Mathematical Methods - Part 1

Vector Algebra

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Levi-Cevita Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } (i,j,k) \in \{(3,2,1),(1,3,2),(2,1,3)\} \\ 0 & \text{otherwise} \end{cases}$$

Scalar Product

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j = a_i b_i$$

Vector Product

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$$

Scalar Triple Product

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \epsilon_{ijk} a_i b_j c_k$$

Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Equation of a Line

For

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equation of a line:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$

Where $\mathbf{a}$ is a position vector for a point on the line, $\mathbf{b}$ is a vector parallel to the line, and λ is an arbitrary scalar.

The equation can also be written as:

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$$

Equation of a Plane

Equation of a plane:

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0$$

Where $\mathbf{a}$ is the position vector of a point on the plane and $\hat{\mathbf{n}}$ is the normal vector to the plane.

Or, alternatively:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

Where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are position vectors of points on the plane.

Vector Spaces

Linear Vector Space

A set of vectors forms a vector space, V, if there are two operations defined on the elements of the set called addition and multiplication by scalars, which obey the following simple rules (the axioms of the vector space):

1. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$. If $\mathbf{v} \in V$ then $\alpha\mathbf{v} \in V$ where $\alpha \in \mathbb{C}$. The vector space is closed with respect to addition and scalar multiplication.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
3. There exists a neutral element, $\mathbf{0}$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$.
4. There exists an inverse element, $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$.
5. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
6. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$

Where α, β are scalars and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $\alpha, \beta \in \mathbb{R}$ then V is called a real vector space. If $\alpha, \beta \in \mathbb{C}$ then V is called a complex vector space.

Linear Combinations

The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \alpha_i\mathbf{v}_i$$

Is called the "span" of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

Linearly Independent Vectors

k vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, are said to be linearly independent if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$$

is satisfied if and only if all $\alpha_i = 0$. Otherwise the vectors are said to be linearly dependent.

Basis

A basis is a set of vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$, which are linearly independent and satisfy $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. Then

1. The numbers of vector in a basis is called the dimension of the space V, $\dim V$.
2. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis of the vector space V, then any vector $\mathbf{v}$ in V can be written as linear combination of the basis vectors. The coefficients of basis vectors are called the components of $\mathbf{v}$ with respect to the basis.

Inner Product

The inner product between vectors in V, $\langle\mathbf{u}|\mathbf{v}\rangle$, satisfies:

1. $\langle\mathbf{u}|\mathbf{v}\rangle = \langle\mathbf{v}|\mathbf{u}\rangle^*$
2. $\langle\mathbf{u}|(\lambda\mathbf{v} + \mu\mathbf{w})\rangle = \lambda\langle\mathbf{u}|\mathbf{v}\rangle + \mu\langle\mathbf{u}|\mathbf{w}\rangle$
3. $\langle\mathbf{u}|\mathbf{u}\rangle > 0$ if $\mathbf{u} \neq 0$

The length of a vector is $|\mathbf{u}| = \sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}$ and two vectors are orthogonal is $\langle \mathbf{u}|\mathbf{w}\rangle = 0$.

Matrices

Linear Operators

An object, $\mathcal{A}$, is a linear operator if acts on the vectors $\mathbf{v}$ and $\mathbf{u}$ like so:

$$\mathcal{A}(\alpha \mathbf{v} + \beta \mathbf{u}) = \alpha(\mathcal{A}\mathbf{v}) + \beta(\mathcal{A}\mathbf{u})$$

Matrix Operations

1. Addition: $(A + B)_{ij} = A_{ij} + B_{ij}$
2. Scalar multiplication: $(\alpha A)_{ij} = \alpha A_{ij}$
3. Multiplication of matrices: $(AB)_{ij} = A_{ik}B_{kj}$
4. Transposition: $(A^T)_{ij} = A_{ji}$ with $(ABC \ldots F)^T = F^T \ldots C^T B^T A^T$
5. Complex conjugation: $(A^*)_{ij} = (A_{ij})^*$
6. Hermitian conjugation (adjoint): $(A^\dagger)_{ij} = (A_{ji})^*$ with $(ABC \ldots F)^\dagger = F^\dagger \ldots C^\dagger B^\dagger A^\dagger$

Determinant of a Square Matrix

For a square matrix, A, the determinant is:

$$|A| = A_{jk}C_{jk}$$

For any row j or

$$|A| = A_{kj}C_{kj}$$

For any column j.

Where

$$C_{mn} = (-1)^{m+n}|A_{mn}|$$

$|A_{mn}|$ is determinant of the matrix obtained by removing the mth row and nth column from the matrix A.

Properties:

1. $|ABC \ldots F| = |A||B||C| \ldots |F|$
2. $|A^T| = |A|$, $|A^*| = |A|^*$, $|A^\dagger| = |A|^*$, $|A^{-1}| = |A|^{-1}$
3. If the rows or columns are linearly dependent then $|A| = 0$
4. If B is obtained from A by multiplying the elements of any row (or column) by a factor α, then $|B| = \alpha|A|$
5. If B is obtained from A by interchanging two rows (or columns), then $|B| = -|A|$
6. If B is obtained from A by adding k times one row (or column) to the other row (or column), then $|A| = |B|$

Elementary Row Operations

1. Multiply any row by a non-zero scalar
2. Interchange any two rows
3. Add some multiple of row to any other row

Inverse of a Square Matrix

The inverse of a matrix A is:

$$A^{-1} = \frac{C^T}{|A|}$$

Where $A^{-1}A = AA^{-1} = I$

Properties:

1. $(ABC \ldots F)^{-1} = F^{-1} \ldots C^{-1} B^{-1} A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T, \; (A^\dagger)^{-1} = (A^{-1})^\dagger$

Trace of a Square Matrix

The trace of a matrix A is:

$$\text{Tr } A = \sum_k A_{kk} \equiv A_{kk}$$

Properties:

1. Trace is a linear operation
2. $\text{Tr } A^T = \text{Tr } A$; $\text{Tr } A^\dagger = (\text{Tr } A)^*$
3. $\text{Tr } (AB) = (AB)_{ii} = A_{ij} B_{ji} = B_{ji} A_{ij} = (BA)_{jj} = \text{Tr } (BA)$
4. $\text{Tr } (ABC) = \text{Tr } (BCA) = \text{Tr } (CAB)$ - cyclical order

Special Matrices

Symmetric Matrices: $A^T = A$

Antisymmetric or Skew Matrices: $A^T = -A$

Any matrix can be written as the sum of a symmetric matrix and an antisymmetric matrix.

Hermitian matrices: $A^\dagger = A$

Antihermitian matrices: $A^\dagger = -A$

Any matrix can be written as the sum of a hermitian matrix and an antihermitian matrix.

Unitary matrices: $A^\dagger = A^{-1}$

Unitary matrices preserve the length of vectors.

Orthogonal matrices: $A^T = A^{-1}$

Orthogonal matrices also preserve the length of vectors.

The Eigenvalue Problem

Eigenvalues

For the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

1. λ is the eigenvalue, $\mathbf{x}$ is the corresponding eigenvector
2. The determinant, $|A - \lambda I|$, is called the characteristic polynomial or degree n
3. The eigenvalue equation has a nontrivial solution if and only if $|A - \lambda I| = 0$
4. The eigenvalues of the matrix A are the roots of the characteristic polynomial
5. The eigenvectors associated to the eigenvalue μ are the vectors $\mathbf{x}$ such that

$$(A - \mu I)\mathbf{x} = 0$$

Eigenvectors associated to different eigenvalues are linearly independent.

If a matrix has an eigenvalue equal to zero, then the matrix is singular since its determinant is zero.

Eigenvalues of Special Matrices

1. Hermitian and symmetric matrices have real eigenvalues
2. Antihermitian and antisymmetric matrices have eigenvalues which are purely imaginary or zero
3. Unitary and orthogonal matrices have eigenvalues whose absolute values are equal to one
4. The eigenvectors of all special matrices are linearly independent. In addition, they can always be chosen in such a way that they form a mutually orthogonal set.

Similar Matrices

Two matrices A and B can be said to be to similar if there exists a matrix S such that

$$B = S^{-1}AS$$

The two matrices represent the same linear operator in different basis.

Diagonalising a Matrix

If the new basis is a set of eigenvectors of A then $B \equiv D$ is diagonal with

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & \lambda_n \end{pmatrix}, \ S = \begin{pmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}$$

Where λ_i are the eigenvalues and $\mathbf{x}_i$ are the corresponding eigenvectors.

Applications:

1. nth power of A:

$$A^n = AA\dots A = (SDS^{-1})(SDS^{-1})\dots(SDS^{-1}) = SD^n S^{-1}$$

2. Exponential of A:

$$e^A = e^{SDS^{-1}} = \sum_{n=0}^{\infty} \frac{(SDS^{-1})^n}{n!} = Se^D S^{-1}$$

Fourier Series

Mathematical Methods - Part 2

1st Order ODEs

Separable ODE

Form:

$$f(x,y) = \frac{\mathrm{d}s}{\mathrm{d}t} = u(x)v(y)$$

Solution:

$$\int \frac{1}{v(y)}\mathrm{d}y = \int u(x)\mathrm{d}x$$

Exact ODE

Form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

Exactness condition:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

If exact, solution is:

$$u(x,y) = \int A(x,y)\mathrm{d}x \quad \text{or} \quad u(x,y) = \int B(x,y)\mathrm{d}y$$

Since

$$A(x,y) = \frac{\partial u}{\partial x} \quad \text{and} \quad B(x,y) = \frac{\partial u}{\partial y}$$

Also since $du = 0$, $u = \text{constant}$

Inexact ODE

Form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

Inexactness condition:

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

Use integrating factor, μ, and assume that it is exact:

$$\mu A(x,y)\mathrm{d}x + \mu B(x,y)\mathrm{d}y = 0$$

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}$$

Integrating factor can be either $\mu = \mu(x)$ or $\mu = \mu(y)$.

Linear 1st Order ODE

Form:

$$a(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x)$$

Or:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

Use integrating factor:

$$\mu(x) = \exp\left(\int P\mathrm{d}x\right)$$

Sub in and solve:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)y\right) = \mu(x)q(x) \quad \Rightarrow \quad y = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x$$

Bernoulli ODE

Form ($n \notin \{0, 1\}$):

$$\frac{dy}{dx} + b(x)y = c(x)y^n$$

Substitution:

$$z = y^{1-n}$$

Homogenous ODE

Form:

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Homogenous functions follow the rule:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Substitutions:

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Isobaric ODE

Form:

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Isobaric functions follow the rule:

$$f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x, y)$$

Assign the weights $y, dy \rightarrow m$ and $x, dx \rightarrow 1$ and solve for m.

Use substitution:

$$y = vx^m$$

2nd Order ODEs

Linear 2nd Order ODE with Constant Coefficients

Form:

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = f(x)$$

Find auxiliary equation and solve for λ:

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

Find complementary function.

For two solutions of λ, λ_1 and λ_2:

$$y_c = A\exp(\lambda_1 x) + B\exp(\lambda_2 x)$$

Or for a repeated root, λ_1:

$$y_c = (A + Bx)\exp(\lambda_1 x)$$

Then find particular solution, y_p, using a trial function and subbing into original ODE.

General solution is:

$$y = y_c + y_p$$

Laplace Transform Method for Linear 2nd Order ODEs with Constant Coefficients

Laplace Transform is defined as:

$$\mathcal{L}[f(x)](s) = \bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

For a second order derivative:

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = s^2 \mathcal{L}[f] - sf(0) - \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=0}$$

And for first order derivative:

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = s\mathcal{L}[f] - f(0)$$

Sub these into ODE, apply boundary conditions and apply inverse Laplace transform using the standard transforms table.

Note that the Laplace transform and its inverse are linear:

$$\mathcal{L}[af(x) + bg(x)] = a\mathcal{L}[f(x)] + b\mathcal{L}[g(x)]$$

$$\mathcal{L}^{-1}[af(x) + bg(x)] = a\mathcal{L}^{-1}[f(x)] + b\mathcal{L}^{-1}[g(x)]$$

Where $a, b \in \mathbb{C}$.

Legendre Linear Equations

Form:

$$a_2(\alpha x + \beta)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(\alpha x + \beta)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

Use substitution $\alpha x + \beta = e^t$ which gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha}{\alpha x + \beta}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\alpha^2}{(\alpha x + \beta)^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

Euler Linear Equations

Form:

$$a_2 x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 x\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

Make substitution $x = e^t$ or, if $f(x) = 0$, you may use $y = x^\lambda$.

Wronskian Method for Linear 2nd Order ODEs

The Wronskian for an nth order ODE is defined as:

$$W(y_1, y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & & \ddots & \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{vmatrix}$$

Hence for a 2nd order ODE the Wronskian is:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Where y_1, y_2 are solutions to the ODE.

We can then find a general solution to the ODE using:

$$y = y_p = k_1 y_1 + k_2 y_2$$

Where

$$k_1 = -\int \frac{y_1 f(x)}{W(y_1, y_2)} \mathrm{d}x$$

$$k_2 = \int \frac{y_2 f(x)}{W(y_1, y_2)} \mathrm{d}x$$

Ordinary and Singular Points

For the 2nd order ODE where $y = y(z)$:

$$y'' + p(z)y' + q(z)y = 0$$

The point $z = z_0$ is an ordinary point if both

$$\lim_{z \to z_0} p(z)$$

and

$$\lim_{z \to z_0} q(z)$$

exist and are finite (i.e. $p(z)$ and $q(z)$ converge at z_0)

If $p(z)$ and/or $q(z)$ diverge at $z = z_0$, it is a singular point.

The point $z = z_0$ is a regular singular point if both

$$\lim_{z \to z_0} (z - z_0)p(z)$$

and

$$\lim_{z \to z_0} (z - z_0)^2 q(z)$$

exist and are finite (i.e. $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ converge at z_0).

If $(z - z_0)p(z)$ and/or $(z - z_0)^2 q(z)$ diverge at $z = z_0$, it is an irregular singular point.

When checking $z_0 = \infty$, use the substitutions $w = \frac{1}{z}$, $w_0 = 0$.

Series Solutions at Ordinary Points

To solve the 2nd order ODE where $y = y(z)$:

$$y'' + p(z)y' + q(z)y = 0$$

Substitute:

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

To get:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + p(z)(n+1) a_{n+1} + q(z) a_n \right] z^n = 0$$

Then for the case where $z^n \neq 0$:

$$(n+2)(n+1) a_{n+2} + p(z)(n+1) a_{n+1} + q(z) a_n = 0$$

Then rearrange to get a recurrence relation.

Legendre's Differential Equation

Form:

$$(1 - x^2) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + \ell(\ell+1) y = 0$$

Solutions are Legendre Polynomials, $P_\ell(x)$.

Rodrigues' Formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} (x^2 - 1)^\ell$$

Associated Legendre Differential Equation

Form:

$$(1 - x^2) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + \left[\ell(\ell+1) - \frac{m^2}{1 - x^2} \right] y = 0$$

Solution:

$$P_\ell^m(x) = (1 - x^2)^{m/2} \frac{\mathrm{d}^m P_\ell}{\mathrm{d}x^m}$$

For $m > 0$.

Partial Differential Equations

Separation of Variables

For PDE of the form:

$$a u_{xx} + b u_{tt} + c u_x + d u_t + e u = 0$$

Assume solution to PDE can be expressed as:

$$u(x, t) = X(x) T(t)$$

Substitute in and rearrange:

$$a \frac{X''}{X} + c \frac{X'}{X} + e = -b \frac{T''}{T} - d \frac{T'}{T}$$

Separate equations into two ODEs and solve:

$$a\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + c\frac{\mathrm{d}X}{\mathrm{d}x} + (e - \mu)X = 0$$

$$b\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + d\frac{\mathrm{d}T}{\mathrm{d}t} + \mu T = 0$$

Separation of a 3D PDE

Assume solution to PDE can be expressed as:

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

Taking an example PDE, the heat equation:

$$\frac{\partial u}{\partial t} = k^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Substituting in the solution:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{k^2}\frac{T'}{T} = \mu$$

Separating:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - lX = 0$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - mY = 0$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - nZ = 0$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \mu k^2 T = 0$$

Where $l + m + n = \mu$.