Surface Differential Geometry, Movable Frame Method

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Movable Frame

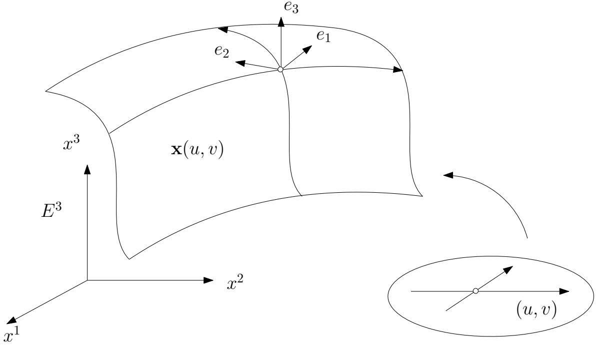


Figure: A parametric surface.

Orthonormal Movable frame

Movable Frame

Suppose a regular surface S is embedded in $\mathbb{R}^3$, a parametric representation is $\mathbf{r}(u, v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let $\mathbf{e}_3$ be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orhonormal frame field* of the surface.

Orthonormal Movalbe frame

Tangent Vector

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$. $d\mathbf{r}$ is orthogonal to the normal vector $\mathbf{e}_3$.

Motion Equation

$$d\mathbf{e}_i = \omega_{i1}\mathbf{e}_1 + \omega_{i2}\mathbf{e}_2 + \omega_{i3}\mathbf{e}_3,$$

where $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$. Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \quad 0 = d\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle + \langle \mathbf{e}_i, d\mathbf{e}_j \rangle$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

Motion Equation

Motion Equation

$$dr = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1\omega_1 + \omega_2\omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1\omega_{31} - \omega_2\omega_{32} = \omega_1\omega_{13} + \omega_2\omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \to \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \to d\mathbf{e}_3, \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 \to \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K\omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gaussian curvature

Weigarten Mapping

$\{\omega_1, \omega_2\}$ form the basis of the cotangent space, therefore ω_{13}, ω_{23} can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so $K = h_{11}h_{22} - h_{12}h_{21}$, the mean curvature $H = \frac{1}{2}(h_{11} + h_{22})$.

Gauss's theorem Egregium

Theorem (Gauss' Theorem Egregium)

The Gaussian curvature is intrinsic, solely determined by the first fundamental form.

Proof.

$$\begin{aligned}
0 =& d^2\mathbf{e}_1\\
=& d(\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3)\\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge d\mathbf{e}_2 + d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge d\mathbf{e}_3\\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) +\\
& d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge (\omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2)\\
=& (d\omega_{12} - \omega_{13} \wedge \omega_{32})\mathbf{e}_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})\mathbf{e}_3
\end{aligned}$$

therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{23} = -K\omega_1 \wedge \omega_2.$$

Lemma

The connection is given by the Riemannian metric:

$$\boxed{\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2}$$

Proof.

$$\begin{aligned}
0 &= d^2\mathbf{r} \\
&= d(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) \\
&= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2 \mathbf{e}_2 - \omega_w \wedge d\mathbf{e}_2 \\
&= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge (\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) + \\
&\quad d\omega_2 \mathbf{e}_2 - \omega_2 \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) \\
&= (d\omega_1 - \omega_2 \wedge \omega_{21})\mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12})\mathbf{e}_2 + \\
&\quad -(\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23})\mathbf{e}_3.
\end{aligned}$$

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet)

Suppose M is a closed orientable C^2 surface, then

$$\int_M K dA = 2\pi\chi(M),$$

where dA is the area element of hte surface, $\chi(M)$ is the Euler characteristic number of M.

Proof.

Construct a smooth vector field v, with isolated zeros $\{p_1, p_2, \cdots, p_n\}$. Choose a small disk $D(p_i, \varepsilon)$. On the surface

$$\bar{M} = M \setminus \bigcup_{i=1}^{n} D(p_i, \varepsilon)$$

□

Gauss-Bonnet Theorem

Proof.

construct orthonormal frame $\{p, e_1, e_2, e_3\}$, where

$$e_1(p) = \frac{v(p)}{|v(p)|}, \quad e_3(p) = n(p).$$

The integration

$$\int_{\bar{M}} KdA = \int_{\bar{M}} K\omega_1 \wedge \omega_2 = -\int_{\bar{M}} d\omega_{12}$$

by Stokes theorem and Poincarère-Hopf theorem, we obtain

$$-\sum_{i=1}^{n} \int_{\partial D(p_i,\varepsilon)} \omega_{12} = 2\pi \sum_{i=1}^{n} \mathrm{Index}(p_i, v) = 2\pi\chi(M).$$

Here by $\omega_{12} = \langle de_1, e_2 \rangle$, ω_{12} is the rotation speed of e_1. Let $\varepsilon \to 0$, the equation holds.

Computing Geodesics

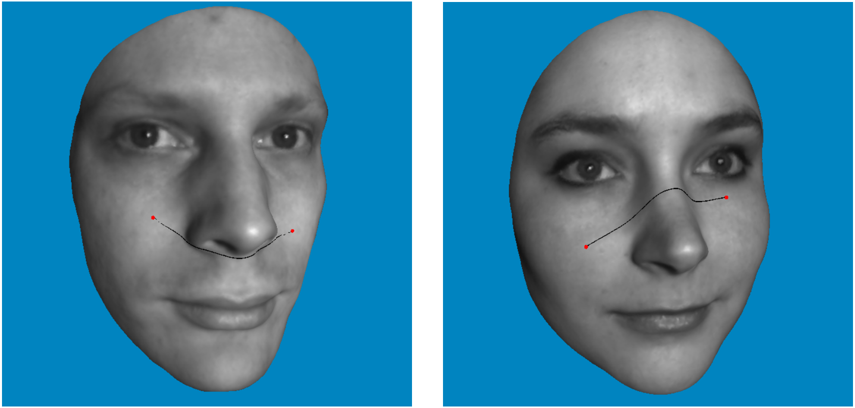


Figure: Geodesics.

Covariant Differential

Definition (Covariant Differentiation)

Covariant differentiation is the generalization of directional derivatives, satisfies the following properties: assume v and w are tangent vector fields on a surface, $f : S \to \mathbb{R}$ is a C^1 function, then

1. $D(\mathbf{v}+\mathbf{w}) = D(\mathbf{v}) + D(\mathbf{w})$,
2. $D(f\mathbf{v}) = df\ \mathbf{v} + fD\mathbf{v}$,
3. $D\langle \mathbf{v}, \mathbf{w}\rangle = \langle D\mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{v}, D\mathbf{w}\rangle$.

By movable framework, the motion equation of the surface is

$$d\mathbf{e_1} = \omega_{12}\mathbf{e_2} + \omega_{13}\mathbf{e_3}, \quad d\mathbf{e_2} = \omega_{21}\mathbf{e_1} + \omega_{23}\mathbf{e_3},$$

We only keep tangential component, and delete the normal part to obtain covariant differential

$$D\mathbf{e_1} = \omega_{12}\mathbf{e_1}, \quad D\mathbf{e_2} = \omega_{21}\mathbf{e_1}.$$

Covariant Differential

Definition (Parallel transport)

Suppose S is a metric surface, $\gamma : [0,1] \to S$ is a smooth curve, $v(t)$ is a vector field along γ, if

$$\frac{Dv}{dt} \equiv 0,$$

then we say the vector field $v(t)$ is parallel transportation along γ.

Given a tangent vector field $v = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, then

$$\begin{aligned} Dv &= df_1\mathbf{e_1} + f_1 D\mathbf{e_1} + df_2\mathbf{e_2} + f_2 D\mathbf{e_2} \\ &= (df_1 - f_2\omega_{12})\mathbf{e_1} + (df_2 + f_1\omega_{12})\mathbf{e_2}. \end{aligned}$$

and

$$\frac{Dv}{dt} = \left(\frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt}\right)\mathbf{e_1} + \left(\frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt}\right)\mathbf{e_2}.$$

where $\frac{\omega_{12}}{dt} = \langle \omega_{12}, \dot{\gamma} \rangle$. If $\omega_{12} = \alpha dx + \beta dy$, then $\frac{\omega_{12}}{dt} = \alpha\dot{x} + \beta\dot{y}$.

Parallel Transport

Parallel Transport Equation

Therefore parallel vector field satisfies the ODE

$$\begin{cases} \frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} & = & 0 \\ \frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} & = & 0 \end{cases}$$

Given an intial condition $v(0)$, the solution uniquely exists.

Levy-Civita Connection

Definition (Levy-Civita Connection)

The connection D is the Levy-Civita connection with respect to the Riemannianmetic $\mathbf{g}$, if it satisfies:

1. compatible with the metric
$$\mathbf{x}\langle \mathbf{y},\mathbf{z}\rangle_{\mathbf{g}} = \langle D_{\mathbf{x}}\mathbf{y},\mathbf{z}\rangle_{\mathbf{g}} + \langle \mathbf{y}, D_{\mathbf{x}}\mathbf{z}\rangle_{\mathbf{g}}$$
2. free of torsion
$$D_{\mathbf{v}}\mathbf{w} - D_{\mathbf{w}}\mathbf{v} = [\mathbf{v},\mathbf{w}]$$

Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vector fields parallel along γ, then

$$\frac{d}{dt}\langle \mathbf{v},\mathbf{w}\rangle_{\mathbf{g}} = \dot{\gamma}\langle \mathbf{v},\mathbf{w}\rangle_{\mathbf{g}} = \langle D_{\dot{\gamma}}\mathbf{v},\mathbf{w}\rangle + \langle \mathbf{v}, D_{\dot{\gamma}}\mathbf{w}\rangle \equiv 0.$$

Namely, parallel transportation preserves inner product.

Geodesic Curvature

Definition (Geodesic Curvature)

Assume $\gamma : [0,1] \to S$ is a C^2 curve on a surface S, s is the arc length parameter. Construct orthonormal frame field along the curve $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e}_3\}$, where $\mathbf{e_1}$ is the tangent vector field of γ, $\mathbf{e}_3$ is the normal field of the surface,

$$k_g := \frac{D\mathbf{e_1}}{ds} = k_g \mathbf{e_2}$$

is called geodesic curvature vector,

$$k_g = \left\langle \frac{D\mathbf{e_1}}{ds}, \mathbf{e_2} \right\rangle = \frac{\omega_{12}}{ds}$$

is called geodesic curvature.

Geodesic Curvature

Geodesic curvature, normal curvature

Given a spacial curve, its curvature vector satisfies

$$\frac{d^2\gamma}{ds^2} = k_g \mathbf{e_2} + k_n \mathbf{e_3},$$

where k_n is the normal curvature of the curve. The curvature of the curve, geodesic curvature and normal curvature satisfy

$$k^2 = k_g^2 + k_n^2.$$

Geodesic curvature k_g only depends on the Riemannian metric of the surface, is independent of the 2nd fundamental form. Therefore k_g is intrinsic, k_n is extrinsic.

Gauss-Bonnet

Theorem

Suppose $(S, \mathbf{g})$ *is an oriented metric surface with boundaries, then*

$$\int_S KdA + \int_{\partial S} k_g ds = 2\pi\chi(S).$$

Proof.

Construct a vector field with isolated zeros $\{p_i\}$, $\mathbf{e}_1$ is tangent to ∂S, small disks $D(p_i, \varepsilon)$. Define $\bar{S} := S \setminus \bigcup_i D(p_i, \varepsilon)$,

$$\begin{aligned}\int_{\bar{S}} KdA &= -\int_{\bar{S}} \frac{d\omega_{12}}{\omega_1 \wedge \omega_2} dA = -\int_{\bar{S}} d\omega_{12} = -\int_{\partial \bar{S}} \omega_{12} \\ &= -\int_{\partial S - \bigcup_i \partial D(p_i, \varepsilon)} \omega_{12} = -\int_{\partial S} \frac{\omega_{12}}{ds} ds + \sum_i \int_{\partial D(p_i, \varepsilon)} \omega_{12} \\ &= -\int_{\partial S} k_g ds + 2\pi \sum_i \mathrm{Index}(p_i) = -\int_{\partial S} k_g ds + 2\pi\chi(S).\end{aligned}$$

Compute Minimal Surface

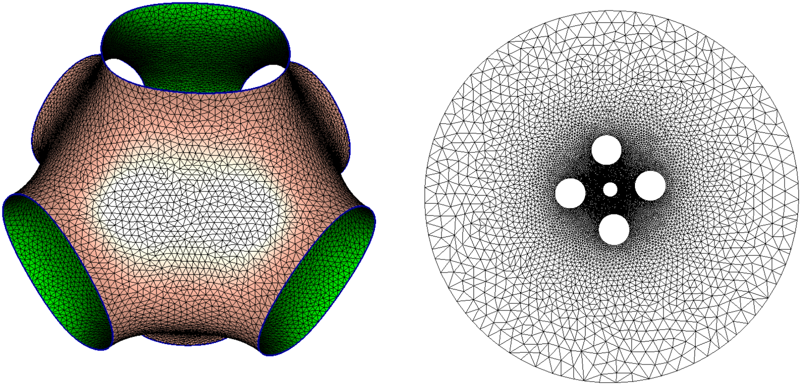


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0$, $\forall v_i \notin \partial M$.

Minimal Surface

Lemma

Given a metric surface $(S, \mathbf{g})$ *embedded in* $\mathbb{R}^3$, *then* $\Delta_{\mathbf{g}}\mathbf{r} = 2H(p)\mathbf{n}$, *where* $\mathbf{r}$, $\mathbf{n}$ *are the position and normal vectors.*

Proof.

We choose isothermal coordinates (x, y). Then $\mathbf{g} =^{2\lambda(x,y)} (dx^2 + dy^2)$, $\omega_{12} = -\lambda_y dx + \lambda_x dy$, $\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$, $\omega_{23} = h_{12}\omega_1 + h_{22}\omega_2$, $\omega_1 = e^\lambda dx$, $\omega_2 = e^\lambda dy$,

$$
\begin{aligned}
\frac{\partial}{\partial x}\mathbf{r_x} &= \frac{\partial}{\partial x} e^\lambda \mathbf{e_1} = e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \frac{\partial}{\partial x}\mathbf{e_1} \\
&= e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \langle d\mathbf{e}_1, \frac{\partial}{\partial x}\rangle = e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \langle \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e_3}, \partial_x\rangle \\
&= e^\lambda \lambda_x \mathbf{e_1} + e^\lambda(-\lambda_y)\mathbf{e_2} + e^\lambda \mathbf{e_3}\langle h_{11}\omega_1, \partial_x\rangle \\
&= e^\lambda \lambda_x \mathbf{e_1} - e^\lambda \lambda_y \mathbf{e_2} + e^{2\lambda} h_{11}\mathbf{e_3}
\end{aligned}
$$

Minimal Surface

Proof.

Similarly,

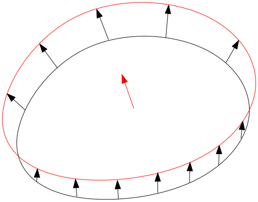
$$\begin{aligned}
\frac{\partial}{\partial y}\mathbf{r_y} &= \frac{\partial}{\partial y}e^{\lambda}\mathbf{e_2} = e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\frac{\partial}{\partial y}\mathbf{e_2} \\
&= e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\langle d\mathbf{e}_2, \frac{\partial}{\partial y}\rangle = e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\langle \omega_{21}\mathbf{e_1} + \omega_{23}\mathbf{e_3}, \partial_y\rangle \\
&= e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}(-\lambda_y)\mathbf{e_2} + e^{\lambda}\mathbf{e_3}\langle h_{22}\omega_2, \partial_y\rangle \\
&= e^{\lambda}\lambda_y\mathbf{e_2} - e^{\lambda}\lambda_x\mathbf{e_1} + e^{2\lambda}h_{22}\mathbf{e_3}
\end{aligned}$$

Therefore

$$\Delta_{\mathbf{g}}\mathbf{r} = \frac{1}{e^{2\lambda}}(\mathbf{r}_{xx} + \mathbf{r}_{yy}) = (h_{11} + h_{22})\mathbf{e_3} = 2H\mathbf{e_3}.$$

$\square$

Surface Area Variation



Lemma

Given a surface S with position vector $\mathbf{r}(x,y)$, perturb the surface along the normal direction

$$\mathbf{r}_{\varepsilon,\varphi}(x,y) = \mathbf{r}(x,y) + \varepsilon\varphi(x,y)\mathbf{n}(x,y),$$

the area variation is given by

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Area(\mathbf{r}_{\varepsilon,\varphi}) = \int_S 2\varphi(x,y)He^{2u(x,y)}dxdy = \int_S 2\varphi H dA.$$

Surface Area Variation

Proof.

We use isothermal coordinate, the first fundamental form:

$$E = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_x + \varepsilon \mathbf{n}_x \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \varepsilon^2 |\mathbf{n}_x|^2$$
$$G = \langle \mathbf{r}_y + \varepsilon \mathbf{n}_y, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_y, \mathbf{n}_y \rangle + \varepsilon^2 |\mathbf{n}_y|^2$$
$$F = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = \varepsilon \langle \mathbf{r}_x, \mathbf{n}_y \rangle + \varepsilon \langle \mathbf{r}_y, \mathbf{n}_x \rangle + \varepsilon^2 \langle \mathbf{n}_x, \mathbf{n}_y \rangle$$

$$EG - F^2 = e^{4u} + 2\varepsilon e^{2u} (\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle) + O(\varepsilon^2)$$
$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sqrt{EG - F^2} = \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle = 2He^{2u}$$

where we use the mean curvature formula

$$2H = \mathrm{Tr}\left(-\frac{II}{I}\right) = -e^{-2u}(\langle \mathbf{r}_{xx}, \mathbf{n} \rangle + \langle \mathbf{r}_{yy}, \mathbf{n} \rangle) = e^{-2u}(\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle r_y, \mathbf{n}_x \rangle)$$

$$\frac{d}{d\varepsilon} Area(\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_S \sqrt{EG - F^2} dxdy = \int_S 2He^{2u} dxdy.$$

Minimal Surface

Lemma

A surface M, $\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, with isothermal coordinates is minimal if and only if x_1, x_2, and x_3 are all harmonic.

Proof.

If M is minimal, then $H = 0$, $\Delta \mathbf{x} = (2H)e^{2\lambda}\mathbf{n} = 0$, therefore x_1, x_2, x_3 are harmonic.

If x_1, x_2, x_3 are harmonic, then $\Delta \mathbf{x} = 0$, $(2H)e^{2\lambda}\mathbf{n} = 0$. Now $\mathbf{n}$ is the unit normal vector, so $\mathbf{n} \neq 0$ and $e^{2\lambda} = \langle x_u, x_u \rangle = |x_u|^2 \neq 0$. So $H = 0$, M is minimal. $\square$

Weierstrass-Ennerper Representation

Lemma

Let $z = u + \sqrt{-1}v$, $\frac{\partial x^j}{\partial z} = \frac{1}{2}(x^j_u - \sqrt{-1}x^j_v)$, *define*

$$\varphi = \frac{\partial \mathbf{x}}{\partial z} = (x^1_z, x^2_z, x^3_z)$$
$$(\varphi)^2 = (x^1_z)^2 + (x^2_z)^2 + (x^3_z)^2$$

if $\mathbf{x}$ *is isothermal, then* $(\varphi)^2 = 0$.

Proof.

$(\varphi^j)^2 = (x^j_z)^2 = \frac{1}{4}((x^j_j)^2 - (x^j_v)^2 - 2ix^j_u x^j_v)$, so
$(\varphi)^2 = \frac{1}{4}(|\mathbf{x}_u|^2 - |\mathbf{x}_v|^2 - 2i\mathbf{x}_u \cdot \mathbf{x}_v)$. If $\mathbf{x}$ is isothermal, then $(\varphi)^2 = 0$. $\square$

Weierstrass-Ennerper Representation

Theorem

Suppose M is a surface with position $\mathbf{x}$. Let $\varphi = \frac{\partial \mathbf{x}}{\partial z}$ and suppose $(\varphi)^2 = 0$. Then M is minimal if and only if φ^j is holomorphic.

Proof.

M is minimal, then x^j is harmonic, therefore $\Delta \mathbf{x} = 0$, therefore

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \mathbf{x}}{\partial z}\right) = \frac{\partial \varphi}{\partial \bar{z}} = 0$$

If φ^j is holomorphic, then $\frac{\partial \varphi}{\partial \bar{z}} = 0$, then $\Delta \mathbf{x} = 0$, x^j is harmonic, hence M is minimal. □

Weierstrass-Ennerper Representation

Lemma

$$x^j(z,\bar{z}) = c_j + \Re\left(\int \varphi^j dz\right).$$

Proof.

$$\varphi^j dz + \bar{\varphi}^j d\bar{z}^j = x^j_u du + x^j_v dv = dx^j.$$

hence

$$x^j = c_j + \int dx^j = c_j + \Re\left(\int \varphi^j dz\right).$$

□

Weierstrass-Ennerper Representation

Let f be a holomorphic function and g be a meromorphic function, such that fg^2 is holomorphic,

$$\varphi^1 = \frac{1}{2} f(1 - g^2), \varphi^2 = \frac{i}{2} f(1 + g^2), \varphi^3 = fg,$$

then

$$(\varphi)^2 = \frac{1}{4} f^2 (1 - g^2)^2 - \frac{1}{4} f^2 (1 + g^2)^2 + f^2 g^2 = 0.$$

Weierstrass-Ennerper Representation

Theorem (Weierstrass-Ennerper)

If f is holomorphic on a domain Ω, g is meromorphic in Ω, and fg^2 is holomorphic on Ω, then a minimal surface is defined by $\mathbf{x}(z,\bar{z}) = (x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z}))$, *where*

$$x^1(z,\bar{z}) = \Re\left(\int f(1-g^2)dz\right)$$

$$x^2(z,\bar{z}) = \Re\left(\int \sqrt{-1}f(1+g^2)dz\right)$$

$$x^3(z,\bar{z}) = \Re\left(\int 2fgdz\right)$$

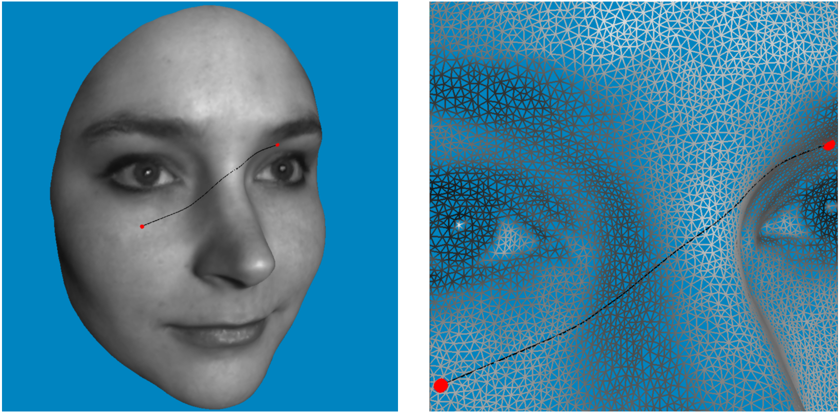


Figure: Geodesic on polyhedral surfaces.

Geodesic on a surface $\gamma : [0,1] \to (S, \mathbf{g})$:

$$D_{\dot{\gamma}}\dot{\gamma} \equiv 0.$$

Compute Geodesics

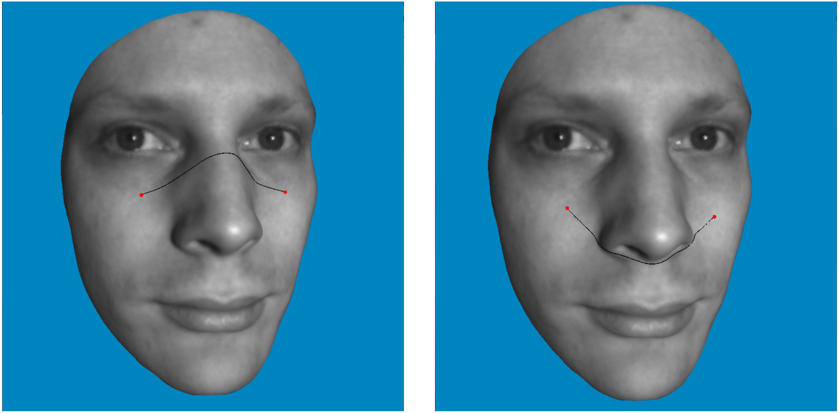
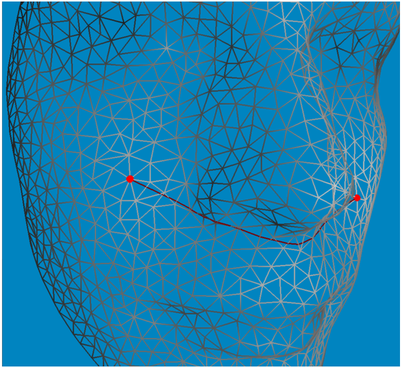


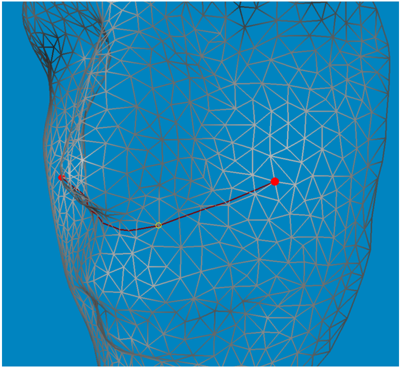
Figure: Conjugate point of geodesics.

Geodesic on a surface $\gamma : [0,1] \to (S, \mathbf{g})$:

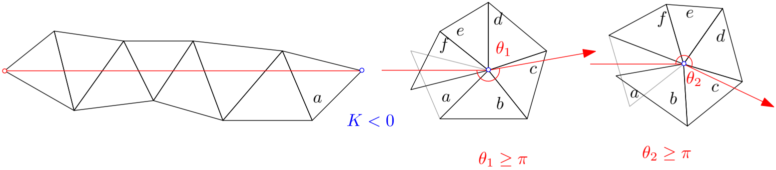
$$D_{\dot{\gamma}}\dot{\gamma} \equiv 0.$$

Discrete Geodesics





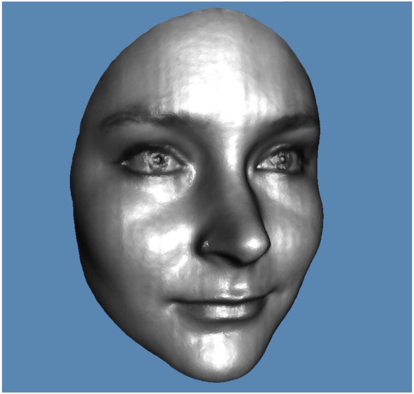
Discrete Geodesics



Suppose γ is a discrete geodesic:

1. isometrically flatten the strip of curve γ onto the plane;
2. when the γ crosses an edge, it is straight;
3. γ never crosses any convex vertex;
4. when γ crosses a concave vertex, if we flatten the neighborhood from right, then $\theta_1 \geq \pi$; flatten from left, $\theta_2 \geq \pi$.

Discrete Harmonic Map



Smooth surface harmonic map $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$, $\Delta_{\mathbf{g}} \varphi \equiv 0$, with Dirichlet boundary condition $\varphi|_{\partial} S = f$. A discrete harmonic ma satisfies $\sum_{v_i \sim v_j} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0$, $\forall v_i \notin \partial M$.

Compute Minimal Surface

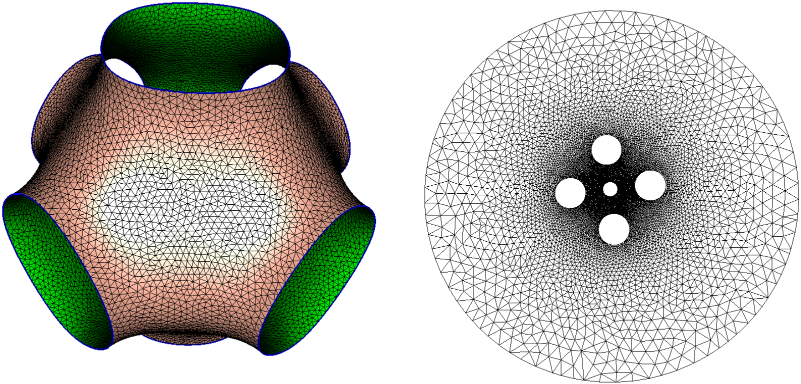
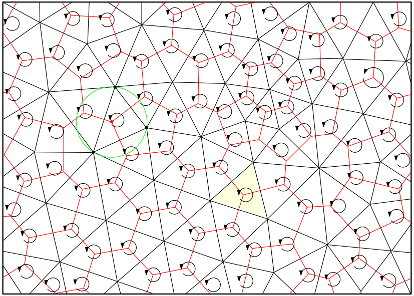
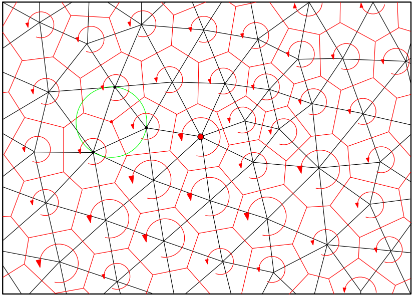


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$.

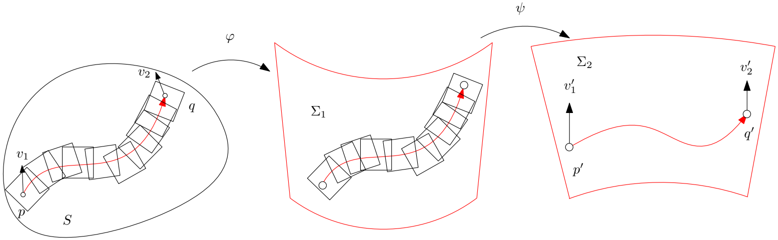


$d\omega = 0$



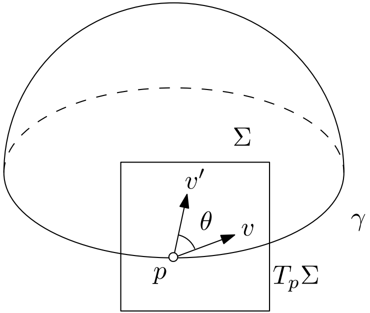
$\delta\omega = 0$

Harmonic map $\varphi : M \to \mathbb{D}^2$; minimal surface $\varphi : M \to \mathbb{R}^3$.



Given $\gamma \subset S$, find an envelope surface Σ_1 of all the tangent planes along γ, $\varphi : \gamma \to \Sigma_1$ isometrically maps γ to Σ_1. Σ_1 is developable, flatten Σ_1 to obtain a planar domain Σ_2, $\psi : \Sigma_1 \to \Sigma_2$. The composition $\psi \circ \varphi$ maps $p, q, v_1 \in T_pS, v_2 \in T_pS$ to p', q', v_1', v_2'. On the plane, translate a tangent vector v_1' from starting point p to the ending point q to get v_2', maps back v_2', $v_2 = (\psi \circ \varphi)^{-1}(v_2')$. Then v_1 is parallelly transported along γ to get v_2.

Gaussian Curvature



Parallel transport v along $\partial\Sigma$, to get v' when returned to the original point p, then the angle difference between v and v' equals to the total Gaussian curvature,

$$\theta = \int_{\Sigma} K dA.$$

Gaussian Curvature

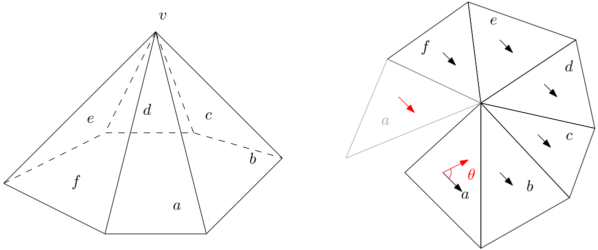
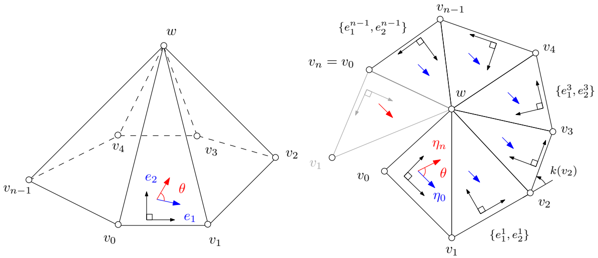


Figure: Discrete parallel transport, $K(v) = \theta$.

Parallel transport a vector, when return to the original position, the difference angle equals to the discrete Gaussian curvature of the interior vertices.

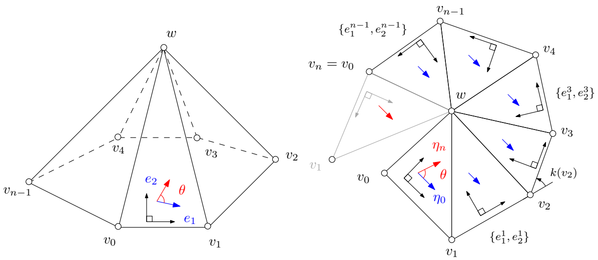
Gaussian Curvature



For each face $[w, v_i, v_{i+1}]$, build a local frame $\{e_1^i, e_2^i\}$, such that e_1^i is parallel to $[v_i, v_{i+1}]$; Connection ω_{12} is defined at edges, $\omega_{12}([w, v_i]) = \measuredangle(e_1^{i-1}, e_1^i) = k(v_i)$. Given a unit vector at $[w, v_0, v_1]$, with angle η_0; parallel transport to $[w, v_1, v_2]$ the angle representation is $\eta_1 = \eta_0 - k_1$; parallel transport to $[w, v_i, v_{i+1}]$,

$$\eta_i = \eta_{i-1} - k_i = \eta_0 - \sum_{j=1}^{i} k_j.$$

Gaussian Curvature



Parallel transport accorss $[w, v_1], [w, v_2], \cdots, [w, v_n]$, where $v_n = v_0$, then

$$\eta_n = \eta_0 - \sum_{i=1}^{n} \omega_{12}([w, v_i]) = \eta_0 - \sum_{i=1}^{n} k(v_i),$$

By Gauss-Bonnet, $K(w) + \sum_{i=1}^{n} k(v_i) = 2\pi$,therefore

$$\eta_n = \eta_0 - 2\pi + K(w).$$

Gaussian Curvature

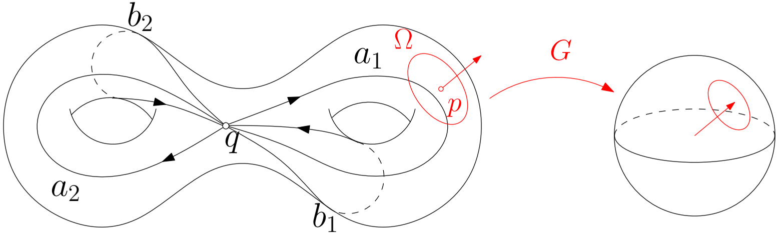


Figure: Gaussian curvature.

Gauss map: $\mathbf{r}(p) \mapsto \mathbf{n}(p)$,

$$K(p) := \lim_{\Omega \to \{p\}} \frac{|G(\Omega)|}{|\Omega|}$$

Gaussian Curvature

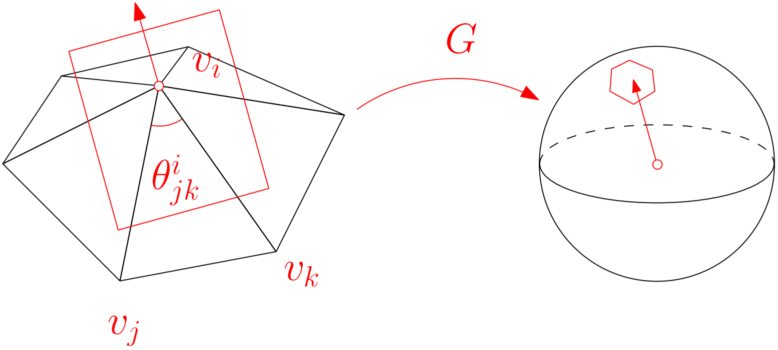


Figure: Discrete Gaussian curvature.

$$G(v_i) := \{\mathbf{n} \in \mathbb{S}^2 | \exists \text{Support plane with normal } \mathbf{n}\}.$$

Gaussian Curvature

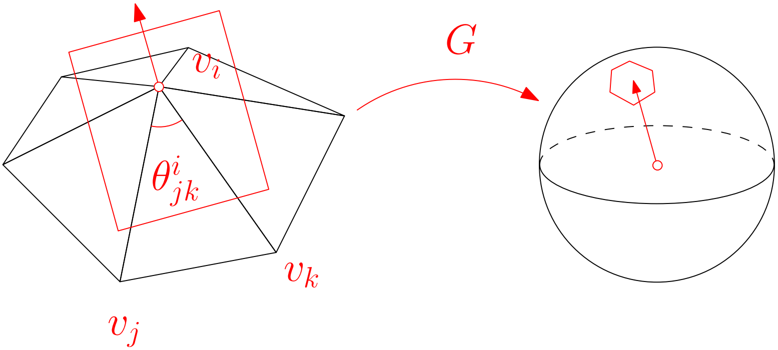


Figure: Discrete Gaussian curvature for convex vertex.

$$K(v_i) := |G(v_i)| = 2\pi - \sum_{jk} \theta^i_{jk}.$$

Gauss-Bonnet

For a closed oriented metric surface $(S, \mathbf{g})$,

$$\int_S K dA = 2\pi\chi(S).$$

For a closed oriented discrete polygonal surface M,

$$\sum_{v_i} K(v_i) = 2\pi\chi(M).$$

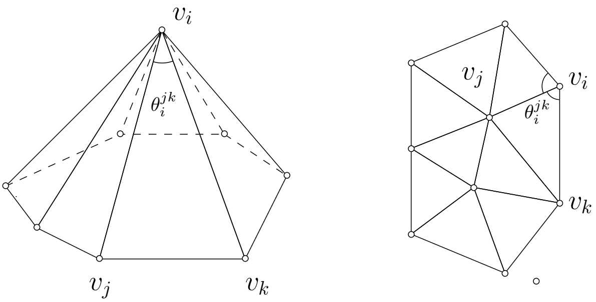


Figure: Discrete Gaussian curvature.

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \tag{1}$$

Gauss-Bonnet

Theorem (Discrete Gauss-Bonnet Theorem)

Given polyhedral surface $(S, V, \mathbf{d})$, *the total discrete curvature is*

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(S),$$

where $\chi(S)$ *is the Euler characteristic number of* S.

Proof.

We denote the polyhedral surface $M = (V, E, F)$, if M is closed, then

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jK} = 2\pi|V| - \pi|F|.$$

Since M is closed, $3|F| = 2|E|$,
$\chi(S) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F| = |V| - \frac{1}{2}|F|.$ $\square$

Discrete Guass-Bonnet

continued.

Assume M has bounary ∂M. Assume the interior vertex set is V_0, boundary vertex set is V_1, then $|V| = |V_0| + |V_1|$; assume interior edge set is E_0, boundary edge set is E_1, then $|E| = |E_0| + |E_1|$. Furthermore, all boundaries are closed loops, hence boundry vertex number equals to the boundary edge number, $|V_1| = |E_1|$. Every interior edge is adjacent to two faces, every boundary edge is adjacent to one face, we have $3|F| = 2|E_0| + |E_1| = 2|E_0| + |v_1|$. We compute the Euler number

$$\chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1| = |V_0| + |F| - |E_0|,$$

by $|E_0| = 1/2(3|F| - |V_1|)$

$$\chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|$$

Discrete Guass-Bonnet

continued.

we have:

$$\begin{aligned}\sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) &= \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk}\right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk}\right) \\ &= 2\pi|V_0| + \pi|V_1| - \pi|F| \\ &= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|\right) \\ &= 2\pi\chi(M).\end{aligned} \tag{2}$$

□.