Dolbeault Theorem and de Rham Theorem

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Fine Sheaf

Fine Sheaf

Definition (Fine Sheaf)

Suppose $\mathcal{S}$ is a sheaf on a Riemann surface M, $\mathcal{W} = \{W_\alpha\}$ is a local finite open cover, if $\phi_\alpha : \mathcal{S} \to \mathcal{S}$ is a sheaf homomorphism, satisfying

1. for any α, there is a closed set $K_\alpha \subset W_\alpha$, such that when $p \notin K_\alpha$, $\phi_\alpha|_{\mathcal{S}_p} = 0$;
2. $\sum_\alpha \phi_\alpha = \mathrm{id}_{\mathcal{S}}$.

then we call $\{\phi_\alpha\}$ is a partition of unity belonging to the open cover $\mathcal{W}$. If for any local finite open cover, there is a partition of unity satisfying the above conditions, then $\mathcal{S}$ is called a fine sheaf.

Any open cover of a Riemann surface has local finite subdivision, for each local finite open cover $\mathcal{W} = \{W_\alpha\}$, there is a partition of unity $\{f_\alpha\}$ associated with $\mathcal{W}$, namely the support of the smooth function f_α is inside W_α, and the summation of f_α's is 1.

Fine Sheaf

The sheaves of smooth function, p-forms, (p, q)-forms, L-valued (p, q)-forms are fine sheaves.

Consider L-valued p-form sheaf $\mathcal{S}^p(L)$, $\phi_\alpha : \mathcal{S}^p(L) \to \mathcal{S}^p(L)$ is

$$\phi_\alpha\left(\left[\sum_i \omega_i \otimes s_i\right]_p\right) = \left[\sum_i f_\alpha \omega_i \otimes s_i\right]_p,$$

ω_i is a local p-form, s_i is the local section of L, $\{\phi_\alpha\}$ is the partition of the unity of $\mathcal{S}^p(L)$ associated with the open cover $\mathcal{W}$.

The sheaves of holomorphic functions, holomorphic sections of holomorphic line bundle are not fine sheaves.

Fine Sheaf

Theorem (Fine Sheaf)

If $\mathcal{S}$ is a fine sheaf, then $H^q(M;\mathcal{S}) = 0$, $\forall q \geq 1$.

Proof.

Suppose $\mathcal{U} = \{U_\alpha\}$ is a locally finite open cover. Suppose $f \in C^1(\mathcal{U};\mathcal{S})$, $\delta f = 0$, we want to show $\exists g \in C^0(\mathcal{U};\mathcal{S})$, such that $f = \delta g$. In fact, assume $\{\phi_\alpha\}$ is the partition of unity of $\mathcal{S}$ associated with $\mathcal{U}$, define $g \in C^0(\mathcal{U};\mathcal{S})$ as follows:

$$g(U_\alpha) = \sum_\gamma \phi_\gamma \circ (f(U_\gamma, U_\alpha)).$$

where $\phi_\gamma \circ (f(U_\gamma, U_\alpha))$ are in $\Gamma(\mathcal{S}, U_\gamma \cap U_\alpha)$, and extended by zero to an element in $\Gamma(\mathcal{S}, U_\alpha)$. $\square$

Fine Sheaf

Continued.

We have

$$\begin{aligned}\delta g(U_\alpha, U_\beta) &= g(U_\beta) - g(U_\alpha)\\ &= \sum_\gamma [\phi_\gamma \circ (f(U_\gamma, U_\beta)) - \phi_\gamma \circ (f(U_\gamma, U_\alpha))]\\ &= \sum_\gamma \phi_\gamma [f(U_\gamma, U_\beta) - f(U_\gamma, U_\alpha)]\\ &= \sum_\gamma \circ (f(U_\alpha, U_\beta))\\ &= f(U_\alpha, U_\beta).\end{aligned}$$

Hence $f = \delta g$, $H^1(\mathcal{U}, \mathcal{S}) = 0$. Therefore $H^1(M, \mathcal{S}) = 0$. $\square$

de Rham Theorem

Fine Sheaf Decomposition

Definition (Fine Sheaf Decomposition)

Suppose $\mathcal{S}$ is a sheaf on a Riemann surface M. If there are fine sheaves $\{\mathcal{S}_i\}_{i\geq 0}$ and exact sequence of sheaf homomorphisms

$$0 \to \; S \to \; S_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots$$

Fine Sheaf Decomposition

Theorem (de Rham)

Suppose $\mathcal{S}$ is a sheaf on the Riemann surface M, if $\mathcal{S}$ has a fine sheaf decomposition

$$0 \to \ S \to \ S_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots$$

and the induced homomorphism sequence is

$$0 \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}_0) \xrightarrow{d_0^*} \Gamma(\mathcal{S}_1) \xrightarrow{d_1^*} \Gamma(\mathcal{S}_2) \xrightarrow{d_2^*} \cdots$$

then there are group isomorphisms

$$H^q(M;\mathcal{S}) \cong \frac{Ker\ d_q^*}{Img\ d_{q-1}^*}, \quad \forall q \geq 1.$$

de Rham Theorem

Proof.

Let $Z_p = \text{Ker } d_p$, we have the short exact sequence of sheaves,

$$0 \to \; S \to \; S_0 \xrightarrow{d_0} Z_1 \xrightarrow{d_1} 0$$

By short-long theorem, we have

$$\begin{aligned} (i) \quad 0 \to H^0(\mathcal{S}) \to H^0(\mathcal{S}_0) &\xrightarrow{d_0^*} H^0(\mathcal{Z}_1) \\ \xrightarrow{\delta_0^*} H^1(\mathcal{S}) \to H^1(\mathcal{S}_0) &\xrightarrow{d_0^*} H^1(\mathcal{Z}_1) \\ \xrightarrow{\delta_1^*} H^2(\mathcal{S}) \to H^2(\mathcal{S}_0) &\xrightarrow{d_0^*} H^2(\mathcal{Z}_1) \cdots \end{aligned}$$

Since $\mathcal{S}_0$ is a fine sheaf, $H^1(\mathcal{S}_0) = 0$, $H^2(\mathcal{S}_0) = 0$, we have

$$(a) \quad 0 \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}_0) \xrightarrow{d_0^*} \Gamma(Z_1) \xrightarrow{\delta_0^*} H^1(M; \mathcal{S}) \to 0$$

$$(b) \quad 0 = H^p(M; \mathcal{S}_0) \xrightarrow{d_0^*} H^p(M; \mathcal{Z}_1) \xrightarrow{\delta_1^*} H^{p+1}(M; \mathcal{S}) \to 0$$

continued.

From the exact sequence:

$$(a) \quad 0 \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}_0) \xrightarrow{d_0^*} \Gamma(Z_1) \xrightarrow{\delta_0^*} H^1(M;\mathcal{S}) \to 0$$

The last map is surjective, hence

$$H^1(M;\mathcal{S}) \cong \mathrm{Img}\delta_0^* \cong \Gamma(Z_1)/\mathrm{Ker}\delta_0^* \cong \Gamma(Z_1)/\mathrm{Img}d_0^* = \boxed{\mathrm{Ker}d_1^*/\mathrm{Img}d_0^*}.$$

From the exact sequence:

$$(b) \quad 0 = H^1(M;\mathcal{S}_0) \xrightarrow{d_0^*} H^1(M;\mathcal{Z}_1) \xrightarrow{\delta_1^*} H^2(M;\mathcal{S}) \to 0$$

We have

$$(c) \quad H^{p+1}(M;\mathcal{S}) \cong H^p(M;\mathcal{Z}_1) \quad p \geq 2$$

Proof.

We have the short exact sequence of sheaves,

$$0 \to \ Z_p \xrightarrow{i} \mathcal{S}_p \xrightarrow{d_p} Z_{p+1} \to 0, \quad p \geq 1$$

By short-long theorem, we have

$$\begin{aligned}(ii) \quad 0 \to H^0(\mathcal{Z}_p) &\xrightarrow{i^*} H^0(\mathcal{S}_p) \xrightarrow{d_p^*} H^0(\mathcal{Z}_{p+1}) \\ &\xrightarrow{\delta_0^*} H^1(\mathcal{Z}_p) \xrightarrow{i^*} {\color{red}H^1(\mathcal{S}_p)} \xrightarrow{d_p^*} H^1(\mathcal{Z}_{p+1}) \\ &\xrightarrow{\delta_1^*} H^2(\mathcal{Z}_p) \xrightarrow{i^*} {\color{red}H^2(\mathcal{S}_p)} \xrightarrow{d_p^*} H^2(\mathcal{Z}_{p+1}) \cdots\end{aligned}$$

Since $\mathcal{S}_p$ is a fine sheaf, ${\color{red}H^1(\mathcal{S}_p)} = 0$, we have

$$(d) \quad 0 \to \Gamma(\mathcal{Z}_1) \to \Gamma(\mathcal{S}_1) \xrightarrow{d_1^*} \Gamma(\mathcal{Z}_2) \xrightarrow{\delta^*} H^1(M; \mathcal{Z}_1) \to 0$$

$$(e) \quad 0 \xrightarrow{d_p^*} H^k(\mathcal{Z}_{p+1}) \xrightarrow{\delta_1^*} H^{k+1}(\mathcal{Z}_p) \xrightarrow{i^*} 0, \quad \forall k \geq 1.$$

de Rham Theorem

continued.

From the exact sequence

$$(d) \quad 0 \to \Gamma(\mathcal{Z}_1) \to \Gamma(\mathcal{S}_1) \xrightarrow{d_1^*} \Gamma(\mathcal{Z}_2) \xrightarrow{\delta^*} H^1(M;\mathcal{Z}_1) \to 0$$

We have

$$H^1(M;\mathcal{Z}_1) \cong \Gamma(\mathcal{Z}_2)/\mathrm{Img}\ d_1^* = \mathrm{Ker}\ d_2^*/\mathrm{Img}\ d_1^*.$$

Hence from (c)

$$H^2(M;\mathcal{S}) = H^1(M;\mathcal{Z}_1) \cong \boxed{\mathrm{Ker}\ d_2^*/\mathrm{Img}\ d_1^*}.$$

□

de Rham Theorem

continued.

From the exact sequence

$$(e) \quad 0 \to H^k(\mathcal{Z}_{p+1}) \xrightarrow{\delta_1^*} H^{k+1}(\mathcal{Z}_p) \xrightarrow{i^*} 0 \quad \forall k \geq 1,$$

We have $\quad H^k(\mathcal{Z}_{p+1}) \cong H^{k+1}(\mathcal{Z}_p)$. From

$$(c) \quad H^p(\mathcal{S}) \cong H^{p-1}(\mathcal{Z}_1) \quad p \geq 2$$

we have $H^p(\mathcal{S}) \cong H^{p-1}(\mathcal{Z}_1) \cong H^{p-2}(\mathcal{Z}_2) \cdots \cong H^1(\mathcal{Z}_{p-1})$,

$$(ii)\ 0 \to H^0(\mathcal{Z}_{p-1}) \xrightarrow{i^*} H^0(\mathcal{S}_{p-1}) \xrightarrow{d_{p-1}^*} H^0(\mathcal{Z}_p) \xrightarrow{\delta_0^*} H^1(\mathcal{Z}_{p-1}) \xrightarrow{i^*} H^1(\mathcal{S}_p)$$

$$H^p(\mathcal{S}) \cong H^1(\mathcal{Z}_{p-1}) \cong \mathrm{Img}\delta_0^* = \frac{\Gamma(\mathcal{Z}_p)}{\mathrm{Ker}\delta_0^*} = \frac{\Gamma(\mathcal{Z}_p)}{\mathrm{Img}d_{p-1}^*} = \boxed{\frac{\mathrm{Ker}d_p^*}{\mathrm{Img}d_{p-1}^*}}.$$

Dolbeault Cohomology

Dolbeault Cohomology

Definition (Dolbeault Cohomology Group)

The differential operator $\bar{\partial} : A^{p,q} \to A^{p,q+1}$, $\bar{\partial}^2 = 0$, define the (p, q) degree Dolbeault cohomology group

$$H_{\bar{\partial}}^{p,q} = \{\omega \in A^{p,q} | \bar{\partial}\omega = 0\} / \{\bar{\partial}\eta | \eta \in A^{p,q-1}\}$$

Dolbeault Lemma

Lemma

Suppose f is a smooth function defined on $\mathbb{C}$ with compact support, then there is a smooth function g on $\mathbb{C}$, such that $\bar{\partial} g = f$.

Proof.

$\bar{\partial} z^{-1} = \delta(0)$, so

$$g(w) = f(z) * \frac{1}{z} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f(z)}{z-w} dz \wedge d\bar{z}.$$

$\square$

Dolbeault Lemma

Lemma

Suppose M is a Riemann surface, ω is a (p,q) form on an open set U, $q \geq 1$. For any point $p \in U$, there is an open neighborhood $V \subset U$, and $(p, q-1)$ form η on V, such that $\omega = \bar{\partial}\eta$.

Proof.

Assume $M = \mathbb{C}$, p is the origin. We only consider $\omega = hd\bar{z}$ and $\omega = hdz \wedge d\bar{z}$. Choose smooth cutoff function ϕ near the origin, in the neighborhood of the origin, $\phi \equiv 1$, and on the boundary of U is zero. Let $f = \phi \cdot h$, f is treated as a smooth function on $\mathbb{C}$ with compact support, there is a function g, such that $\bar{\partial} g = f$, near the origin

$$\bar{\partial} g = f d\bar{z} = \phi \cdot h d\bar{z} = h d\bar{z}.$$

$\square$

Dolbeault Theorem

Theorem (Dolbeault)

On a Riemann surface M, the following cohomology groups are isomorphic

$$H^1(M;\mathcal{O}) \cong H^{0,1}_{\bar{\partial}}(M), \quad H^1(M;\Omega^1) \cong H^{1,1}_{\bar{\partial}}(M).$$

Proof.

By Dolbeault lemma, $\bar{\partial}_0 : \mathcal{S}^0 \to \mathcal{S}^{0,1}$ is surjective, the inclusion map $i : \mathcal{O} \to \mathcal{S}^0$ is injective, so we obtain the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{S}^0 \xrightarrow{\bar{\partial}_0} \mathcal{S}^{0,1} \xrightarrow{\bar{\partial}_1} 0,$$

where $\mathcal{O}$ is the holomorphic function sheaf. By de Rham theorem,

$$H^1(M;\mathcal{O}) = Ker\bar{\partial}_1 / Img\bar{\partial}_0 = H^{0,1}_{\bar{\partial}}(M).$$

$\square$

Dolbeault Theorem

continued.

By Dolbeault lemma, $\bar{\partial} : \mathcal{S}^{0,1} \to \mathcal{S}^{1,1}$ is surjective, the inclusion map $i : \Omega^1 \to \mathcal{S}^{1,0}$ is injective, so we obtain the short exact sequence

$$0 \to \Omega^1 \to \mathcal{S}^{1,0} \xrightarrow{\bar{\partial}_0} \mathcal{S}^{1,1} \xrightarrow{\bar{\partial}_1} 0,$$

where Ω^1 is the holomorphic 1-form sheaf. By de Rham theorem,

$$H^1(M;\Omega^1) = Ker\bar{\partial}_1 / Img\bar{\partial}_0 = H^{1,1}_{\bar{\partial}}(M).$$

$\square$

Doleault Theorem

$\Omega^0(L)$ is the sheaf of holomorphic section of L; Ω^1 is the sheaf of L-valued $(1,0)$-form; $\mathcal{S}^{p,q}(L)$ is the sheaf of L-valued smooth (p,q)-form.

Definition (L-valued (p,q) form)

For $p \in M$, in the neighborhood of p,

$$\omega = \sum_i \omega_i \otimes s_i = \sum_i \omega_i s_i,$$

ω_i is a local (p,q)-form, s_i is the local holomorphic section of L. Holomorphic line bundle L has local trivialization, s is a holomorphic section non-zero everywhere, each s_i can be represented as $s_i = f_i s$, where f_i is a local smooth function, namely

$$\sum_i \omega_i s_i = \sum_i f_i \omega_i s = \omega s.$$

Doleault Theorem

Definition ($\bar{\partial}$ operator)

The operator $\bar{\partial} : \mathcal{S}^{p,q}(L) \to \mathcal{S}^{p,q+1}(L)$

$$\bar{\partial}\left(\sum_i \omega_i s_i\right) = \bar{\partial}\left(\sum_i f_i \omega_i s\right) = (\bar{\partial}\omega)s.$$

Suppose t is another local holomorphic section no-zero everywhere, and omega s $= \eta t$. Then there is a holomorphic function f, such that $t = f \cdot s$, then $\omega = f\eta$,

$$(\bar{\partial}\omega)s = (\bar{\partial}f\eta)s = (\bar{\partial}f \cdot \eta + f\bar{\partial}\eta)s = f(\bar{\partial}\eta)s = (\bar{\partial}\eta)t.$$

Dolbeault Theorem

Theorem (Dolbeault)

Suppose L is a holomorphic line bundle on a Riemann surface M, then the following cohomology group isomorphisms hold: $\forall p, q \geq 0$,

$$H^q(M, \Omega^p(L)) \cong \frac{\{\bar{\partial}\text{-closed } L\text{-valued } (p,q) \text{ form}\}}{\{\bar{\partial}\text{-exact } L\text{-valued } (p,q) \text{ form}\}}$$

particularly, when $p + q > 2$ $H^q(M, \Omega^p(L)) = 0$.

Dolbeault Theorem

Example

By the short exact sequence,

$$0 \to \Omega^0(L) \to \mathcal{S}^0(L) \xrightarrow{\bar{\partial}} \mathcal{S}^{0,1}(L) \to 0,$$

By de Rham theorem, we obtain

$$H^1(M, \Omega^0(L)) \cong \frac{\{\bar{\partial}\text{-closed L-valued } (0,1) \ \textit{form}\}}{\{\bar{\partial}\text{-exact L-valued } (0,1) \ \textit{form}\}}$$

Dolbeault Theorem

Example

By the short exact sequence,

$$0 \to \Omega^1(L) \to \mathcal{S}^{1,0}(L) \xrightarrow{\bar{\partial}} \mathcal{S}^{1,1}(L) \to 0,$$

By de Rham theorem, we obtain

$$H^1(M, \Omega^1(L)) \cong \frac{\{\bar{\partial}\text{-closed L-valued } (1,1) \textit{ form}\}}{\{\bar{\partial}\text{-exact L-valued } (1,1) \textit{ form}\}}$$

when $p + q > 2$, then a (p, q)-form must be zero.