Holomorphic Line Bundle

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Holomorphic and Meromorphic Differentials

Holomorphic or meromorphic differentials play fundamental roles in many applications:

1. Holomorpic 1-forms: conformal mappings;
2. Holomorphic quadratic differentials: surface foliations, Teichmüller maps;
3. Holomorphic cubic differentials: surface projective structures;
4. Meromorphic quatic differentials: surface quadrilateral meshes.

All of them can be treated as global sections of holomorpic line bundles over a Riemann surface.

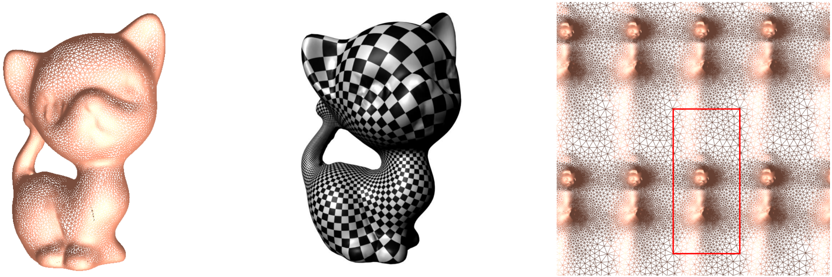

Figure: Conformal mapping of the kitten model.

Holomorphic Quadratic Differentials

Holomorphic Quadratic Differentials

Figure: Foliations of the cat model.

Meromorphic Quartic Differentials

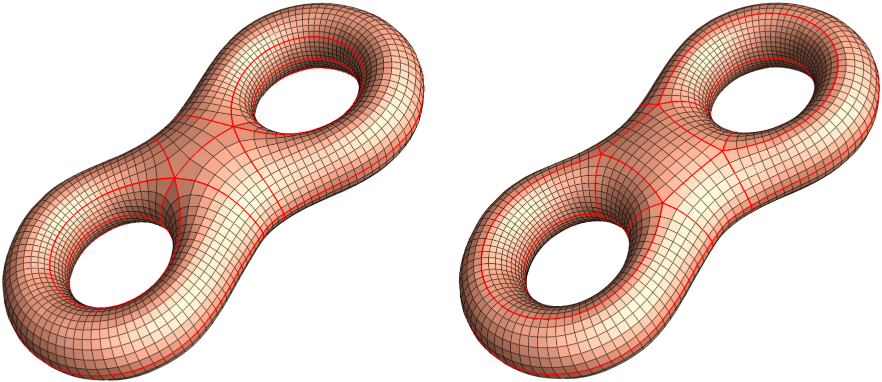

Figure: Quad-meshes on the eight model.

Holomorphic Line Bundle

Definition (Holomorphic Line Bundle)

Suppose M is a Riemann surface, L is a 2 dimensional complex manifold, $\pi : L \to M$ is a surjective map. If there is an open covering $\{U_\alpha\}$ and biholomorphic map $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ satisfying:

1. $\psi_\alpha(\pi^{-1}(p)) = \{p\} \times \mathbb{C}, \forall p \in U_\alpha$.
2. when $U_\alpha \cap U_\beta \neq \emptyset$, there is a holomorphic function $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, such that

$$\psi_\beta \circ \psi_\alpha^{-1}(p, a) = (p, g_{\beta\alpha}(p) \cdot a), \quad \forall p \in U_\alpha \cap U_\beta, \quad a \in \mathbb{C},$$

then we say L is a holomorphic line bundle over M, π is the bundle projection.

ψ_α is called *local trivialization*, $g_{\beta\alpha}$ *transition function*, $\pi^{-1}(p)$ *fiber*.

Holomorphic Line Bundle

A holomorphic line bundle is determined by the open covering $\{U_\alpha\}$ and the transition functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, satisfying $\forall U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$,

$$g_{\alpha\alpha} = 1, \quad g_{\alpha\beta} = (g_{\beta\alpha})^{-1}, \quad g_{\beta\alpha} \cdot g_{\alpha\gamma} \cdot g_{\gamma\beta} = 1. \tag{1}$$

The holomorphic line bundle $L = (\{U_\alpha\}, \{g_{\beta\alpha}\})$ is defined as

$$L = \coprod_\alpha (U_\alpha \times \mathbb{C}) / \sim,$$

where the equivalence relation $\sim$ is defined as: for any $(p, a) \in U_\alpha \times \mathbb{C}$, $(q, b) \in U_\beta \times \mathbb{C}$,

$$(p, a) \sim (q, b) \iff p = q, b = g_{\beta\alpha}(p)a.$$

Holomorphic Section

Definition (Holomorphic Section)

let L be a holomorphic line bundle over a Riemann surface M with projection $\pi : L \to M$. A holomorphic section s of L is a map $s : M \to L$ such that:

1. $\pi(s(p)) = p$ for all $p \in M$. In other words, $s(p)$ lies in the fiber $L_p = \pi^{-1}(p)$ for each point p in M.
2. s is holomorphic, which means that for each local trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{C}$ of L, the map $\psi \circ s : U \to \mathbb{C}$ is holomorphic, where U is an open set in M.

All the holomorphic sections of L is denoted as $\Gamma_h(L)$. If M is a compact Riemann surface, then $\Gamma_h(L)$ is a finite dimensional complex vector space.

Holomorphic Section

Given a holomorphic line bundle $L = (\{U_\alpha\}, \{g_{\beta\alpha}\})$, $\{\psi_\alpha\}$ is the local trivialization, $s : M \to L$ is a holomorphic section, the local representation of the section is

$$\psi_\alpha(s(p)) = (p, s_\alpha(p)), \quad \forall p \in U_\alpha.$$

where $s_\alpha : U \to \mathbb{C}$ is a holomorphic function. When $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$s_\beta(p) = g_{\beta\alpha}(p) \cdot s_\alpha(p), \quad \forall p \in U_\alpha \cap U_\beta. \tag{2}$$

Therefore, a holomorphic section has the local representation

$$(L, s) = (\{U_\alpha\}, \{g_{\beta\alpha}\}, \{s_\alpha\}).$$

Meromorphic Section

Definition (Meromorphic Section)

Given a set of meromorphic function $s_\alpha : U_\alpha \to \mathbb{C}$ satisfying

$$s_\beta(p) = g_{\beta\alpha}(p) \cdot s_\alpha(p), \quad \forall p \in U_\alpha \cap U_\beta. \tag{3}$$

then $s = \{s_\alpha\}$ is called a meromorphic section of the holomorphic line bundle $L = (\{U_\alpha\}, \{g_{\beta\alpha}\})$.

The set of all meromorphic sections is denoted as $\mathfrak{M}(L)$.

Picard Group

Bundle Homomorphism

Definition (bundle homomorphism (bundle map))

Suppose L_1, L_2 are holomorphic line bundles over M and N respectively, π_1 and π_2 are bundle projections. If the pair of holomorphic mappings $(F, f) : (L_1, M) \to (L_2, N)$ satisfy the conditions:

1. $\pi_2 \circ F = f \circ \pi_1$, namely $F(\pi_1^{-1}(p)) \subset \pi_2^{-1}(f(p)), \forall p \in M$
2. the restriction of F on every fiber $\pi^{-1}(p)$ is a linear homomorphism,

then (F, f) is called a bundle homomorphism between L_1 and L_2.

$$\begin{tikzcd} L_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & L_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Bundle Homomorphism

Definition (bundle Isomorphism)

Suppose (F, f) is a bundle homomorphism between holomorphic line bundles L_1, L_2. If there is a bundle homomorphism (G, g) from L_2 to L_1, such that F, G are inverse biholomorphic maps, f, g are inverse biholomorphic maps, then we say L_1 and L_2 are isomorphic, $(F, f), (G, g)$ are bundle isomorphisms.

$$\begin{array}{ccccccc} L_1 & \xrightarrow{F} & L_2 & \quad & L_2 & \xrightarrow{G} & L_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow[f]{} & N & & N & \xrightarrow[g]{} & M \end{array}$$

Bundle Homomorphism

Suppose (F, f) is a bundle isomorphism between $L_1 = (\{U_\alpha\}, \{g_{\beta\alpha}\})$ and $L_2 = (\{U_\alpha\}, \{h_{\beta\alpha}\})$, φ_α and ψ_α are local trivializations of $\{U_\alpha\}$. F has local representation $F_\alpha = \psi_\alpha \circ F \circ \varphi_\alpha^{-1} : U_\alpha \times \mathbb{C} \to U_\alpha \times \mathbb{C}$:

$$F_\alpha(p, a) = (p, f_\alpha(p) \cdot a), \quad \forall p \in U_\alpha, a \in \mathbb{C},$$

where $f_\alpha : U_\alpha \to \mathbb{C}^*$ is a holomorphic function, when $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$g_{\beta\alpha} = f_\beta^{-1} h_{\beta\alpha} f_\alpha. \tag{4}$$

Corollary (Trivial Bundle)

Suppose a holomorpic line bundle L is isomorphic to a trivial bundle, if and only if there is a local trivialization $L = (\{U_\alpha\}, \{g_{\beta\alpha}\})$ and holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}^*$, such that the transition function $g_{\beta\alpha} = f_\beta^{-1} f_\alpha$.

Line Bundle Class Group - Picard Group

All the holomorphic line bundles over a Riemann surface are classified by up to bundle isomorphism. Each class is denoted as $[L]$.
The set of all the isomorphism classes of holomorphic line bundles is denoted as $\mathcal{L}(M)$. $\mathcal{L}(M)$ forms a group, the so-called *Picard group* $Pic(M)$

1. the trivial bundle is the unit element;
2. the inverse of $L = (\{U_\alpha\}, \{g_{\beta\alpha}\})$ is $-L = (\{U_\alpha\}, \{g_{\beta\alpha}^{-1}\})$
3. the product of $L_1 = (\{U_\alpha\}, \{g_{\beta\alpha}^1\})$ and $L_2 = (\{U_\alpha\}, \{g_{\beta\alpha}^2\})$ is

$$L_1 + L_2 = (\{U_\alpha\}, \{g_{\beta\alpha}^1 \cdot g_{\beta\alpha}^2\}).$$

The Picard group is Abelian.

Divisor and Line Bundle

Divisor

Definition (Divisor)

Assume M is a Riemann surface, consider a map $D : M \to \mathbb{Z}$, if $D(p) = 0$ for almost all the points $p \in M$, except for a finite number of points, then D is called a *divisor* on M.

Representation

A divisor D is represented as a formal summation $D = \sum_{p \in M} D_1(p) \cdot p$. If $D(p) \geq 0$, $\forall p \in M$, then D is called an *effective divisor*, denoted as $D \geq 0$.

Divisor Group

Definition (Divisor Group)

The set of all divisors is denoted as $\mathcal{D}$. Introduce addition operators in $\mathcal{D}$: assume $D_1 = \sum_{p\in M} D_1(p)\cdot p$, $D_2 = \sum_{p\in M} D_2(p)\cdot p$

$$D_1 \pm D_2 = \sum_{p\in M} (D_1(p) \pm D_2(p))\cdot p.$$

$\mathcal{D}$ is an Abelian group under the addition operator, which is called the *divisor group*.

Definition (Divisor Degree)

Define a homomorphism $d:\mathcal{D}\to\mathbb{Z}$, $d(D) = \sum_{p\in M} D(p)$ is called the degree of the divisor D.

Principle Divisor

Definition (Principle Divisor)

Suppose f is a meromorphic function on M, then ω determines a divisor

$$(f) = \sum_{p \in M} \nu_p(f) \cdot p,$$

where $\nu_p(f)$ is the order of f at p. (f) is called a principle divisor.

Principle divisor has the following properties:

- if M is a compact Riemann surface, then $d((f)) = 0$.
- $(f \cdot g) = (f) + (g)$, $(f/g) = (f) - (g)$.

Principle Divisor

Definition (Principle Divisor Group)

Denote the subgroup of $\mathcal{D}$,

$$\mathcal{P} := \{(f) | f \in \mathcal{M}(M)\},$$

$\mathcal{P}$ is called the principle divisor group.

Definition (Divisor Class Group)

For any $D_1, D_2 \in \mathcal{D}$, if $D_1 - D_2 \in \mathcal{P}$, then D_1 and D_2 are linearly equivalent, denoted as $D_1 \cong D_2$. The equivalence classes form the divisor class group, $\mathcal{D}/\mathcal{P}$, still denoted as $\mathcal{D}$.

Canonical Divisor

Definition (Canonical Divisor)

Suppose ω is a meromorphic differential on M, then ω determines a divisor

$$(\omega) = \sum_{p \in M} \nu_p(\omega) \cdot p,$$

where $\nu_p(\omega)$ is the order of ω at p. (ω) is called a canonical divisor.

Definition (Canonical Divisor Class)

Suppose ω_1, ω_2 are meromorphic differentials, then

$$(\omega_1) - (\omega_2) = (\omega_1/\omega_2),$$

where ω_1/ω_2 is a meromorphic function, hence (ω_1) and (ω_2) are linearly equivalent. Their equivalence class is denoted as K, called the *canonical divisor class*.

Principle Divisor Space

Definition (Meromorphic Function/Differential Spaces)

Suppose M is a compact Riemann surface, D is a divisor on M, let

$$l(D) = \{f \in \mathfrak{M}(M) | (f) + D \geq 0\}$$
$$i(D) := \{\omega \in \mathfrak{K}(M) | (\omega) - D \geq 0\}$$

The Riemann-Roch theorem can be formulated as

$$\boxed{\dim\ l(D) - \dim\ i(D) = \deg(D) + 1 - g}.$$

Note: the notations $l(D)$ and $i(D)$ are equivalent to $L(-D)$ and $\Omega(D)$ before.

$$\boxed{\dim\ L(-D) = \dim\ \Omega(D) + \deg(D) + 1 - g}.$$

Principle Divisor Space

Lemma

$l(D)$ and $i(D)$ have the following properties:

1. *$l(D)$ and $i(D)$ are complex vector spaces. If $D_1 \cong D_2$, then $l(D_1)$ and $l(D_2)$ are linearly isomorphic, $i(D_1)$ and $i(D_2)$ are linearly isomorphic.*
2. *$l(D)$ and $i(D)$ are finite dimensional complex vector space.*
3. *if K is a canonical divisor, then $i(D)$ and $l(K-D)$ are linearly isomorphic.*

Principle Divisor Space

Proof.

1. Assume $f, g \in l(D)$, then by $\nu_p(f+g) \geq \min\{\nu_p(f), \nu_p(g)\}$, and $(f)+D \geq 0$, $(g)+D \geq 0$, we obtain $\nu_p(f+g)+D(p) \geq 0$, $\forall p \in M$, hence $f+g \in l(D)$. For any $\lambda \in \mathbb{C}$, if $f \in l(D)$, then $\lambda f \in l(D)$. Therefore $l(D)$ is a complex vector space.
If $D_1 - D_2 = (f_0)$, $f_0 \in \mathfrak{M}(M)$, then

$$(f)+D_1 \geq 0 \iff (f)+D_2+(f_0) \geq 0 \iff (ff_0)+D \geq 0.$$

$\varphi : l(D_1) \to l(D_2)$, $f \mapsto ff_0$ is the linear isomorphism. $\square$

Principle Divisor Space

Proof.

2. Write D as the difference between two effective divisors $D = D_1 - D_2$, $D_1 \geq 0$, $D_2 \geq 0$. It is obvious $l(D) \subset l(D_1)$. So we only need to show for effective divisor D_1, $dim\ l(D_1) < \infty$. By induction on $d(D_1)$. When $d(D_1) = 0$, $D_1 = 0$, hence

$$l(D_1) = l(0) = \{f \in \mathfrak{M}(M) | (f) \geq 0\} = \{f \in \mathcal{O}(M)\} = \mathbb{C}$$

Assume $d(D_1) = m$, $dim\ l(D_1) < \infty$, when $d(D_1) = m + 1$, suppose

$$D_1 = n \cdot p + \cdots, \ n > 0.$$

Suppose

$$A_{D_1} = \{f \in l(D_1) | \nu_p(f) > -n\}, \quad B_{D_1} = \{f \in l(D_1) | \nu_p(f) = -n\}$$

then $l(D_1) = A_{D_1} \cup B_{D_1}$, and $A_{D_1} \subset l(D_1 - p)$. $\square$

Principle Divisor Space

continued.

2. By induction assumption, $dim\ A_{D_1} < \infty$. If $B_{D_1} \neq \emptyset$, choose $f_0 \in B_{D_1}$. Let z be a local coordinates near p, $z(p) = 0$. f_0 has expansion near p,

$$f_0(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \cdots, a_{-n} \neq 0.$$

For any $f \in B_{D_1}$, f has similar expansion, there is a $\lambda \in \mathbb{C}$, such that $f - \lambda f_0 \in A_{D_1}$. This shows $l(D_1) = span\{f_0, A_{D_1}\}$, particularly

$$\dim\ l(D_1) \leq \dim\ A_{D_1} + 1 < \infty.$$

By induction, $l(D)$ is of finite dimensional. $\square$

Principle Divisor Space

Proof.

3. Suppose the canonical divisor K is induced by a non-zero meromorphic differential ω, then we have

$$(\eta) - D \geq 0 \iff (\eta) - (\omega) + (\omega) - D \geq 0 \iff (\eta/\omega) + K - D \geq 0.$$

Hence the map $\psi : i(D) \to l(K - D)$

$$\eta \mapsto \eta/\omega$$

is the linear isomorphism. $\square$

Divisor & Line Bundle

Definition ($\lambda(D)$)

Suppose $D = \sum_i n_i \cdot p_i$ is a divisor on M, choose an local coordinate chart cover $\{U_\alpha\}$, for each chart U_α, select a meromorphic function f_α, such that $(f_\alpha) = D \cap U_\alpha$. If $U_\alpha \cap U_\beta \neq \emptyset$, then on $U_\alpha \cap U_\beta$, $f_{\beta\alpha} = f_\beta / f_\alpha$ has no poles or zeros, so it is a holomorphic function nowhere zero. $\{f_{\beta\alpha}\}$ satisfies the requirements of transition functions, then determines a holomorphic line bundle, denoted as $\lambda(D)$.

Divisor & Line Bundle

λ is well defined

Suppose we choose another meromorphic function g_α on U_α, such that $(g_\alpha) = D \cap U_\alpha$, then $h_\alpha = f_\alpha / g_\alpha$ has no zero or pole on U_α, therefore h_α is a holomorphic function nowhere zero. The transition function $g_{\beta\alpha} = g_\beta / g_\alpha$, then

$$g_{\beta\alpha} = h_\beta^{-1} f_{\beta\alpha} h_\alpha,$$

therefore the holomorphic line bundles determined by $\{g_{\beta\alpha}\}$ and $\{f_{\beta\alpha}\}$ are bundle isomorphic.

Divisor & Line Bundle

$\lambda(D)$ trivial iff D principle

If $D = (f)$ is principle, then we can choose $f_\alpha = f|_{U_\alpha}$, then

$$f_{\beta\alpha} = f_\beta / f_\alpha \equiv 1$$

therefore $\lambda(D)$ is trivial. Inversely, if $\lambda(D)$ is trivial, then there exists a trivialization open cover $\{U_\alpha\}$ and holomorphic functions $\varphi : U_\alpha \to \mathbb{C}^*$, such that on $U_\alpha \cap U_\beta$, $f_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$. On $U_\alpha \cap U_\beta$,

$$f_{\beta\alpha} = f_\beta / f_\alpha = \varphi_\alpha / \varphi_\beta,$$

therefore $f_\beta \varphi_\beta = f_\alpha \varphi_\alpha$, $\{f_\alpha \varphi_\alpha\}$ defines a globally defined meromorphic function on M, then

$$(f_\alpha \varphi_\alpha) = (f_\alpha) + (\varphi_\alpha) = (f_\alpha) = D \cap U_\alpha,$$

hence $(f\varphi_\alpha) = D$, D is a principle divisor.

Divisor & Line Bundle

$\lambda : \mathcal{D} \to \mathcal{L}$ homomorphism

Because

$$\lambda(-D) = -\lambda(D), \lambda(D_1) + \lambda(D_2) = \lambda(D_1 + D_2)$$

$\lambda : \mathcal{D} \to \mathcal{L}$ induces a homomorphism from the divisor class group $\mathcal{D}$ to the holomorphic line bundle class group $\mathcal{L}$. $\lambda(D)$ is trivial if and only if D is principle. This means λ is injective. In fact, λ is an isomorpism.

Picard Group

Theorem (Divisor Class Group and Line Bundle Class Group)

λ induces an isomorphism between divisor class group and the line bundle class group, $\lambda : \mathcal{D} \to \mathcal{L}$.

Divisors and Line Bundles

Lemma

For any divisor D, there is a linear isomorphism $l(D) \cong \Gamma_h(\lambda(D))$.

Proof.

Select an arbitrary holomorphic section s of $\lambda(D)$, s has local representation $\{s_\alpha : U_\alpha \to \mathbb{C}\}$, on $U_\alpha \cap U_\beta$, s_α satisfies

$$s_\beta = f_{\beta\alpha} s_\alpha = \frac{f_\beta}{f_\alpha} s_\alpha, \quad \frac{s_\beta}{f_\beta} = \frac{s_\alpha}{f_\alpha},$$

Therefore s_α / f_α determines a global meromorphic function on M, denoted as $i(s)$. On every U_α, we have

$$[(i(s)) + D] \cap U_\alpha = (s_\alpha) - (f_\alpha) + D \cap U_\alpha = (s_\alpha) \geq 0.$$

Hence $i(s) \in l(D)$, we have defined a linear map $i : \Gamma_h(\lambda(D)) \to l(D)$. □

Divisors and Line Bundles

Continued.

Inversely, given a meromorphic function $f \in l(D)$, let $s_\alpha = f \cdot f_\alpha$, then

$$(s_\alpha) = (f) \cap U_\alpha + (f_\alpha) = (f) \cap U_\alpha + D \cap U_\alpha = [(f) + D] \cap U_\alpha \geq 0.$$

s_α is holomorphic on U_α, satisfies the cocycle condition Eqn. 2, therefore $\{s_\alpha\}$ determines a holomorphic section of the line bundle $\lambda(D)$, denoted as $j(f)$. We obtain a linear map $j : l(D) \to \Gamma_h(\lambda(D))$. i, j are inverse linear maps of each other, hence linear isomorphism. $\square$

Divisor induced by Meromorphic Section

We define the divisor induced by a meromorphic section as follows: suppose s is a meromorphic section of a holomorphic line bundle L with local representation $\{s_\alpha\}$. On U_α, s_α induces a divisor (s_α). On $U_\alpha \cap U_\beta$, transition function $g_{\beta\alpha}$ is holomorphic, hence we have

$$(s_\beta) = (g_\beta \alpha \cdot s_\alpha) - (g_{\beta\alpha}) + (s_\alpha) = (s_\alpha).$$

This means there is a divisor D on M, such that $D \cap U_\alpha = (s_\alpha)$. D is called the divisor induced by the meromorphic section s, denoted as (s).

Divisors and Line Bundles

Lemma

If L is a holomorphic line bundle, then there is a linear isomorphicsm

$$\Gamma_h(L - \lambda(D)) \cong \{s \in \mathfrak{M}(L) | (s) - D \geq 0\}. \quad (5)$$

Proof.

Choose a trivialization open covering $\{U_\alpha\}$, suppose the transition functions are $f_{\beta\alpha}$. Select a meromorphic function $g_\alpha : U_\alpha \to \mathbb{C}$, such that $(g_\alpha) = D \cap U_\alpha$. The transition functions of $\lambda(D)$ are $g_{\beta\alpha} = g_\beta / g_\alpha$,then the transition functions of $L - \lambda(D)$ are

$$f_{\beta\alpha} g_\alpha / g_\beta.$$

Select a holomorphic section s of $L - \lambda(D)$ with local representation $\{s_\alpha\}$, on $U_\alpha \cap U_\beta$

$$s_\beta = f_{\beta\alpha} g_\alpha / g_\beta s_\alpha \implies s_\beta g_\beta = f_{\beta\alpha}(s_\alpha g_\alpha)$$

Continued.

$\{s_\alpha g_\alpha\}$ is a meromorphic section of L, denoted as $i(s)$, $i(s) \in \mathfrak{M}(L)$. Furthermore,

$$(s_\alpha g_\alpha) - D \cap U_\alpha = (s_\alpha) + (g_\alpha) - D \cap U_\alpha = (s_\alpha) + D \cap u_\alpha - D \cap U_\alpha = (s_\alpha) \geq 0,$$

therefore $(s) - D \geq 0$, hence $i(s) \in \mathfrak{M}(L)$. This gives a linear homomorphism $i : \Gamma_h(L - \lambda(D)) \to \{s \in \mathfrak{M}(L) | (s) - D \geq 0\}$. Inversely, suppose we choose a meromorphic section s of L, such that $(s) - D \geq 0$, then s has local representation $\{s_\alpha\}$, where s_α is a meromorphic function defined on U_α, $s_\beta = f_{\beta\alpha} s_\alpha$, furthermore $(s_\alpha) - D \cap U_\alpha \geq 0$,

$$(s_\alpha / g_\alpha) = (s_\alpha) - (g_\alpha) = (s_\alpha) - D \cap U_\alpha \geq 0,$$

therefore (s_α / g_α) is holomorphic, $\square$

Divisors and Line Bundles

Continued.

$$\frac{s_\beta}{g_\beta} = \frac{f_{\beta\alpha} s_\alpha}{g_\beta} = \frac{s_\alpha}{g_\alpha} \frac{g_\alpha}{g_\beta} f_{\beta\alpha}$$

This shows $\{s_\beta / g_\beta\}$ is a holomorphic section $j(s)$ of the line bundle $L - \lambda(D)$ with transition functions $\{f_{\beta\alpha} g_\alpha / g_\beta\}$. The linear homorphism $j : \{s \in \mathfrak{M}(L) | (s) - D \geq 0\} \to \Gamma_h(L - \lambda(D))$ is the inverse of i. Therefore Eqn. 5 holds. □

Holomorphic Sections

Corollary

$$i(D) \cong \Gamma_h(T_h^*M - \lambda(D))$$

Proof.

By Eqn. (5), $\Gamma_h(T_h^*M - \lambda(D)$ is isomorphic to $\{s \in \mathfrak{M}(T_h^*M) | (s) - D \geq 0\} = i(D)$, namely meromorphic differentials on M, whose divisors are greater or equal to D. □

Hence the Riemann-Roch theorem can be reformulated as

$$\boxed{\dim \Gamma_h(\lambda(D)) - \dim \Gamma_h(T_h^*M - \lambda(D)) = \deg(D) + 1 - g}.$$

Meromorphic Sections

Lemma

Suppose s is a meromorphic section of holomorphic line bundle L, then $L = \lambda((s))$; inversely, for any divisor D, there is a meromorphic section s of $\lambda(D)$, such that $D = (s)$.

Proof.

Suppose s is a non-zero meromorphic section of L with local representation $\{s_\alpha\}$, by Eqn. (2),

$$s_\beta = f_{\beta\alpha} s_\alpha \implies s_\beta / s_\alpha = f_{\beta\alpha},$$

the transition functions of $\lambda((s))$ are $\{s_\beta / s_\alpha\}$ therefore $L = \lambda((s))$. $\square$

Meromorphic Sections

Continued.

Inversely, for any divisor D, construct meromorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$, $(f_\alpha) = D \cap U_\alpha$, the transition functions of $\lambda(D)$ are $\{f_{\beta\alpha} = f_\beta / f_\alpha\}$. The family of meromorphic functions $\{f_\alpha\}$ satisfy the condition

$$f_\beta = \frac{f_\beta}{f_\alpha} f_\alpha = f_{\beta\alpha} f_\alpha,$$

therefore $\{f_\alpha\}$ determines a meromorphic section s of $\lambda(D)$, by construction $(s) = D$. $\square$