Mathematical Methods II
Lecture 10

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Key Points

- Revision Lecture

Revision

- **Common terms in y_c and y_p:** Let's say you have the following solution to a homogeneous ODE $y_c = c_1x + c_2$. Let's also say the inhomogeneous RHS is x^2. So you decide your $y_p = ax^2 + bx + c$. But as you want to avoid terms already found in y_c you multiply by x^2. Now $y_p = ax^4 + bx^3 + cx^2$, *not* $y_p = ax^4 + bx^3 + cx^2 + dx + e$. This is the equation you would use if you started with a quartic RHS. The d and e terms won't invalidate the solution, they will just add to the c_1 and c_2 terms in y_c. These terms are unecessary and just cause more work if included.

- **1st Order Isobaric ODE:** Solve

$$\frac{dy}{dx} = -\frac{1}{2yx}\left(y^2 + \frac{2}{x}\right)$$

Rearrange

$$\left(y^2 + \frac{2}{x}\right)dx + 2yxdy = 0$$

$$y^2dx + \frac{2}{x}dx + 2yxdy = 0$$

We would like to compare the relative contribution of powers of x and y to the result on the RHS, so lets say that every power of x or dx is normalised to a value of 1 and every power of y and dy is some value m. What is the $1 : m$ ratio? Compare the powers of each term on the LHS.

$$(2m + 1), (1 - 1), (m + 1 + m)$$

$$(2m + 1), (0), (2m + 1)$$

Let's assume each term makes an equal contribution to the RHS and set our weightings equal to each other

$$2m + 1 = 0 = 2m + 1$$

Clearly this is true if $2m + 1 = 0$ so

$$m = -\frac{1}{2}$$

So we know that the power ratio for x and y is $1 : -\frac{1}{2}$. Now we can make the substitution for $y = vx^m = vx^{-1/2}$. We are essentially claiming that y is equal to some power of x, adjusted by some scaling factor v. We know the power, and the scaling factor can be eliminated. Find dy/dx

$$\frac{dy}{dx} = \frac{dv}{dx}x^{-1/2} - \frac{1}{2}vx^{-3/2}$$

$$dy = x^{-1/2}dv - \frac{1}{2}vx^{-3/2}dx$$

Sub into the equation

$$(vx^{-1/2})^2 dx + \frac{2}{x}dx + 2vx^{1/2}\left(x^{-1/2}dv - \frac{1}{2}vx^{-3/2}dx\right) = 0$$

$$\frac{v^2}{x}dx + \frac{2}{x}dx + 2vdv - \frac{v^2}{x}dx = 0$$

$$\frac{1}{x}dx + vdv = 0$$

$$\int \frac{1}{x}dx + \int vdv = 0$$

$$\ln x + \frac{1}{2}v^2 = c$$

Sub back in for y

$$\ln x + \frac{1}{2}y^2 x = c$$

- **Green's equations:** Step by step guide to Green's function method:

  Solve

  $$y'' = x$$

  subject to the conditions $f(0) = f'(0) = 0$.

  (1) Write down two solutions to the homogeneous equation. i.e. These solutions should be able to give $RHS = 0$ when substituted into the equation.

  $$G(x,z) = \begin{cases} ax + b & \text{for } x < z \\ cx + d & \text{for } x > z. \end{cases}$$

  (2) Apply the boundary conditions. Conditions should be applied as $z_1 < z < z_2$. If both conditions given can only apply to one half of the discontunity then you can eliminate one equation. We have $f(0) = f'(0) = 0$, which implies that we only have either z_1 or z_2. Let's say that $z_1 = 0$ is our lower bound. $f(0) = 0$ would therefore

mean that $b = 0$, as $a \times 0 + 0 = 0$. And f'(0)=0 means that a=0, as differentiating leaves us with just $a = 0$. Hence

$$G(x,z) = \begin{cases} 0 & = G_1, \text{ for } x_1 \leq x \leq z \\ cx + d & = G_2, \text{ for } z \leq x \leq x_2. \end{cases}$$

We can write $\leq$ instead of $<$ as G is continuous at $x = z$ even though its derivative is not.

(3) Enforce the condition that $G_2 - G_1 = 0$ at $x = z$. The order is important; function after z - function before z. This comes from the integral definitions of the restraining conditions.

$$(cz + d) - (0) = 0$$
$$cz + d = 0$$

(4) Enforce the condition that $G_2' - G_1' = 1$ at $x = z$.

$$(c) - (0) = 1$$
$$c = 1$$

Solving for d gives $d = -z$. Hence

$$G(x,z) = \begin{cases} 0 & = G_1, \text{ for } x_1 \leq x \leq z \\ x - z & = G_2, \text{ for } z \leq x \leq x_2. \end{cases}$$

(5) Integrate to find $y(x)$. But be aware that we are integrating w.r.t z, not x, so the limits of the integrals appear to switch when compared to the inequalities above. Basically, set up as above, the top equation is always your higher integral in z and the bottom equation is always your lower integral in z.

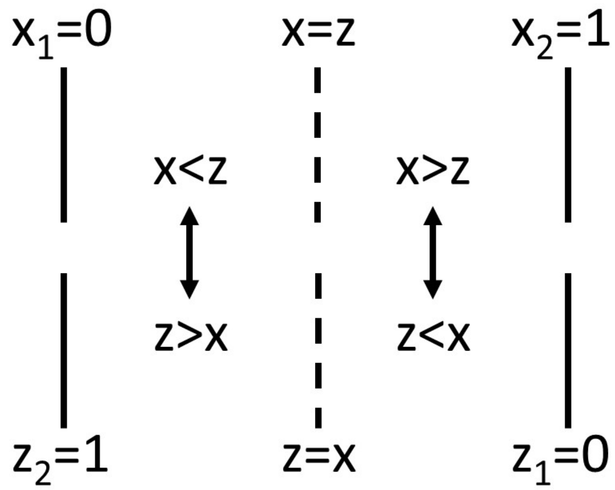
$f(z)$ is your particular integral at $x = z$. i.e. the RHS of your ODE, but in terms of z.

$$\begin{aligned} y(x) &= \int_{z_1}^{z_2} G(x,z) f(z) dz \\ &= \int_{z_1}^{z=x} (x - z) f(z) dz + \int_{z=x}^{z_2} 0 \times f(z) dz \\ &= \int_0^x (x - z) z dz = \int_0^x xz dz - \int_0^x z^2 dz \\ &= \left[\frac{xz^2}{2}\right]_0^x - \left[\frac{z^3}{3}\right]_0^x \\ &= \frac{x^3}{2} - \frac{x^3}{3} = \frac{x^3}{6} \end{aligned}$$

If we test this solution,

$$\begin{aligned} y(x) &= \frac{x^3}{6} \\ y'(x) &= \frac{x^2}{2} \\ y''(x) &= x \end{aligned}$$

which agrees with our ODE.



- **Singular points at ∞:** Show that Legendre's equation has a regular singular point at $|z| \to \infty$.

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0$$

Let $w = 1/z$. We need to eliminate z from the derivatives, expressing them in terms of w

$$\frac{dy}{dz} = \frac{dy}{dw}\frac{dw}{dz} = \frac{dy}{dw}\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$\begin{aligned}
\frac{d^2y}{dz^2} &= \frac{d}{dw}\left(\frac{dy}{dz}\right)\frac{dw}{dz} \\
&= \frac{d}{dw}\left(-w^2\frac{dy}{dw}\right) \times \frac{-1}{z^2} \\
&= \left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right) \times -w^2 \\
&= w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right)
\end{aligned}$$

Sub into the ODE

$$\left(1-\frac{1}{w^2}\right)w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right) + 2\frac{1}{w}w^2\frac{dy}{dw} + \ell(\ell+1)y = 0$$

Simplifying

$$w^2(w^2-1)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + \ell(\ell+1)y = 0$$

Dividing by $w^2(w^2-1)$ we find

$$p(w) = \frac{2w}{w^2-1}, \qquad q(w)\frac{\ell(\ell+1)}{w^4-w^2}$$

$p(0) = 0$ but $q(0)$ diverges, so $|z| \to \infty$ is a singular point. Testing wp and w^2p we find both converge at $w = 0$, so $|z| \to \infty$ is a regular singular point.

Name of ODE / method	**Form/Condition**	**Order**	**Coeff.**	**Notes**
Separable	$dy/dx = u(x)v(y)$	1	Var	Integrate independently
Exact	$du = A(x,y)dx$ $+B(x,y)dy = 0$ Test if $\partial A/\partial y = \partial B/\partial x$ $\partial u/\partial x = A, \partial u/\partial y = B$	1	Var	Find $u(x,y) = C$ by integrating A or B, use other to find $F(x$ or $y)$ from integral.
Integrating factor	$\mu(x,y)A(x,y)dx+$ $\mu(x,y)B(x,y)dy = 0$	1	Var	For inexact eqns
Homogeneous	$A(x,y)dx = B(x,y)dy$ $f(\lambda x, \lambda y) = \lambda^n f(x,y)$. Sub $y = vx$	1	Var	
Isobaric	$A(x,y)dx = B(x,y)dy$ $f(\lambda x, \lambda^m y) = \lambda^{m-1} f(x,y)$. Sub $y = vx^m$	1	Var	Set powers of: $x, dx = 1, y, dy = m$
Linear 1st order	$dy/dx + p(x)y = q(x)$ $y = 1/\mu(x) \int \mu(x)q(x)dx$ $\mu(x) = e^{\int p(x)dx}$	1	Var	
Bernoulli	$dy/dx + b(x)y = c(x)y^n$ $z = y^{1-n}$	1	Var	Solve as linear 1st order
Linear nth order	$a_n(x)d^ny/dx^n$ $+a_{n-1}(x)d^{n-1}y/dx^{n-1} + ...$ $+a_1(x)dy/dx + a_0(x)y = f(x)$	n	Var	
Linear 2nd order	$y'' + p(z)y' + q(z)y = f(z)$	2	Var	
Complementary function (linear superposition)	$y_c = c_1y_1(x) + c_2y_2(x)$	2+	Const	Solve as RHS=0. y_1 and y_2 must be linearly independent
Auxiliary equation	Sub $y = Ae^{\lambda x}$ Real: $c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ Repeat: $(c_1 + c_2x)e^{\lambda_1 x}$ Complex: $c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$	2+	Const	Identify roots
Particular integral / trial functions	$y_p = be^{rx}$ or $b_1 sinrx + b_2 cosrx$ or $b_0 + b_1x + ... + b_Nx^N$	2+	Const	To find $RHS \neq 0$
General solution	$y = y_c + y_p$	2+	Const	
Laplace transform	$\bar{f}(s) \equiv \int_0^\infty e^{-sx}f(x)dx$ $\bar{f^n}(s) = s^n\bar{f}(s) - s^{n-1}f(0)$ $-s^{n-2}f'(0) - ...$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$	2+	Const	

Name of ODE / method	Form/Condition	Order	Coeff.	Notes
Legendre linear eqns	$a_n(\alpha x+\beta)^n\frac{d^ny}{dx^n}+...$ $+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0y=f(x)$ Sub $\alpha x+\beta=e^t$	n	Var	Make coeffs. const. with sub.
Euler linear eqns	$a_nx^n\frac{d^ny}{dx^n}+...+a_1x\frac{dy}{dx}+a_0y=f(x)$ Sub $x=e^t$	n	Var	Make coeffs. const. with sub.
Wronskian	$W=y_1y_2'-y_1'y_2$	2+	Var	Check for linear independence
Wronskian method / variation of parameters	$y_p(x)=$ $k_1(x)y_1(x)+k_2(x)y_2(x)$ $k_1'=\frac{-f(x)}{W(x)}y_2$ $k_2'=\frac{f(x)}{W(x)}y_1$	2+	Var	Find y_c as usual. $y=y_p$ as y_c is implicit in y_p
Dirac δ function	$\delta(t)=0$ for $t\neq 0$ $\int\delta(t-a)f(t)dt=f(a)$	-	-	
Green's function	$LG(x,z)=\delta(x-z)$ $G_2-G_1=0$ $G_2'-G_1'=1$ $y(x)=\int_a^b G(x,z)f(z)dz$	2+	Var	Find G_c form that gives $RHS=0$, use boundary conditions to restrict G_c, integrate from $z=a$ to $z=x$ and $z=x$ to $z=b$
Ordinary and singular points	p and q finite $\rightarrow$ ordinary p or q infinite $\rightarrow$ singular $(z-z_0)p$ and $(z-z_0)^2q$ finite $\rightarrow$ regular singular $(z-z_0)p$ or $(z-z_0)^2q$ infinite $\rightarrow$ irregular singular	2+	Var	
Taylor series	$y(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ $=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ $y'=\sum_{n=0}^{\infty}na_nz^{n-1}$ $y''=\sum_{n=0}^{\infty}n(n-1)a_nz^{n-2}$	2+	Var	Requires ordinary point. Shift index by adding to n terms. Determine recurrance relation(s) for a_n.
Legendre's DE	$(1-x^2)y''-2xy'$ $+\ell(\ell+1)y=0$ $P_\ell(x)=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$	2	Var	Determine ℓ, solve with Rodrigues' formula