Stars and Galaxies - Summary

Observational Techniques

Introduction

Magnitudes

Intensity of a source of luminosity, L, at a distance r is given by

$$I = \frac{L}{4\pi r^2}$$

The apparent magnitude is then given as:

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{I_1}{I_2} \right)$$

For an object with apparent magnitude, m, at distance, D, the absolute magnitude, M, is given by

$$m - M = 5 \log_{10} D - 5$$

Where D is in parsecs.

Telescopes

Real and Virtual Images

Real images are formed when light rays from an object actually cross. A virtual image is formed when the light rays only appear to come from a point, so they never cross.

Lensmaker's Equation

The Lensmaker's Equation is given as

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

where u is the object distance, v is the image distance, and f is the focal length.

Sign Convention:

Quantity	***+ve***	***-ve***
u	Real object	Virtual object
v	Real image	Virtual image
f	Converging lens/mirror	Diverging lens/mirror

The focal length is related to the radii of curvature by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Where n is the refractive index. The radii of curvature, R, are positive for a convex surface.

For mirrors,

$$\frac{1}{f} = -\frac{2}{R}$$

Magnification

The linear magnification is given by

$$M_l = -\frac{v}{u}$$

If the magnification is negative, the image is inverted relative to the object.

Compound Lenses

For a system of two lenses of focal lengths f_1 and f_2, the angular magnification is given by:

$$M_\theta = \frac{\beta}{\alpha} = \frac{f_1}{f_2}$$

Reflecting vs Refracting Telescopes

Telescopes that use a lens are called **refractors**. Large lenses are difficult to make, and hence expensive. Since the refractive index depends on wavelength, the focal point depends on wavelength (chromatic aberration), leading to blurred images.

Telescopes which use mirrors are called **reflectors**. A large mirror may weigh several tons and will bend under its own weight, distorting the images.

Different parts of the mirrors may expand or contract at different rates when heated or cooled during an observing night. To avoid these problems, mirror and support structures are made of materials that limit thermal distortions. This limits the mirror size to around 4 meters.

A plane wave reflected by a parabolic mirror is focused to a point called prime focus. The distance from prime focus to mirror is called the focal length. The diameter of the mirror is called aperture.

The focal ratio is given by

$$\text{focal ratio} = \frac{\text{focal length}}{\text{aperture}}$$

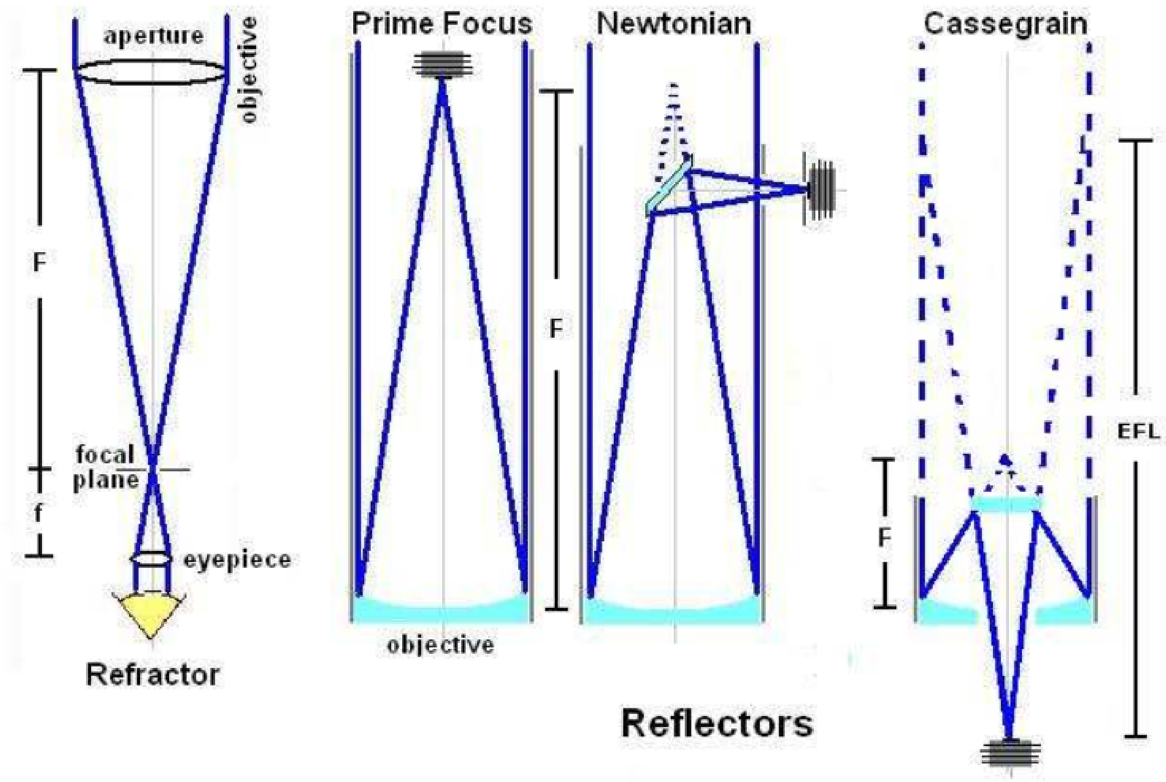
Telescope Focii

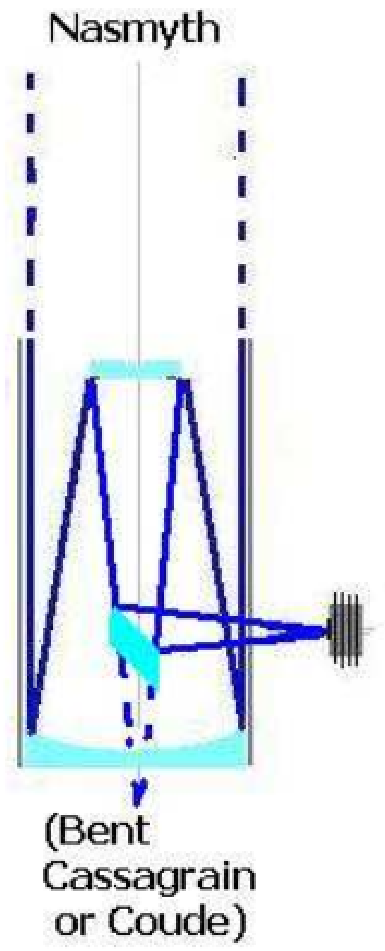
- **Prime Focus:** Put detector at the focal point. Has the advantage that the instrument will have a large field of view, and the fewest mirrors. This is good because each extra mirror leads to some loss of light.

  Since the prime focus lies directly above the mirror, a large, heavy instrument will block light and will have to move with the telescope. Only stars perfectly aligned with the telescope axis will be imaged in the prime focus.

- **Schmidt Focus:** Uses a spherical (as opposed to parabolic) mirror, which focuses the light after it has passed through a glass lens (called a corrector), with a camera in the prime focus. Allows for a very large field of view. This design is only used for relatively small telescopes

- **Cassegrain Focus:** Most small telescopes use this focus. Involves a parabolic primary and convex, hyperbolic secondary mirror. Design is compact, but has a long focal length. The eyepiece of a telescope is usually at the Cassegrain focus. However, the field of view in limited to 10–20 arcminutes.
- **Naysmith Focus:** Used for large, bulky instruments. This focii sits on a platform that rotates with the telescope and so is stable.
- **Coude Focus:** Used if very high stability is required. The light is folded down to a fixed (stationary) position in the observatory. The major disadvantage is that the field of view is small and the light needs to be folded through several mirrors, resulting in light losses





Telescope Mounts

- **Equatorial Mount:**
  - *Advantages:*
    - Only one axis must be controlled (RA)
    - Tracking rate is constant (360° per sidereal day)
    - Star field does not rotate with time
  - *Disadvantages:*
    - Large, bulky & Expensive
    - Gravity vector hard to predict
- **Alt-Az Mount:**
  - *Advantages:*
    - Simple and more compact to construct
    - Naysmith platform available
  - *Disadvantages:*
    - Non uniform tracking speed
    - Requires two axis to be controlled
    - Requires image derotator

Angular Resolution

The diffraction limited angular resolution of a telescope is:

$$\text{diffraction limited resolution} = 1.22\frac{\lambda}{D}$$

Where λ is the light wavelength and D is the telescope aperture.

Plate Scale

For small angles, the plate scale is given as

$$\text{plate scale} = \frac{u}{s} = \frac{1}{f}$$

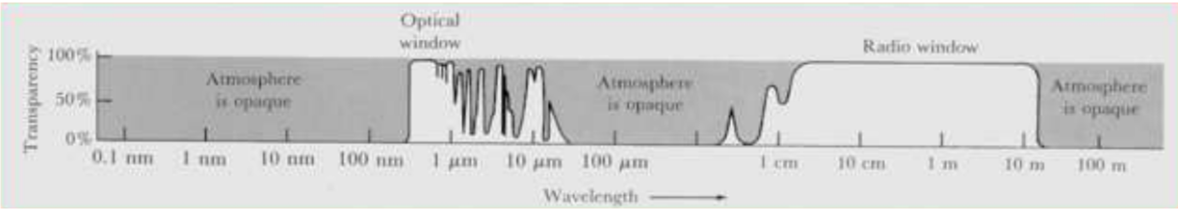
Focal Ratio or f-Number

The focal ratio or f-number of a telescope of diameter D and focal length f is:

$$f/\# = \frac{f}{D}$$

The Turbulent Atmosphere

Transparency



Observations can only be made in the optical and radio bands from the Earth's surface.

Dispersion

The amplitude of the spread of an object's position is given by:

$$A = (n_{\text{blue}} - n_{\text{red}}) \tan z_0$$

Where z_0 is the angular distance of the object from the zenith and n_{blue} and n_{red} are the refractive indices of the atmosphere in their respective wavelengths.

Atmospheric Light

Sources of atmospheric light include:

- fluorescent emissions called **air glow**
- **scattered light** e.g. from the Moon
- **light pollution** from the ground, satellites or aircraft
- **zodiacal light** scattered by interplanetary dust

Telescope Sites

- Clear nights
- Good seeing conditions
- Dark skies
- Little water vapour (good for IR)

Detectors

CCD Detectors

The quantum efficiency of a CCD is given by:

$$QE = \frac{\text{Number of electrons generated}}{\text{Number of incoming photons}}$$

Noise in Detectors

The gain of a detector, g, is an analogue to digital conversion and is measured in *ADU per electron*.

Some sources of noise include:

- **Sky background:** light pollution coming from the aforementioned atmospheric light sources
- **Cosmic rays:** External sources are referred to as cosmic rays, N_{ext}.
- **Sensitivity:** a, varies from pixel to pixel and is wavelength dependent
- **Bias:** The amplifier that boosts the signal also adds an offset, called bias, N_0
- **Dark current:** Pixels will build-up an electric charge generated by thermal motions, which will then be amplified and contribute to the counts, $gN_d t$.
- **Read-out noise:** the electronics add noise when reading out the CCD with mean, R, and variance, σ_R.

The number of photons, N_γ, number of electrons, N_e, and number of counts, N_c, can then be written as:

$$N_\gamma = (\dot{N}_\gamma(\text{source}) + \dot{N}_\gamma(\text{background}))t$$

$$N_e = aN_\gamma + N_{\text{ext}} + \dot{N}_d t$$

$$N_c = N_0 + f(gN_e) + gR$$

The registered counts may have an offset, be non-linear in the number of electrons (function f), and contain noise.

Detector Limits

- **Depth of potential well:** maximum number of electrons on pixel before it saturates
- **Resolution:** The number of counts is stored by the amplifier as a binary number with a given number of bits
- **Bandwidth:** the range of wavelengths the CCD is sensitive to

Photometry

Calibration and Detector Noise

- Readout noise is **Gaussian**
- Photon counting noise (from stars or galaxies) is **Poisson**
- Dark current noise is **Poisson**
- Bias noise is zero
- Sky background noise is **Poisson**

To reduce the image, we use the following equation:

$$\text{result} = \frac{N_c - (\text{dark} + \text{bias})}{\text{flat field}}$$

Signal to Noise Ratio

The SNR is given by:

$$\frac{S}{N} = \frac{\sum N_\gamma}{\sqrt{\sum(N_\gamma + N_d + N_b + R^2)}}$$

Alternatively, the SNR can written as:

$$\text{SNR} = \frac{S}{\sqrt{S + B + D + \sigma^2}}$$

Where S represents the signal, B the background, D the dark and σ^2 the readout.

Conversion to Magnitudes

The signal can be converted to a magnitude using a star of known magnitude, m_s and signal, S_s, by:

$$m = m_s - 2.5\log\left(\frac{S}{S_s}\right)$$

Spectroscopy

Resolving Power of a Diffraction Grating

Minimum distance between two lines which can be resolved is:

$$\Delta\lambda_{\text{min}} = \frac{\lambda}{nN}$$

For a diffraction grating with N slits at the nth order maximum.

The resolving power is hence defined as:

$$R = \frac{\lambda}{\Delta\lambda_{\text{min}}} = nN$$

Differentiating Snell's Law gives the angular dispersion:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{n}{d\cos\theta}$$

Then using the plate scale, $\mathrm{d}\theta/\mathrm{d}x$, the reciprocal linear dispersion is:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = \frac{1}{f_{\text{camera}}}\frac{d\cos\theta}{n}$$

Spectral Resolution Through a Finite Slit Width

$$R = \frac{n\rho\lambda W}{\chi D_T}$$

where n is the diffraction order, ρ is the line density, λ is the wavelength, W is the grating size, χ is the angular size of the image of a star on the slit (which is usually the seeing) and D_T is the telescope size.

Reflection Gratings

The equation for a reflection grating is given by:

$$n\lambda\rho = \sin\alpha + \sin\beta$$

Measuring Stars

Blackbody Radiation

The Wien distribution (for low temperatures) is given by:

$$I_\lambda(T) = \left(\frac{2hc^2}{\lambda^5}\right)e^{-hc/\lambda kT}$$

The Rayleigh-Jeans distribution (for high temperatures) is given by:

$$I_\lambda(T) = \frac{2ckT}{\lambda^4}$$

Where I_λ is the intensity of the black-body spectrum at wavelength, λ.

The Wien displacement law is:

$$\lambda_{\text{max}}T = 2.898\times10^{-3}\ \text{mK}$$

From this, the flux of the star is:

$$F = \sigma T^4$$

And the luminosity is:

$$L = 4\pi R^2 \sigma T^4$$

Stellar Distances

The distance of a star can be found its parallax angle, p, with:

$$d = \frac{206265}{p} \text{ AU}$$

Resolution of an Interferometer

For two telescopes separated by a distance, D, the angular resolution of the interferometer is:

$$\phi = \frac{\lambda}{D}$$

Stars

Note: unlike Obs Techniques, the Stars and Galaxies portions of this summary are based upon the respective equations lists the can be found on DUO. Where indicated on the equations list, a derivation will be supplied.

Production of Spectral Lines

The Boltzmann equation for spectral lines is given as:

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)kT}$$

Where N_a and N_b are the number atoms in states a and b respectively, g_a and g_b are statistical weights which account for degeneracy (for hydrogen, $g_n = 2n^2$) and $e^{-(E_b - E_a)kT}$ is what's known as the Boltzmann factor. E_a and E_b are the energies of the states a and b and T is the temperature of the system.

Kepler's Generalised Third Law

Kepler's third law for a binary system is given as:

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

Where P is the orbital period, a is sum of the semi-major axes of the orbiting bodies ($a = a_1 + a_2$), and m_1 and m_2 are the masses of the orbiting bodies.

This is a generalised form of Kepler's Third Law:

$$P^2 \propto a^3$$

Binary Star Mass Ratio Relationships

The masses, m_i, semi-major axes, a_i, angles, α_i, and velocities, v_i of two stars in a binary system are related by:

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1} = \frac{v_2}{v_1}$$

Stellar Mass-Luminosity Relationship

An approximate relationship between stellar mass and luminosity is given as:

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^\alpha$$

Where $\alpha \sim 3$ to 4.

Hydrostatic Equilibrium

The hydrostatic equilibrium equation is given as:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho$$

Derivation:

Take $P = P(r)$ as the pressure, M_r as the mass in a sphere of radius r, and $\rho = \rho(r)$ as the density.

The pressure force on a volume element is:

$$F_P = [P(r + \mathrm{d}r) - P(r)]\mathrm{d}A$$

Providing $\mathrm{d}r$ is small, we can write this as:

$$F_P = \mathrm{d}P\mathrm{d}A$$

The gravitational force on the volume element is:

$$F_g = \rho\mathrm{d}r\mathrm{d}Ag$$

Where for a spherical body:

$$g = -\frac{GM_r}{r^2}$$

The condition for hydrostatic equilibrium is that **the gravitational force equals the pressure force**, so we set $F_P = F_G$:

$$\mathrm{d}P\mathrm{d}A = -\frac{GM_r}{r^2}\rho\mathrm{d}r\mathrm{d}A$$

"Dividing" through by $\mathrm{d}r\mathrm{d}A$ gives:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho$$

Relating M_r to r

The relation of M_r to r is:

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho$$

(Differentiate the volume of a sphere with respect to r)

Central Core Pressure of a Star

The central core pressure of a star can be calculated using the equation:

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Derivation:

Integrating the M_r to r relation with constant density gives:

$$M_r = \frac{4}{3}\pi r^3 \rho$$

Substituting into the hydrostatic equilibrium equation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4}{3}\pi G \rho^2 r$$

Integrating again from $r = 0$ to r and $P = P_c$ to P gives:

$$P = P_c - \frac{2\pi}{3} G \rho^2 r^2$$

Since surface pressure is zero, we know that when $r = R$, $P = 0$. Substituting in and rearranging gives:

$$P_c = \frac{2\pi}{3} G \rho^2 R^2$$

Now, substituting for ρ:

$$\begin{aligned} P_c &= \frac{2\pi}{3} G R^2 \left(\frac{3M}{4\pi r^3} \right)^2 \\ &= \frac{3}{8\pi} \frac{GM^2}{R^4} \end{aligned}$$

Pressure Inside a Star

The pressure inside a star is:

$$P = P_{\text{gas}} + P_{\text{rad}} = nkT + aT^4$$

Where P_{gas} is the gas pressure, P_{rad} is the radiation pressure, n is the number of particles per m^{-3}, k is the Boltzmann constant, a is the radiation density constant ($7.57 \times 10^{-16}\ \mathrm{Jm^{-3}K^{-4}}$).

In general $P_{\text{rad}}/P_{\text{gas}} \sim 10^{-4}$.

Virial Theorem

The Virial Theorem is stated as:

$$K = -\frac{1}{2} U$$

Where K is the internal energy of the star, and U is the gravitational potential of the star.

Derivation:

Starting with the hydrostatic equilibrium and multiplying it by the volume of a sphere gives:

$$V \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2} \frac{4\pi r^3}{3}$$

Substituting in the equation for $\mathrm{d}M/\mathrm{d}r$:

$$V \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{3} \frac{GM}{r} \frac{\mathrm{d}M}{\mathrm{d}r}$$

Integrating:

$$\int_0^{P(R)} V \mathrm{d}P = -\frac{1}{3} \int_0^M \frac{GM}{r} \mathrm{d}M$$

The integral on the RHS will give the gravitational potential of the star, U.

Integrating the LHS by parts:

$$\int_0^{P(R)} V \mathrm{d}P = PV \Big|_0^R - \int_0^{V(R)} P \mathrm{d}V$$

Since $P(R) = 0$ and $V(0) = 0$ the first term on the RHS vanishes and so:

$$-3\int_0^{V(R)} P\mathrm{d}V = U$$

Since $\mathrm{d}V = \mathrm{d}m/\rho$:

$$-3\int_0^{M} \frac{P}{\rho}\mathrm{d}m = U$$

Assuming the ideal gas law:

$$P = nkT = \frac{\rho kT}{\mu m_{\mathrm{H}}}$$

And since the kinetic energy per particle is $\frac{3}{2}kT$:

$$E_{\mathrm{KE}} = \frac{3}{2}\frac{kT}{\mu m_{\mathrm{H}}} = \frac{3}{2}\frac{P}{\rho}$$

Is the kinetic energy per kilogram.

Then:

$$\int_0^{M} E_{\mathrm{KE}}\mathrm{d}m = -\frac{1}{2}U$$

Since the integral on the LHS is the internal energy, K, we get:

$$K = -\frac{1}{2}U$$

Energy from Gravitational Collapse

The energy of a gravitational collapse from radius R_{initial} to R is given as:

$$E = \frac{3GM^2}{10R}\left[\frac{1}{R} - \frac{1}{R_{\mathrm{initial}}}\right]$$

Derivation:

The gravitational potential of a point mass is:

$$\mathrm{d}U_{g,i} = -\frac{GM_r\mathrm{d}m_i}{r}$$

If we assume a shell of thickness $\mathrm{d}r$ and mass $\mathrm{d}m$ then

$$\mathrm{d}m = 4\pi r^2\rho\mathrm{d}r$$

and therefore:

$$\mathrm{d}U_g = -\frac{GM_r 4\pi r^2\rho}{r}\mathrm{d}r$$

Integrating over all shells, assuming a constant density:

$$U_g = -4\pi G\int_0^{R} M_r\rho r\mathrm{d}r$$

Where

$$M_r = \frac{4}{3}\pi r^3\bar{\rho}$$

Which gives:

$$U_g = -\frac{16\pi^2}{15} G\bar{\rho}^2 R^5$$

Converting from density back to mass gives:

$$U_g \sim -\frac{9}{15}\frac{GM^2}{R}$$

This is the total energy from the gravitational collapse.

Note: *the reason as to why the $=$ turns to a $\sim$ in the notes is unclear, as no approximations (besides the constant density assumption) are made between the last two steps.*

For a system in equilibrium, the virial theorem applies gives us:

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle \sim \frac{3GM^2}{10}\frac{1}{R}$$

We can then find the radiated energy as:

$$E \sim \frac{3GM^2}{10}\left[\frac{1}{R} - \frac{1}{R_{\text{initial}}}\right]$$

For $R \ll R_{\text{initial}}$:

$$E \sim \frac{3}{10}\frac{GM^2}{R}$$

Radiative Timescale

An approximation for the lifetime of a star can be made using the following:

$$t_{\text{lifetime}} \sim \frac{E}{L}$$

Where E is the energy release due to the process which powers the star, and L is the luminosity of the star.

Classical Temperature for Nuclear Reaction

The classical temperature for a nuclear reaction is given by:

$$T_{\text{classical}} = \frac{Z_1 Z_2 e_c^2}{6\pi\epsilon_0 kr}$$

Derivation:

Relate thermal energy to the Coulomb barrier energy:

$$\frac{3}{2}kT_{\text{classical}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e_c^2}{r}$$

Where Z_1 and Z_2 are the number of protons in each interacting particle 1 and 2, e_c is the elementary electrical charge and r is the distance of separation.

Rearranging gives:

$$T_{\text{classical}} = \frac{Z_1 Z_2 e_c^2}{6\pi\epsilon_0 kr}$$

Quantum Temperature for Nuclear Reaction

The quantum temperature for a nuclear reaction is given by:

$$T_{\text{quantum}} = \frac{Z_1 Z_2 e_c^4 \mu_m}{12\pi^2\epsilon_0^2 kh^2}$$

Where μ_m is the reduced mass and h is the Planck constant.

Derivation:

The kinetic energy equation can be written in terms of momentum and λ:

$$\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m} = \frac{h^2}{\lambda^2}\frac{1}{2\mu_m}$$

This can be done by using the de Broglie wavelength:

$$\lambda = \frac{h}{p}$$

We can relate the particle kinetic energy to the Coulomb barrier energy by:

$$\frac{h^2}{\lambda^2}\frac{1}{2\mu_m} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e_c^2}{r}$$

Then rearranging for λ:

$$\lambda = \frac{4\pi\epsilon_0 h^2}{Z_1 Z_2 e_c^2 2\mu_m}$$

Note: *for some godforsaken reason, the square on the λ has gone missing. The notes offer no explanation for this.*

Then. replacing r with λ in $T_{\text{classical}}$:

$$T_{\text{quantum}} = \frac{Z_1 Z_2 e_c^4 \mu_m}{12\pi^2\epsilon_0^2 k h^2}$$

Energy Release from Nuclear Reactions

The amount of energy released from a nuclear reaction per kilogram of material per second ($\mathrm{W\,kg^{-1}}$) can be found as:

$$\epsilon_{ix} = \epsilon_0' X_i X_x \rho^{\alpha'-1} T^{\beta}$$

Where ϵ_0' is the amount of energy released per reaction, X_i and X_x are the mass fractions of the particles, ρ is the density, T is the temperature and α' and β are determined from the power-law expansion of the reaction-rate equations. $\alpha' = 2$ for a two-body collision.

Energy Conservation

The equation for energy conservation in a star is:

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon_{ix}$$

Where ϵ_{ix} is the amount of energy released from a nuclear reaction per kilogram of material per second.

Mean Free Path

The mean free path of an photon (average distance travelled before one collision or interaction) is:

$$\ell = \frac{1}{n\sigma}$$

Where n is the number of atoms per unit volume and σ is the collision cross-section of the atom.

Number of Scattering Events

For a random walk of photons, the number of scattering events for a photon to cover the distance d is given as:

$$N = \left(\frac{d}{\ell}\right)^2$$

Where ℓ is the mean free path.

Temperature Gradient in a Star

The temperature gradient of a star can be expressed as:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho F_{\mathrm{rad}}}{T^3} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{T^3}\frac{L_r}{r^2}$$

Where κ is the opacity, ρ is the gas density, F_{rad} is the radiative flux, a is the radiation density constant and L_r is the luminosity.

Derivation:

Taking the expression for radiation pressure:

$$P_{\mathrm{rad}} = \frac{1}{3}aT^4$$

And differentiating:

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = \frac{4aT^3}{r}\frac{\mathrm{d}T}{\mathrm{d}r}$$

The pressure differential can also be expressed as:

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\kappa\rho F_{\mathrm{rad}}}{c}$$

Combining the two equations allows us to determine a temperature differential:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho F_{\mathrm{rad}}}{T^3}$$

Or in luminosity units:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{T^3}\frac{L_r}{r^2}$$

Optical Depth

The optical depth or impact of opacity on intensity is given as:

$$I_\lambda = I_{\lambda,0}e^{-\kappa_\lambda \rho s}$$

Where I_λ is the photon intensity at wavelength λ, κ_λ is the opacity at that wavelength, ρ is the gas density and s is the distance.

Derivation:

The change in intensity over the distance $\mathrm{d}s$ is calculated from the opacity and the gas density as:

$$\mathrm{d}I_\lambda = -\kappa_\lambda \rho I_\lambda \mathrm{d}s$$

To determine the final intensity of the beam of photons, integrate through the column density of gas:

$$\int_{I_{\lambda,0}}^{I_{\lambda,f}} \frac{\mathrm{d}I_\lambda}{I_\lambda} = -\int_0^s \kappa_\lambda \rho \mathrm{d}s$$

The observed intensity will therefore be:

$$I_\lambda = I_{\lambda,0}e^{-\int_0^s \kappa_\lambda \rho \mathrm{d}s}$$

Which for the case of a uniform gas density is:

$$I_\lambda = I_{\lambda,0}e^{-\kappa_\lambda \rho s}$$

General Form of Opacity

The general form for the opacity of a gas is:

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

Where α and β are arbitrary constants to be determined from a log-log graph.

Condition for Convection to Occur

In order for convection to occur, the following condition must be satisfied:

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{sur}} > \left(\frac{\gamma_{\mathrm{ad}} - 1}{\gamma_{\mathrm{ad}}}\right)\frac{T}{P}\left|\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{\mathrm{sur}}$$

Where γ is the ratio of specific heats.

Derivation:

The adiabatic gas law for gas pressure in a mass element, Δm, is

$$P = K_a \rho^\gamma$$

Where K_a is a constant.

The ideal gas law for the surrounding gas is defined as:

$$P = \frac{\rho k T}{\mu m_H}$$

So:

$$P \propto \rho T$$

And therefore:

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T}$$

So the logarithmic derivative of the ideal gas equation is:

$$\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}P}{P} - \frac{\mathrm{d}T}{T}$$

The adiabatic gas law can written as:

$$P \propto \rho^\gamma$$

And therefore:

$$\frac{\mathrm{d}P}{P} = \gamma \frac{\mathrm{d}\rho}{\rho}$$

And so:

$$\gamma = \frac{\rho}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)$$

Unstable against convection (convection prone to occur) requires:

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{\mathrm{sur}} > \left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{\mathrm{ad}}$$

Where 'sur' means the surrounding gas in the star and 'ad' means the mass element.

Multiply by ρ/P on both sides:

$$\frac{\rho}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{\mathrm{sur}} > \frac{\rho}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{\mathrm{ad}}$$

And equate to the specific heat ratio for the adiabatic component:

$$\frac{\rho}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{\mathrm{sur}} > \gamma_{\mathrm{ad}}$$

Or:

$$\frac{P}{\mathrm{d}P}\left(\frac{\mathrm{d}\rho}{\rho}\right) < \frac{1}{\gamma_{\mathrm{ad}}}$$

Replace $\mathrm{d}\rho/\rho$ with the logarithmic derivative of the ideal gas equation:

$$\frac{P}{\mathrm{d}P}\left(\frac{\mathrm{d}P}{P} - \frac{\mathrm{d}T}{T}\right) < \frac{1}{\gamma_{\mathrm{ad}}}$$

Expand and show the condition where convection is prone to occur:

$$\frac{T}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)_{\mathrm{sur}} < \frac{\gamma_{\mathrm{ad}}}{\gamma_{\mathrm{ad}} - 1}$$

Dividing by $\mathrm{d}r$ and rearranging for the temperature gradient gives:

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{sur}} > \left(\frac{\gamma_{\mathrm{ad}} - 1}{\gamma_{\mathrm{ad}}}\right)\frac{T}{P}\left|\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{\mathrm{sur}}$$

Convection Mixing Length

The convection mixing length is given as:

$$\ell = \alpha H_P$$

Where H_P is the scale height and α is a constant typically found to be ~ 0.5 to 3.

Maximum Mass of Stars

The maximum mass of a star is given by:

$$\frac{M_{\mathrm{max}}}{M_\odot} = \sqrt[\alpha-1]{\frac{4\pi c G M_\odot}{\kappa L_\odot}}$$

Derivation:

An upper limit to the mass of stars can be place from the violation of hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2}$$

Which will occur if the internal pressure exceeds the gravitational force. This happens when the photon pressure on the gas exceeds the gravitational force:

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\kappa\rho F_{\mathrm{rad}}}{c} = -\frac{\kappa\rho L_r}{4\pi r^2 c}$$

Equating the two equations gives the maximum luminosity before hydrostatic equilibrium is violated, the **Eddington luminosity**:

$$L_r = \frac{4\pi c G M_r}{\kappa} = L_{\mathrm{Edd}}$$

This is the point at which radiation pressure equals the gravitational force.

We know that:

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^\alpha$$

Where $\alpha \sim 3$ to 4.

Calibrating to the mass and luminosity of the Sun:

$$\frac{L_{\text{Edd}}}{L_\odot} = \frac{4\pi c G M_\odot}{\kappa L_\odot}\frac{M}{M_\odot}$$

And plugging into the luminosity-mass relationship we get:

$$\frac{M_{\text{max}}}{M_\odot} = \sqrt[\alpha-1]{\frac{4\pi c G M_\odot}{\kappa L_\odot}}$$

Stellar Pulsation Period

The pulsation period of a variable star is given by

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

Where γ is the ratio of the specific heats and ρ is the gas density.

Derivation:

Using an adiabatic approximation, the speed of sound in the star is:

$$v_s = \sqrt{\frac{\gamma P}{\rho}}$$

To find the pressure, use the equation of hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2}$$

Assuming the star has constant density:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)\rho}{r^2} = -\frac{4}{3}\pi G\rho^2 r$$

Integrating with the boundary condition that $P(R) = 0$ at the surface:

$$P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$$

The pulsation period will be approximately the distance divided by the speed, or:

$$\begin{aligned}
\Pi &\approx 2\int_0^R \frac{\mathrm{d}r}{v_s} \\
&\approx 2\int_0^R \frac{\mathrm{d}r}{\sqrt{\frac{2}{3}\gamma\pi G\rho\left(R^2 - r^2\right)}} \\
&\approx 2\sqrt{\frac{3}{2\gamma\pi G\rho}}\int_0^R \frac{\mathrm{d}r}{\sqrt{R^2 - r^2}} \\
&\approx 2\sqrt{\frac{3}{2\gamma\pi G\rho}}\cdot \sin^{-1}\left(\frac{r}{R}\right)\Big|_0^R
\end{aligned}$$

So:

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

Condition Required for Gravitational Collapse

The minimum mass required for gravitational collapse, or Jeans mass, is:

$$M_J \cong \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}$$

Where μ is the reduced mass, is the mass of a Hydrogen atom and ρ_0 is the initial density.

Derivation:

Assuming a spherical cloud of constant density, the gravitational potential energy is:

$$U \sim -\frac{3}{5}\frac{GM_c^2}{R_c}$$

The cloud's kinetic energy is:

$$K = \frac{3}{2}NkT$$

Where:

$$N = \frac{M_c}{\mu m_H}$$

Using the virial theorem, $2K < |U|$:

$$\frac{3M_c kT}{\mu m_H} < \frac{3}{5}\frac{GM_c^2}{R_c}$$

Since constant density was assumed, R_c can be replaced with the initial density of the cloud, ρ_0:

$$R_c = \left(\frac{3M_c}{4\pi\rho_0}\right)^{\frac{1}{3}}$$

Subbing back into the original equation, we find that the condition for spontaneous collapse is the Jeans criterion:

$$M_c > M_J$$

Where

$$M_J \cong \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}$$

Nuclear Fusion Lifetime

The lifetime that a star undergoes nuclear fusion for a given fusion process is:

$$t = \frac{X\xi Mc^2}{L}$$

Where X is the fraction of the mass in the star that will be used in the fusion process, ξ is the mass-to-light efficiency conversion, M is the mass of the star, and L is the luminosity of the star.

Mass-dependent Main Sequence Lifetime

The time that a star will stay on the main sequence is:

$$t = 10^{10}\left(\frac{M_\odot}{M}\right)^{\alpha-1} \text{ years}$$

Where $\alpha \sim 3$ to 4.

Electron-Degeneracy Pressure

The electron degeneracy pressure in a white dwarf is given by:

$$P = \frac{\hbar^2}{m_e}\left[\frac{Z}{A}\frac{\rho}{m_H}\right]^{\frac{5}{3}}$$

Where Z is the number of electrons, and A is the number of nucleons.

Derivation:

The pressure at the centre of a white dwarf can be calculated from:

$$P \sim \frac{1}{3}n_e p v$$

Where p is the momentum and v is the velocity.

The electron number density is:

$$n_e = \frac{Z}{A}\frac{\rho}{m_H} = \frac{\text{no. electrons}}{\text{no. nucleons}}\frac{\text{no. nucleons}}{\text{volume}}$$

The uncertainty in the position is therefore:

$$\Delta x \sim n_e^{-\frac{1}{3}}$$

And hence

$$p_x \sim \hbar n_e^{\frac{1}{3}}$$

Since $p_x \sim \hbar/\Delta x$.

However, in 3 dimensions each direction is equally likely and so:

$$p^2 = p_x^2 + p_y^2 + p_z^2 = 3p_x^2$$

And therefore

$$p = \sqrt{3}p_x$$

Rewriting the pressure equation $P \sim \frac{1}{3}n_e p v$ in terms of electrons:

$$p \sim \sqrt{3}\hbar\left[\frac{Z}{A}\frac{\rho}{m_H}\right]^{\frac{1}{3}}$$

If the electrons are non-relativistic then $p = v m_e$ so:

$$v \sim \frac{\sqrt{3}\hbar}{m_e}\left[\frac{Z}{A}\frac{\rho}{m_H}\right]^{\frac{1}{3}}$$

Hence the pressure will be:

$$P = \frac{\hbar^2}{m_e}\left[\frac{Z}{A}\frac{\rho}{m_H}\right]^{\frac{5}{3}}$$

Period-density Relationship

The minimum period of a star is given as:

$$P_{\text{min}} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$$

Derivation:

The maximum angular velocity can be found by equating the gravitational and centripetal accelerations at the equator:

$$\omega_{\max}^2 R = \frac{GM}{R^2}$$

Assuming uniform density, then:

$$\omega_{\max}^2 R = \frac{G}{R^2}\rho\frac{4}{3}\pi R^3$$

Since $\omega_{\max} = 2\pi/P_{\min}$:

$$\frac{4\pi^2 R}{P_{\min}^2} = \frac{4}{3}G\rho\pi R$$

Rearranging:

$$P_{\min} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$$

Schwarzschild Radius

The Schwarzschild radius of a black hole is:

$$R_S = \frac{2GM}{c^2} = 2.96\left(\frac{M}{M_\odot}\right)\ \mathrm{km}$$

Galaxies

Surface Brightness

The surface brightness of a galaxy, the flux received per unit solid angle, is given by:

$$\frac{\mathrm{d}F}{\mathrm{d}\Omega} = \frac{I}{4\pi}$$

Where F is the flux, Ω is the solid angle ($\mathrm{d}\Omega = \mathrm{d}S/d^2$ is the solid angle the surface area $\mathrm{d}S$ extends on the sky) and I is the intensity or luminosity per unit area ($I = \mathrm{d}L/\mathrm{d}S = \sigma L$ where σ is the surface density in stars per unit area). **Surface brightness is independent of distance.**

Absorption

The decrease in intensity due to the presence of intergalactic dust is according to:

$$\frac{\mathrm{d}I}{\mathrm{d}r} = -AI$$

Where A is a constant which is dependent on the number density of dust grains and their size.

Magnitude-Absorption Relation

The relation between apparent and absolute magnitudes and absorption is:

$$m - M = 5\log(r) - 5 + \hat{A}r$$

Derivation:

Solving the absorption equation gives:

$$I(r) = I_0 \exp(-Ar)$$

So in terms of magnitudes:

$$
\begin{aligned}
\Delta m &= -2.5\log\left(\frac{I}{I_0}\right) \\
&= -2.5\log(\exp(-Ar)) \\
&= (2.5\log e)Ar \\
&= \hat{A}r
\end{aligned}
$$

Where $\hat{A} = (2.5\log e)A$ and has units of magnitude per unit length.

The relation between apparent and absolute magnitudes then becomes:

$$m - M = 5\log(r) - 5 + \hat{A}r$$

Density Distribution of Stars in a Disc

The density distribution of stars in an disc in cylindrical coordinates is:

$$n(R, \phi, z) \propto \exp\left(-\frac{R}{R_h}\right)\exp\left(-\frac{|z|}{z_h}\right)$$

Where R_h is the scale-length and z_h is the scale-height.

Luminosity Profile of the Bulge $\circ\omega\circ$

The luminosity of a bulge of the Milky Way is a *de Vaucouleurs* or $r^{1/4}$ profile:

$$I(r) = I_e \exp[-7.67(r/r_e)^{1/4} - 1]$$

Density of the Stellar Halo

The number density of halo stars and globular clusters is:

$$n(r) = n_0\left(\frac{r}{r_0}\right)^{-3.5}$$

Metallicity of a Star

The metallicity of a star is defined as:

$$[\mathrm{Fe/H}] \equiv \log_{10}\left[\frac{M_{\mathrm{Fe}}/M_{\mathrm{H}}}{(M_{\mathrm{Fe}}/M_{\mathrm{H}})_{\odot}}\right]$$

Where $M_{\mathrm{Fe}}/M_{\mathrm{H}}$ is the ratio of iron to hydrogen by mass.

Speed of the Ionisation Front

The speed of the ionisation front in a gas cloud is:

$$\dot{R}(t) = \frac{\dot{N}_\gamma}{4\pi n R^2}$$

Where $\dot{N}_\gamma$ is the number ionising photons emitted per second.

Derivation:

The radius of the front at time t, $R(t)$, follows from requiring each of the $(4\pi/3)R^3 n$ hydrogen atoms (the number of atoms in a sphere of radius R) has interacted with a photon. Since the number of photons emitted in time t is $\dot{N}_\gamma t$, this results in

$$\dot{N}_\gamma t = \frac{4\pi}{3} n R(t)^3$$

Taking the time derivative equation and rearranging gives:

$$\dot{R}(t) = \frac{\dot{N}_\gamma}{4\pi n R^2}$$

Keplerian Circular Orbit Velocity

The velocity of a cluster in circular motion around a galaxy at distance R is:

$$\frac{V_C^2}{R} = \frac{GM(< R)}{R^2}$$

Oort's Constants

The line-of-sight velocity of a cluster is:

$$V_r = Ad\sin(2l)$$

Where A is one of Oort's constants.

Derivation:

The rotation curve V(R) can be inferred by measuring $V_r(d, l)$ as follows. First, use trigonometry to show that

$$V_r = V\cos(\alpha) - V_0\sin(l)$$

Again, using trigonometry in the indicated right-angled triangle,

$$\begin{aligned} d + R\sin(\alpha) &= R_0\cos(l) \\ R\cos(\alpha) &= R_0\sin(l) \\ R_0 &= d\cos(l) + R\cos(\beta) \approx d\cos(l) + R \end{aligned}$$

When $d \ll R_0$ and β is the angle Sun - Milky Way centre - Star. The last step assumes that $\beta \approx 0$.

We the find that

$$V_r = (\Omega - \Omega_0)R_0\sin(l)$$

Where $\Omega_0 \equiv V_0/R_0$ is the angular velocity of the Sun and $\Omega \equiv V/R$ is the angular velocity of the star. For nearby stars, we can expand $\Omega(R)$ as a Taylor series about $R = R_0$, keeping only the first terms:

$$\Omega(R) \approx \Omega(R_0) + \left.\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right|_{R=R_0}$$

Notice that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{V}{R}\right) = \frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}R} - \frac{V}{R^2}$$

Define Oort's constant A by:

$$A \equiv -\frac{1}{2}\left[\left.\frac{\mathrm{d}V}{\mathrm{d}R}\right|_{R=R_0} - \frac{V_0}{R_0}\right] \approx 14.4 \pm 1.2 \text{ km s}^{-1}\text{ kpc}^{-1}$$

We can then find that

$$V_r = Ad\sin(2l)$$

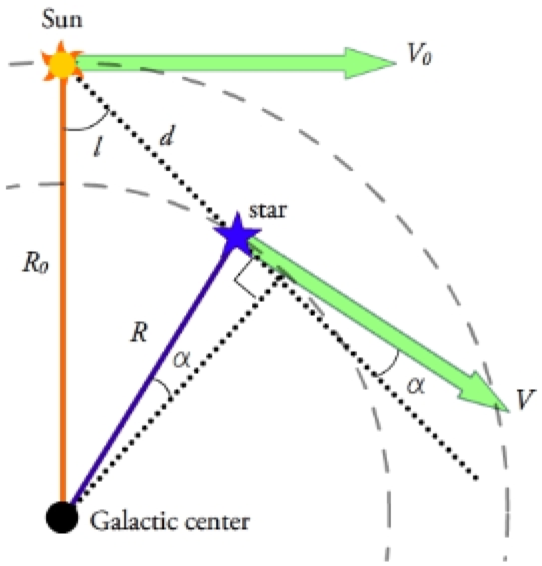


Figure 5.1: (Taken from wikipedia) The observer at the Sun is moving on a circular orbit with velocity V_0 and radius R_0. The observed star is at distance d and has galactic longitude l (with $b = 0$ since it is in the disc). The star is on a circular orbit with radius R and speed V.

Density Distribution for a Flat Rotation Curve

The density distribution for a flat rotation curve is given by:

$$\rho(R) = \frac{V^2}{4\pi G R^2}$$

Derivation:

For a flat rotation curve, $V \sim \text{const}$ we first the derivative with respect to R of:

$$V^2 R = GM$$

Since V is constant:

$$V^2 = G\frac{\mathrm{d}M}{\mathrm{d}R}$$

We can now use that $\mathrm{d}M/\mathrm{d}v = \rho(R)$ (where v represents volume) to get:

$$V^2 = G\rho(R)\frac{\mathrm{d}v}{\mathrm{d}R}$$

For a sphere:

$$\frac{\mathrm{d}v}{\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2$$

Hence, substituting into the equation and rearranging gives:

$$\rho(R) = \frac{V^2}{4\pi G R^2}$$

Escape Speed - Circular Speed Relation

The relation of escape speed to circular speed in a dark matter halo of radius R_h is:

$$v_e^2 = 2V_c^2[1 + \ln(R_h/R)]$$

Derivation:

For $R < R_h$ the mass distribution with a flat rotation curve is:

$$M(R) = \frac{V_c^2 R}{G}$$

The gradient of the gravitational potential is the force per unit mass:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = \frac{V_c^2}{R} = \frac{GM}{R^2}$$

Integrating this between $R \leq R_h$ and R_h with the boundary conditions $\Phi(R_h) = -GM/R_h = -V_c^2$ we get:

$$\Phi(R) = -v_c^2[1 + \ln(R_h/R)]$$

We know that

$$0 = E = \frac{1}{2}v_e^2 + \Phi$$

Where v_e is the escape speed.

Substituting and rearranging we get:

$$v_e^2 = 2V_c^2[1 + \ln(R_h/R)]$$

Local Group Timing Argument

A relation between the velocity of Andromeda, v, the time since the Big Bang, t, and the distance between the Milky Way and Andromeda, r, can be found as:

$$\frac{vt}{r} = \frac{\sin\theta(\theta - \sin\theta)}{(1 - \cos\theta)^2}$$

Where θ is an arbitrary parameter.

Derivation:

Assuming the orbit to be radial and Andromeda and the Milky Way to be point masses, we have:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM_{\text{total}}}{r^2}$$

Where $r(t = 0) = 0$.

The solution can be written in parametric form as:

$$r = \frac{R_{\max}}{2}(1 - \cos\theta)$$

$$t = \left(\frac{R_{\max}^3}{8GM_{\text{total}}}\right)^{1/2}(\theta - \sin\theta)$$

The relative velocity can be found from taking the derivative and using the chain rule:

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta} \div \frac{\mathrm{d}t}{\mathrm{d}\theta} = \left(\frac{2GM_{\text{total}}}{R_{\max}}\right)^{1/2}\left(\frac{\sin\theta}{1 - \cos\theta}\right)$$

Combining the last three equations eliminates $R_{\max}$, G and M_{total} to give:

$$\frac{vt}{r} = \frac{\sin\theta(\theta - \sin\theta)}{(1 - \cos\theta)^2}$$

Note: *equations 16 to 18 in Theuns' list don't appear in his notes or lecture slides and so have been omitted*

Mass-Velocity Dispersion Relation

The relation of the mass of a galaxy cluster to its velocity dispersion is:

$$M = \frac{\sigma^2 R}{G}$$

Derivation:

Assuming that all cluster galaxies have the same mass, m, and there are N galaxies in the cluster, the kinetic energy of the system is:

$$K = \frac{1}{2} \sum mv^2 = \frac{1}{2} M\sigma^2$$

Where $M = Nm$ is the total mass of the cluster, and the velocity dispersion is defined by:

$$\sigma^2 = \frac{\sum mv^2}{M}$$

The potential energy in the system is of the order:

$$U \sim -\frac{GM^2}{R}$$

Where R is a measure of the size of the cluster.

Using the virial theorem (when the system is in virial equilibrium), $2K = |U|$:

$$M\sigma^2 \sim \frac{GM^2}{R}$$

Rearranging:

$$M \sim \frac{\sigma^2 R}{G}$$

Temperature of a Cluster

The temperature of a cluster is given by:

$$kT = \frac{2}{3} \frac{\mu m_p GM}{R}$$

Derivation:

Since energy is conserved along the orbit of a parcel of gas:

$$0 = E = \frac{1}{2} v^2 - \frac{GM}{R}$$

The gas then converts its kinetic energy into thermal energy, and so gets heated to a temperature, T:

$$\frac{1}{2} mv^2 = \frac{3}{2} m \frac{kT}{\mu m_p}$$

Where μm_p is the mean molecular weight per particle. Combining the last two equations gives:

$$kT = \frac{2}{3} \frac{\mu m_p GM}{R}$$

Tully-Fisher Relation

For **spiral galaxies**, the Tully-Fisher relation is given by:

$$L \propto V_c^4$$

Where L is the luminosity and V_c is the circular speed.

Faber-Jackson Relation

For **elliptical galaxies**, the Faber-Jackson relation is given by:

$$L \propto \sigma^4$$

Where L is the luminosity and σ is the velocity dispersion.

Radiative Efficiency of an AGN

The radiative efficiency of an AGN is given by:

$$\epsilon = \frac{L}{\dot{M}c^2}$$

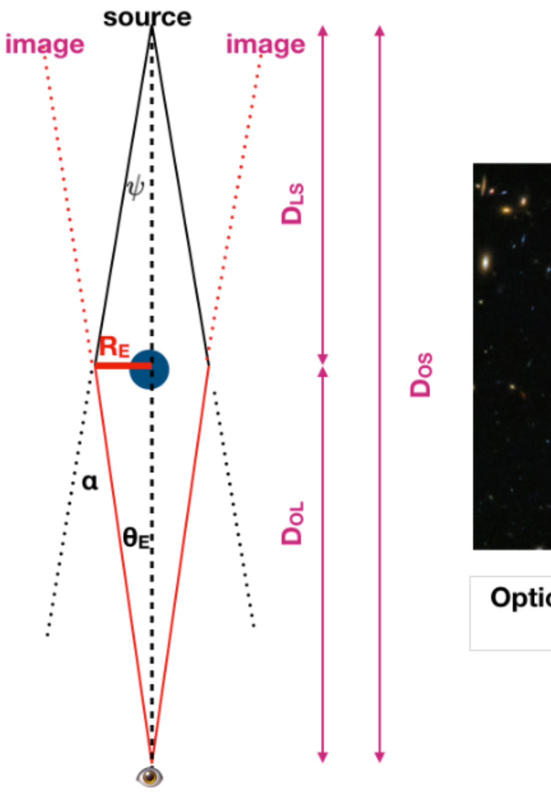
Where L is the luminosity and $\dot{M}$ is the rate of accretion of mass onto the AGN.

Einstein Radius

The (angular) radius of an Einstein lens is given by:

$$\alpha = \frac{4GM}{bc^2}$$

Where b is the impact parameter.



Optical Einstein ring around galaxy cluster SDSS J0146-0929. (HST)