Stars and Galaxies - Summary

Observational Techniques

Introduction

Magnitudes

Intensity of a source of luminosity, L, at a distance r is given by

$$I = \frac{L}{4\pi r^2}$$

The apparent magnitude is then given as:

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{I_1}{I_2}\right)$$

For an object with apparent magnitude, m, at distance, D, the absolute magnitude, M, is given by

$$m - M = 5 \log_{10} D - 5$$

Where D is in parsecs.

Telescopes

Real and Virtual Images

Real images are formed when light rays from an object actually cross. A virtual image is formed when the light rays only appear to come from a point, so they never cross.

Lensmaker's Equation

The Lensmaker's Equation is given as

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

where u is the object distance, v is the image distance, and f is the focal length.

Sign Convention:

Quantity	***+ve***	***-ve***
u	Real object	Virtual object
v	Real image	Virtual image
f	Converging lens/mirror	Diverging lens/mirror

The focal length is related to the radii of curvature by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Where n is the refractive index. The radii of curvature, R, are positive for a convex surface.

For mirrors,

$$\frac{1}{f} = -\frac{2}{R}$$

Magnification

The linear magnification is given by

$$M_l = -\frac{v}{u}$$

If the magnification is negative, the image is inverted relative to the object.

Compound Lenses

For a system of two lenses of focal lengths f_1 and f_2, the angular magnification is given by:

$$M_\theta = \frac{\beta}{\alpha} = \frac{f_1}{f_2}$$

Reflecting vs Refracting Telescopes

Telescopes that use a lens are called **refractors**. Large lenses are difficult to make, and hence expensive. Since the refractive index depends on wavelength, the focal point depends on wavelength (chromatic aberration), leading to blurred images.

Telescopes which use mirrors are called **reflectors**. A large mirror may weigh several tons and will bend under its own weight, distorting the images.

Different parts of the mirrors may expand or contract at different rates when heated or cooled during an observing night. To avoid these problems, mirror and support structures are made of materials that limit thermal distortions. This limits the mirror size to around 4 meters.

A plane wave reflected by a parabolic mirror is focused to a point called prime focus. The distance from prime focus to mirror is called the focal length. The diameter of the mirror is called aperture.

The focal ratio is given by

$$\text{focal ratio} = \frac{\text{focal length}}{\text{aperture}}$$

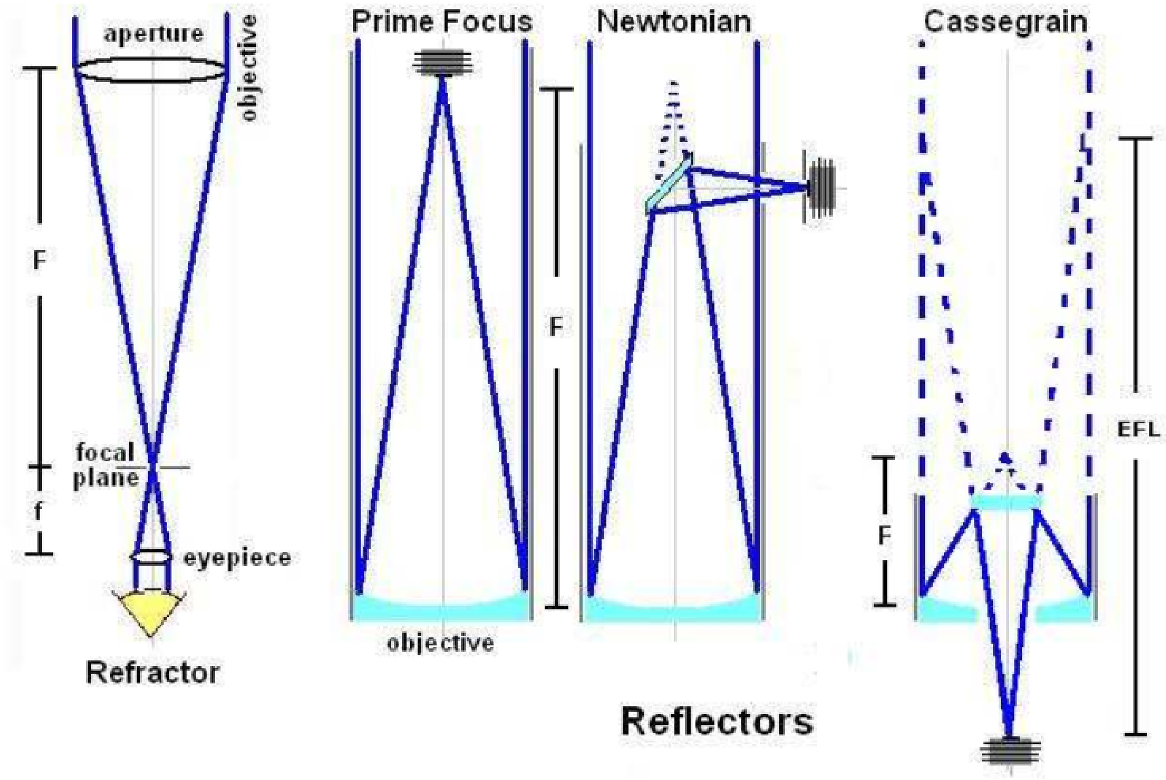
Telescope Focii

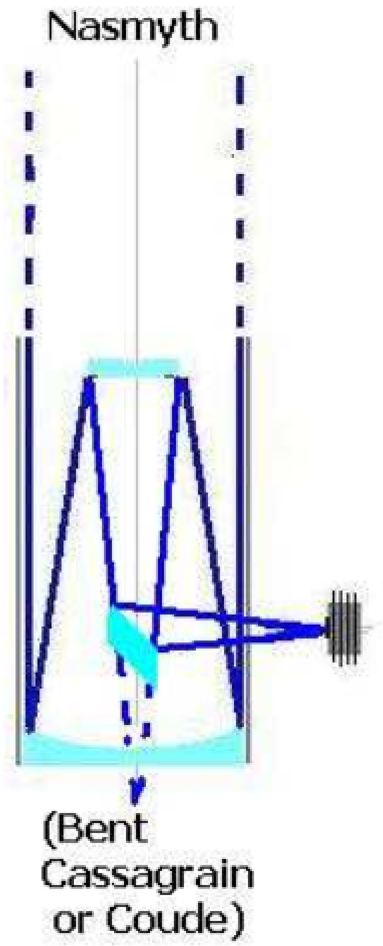
- **Prime Focus:** Put detector at the focal point. Has the advantage that the instrument will have a large field of view, and the fewest mirrors. This is good because each extra mirror leads to some loss of light.

  Since the prime focus lies directly above the mirror, a large, heavy instrument will block light and will have to move with the telescope. Only stars perfectly aligned with the telescope axis will be imaged in the prime focus.

- **Schmidt Focus:** Uses a spherical (as opposed to parabolic) mirror, which focuses the light after it has passed through a glass lens (called a corrector), with a camera in the prime focus. Allows for a very large field of view. This design is only used for relatively small telescopes

- **Cassegrain Focus:** Most small telescopes use this focus. Involves a parabolic primary and convex, hyperbolic secondary mirror. Design is compact, but has a long focal length. The eyepiece of a telescope is usually at the Cassegrain focus. However, the field of view in limited to 10–20 arcminutes.
- **Naysmith Focus:** Used for large, bulky instruments. This focii sits on a platform that rotates with the telescope and so is stable.
- **Coude Focus:** Used if very high stability is required. The light is folded down to a fixed (stationary) position in the observatory. The major disadvantage is that the field of view is small and the light needs to be folded through several mirrors, resulting in light losses





Telescope Mounts

- **Equatorial Mount:**
  - *Advantages:*
    - Only one axis must be controlled (RA)
    - Tracking rate is constant (360° per sidereal day)
    - Star field does not rotate with time
  - *Disadvantages:*
    - Large, bulky & Expensive
    - Gravity vector hard to predict
- **Alt-Az Mount:**
  - *Advantages:*
    - Simple and more compact to construct
    - Naysmith platform available
  - *Disadvantages:*
    - Non uniform tracking speed
    - Requires two axis to be controlled
    - Requires image derotator

Angular Resolution

The diffraction limited angular resolution of a telescope is:

$$\text{diffraction limited resolution} = 1.22\frac{\lambda}{D}$$

Where λ is the light wavelength and D is the telescope aperture.

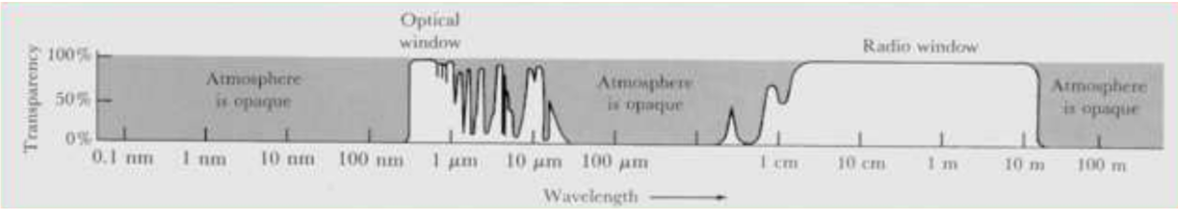
Plate Scale

For small angles, the plate scale is given as

$$\text{plate scale} = \frac{u}{s} = \frac{1}{f}$$

The Turbulent Atmosphere

Transparency



Observations can only be made in the optical and radio bands from the Earth's surface.

Dispersion

The amplitude of the spread of an object's position is given by:

$$A = (n_{\text{blue}} - n_{\text{red}}) \tan z_0$$

Where z_0 is the angular distance of the object from the zenith and n_{blue} and n_{red} are the refractive indices of the atmosphere in their respective wavelengths.

Atmospheric Light

Sources of atmospheric light include:

- fluorescent emissions called **air glow**
- **scattered light** e.g. from the Moon
- **light pollution** from the ground, satellites or aircraft
- **zodiacal light** scattered by interplanetary dust

Telescope Sites

- Clear nights
- Good seeing conditions
- Dark skies
- Little water vapour (good for IR)

Detectors

CCD Detectors

The quantum efficiency of a CCD is given by:

$$QE = \frac{\text{Number of electrons generated}}{\text{Number of incoming photons}}$$

Noise in Detectors

The gain of a detector, g, is an analogue to digital conversion and is measured in *ADU per electron*.

Some sources of noise include:

- **Sky background:** light pollution coming from the aforementioned atmospheric light sources

- **Cosmic rays:** External sources are referred to as cosmic rays, N_{ext}.
- **Sensitivity:** a, varies from pixel to pixel and is wavelength dependent
- **Bias:** The amplifier that boosts the signal also adds an offset, called bias, N_0
- **Dark current:** Pixels will build-up an electric charge generated by thermal motions, which will then be amplified and contribute to the counts, gN_dt.
- **Read-out noise:** the electronics add noise when reading out the CCD with mean, R, and variance, σ_R.

The number of photons, N_γ, number of electrons, N_e, and number of counts, N_c, can then be written as:

$$N_\gamma = (\dot{N}_\gamma(\text{source}) + \dot{N}_\gamma(\text{background}))t$$

$$N_e = aN_\gamma + N_{\text{ext}} + \dot{N}_dt$$

$$N_c = N_0 + f(gN_e) + gR$$

The registered counts may have an offset, be non-linear in the number of electrons (function f), and contain noise.

Detector Limits

- **Depth of potential well:** maximum number of electrons on pixel before it saturates
- **Resolution:** The number of counts is stored by the amplifier as a binary number with a given number of bits
- **Bandwidth:** the range of wavelengths the CCD is sensitive to

Photometry

Calibration and Detector Noise

- Readout noise is **Gaussian**
- Photon counting noise (from stars or galaxies) is **Poisson**
- Dark current noise is **Poisson**
- Bias noise is zero
- Sky background noise is **Poisson**

To reduce the image, we use the following equation:

$$\text{result} = \frac{N_c - (\text{dark} + \text{bias})}{\text{flat field}}$$

Signal to Noise Ratio

The SNR is given by:

$$\frac{S}{N} = \frac{\sum N_\gamma}{\sqrt{\sum(N_\gamma + N_d + N_b + R^2)}}$$

Alternatively, the SNR can written as:

$$\text{SNR} = \frac{S}{\sqrt{S + B + D + \sigma^2}}$$

Where S represents the signal, B the background, D the dark and σ^2 the readout.

Conversion to Magnitudes

The signal can be converted to a magnitude using a star of known magnitude, m_s and signal, S_s, by:

$$m = m_s - 2.5\log\left(\frac{S}{S_s}\right)$$

Spectroscopy

Resolving Power of a Diffraction Grating

Minimum distance between two lines which can be resolved is:

$$\Delta\lambda_{\text{min}} = \frac{\lambda}{nN}$$

For a diffraction grating with N slits at the nth order maximum.

The resolving power is hence defined as:

$$R = \frac{\lambda}{\Delta\lambda_{\text{min}}} = nN$$

Differentiating Snell's Law gives the angular dispersion:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{n}{d\cos\theta}$$

Then using the plate scale, $\mathrm{d}\theta/\mathrm{d}x$, the reciprocal linear dispersion is:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = \frac{1}{f_{\text{camera}}}\frac{d\cos\theta}{n}$$

Spectral Resolution Through a Finite Slit Width

$$R = \frac{n\rho\lambda W}{\chi D_T}$$

where n is the diffraction order, ρ is the line density, λ is the wavelength, W is the grating size, χ is the angular size of the image of a star on the slit (which is usually the seeing) and D_T is the telescope size.

Reflection Gratings

The equation for a reflection grating is given by:

$$n\lambda\rho = \sin\alpha + \sin\beta$$

Measuring Stars

Blackbody Radiation

The Wien distribution (for low temperatures) is given by:

$$I_\lambda(T) = \left(\frac{2hc^2}{\lambda^5}\right)e^{-hc/\lambda kT}$$

The Rayleigh-Jeans distribution (for high temperatures) is given by:

$$I_\lambda(T) = \frac{2ckT}{\lambda^4}$$

Where I_λ is the intensity of the black-body spectrum at wavelength, λ.

The Wien displacement law is:

$$\lambda_{\text{max}}T = 2.898 \times 10^{-3}\ \text{mK}$$

From this, the flux of the star is:

$$F = \sigma T^4$$

And the luminosity is:

$$L = 4\pi R^2\sigma T^4$$

Stellar Distances

The distance of a star can be found its parallax angle, p, with:

$$d = \frac{206265}{p}\ \mathrm{AU}$$

Resolution of an Interferometer

For two telescopes separated by a distance, D, the angular resolution of the interferometer is:

$$\phi = \frac{\lambda}{D}$$

Stars

Note: unlike Obs Techniques, the Stars and Galaxies portions of this summary are based upon the respective equations lists the can be found on DUO. Where indicated on the equations list, a derivation will be supplied.

Production of Spectral Lines

The Boltzmann equation for spectral lines is given as:

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)kT}$$

Where N_a and N_b are the number atoms in states a and b respectively, g_a and g_b are statistical weights which account for degeneracy (for hydrogen, $g_n = 2n^2$) and $e^{-(E_b - E_a)kT}$ is what's known as the Boltzmann factor. E_a and E_b are the energies of the states a and b and T is the temperature of the system.

Kepler's Generalised Third Law

Kepler's third law for a binary system is given as:

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

Where P is the orbital period, a is sum of the semi-major axes of the orbiting bodies ($a = a_1 + a_2$), and m_1 and m_2 are the masses of the orbiting bodies.

This is a generalised form of Kepler's Third Law:

$$P^2 \propto a^3$$

Binary Star Mass Ratio Relationships

The masses, m_i, semi-major axes, a_i, angles, α_i, and velocities, v_i of two stars in a binary system are related by:

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1} = \frac{v_2}{v_1}$$

Stellar Mass-Luminosity Relationship

An approximate relationship between stellar mass and luminosity is given as:

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^\alpha$$

Where $\alpha \sim 3$ to 4.

Hydrostatic Equilibrium

The hydrostatic equilibrium equation is given as:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho$$

Derivation:

Take $P = P(r)$ as the pressure, M_r as the mass in a sphere of radius r, and $\rho = \rho(r)$ as the density.

The pressure force on a volume element is:

$$F_P = [P(r + \mathrm{d}r) - P(r)]\mathrm{d}A$$

Providing $\mathrm{d}r$ is small, we can write this as:

$$F_P = \mathrm{d}P\mathrm{d}A$$

The gravitational force on the volume element is:

$$F_g = \rho\mathrm{d}r\mathrm{d}Ag$$

Where for a spherical body:

$$g = -\frac{GM_r}{r^2}$$

The condition for hydrostatic equilibrium is that **the gravitational force equals the pressure force**, so we set $F_P = F_G$:

$$\mathrm{d}P\mathrm{d}A = -\frac{GM_r}{r^2}\rho\mathrm{d}r\mathrm{d}A$$

"Dividing" through by $\mathrm{d}r\mathrm{d}A$ gives:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho$$

Relating M_r to r

The relation of M_r to r is:

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho$$

(Differentiate the volume of a sphere with respect to r)

Central Core Pressure of a Star

The central core pressure of a star can be calculated using the equation:

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Derivation:

Integrating the M_r to r relation with constant density gives:

$$M_r = \frac{4}{3}\pi r^3\rho$$

Substituting into the hydrostatic equilibrium equation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4}{3}\pi G\rho^2 r$$

Integrating again from $r = 0$ to r and $P = P_c$ to P gives:

$$P = P_c - \frac{2\pi}{3}G\rho^2 r^2$$

Since surface pressure is zero, we know that when $r = R$, $P = 0$. Substituting in and rearranging gives:

$$P_c = \frac{2\pi}{3}G\rho^2 R^2$$

Now, substituting for ρ:

$$\begin{aligned} P_c &= \frac{2\pi}{3}GR^2\left(\frac{3M}{4\pi r^3}\right)^2 \\ &= \frac{3}{8\pi}\frac{GM^2}{R^4} \end{aligned}$$

Pressure Inside a Star

The pressure inside a star is:

$$P = P_{\text{gas}} + P_{\text{rad}} = nkT + aT^4$$

Where P_{gas} is the gas pressure, P_{rad} is the radiation pressure, n is the number of particles per m^{-3}, k is the Boltzmann constant, a is the radiation density constant ($7.57 \times 10^{-16}\ \text{Jm}^{-3}\text{K}^{-4}$).

In general $P_{\text{rad}}/P_{\text{gas}} \sim 10^{-4}$.

Virial Theorem

The Virial Theorem is stated as:

$$K = -\frac{1}{2}U$$

Where K is the internal energy of the star, and U is the gravitational potential of the star.

Derivation:

Starting with the hydrostatic equilibrium and multiplying it by the volume of a sphere gives:

$$V\frac{\text{d}P}{\text{d}r} = -\frac{GM\rho}{r^2}\frac{4\pi r^3}{3}$$

Substituting in the equation for $\text{d}M/\text{d}r$:

$$V\frac{\text{d}P}{\text{d}r} = -\frac{1}{3}\frac{GM}{r}\frac{\text{d}M}{\text{d}r}$$

Integrating:

$$\int_0^{P(R)} V\text{d}P = -\frac{1}{3}\int_0^M \frac{GM}{r}\text{d}M$$

The integral on the RHS will give the gravitational potential of the star, U.

Integrating the LHS by parts:

$$\int_0^{P(R)} V\text{d}P = PV\Big|_0^R - \int_0^{V(R)} P\text{d}V$$

Since $P(R) = 0$ and $V(0) = 0$ the first term on the RHS vanishes and so:

$$-3\int_0^{V(R)} P\text{d}V = U$$

Since $\text{d}V = \text{d}m/\rho$:

$$-3\int_0^M \frac{P}{\rho}\mathrm{d}m = U$$

Assuming the ideal gas law:

$$P = nkT = \frac{\rho kT}{\mu m_{\mathrm{H}}}$$

And since the kinetic energy per particle is $\frac{3}{2}kT$:

$$E_{\mathrm{KE}} = \frac{3}{2}\frac{kT}{\mu m_{\mathrm{H}}} = \frac{3}{2}\frac{P}{\rho}$$

Is the kinetic energy per kilogram.

Then:

$$\int_0^M E_{\mathrm{KE}}\mathrm{d}m = -\frac{1}{2}U$$

Since the integral on the LHS is the internal energy, K, we get:

$$K = -\frac{1}{2}U$$

Energy from Gravitational Collapse

The energy of a gravitational collapse from radius R_{initial} to R is given as:

$$E = \frac{3GM^2}{10R}\left[\frac{1}{R} - \frac{1}{R_{\mathrm{initial}}}\right]$$

Derivation: