

Circle Domain Mapping: Koebe's Theorem

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Motivation

Conformal Module for Poly-annulus

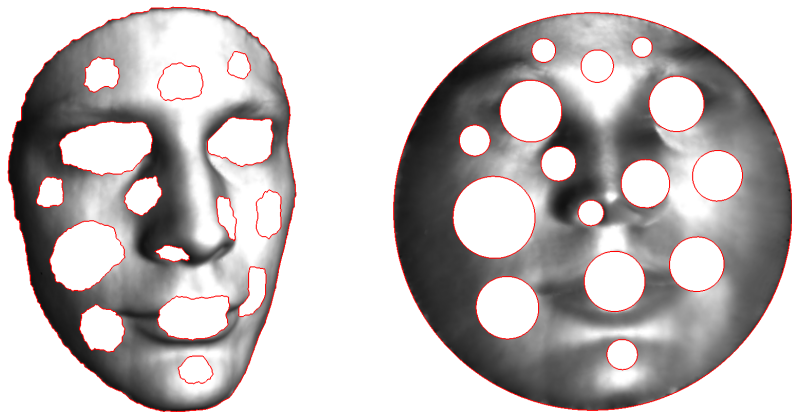


Figure: Conformal mapping from a poly-annulus to a circle domain.

Definition (Circle Domain)

Suppose $\Omega \subset \hat{\mathbb{C}}$ is a planar domain, if $\partial\Omega$ has finite number of connected components, each of them is either a circle or a point, then Ω is called a circle domain.

Theorem (Koebe)

Suppose S is of genus zero, ∂S has finite number of connected components, then S is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.

Schwartz Reflection Principle

Definition (Mirror Reflection)

Given a circle $\Gamma : |z - z_0| = \rho$, the reflection with respect to Γ is defined as:

$$\varphi_\Gamma : re^{i\theta} + z_0 \mapsto \frac{\rho^2}{r}e^{i\theta} + z_0. \quad (1)$$

Two planar domains S and S' are symmetric about Γ , if $\varphi_\Gamma(S) = S'$.

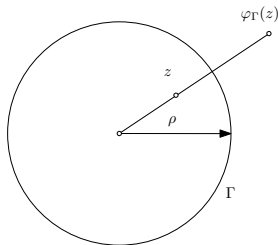


Figure: Reflection about a circle.

Schwartz Reflection Principle

Definition (Reflection)

Suppose Γ is an analytic curve, domain S, S' and Γ are included in a planar domain Ω . There is a conformal map $f : \Omega \rightarrow \hat{\mathbb{C}}$, such that $f(\Gamma)$ is a canonical circle, $f(S)$ and $f(S')$ are symmetric about $f(\Gamma)$, then we say S and S' are symmetric about Γ , and denoted as

$$S|S' \quad (\Gamma).$$

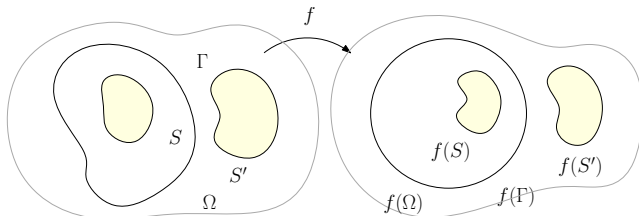


Figure: General symmetry.

Schwartz Reflection Principle

Theorem (Schwartz Reflection Principle)

Assume f is an analytic function, defined on the upper half disk $\{|z| < 1, \Im(z) > 0\}$. If f can be extended to a real continuous function on the real axis, then f can be extended to an analytic function F defined on the whole disk, satisfying

$$F(z) = \begin{cases} f(z), & \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \Im(z) < 0 \end{cases}$$

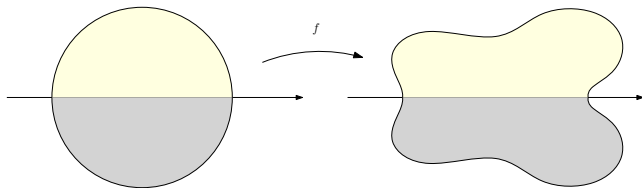
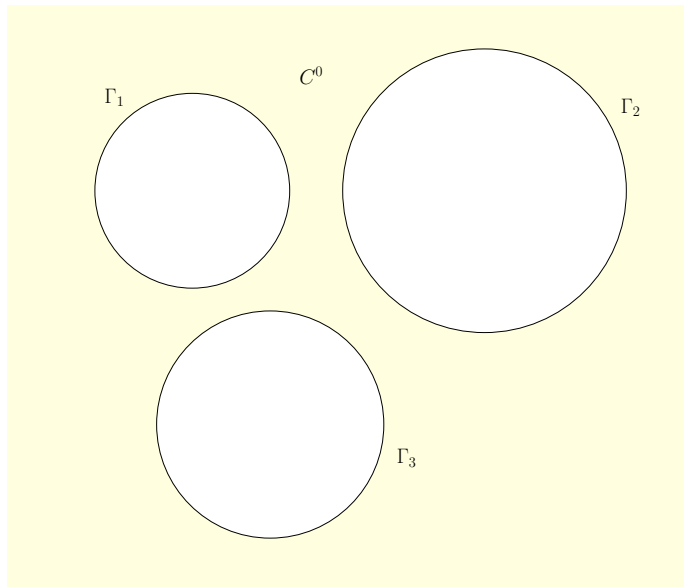
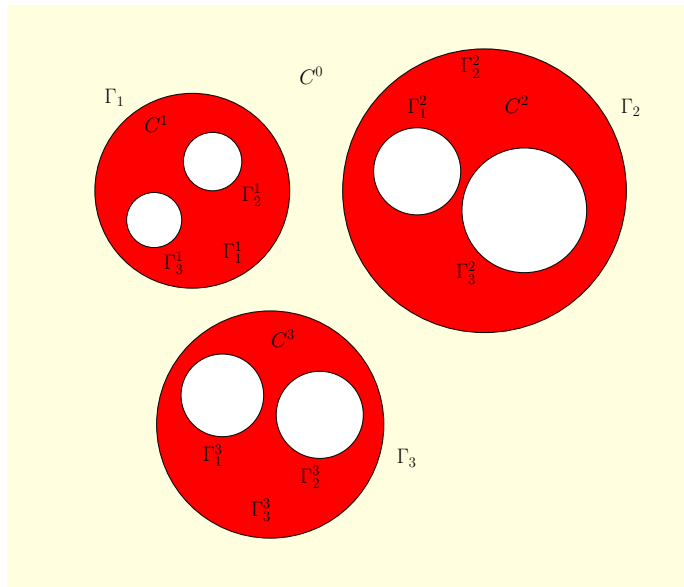


Figure: Schwartz reflection principle.

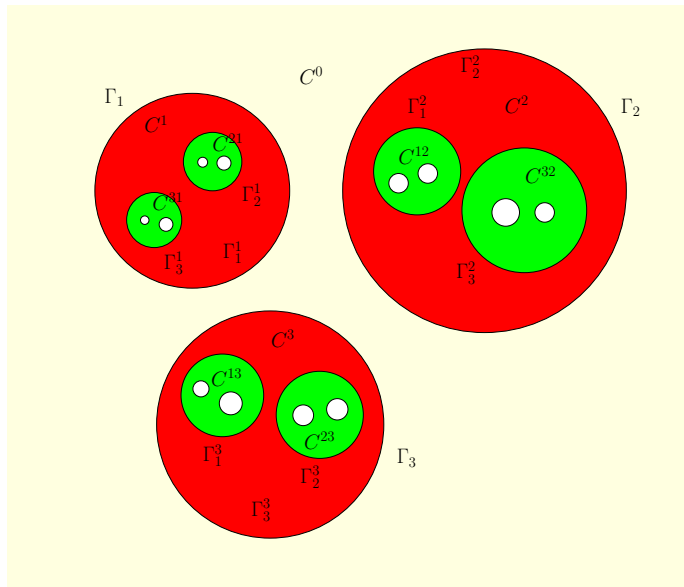
Multiple Reflection



Multiple Reflection



Multiple Reflection



Multiple Reflection

- ① Initial circle domain C^0 : complex plane remove three disks, its boundary is $\{\Gamma_1, \Gamma_2, \Gamma_3\}$;
- ② First level reflection: C^0 is reflected about Γ_{i_1} to C^{i_1} , $i_1 = 1, 2, 3$;

$$\partial C^{i_1} = \Gamma_{i_1}^{i_1} - \sum_{j \neq i_1} \Gamma_j^{i_1},$$

where $\Gamma_{i_1}^{i_1} = \Gamma_{i_1}$.

- ③ Second level reflection: C^{i_1} is reflected about Γ_{i_2} to $C^{i_1 i_2}$, $i_1 \neq i_2$; the boundary of $C^{i_1 i_2}$ are $\Gamma_j^{i_1 i_2}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$.

$$\partial C^{i_1 i_2} = \Gamma_{i_1}^{i_2} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2}$$

when $j = i_1$, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$;

Multiple Reflection

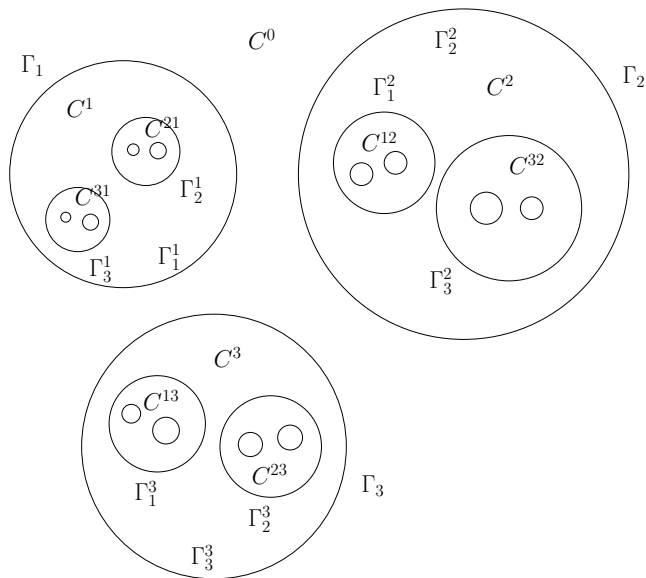
- ④ Third level reflection: $C^{i_1 i_2}$ is reflected about Γ_{i_3} to $C^{i_1 i_2 i_3}$, $i_1 \neq i_2$, $i_2 \neq i_3$; the boundary of $C^{i_1 i_2 i_3}$ are $\Gamma_j^{i_1 i_2 i_3}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3}$.

$$\partial C^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 i_3}.$$

- ⑤ The m -level reflection: $C^{i_1 i_2 \dots i_{m-1}}$ is reflected about Γ_{i_m} to $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$; the boundary of $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$ are $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 \dots i_{m-1} i_m} = \Gamma_{i_1}^{i_2 \dots i_{m-1} i_m}$ is an interior boundary,

$$\partial C^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 i_3 \dots i_m} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 \dots i_m}.$$

Multiple Reflection



Multiple Reflection

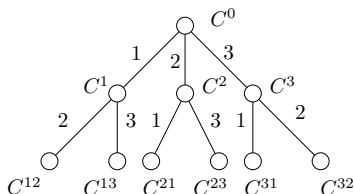


Figure: Reflection tree.

- Each node represents a domain $C^{i_1 i_2 \dots i_m}$;
- Each edge represents a circle Γ_k , $k = 1, \dots, n$;
- Father and Son share an edge i_1

$$\Gamma_{i_1}^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 \dots i_m}.$$

- Each node $C^{(i)}$, $(i) = i_1 i_2 \dots i_m$ is the path from the root to $C^{(i)}$,

$$C^{(i)} = \varphi_{\Gamma_{i_m}} \circ \varphi_{\Gamma_{i_{m-1}}} \cdots \varphi_{\Gamma_{i_1}}(C^0).$$

Multiple Reflection

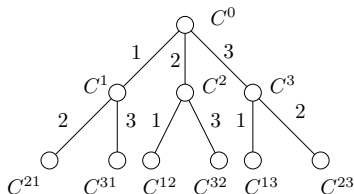


Figure: Embedding tree.

- Father node $C^{i_2 \cdots i_m}$ and child node $C^{i_1 i_2 \cdots i_m}$ is connected by edge i_1 , the exterior boundary of child equals to an interior boundary of the father

$$\Gamma_{i_1}^{i_1 i_2 \cdots i_m} = \Gamma_{i_1}^{i_2 \cdots i_m}.$$

- From the root C^0 to $C^{i_1 \cdots i_m}$, the path is inverse to the index

$$(i)^{-1} = i_m i_{m-1} \cdots i_2 i_1,$$

starting from C^0 crosses Γ^{i_m} to C^{i_m} , crosses $\Gamma_{i_{m-1}}^{i_m}$ to $C^{i_{m-1} i_m}$; when arrives at $C^{i_{k-1} \cdots i_1}$, crosses $\Gamma_{i_k}^{i_{k-1} \cdots i_1}$ to $C^{i_k i_{k-1} \cdots i_1}$; and eventually reach $C^{(i)}$.

Lemma

Suppose $C^{(i)}$ is an interior node in the reflection tree,

$$(i) = i_1 i_2 \cdots i_m,$$

its exterior boundary is $\Gamma_{i_1}^{(i)}$, interior boundaries are $\Gamma_j^{(i)}$, $j \neq i_1$, we have the estimate:

$$\sum_{j \neq i_1} \alpha(\Gamma_j^{(i)}) \leq \mu^4 \alpha(\Gamma_{i_1}^{(i)}).$$

Hole Area Estimation

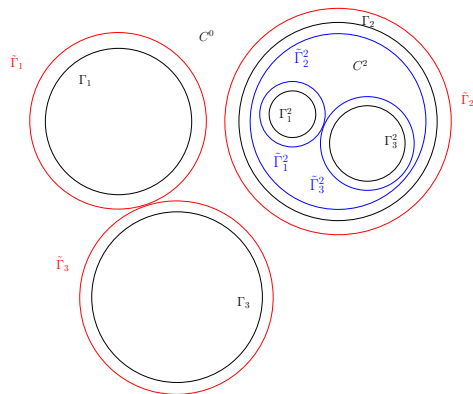


Figure: Hole area estimation.

Enlarge all Γ_k 's by factor μ^{-1} to $\tilde{\Gamma}_k$, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_3$ touch each other; reflect C^0 about Γ_2

- $\Gamma_k | \Gamma_k^2 \quad (\Gamma_2)$.
- $\tilde{\Gamma}_k | \tilde{\Gamma}_k^2 \quad (\Gamma_2)$.

$$\alpha(\tilde{\Gamma}_1^2) = \mu^{-2} \alpha(\Gamma_1^2)$$

$$\alpha(\tilde{\Gamma}_3^2) = \mu^{-2} \alpha(\Gamma_3^2)$$

$$\alpha(\tilde{\Gamma}_2^2) = \mu^2 \alpha(\Gamma_2^2)$$

$$\alpha(\Gamma_1^2) + \alpha(\Gamma_3^2) = \mu^2 (\alpha(\tilde{\Gamma}_1^2) + \alpha(\tilde{\Gamma}_3^2)) \leq \mu^2 \alpha(\tilde{\Gamma}_2^2) = \mu^4 \alpha(\Gamma_2^2).$$

Hole Area Estimation

Lemma

Suppose the boundaries of the initial circle domain C^0 are $\Gamma_1, \Gamma_2, \dots, \Gamma_n$, consider the reflection tree with m layers, then the total area of the holes bounded by the interior boundaries of leaf nodes is no greater than μ^{4m} times the area bounded by Γ_k 's,

$$\sum_{(i)=i_1 i_2 \dots i_m} \sum_{k \neq i_1} \alpha(\Gamma_k^{(i)}) \leq \mu^{4m} \sum_{i=1}^n \alpha(\Gamma_i). \quad (2)$$

Proof.

By induction on m . The area bounded by the exterior boundaries of the nodes in the $k+1$ -layer is no greater than μ^4 times that of the k -layer. The total area of the interior boundaries of leaf nodes is no greater than the area bounded by the exterior boundaries of leaf nodes. \square

Uniqueness

Theorem (Uniqueness)

Given two circle domains $C_1, C_2 \subset \hat{\mathbb{C}}$, $f : C_1 \rightarrow C_2$ is a univalent holomorphic function, then f is a linear rational, namely a Möbus transformation.

Proof.

Assume both C_1 and C_2 include ∞ , and $f(\infty) = \infty$. Since f is holomorphic, it maps the boundary circles of C_1 to those of C_2 . By Schwartz reflection principle, f can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. 2, the multiple reflected domains cover the whole $\hat{\mathbb{C}}$, hence f can be extended to the whole $\hat{\mathbb{C}}$, since $f(\infty) = \infty$, f is a linear function. If $f(\infty) \neq \infty$, we can use a Möbius map to transform $f(\infty)$ to ∞ . □

Definition (Kernel)

Suppose $\{B_n\}$ is a family of domains on the complex plane, $\infty \in B_k$ for all k . Suppose B is the maximal set: $\infty \in B$, and for any closed set $K \subset B$, there is an N , such that for any $n > N$, $K \subset B_n$. Then B is called the kernel of $\{B_n\}$.

Definition (Domain Convergence)

We say a sequence $\{B_n\}$ converges to its kernel B , if any sub-sequence $\{B_{n_k}\}$ of $\{B_n\}$ has the same kernel B . We denote $B_n \rightarrow B$.

Theorem (Goluzin)

Let $\{A_n\}$ be a sequence of domains on the complex domain. Any domain A_n includes ∞ , $n = 1, 2, \dots$. Assume $\{A_n\}$ converges to its kernel A . Let $\{f_n(z)\}$ be a family of analytic function, for all n , $f_n(z)$ maps A_n to B_n surjectively, such that $f_n(\infty) = \infty$, $f'_n(\infty) = 1$. Then $\{f_n(z)\}$ uniformly converges to a univalent analytic function $f(z)$ in the interior of A , if and only if $\{B_n\}$ converges to its kernel B , then the univalent analytic function $f(z)$ maps A to B surjectively.

Theorem (Existence)

On the z -plane, every n -connected domain Ω can be mapped to a circle domain on the ζ -plane by a univalent holomorphic function. Choose a point $a \in \Omega$, there is a unique map which maps a to $\zeta = \infty$, and in a neighborhood of $z = a$, the map has the power series

$$\frac{1}{z - a} + a_1(z - a) + \cdots \text{ if } a \neq \infty$$
$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$

Proof.

According to Hilbert theorem, all n -connected domains are conformally equivalent to slit domains. We can assume Ω is a slit domain. We use \mathcal{S} to represent all the n -connected slit domains with horizontal slits, and \mathcal{C} the n -connected circle domains. We label all the boundaries of the domains, $\partial\Omega = \bigcup_{k=1}^n \gamma_k$. For each slit γ_k , we represent it by the starting point p_k and the length l_k , then we get the coordinates of the slit domain Ω

$$(p_1, l_1, p_2, l_2, \dots, p_n, l_n).$$

Hence \mathcal{S} is a connected open set in \mathbb{R}^{3n} . Similarly, consider a circle domain $\mathcal{D} \in \mathcal{C}$, we use the center and the radius to represent each circle (q_k, r_k) , and the coordinates of \mathcal{D} are given by,

$$(q_1, r_1, q_2, r_2, \dots, q_n, r_n).$$

\mathcal{C} is also a connected open set in \mathbb{R}^{3n} . □

continued

Consider a normalized univalent holomorphic function $f : \Omega \rightarrow \mathcal{D}$, $\Omega \in \mathcal{S}$ and $\mathcal{D} \in \mathcal{C}$, f maps the k -th boundary curve γ_k to the k -th circular boundary of \mathcal{D} . By the existence of slit mapping and the uniqueness of circle domain mapping, we have

- 1 Every circle domain $\mathcal{D} \in \mathcal{C}$ corresponds to a unique slit domain $\Omega \in \mathcal{S}$;
- 2 Every slit domain $\Omega \in \mathcal{S}$ corresponds to at most one circle domain $\mathcal{D} \in \mathcal{C}$.

Then we establish a mapping from circle domains to slit domains $\varphi : \mathcal{C} \rightarrow \mathcal{S}$.

continued

Assume $\{\mathcal{D}_n\}$ is a family of circle domains, converge to the kernel \mathcal{D}^* . The domain convergence definition is consistent with the convergence of coordinates, namely, the boundary circles of \mathcal{D}_n converge to the corresponding boundary circles of \mathcal{D}^* , denoted as $\lim_{n \rightarrow \infty} \mathcal{D}_n = \mathcal{D}^*$. The convergence of slit domains can be similarly defined. By Goluzin's theorem, we obtain the mapping $\varphi : \mathcal{C} \rightarrow \mathcal{S}$ is continuous:

$$\varphi\left(\lim_{n \rightarrow \infty} \mathcal{D}_n\right) = \lim_{n \rightarrow \infty} \varphi(\mathcal{D}_n).$$

By the uniqueness of circle domain mapping, we obtain φ is injective. We will prove the mapping φ is surjective.

continued

\mathcal{C} is an open set in Euclidean space $\varphi : \mathcal{C} \rightarrow \mathcal{S}$ is injective continuous map. According to invariance of domain theorem, $\varphi(\mathcal{C})$ is an open set, $\varphi : \mathcal{C} \rightarrow \varphi(\mathcal{C})$ is a homeomorphism.

Choose a circle domain $\mathcal{D}_0 \in \mathcal{C}$, its corresponding slit domain is $\varphi(\mathcal{D}_0) = \Omega_0 \in \mathcal{S}$, then $\Omega_0 \in \varphi(\mathcal{C})$. Choose another slit map $\Omega_1 \in \mathcal{S}$, we don't know if Ω_1 is in $\varphi(\mathcal{C})$ or not. We draw a path $\Gamma : [0, 1] \rightarrow \mathcal{S}$, $\Gamma(0) = \Omega_0$ and $\Gamma(1) = \Omega_1$. Let

$$t^* = \sup\{t \in [0, 1] \mid \forall 0 \leq \tau \leq t, \Gamma(\tau) \in \varphi(\mathcal{C})\},$$

namely Γ from starting point to t^* belongs to $\varphi(\mathcal{C})$.

continued

By the definition of domain convergence,

$$\lim_{n \rightarrow \infty} \Gamma(t_n) \rightarrow \Gamma(t^*).$$

By $\{\Gamma(t_n)\} \subset \varphi(\mathcal{C})$, there is a family of circle domains $\{\mathcal{D}_n\} \subset \mathcal{C}$, $\varphi(\mathcal{D}_n) = \Gamma(t_n)$. Let $\lim_{n \rightarrow \infty} \mathcal{D}_n = \mathcal{D}^*$, by domain limit theorem, we have

$$\varphi(\mathcal{D}^*) = \varphi(\lim_{n \rightarrow \infty} \mathcal{D}_n) = \lim_{n \rightarrow \infty} \varphi(\mathcal{D}_n) = \lim_{n \rightarrow \infty} \Gamma(t_n) = \Gamma(t^*),$$

namely $\varphi(\mathcal{D}^*) = \Gamma(t^*)$, hence $\Gamma(t^*) \in \varphi(\mathcal{C})$. But $\varphi(\mathcal{C})$ is an open set, hence if $t^* < 1$, t^* can be further extended. This contradicts to the choice of t^* , hence $t^* = 1$. Therefore $\Omega_1 \in \varphi(\mathcal{C})$. Since Ω_1 is arbitrarily chosen, hence $\varphi : \mathcal{C} \rightarrow \mathcal{S}$ is surjective. This proves the existence of the circle domain mapping.