Mathematical Methods II
Lecture 7

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Key Points

- Power series
- Finding singular points

Series solutions to linear ODEs

- **Introduction:** Up to now we have examined a handful of techniques designed to solve homogeneous and inhomogeneous linear ODEs with constant or variable coefficients, by writing the solutions as elementary functions or integrals. However, in general those with variable coefficients cannot be solved this way.

  An alternative approach is to obtain solutions in the form of convergent series, which can be evaluated numerically. It is worth noting that there is no distinct boundary between series solutions and using elementary functions, as those functions can be written as convergent series themselves (i.e. using the relevant Taylor series). In fact, some series are so common that they are given their own names, e.g. $\sin x$, $\cos x$ and e^x.

- **Aside: Power series -** An power series is an infinite polynomial and has the form

  $$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

  were a_n are constants. They occur often in physics and are useful as for $|x| < 1$ the later terms become small and may be discarded. e.g. if $x = 0.1$ for the series

  $$S(x) = 1 + x + x^2 + x^3 + ...$$

  then the value of the series is 1 for one term, 1.1 for two terms, 1.11 for three, etc. If you make a measurement with an accuracy limited to 2 decimal places then any terms after x^2 can be ignored.

  When dealing with infinite series it is important to consider if the series will converge or diverge at a given point, as this will affect which method we use to solve the equation at this point. Taylor series solutions work well for 'ordinary points' but 'singular points' require more advanced techniques.

- **Convergence:** The property of an infinite series to approach a limit (fixed value) as the series progresses - the sum of the series. In order for a series to converge on a value the individual terms in a series must themselves converge on zero (this condition is necessary, but does not guarantee a series will converge). e.g. The reciprocals of factorials are a convergent series

$$\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+...=e \text{ (Euler's number)}$$

- **Divergence:** If an infinite series does not converge, it diverges towards ∞. e.g. The reciprocals of positive integers are a divergent series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...=\infty$$

- **Testing convergence:** One way of testing whether a series converges is to apply D'Alembert's ratio test. This ratio is a simple way of comparing two consecutive terms in a series in the limit as $n \to \infty$. Assume for each term in a series (u_n, where all $u_n > 0$ and $0 \leq n \leq \infty$), there exists a value r (the radius of convergence) such that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r \begin{cases} \text{If } r < 1 \text{ power series converges.} \\ \text{If } r = 1 \text{ power series may diverge or converge - inconclusive.} \\ \text{If } r > 1 \text{ power series diverges.} \end{cases}$$

  e.g. For the above reciprocals factorial series we find as $n \to \infty$, $r \to 0$, hence the series converges.
  For the reciprocals of positive integers series we find as $n \to \infty$, $r \to 1$. In this case we would need to use another method to test the convergence of the series.

  Consider the general power series

$$P(x) = a_0 + a_1 x + a_2 x^2 + ...$$

  D'Alembert's ratio test tells us that P(x) converges if

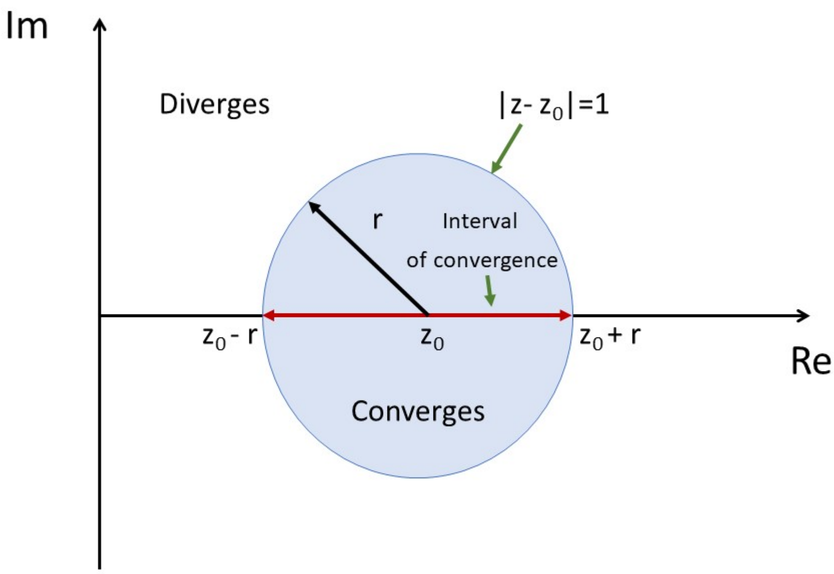
$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}|x|}{a_n}\right| = |x| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

  Note that there is an x term, because each $(n+1)^{\text{th}}$ term will have a power of x one greater than the n^{th} term, cancelling in the fraction to a single x. So the convergence of $P(x)$ depends on the value of x; there will be a range of values of x for which $P(x)$ will converge, an *interval of convergence.* Section 4.3 of Riley explains how to determine this interval.

- **Radius of convergence:** This is the radius of the largest disc in which the series will converge (i.e. all points within the radius do converge, some outside of the radius may). For a real function with real variables we can describe an interval of convergence (1-dimensional), say $-1 \leq x \leq 1$. But if the function is complex then the interval is best described in 2 dimensions, hence as the radius of the largest disc of convergence.

For a radius of convergence r, centred on the point $z = z_0$

$$|z - z_0| \begin{cases} < r \text{ series converges.} \\ > r \text{ series diverges.} \end{cases}$$



– **Definition of a power series:** Power series are best defined in complex terms

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where z_0 is a complex constant, the centre of the disc of convergence, c_n is the n^{th} complex coefficient (which may be a variable function in z) and z is a complex variable.

- **Complex variables:** Let's look at the canonical form of a 2^{nd} order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0$$

Up to now we have assumed that $y(x)$ is a real function of a real variable x, but this is not always the case. We can easily generalise the homogeneous linear ODE to include functions of a complex variable z

$$y'' + p(z)y' + q(z)y = 0$$

where $y = y(z)$ and $y' = dy/dz$. We may treat differentiation w.r.t z just as we would w.r.t x.

- **Ordinary and singular points:** Let's take some point $z = z_0$. We are interested in evaluating the nature of $p(z)$ and $q(z)$ at this point. Expressed as complex power series the functions $p(z)$ and $q(z)$ have the following forms

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n, \qquad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n.$$

If $p(z)$ and $q(z)$ both converge to a finite value at $z = z_0$ it is an ordinary point. If $p(z)$ or $q(z)$ or both diverge at $z = z_0$ then it is a singular point.

If the ODE has a singular point it may still have a finite solution at that point (regular singular point), or it may not have a solution at that point (irregular singular point). To test if a singular point is regular or not we check if the following conditions converge or diverge,

$$(z - z_0)p(z) \qquad \text{and} \qquad (z - z_0)^2 q(z).$$

If both of these conditions converge to a finite value we have a regular singular point. If one or both of these conditions diverge then we have an irregular singular point.

e.g. 7.1 Legendre's equation has the form

$$(1 - z^2)y'' - 2zy' + \ell(\ell + 1)y = 0$$

where ℓ is a constant. Show that $z = 0$ is an ordinary point and $z = \pm 1$ are regular singular points of this equation.

First, let's rearrange the equation into the canonical form by dividing by the coefficient of the 2$^{\text{nd}}$ order derivative

$$y'' - \frac{2z}{1 - z^2}y' + \frac{\ell(\ell + 1)}{(1 - z^2)}y = 0$$

Now we identify $p(z)$ and $q(z)$

$$p(z) = \frac{-2z}{1 - z^2} = \frac{-2z}{(1 + z)(1 - z)}$$

$$q(z) = \frac{\ell(\ell + 1)}{1 - z^2} = \frac{\ell(\ell + 1)}{(1 + z)(1 - z)}$$

Now we can check the nature of the points at the first boundary condition, $z = 0$

$$p(0) = \frac{-2 \times 0}{(1 + 0)(1 - 0)} = \frac{0}{1} = 0$$

$$q(0) = \frac{\ell(\ell + 1)}{(1 + 0)(1 - 0)} = \frac{\ell(\ell + 1)}{1} = const$$

Both $p(0)$ and $q(0)$ converge, hence $z = 0$ is an ordinary point.

Now we can check the nature of the points at the second boundary condition, $z = 1$

$$p(1) = \frac{-2 \times 1}{(1+1)(1-1)} = \frac{-2}{0} = \infty$$

$$q(1) = \frac{\ell(\ell+1)}{(1+1)(1-1)} = \frac{\ell(\ell+1)}{0} = \infty$$

Both $p(1)$ and $q(1)$ diverge, hence $z = 1$ is a singular point.

Now we can check for the nature of the singular point at $z = 1$ using our test conditions.

$$(z-1)p(1) = \frac{-2z(z-1)}{(1+z)(1-z)} = \frac{2z}{(1+z)} = \frac{2 \times 1}{(1+1)} = 1$$

$$(z-1)^2 q(1) = \frac{\ell(\ell+1)(z-1)^2}{(1+z)(1-z)} = \frac{-\ell(\ell+1)(z-1)}{(1+z)} = \frac{-\ell(\ell+1)(1-1)}{(1+1)} = 0$$

Both $(z-1)p(1)$ and $(z-1)^2 q(1)$ converge, hence $z = 1$ is a regular singular point.

To finish, repeat the steps for the final boundary condition, $z = -1$.

$$p(-1) = \frac{-2 \times -1}{(1-1)(1+1)} = \frac{2}{0} = \infty$$

$$q(-1) = \frac{\ell(\ell+1)}{(1-1)(1+1)} = \frac{\ell(\ell+1)}{0} = \infty$$

Both $p(-1)$ and $q(-1)$ diverge, hence $z = -1$ is a singular point.

$$(z+1)p(-1) = \frac{-2z(z+1)}{(1+z)(1-z)} = \frac{-2z}{(1-z)} = 1$$

$$(z+1)^2 q(-1) = \frac{\ell(\ell+1)(z+1)^2}{(1+z)(1-z)} = \frac{\ell(\ell+1)(z+1)}{(1-z)} = 0$$

Both $(z+1)p(-1)$ and $(z+1)^2 q(-1)$ converge, hence $z = -1$ is a regular singular point.

Note: with the Legendre equation it was easy to sub in values for z and arrive at sound conclusions regarding the nature of the points. In general this will not be true.

The general approach is to examine what happens in the limit $z \to z_0$. The weekly problems examine the Bessel equation. If you try to sub in values for one of the singularities you will get $0 \div 0$, which is undefined. However, the $\lim_{z \to 0}$ will give an answer of 1.

- **Singular points at infinity:** For testing singular points at infinity you can make a change of variable $x = 1/\omega$ and take the limit where $\omega \to 0$.