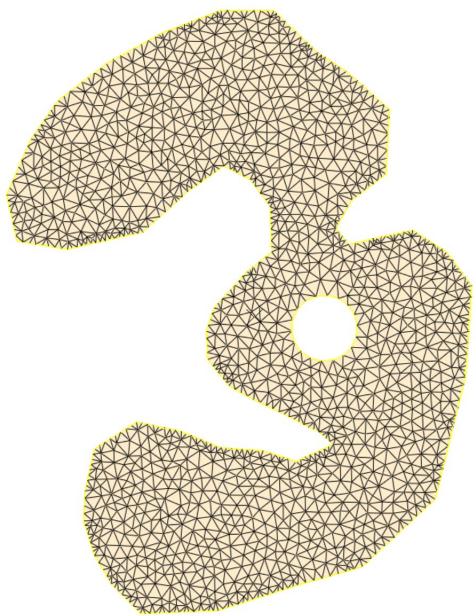
Brenier Potential Singularity



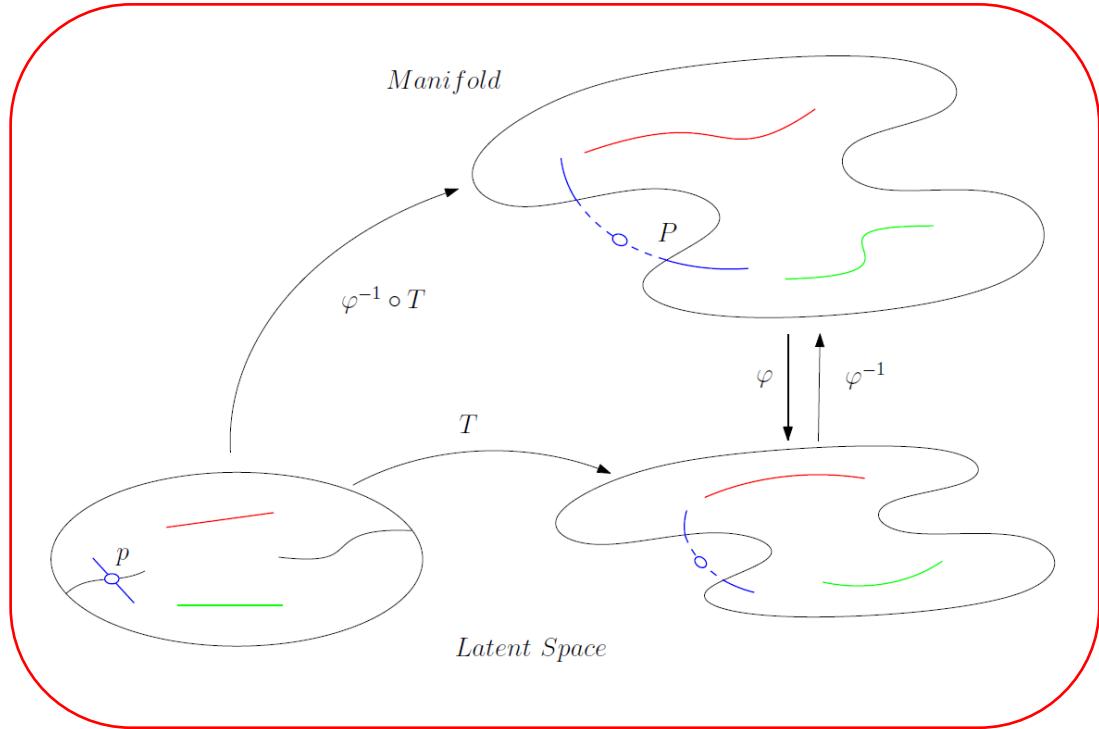
Brenier Potential Singularity



Brenier Potential Singularity



Regularity Theory and Mode Collapse

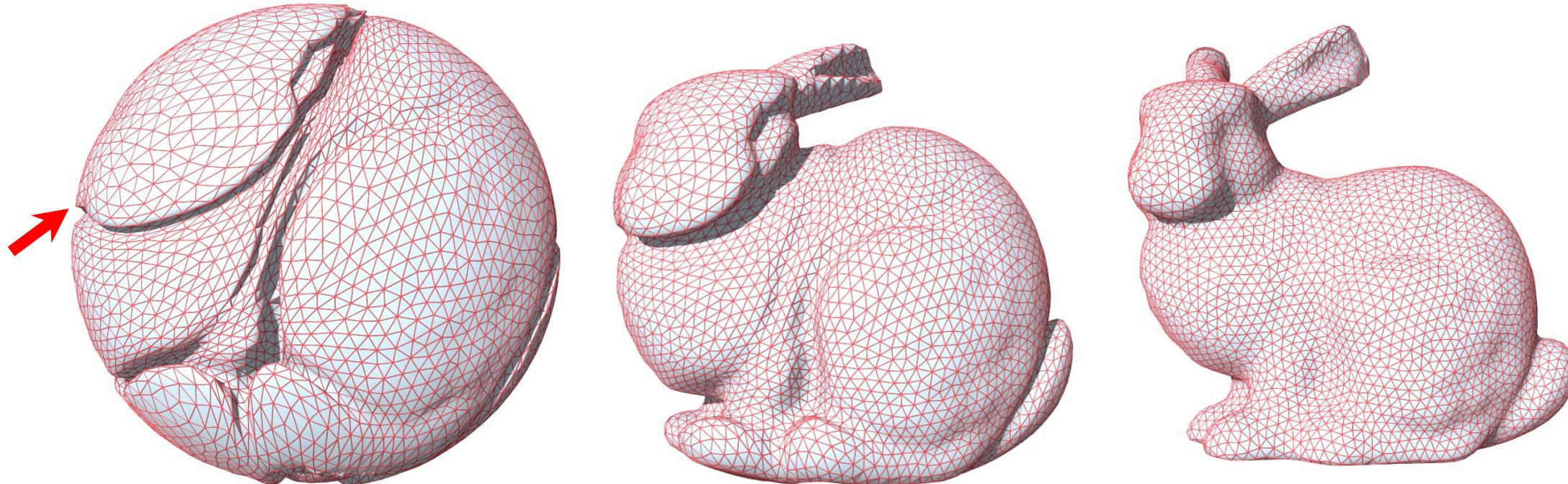


Paths on human facial photo manifold



Implication of Figalli's Regularity Theorem

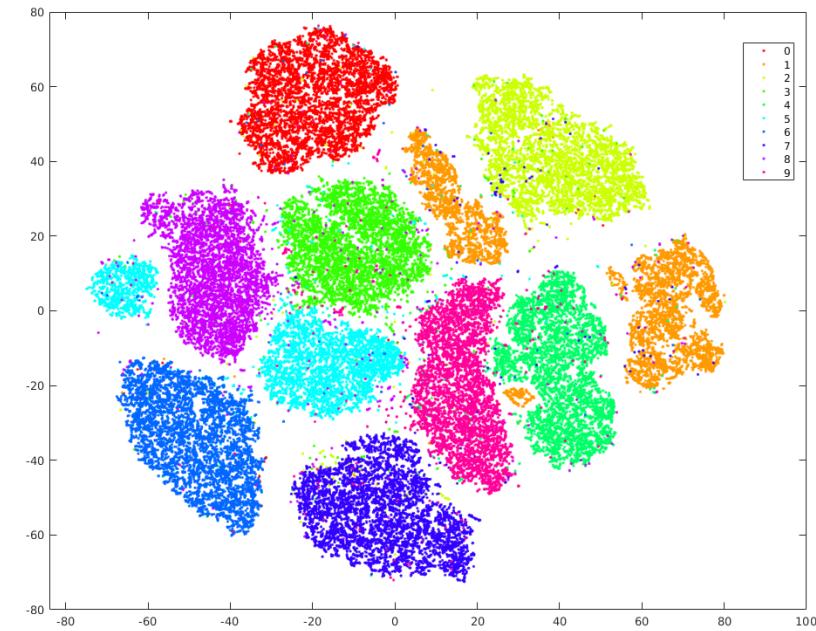
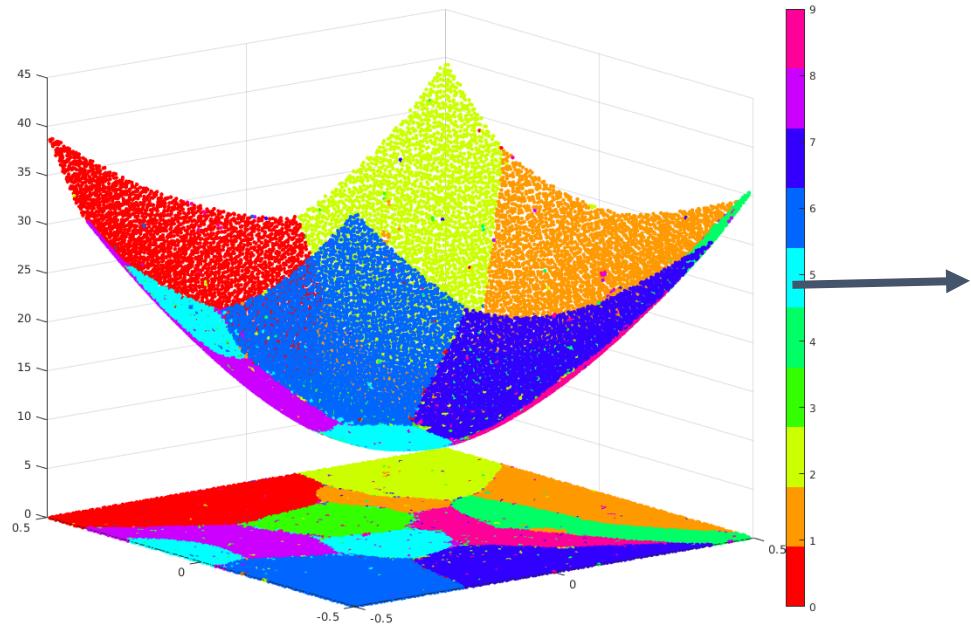
Deep neural networks can only represent **continuous** maps.
This intrinsic conflict causes mode collapses and mode mixture.



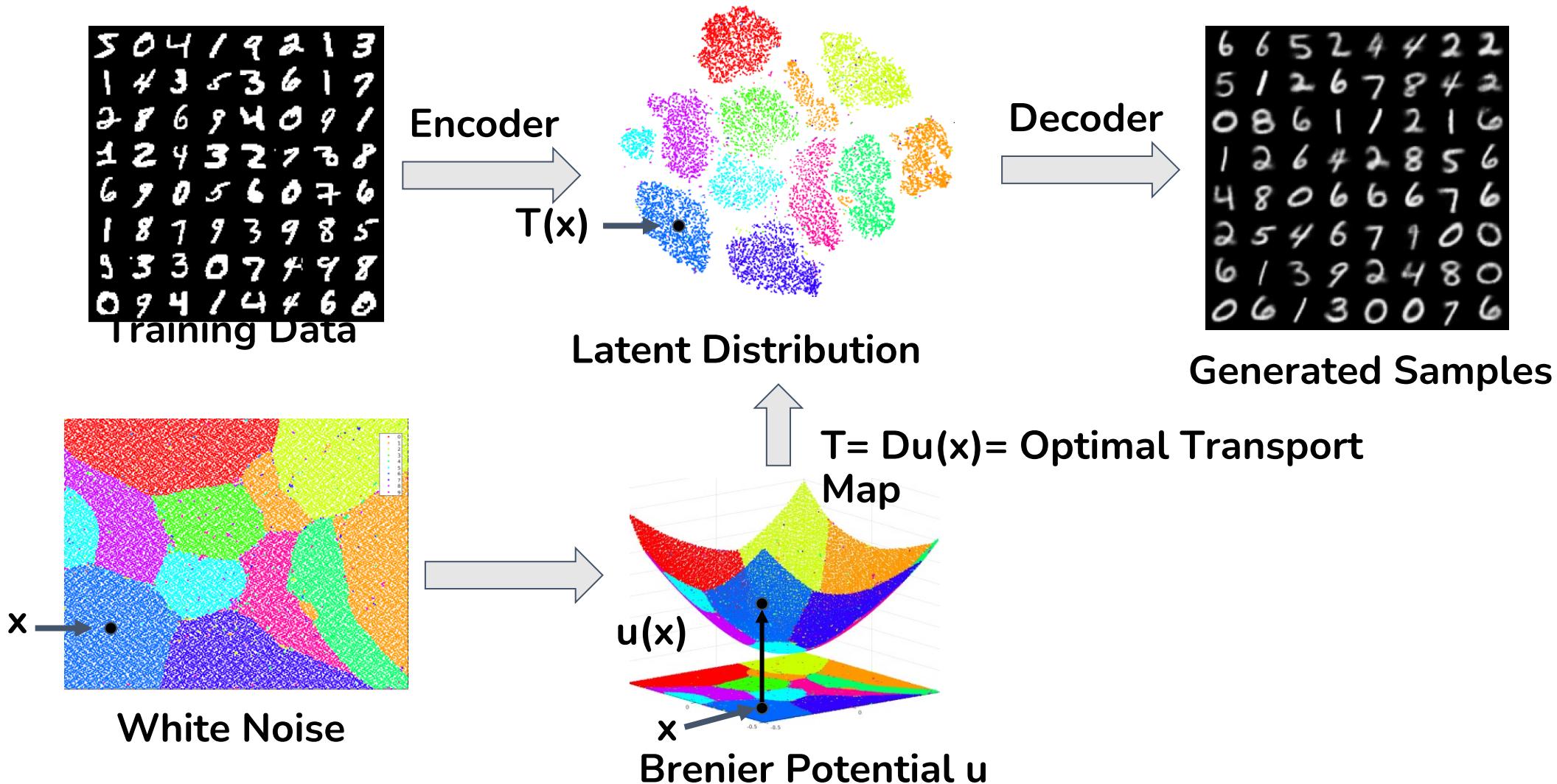
Eliminate Mode Collapse by Brenier Map

- Compute Brenier's potential (continuous), whose gradient (possibly non-continuous) gives the optimal transport map.

Brenier's potential



Optimal Transport Generative Model



GAN based on Brenier Theory



Conclusion

Modern Geometry is essential to tackle fundamental problems in different fields in engineering and medicine.

- Computer Graphics
- Computer Vision
- Geometric Modeling (CAD/CAE)
- Wireless Sensor networks
- Medical Imaging
- Digital geometry processing
- Artificial Intelligence

Thank You