

Mathematical Methods in Physics

Weekly Problems 8. Solution

8.1

- a) The straight line between A and B could be represented as follows (mind you, it is not unique)

$$\mathbf{r}(t) = (1, 2, 3) + t((4, 5, 9) - (1, 2, 3)) \longrightarrow \mathbf{r}(t) = (1+3t)\mathbf{i} + (2+3t)\mathbf{j} + (3+6t)\mathbf{k} \quad 0 \leq t \leq 1.$$

1 mark

Then

$$\mathbf{a}(\mathbf{r}(t)) = (4 + 9t)\mathbf{i} + (5 + 9t)\mathbf{j} - \mathbf{k}, \quad \mathbf{r}'(t) = 3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}.$$

The line integral is given by

$$I = \int_C \mathbf{a} \cdot d\mathbf{r} = \int_0^1 (\mathbf{a} \cdot \mathbf{r}') dt = \int_0^1 (54t + 21) dt = 48. \quad \text{1 mark}$$

- b) Set $t = x$ then $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ with $0 \leq x \leq 2$.

1 mark

In addition $\mathbf{a}(\mathbf{r}(t)) = (t + t^3)\mathbf{i} + (t^3 + t^2)\mathbf{j} + (t - t^2)\mathbf{k}$ and $\mathbf{r}' = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$.

The line integral is

$$I = \int_0^2 ((t + t^3) + 2t(t^3 + t^2) + 3t^2(t - t^2)) dt = \frac{98}{5}. \quad \text{2 marks}$$

8.2

The vector area element is given by

$$d\mathbf{S} = \left(\frac{d\mathbf{r}}{d\phi} \times \frac{d\mathbf{r}}{dz} \right) d\phi \times \begin{pmatrix} \cos \phi / (2\sqrt{z}) \\ \sin \phi / (2\sqrt{z}) \\ 1 \end{pmatrix} d\phi dz = \begin{pmatrix} \sqrt{z} \cos \phi \\ \sqrt{z} \sin \phi \\ -1/2 \end{pmatrix} d\phi dz,$$

hence the scalar area element is $dS = (z + 1/4)^{1/2} d\phi dz$.

1 mark

a)

$$I_1 = \int_S dS = 2\pi \int_0^2 dz \sqrt{z + \frac{1}{4}}.$$

Set $t = (z + 1/4)^{1/2}$. Then $dt = dz/2t$ and the integral becomes

$$I_2 = 2\pi \int_{1/2}^{3/2} 2t^2 dt = \frac{13\pi}{3}. \quad \boxed{2 \text{ marks}}$$

b)

$$\nabla \times \mathbf{a} = -2\mathbf{k},$$

and the integral is

$$I_2 = \int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \int_0^2 dz \int_0^{2\pi} d\phi = 4\pi. \quad \boxed{2 \text{ marks}}$$