Theoretical Physics 2019/20 — Problem QT2.3

Preparation: Part 2 and Section 6.1 of Part 6 of the notes, and Section 5.5 of Professor Cole's notes. This problem is a direct continuation of the previous homework problem for this course, Problem QT2.2. For convenience and clarity, all the information given at the beginning of the latter is repeated here.

The two vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

form an orthonormal basis for the Hilbert space of 2-component complex column vectors. Any spin state of an electron can be represented by an element of that space, i.e., by a column vector of the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α and β are two complex numbers (α and β cannot be both zero, the zero vector never represents a quantum state). Because electrons have a spin-dependent magnetic moment, their spin state may be modified by a magnetic field.

Consider an electron at rest in a magnetic field $\mathbf{B}$ parallel to the $x-z$ plane (the y-component of $\mathbf{B}$ is zero). The Hamiltonian of this system can be represented by a 2×2 matrix. This matrix is given by the following equation in the $\{\chi_+, \chi_-\}$ basis:

$$H = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}, \tag{2}$$

where γ is a constant and B_x and B_z are the x- and z-components of $\mathbf{B}$. We assume that $B_x \neq 0$ (recall that we also assume that the y-component of $\mathbf{B}$ is zero). As found in part (b) of Problem QT2.2, the two column vectors χ_a and χ_b defined by the following equations are eigenvectors of the matrix H and correspond, respectively, to the eigenvalues $E_a = \mu|\mathbf{B}|$ and $E_b = -\mu|\mathbf{B}|$:

$$\chi_a = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} -k \\ 1 \end{pmatrix}, \qquad \chi_b = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} 1 \\ k \end{pmatrix},$$

with $k = B_x/(B_z + |\mathbf{B}|)$. These two column vectors form an orthonormal basis of the Hilbert space these spin states belong to.

(a) Why can you be sure that this electron cannot be in a spin state represented by a column vector orthogonal both to χ_a and χ_b?

(b) Let us represent the spin state of this electron by the following time-dependent column vector,

$$\chi(t) = c_a\,\chi_a \exp(-iE_a t/\hbar) + c_b\,\chi_b \exp(-iE_b t/\hbar), \tag{3}$$

where c_a and c_b are two constants. (You have seen similar superpositions of energy eigenstates in Term 1, although perhaps not for spin states.) Show that $\chi(t)$ is a solution of the time-dependent Schrödinger equation. I.e., show that

$$i\hbar\frac{\partial \chi}{\partial t} = H\chi(t).$$

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(c) Show that $\chi(t)$ is normalized if $|c_a|^2 + |c_b|^2 = 1$.

(d) Suppose the electron is in the spin state χ_+ at time $t = 0$ (i.e., $\chi(t = 0) = \chi_+$). Determine the constants c_a and c_b. (As a check of your results, show that $|c_a|^2 + |c_b|^2 = 1$.) [Hint: What do you get if you calculate the inner product of χ_a and $\chi(t)$ at $t = 0$?]

(e) Suppose that a measurement of the spin state of this electron is made at a time $t > 0$. Let us denote by $P_+(t)$ the probability that the electron is found to be in the spin state represented by χ_+. Show that $P_+(t) = \cos^2(\mu|\mathbf{B}|t/\hbar)$ when $B_z = B_y = 0$. [Hint: Since χ_+ and $\chi(t)$ are normalized, $P_+(t) = |(\chi_+, \chi(t))|^2$, where $(\chi_+, \chi(t))$ is the inner product of χ_+ and $\chi(t)$.]

(f) Let us also denote by $P_-(t)$ the probability that the electron is found to be in the spin state represented by χ_-. For which direction(s) of $\mathbf{B}$ would $P_-(t)$ be equal to $P_+(t)$ at all times?