Hermite Metric and Chern Class

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Hermite Metric, Connection and Curvature

Hermite Inner Product

Definition (Hermite Inner Product)

Suppose V is a complex linear space, the map $\langle , \rangle \to \mathbb{C}$ satisfying the following conditions:

1. $\langle \lambda v_1 + \mu v_2, w \rangle = \lambda \langle v_1, w \rangle + \mu \langle v_2, w \rangle$,
2. $\overline{\langle v, w \rangle} = \langle w, v \rangle$,
3. $\langle v, v \rangle \geq 0$, equality holds if and only if $v = 0$,

then is called a Hermite inner product on V.

A Hermite inner product induces another Hermite inner product on the dual space V^*.

Hermite Metric

Definition (Hermite Metric)

Suppose M is a Riemann surface, $T_h M$ is the tangent bundle. If on each holomorphic tangent space $T_{hp}M$, a Hermite inner product is assigned $h_p = \langle , \rangle_p$, and for any two smooth sections X_1, X_2, the function on M $h(X_1, X_2)$, $p \mapsto \langle X_1(p), X_2(p) \rangle_p$ is a smooth function, then h is called a Hermit metric on M (or $T_h M$).

Suppose h is a Hermite metric, U_α is any local coordinates $z_\alpha = x_\alpha + \sqrt{-1} y_\alpha$, $h_\alpha = \langle \partial_{z_\alpha}, \partial_{z_\alpha} \rangle$, then h_α is a smooth function,

$$h = h_\alpha dz_\alpha \otimes d\bar{z}_\alpha$$

Two vector fields $X = a_\alpha \partial_{z_\alpha}$, $Y = b_\alpha \partial_{z_\alpha}$, then

$$h(X, Y) = a_\alpha h_\alpha \bar{b}_\alpha.$$

Local Representation of Hermite Metric

Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$h_\beta = h(\partial_{z_\beta}, \partial_{z_\beta}) = h\left(\frac{\partial z_\beta}{\partial z_\alpha}\frac{\partial}{\partial z_\alpha}, \frac{\partial z_\beta}{\partial z_\alpha}\frac{\partial}{\partial z_\alpha}\right)$$
$$= \left|\frac{\partial z_\beta}{\partial z_\alpha}\right|^2 h(\partial_{z_\alpha}, \partial_{z_\alpha}) = \left|\frac{\partial z_\beta}{\partial z_\alpha}\right|^2 h_\alpha$$

Inversely, if there is a family of positive smooth functions $\{h_\alpha\}$ satisfying the above equation, the it defines a Hermite metric on M. We call $\{h_\alpha\}$ as the local representation of the Hermite metric.

Volumetric Form

Given an Hermite metric with local representation $\{h_\alpha\}$, then the $(1,1)$ form

$$\Omega = \frac{\sqrt{-1}}{2} h_\alpha dz_\alpha \wedge d\bar{z}_\alpha = h_\alpha dx_\alpha \wedge dy_\alpha$$

is called the volume (area) form of the metric h on M.

The total area of the Riemann surface is given by

$$\int_M \Omega$$

Curvature Form

Given an Hermite metric with local representation $\{h_\alpha\}$, then the $(1,1)$ form $\Theta_\alpha = \bar{\partial}\partial \log h_\alpha$ is called the curvature form of the metric h on M. If $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\begin{aligned}
\Theta_\beta = \bar{\partial}\partial \log h_\beta &= \bar{\partial}\partial \log \left|\frac{\partial z_\beta}{\partial z_\alpha}\right|^2 h_\alpha \\
&= \bar{\partial}\partial \log h_\alpha + \bar{\partial}\partial \log \frac{\partial z_\beta}{\partial z_\alpha} + \bar{\partial}\partial \log \overline{\frac{\partial z_\beta}{\partial z_\alpha}} \\
&= \bar{\partial}\partial \log h_\alpha = \Theta_\alpha.
\end{aligned}$$

So the curvature $(1,1)$ form is globally defined. It can be represented as

$$\Theta = \frac{K}{\sqrt{-1}}\Omega$$

where K is called the Gaussian curvature of the metric h on M.

Gaussian Curvature

The Gaussian curvature has local representation

$$K = -\frac{2}{h_\alpha} \frac{\partial^2 \log h_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha}$$

Theorem (Gauss-Bonnet)

Suppose M is a compact Riemann surface with an Hermite metric h, then the total Gaussian curvature is

$$\int_M K\Omega = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic number of M.

Hermite Metric for a Line Bundle

Definition (Bundle Hermite Metric)

Suppose L is a holomorphic line bundle over a Riemann surface M. For each fiber L_p, an Hermite inner product is assigned $g_p = \langle , \rangle_p$, and for any smooth sections s_1, s_2, the function on M, $g(p) = \langle s_1, s_2 \rangle_p$ is smooth, then g is called an Hermite metric of the bundle L.

Hermite Metric for a Line Bundle

Suppose L on U_α has local trivialization ψ_α, then on U_α there is a local holomorphic section s_α, non-zero everywhere, $s_\alpha(x) = \psi_\alpha^{-1}(x,1)$, $x \in U_\alpha$. Denote $g_\alpha = g(s_\alpha, s_\alpha)$, then g_α is a positive smooth function on U_α. When $U_\alpha \cap U_\beta \neq \emptyset$, $s_\alpha = f_{\beta\alpha} s_\beta$, where $f_{\beta\alpha}$ is the transition function of L, then

$$g_\alpha = g(s_\alpha, s_\alpha) = g(f_{\beta\alpha} s_\beta, f_{\beta\alpha} s_\beta) = |f_{\beta\alpha}|^2 g_\beta,$$

Inversely, if $\{g_\beta\}$ satisfies the condition, then it gives a Hermite metric on L. $\{g_\beta\}$ is called the local representation of the Hermite metric g.

Suppose g is an Hermite metric on the holomorphic line bundle L with local representation $\{g_\alpha\}$. In the local trivialization neighborhood U_α define a $(1,0)$ form

$$\theta_\alpha := \partial \log g_\alpha,$$

On $U_\alpha \cap U_\beta$,

$$\begin{aligned}\theta_\alpha &= \partial \log f_{\beta\alpha} + \partial \log \bar{f}_{\beta\alpha} + \partial \log g_\beta \\ &= f_{\beta\alpha}^{-1} \partial f_{\beta\alpha} + 0 + \theta_\beta\end{aligned}$$

$$\boxed{\theta_\alpha = f_{\beta\alpha}^{-1} \partial f_{\beta\alpha} + \theta_\beta,}$$

Hence θ_α is not globally defined.

Connection Form

Definition (Connection)

Given a smooth section s of L, on a local trivialization neighborhood U_α, $s = f_\alpha s_\alpha$, where $f_\alpha : U_\alpha \to \mathbb{C}$ is a local smooth function. Define

$$Ds = (df_\alpha + f_\alpha \theta_\alpha) s_\alpha,$$

D is called the connection of L.

Connection Form

On $U_\alpha \cap U_\beta$,

$$s = f_\alpha s_\alpha = f_\beta s_\beta \implies f_\alpha \cdot f_{\beta\alpha} = f_\beta,$$

then

$$\begin{aligned}
df_\beta &= (\partial + \bar{\partial})(f_{\beta\alpha} f_\alpha) \\
&= \partial f_{\beta\alpha} f_\alpha + f_{\beta\alpha}\partial f_\alpha + \bar{\partial} f_{\beta\alpha} f_\alpha + f_{\beta\alpha}\bar{\partial} f_\alpha \\
&= \partial f_{\beta\alpha} f_\alpha + f_{\beta\alpha}\partial f_\alpha + 0 + f_{\beta\alpha}\bar{\partial} f_\alpha \\
&= \partial f_{\beta\alpha} f_\alpha + f_{\beta\alpha}\partial f_\alpha + f_{\beta\alpha}\bar{\partial} f_\alpha \\
&= \partial f_{\beta\alpha} f_\alpha + f_{\beta\alpha}(\partial f_\alpha + \bar{\partial} f_\alpha) \\
&= \partial f_{\beta\alpha} f_\alpha + f_{\beta\alpha} df_\alpha
\end{aligned}$$

Connection Form

$$
\begin{aligned}
(df_\alpha + f_\alpha\theta_\alpha)s_\alpha &= (df_\alpha + f_\alpha\theta_\alpha)f_{\beta\alpha}s_\beta \\
&= (f_{\beta\alpha}df_\alpha + f_{\beta\alpha}f_\alpha\theta_\alpha)s_\beta \\
&= (f_{\beta\alpha}df_\alpha + f_{\beta\alpha}f_\alpha(f_{\beta\alpha}^{-1}\partial f_{\beta\alpha} + \theta_\beta))s_\beta \\
&= (f_{\beta\alpha}df_\alpha + f_\alpha\partial f_{\beta\alpha} + f_{\beta\alpha}f_\alpha\theta_\beta))s_\beta \\
&= ((f_{\beta\alpha}df_\alpha + f_\alpha\partial f_{\beta\alpha}) + f_{\beta\alpha}f_\alpha\theta_\beta)s_\beta \\
&= (df_\beta + f_\beta\theta_\beta)s_\beta = Ds
\end{aligned}
$$

Therefore Ds is globally defined.

Connection

The connection Ds has the following properties:

1. $D(s_1 + s_2) = Ds_1 + Ds_2$, $\forall s_1, s_2 \in A^0(L)$
2. $D(fs) = dfs + fDs$, $\forall f \in A^0(M), s \in A^0(L)$

A linear operator satisfies the first and the second conditions is called a connection of L.

Connection

The connection Ds compatible with the Hermite metric has the following properties:

1. $d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$

$$d\langle s_\alpha, s_\alpha\rangle = dg_\alpha = \theta_\alpha g_\alpha + \bar{\theta}_\alpha g_\alpha = \langle Ds_\alpha, s_\alpha\rangle + \langle s_\alpha, Ds_\alpha\rangle$$

2. if s is a (local) global holomorphic section of L, then $Ds \in A^{1,0}(L)$

$$Ds = (df_\alpha + f_\alpha \cdot \theta_\alpha)s_\alpha = (\partial f_\alpha + f_\alpha \theta_\alpha)s_\alpha$$

hence

$$Ds_\alpha = D(1 \cdot s_\alpha) = \theta_\alpha s_\alpha$$

Connection

We generalize the connection operator to any L-valued differential form. Let ω is a L-valued p-form, with local representation $\omega_\alpha \otimes s_\alpha$ (or ωs_α) where ω_α is a local p-form on M. Let

$$D\omega := (d\omega_\alpha + (-1)^p \omega_\alpha \wedge \theta_\alpha) s_\alpha,$$

this defines an operator $D : A^p(L) \to A^{p+1}(L)$ with properties

1. $D(\omega_1 + \omega_2) = D\omega_1 + D\omega_2$
2. $D(f\omega) = df \wedge \omega + fD\omega$, $\forall f \in A^0(M)$

Curvature Form

Suppose $\{\theta_\alpha\}$ is the connection form, then

$$\Theta_\alpha = d\theta_\alpha = (\partial + \bar{\partial})\partial \log g_\alpha = \bar{\partial}\partial \log g_\alpha = \bar{\partial}\theta_\alpha$$

On $U_\alpha \cap U_\beta$,

$$\begin{aligned}
\Theta_\alpha &= d\theta_\alpha = \bar{\partial}\theta_\alpha \\
&= \bar{\partial}(f_{\beta\alpha}^{-1}\partial f_{\beta\alpha} + \theta_\beta) \\
&= \bar{\partial}(f_{\beta\alpha}^{-1}\partial f_{\beta\alpha}) + \bar{\partial}\theta_\beta \\
&= (\bar{\partial} f_{\beta\alpha}^{-1})\partial f_{\beta\alpha} + f_{\beta\alpha}^{-1}(\bar{\partial}\partial f_{\beta\alpha}) + \bar{\partial}\theta_\beta \\
&= \bar{\partial}\theta_\beta = \Theta_\beta,
\end{aligned}$$

where we use the fact that $\bar{\partial} f_{\beta\alpha}^{-1} = 0$, and $\Delta f_{\beta\alpha} = 0$.
So the curvature form $\Theta = \Theta_\alpha = \Theta_\beta$ is globally defined $(1,1)$ form.

Chern Class of the Line Bundle

Definition (The first Chern class)

The $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\Theta$ is a cohomology class in the de Rham cohomology $H^2_{dR}(M,\mathbb{C})$, this class is independent of the choice of the Herminte metric g, and called the first Chern class of the holomorpic line bundle L, denoted as $c_1(L)$.

Chern Class of the Line Bundle

Proof.

Assume there is another Hermite metric g' on L, g and g' have local representations $\{g_\alpha\}$ and $\{g'_\alpha\}$, satisfying

$$g_\alpha = |f_{\beta\alpha}|^2 g_\beta, \quad g'_\alpha = |f_{\beta\alpha}|^2 g'_\beta.$$

Then $f = g'_\alpha / g_\alpha$ is a global smooth positive real function defined on M,

$$\begin{aligned} \Theta' - \Theta &= \bar{\partial}\partial \log g'_\alpha - \bar{\partial}\partial \log g_\alpha \\ &= \bar{\partial}\partial \log(g'_\alpha / g_\alpha) \\ &= \bar{\partial}\partial \log f = d\partial \log f \end{aligned}$$

Therefore Θ' is cohomological to Θ. $\square$

Gauss-Bonnet Theorem

Lemma

The Chern class $c_1 : \mathcal{L} \to H^2_{dR}(M, \mathbb{C})$ *is a homomorphism.*

Proof.

Suppose $L_1, L_2 \in \mathcal{L}$ with the same local trivialization $\{U_\alpha\}$. Assume $\{g_\alpha\}$ and $\{h_\alpha\}$ are Hermite metrics on L_1 and L_2 respectively.Then $\{g_\alpha h_\alpha\}$ and $\{g_\alpha / h_\alpha\}$ are the Hermite metrics of $L_1 + L_2$ and $L_1 - L_2$ respectively. The curvature form of $L_1 + L2$ is

$$\bar{\partial}\partial \log(g_\alpha h_\alpha) = \bar{\partial}\partial \log g_\alpha + \bar{\partial}\partial_\alpha = \Theta_1 + \Theta_2,$$

where Θ_1 and Θ_2 are the curvature forms of L_1 and L_2 respectively. Similarly, $\Theta_1 - \Theta_2$ is the curvature form of $L_1 - L_2$. □

Gauss-Bonnet Theorem

Lemma (Divisor of Holomorphic Line Bundle)

If $D \in \mathcal{D}$, then there is a meromorphic section $s \in \mathfrak{M}(\lambda(D))$ such that $(s) = D$. Inversely, if $L \in \mathcal{L}$ and $s \in \mathfrak{M}(L)$, then $L = \lambda((s))$.

Gauss-Bonnet Theorem

Lemma (Curvature Form)

Suppose L is a holomorphic line bundle, $\{g_\alpha\}$ is a Hermite metric with curvature form Θ. s is a holomorphic section nowhere zero on U, with local representation s_α on $U_\alpha \cap U$. The norm of s is given by $|s|^2 = s_\alpha \bar{s}_\alpha g_\alpha$, then

$$\Theta = \bar{\partial}\partial \log |s|^2.$$

Proof.

$$\begin{aligned}
\bar{\partial}\partial \log |s|^2 &= \bar{\partial}\partial(\log g_\alpha + \log s_\alpha + \log \bar{s}_\alpha) \\
&= \bar{\partial}\partial \log g_\alpha + \bar{\partial}\partial \log s_\alpha + \bar{\partial}\partial \log \bar{s}_\alpha) \\
&= \bar{\partial}\partial \log g_\alpha + 0 + 0 \\
&= \Theta.
\end{aligned}$$

□

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet)

Suppose D is a divisor on a compact Riemann surface M, g is an Hermite metric on the holomorphic line bundle $\lambda(D)$, then

$$\frac{\sqrt{-1}}{2\pi}\int_M \Theta = deg(D) = \chi(L) - \frac{1}{2}\chi(M),$$

where $\chi(L)$ is the Euler-characteristic number of the bundle L, $\chi(M)$ is the Euler number of the surface M.

Gauss-Bonnet Theorem

Proof.

Suppose $L = \lambda(D)$, $D = \sum_i n_i p_i$, $\lambda : \mathcal{D} \to \mathcal{L}$ is a homomorphism, hence

$$\lambda(D) = \lambda(\sum_i n_i p_i) = \sum_i n_i \lambda(p_i).$$

$c_1 : \mathcal{L} \to H^2_{dR}(M, \mathbb{C})$ is also homomorphism,

$$c_1(\lambda(D)) = \sum_i n_i c_1(\lambda(p_i)).$$

Therefore

$$\int_M c_1(\lambda(D)) = \sum_i n_i \int_M c_1(\lambda(p_i)).$$

It is surficient to prove

$$\int_M c_1(\lambda(p)) = 1.$$

Gauss-Bonnet Theorem

Proof.

By lemma of divisor of holomorphic line bundle, there is a mermorphic section s of $\lambda(p)$, $(s) = p$, so s is a holomorphic section, $s \in \Gamma_h(\lambda(p))$, s has a simple zero at p, and

$$s(q) \neq 0, \quad \forall q \neq p. \tag{1}$$

Suppose $p \in U_\alpha$, $s_\alpha = ze_\alpha$, where e_α is a holomorphic section nowhere zero on U_α. z is a holomorphic function on U_α with a simple zero p, $z(p) = 0$. Without loss of generality, we can assume z is a coordinate function on U_α. Let

$$B(\delta) = \{x \in U_\alpha | \; |z(x)| < \delta\},$$

By Eqn. (1), s is non-zero on $U_\alpha - B(\delta)$, □

Gauss-Bonnet Theorem

Proof.

$$
\begin{aligned}
\int_M c_1(\lambda(p)) &= \frac{\sqrt{-1}}{2\pi}\int_M \Theta = \frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{M-B(\delta)} \Theta \\
&= \frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{M-B(\delta)} \bar{\partial}\partial \log |s|^2 \\
&= \frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{M-B(\delta)} (\partial+\bar{\partial})\partial \log |s|^2 \\
&= \frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{M-B(\delta)} d\partial \log |s|^2 \\
&= -\frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{\partial B(\delta)} \partial \log |s|^2
\end{aligned}
$$

$\square$

Gauss-Bonnet Theorem

Proof.

$$\int_M c_1(\lambda(p)) = -\frac{\sqrt{-1}}{2\pi} \lim_{\delta\to 0} \int_{\partial B(\delta)} \partial \log |s|^2$$
$$= -\frac{\sqrt{-1}}{2\pi} \lim_{\delta\to 0} \int_{\partial B(\delta)} \partial \log z + \partial \log \bar{z} + \partial \log |e_\alpha|^2$$

$\partial \log \bar{z} = 0$. $\log |e_\alpha|^2$ is a C^∞ function, hence

$$\lim_{\delta\to 0} \int_{\partial B(\delta)} \partial \log |e_\alpha|^2 = 0.$$

$\square$

Gauss-Bonnet Theorem

Proof.

$$\begin{aligned}
\int_M c_1(\lambda(p)) &= -\frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{\partial B(\delta)} \partial \log z \\
&= -\frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}\int_{\partial B(\delta)} \frac{dz}{z} \\
&= -\frac{\sqrt{-1}}{2\pi}\lim_{\delta\to 0}(2\pi\sqrt{-1}) \\
&= 1.
\end{aligned}$$

$\square$

Characteristic Class

Characteristic Class

- $\lambda(D_1)$ and $\lambda(D_2)$ are differential isomorphic, iff they have the same Chern class, $\deg(D_1) = \deg(D_2)$;
- $\lambda(D_1)$ and $\lambda(D_2)$ are holomorphic isomorphic, iff $D_1 - D_2 = (f)$ for a meromorphic function $f \in \mathfrak{M}(M)$.

This explains the difference between cross fieldes and quad-meshes on a surface. There is a cross field on a torus with two singularities with indices $+1$ and -1 respectively; there is no quad-mesh on a torus with two singularities with valence 3 and 5 respectively.