University of Durham

EXAMINATION PAPER

May/June 2015 Examination code: PHYS2591WE01

FOUNDATIONS OF PHYSICS 2B

SECTION A. Thermodynamics
SECTION B. Condensed Matter Physics
SECTION C. Modern Optics

Time allowed: 3 hours

Additional material provided: None

Materials permitted: None

Calculators permitted: Yes **Models permitted:** Casio fx-83 GTPLUS or Casio fx-85 GTPLUS

Visiting students may use dictionaries: No

Instructions to candidates:

- Answer the compulsory question that heads each of sections A, B and C. These **three** questions have a total of 15 parts and carry 50% of the total marks for the paper. Answer **one** other question from **each** section. If you attempt more than the required number of questions only those with the lowest question number compatible with the rubric will be marked: **clearly delete** those that are not to be marked. The marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.
- **ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK**
- Do **not** attach your answer booklets together with a treasury tag, unless you have used more than one booklet for a single section.

Information

A list of physical constants is provided on the next page.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Electron mass:	$m_\mathrm{e} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_\mathrm{p} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_\mathrm{A} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_\odot = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_\odot = 3.84 \times 10^{26}$ W

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SECTION A. THERMODYNAMICS

Answer question 1 and **either** question 2 **or** question 3.

1. (a) The enthalpy is defined as $H = U + pV$. Show that enthalpy has natural variables of entropy and pressure, i.e. $H = H(S, p)$, and hence determine the type of interaction it represents in an isobaric process. Furthermore, derive the following Maxwell Relation

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S .$$

Here, U is the internal energy and all other symbols have their usual meanings. [4 marks]

(b) Define the *Nernst Heat Theorem* and show that it implies that all heat capacities must tend to zero as the temperature approaches absolute zero. [4 marks]

(c) A simple harmonic oscillator has energy levels given by $\varepsilon_n = (n + \frac{1}{2})\hbar\omega$. Show that its partition function can be written as

$$Z = \frac{1}{2\sinh\left(\hbar\omega/2k_{\mathrm{B}}T\right)}.$$

[Hint: For a geometric progression, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.]

[4 marks]

(d) Write down the Boltzmann distribution for the number of distinguishable particles occupying an energy state ε_j. A certain system has the following distribution of particles in its lowest two energy levels. What is the system's temperature? Comment on its significance.

Energy	Population
10.0 meV	100
40.0 meV	800

[4 marks]

2. (a) Provide a definition of *entropy* in terms of macroscopic and statistical thermodynamics. Explain how entropy describes reversible and irreversible processes via the Clausius Inequality, and how it describes the quality of energy and its availability. [4 marks]

   (b) Calculate the entropy change of the Universe as 1.00 kg of water at 20.0 °C is brought to a temperature of 95.0 °C by connecting it to a heat reservoir, before the water is allowed to cool back to its original temperature. For water, $c_p = 4200$ J kg^{-1} K^{-1}. [3 marks]

   (c) Making use of the Maxwell Relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V,$$

   explicitly derive the following relation by considering one mole of ideal gas

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{C_V}{T},$$

   where C_V is the heat capacity at constant volume and all other symbols have their usual meanings. [6 marks]

   (d) A system of N distinguishable particles is initially arranged in two energy states, with n_1 particles in state 1 and n_2 particles in state 2. Determine the entropy change when the system is placed in contact with a heat reservoir such that $n_1 \to 2n_1$ and $n_2 \to n_2 - n_1$. Using your result, calculate the entropy change if initially $n_1 = N_{\mathrm{A}}$ and $n_2 = 3N_{\mathrm{A}}$. [7 marks]

   [Hint: You may wish to consider using Stirling's Approximation.]

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3. (a) Describe, using appropriate thermodynamic terminology, the processes that are required for an ideal, reversible heat engine. [4 marks]

   (b) A heat engine is initially in a thermodynamic state with coordinates (p_i, V_i, T_i) and the working fluid is one mole of ideal gas. It consists of the following thermodynamic processes:

   – Isochoric heating, which causes the pressure to double;
   – Isothermal expansion back to the original pressure;
   – Isobaric compression to return the engine to its initial state.

   (i) Draw fully labelled pV and TS diagrams for this engine, where all symbols have their usual meanings. You must explicitly determine the mathematical shapes of any non-horizontal or non-vertical lines. [7 marks]

   (ii) Calculate the efficiency of this engine and compare it to a Carnot cycle if the ideal gas used as the working fluid is diatomic. [6 marks]

   (c) It is proposed to design an engine that will take in heat $4Q$ from a reservoir at a temperature $4T$ and heat $2Q$ from a reservoir at $2T$, whilst dumping heat Q to a reservoir at T. During the process an amount $-5Q$ of work will be done. Comment on the viability, or otherwise, of this engine. [3 marks]

SECTION B. CONDENSED MATTER PHYSICS
Answer question 4 and **either** question 5 **or** question 6.

4. (a) For a simple cubic lattice, sketch the set of planes with the Miller indices (110) and (121). Include the x, y and z axes in your diagram. If the lattice constant a is 0.5 nm, determine the spacing for each of these two families of planes. [4 marks]

   (b) For both *acoustic* and *optical* phonons, describe the relative motion of the two atoms in the crystalline basis. For the ionic compound NaCl determine the angular frequency, and hence energy, of the optical phonon mode at the centre of the Brillouin zone. You may assume the atomic masses of Na and Cl are 23 atomic mass units and 35 atomic mass units respectively, and that the force constant between the atoms is 5 N m^{-1}. [4 marks]

   (c) Using the information that copper has a density of 8950 kg m^{-3} and an atomic mass of 63.5 atomic mass units, determine the free electron density of Cu. State clearly any assumptions you made. Using the Drude model determine the mean relaxation time τ for electrons in Cu if the conductivity is $\sigma = 6 \times 10^7$ S m^{-1}. [4 marks]

   (d) State the relationship between the *effective mass* of an electron in a nearly-free energy band and the energy dispersion relation $E(k)$. Describe, with the aid of an appropriate sketch, how the effective mass of an electron varies across a nearly-free electron energy band as the wavevector increases from zero to the first Brillouin zone boundary. [4 marks]

   (e) Explain how the hydrogenic model is used to predict the binding energy of shallow donors in a semiconductor. The binding energy of hydrogen is given by

$$-\frac{e^4 m_{\mathrm{e}}}{2(4\pi\epsilon_0\hbar)^2}.$$

   Using this expression determine the binding energy for a shallow donor in a semiconductor with an effective mass of 0.05 m$_e$ and a relative permittivity of $\epsilon_r = 14$. Illustrate the position of this energy level on an appropriate diagram. [4 marks]

5. (a) Describe the physical origin of both the repulsive and attractive forces in a covalent bond, stating clearly how the symmetry of the electron wave-function relates to the formation of a stable bond. [4 marks]

(b) An interatomic potential curve is approximated by the Morse potential given by the function:

$$V(r) = V_0 \left[\exp\left(\frac{-2\,(r - r_0)}{a} \right) - 2\exp\left(\frac{-\,(r - r_0)}{a} \right) \right]$$

where r is the separation between the two atoms and V_0, r_0 and a are constants. Draw a plot of $V(r)/V_0$ versus r/r_0. Fully annotate your plot with key features and qualitative behaviour. What is the significance of the constant r_0? Indicate this on your graph. Indicate the bond dissociation energy on your graph. [10 marks]

(c) When an atom described by the Morse potential above is displaced slightly from its equilibrium position it experiences a restoring force. Show that the restoring force is given by:

$$F = -\frac{2V_0\,(r - r_0)}{a^2}.$$

Describe qualitatively the motion of an atom which has been displaced from equilibrium, explaining your answer. [6 marks]

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6. (a) What does the *energy density of states* function $n(E)$ describe? How does $n(E)$ vary with energy in three dimensions? Illustrate your answer with a diagram. [4 marks]

   (b) Explain the significance of the *Fermi energy* in a metal. Draw a diagram to illustrate the behaviour of the Fermi-Dirac distribution function at room temperature in a typical metal. [4 marks]

   (c) Silver has a free electron density of 6×10^{28} m^{-3} at 300 K. Determine the Fermi energy and Fermi velocity of free electrons in silver. [5 marks]

   (d) Compare the values obtained in part (c) with the thermal energy and velocity of free electrons at 300 K and explain the reasons for the differences. [4 marks]

   (e) Describe what changes will occur in the Fermi energy when the metal is cooled down from 300 K giving an explanation for your answer. [3 marks]

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SECTION C. MODERN OPTICS

Answer question 7 and **either** question 8 **or** question 9.

7. (a) The Fourier transforms of $g(x)$ and $h(x)$ are

$$G(u) = \int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{i}2\pi ux}\mathrm{d}x \quad \text{and} \quad H(u) = \int_{-\infty}^{\infty} h(x)\mathrm{e}^{-\mathrm{i}2\pi ux}\mathrm{d}x \ ,$$

respectively. Give expressions for the Fourier transforms of: (i) $g(x)+h(x)$, (ii) $g(x-d)+h(x/a)$, (iii) $g(x)h(x)$ and (iv) $\cos 2\pi x/\lambda_0$ where λ_0 is a constant. [4 marks]

(b) The gauss function is defined as $\text{gauss}(\rho/w) = \mathrm{e}^{-\rho^2/w^2}$, where w is a constant and $\rho^2 = x^2 + y^2$. Show that $\text{gauss}(\rho/w)$ is 'cartesian separable'. Write an expression for the Fourier transform

$$F(k_x, k_y) = \mathcal{F}[\text{gauss}(\rho/w)](k_x, k_y) \ ,$$

where k_x and k_y are the angular spatial frequencies corresponding to the x and y directions. [4 marks]

[Hint: $\int_{-\infty}^{\infty} \mathrm{e}^{-\xi^2}\mathrm{d}\xi = \sqrt{\pi}$.]

(c) Sketch the far-field diffraction pattern produced by a laser beam with beam radius w propagating along the z axis incident on (i) an opaque screen containing a single vertical slit with width a ($a \ll w$) aligned along the y axis, and (ii) a wire with radius a also aligned along the y axis. Which pattern is wider? [4 marks]

(d) Monochromatic light from two distant sources with angular separation θ is incident on a lens with focal length f and diameter D. Sketch the intensity distribution on the lens and in the focal plane assuming that the sources are coherent and that $\theta > 1.22\lambda/D$. [4 marks]

(e) A laser with wavelength 1 micron and beam size 1 mm is mounted on an aeroplane and used for remote mapping of sea ice. If the aeroplane flies at 10 km above the ice surface what is the approximate spatial resolution of the map? [4 marks]

(f) Sketch the wavefronts of a TEM_{00} laser mode inside a plano-convex laser cavity. Include the mirrors and an arrow to indicate the position of the beam waist. [4 marks]

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8. (a) Sketch the $4f$ set up used in spatial filtering. Label your sketch to indicate all relevant planes and distances. [4 marks]

(b) The field in the input plane of a $4f$ spatial filter is $\mathcal{E}_0 f(x', y')$ where

$$f(x', y') = \text{gauss}\left(\frac{\rho'}{w_0}\right) \otimes [\delta(x' - x_0) - \delta(x' + x_o)] \ ,$$

where $\rho'^2 = x'^2 + y'^2$ and $x_0 > w_0$. Sketch the field along the x' axis. [2 marks]

(c) The field in the Fourier plane is given by

$$\mathcal{E}(k_x, k_y) = \frac{\mathcal{E}_0 \mathrm{e}^{\mathrm{i}2kf}}{\mathrm{i}\lambda f} \mathcal{F}[f(x', y')](k_x, k_y) \ ,$$

where f is the focal length of the lens and λ is the optical wavelength. Rewrite the Fourier transform as an integral. [2 marks]

(d) Write an expression for the field distribution in the Fourier plane as a function of the x and y displacement from the optical axis. [4 marks]

(e) Sketch the field amplitude in the Fourier plane along the x axis. [2 marks]

(f) A filter is placed in the Fourier plane that blocks light with $|x| > f\lambda/2x_0$. Write an expression for the filtered field. [2 marks]

(g) Write an expression for the field in the output plane and sketch the field along the x axis. [4 marks]

9. (a) Explain, briefly, under what conditions you might expect to observe Fraunhofer diffraction. Comment on when it is reasonable to assume that the electromagnetic field can be represented by a scalar field $\mathcal{E}$. [4 marks]

(b) An opaque screen in the $z = 0$ plane is illuminated normally by a monochromatic plane wave with wavelength λ. Write an expression for the field distribution in the Fraunhofer limit in terms of the aperture function $f(x', y')$. [2 marks]

(c) The aperture function is given by

$$f(x', y') = \text{rect}\left(\frac{x'}{a}\right) \text{rect}\left(\frac{y'}{b}\right) \otimes \text{comb}_4\left(\frac{x'}{d}\right) \ ,$$

where $b > d > a$ and

$$\text{comb}_N\left(\frac{x'}{d}\right) = \sum_{n=-(N-1)/2}^{(N-1)/2} \delta(x' - nd) \ .$$

Write an expression for the Fourier transform of $f(x', y')$ and hence the intensity distribution corresponding to the Fraunhofer diffraction pattern. [4 marks]

(d) Sketch the intensity distribution corresponding to the Fraunhofer diffraction pattern. [4 marks]

(e) Draw a phasor diagram corresponding to the first zero and second zero intensity points along the positive x axis. [2 marks]

(f) The aperture is replaced by a new screen with

$$f(x', y') = \text{rect}\left(\frac{x'}{a}\right) \text{rect}\left(\frac{y'}{b}\right) \otimes \left[\delta\left(\frac{x'}{d}\right) + \text{comb}_4\left(\frac{x'}{d}\right)\right] \ .$$

By what factor is the peak intensity of the diffraction pattern increased? [2 marks]

(g) Draw the new phasor diagram corresponding to the first zero along the positive x axis. [2 marks]