Abel-Jacobi Theory

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Smooth Manifold

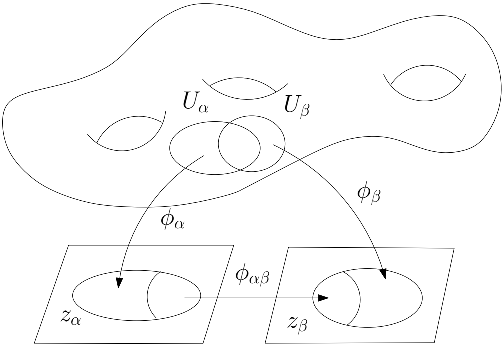


Figure: A manifold.

Smooth Manifold

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Riemann Surface

Definition (Riemann Surface)

A two dimensional manifold S is a Riemann surface, if the chart transition maps

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are biholomorphic. On each local chart $(U_\alpha, \varphi_\alpha)$, we use z_α to denote the local complex coordinate. The atlas $\{(U_\alpha, z_\alpha)\}$ is called a conformal structure of the surface S.

Definition (Holomorphic Function)

Suppose C is a Riemann surface, $\{(U_i, z_i)\}$ is a holomorphic coordinate covering. A meromorphic (holomorphic) function on C is given by a family of map $f_i : U_i \to \mathbb{C}$, such that

1. If $U_i \cap U_j \neq \emptyset$, on $U_i \cap U_j$ we have
$$f_i = f_j;$$
2. $\forall i$, $f_i \circ z_i^{-1}$ is a meromorphic (holomorphic) function.

All the meromorphic functions on C form a field, denoted as $K(C)$, called the meromorphic function field on C.

Definition (Zeros and Poles)

Suppose C is a compact Riemann surface, $f \in K(C)$, $p \in C$. Choose a local coordinates z of the neighborhood of p, such that $z(p) = 0$, then in the neighborhood

$$f(z) = z^{\nu} h(z),$$

where $h(z)$ is a holomorphic function, $h(0) \neq 0$, $\nu \in \mathbb{Z}$. ν is called the order of f at p, denoted as $\nu_p(f)$. when $\nu_p(f) > 0$, $p$is called a zero of f, $\nu_p(f)$ is called the order of the zero p; when $\nu_p(f) < 0$, p is called a pole of f, $|\nu_p(f)|$ is called the order of the pole p.

Holomorphic Differential

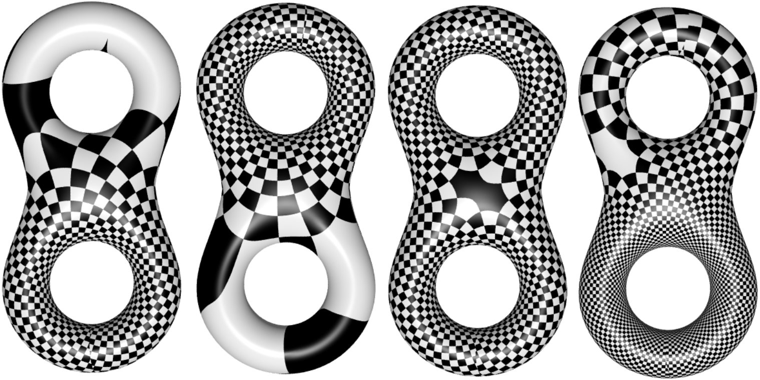


Figure: Holomorphic 1-form on a genus two surface.

Definition (Meromorphic Differential)

Suppose S is a Riemann surface with a conformal structure $\{(U_\alpha, z_\alpha)\}$, a complex differential 1-form ω is called a meromorphic (holomorphic) 1-form (meromorphic differential), if on each local chart $(U_\alpha, \varphi_\alpha)$, its local representation is

$$\omega = f_\alpha(z_\alpha) dz_\alpha,$$

where f_α is a meromorphic (holomorphic) function, and on the other chart $\omega = f_\beta(z_\beta) dz_\beta$,

$$f_\alpha(z_\alpha) = f_\beta(z_\beta(z_\alpha)) \frac{dz_\beta}{dz_\alpha}.$$

The zeros and poles of ω are those of f_α's.

All the meromrophic (holomorhic) 1-forms on C is denoted as $K^1(C)(\Omega^1(C))$.

Residue Theorem

Definition (Residue)

Let C be a Riemann surface, $\omega \in K^1(C)$, $p \in C$, γ_p is a small circle around the point p, ω has no other pole except p (p itself may be or may be not a pole). Then the residue of ω at p is defined as

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \oint_{\gamma_p} \omega.$$

Locally, $p \in U_j$, $\gamma_p \subset U_j$, we have

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \oint_{\gamma_p} \omega = \frac{1}{2\pi i} \oint f_j(z_j) dz_j = \mathrm{Res}_p(f_j(z_j) dz_j).$$

Residue Theorem

Theorem (Residue)

Suppose C is a compact Riemann surface, for $\omega \in K^1(C)$, we have

$$\sum_{p\in C} Res_p(\omega) = 0.$$

Proof.

Since C is compact, ω has finite number of poles on C, denoted as $p_1, p_2, \dots, p_m$. Choose small disks $\Delta_1, \Delta_2, \dots, \Delta_m$ surrounding these poles. Denote

$$\Omega = C \setminus \bigcup_i \Delta_i, \quad \partial\Omega = -\bigcup_i \partial\Delta_i.$$

By Stokes, we have

$$2\pi i \sum_{p\in C} \text{Res}_p(\omega) = 2\pi i \sum_{j=1}^m \text{Res}_{p_j}(\omega) = \sum_{j=1}^m \int_{\partial\Delta_j} \omega = -\int_{\partial\Omega} \omega = -\int_{\Omega} d\omega = 0.$$

Residue Theorem

Theorem (Meromorphic Function)

If $f \in K(C)$ is not a constant function, then

$$\sum_{p \in C} \nu_p(f) = 0.$$

Proof.

Construct

$$\omega = \frac{df}{f} \in K^1(C),$$

then the residue of ω is zero. Then means

$$\#\{zeros\ of\ f\} = \#\{poles\ of\ f\}.$$

$\square$

Principle Divisor

Theorem

If $f \in K(C)$ is not a constant, then

$$deg(f) = \sum_{p \in C} \nu_p(f) = 0.$$

Proof.

The meromorphic function f on C induces a conformal map $f : C \to \mathbb{S}^2$, suppose the mapping degree is k, then the preimages of the south pole are the zeros of f, the preimages of the north pole are the poles of f. The number of zeros equals to the mapping degree k, the number of poles equals to the mappping degree k as well. □

Jacobi Variety

Suppose C is a $g \geq 1$ compact Riemann surface. $H_1(C, \mathbb{Z})$ is a rank $2g$ free Abel group. Choose a canonical basis of $H_1(C, \mathbb{Z})$ $\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\}$,

$$\gamma_i \cdot \gamma_{g+i} = 1, \quad \gamma_{g+i} \cdot g_i = -1,$$

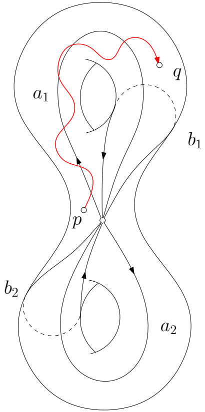
and the other algebraic intersection numbers are zeros. $\{\omega_1, \omega_2, \cdots, \omega_g\}$ is a set of basis of $\Omega^1 C$,

Definition (Period Vector)

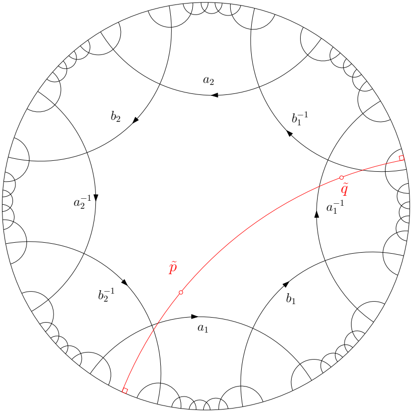
For each γ_i,

$$\pi_j = \begin{pmatrix} \int_{\gamma_j} \omega_1 \\ \int_{\gamma_j} \omega_2 \\ \vdots \\ \int_{\gamma_j} \omega_g \end{pmatrix} \in \mathbb{C}^g \quad (j = 1, 2, \ldots, 2g)$$

Hyperbolic Geodesic



geodesic on surface



Poincaré's disk model

Period Matrix

Definition (Period Matrix)

The matrix

$$\Pi := (\pi_1, \pi_2, \cdots, \pi_{2g})_{g\times 2g}$$

is called the period matrix of the Riemann surface.

Definition (Jacobi Variety)

The period vectors generate a lattice

$$\Lambda := \left\{ \sum_{j=1}^{2g} m_j \pi_j \mid m_j \in \mathbb{Z} \right\} \quad \subset \mathbb{C}^g$$

The quotient space $\mathbb{C}^g/\Lambda$ is a g dimensional complex torus, and called the Jacobi variety of C, denoted as $J(C)$.

Riemann Bilinear Relation

Suppose $\gamma \subset C$ is a closed loop, slice C along γ to obtain $\bar{C} = C \setminus \{\gamma\}$. $\partial \bar{C} = \gamma^+ - \gamma^-$. Set a function $f : \bar{C} \to \mathbb{R}$, such that $f|_{\gamma^+} = +1$ and $f|_{\gamma^-} = 0$. The $\omega_\gamma = df$ is a closed 1-form on C, which is called the 1-form corresponding to γ, such that for any loop τ,

$$\tau \cdot \gamma = \int_\tau \omega_\gamma.$$

Suppose $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$ is a set of canonical basis of $H_1(C, \mathbb{Z})$, α_k is corresponding to b_k, $-\beta_k$ corresponding to a_k, then

$$a_k \cdot b_k = \int_{a_k} \alpha_k = \int\int \alpha_k \wedge \beta_k = 1$$

$$b_k \cdot a_k = -\int_{b_k} \beta_k = -\int\int \alpha_k \wedge \beta_k = -1$$

Namely, the period of α_k along a_k is 1, the period of β_k along b_k is 1. The other integrations equal to zero.

Riemann Bilinear Relation

Lemma

$(\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g)$ is a basis of $H^1_\Delta(C, \mathbb{R})$. For any closed 1-form ω, we have the decomposition:

$$\omega = \sum_{i=1}^{g} A_i \alpha_i + \sum_{i=1}^{g} B_i \beta_i + df,$$

where A_i's are A-periods, B_j's are B-periods of ω, $A_i = \int_{a_i} \omega$ and $B_j = \int_{b_j} \omega$.

Lemma

Suppose θ and ω are closed 1-forms, then

$$\int\int_C \theta \wedge \omega = \sum_{i=1}^{g} \left[\int_{a_i} \theta \int_{b_i} \omega - \int_{a_i} \omega \int_{b_i} \theta \right].$$

Riemann Bilinear Relation

Assume the A-period of θ is $(A_1, \cdots, A_g)$, the B-period of θ is $(B_1, \cdots, B_g)$, the A-period of ω is $(A'_1, \cdots, A'_g)$, the B-period of ω is $(B'_1, \cdots, B'_g)$, then

$$\theta = \sum_{i=1}^{g} A_i \alpha_i + \sum_{j=1}^{g} B_j \beta_j + df, \omega = \sum_{i=1}^{g} A'_i \alpha_i + \sum_{j=1}^{g} B'_j \beta_j + dh,$$

Note that $d(f\theta) = df \wedge d\theta + fd\theta$ and

$$\int_C df \wedge \theta = \int_{\partial C} f\theta = 0 \quad \int\int_C \alpha_i \wedge \beta_i = 1,$$

the others are 0, by direct computation

$$\int\int_C \theta \wedge \omega = \sum_{i=1}^{g} (A_i B'_i - A'_i B_i) = \sum_{i=1}^{g} \left[\int_{a_i} \theta \int_{b_i} \omega - \int_{a_i} \omega \int_{b_i} \theta \right].$$

Riemann Bilinear Relation

Theorem (Riemann Bilinear Relation I)

Suppose φ *and* φ' *are holomorphic 1-forms. The A-period and B-period for* φ *are* A_i *and* B_i*, those for* φ' *are* A_i' *and* B_i'*,* $(1 \leq i \leq g)$*, then*

$$\sum_{i=1}^{g}(A_i B_i' - B_i A_i') = 0.$$

Proof.

$$0 = \int\int \varphi \wedge \varphi' = \sum_{i=1}^{g}(A_i B_i' - A_i' B_i). \tag{1}$$

□

Riemann Bilinear Relation

Theorem (Riemann Bilinear Relation II)

Suppose φ is a holomorphic 1-forms. The A-period and B-period for φ are A_i and B_i, then

$$\sqrt{-1}\sum_{i=1}^{g}(A_i\bar{B}_i - B_i\bar{A}_i) \geq 0.$$

Proof.

$$\|\varphi\| = (\varphi,\varphi) = i\int\int \varphi\wedge\bar{\varphi} = \sum_{i=1}^{g}(A_i\bar{B}_i - A_i\bar{B}_i) \geq 0. \tag{2}$$

□

Period Matrix

Theorem (Period Matrix)

Suppose C is a compact Riemann surface, the period matrix Π under a canonical basis of $H_1(C,\mathbb{Z})$ and a basis of $\Omega^1(C)$ is

$$\Pi_{g\times 2g} = \left(A_{g\times g}, B_{g\times g}\right),$$

then we have

1. $AB^T = BA^T$
2. $\sqrt{-1}(A\bar{B}^T - B\bar{A}^T)$ *is a Hermite positive definite matrix.*

Period Matrix

Proof.

$$A=\begin{pmatrix}\int_{a_1}\varphi_1 & \int_{a_2}\varphi_1 & \cdots & \int_{a_g}\varphi_1\\ \int_{a_1}\varphi_2 & \int_{a_2}\varphi_2 & \cdots & \int_{a_g}\varphi_2\\ \vdots & \vdots & & \vdots\\ \int_{a_1}\varphi_g & \int_{a_2}\varphi_g & \cdots & \int_{a_g}\varphi_g\end{pmatrix} B=\begin{pmatrix}\int_{b_1}\varphi_1 & \int_{b_2}\varphi_1 & \cdots & \int_{b_g}\varphi_1\\ \int_{b_1}\varphi_2 & \int_{b_2}\varphi_2 & \cdots & \int_{b_g}\varphi_2\\ \vdots & \vdots & & \vdots\\ \int_{b_1}\varphi_g & \int_{b_2}\varphi_g & \cdots & \int_{b_g}\varphi_g\end{pmatrix}$$

$$(AB^T)_{i,j}=\sum_{k=1}^{g}\int_{a_k}\varphi_i\int_{b_k}\varphi_j \quad (BA^T)_{i,j}=\sum_{k=1}^{g}\int_{b_k}\varphi_i\int_{a_k}\varphi_j$$

By Riemann bilinear relation:

$$\sum_{k=1}^{g}\left(\int_{a_k}\varphi_i\int_{b_k}\varphi_j-\int_{b_k}\varphi_i\int_{a_k}\varphi_j\right)=0,$$

hence $AB^T=BA^T$. □

Period Matrix

Proof.

Let $\omega = \lambda_1\varphi_1 + \lambda_2\varphi_2 + \cdots + \lambda_g\varphi_g$, then

$$(\omega, \omega) = \sqrt{-1}\int \omega \wedge \bar{\omega} =$$

$$= \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_g \end{pmatrix} \sqrt{-1}(A\bar{B}^T - B\bar{A}^T) \begin{pmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \\ \vdots \\ \bar{\lambda}_g \end{pmatrix}$$

$$\geq 0.$$

Hence $\sqrt{-1}(A\bar{B}^T - B\bar{A}^T) \geq 0$. □

Period Matrix

We can change the basis of $\Omega^1(C)$ by A^{-T} to obtain the normalized period matrix

$$\Pi = (I_g \; Z)$$

then the Riemann bilinear relation becomes

1. $Z = Z^T$;
2. The imginary part of Z $\mathrm{Img}(Z)$ is a real positive definite matrix.

Period Matrix

Theorem (Torelli)

Two compact Riemann surfaces C and C' are conformal equivalent, if and only if they share the same normalized period matrix under appropriate canonical homology basis.

Problem (Schotty)

Suppose $Z = Z^T$, and the imaginary part of Z is positive definite, under what other conditions such that $(I_g\ Z)$ is a period matrix of some Riemann surface ?

Divisor

Definition (Divisor)

Suppose C is a compact Riemann surface, a divisor is a finite form of sum

$$D = m_1 p_1 + m_2 p_2 + \cdots + m_l p_l,$$

where $m_j \in \mathbb{Z}$, $p_j \in C$ $(j = 1, 2, \dots, l)$. The degree of D is defined as

$$\deg(D) = \sum_{j=1}^{l} m_j.$$

All the divisors under the addition form an Abelian group, the so-called divisor group.

Principle Divisor

Definition (Principle Divisor)

Suppose C is a compact Riemann surface, $f \in K(C)$ is a meromorphic function, the divisor of f is defined by

$$(f) = \sum_{p \in C} \nu_p(f) p$$

which is called a principle divisor.

Definition (Zero Degree Divisor Group)

Suppose C is a compact Riemann surface, $\mathrm{Div}(C)$ is the divisor group of C, then

$$\mathrm{Div}^0(C) := \{D \in \mathrm{Div}(C) : \mathrm{deg} D = 0\}$$

Abel-Jacobi Map

Definition (Abel-Jacobi Map)

Suppose C is a compact Riemann surface, choose a base point $q \in C$, the Abel-Jacobi map

$$\mu : \mathrm{Div}(C) \to J(C)$$

is given by

$$\mu(D) = \begin{pmatrix} \sum_{i=1}^{k} n_i \int_q^{p_i} \omega_1 \\ \sum_{i=1}^{k} n_i \int_q^{p_i} \omega_2 \\ \vdots \\ \sum_{i=1}^{k} n_i \int_q^{p_i} \omega_{g-1} \\ \sum_{i=1}^{k} n_i \int_q^{p_i} \omega_g \end{pmatrix} / \Lambda$$

where $D = \sum_{i=1}^{k} n_i p_i \in \mathrm{Div}(C)$.

Theorem (Abel)

The homomorphism sequence

$$K^*(C) \xrightarrow{()} Div^0(C) \xrightarrow{\mu} J(C) \longrightarrow 0$$

is exact, namely

$$Img(\ ) = Ker\ \mu$$

and μ is surjective.

Abel-Jacobi Theorem

Definition (Picard variety)

The quotient group

$$\mathrm{Pic}(C) := \frac{\mathrm{Div}^0(C)}{\mathrm{Img}\ ()}$$

is called the Picard variety of C.

Theorem (Abel)

The Abel-Jacobi map μ induces an isomorphism

$$Pic(C) \xrightarrow{\sim} J(C).$$

Abel-Jacobi Theorem

Lemma

$Img() \subset Ker\mu$, namely, for any $f \in K^*(C)$, denote $D = (f)$, then

$$\mu(D) = 0$$

.

Lemma

$ker\mu \subset Img()$, namely, if $\mu(D) = 0$, where $D \in Div^0(C)$, then there exists an $f \in K^*(C)$, such that

$$(f) = D$$

.

Lemma

The Abel-Jacobi map $\mu : Div^0(C) \to J(C)$ is surjective.

Proof for $\mu((f)) = 0$, Img() $\subset$ Kerμ

Assume $f \in K^*(C)$, for any $t \in \mathbb{C} \cup \{\infty\}$, let

$$D_t = f^{-1}(t) \in \mathrm{Div}(C).$$

Obvious

$$D = (f) = f^{-1}(0) - f^{-1}(\infty) = D_0 - D_\infty$$

we are going to prove $\mu(D_t) = \text{const}, \in \mathbb{C} \cup \{\infty\}$, then

$$\mu(D) = \mu(D_0) - \mu(D_\infty) = 0,$$

this proves the lemma. In order to prove $\mu(D_t) = \text{const}$, we consider its derivative

$$\frac{d}{dt}\mu(D_t) = \frac{d}{dt}\begin{pmatrix} \sum_{j=1} \int_q^{p_j(t)} \omega_1 \\ \vdots \\ \sum_{j=1} \int_q^{p_j(t)} \omega_g \end{pmatrix}$$

Proof for $\mu((f)) = 0$,Img() $\subset$ Kerμ

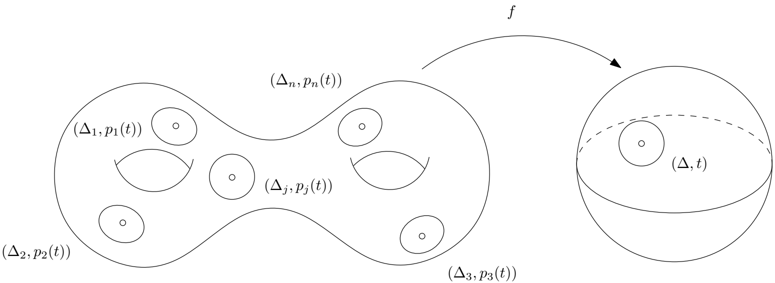


Figure: Proof for $\mu((f)) = 0$

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Proof for $\mu((f)) = 0$,Img() $\subset$ Kerμ

For $t_0 \in \mathbb{S}^2$, if $f^{-1}(t_0)$ has no branching point, then there exists a disk $\Delta \subset \mathbb{S}^2$ surrounding t_0, and n disks $\Delta_1, \Delta_2, \cdots, \Delta_n \subset C$ surrounding $p_1(t_0), p_2(t_0), \cdots, p_n(t_0)$, such that for any $j = 1, 2, \cdots, n$,

$$f : \Delta_j \to \Delta$$

is biholomorphic. So we can use $z(p) = f(p)$ as the local coordinates of Δ_j. Assume in this coordinates,

$$\omega_\alpha = h_{\alpha\ j}(z)dz,$$

then

$$\begin{aligned}\frac{d}{dt}\int_q^{p_j(t)} \omega_\alpha &= \frac{d}{dt}\int_q^{p_j(t_0)} \omega_\alpha + \frac{d}{dt}\int_{p_j(t_0)}^{p_j(t)} \omega_\alpha \\ &= \frac{d}{dt}\int_q^{p_j(t_0)} \omega + \frac{d}{dt}\int_{t_0}^{t} h_{\alpha\ j}(z)dz = h_{\alpha\ j}(t).\end{aligned}$$

Proof for $\mu((f)) = 0, \text{Img}() \subset \text{Ker}\mu$

On the other hand, in the neighborhood of $p_j(t)$, on the selected local coordinates on Δ_j, we construct the meromorophic 1-form:

$$\frac{\omega_\alpha}{f-t} = \frac{h_{\alpha\ j}(z)dz}{z-t}$$

By direct computation

$$2\pi\sqrt{-1}\text{Res}_{p_j(t)}\frac{\omega_\alpha}{f-t} = \oint_{\partial\Delta_j}\frac{\omega_\alpha}{f-t} = \oint_{\partial\Delta_j}\frac{h_{\alpha\ j}(z)dz}{z-t} = 2\pi\sqrt{-1}h_{\alpha\ j}(t).$$

By the meromorphic differential residue theorem, we have

$$\frac{d}{dt}\mu(D_t) = \frac{d}{dt}\sum_{j=1}^n\int_q^{p_j(t)}\omega_\alpha = \sum_{j=1}^n h_{\alpha\ j}(t) = \sum_{j=1}^n \text{Res}_{p_j(t)}\frac{\omega_\alpha}{f-t} = 0.$$

Proof for $\mu((f)) = 0$,Img() $\subset$ Kerμ

We use R to represent the set of the branching points of f, then $\mu(D_t)$ is holomorphic outside the finite set $f(R)$, and

$$\frac{d}{dt}\mu(D_t) = 0.$$

It is obvious that $\mathbb{S}^2 \setminus f(R)$ is connected, therefore at $t \in \mathbb{S}^2 \setminus f(R)$ we have

$$\mu(D_t) = \text{const},$$

by Riemann extension theorem, we have $\mu(D_t) = \text{const}$ on the whole sphere $\mathbb{S}^2 = \mathbb{P}^1$, hence

$$\mu((f)) = \mu(D_0) - \mu(D_\infty) = 0.$$

Proof for Ker$\mu \subset$ Img()

If $D \in \text{Div}^0(C)$, $\mu(D) = 0$, we would like to find a meromorphic function $f \in K^*(C)$, such that $(f) = D$. Assume

$$D = \sum_{i=1}^{k} n_i p_i \in \text{Div}^0(C),$$

if there is $f \in K^*(C)$, such that $(f) = D$, let

$$\varphi = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f} \in K^1(C).$$

Then φ must satisfiy

$$\begin{cases} a)\ (\varphi)_\infty = \sum_{i=1}^{k} p_i, \varphi \text{ only has simple poles} \\ b)\ \text{Res}_{p_i}\varphi = \frac{n_i}{2\pi\sqrt{-1}}, \quad n_i \in \mathbb{Z}; \\ c)\ \int_{\gamma_i} \varphi \in \mathbb{Z} \end{cases} \tag{3}$$

Proof for Ker$\mu \subset$ Img()

Eqn. (3) item c) holds, since

$$\int_{\gamma_i} \varphi = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_i} \frac{df}{f} = \frac{1}{2\pi\sqrt{-1}} \int d(\sqrt{-1}\arg f) \in Z.$$

Lemma (Meromorphic Differential)

If $\varphi \in K^1(C)$, satisfying Eqn. (3). Assume q is a fixed based point on C, let

$$f(p) = \exp\left(2\pi\sqrt{-1}\int_q^p \varphi\right),$$

the integration path doesn't go through any pole of φ, then f is a meromorphic function on C, satisfying

$$(f) = \sum_{i=1}^{k} n_i p_i = D,$$

where p_i, n_i are given in Eqn. (3) a) and b).

Proof for Ker$\mu \subset$ Img()

Note that, based on Residue theorem $\sum_{i=1}^{k} n_i = 0$, namely $D \in \text{Div}^0(C)$.

Proof.

Choose two paths γ and γ' from q to p, such that $\gamma - \gamma' = \sum_{i=1}^{2g} n_i \gamma_i + \partial B$, where B is a domain on C, therefore

$$\int_\gamma \varphi - \int_{\gamma'} \varphi = \sum_{i=1}^{2g} n_i \int_{\gamma_i} \varphi + 2\pi\sqrt{-1} \sum_j \text{Res}_{p_j} \varphi \quad \in \mathbb{Z},$$

since $\text{Res}_{p_j} \varphi = n_j / 2\pi\sqrt{-1}$, therefore $f(p)$ is independent of the choice of the integration path,

$$\exp\left(2\pi\sqrt{-1} \int_\gamma \varphi\right) = \exp\left(2\pi\sqrt{-1} \int_{\gamma'} \varphi\right),$$

$f(p)$ is a well defined function on C. $\square$

Proof for Ker$\mu \subset$ Img()

continued.

Since φ satisfies Eqn (3) a), f is holomorphic on C excepts on p_i's. In a neighborhood of p_i with local coordinates z, $z(p_i) = 0$, then

$$\varphi(z) = \frac{n_i}{2\pi\sqrt{-1}}\frac{dz}{z} + h(z)dz,$$

where $h(z)$ is holomorphic. Choose another point p_0 ($p_0 \neq p_i$) in the neighborhood of p_i, suppose $z(p_0) = z_0$, then

$$\begin{aligned} f(z) &= \exp\left(2\sqrt{-1}\pi\int_q^p \varphi\right) = \exp\left(2\sqrt{-1}\pi\left(\int_q^{p_0}\varphi + \int_{p_0}^{p}\varphi\right)\right) \\ &= \exp\left(2\sqrt{-1}\pi\left(\int_q^{p_0}\varphi + \int_{z_0}^{z}\frac{n_i}{2\sqrt{-1}\pi}\frac{dz}{z} + \int_{z_0}^{z} h(z)dz\right)\right) \end{aligned}$$

$\square$

continued.

$$= \exp\left(2\sqrt{-1}\pi\left(\int_q^{p_0}\varphi - \frac{n_i}{2\sqrt{-1}\pi}\ln z_0 + \frac{n_i}{2\sqrt{-1}\pi}\ln z + \int_{z_0}^{z} h(z)dz\right)\right)$$
$$= cz^{n_i}H(z),$$

where

$$c = \exp\left(2\sqrt{-1}\pi\left(\int_q^{p_0}\varphi - \frac{n_i}{2\sqrt{-1}\pi}\ln z_0\right)\right)$$

is a non-zero constant,

$$H(z) = \exp\left(2\sqrt{-1}\pi\int_{z_0}^{z} h(z)dz\right)$$

is a non-zero holomorphic function. Hence $(f) = \sum_{i=1}^{k} n_i p_i = D$. □

Proof for Ker$\mu \subset$ Img()

Definition (Abelian Differential of The Third Kind)

If $\varphi \in K^1(C)$ has at most simple poles, then φ is called a third type of differential. For any $p, q \in C$, $p \neq q$, $\varphi = \varphi_{pq} \in K^1(C)$ is called a third type of elementary differential, if

$$(\varphi)_\infty = p + q$$

and

$$\text{Res}_p\varphi = \frac{1}{2\sqrt{-1}\pi}, \quad \text{Res}_q\varphi = -\frac{1}{2\sqrt{-1}\pi}.$$

Theorem (Existence of Abelian Differential of the Third Kind)

For any $p, q \in C$, $p \neq q$, there is a normal Abelian differential of the third kind $\varphi_{pq} \in K^1(C)$, such that $(\varphi)_\infty = p + q$ and

$$Res_p\varphi = (2\sqrt{-1}\pi)^{-1}, \quad Res_q\varphi = -(2\sqrt{-1}\pi)^{-1}.$$

Abel Differential of the Third Type

Proof.

Set the divisor $D = -p - q$, then by Riemann-Roch formula

$$\dim l(-D) = \dim i(D) + d(D) + 1 - g,$$

$-D \geq 0$, so $f \in l(-D)$ must be holomorphic, therefore $f \equiv const$, $(f) = 0$, hence $\dim l(-D) = 0$. Therefore

$$0 = \dim i(D) - 2 + 1 - g \implies \dim i(D) = g + 1.$$

Therefore we can pick $\omega \in i(D)$, then ω has poles at p and q only. $\square$

Proof for Ker$\mu \subset$ Img()

Proof.

For any divisor

$$D = \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} q_i \in \text{Div}^0(C)$$

(p_i or q_i may be repeated), there are k elementary Abelian differentials of the 3rd kind $\varphi_1, \varphi_2, \cdots, \varphi_k$, where φ_i has simple poles at q_i^+ and q_i^- with residues

$$\text{Res}_{q_i^+}\varphi_i = (2\sqrt{-1}\pi)^{-1} \quad \text{Res}_{q_i^-}\varphi_i = -(2\sqrt{-1}\pi)^{-1}.$$

Let

$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_k.$$

Choose a canonical basis of $H_1(C,\mathbb{Z})$ $\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\}$, which do not go through any pole of φ; $\omega_1, \omega_2, \cdots, \omega_g$ is a basis of $\Omega^1(C)$, such that the period matrix is normalized to be $(I\ Z)$. □

Proof for Ker$\mu \subset$ Img()

continued.

Let

$$\varphi' = \varphi - \sum_{\alpha=1}^{g} \left(\int_{\gamma_\alpha} \varphi \right) \omega_\alpha$$

Then φ' has the same poles and residues as φ, and the periods of φ' on γ_j's are zeros, $\pi_j(\varphi') = 0$, for $j = 1, 2, \cdots, g$. $\square$

Lemma (Bilinear Relation between I and III Abel Differentials)

Suppose $\omega \in \Omega^1(C)$ *is a holomorphic 1-form, then*

$$\sum_{i=1}^{k} \int_{q_i}^{p_i} \omega = \sum_{i=1}^{g} \pi_i(\omega)\pi_{g+i}(\varphi'). \tag{4}$$

Proof.

Suppose the fundamental polygon is

$$\Omega = C - \bigcup_{i=1}^{2g} \gamma_i,$$

choose a base point $b \in \Omega$, define a holomorphic function by integrating ω inside Ω,

$$\nu(p) := \int_b^p \omega \quad (p \in \Omega).$$

continued.

Then $\nu\varphi'$ is a meromorphic differential, whose poles are the same as φ', by Residue theorem

$$2\sqrt{-1}\pi\sum_{i=1}^{k}(\text{Res}_{p_i}(\nu\varphi')+\text{Res}_{q_i}(\nu\varphi'))=\int_{\partial\Omega}\nu\varphi' \tag{5}$$

The left hand side of Eqn. (5) equals to

$$\sum_{i=1}^{k}(\nu(p_i)-\nu(q_i))=\sum_{i=1}^{k}\int_{q_i}^{p_i}\omega$$

The right hand side of Eqn. (5) is ($\pi_i(\varphi')=0$, $i=1,\dots,g$)

$$\int_{\partial\Omega}\nu\varphi'=\sum_{i=1}^{g}(\pi_i(\omega)\pi_{g+i}(\varphi')-\pi_i(\phi')\pi_{g+i}(\omega))=\sum_{i=1}^{g}\pi_i(\omega)\pi_{g+i}(\varphi')$$

Bilinear Relation between I and III Abel Differentials

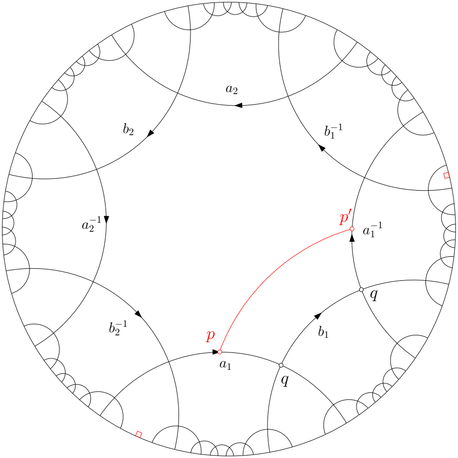

Figure: $\int_{a_1} \nu\varphi + \int_{a_1^{-1}} \nu\varphi = -\pi_{b_1}(\omega)\pi_{a_1}(\varphi)$, $\nu = \int \omega$.

continued.

$$\int_{\partial\Omega} \nu\varphi' = \sum_{i=1}^{g} \left(\int_{\gamma_i} \nu\varphi' + \int_{\gamma_i^{-1}} \nu\varphi' + \int_{\gamma_{g+i}} \nu\varphi' + \int_{\gamma_{g+i}^{-1}} \nu\varphi' \right).$$

Choose $p \in \gamma_i$, the same point $p' \in \gamma_i^{-1}$, then

$$\int_{\gamma_i} \nu\varphi' + \int_{\gamma_i^{-1}} \nu\varphi' = \int_{\gamma_i} (\nu(p) - \nu(p'))\varphi' = -\pi_{g+i}(\omega)\pi_i(\varphi').$$

$$\nu(p) - \nu(p') = \int_{p'}^{p} \omega = \int_{p'}^{q} \omega - \int_{\gamma_{g+i}} \omega + \int_{q}^{p} \omega = -\int_{\gamma_{g+i}} \omega = -\pi_{g+i}(\omega).$$

similarly

$$\int_{\gamma_{g+i}} \nu\varphi' + \int_{\gamma_{g+i}^{-1}} \nu\varphi' = \int_{\gamma_{g+i}} (\nu(p) - \nu(p'))\varphi' = \pi_i(\omega)\pi_{g+i}(\varphi').$$

Proof for Ker$\mu \subset$ Img()

continued.

By Eqn. (4), let $\omega = \omega_\alpha$, $\alpha = 1, 2, \cdots, g$

$$\sum_{i=1}^{k} \int_{q_i}^{p_i} \omega_\alpha = \sum_{\beta=1}^{g} \pi_\beta(\omega_\alpha) \pi_{g+\beta}(\varphi'),$$

Since the period matrix id $(I\ \ Z)$, $\pi_\beta(\omega_\alpha) = \delta_{\alpha\beta}$, the right hand side is $\pi_{g+\alpha}(\varphi')$.

$$\sum_{i=1}^{k} \int_{q_i}^{p_i} \omega_\alpha = \pi_{g+\alpha}(\varphi'). \tag{6}$$

□

Proof for Ker$\mu \subset$ Img()

continued.

By the assumption, $D \in \text{Ker}\mu$, the left hand side is

$$(\mu(D))_\alpha = \sum_{i=1}^{k} \int_q^{p_i} \omega_\alpha - \sum_{i=1}^{k} \int_q^{q_i} \omega_\alpha = \sum_{i=1}^{k} \int_{q_i}^{p_i} \omega_\alpha = 0 \pmod{\Lambda}$$

We obtain left hand side of Eqn. (6) becomes ($\alpha = 1, 2, \cdots, g$) $\int_{\gamma_\beta} \omega_\alpha = \delta_\alpha^\beta$,

$$\sum_{\beta=1}^{g} \left(m_\beta \int_{\gamma_\beta} \omega_\alpha + m_{g+\beta} \int_{\gamma_{g+\beta}} \omega_\alpha \right) = m_\alpha + \sum_{\beta=1}^{g} m_{g+\beta} \int_{\gamma_{g+\beta}} \omega_\alpha$$

where m_β, $m_{g+\beta}$, $\beta = 1, 2, \cdots, g$ are integers independent of α. $\square$

Proof for Ker$\mu \subset$ Img()

continued.

By Riemann bilinear relation $Z^T = Z$, we have

$$\int_{\gamma_{g+\beta}} \omega_\alpha = \int_{\gamma_{g+\alpha}} \omega_\beta.$$

The left hand side of Eqn. (6) becomes

$$m_\alpha + \sum_{\beta=1}^{g} m_{g+\beta} \int_{\gamma_{g+\alpha}} \omega_\beta,$$

hence Eqn. (6) becomes

$$m_\alpha + \sum_{\beta=1}^{g} m_{g+\beta} \int_{\gamma_{g+\alpha}} \omega_\beta = \pi_{g+\alpha}(\varphi')$$

Proof for Ker$\mu \subset$ Img()

Proof.

Then we define

$$\varphi'' := \varphi' - \sum_{\beta=1}^{g} m_{g+\beta}\omega_\beta,$$

Then φ'' has the same poles and residues as φ', so as φ,

$$(\varphi'')_\infty = \sum_{i=1}^{k} q_i^+ + q_i^- \quad \text{Res}_{p_i^+}\varphi'' = (2\sqrt{-1}\pi)^{-1} \quad \text{Res}_{p_i^-}\varphi'' = 2\sqrt{-1}\pi)^{-1}.$$

Now $\alpha = 1, 2, \cdots, g$

$$\pi_\alpha(\varphi'') = \pi_\alpha(\varphi') - \sum_{\beta=1}^{g} m_{g+\beta}\pi_\alpha(\omega_\beta) = 0 - \sum_{\beta=1}^{g} m_{g+\beta}\delta_{\alpha\beta} = -m_{g+\alpha}.$$

$$\pi_{g+\alpha}(\varphi'') = \pi_{g+\alpha}(\varphi') - \sum_{\beta=1}^{g} m_{g+\beta}\pi_{g+\alpha}(\omega_\beta) = m_\alpha$$

Proof for Ker$\mu \subset$ Img()

continued.

Since φ'' satisfies all three conditions in Eqn. (3), it is the desired meromorphic differential, by the lemma of Meromrophic differential, we construct the meromorphic function

$$f(p) = \exp\left(2\sqrt{-1}\pi \int_q^p \varphi''\right),$$

then

$$(f) = D.$$

Hence Ker$\mu \subset$ Img(). Therefore Ker$\mu =$ Img(). $\square$

Jacobi Theorem

Lemma (Special Holomorphic Differential Basis)

Suppose C is a compact genus g Riemann surface, (U,z) is a local coordinate chart of C, then there are g distinct points $p_1, p_2, \cdots, p_g$ in U, and a basis of $\Omega^1(C)$ holomorphic differentials, such that the matrix

$$\begin{pmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(p_g) \\ f_2(p_1) & f_2(p_2) & \cdots & f_2(p_g) \\ \vdots & \vdots & & \vdots \\ f_g(p_1) & f_g(p_2) & \cdots & f_g(p_g) \end{pmatrix}$$

is non-degenerated, where $f_i dz$ is the local representation of φ_i.

Jacobi Theorem

Proof.

Choose a non-zero holomorphic 1-form φ_1, since $\varphi_1 \not\equiv 0$ in U, there is a point $p_1 \in U$, such that $\varphi_1(p_1) \neq 0$. By Riemann-Roch, let $D = p_1$

$$\dim l(-p_1) = \dim i(p_1) + deg(p_1) + 1 - g,$$

suppose $f \in K(C)$, $(f) \geq -p_1$. Any meromorphic (non-holomorphic) function must have multiple poles, (since genus is non-zero), so f is holomorphic, $f \equiv const$, so dim $l(-p_1) = 1$.

$$1 = \dim\ i(p_1) + 1 + 1 - g \implies \dim\ i(p_1) = g - 1.$$

We can choose a holomorphic 1-form $\varphi_2 \in i(p_1)$, such that at some point $p_2 \in U$,

$$\varphi_2(p_1) = 0; \quad \varphi_2(p_2) \neq 0.$$

$\square$

Jacobi Theorem

continued.

By Riemann-Roch, {holomorphic functions} $\subset l(-p_1 - p_2)$,

$$\dim\ l(-p_1 - p_2) = \dim\ i(p_1 + p_2) + 2 + 1 - g.$$

Since $\dim\ i(p_1 + p_2) \leq \dim\ i(p_1) - 1 = g - 2$, $\dim\ l(-p_1 - p_2) \leq 1$,
therefore $\dim\ l(-p_1 - p_2) = 1$, $\dim\ i(p_1 + p_2) = g - 2$.
we can choose another holomorphic 1-form $\varphi_3 \in i(p_1 + p_2)$, such that φ_3 is non-zero at some point $p_3 \in U$, $\varphi_3(p_3) \neq 0$.

$$\varphi_3(p_1) = 0, \quad \varphi_3(p_2) = 0; \quad \varphi_3(p_3) \neq 0.$$

Similarly

$$\dim l(-p_1 - p_2 - p_3) = \dim i(p_1 + p_2 + p_3) + 3 + 1 - g.$$

$\dim i(p_1 + p_2 + p_3) \leq \dim i(p_1 + p_2) - 1 = g - 3$, $\dim l(-p_1 - p_2 - p_3) \leq 1$,
therefore $\dim l(-p_1 - p_2 - p_3) = 1$, $\dim i(p_1 + p_2 + p_3) = g - 3$ $\square$

Jacobi Theorem

continued.

By repeating this procedure, we can obtain g points $p_1, p_2, \cdots, p_g \in U$ and g non-zero holomorphic 1-forms $\varphi_1, \varphi_2, \cdots, \varphi_g$, such that

$$\varphi_i(p_j) = 0, j = 1, 2, \cdots, i-1;\ \ \varphi_i(p_i) \neq 0.$$

If in U, $\varphi_i = f_i dz$ $(i = 1, 2, \cdots, g)$, then the matrix

$$(f_i(p_j))_{g\times g}$$

is triangular, and the diagonal elements are non-zeros. Therefore the matrix is non-degenerated, $\{\varphi_i\}$ form a basis of $\Omega^1(C)$. □

Special Holomorphic Differential Basis

$$\begin{pmatrix} f_1(p_1) & f_1(p_2) & f_1(p_3) & \cdots & f_1(p_{g-1}) & f_1(p_g) \\ 0 & f_2(p_2) & f_2(p_3) & \cdots & f_2(p_{g-1}) & f_2(p_g) \\ 0 & 0 & f_3(p_3) & \cdots & f_3(p_{g-1}) & f_3(p_g) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_g(p_g) \end{pmatrix}$$

Jacobi Theorem

Suppose $p_1, p_2, \cdots, p_g$ are g points in the lemma of special holomorphic differential basis, $C^g := C \times C \times, \cdots, \times C$, define

$$\Psi : C^g \to \mathrm{Pic}(C), \quad \Psi(x_1, x_2, \cdots, x_g) = \sum_{i=1}^{g} (x_i - p_i) \mod \mathcal{P},$$

where $\mathcal{P}$ is the set of principle divisors. Denote the composition map $\mu \circ \Psi$ as J.

$$J : C^g \xrightarrow{\Psi} \mathrm{Pic}(C) \xrightarrow{\mu} J(C).$$

Theorem (Jacobi)

The map $\Psi : C^g \to Pic(C)$ is surjective, $\mu : Pic(C) \to J(C)$ is an isomorphism, hence $J : C^g \to J(C)$ is surjective.

Jacobi Theorem

Proof.

Suppose D is a zero degree divisor. Consider the degree g divisor,

$$D' = D + p_1 + p_2 + \cdots + p_g.$$

By Riemann-Roch formula, we have

$$\dim l(-D') = \dim i(D') + d(D') + 1 - g \geq d(D') + 1 - g = 1,$$

therefore there is a non-zero meromorphic function $f \in l(-D')$, $(f) + D' \geq 0$. $deg((f) + D') = deg((f)) + deg(D) + g = g$, hence

$$(f) + D' = x_1 + x_2 + \cdots + x_g, \quad x_i \in C, i = 1, 2, \cdots, g.$$

Namely $(f) + D = \sum_{i=1}^{g}(x_i - p_i) = \Psi(x_1, x_2, \cdots, x_g)$. This means $\Psi(x_1, x_2, \cdots, x_g) = [D] \in \mathrm{Pic}(C)$, namely Ψ is surjective. $\square$

continued.

By Abel theorem, μ is injective. In order to show μ is isomorphic, it is surficient to show the image of μ contains an open set of $[0] \in J(C)$, in turn, we only need to show the image of $J = \mu \circ \Psi$ contains such an open set. Select $\{\varphi_i\}$ as the set of holomorphic 1-form basis in lemma of special holomorphic differential basis. Choose disjoint small disks $B_i \subset U$ centered at p_i, the local coordinate on B_i is z. In each B_i, choose $z_i \in B_i$, then

$$\lambda = (z_1, z_2, \cdots, z_g) \in C^g.$$

The local representation of J is

$$J(z_1, z_2, \cdots, z_g) = \left(\sum_{j=1}^{g} \int_{p_j}^{z_j} f_1 dz, \sum_{j=1}^{g} \int_{p_j}^{z_j} f_2 dz, \cdots, \sum_{j=1}^{g} \int_{p_j}^{z_j} f_g dz \right),$$

where the integration paths are contained in each disk B_i's. □

Jacobi Theorem

continued.

The i-th component of J is $u_i(z_1, z_2, \ldots, z_g)$, then

$$\frac{\partial u_i}{\partial z_j} = f_i(z_j).$$

According to lemma of special holomorphic differential basis, the Jacobi matrix of J at $(p_1, p_2, \cdots, p_g)$ is non-degenerated. By inverse mapping theorem, we know the image of J contains an open set. This completes the proof. $\square$