Theoretical Physics 2 - Summary

Classical Mechanics

Generalised Description of Mechanical Systems

Dynamical Variables

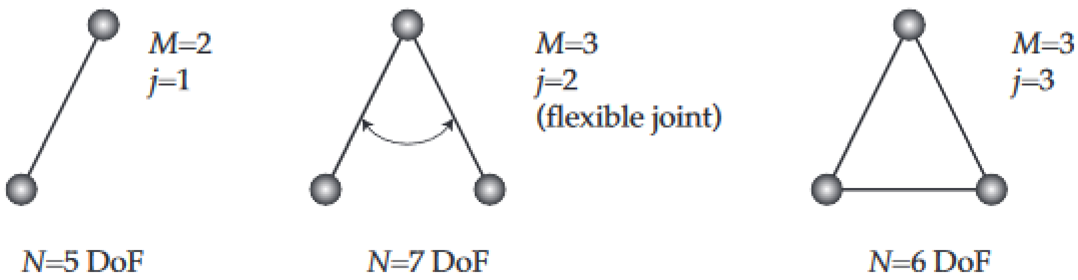
These are any set of variables that completely describe the configuration of a mechanical system. In 3D space possible sets include (x, y, z), (r, ϕ, z) and (r, θ, ϕ).

The **equation of motion** of a system specifies the dynamical variables as functions of time.

Degrees of Freedom (DoF)

The motion of a point mass $\mathbf{r}(t)$ is described by three *independent* dynamical variables, i.e. the point mass has $N = 3$ DoF.

A system of M point masses has $N = 3M$ DoF, but the existence of j *independent* constraints reduced this number to $N = 3M - j$ DoF.



Types of constraints

If constraints on a system of M point masses can be expressed in the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M, t) = 0$$

Then the constraints are called **holonomic**:

- **Rheonomic** constraints have an explicit time dependence.
- **Scleronomic** constraints do no, i,e, $f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M) = 0$

Nonholonomic constraints involve either differential equations or inequalities (rather than algebraic equations).

Generalised Coordinates

The existence of j constraints means that the coordinates are no longer independent.

We require only as many generalised coordinates q_k as there are DoF.

If the constraints are holonomic, then the position of the ith part of the system can be expressed as:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_k, \ldots, q_N, t)$$

The generalised velocities are:

$$\dot{q}_k = \frac{\mathrm{d}q_k}{\mathrm{d}t}$$

q_k and $\dot{q}_k$ are independent variables, so

$$\frac{\partial q_k}{\partial \dot{q}_k} = \frac{\partial \dot{q}_k}{\partial q_k} = 0$$

By the chain rule, we can derive the following:

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \implies \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k}$$

This last step is known as "cancelling the dots".

The Lagrangian

Lagrange's Formulation of Mechanics

Make three assumptions:

1. Holonomic constants
2. Constraining forces do no virtual work
3. Applied forces are conservative such that a scalar potential energy function exists

D'Alembert's Principle

Newton's 2nd Law for particle i is:

$$\dot{\mathbf{p}}_i = \mathbf{F}_i$$

Implying that:

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Where $\delta \mathbf{r}_i$ represents an arbitrary virtual displacement (a hypothetical change of coordinates at one instant in time that is compatible with the constraints).

The force includes both the constraint and applied forces such that:

$$\mathbf{F}_i = \mathbf{F}_i^{(c)} + \mathbf{F}_i^{(a)}$$

But since the constraining forces do no virtual work, D'Alembert's Principle is given as:

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Generalised Equations of Motion

The Lagrangian is defined as:

$$\mathcal{L} = T - V$$

The Euler-Lagrange equations are then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

Where $\mathcal{L} = \mathcal{L}(q_k, \dot{q}_k, t)$.

Ignorable Coordinates

If the time derivative of a coordinate appears in the Lagrangian, but the coordinate itself does not, then this is an ignorable coordinates. Since

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0$$

the Euler-Lagrange equation implies that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = 0$$

Thus, the **canonically conjugate momentum** to the coordinates, defined as

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$$

is a constant of the motion.

Variational Calculus

Calculus of Variations

To find the path $y(x)$ which yields the extreme value of the functional:

$$I[y] = \int_{x_1}^{x_2} f(y, y', x)\mathrm{d}x$$

where $y' = \mathrm{d}y/\mathrm{d}x$, the Euler-Lagrange equation must be used:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

Hamilton's Principle of Least Action

The integral

$$S[\mathbf{q}(t)] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\mathrm{d}t$$

Is known as the **action functional**. Hamilton's Principle states that a mechanical system moves in such a way as to minimise its action.

Hamilton's Principle can written as $\delta S = 0$, where the δ represents a change with respect to the path between the end points of the integral.

The variational derivative of $\mathcal{L}$ with respect to q is the Euler-Lagrange equation.

Linear Oscillators

Equilibrium

A mechanical system that remains at rest is in equilibrium. This occurs at points in configuration space where all generalise forces $\mathcal{F}_k$ vanish.

For a conservative system, this corresponds to configurations where the potential energy $V(q_1, \ldots, q_N)$ is stationary.

Simple Harmonic Oscillator (SHO)

The Lagrangian for a simple harmonic oscillator can be written as:

$$\mathcal{L} = \frac{1}{2}m\dot{q} - \frac{1}{2}kq^2$$

$\ddot{q} + \omega^2 q = 0$ where $\omega = \sqrt{k/m}$ is a linear, homogenous differential equation and can be solved to give:

$$q(t) = q(0)\cos(\omega t) + \frac{\dot{q}(0)}{\omega}\sin(\omega t)$$

Damping Force

A frictional force F_d acts to suppress motion.

The simplest such force is

$$F_d = -\gamma\dot{q} = -\frac{m\omega}{Q}\dot{q}$$

Where Q is the dimensionless **quality factor** of the oscillator.

Damped Simple Harmonic Oscillator

The (non-conservative) damping force is incorporated into the Euler-Lagrange equation like so:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} = F_d$$

The result is a linear homogenous differential equation:

$$\ddot{q} + \frac{\omega}{Q}\dot{q} + \omega^2 q = 0$$

Driven Oscillators

Oscillator Driven by an External Force

Considering an undamped oscillator driven by a time-dependent external force $F(t)$ with no spatial dependence (where the driving force is not part of the dynamical system), the corresponding Lagrangian is given by:

$$\mathcal{L} = T - V = \frac{m\dot{q}^2}{2} - \left[\frac{m\omega^2 q^2}{2} - F(t)q\right]$$

Which yields the second order linear inhomogenous differential equation:

$$\ddot{q} + \omega^2 q = \frac{F(t)}{m}$$

Dirac δ-functions

To express an **impulsive force** mathematically we use Dirac δ-functions.

The effect of $\delta(t - t')$ is defined as:

$$f(t') = \int_{-\infty}^{\infty} \delta(t - t')f(t)\mathrm{d}t$$

Where $\delta(t - t')$ has units of inverse time. We can define a single impulsive force at time t' as

$$F(t) = K\delta(t - t')$$

Where K is the total impulse provided.

Response of an Oscillator to a q-Independent Impulsive Force

With $F(t) = K\delta(t - t')$, integrating the differential equation for the driven oscillator around $t = t'$ yields:

$$\int_{t'-\epsilon}^{t'+\epsilon} (\ddot{q} + \omega^2 q)\mathrm{d}t = \int_{t'-\epsilon}^{t'+\epsilon} \frac{K}{m}\delta(t-t')\mathrm{d}t$$

$$\Rightarrow \dot{q}(t'+\epsilon) - \dot{q} + \omega^2 \int_{t'-\epsilon}^{t'+\epsilon} q(t)\mathrm{d}t = \frac{K}{m}$$

Where the integral in blue tends to zero as $\epsilon \to 0$ unless $q(t') \to \infty$.

Hence there is an instantaneous change in the velocity:

$$\dot{q}(t'_+) = \dot{q}(t'_-) + \frac{K}{m}$$

Where

$$t'_\pm = \lim_{\epsilon \to 0}(t' \pm \epsilon)$$

Note that for a finite velocity:

$$q(t'_+) = q(t'_-)$$

i.e. there is no instantaneous change in position.

Time-Evolution Sequence of an SHO with an Impulsive Force

1. $t < t'$: ($F(t) = 0$) system evolves as an SHO to position $q(t'_-)$, velocity $\dot{q}(t'_-)$
2. $t = t'$: ($F(t) \neq 0$) instantaneous jump in velocity
3. $t > t'$: ($F(t) = 0$) system evolves as an SHO:

$$q(t) = q(t'_-)\cos(\omega[t-t']) + \frac{\dot{q}(t'_-) + K/m}{\omega}\sin(\omega[t-t'])$$

Causal Green's Function G

The Green's function for this system is the solution to the differential equation:

$$\ddot{G}(t-t') + \omega^2 G(t-t') = \delta(t-t')$$

And represents the response of the oscillator to a single unit-sized impulsive force, scaled as

$$G(t-t') = q(t)\frac{m}{K}$$

So that the differential equation only has a δ-function on the right-hand side.

Considering the oscillator to be at rest prior to the application of the impulsive force, the solution for G is:

For $t - t' \leq 0$:

$$G(t-t') = 0$$

For $t - t' \geq 0$:

$$G(t-t') = \frac{1}{\omega}\sin(\omega[t-t'])$$

General Driving Force $F(t)$

Using the definition of the δ-function, the equation of motion for the driven oscillator can be written as:

$$\ddot{q} + \omega^2 q = \frac{1}{m}\int_{-\infty}^{\infty} \mathrm{d}t'\ F(t')\delta(t-t')$$

Substituting in the differential equation for the Green's function G:

$$\ddot{q}+\omega^2 q=\frac{1}{m}\int_{-\infty}^{\infty}\mathrm{d}t'\ F(t')[\ddot{G}(t-t')+\omega^2 G(t-t')]$$

Which can be written as:

$$\ddot{q}+\omega^2 q=\frac{1}{m}\int_{-\infty}^{\infty}\mathrm{d}t'\ F(t')\ddot{G}(t-t')+\omega^2\frac{1}{m}\int_{-\infty}^{\infty}\mathrm{d}t'\ F(t')G(t-t')$$

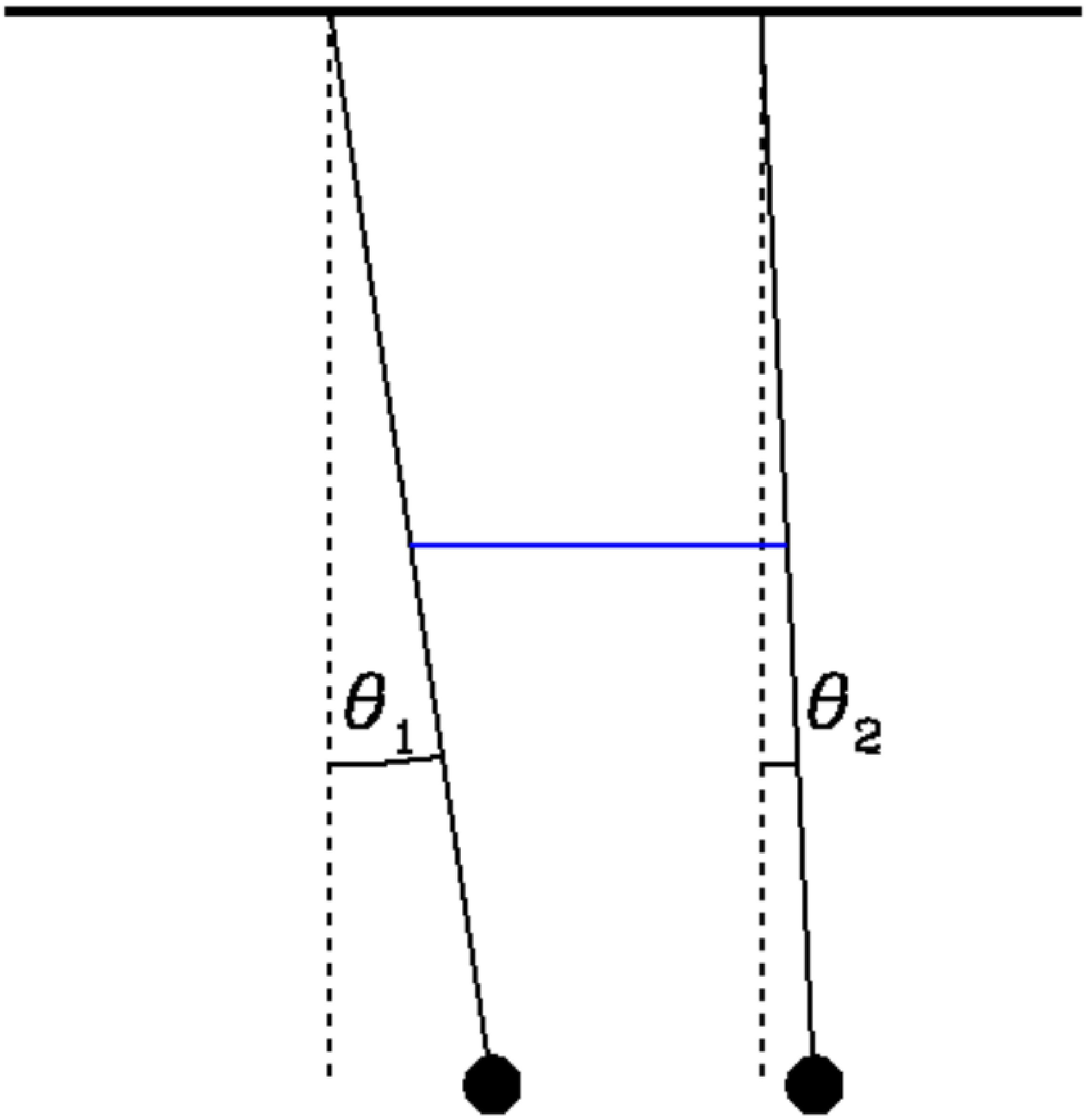
Comparing terms and substituting in the solution for $G(t-t')$ gives:

$$q(t)=\frac{1}{m}\int_{-\infty}^{t}F(t')G(t-t')\mathrm{d}t'=\frac{1}{m\omega}\int_{-\infty}^{t}F(t')\sin(\omega[t-t'])\mathrm{d}t'$$

Hence, the displacement at time t is found by summing over the displacements caused by all the preceding impulsive forces.

Coupled Small Oscillations

Two Coupled Pendulums



Consider a system with two pendula of length l connected half-way down by a spring with constant k and unstretched length equal to the horizontal separation of the pivot points.

There are two DoF with θ_1 and θ_2 as generalised coordinates. If the rigid pendula have all of their mass, m, contained in the bob and the spring is massless, then the kinetic energy is:

$$T = \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

Without coupling and assuming small oscillations:

$$V = \frac{mgl}{2}(\theta_1^2 + \theta_2^2)$$

The spring's coupling potential is:

$$V_{\text{coupling}} = \frac{k}{2}\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2$$

Hence

$$\mathcal{L} = \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}\left(mgl(\theta_1^2 + \theta_2^2) + k\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2\right)$$

To remove the $\theta_1\theta_2$ cross-term, transform to **centre of mass** $\theta_c = (\theta_1 + \theta_2)/2$ and **relative** $\theta_r = \theta_2 - \theta_1$ **coordinates**.

Defining $\omega_0 = \sqrt{g/l}$ and the coupling constant $\eta = kl/(4mg)$:

$$\mathcal{L} = ml^2\left[\left(\dot{\theta}_c^2 - \omega_0^2\theta_c^2\right) + \frac{1}{4}\left(\dot{\theta}_r^2 - \omega_0^2(1+2\eta)\theta_r^2\right)\right]$$

The Euler-Lagrange equations yield two separated harmonic oscillator EoMs:

$$\ddot{\theta}_c + \omega_0^2\theta_c = 0$$

And:

$$\ddot{\theta}_r + \omega_0^2(1+2\eta)\theta_r = 0$$

A More General Recipe

1. Find an equilibrium configuration, i.e. vales of the generalised coordinates where all generalised forces $\mathcal{F}_k = -\partial V/\partial q_k = 0$
2. Taylor expand the Lagrangian $\mathcal{L}$ to second order in the generalised coordinates and velocities, around values for the generalised coordinates q_k given by the equilibrium configuration, with the values of the generalised velocities $\dot{q}_k = 0$
3. For N DoF, the Taylor expansion must in principal be in $2N$ variables.

Matrix Form of $\mathcal{L}$

If a_k are the equilibrium values of an initially used set of generalised coordinates q'_k then a new set of generalised coordinates $q_k = q'_k - a_k$ can be defined.

These q_k will have equilibrium values $= 0$ and $\mathcal{L}$ can take the form:

$$\mathcal{L} = T - V = \dot{\mathbf{q}}^T\hat{\tau}\dot{\mathbf{q}} - \mathbf{q}^T\hat{v}\mathbf{q} + c$$

Where c is a constant, $\mathbf{q}$ is a column vector and its transpose $\mathbf{q}^T = (q_1, \ldots, q_N)$ is a row vector.

The matrices $\hat{\tau}$ and $\hat{v}$ are symmetric with matrix elements given by:

$$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}\bigg|_{\dot{q}_j, \dot{q}_k = 0}$$

$$v_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k}\bigg|_{q_j, q_k = 0}$$

Equations of Motion

Note that partial differentiation of the Lagrangian $\mathcal{L}$ with respect to the generalised coordinates and velocities yields:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 2 \sum_{k=1}^{N} \tau_{jk} \dot{q}_k = 2(\hat{\tau}\dot{\mathbf{q}})_j$$

$$\frac{\partial \mathcal{L}}{\partial q_j} = -2 \sum_{k=1}^{N} v_{jk} q_k = -2(\hat{v}\mathbf{q})_j$$

The E-L equations therefore yield:

$$\hat{\tau}\ddot{\mathbf{q}} + \hat{v}\mathbf{q} = 0$$

Normal Modes