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# RECURSIVE CALCULATIONS FOR PROCESSES WITH *n* GLUONS

F.A. BERENDS and W.T. GIELE*

*Instituut-Lorentz, University of Leiden, P.O.B. 9506, 2300 RA Leiden, The Netherlands*

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A method is presented in which multigluon processes are calculated recursively. The technique is explicitly developed for processes where only gluons are produced and processes where in addition to the gluons also a quark-antiquark pair with or without a vector boson or $e^+e^-$ pair are present. The recursion relations are used to derive rigorously amplitudes for certain configurations, where most of the gluons have the same helicities. This proves a number of conjectures in the literature. Also expressions for amplitudes with collinear or soft gluons are derived.

## 1. Introduction

Cross sections for processes involving a number of partons possibly together with a vector boson like W, Z or $\gamma^*$ are of importance for present and future colliders. Although one may primarily be concerned with hadron colliders, multijet events are also relevant for $e^+e^-$ and $e^-P$ collisions.

Amongst the parton processes the pure gluon processes play a special role. On the one hand, when one has techniques to calculate this process it is not too difficult to incorporate a quark-antiquark pair with or without a vector boson. On the other hand, in hadron collisions the gluons have the largest parton luminosity and have the largest parton cross sections. Thus one often focusses on the gluon processes [1]. These processes are known up to six gluons. A number of authors [2] derived involved expressions which they could evaluate numerically. A more systematic answer to the problem of the six-gluon amplitude was subsequently found [3, 4]. A major difference is the colour split-up of the amplitude.

The amplitude is written as

$$\mathscr{M}(1,\dots,n) \sim \sum_{P(1,\dots,n-1)} \operatorname{Tr}(T^{a_1}\dots T^{a_n})\mathscr{C}(1,\dots,n), \tag{1.1}$$

where $a_1,\dots,a_n$ denote the colours of the $n$ gluons, $T^{a_i}$ are the colour matrices in the fundamental representation. Moreover $\mathscr{C}$ is a function depending on the

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momenta and helicities of the gluons. Even with these simpler expressions it remains a formidable task to evaluate amplitudes with more than six gluons. In practice estimates [5] of cross sections involving more than six gluons have been based on conjectured formulae [6,7]. The conjecture of ref. [6] is one for specific helicity amplitudes: those, where for all particles the helicity is the same except for two particles.

In this paper we propose to use recursion relations as a technique to evaluate the exact parton amplitudes. In fact, in first instance a matrix element for $n$ gluons is calculated, one of which is off-shell. From this current $J$ the function $\mathscr{C}(1,\dots,n)$ is obtained. The advantage is that for the calculation of an $n+1$ gluon process one can use the calculation of the $n$-gluon process. Both for analytic and numerical evaluation this is an asset. A numerical evaluation of the 7 and 8 gluon process now becomes possible, although at present a straightforward use of the recursion relations without further tricks remains time consuming. On the other hand, the recursion relation takes automatically into account all Feynman diagrams. Writing down those diagrams would be a problem in itself, which is now avoided.

Another advantage of the recursion relations is that one can use them for the proof of certain properties of the amplitudes. Consider for instance the above conjecture [6] for certain helicity amplitudes. A rather similar property of helicity amplitudes was known to hold for QED amplitudes [8] in the process $e^+e^- \to n\gamma$, $e^+e^- \to \mu^+\mu^- n\gamma$ and $e^+e^- \to e^+e^- n\gamma$. The QED diagrammatic structure is simple enough to prove such a property. In QCD the structure seems to be too forbidding to give an analogous proof for the conjecture of ref. [6]. However by means of the recursion relation we show the conjecture of ref. [6] and other similar conjectures [9] to be valid. Also certain properties of the $\mathscr{C}$-functions, such as cyclic symmetry, subcyclic relations, gauge invariance and factorization for soft and collinear gluons will be proven by means of the recursion relations.

Once one knows the gluon currents $J$, they can be used as building blocks for those reactions, where besides $n$ gluons, a quark-antiquark pair with or without a vector boson is produced. Again a certain number of properties of the amplitudes, analogous to the ones mentioned above can be shown to exist.

Thus the paper introduces a recursive calculational technique for parton processes, which is suitable for analytical and numerical evaluation. Moreover a number of properties of processes involving $n$ gluons are proved, some of which existed sofar only as conjectures.

The actual outline of the paper is as follows. The recursion relation relevant for the pure gluonic processes is derived in sect. 2, whereas the extension to processes with a quark-antiquark pair is given in sect. 3. Sect. 4 shows how to obtain the amplitudes and what the expressions for their squares are, when summed over colour and when the terms of leading order in $N$ are considered. Solutions to the recursion relations are obtained in sect. 5 for a number of specific helicity configurations. The related amplitudes are listed in sect. 6. Sect. 7 discusses special kinematical situations, those of soft and collinear gluons. In those cases the $\mathscr{C}$-functions show factorization properties. Sect. 8 summarizes our conclusions. Finally, appendix A collects some conventions and often used formulae, whereas appendices B and C prove various assertions made in the main text.

## 2. Gluon recursion relation

In this section an expression will be derived for a matrix element for $n+1$ outgoing gluons, where one gluon is off shell. This quantity will be called an $n$-gluon current and is denoted by $\hat{J}^x_\xi(1,2,\dots,n)$, where $x$ and $\xi$ denote the colour and vector index of the off-shell gluon. The $(n+1)$-particle amplitude can be obtained from this current by a suitable contraction with a polarization vector of the last – $n+1$ – gluon.

The current will first be introduced for 1, 2 and 3 gluons. It turns out that the current $\hat{J}^x_\xi$ can be decomposed in a colour part and a space-time part $J_\xi$. The latter has a number of symmetry properties, related to permutations of the gluons and is moreover a conserved current. For $n$ gluons $\hat{J}^x_\xi(1,2,\dots,n)$ is again related to $J_\xi(1,2,\dots,n)$.

For the latter a recursion relation holds, which relates it to all $J_\xi(1,2,\dots,m)$ with $m<n$. Again, this current is conserved and obeys certain symmetry properties.

For one gluon we define

$$\hat{J}^x_\xi(1) = \delta^{a_1x} e_\xi = \delta^{a_1x} J_\xi(1) = 2\,\mathrm{Tr}(T^{a_1}T^x) J_\xi(1)\,, \tag{2.1}$$

where $e_\xi$ is the polarization vector of the gluon, depending on the helicity and momentum $K_1$ of the particle. The colour is $a_1$, the indices $x$ and $\xi$ are summation indices. The SU($N$) matrices in the fundamental representation are denoted by $T$.

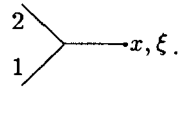
Obviously we have

$$K_1 \cdot J(1) = 0\,. \tag{2.2}$$

For two gluons we use the 3-vertex and introduce a propagator

$$\hat{J}^x_\xi(1,2) = \frac{ig}{(K_1+K_2)^2} f^{a_1a_2x} V^{\alpha_1\alpha_2}{}_\xi\big(K_1, K_2, -(K_1+K_2)\big) J_{\alpha_1}(1) J_{\alpha_2}(2)\,. \tag{2.3}$$

In terms of a diagram we have



Our conventions for the vertex, colour matrices in the fundamental representation, etc. are listed in appendix A, together with some useful relations.

It is convenient to introduce

$$J_\xi(1,2) = \frac{1}{(K_1+K_2)^2} V^{\alpha_1\alpha_2}{}_\xi(K_1, K_2, -(K_1+K_2)) J_{\alpha_1}(1) J_{\alpha_2}(2)$$

$$= \frac{1}{(K_1+K_2)^2}\big[2K_2\cdot J(1) J_\xi(2) - 2K_1\cdot J(2) J_\xi(1)$$

$$+ (K_1-K_2)_\xi J(1)\cdot J(2)\big], \qquad (2.4)$$

which obeys (see eq. (A.11))

$$(K_1+K_2)\cdot J(1,2) = 0, \qquad (2.5)$$

$$J_\xi(1,2) = -J_\xi(2,1). \qquad (2.6)$$

Because of the antisymmetry property we introduce the suggestive notation

$$J_\xi(1,2) = \frac{1}{(K_1+K_2)^2}[J(1), J(2)]_\xi, \qquad (2.7)$$

the bracket is however not a commutator.

With this definition and eq. (A.8) we now have

$$\hat{J}^x_\xi(1,2) = 2g\sum_{P(1,2)} \mathrm{Tr}(T^{a_1}T^{a_2}T^x) J_\xi(1,2)$$

$$= 2g\sum_{P(1,2)} (a_1 a_2 x) J_\xi(1,2), \qquad (2.8)$$

with a sum over the permutations of (1,2).

In the case of three gluons the four diagrams in fig. 1 should be considered.

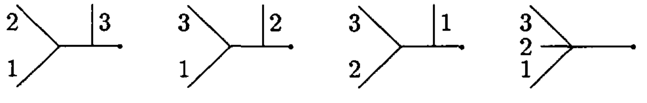



Fig. 1.

The first three diagrams give

$$\hat{J}^x_\xi(1,2,3) = \frac{ig}{(K_1+K_2+K_3)^2}$$

$$\times \sum_{C(1,2,3)} f^{ya_3x} V^{\eta\alpha_3}{}_\xi(K_1+K_2, K_3, -(K_1+K_2+K_3)) \hat{J}^y_\eta(1,2) J_{\alpha_3}(3)$$

$$= \frac{(2g)^2}{[\kappa(1,3)]^2} \sum_{C(1,2,3)} [(ya_3x)-(a_3yx)] \sum_{P(1,2)} (a_1a_2y)[J(1,2), J(3)]_\xi$$

$$= 2g^2 \sum_{P(1,2,3)} (a_1a_2a_3x) J_\xi(1,2,3), \tag{2.9}$$

where $C(1,2,3)$ denotes a summation over the cyclic permutations of $(1,2,3)$. Moreover

$$\kappa(m,n) = K_m + K_{m+1} + \cdots + K_n, \tag{2.10}$$

and

$$J_\xi(1,2,3) = \frac{1}{[\kappa(1,3)]^2}\left([J(1), J(2,3)]_\xi + [J(1,2), J(3)]_\xi\right). \tag{2.11}$$

The fourth diagram, when using eqs. (2.1), (A.6) and (A.9) gives

$$\hat{J}^x_\xi(1,2,3) = \frac{2g^2}{[\kappa(1,3)]^2} \sum_{C(1,2,3)} \{(a_1a_2a_3x)-(a_2a_1a_3x)+(a_3a_2a_1x)-(a_3a_1a_2x)\}$$

$$\times K(\alpha_1,\alpha_2;\alpha_3,\xi) J^{\alpha_1}(1) J^{\alpha_2}(2) J^{\alpha_3}(3)$$

$$= \frac{2g^2}{[\kappa(1,3)]^2} \sum_{P(1,2,3)} (a_1a_2a_3x)\{J(1), J(2), J(3)\}_\xi. \tag{2.12}$$

The definition

$$\{J(1), J(2), J(3)\}_\xi = J(1)\cdot\left(J(3)J_\xi(2) - J(2)J_\xi(3)\right)$$
$$- J(3)\cdot\left(J(2)J_\xi(1) - J(1)J_\xi(2)\right), \tag{2.13}$$

shows the same symmetry properties as eq. (2.11).

The full three-gluon current for eq. (2.9) now is

$$J(1,2,3)=\frac{1}{[\kappa(1,3)]^2}\left([J(1),J(2,3)]+[J(1,2),J(3)]+\{J(1),J(2),J(3)\}\right), \tag{2.14}$$

where we have suppressed the index $\xi$.

The current has the following properties

$$J(3,2,1)=J(1,2,3), \tag{2.15}$$

$$J(1,2,3)+J(2,3,1)+J(3,1,2)=0, \tag{2.16}$$

$$(K_1+K_2+K_3)\cdot J(1,2,3)=0. \tag{2.17}$$

Properties (2.15) and (2.16) easily follow from the symmetry properties manifest in eqs. (2.11) and (2.13). In appendix B current conservation is shown to hold.

The $n$-gluon current is a generalization of eqs. (2.1), (2.8), (2.9) and (2.12)

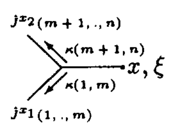
$$\hat{J}^x_\xi(1,2,\dots,n)=2g^{n-1}\sum_{P(1,2,\dots,n)}(a_1a_2\dots a_nx)J_\xi(1,2,\dots,n), \tag{2.18}$$

where $J_\xi$ is a generalization of eqs. (2.7) and (2.14)

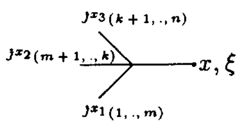
$$J(1,2,\dots,n)$$
$$=\frac{1}{\kappa(1,n)^2}\left(\sum_{m=1}^{n-1}[J(1,\dots,m),J(m+1,\dots,n)]\right.$$
$$\left.+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\{J(1,\dots,m),J(m+1,\dots,k),J(k+1,\dots,n)\}\right). \tag{2.19}$$

The correctness of eqs. (2.18) and (2.19) follows by induction. So assume the validity for $m<n$, then the $n$-gluon current is obtained from considering a 3-vertex and a 4-vertex with all possible currents attached.

$$\hat{J}^x_\xi(1,\dots,n)=\frac{1}{[\kappa(1,n)]^2}\sum_{P(1,\dots,n)}\left(\sum_{m=1}^{n-1}\frac{1}{2!}\frac{1}{m!}\frac{1}{(n-m)!}\right.$$

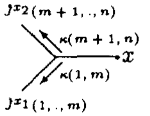



$$\left.+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\frac{1}{3!}\frac{1}{m!}\frac{1}{(k-m)!}\frac{1}{(n-k)!}\right). \tag{2.20}$$



In eq. (2.20) a summation over all permutations of the $n$ gluons is performed. In order to avoid multiple counting factors like $1/m!$ are introduced, since $\hat{J}(1,\ldots,m)$ contains all $m!$ permutations of the particles.

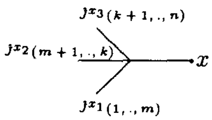
Using the expression (2.18) one rewrites the terms in (2.20)




$$= \sum_{P(1,\ldots,m)} \sum_{P(m+1,\ldots,n)}$$

$$\times 8([x_1,x_2]x)(\Omega_1 x_1)(\Omega_2 x_2)[J(1,\ldots,m),J(m+1,\ldots,n)], \quad (2.21)$$

where $\Omega_1$ and $\Omega_2$ are the strings of $T$ matrices with labels $a_1 \ldots a_m$ and $a_{m+1} \ldots a_n$. Carrying out the $x_1, x_2$ summation the first term of eq. (2.19) is obtained. The 4-vertex term becomes




$$= \sum_{P(1,\ldots,m)} \sum_{P(m+1,\ldots,k)} \sum_{P(k+1,\ldots,n)} \Bigg\{ \sum_{C(1,2,3)} 2[(\Omega_1\Omega_2\Omega_3 x) - (\Omega_2\Omega_1\Omega_3 x)$$

$$+ (\Omega_3\Omega_2\Omega_1 x) - (\Omega_3\Omega_1\Omega_2 x)] K(\alpha_1,\alpha_2;\alpha_3,\xi)\Bigg\}$$

$$\times J^{\alpha_1}(1,\ldots,m) J^{\alpha_2}(m+1,\ldots,k) J^{\alpha_3}(k+1,\ldots,n), \quad (2.22)$$

where $\Omega_1$, $\Omega_2$ and $\Omega_3$ are strings of $T$ matrices with labels $a_1 \ldots a_m$, $a_{m+1} \ldots a_k$ and $a_{k+1} \ldots a_n$. The 4-vertex term contains a cyclic permutation over $(\Omega_i, \alpha_i)$ inside the brackets. This particular expression follows from eqs. (A.6) and (A.9) and can be rewritten as

$$\{\cdots\} = 2 \sum_{P(1,2,3)} (\Omega_1\Omega_2\Omega_3 x)[K(\alpha_1,\alpha_2;\alpha_3,\xi) + K(\alpha_3,\alpha_2;\alpha_1,\xi)]. \quad (2.23)$$

From this, the second term in (2.19) easily follows.

The current $J(1,2,\dots,n)$ has the properties

$$J(n,n-1,\dots,1)=(-1)^{n-1}J(1,2,\dots,n), \tag{2.24}$$

$$\sum_{C(1,\dots,n)} J(1,2,\dots,n)=0, \tag{2.25}$$

$$\kappa(1,n)\cdot J(1,2,\dots,n)=0. \tag{2.26}$$

These properties are proved in appendix B.

One may wonder how the $n$-gluon current is affected when we change the gauge of a specific gluon, e.g.

$$\tilde{J}_1=J_1+\rho K_1. \tag{2.27}$$

In order to obtain the additional part to the $n$-gluon current, we replace $J_1$ by $K_1$ in the recursion relation (2.19). After evaluating the current for a few cases one is led to a general answer for the current with $J_1=K_1$:

$$J(1,2,\dots,n)=-\kappa(1,n)\frac{\kappa(1,n)\cdot J'(2,\dots,n)}{[\kappa(1,n)]^2}+J'(2,\dots,n), \tag{2.28}$$

with

$$J'(2)=J(2), \tag{2.29}$$

$$J'(2,\dots,n)=J(2,\dots,n)-\sum_{m=2}^{n-1}\frac{\kappa(1,m)\cdot J'(2,\dots,m)}{[\kappa(1,m)]^2}J(m+1,\dots,n). \tag{2.30}$$

Using induction, this form can be proven to be correct. Notice that the current (2.28) satisfies current conservation.

It is sometimes useful to describe the successive terms in eq. (2.19) by means of pictures. For instance, for $J(1,2,3)$ and $J(1,2,3,4)$ we find the diagrams of figs. 2 and 3.

These diagrams have a clock-wise orientation for the labels $1,2,3,\dots,n$. This is necessary since $J(1,2,\dots,m)$ is not symmetric in the indices.

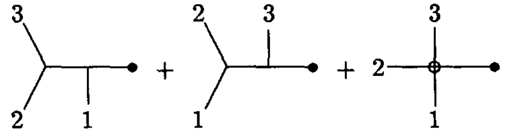



Fig. 2.

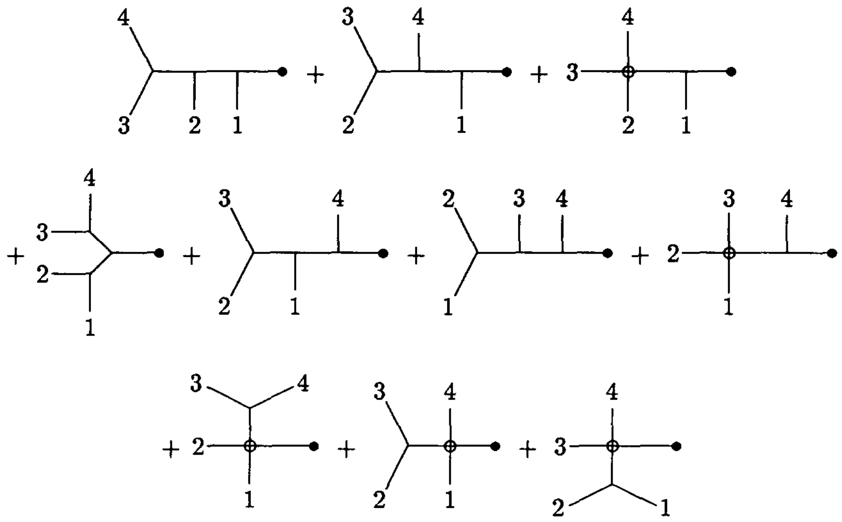



Fig. 3.

## 3. Spinorial recursion relations

In this section an expression will be derived for a matrix element, where a quark-antiquark pair and $n$ gluons are outgoing. The antiquark is off shell such that we have a spinorial current $\hat{J}_c^j(Q;1,2,\ldots,n)$. In this notation $Q$ stands for the quark momentum whereas the quark helicity is suppressed. Moreover the numbers $1,2,\ldots,n$ denote the $n$ gluons and their momentum and helicity. The colour $j$ and the spinor index $c$ of the off-shell anti-quark are written explicitly, but will often be suppressed. The other colour indices will be manifest in the explicit formulae. As in section 2 the current $\hat{J}$ will be expressed in a sum of terms consisting of a colour factor and a current $J_c(Q;1,\ldots,n)$ which is independent of colour and for which a recursion relation holds.

For a single quark and no gluon we have

$$\hat{J}_c^j(Q)=\delta_{ij}\bar{u}_c(Q) \qquad \text{or} \qquad \hat{J}(Q)=\delta_{ij}\bar{u}(Q)=\delta_{ij}J(Q). \tag{3.1}$$

The one-gluon spinorial current is

$$\hat{J}(Q;1)=igT_{ij}^{a_1}\bar{u}(Q)\not{\epsilon}_1\frac{i}{\not{Q}+\not{K}-m} \tag{3.2}$$

$$=gT_{ij}^{a_1}J(Q;1)$$

$$=g(a_1)_{ij}J(Q;1), \tag{3.3}$$

with

$$J(Q;1)=-J(Q)\not{J}(1)\frac{1}{\not{Q}+\not{K}-m}. \tag{3.4}$$

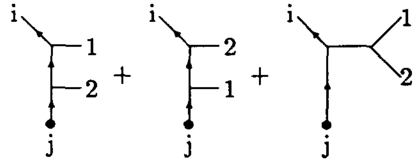



Fig. 4.

For two gluons the diagrams of fig. 4 should be considered.

$$\hat{J}(Q;1,2) = ig^2\{(a_1a_2)_{ij}J(Q;1)\not{J}(2) + (a_2a_1)_{ij}J(Q;2)\not{J}(1)$$
$$+2(x)_{ij}[(a_1a_2x)J(Q)\not{J}(1,2) + (a_2a_1x)J(Q)\not{J}(2,1)]\}$$
$$\times \frac{i}{\not{Q} + \not{K}_1 + \not{K}_2 - m}. \tag{3.5}$$

The one- and two-gluon currents eqs. (2.1) and (2.8) have been used in eq. (2.6). The summation over $x$ is carried out with the help of eq. (A.3), giving

$$\hat{J}(Q;1,2) = g^2 \sum_{P(1,2)} (a_1a_2)_{ij}J(Q;1,2), \tag{3.6}$$

with

$$J(Q;1,2) = -(J(Q;1)\not{J}(2) + J(Q)\not{J}(1,2))\frac{1}{\not{Q} + \not{K}_1 + \not{K}_2 - m}. \tag{3.7}$$

The $n$-gluon spinorial current has the form

$$\hat{J}(Q;1,2,\dots,n) = g^n \sum_{P(1,\dots,n)} (a_1a_2\dots a_n)_{ij}J(Q;1,2,\dots,n), \tag{3.8}$$

where

$$J(Q;1,2,\dots,n) = -\sum_{m=0}^{n-1} J(Q;1,2,\dots,m)\not{J}(m+1,\dots,n)\frac{1}{\not{Q} + \not{K}(1,n) - m}. \tag{3.9}$$

This is proven by induction

$$\hat{J}(Q;1,\dots,n)=2ig^n\sum_{P(1,\dots,n)}\sum_{m=0}^{n-1}\frac{1}{m!}\frac{1}{(n-m)!}\sum_{P(1,\dots,m)}\sum_{P(m+1,\dots,n)}$$
$$\times(a_1\dots a_m)_{il}(x)_{lj}(a_{m+1}\dots a_n x)J(Q;1,\dots,m)\not{J}(m+1,\dots,n)$$
$$\times\frac{i}{\not{Q}+\not{K}(1,n)-m}. \tag{3.10}$$

The colour sum over $x$ gives a term $\frac{1}{2}(a_1\dots a_n)_{ij}$ and

$$-(1/2N)(a_1\dots a_m)_{ij}(a_{m+1}\dots a_n).$$

The second term does not contribute since for each choice of the labels $1,\dots,m$ we perform a summation over all permutations of the remaining $m+1,\dots,n$ labels which implies sums over cyclic permutations of the pure gluon current. Due to eq. (2.25) the result for these terms is zero. In the special case where we have $-(1/2N)(a_1\dots a_{n-1})_{ij}(a_n)$ this contribution also vanishes. Thus we are left with the $(a_1\dots a_n)_{ij}$ terms which lead to the eqs. (3.8) and (3.9).

In the following it will be useful to have a spinorial current where the outgoing quark instead of the antiquark is off-shell. In exactly the same fashion we derive

$$\hat{J}^j(1,\dots,n;P)=g^n\sum_{P(1,\dots,n)}(a_1a_2\dots a_n)_{ji}J(1,\dots,n;P), \tag{3.11}$$

where $j$ is the colour index of the off-shell outgoing quark and $P$ the momentum of the outgoing antiquark. The current $J$ is given by

$$J(P)=v(P) \tag{3.12}$$

and in general

$$J(1,\dots,n;P)=\frac{1}{\not{P}+\not{K}(1,n)+m}\sum_{m=1}^{n}\not{J}(1,\dots,m)J(m+1,\dots,n;P). \tag{3.13}$$

For the two-gluon case the diagrams are depicted in fig. 5.

The notation for the currents (3.8) and (3.11) is such that the position of the momentum $Q$ or $P$ determines whether one deals with an adjoint spinor or a spinor.

By means of the charge conjugation matrix $C$ for which

$$Cv_{\pm}=-\bar{u}_{\pm}^{\mathrm{T}},\qquad \bar{u}_{\pm}C^{-1}=v_{\pm}^{\mathrm{T}} \tag{3.14}$$

where

$$v_{\pm}=\tfrac{1}{2}(1\mp\gamma_5)v,\qquad u_{\pm}=\tfrac{1}{2}(1\pm\gamma_5)u, \tag{3.15}$$

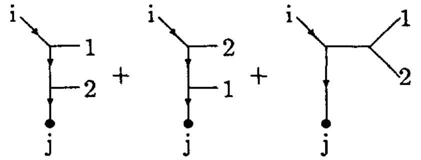



Fig. 5.

one has

$$CJ(1,\dots,n;P\pm)=(-1)^{n-1}J^{\mathrm{T}}(P\pm;n,\dots,1) \tag{3.16}$$

or

$$J(P\pm;1,\dots,n)C^{-1}=-(-1)^{n-1}J^{\mathrm{T}}(n,\dots,1;P\pm). \tag{3.17}$$

The $\pm$ sign in eqs. (3.16) and (3.17) denotes the helicity of the outgoing positron or electron.

## 4. From currents to amplitudes and cross sections

From the pure $n$-gluon current and the spinorial currents the amplitudes and cross sections for the parton processes with a large number of gluons can be obtained. We discuss in this section the most relevant processes i.e. those with only gluons and processes where in addition to $n$ gluons a quark-antiquark pair is produced, possibly together with a W, Z or a virtual photon. The last possibility is of relevance for $e^+e^-$ collisions and $e^-P$ inelastic scattering.

### 4.1. SCATTERING OF $n$ GLUONS

The amplitude for $n$-gluon scattering is obtained from the $n-1$ gluon current $\hat{J}^x_\xi(1,\dots,n-1)$ by removing the propagator of the off-shell gluon, by contracting the current with the polarization vector of the $n$th gluon i.e. $\hat{J}^x_\xi(n)$ and by demanding overall momentum conservation $\kappa(1,n)=0$.

$$\begin{aligned}\mathcal{M}(1,\dots,n)&=\hat{J}^x(1,\dots,n-1)\cdot\hat{J}_x(n)i[\kappa(1,n-1)]^2\big|_{\kappa(1,n)=0}\\&=2ig^{n-2}\sum_{P(1,\dots,n-1)}(a_1\dots a_n)\\&\qquad\times J(1,\dots,n-1)\cdot J(n)[\kappa(1,n-1)]^2\Big|_{\kappa(1,n)=0}\\&=2ig^{n-2}\sum_{P(1,\dots,n-1)}(a_1\dots a_n)\mathscr{C}(1,2,\dots,n),\end{aligned} \tag{4.1}$$

with

$$\mathscr{C}(1,2,\ldots,n)=[\kappa(1,n-1)]^2 J(1,\ldots,n-1)\cdot J(n)\big|_{\kappa(1,n)=0}. \tag{4.2}$$

This function $\mathscr{C}(1,\ldots,n)$ equals the $\mathscr{C}$-function introduced in ref. [3] times a factor $-i2^{-1-n}$. The difference in normalization is due to the explicit factor of $2i$ in eq. (4.1) and to the fact that we use here traces of $SU(N)$ matrices in the fundamental instead of the adjoint representation.

From the definition (4.2) and the definition (2.19) of the gluon current together with the properties (2.24)–(2.26) it is clear the function $\mathscr{C}(1,\ldots,n)$ has the following properties:

(i) $\mathscr{C}$ is invariant under cyclic permutations,

$$\mathscr{C}(1,\ldots,n)=\mathscr{C}(m+1,\ldots,n,1,\ldots,m). \tag{4.3}$$

(ii) $\mathscr{C}$ has a reflective property,

$$\mathscr{C}(1,\ldots,n)=(-1)^n\mathscr{C}(n,\ldots,1). \tag{4.4}$$

(iii) The sub-cyclic sum equals zero,

$$\sum_{\mathscr{C}(1,\ldots,n-1)}\mathscr{C}(1,\ldots,n)=0. \tag{4.5}$$

(iv) The quantity $\mathscr{C}$ is gauge invariant.

The first property stems from the fact, that one can obtain $\mathscr{C}(1,\ldots,n)$ from any $n-1$ gluon currents $\hat{J}(m+1,\ldots,n,1,\ldots,m-1)$ by contraction with $\hat{J}(m)$, as can be seen from figs. 2 and 3. In eq. (4.5) we keep $n$ fixed, but one could also fix another label $m$, using eq. (4.3). The property of gauge invariance here means that another gauge choice for a gluon, like in eq. (2.27) leads to the same $\mathscr{C}$-function. When we call this gluon $n$, which is always possible due to the cyclic property (4.3), it means an additional term $\rho K_n$ in eq. (4.1). Since momentum conservation relates this to $\rho\kappa(1,n-1)$ we can use the current conservation of $J(1,\ldots,n-1)$ to show the vanishing of this additional term. Alternatively, giving the gluon of which the polarization is changed the label 1, we obtain as additional term in eq. (4.1) the current $J(1,\ldots,n-1)$ of eq. (2.28) contracted with $J(n)$. Momentum conservation and the conservation of $J(n)$ now yields a vanishing additional term.

For the cross section one must square the amplitude (4.1) and sum over all colours. In general this becomes complicated. However the terms in the cross section of leading order in $N$ can be obtained for any number of gluons.

An arbitrary term in the cross section contains the colour term

$$\sum_{a_n}(a_1 \ldots a_n)(a_n a_{m_{n-1}} \ldots a_{m_1}) = \tfrac{1}{2}(a_1 \ldots a_{n-1} a_{m_{n-1}} \ldots a_{m_1})$$
$$- \frac{1}{2N}(a_1 \ldots a_{m_{n-1}})(a_{m_{n-1}} \ldots a_{m_1}). \quad (4.6)$$

In leading order in $N$ the second term can be omitted. The remaining term is now summed over $a_{n-1}$. Two different type of structures can occur

$$\sum_{a_{n-1}} (\Omega_1 a_{n-1} a_{n-1} \Omega_2) = \tfrac{1}{2}N(\Omega_1 \Omega_2) - \frac{1}{2N}(\Omega_1 \Omega_2), \quad (4.7)$$

$$\sum_{a_{n-1}} (\Omega_1 a_{n-1} \Omega_2 a_{n-1} \Omega_3) = \tfrac{1}{2}(\Omega_1 \Omega_3)(\Omega_2) - \frac{1}{2N}(\Omega_1 \Omega_2 \Omega_3). \quad (4.8)$$

So in leading order in $N$ only the first structure is of relevance leading to $\frac{1}{2}N(\Omega_1\Omega_2)$. When summing over $a_{n-2}$ the terms with neighbouring matrices $a_{n-2}$ are again leading. Repeating this process we find

$$\sum_{a_1,\ldots,a_n} |\mathscr{M}(1,\ldots,n)|^2$$
$$= 4g^{2n-4}\Bigg(\sum_{P(1,\ldots,n-1)} (a_1 \ldots a_n)\mathscr{C}(1,\ldots,n)\Bigg)$$
$$\times \Bigg(\sum_{P(1,\ldots,n-1)} (a_n a_{n-1} \ldots a_1)\mathscr{C}^*(1,\ldots,n)\Bigg)$$
$$= 2^{2-n} g^{2n-4} N^{n-2}(N^2-1)\Bigg(\sum_{P(1,\ldots,n-1)} |\mathscr{C}(1,\ldots,n)|^2 + \mathcal{O}\left(\frac{1}{N^2}\right)\Bigg). \quad (4.9)$$

The last summation concerns $(a_1 a_1)$ which according to eq. (A.1) gives $\frac{1}{2}(N^2-1)$. Up to 5 gluons the $\mathcal{O}(N^{-2})$ term is zero. From 6 gluons onward interference terms between different $\mathscr{C}$ functions are present in the cross section, but are suppressed by colour.

#### 4.2. THE PROCESS PRODUCING $q\bar{q}$ AND $n$ GLUONS

The amplitude for a process with $n$ outgoing gluons, a quark with momentum $Q$, colour $i$ and antiquark with momentum $P$, colour $j$ can be obtained from either one of the spinorial currents.

When we use eq. (3.8) we find

$$\mathcal{M}(Q;1,2,\dots,n;P) = -i\hat{J}(Q;1,\dots,n)[\not{Q} + \not{\kappa}(1,n) - m]\,v(P)\big|_{P+Q+\kappa(1,n)=0}$$

$$= -ig^n \sum_{P(1,\dots,n)} (a_1 \dots a_n)_{ij} \mathcal{D}(Q;1,\dots,n;P), \qquad (4.10)$$

with

$$\mathcal{D}(Q;1,\dots,n;P) = J(Q;1,\dots,n)[\not{Q} + \not{\kappa}(1,n) - m]\,v(P)\big|_{P+Q+\kappa(1,n)=0}. \qquad (4.11)$$

For the matrix element squared, summed over the colours of the partons we have

$$\sum_{i,j,\{a_m\}} |\mathcal{M}(Q;1,\dots,n;P)|^2$$

$$= g^{2n}\Bigg(\sum_{P(1,\dots,n)} (a_1 \dots a_n)_{ij} \mathcal{D}(Q;1,\dots,n;P)\Bigg)$$

$$\times\Bigg(\sum_{P(1,\dots,n)} (a_n \dots a_1)_{ji} \mathcal{D}^*(Q;1,\dots,n;P)\Bigg) \qquad (4.12)$$

$$= 2^{-n} g^{2n} N^{n-1}(N^2-1)\Bigg(\sum_{P(1,\dots,n)} |\mathcal{D}(Q;1,\dots,n;P)|^2 + \mathcal{O}(N^{-2})\Bigg), \qquad (4.13)$$

where the last expression shows the leading $N$ behaviour.

### 4.3. THE PROCESS PRODUCING $q\bar{q}$, $n$ GLUONS AND A VECTOR BOSON

The typical structure for this process is depicted in fig. 6. The index $\mu$ denotes a vertex $\Gamma_\mu$ to which a vector boson $V_\mu$ can be attached. Depending on whether this vector boson represents a $W$, $Z$ or timelike or spacelike photon the actual form of $\Gamma_\mu$ will differ. In this vertex the colour structure is $\delta_{kl}$.

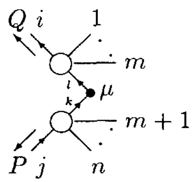



Fig. 6.

The matrix element takes the form

$$\begin{aligned}
&\mathcal{M}(Q, P; V; 1, \ldots, n)\\
&\quad = \sum_{P(1,\ldots,n)} \sum_{m=0}^{n} \frac{1}{m!} \frac{1}{(n-m)!} \hat{J}(Q; 1, \ldots, m) \Gamma_\mu V^\mu \delta_{kl} \hat{J}(m+1, \ldots, n; P)\\
&\quad = g^n \sum_{P(1,\ldots,n)} (a_1 \ldots a_n)_{ij} \mathcal{S}_\mu(Q; 1, \ldots, n; P) V^\mu,
\end{aligned} \tag{4.14}$$

with

$$\mathcal{S}_\mu(Q; 1, \ldots, n; P) = \sum_{m=0}^{n} J(Q; 1, \ldots, m) \Gamma_\mu J(m+1, \ldots, n; P). \tag{4.15}$$

Introducing the $C$ matrix of eq. (3.14) for which

$$C^{-1} \gamma_\mu C = -\gamma_\mu^T, \tag{4.16}$$

we have for $\Gamma_\mu = \gamma_\mu$

$$\mathcal{S}_\mu(Q; 1, \ldots, n; P) = -(-1)^{n-1} \mathcal{S}_\mu(P; n, \ldots, 1; Q). \tag{4.17}$$

The matrix element squared, summed over all colours is again obtained in leading order in $N$

$$|\mathcal{M}(Q, P; V; 1, \ldots, n)|^2 = 2^{-n} g^{2n} N^{n-1} (N^2 - 1) \Big( \sum_{P(1,\ldots,n)} |\mathcal{S}_\mu V^\mu|^2 + \mathcal{O}(N^{-2}) \Big). \tag{4.18}$$

The formulae in this section express cross sections in terms of currents, which can be calculated recursively. A number of explicit expressions for currents will be given in the next section.

## 5. Solutions of the recursion relation for special helicity configurations

In sect. 2 recursion relations for several currents were derived. They can be used to calculate step by step any current with a certain number of gluons having a specific helicity configuration. For some special helicity configurations the answers become so simple and systematic that a generalization to an arbitrary number of gluons presents itself. These generalizations can subsequently be shown to satisfy the recursion relation, so that we have the explicit solution of the recursion relation for an arbitrary $n$. The helicity configurations for which this is possible are those in which all gluon helicities are the same or all but one are the same. In this way we find $J(1+,2+,\dots,n+)$ and $J(1-,2+,3+,\dots,n+)$. With a simple algebraic relation between these currents and the spinorial currents $J(Q;1+,2+,\dots,n+)$ and $J(Q;1-,2+,\dots,n+)$ the latter can also be derived. Furthermore, the quantities $\mathscr{D}(Q;1+,2+,\dots,n+;P)$, $\mathscr{D}(Q;1-,2+,\dots,n+)$ and $\mathscr{S}_\mu(Q;1+,2+,\dots,n+;P)$ can be obtained. It should be noted that the relation between gluonic and spinorial currents is proven without the help of an imbedding of QCD in an $N=1$ supersymmetric theory [10].

Before deriving the explicit expressions for the above currents, it is useful to explain the various steps in the proof for a simpler case. That is the QED case for the emission of $n$ photons with the same helicity in a number of processes [8].

### 5.1. QED EXAMPLES FOR $n$ PHOTON EMISSION

Firstly a recursion relation for a QED spinorial current will be derived. Then it will be solved for the equal helicity configuration of the photons. Analogously to the $n$-gluon spinorial current we introduce an $n$-photon spinorial current with an off-shell positron

$$J(Q;1,\dots,n) = -e\sum_{m=1}^{n} J(Q;1,\dots,m-1,m+1,\dots,n)\not\epsilon(m)\frac{1}{\not Q+\not K(1,n)}, \quad (5.1)$$

where the electron mass is taken to be zero. The no-photon and 1-photon currents are given by

$$J(Q)=\bar{u}(Q), \quad (5.2)$$

$$J(Q;1) = -eJ(Q)\not\epsilon(1)\frac{1}{\not Q+\not K_1}. \quad (5.3)$$

Specifying the electron helicity to $\pm 1$ gives a simple modification of the current

$$J(Q\pm;1,\dots,n)=J(Q;1,\dots,n)\tfrac{1}{2}(1\mp\gamma_5). \quad (5.4)$$

For the current where the electron is off shell and the positron on shell we have

$$J(1,\dots,n;P)=e\frac{1}{\not P+\not K(1,n)}\sum_{m=1}^{n}\not\epsilon(m)J(1,\dots,m-1,m+1,\dots,n;P) \quad (5.5)$$

with $J(P)=v(P)$. Again the helicity of the outgoing positron can be easily specified

$$J(1,\dots,n;P\pm)=\tfrac{1}{2}(1\mp\gamma_5)J(1,\dots,n;P). \tag{5.6}$$

To incorporate the photon helicities it is convenient to translate the currents into the Weyl-van der Waerden spinor formalism. Details of this approach can be found in ref. [3]. For electron helicity $+\frac{1}{2}$ and positron helicity $-\frac{1}{2}$ one has

$$J_{\dot{A}}(Q+)=-iq_{\dot{A}}, \tag{5.7}$$

$$J_{\dot{A}}(Q+;1)=ieq_{\dot{C}}e^{\dot{C}B}(1)\frac{(Q+K_1)_{\dot{A}B}}{(Q+K_1)^2}, \tag{5.8}$$

$$J_{\dot{A}}(Q+;1,\dots,n)=-e\frac{(Q+\kappa(1,n))_{\dot{A}B}}{(Q+\kappa(1,n))^2}\times\sum_{m=1}^{n}J_{\dot{C}}(Q+;1,\dots,m-1,m+1,\dots,n)e^{\dot{C}B}(m), \tag{5.9}$$

$$J_B(P-)=p_B, \tag{5.10}$$

$$J_B(1,\dots,n;P-)=e\frac{(P+\kappa(1,n))_{\dot{C}B}}{(P+\kappa(1,n))^2}\times\sum_{m=1}^{n}J_D(1,\dots,m-1,m+1,\dots,n;P-)e^{\dot{C}D}(m). \tag{5.11}$$

In these expressions the $q$ and $p$ denote the Weyl-van der Waerden spinors related to the null four-vectors $Q$ and $P$. The polarization spinors of the photon take the form

$$e_+^{\dot{A}B}=\sqrt{2}\,\frac{k^{\dot{A}}g^B}{\langle kg\rangle}, \tag{5.12}$$

$$e_-^{\dot{A}B}=\sqrt{2}\,\frac{g^{\dot{A}}k^B}{\langle kg\rangle^*}, \tag{5.13}$$

where the normalization is fixed by $e_\pm\cdot e_\mp=-1$ or $\{e_\pm,e_\mp\}\equiv e_{\pm\dot{A}B}e_\mp^{\dot{A}B}=-2$. The spinor $g$ is the arbitrary gauge spinor. A current like (4.15) can be introduced as well

$$\mathcal{S}_\mu(Q+;1,2,\ldots,n;P-)$$
$$=ie\sum_{P(1\cdots n)}\sum_{m=0}^{n} J(Q+;1\cdots)\gamma_\mu J(m+1,\ldots,n;P-), \quad (5.14)$$

or in spinor language

$$\mathcal{S}_{\dot{A}B}(Q+;1,2,\ldots,n;P-)$$
$$=\sigma^\mu_{\dot{A}B}\mathcal{S}_\mu(Q+;1,\ldots,n;P-)$$
$$=-2e\sum_{P(1\cdots n)}\sum_{m=0}^{n} J_{\dot{A}}(Q+;1,\ldots,m)J_B(m+1,\ldots,n;P-). \quad (5.15)$$

The helicity amplitude for the production of $n$ photons and an electron-positron pair is given by

$$\mathcal{M}(P_-+;1,\ldots,n;P_+-)=\tfrac{1}{2}\mathcal{S}_{\dot{A}B}(P_-+;1,\ldots,n-1;P_+-)e^{\dot{A}B}(n)\big|_{P_++P_-+\kappa(1,n)=0}. \quad (5.16)$$

For the reaction

$$e^+(P_+)+e^-(P_-)\to\mu^+(Q_+)+\mu^-(Q_-)+\gamma(K_1)+\cdots+\gamma(K_n) \quad (5.17)$$

with the photon emission off the muons the amplitude reads

$$\mathcal{M}(P_+-;P_-+;Q_+-;Q_-+;1,\ldots,n)=e\frac{p_{+\dot{A}}p_{-B}}{(P_++P_-)^2}\mathcal{S}^{\dot{A}B}(Q_-+;1,\ldots,n;Q_+-), \quad (5.18)$$

where

$$P_++P_-=Q_++Q_-+\kappa(1,n). \quad (5.19)$$

In order to evaluate (5.16) and (5.18) for certain helicity combinations, the currents $J(P_-+;1+,\ldots,n+)$, $J(1+,\ldots,n+;P_+-)$ and $\mathcal{S}_\mu(P_-+;1+,\ldots,n+;P_+-)$ are required. The actual calculation of the latter will be simplified by choosing as gauge-spinor $p_+$, such that

$$e_+^{\dot{A}B}(m) = \sqrt{2}\,\frac{k_m^{\dot{A}} p_+^B}{\langle k_m p_+ \rangle}\,. \tag{5.20}$$

The one-photon currents for these specific helicities are

$$J_{\dot{A}}(P_-+;1+) = i\sqrt{2}\,e\frac{(P_-+K_1)_{\dot{A}B}}{(P_-+K_1)^2}\frac{k_1^{\dot{C}}p_+^B}{\langle k_1 p_+ \rangle}p_{-\dot{C}}$$

$$= i\sqrt{2}\,e\frac{(P_-+K_1)_{\dot{A}B}p_+^B}{\langle p_+ k_1 \rangle \langle k_1 p_- \rangle}\,, \tag{5.21}$$

$$J_B(1+;P_+-) = 0\,, \tag{5.22}$$

$$\mathscr{S}_{\dot{A}B}(P_-+;1+;P_+-) = -2eJ_{\dot{A}}(P_-+;1+)p_{+B} \tag{5.23}$$

$$= -2i\sqrt{2}\,e^2\frac{(P_-+K_1)_{\dot{A}C}p_+^C p_{+B}}{\langle p_+ k_1 \rangle \langle k_1 p_- \rangle}\,. \tag{5.24}$$

Inserting (5.21) in (5.9) gives the two-photon current

$$J_{\dot{A}}(P_-+;1+,2+) = -e\frac{(P_-+K_1+K_2)_{\dot{A}B}}{(P_-+K_1+K_2)^2}$$

$$\times\left[J_{\dot{C}}(P_-+;1+)e^{\dot{C}B}(2) + J_{\dot{C}}(P_-+;2+)e^{\dot{C}B}(1)\right]$$

$$= -i(\sqrt{2}\,e)^2\frac{(P_-+K_1+K_2)_{\dot{A}B}}{(P_-+K_1+K_2)^2}$$

$$\times\left\{\frac{(P_-+K_1)_{\dot{C}D}p_+^D k_2^{\dot{C}} p_+^B \langle k_2 p_- \rangle}{\langle p_+ k_1 \rangle \langle k_1 p_- \rangle \langle k_2 p_+ \rangle \langle k_2 p_- \rangle}\right.$$

$$\left. + \frac{(P_-+K_2)_{\dot{C}D}p_+^D k_1^{\dot{C}} p_+^B \langle k_1 p_- \rangle}{\langle p_+ k_2 \rangle \langle k_2 p_- \rangle \langle k_1 p_+ \rangle \langle k_1 p_- \rangle}\right\}. \tag{5.25}$$

Using

$$(P_-+K_i)_{\dot{C}D}k_j^{\dot{C}} = (P_-+K_i+K_j)_{\dot{C}D}k_j^{\dot{C}}\,, \tag{5.26}$$

and (cf. eq. (A.14))

$$(P_- + K_1 + K_2)_{\dot{C}D} p_+^D (K_1 + K_2)^{\dot{C}E} p_{-E}$$

$$= (P_- + K_1 + K_2)_{\dot{C}D} (P_- + K_1 + K_2)^{\dot{C}E} p_+^D p_{-E} \tag{5.27}$$

$$= -(P_- + K_1 + K_2)^2 \langle p_+ p_- \rangle \tag{5.28}$$

we find for eq. (5.25)

$$J_{\dot{A}}(P_-+;1+,2+) = i(\sqrt{2}\,e)^2 (P_- + K_1 + K_2)_{\dot{A}B} p_+^B \frac{\langle p_+ p_- \rangle}{\prod_{i=1}^2 \langle p_+ k_i \rangle \langle k_i p_- \rangle}. \tag{5.29}$$

For the other two-photon currents we find

$$J_B(1+,2+;P_+-) = 0, \tag{5.30}$$

and

$$\mathscr{S}_{\dot{A}B}(P_-+;1+,2+;P_+-)$$

$$= -i2(\sqrt{2})^2 e^3 (P_- + K_1 + K_2)_{\dot{A}C} p_+^C p_{+B} \frac{\langle p_+ p_- \rangle}{\prod_{i=1}^2 \langle p_+ k_i \rangle \langle k_i p_- \rangle}. \tag{5.31}$$

A generalization of these results for arbitrary $n$ presents itself in the form

$$J_{\dot{A}}(P_-+;1+,2+,\ldots,n+)$$

$$= i(\sqrt{2}\,e)^n (P_- + \kappa(1,n))_{\dot{A}B} p_+^B \frac{\langle p_+ p_- \rangle^{n-1}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle}, \tag{5.32}$$

$$J_B(1+,2+,\ldots,n+;P_+-) = 0, \tag{5.33}$$

$$\mathscr{S}_{\dot{A}B}(P_-+;1+,\ldots,n+;P_+-)$$

$$= -i2(\sqrt{2})^n e^{n+1} (P_- + \kappa(1,n))_{\dot{A}C} p_+^C p_{+B} \frac{\langle p_+ p_- \rangle^{n-1}}{\prod_{i=1}^n \langle p_+ k_i \rangle \langle k_i p_- \rangle}. \tag{5.34}$$

The conjecture (5.32) requires a proof, the other two are obvious. Assume eq. (5.32) to be valid for $l < n$, then for $n$ photons we have from eq. (5.9):

$$J_{\dot A}(P_-+;1+,\dots,n+)$$

$$= -e\frac{(P_-+\kappa(1,n))}{(P_-+\kappa(1,n))^2}\sum_{m=1}^{n} J_{\dot C}(P_-;1,\dots,m-1,m+1,\dots,n)e^{\dot C B}(m)$$

$$= i(\sqrt{2}\,e)^n\frac{(P_-+\kappa(1,n))_{\dot A B}}{(P_-+\kappa(1,n))^2}p_+^B(P_-+\kappa(1,n))_{\dot C D}\sum_{m=1}^{n} p_+^D k_m^{\dot C}\langle k_m p_-\rangle$$

$$\times\frac{\langle p_+p_-\rangle^{n-2}}{\prod_{i=1}^n\langle p_+k_i\rangle\langle k_ip_-\rangle} \tag{5.35}$$

$$= i(\sqrt{2}\,e)^n(P_-+\kappa(1,n))_{\dot A B}p_+^B\frac{\langle p_+p_-\rangle^{n-1}}{\prod_{i=1}^n\langle p_+k_i\rangle\langle k_ip_-\rangle}\,. \tag{5.36}$$

To obtain eq. (5.35) use has been made of eq. (5.26). For eq. (5.36) steps (5.27) and (5.28) have been used for the case of $\kappa(1,n)$ instead of $\kappa(1,2)$.

Insertion in the amplitudes (5.16) and (5.18) gives

$$\mathscr{M}(P_-+;1+,\dots,n+;P_+-)=0\,, \tag{5.37}$$

$$\mathscr{M}(P_-+;1+,\dots,(n-1)+,n-;P_+-)$$

$$= -i(\sqrt{2})^n e^n(-P_+-K_n)_{\dot A C}\frac{p_+^C p_{+B}p_+^{\dot A}k_n^B}{\langle k_np_+\rangle^*}\frac{\langle p_+p_-\rangle^{n-2}}{\prod_{i=1}^{n-1}\langle p_+k_i\rangle\langle k_ip_-\rangle}$$

$$= -i(\sqrt{2}\,e)^n\frac{\langle p_+k_n\rangle^2\langle p_+p_-\rangle^{n-2}}{\prod_{i=1}^{n-1}\langle p_+k_i\rangle\langle k_ip_-\rangle}\,, \tag{5.38}$$

$$\mathscr{M}(P_+-;P_-+;Q_+-;Q_-+;1+,\dots,n+)$$

$$= -i2(\sqrt{2})^n e^{n+2}(-Q_++P_++P_-)_{\dot A C}\frac{q_+^C q_{+B}p_+^{\dot A}p_-^B}{|\langle p_+p_-\rangle|^2}\frac{\langle q_+q_-\rangle^{n-1}}{\prod_{i=1}^n\langle q_+k_i\rangle\langle k_iq_-\rangle}$$

$$= i(\sqrt{2}\,e)^{n+2}\frac{\langle p_-q_+\rangle^2}{\langle p_-p_+\rangle}\frac{\langle q_+q_-\rangle^{n-1}}{\prod_{i=1}^n\langle q_+k_i\rangle\langle k_iq_-\rangle}\,. \tag{5.39}$$

Changing the fermion helicities in (5.38) and (5.39) to opposite values amounts to the replacements $\langle p_+k_n\rangle \to \langle p_-k_n\rangle$ and $\langle p_-q_+\rangle \to \langle p_+q_-\rangle$. The results are in agreement with those of ref. [8].

For the cross section of $e^+e^- \to n$ photons or $e^+e^- \to \mu^+\mu^- + n$ photons, we can just square the expressions (5.38) and (5.39). In the former the momenta $P_+, P_-$ should be changed into $-P_+, -P_-$ and the helicities of the incoming electron and positron are $-1$ and $+1$ respectively.

### 5.2. THE GLUON RECURSION RELATIONS

The gluon recursion relation (2.19) will be solved for two special helicity configurations. The configurations are those where all helicities are the same (here $+1$) or all but one are the same. One can then choose a gauge for the helicity spinors such that for all polarization vectors $(e_i \cdot e_j) = 0$. Through the recursion relation (2.19) the currents keep this orthogonality property, thus the 4-vertex contributions vanish. As in the previous subsection we use the Weyl-van der Waerden spinor calculus. We have

$$J_{\dot{A}B}(1,\dots,n) = \frac{1}{2\kappa(1,n)^2} \sum_{m=1}^{n-1} \left[ J(1,\dots,m), J(m+1,\dots,n) \right], \tag{5.40}$$

where in eq. (2.4) and (2.7) the inner product between 4-vectors has been replaced by spinor contractions e.g.

$$\begin{aligned} \kappa(m+1,n) \cdot J(1,\dots,m) &\to \{ \kappa(m+1,n), J(1,\dots,m) \} \\ &\equiv \kappa_{\dot{C}D}(m+1,n) J^{\dot{C}D}(1,\dots,m), \end{aligned} \tag{5.41}$$

which necessitates the factor $\frac{1}{2}$ in eq. (5.40).

A choice of gauge spinors which gives $\{e_i, e_j\} = 0$ is for $J(1^+2^+ \dots n^+)$

$$e^+_{\dot{A}B}(i) = -\sqrt{2}\, \frac{k_{i\dot{A}} b_{+B}}{\langle i+ \rangle}, \qquad 1 \leqslant i \leqslant n, \tag{5.42}$$

and for $J(1^-2^+ \dots n^+)$

$$e^-_{\dot{A}B}(1) = -\sqrt{2}\, \frac{k_{2\dot{A}} k_{1B}}{\langle 12 \rangle^*}, \tag{5.43}$$

$$e^+_{\dot{A}B}(i) = -\sqrt{2}\, \frac{k_{i\dot{A}} k_{1B}}{\langle i1 \rangle}, \qquad 2 \leqslant i \leqslant n, \tag{5.44}$$

where $b_+$ is at the moment an arbitrary spinor and where

$$\langle i+ \rangle = \langle k_i b_+ \rangle, \qquad \langle ij \rangle = \langle k_i k_j \rangle. \tag{5.45}$$

The property $J_{\dot{A}B}(i) = X_{\dot{A}}(i) b_B$ with $b = b_+$ or $b = k_1$ extends through the recursion relation (5.40) to the $n$-gluon current

$$J_{\dot{A}B}(1,\ldots,n) = X_{\dot{A}}(1,\ldots,n) b_B . \tag{5.46}$$

The recursion relation takes the simple form

$$J_{\dot{A}B}(1,\ldots,n) = \frac{1}{\kappa(1,n)^2} \sum_{m=1}^{n-1} \big( \{\kappa(m+1,n), J(1,\ldots,m)\} J_{\dot{A}B}(m+1,\ldots,n) - \{\kappa(1,m), J(m+1,\ldots,n)\} J_{\dot{A}B}(1,\ldots,m) \big) . \tag{5.47}$$

Consider the equal helicity case in more detail

$$\begin{aligned} J_{\dot{A}B}(1+,2+) &= \frac{1}{\kappa(1,2)^2} \left( \{K_2, e(1)\} e_{\dot{A}B}(2) - \{K_1, e(2)\} e_{\dot{A}B}(1) \right) \\ &= 2 b_{+B} \frac{\langle 21\rangle^* \langle 2+\rangle k_{2\dot{A}} - \langle 12\rangle^* \langle 1+\rangle k_{1\dot{A}}}{\langle 1+\rangle \langle 2+\rangle \langle 12\rangle \langle 12\rangle^*} \\ &= 2 \frac{\kappa_{\dot{A}C}(1,2) b_+^C b_{+B}}{\langle +1\rangle \langle 12\rangle \langle 2+\rangle} . \end{aligned} \tag{5.48}$$

This leads to the conjecture

$$J_{\dot{A}B}(1+,2+,\ldots,m+) = (\sqrt{2})^m \frac{\kappa_{\dot{A}C}(1,m) b_+^C b_{+B}}{\langle\langle +1, m+\rangle\rangle} , \tag{5.49}$$

where $\langle\langle +1, m+\rangle\rangle = \langle +1\rangle \langle 12\rangle \cdots \langle m-1, m\rangle \langle m+\rangle$. For one gluon eq. (5.49) reduces to eq. (5.42), which explains the introduction of a minus sign in the definition of the polarization vector. The conjecture is proven by induction. Suppose (5.49) to be valid for $m < n$, then using (5.47) we find (cf. eq. (A.14))

$$\begin{aligned} J_{\dot{A}B}&(1+,2+,\ldots,n+) \\ &= \frac{1}{\kappa(1,n)^2} \sum_{m=1}^{n-1} \left( \kappa_{\dot{C}D}(m+1,n) (\sqrt{2})^m \frac{\kappa_E^{\dot{C}}(1,m) b_+^E b_+^D}{\langle\langle +1, m+\rangle\rangle} J_{\dot{A}B}(m+1,\ldots,n) \right. \\ &\qquad \left. - \kappa_{\dot{C}D}(1,m) (\sqrt{2})^{n-m} \frac{\kappa_E^{\dot{C}}(m+1,n) b_+^E b_+^D}{\langle\langle +(m+1), n+\rangle\rangle} J_{\dot{A}B}(1,\ldots,m) \right) \end{aligned} \tag{5.50}$$

$$= (\sqrt{2})^n \frac{1}{\kappa(1,n)^2} \kappa_{\dot{A}C}(1,n) b_+^C b_{+B} P_1^{n-1} , \tag{5.51}$$

where

$$P_1^{n-1} = -\sum_{m=1}^{n-1} \frac{\kappa_{\dot{C}D}(1,m) b_+^D \kappa_E^{\dot{C}}(m+1,n) b_+^E}{\langle\langle +1, m+\rangle\rangle\langle\langle +(m+1), n+\rangle\rangle}. \tag{5.52}$$

When it is shown that

$$P_1^{n-1} = \frac{\kappa(1,n)^2}{\langle\langle +1, n+\rangle\rangle}, \tag{5.53}$$

we have proven the conjecture (5.49).

The validity of (5.53) can again be shown by induction. For $n=2$ it is easily verified to be correct. Suppose $P_1^{n-2}$ is valid, then

$$\begin{aligned} P_1^{n-1} = & -\sum_{m=1}^{n-2} \frac{\kappa_{\dot{C}D}(1,m) b_+^D \kappa_E^{\dot{C}}(m+1,n-1) b_+^E}{\langle\langle +1, m+\rangle\rangle\langle\langle +(m+1),(n-1)+\rangle\rangle} \frac{\langle n-1, +\rangle}{\langle n-1, n\rangle\langle n+\rangle} \\ & -\sum_{m=1}^{n-1} \frac{\kappa_{\dot{C}D}(1,m) b_+^D K_{nE}^{\dot{C}} b_+^E}{\langle\langle +1, n+\rangle\rangle} \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle} \\ = & \frac{1}{\langle\langle +1, n+\rangle\rangle} \\ & \times\left(\kappa(1,n-1)^2 - \sum_{m=1}^{n-1} \kappa_{\dot{C}D}(1,m) b_+^D K_{nE}^{\dot{C}} b_+^E \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle}\right) \\ = & \frac{1}{\langle\langle +1, n+\rangle\rangle} \\ & \times\left(\kappa(1,n-1)^2 - \sum_{i=1}^{n-1}\left(\sum_{m=i}^{n-1} K_{i\dot{C}D} b_+^D K_{nE}^{\dot{C}} b_+^E \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle}\right)\right). \end{aligned} \tag{5.54}$$

Since (cf. eq. (A.13))

$$\frac{\langle ab\rangle}{\langle a+\rangle\langle +b\rangle} + \frac{\langle bc\rangle}{\langle b+\rangle\langle +c\rangle} = \frac{\langle ac\rangle}{\langle a+\rangle\langle +c\rangle}, \tag{5.55}$$

we have

$$\sum_{m=i}^{n-1} \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle} = \frac{\langle in\rangle}{\langle i+\rangle\langle +n\rangle}, \tag{5.56}$$

$$\sum_{m=i}^{n-1} K_{i\dot{C}D} b_+^D K_{nE}^{\dot{C}} b_+^E \frac{\langle m, m+1\rangle}{\langle m+\rangle\langle +, m+1\rangle} = -2K_i \cdot K_n, \tag{5.57}$$

such that eq. (5.54) indeed reduces to (5.53).

In the case of $J(1-2+\cdots n+)$ we use as starting point the currents

$$J_{\dot{A}B}(1-) = e^-_{\dot{A}B}(1) = -\sqrt{2}\,\frac{k_{2\dot{A}} k_{1B}}{\langle 12\rangle^*}, \tag{5.58}$$

$$J_{\dot{A}B}(i+\cdots m+) = (\sqrt{2})^{m-i+1} \frac{\kappa_{\dot{A}C}(i, m) k_{1B} k_1^C}{\langle\langle 1i, m1\rangle\rangle}. \tag{5.59}$$

Using these currents and eq. (5.47) we find

$$J(1-2+) = \frac{1}{2\kappa(1,2)^2}[J(1-), J(2+)] = 0, \tag{5.60}$$

$$J(1-2+3+) = \frac{1}{\kappa(1,3)^2}\{\kappa(2,3), J(1-)\} J(2+3+)$$

$$= -\sqrt{2}\, J(2+3+) \frac{\langle 31\rangle k_{3\dot{C}} k_{1D} K_2^{\dot{C}D}}{\kappa(1,2)^2 \kappa(1,3)^2}. \tag{5.61}$$

The induction conjecture for $l \geqslant 3$ is given by

$$J(1-2+\cdots l+) = -\sqrt{2}\, J(2+\cdots l+) c_l \tag{5.62}$$

where

$$c_l = \sum_{m=3}^{l} \lambda_m, \tag{5.63}$$

$$\lambda_m = \frac{\langle m1\rangle k_{m\dot{C}} k_{1D} \kappa^{\dot{C}D}(2, m)}{\kappa(1, m-1)^2 \kappa(1, m)^2}. \tag{5.64}$$

With eq. (5.62) for $l < n$ in the recursion relation (5.47) we have

$$\begin{aligned}
&J(1-2+\cdots n+) \\
&= \frac{1}{\kappa(1,n)^2}\Bigg(\{\kappa(2,n), J(1-)\}J(2+\cdots n+) \\
&\qquad + \sum_{m=3}^{n-1}[J(1-2+\cdots m+), J((m+1)+\cdots n+)]\Bigg) \qquad (5.65)\\
&= -\sqrt{2}\,\frac{J(2+\cdots n+)}{\kappa(1,n)^2}\,\frac{\kappa_{\dot{C}D}(3,n)k_2^{\dot{C}}k_1^{D}}{\langle 12\rangle^*} \\
&\quad -\sqrt{2}\sum_{m=3}^{n-1}\frac{c_m}{\kappa(1,n)^2}\big(\{\kappa(m+1,n), J(2+\cdots m+)\}J((m+1)+\cdots n+) \\
&\qquad -\{\kappa(1,m), J((m+1)+\cdots n+)\}J(2+\cdots m+)\big) \qquad (5.66)\\
&= -\sqrt{2}\,\frac{J(2+\cdots n+)}{\kappa(1,n)^2}\Bigg(\frac{\kappa_{\dot{C}D}(3,n)k_2^{\dot{C}}k_1^{D}}{\langle 12\rangle^*} \\
&\qquad - \sum_{m=3}^{n-1}c_m\kappa_{\dot{C}D}(2,m)k_1^{D}\kappa_E^{\dot{C}}(m+1,n)k_1^{E}\frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}\Bigg). \qquad (5.67)
\end{aligned}$$

The way in which eq. (5.67) is obtained is similar to that of arriving at eq. (5.51). The proportionality factor in eq. (5.67) is $c_{n-1}$ for $J(2+\cdots(n-1)+)$ from the induction hypothesis and it should be proven that it is $c_n$ for $J(2+\cdots n+)$. The difference of these proportionality factors is from eq. (5.63) known to be

$$\kappa(1,n)^2c_n - \kappa(1,n-1)^2c_{n-1} = \kappa(1,n)^2\lambda_n + 2K_n\cdot\kappa(1,n-1)c_{n-1}, \qquad (5.68)$$

whereas from eq. (5.67) it follows

$$\begin{aligned}
&\kappa(1,n)^2c_n - \kappa(1,n-1)^2c_{n-1} \\
&\quad = \frac{K_{n\dot{C}D}k_2^{\dot{C}}k_1^{D}}{\langle 12\rangle^*} - \sum_{m=3}^{n-1}c_m\kappa_{\dot{C}D}(2,m)k_1^{D}K_{nE}^{\dot{C}}k_1^{E}\frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}. \qquad (5.69)
\end{aligned}$$

In order to prove that eq. (5.69) reduces to eq. (5.68) we carry out the summation in the second term of eq. (5.69), using eqs. (5.56), (5.57), (5.63), (5.64), (A.13) and (A.14)

$$\sum_{m=3}^{n-1}\left(\sum_{i=3}^{m}\lambda_i\right)\kappa_{\dot{C}D}(2,m)k_1^D K_{nE}^{\dot{C}}k_1^E\frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}$$
$$=\sum_{i=3}^{n-1}\lambda_i\left(\sum_{m=i}^{n-1}\kappa_{\dot{C}D}(2,m)k_1^D K_{nE}^{\dot{C}}k_1^E\frac{\langle m,m+1\rangle}{\langle m1\rangle\langle 1,m+1\rangle}\right)$$
$$=\sum_{m=3}^{n-1}\lambda_m\left(-\kappa_{\dot{A}B}(1,m-1)k_n^{\dot{A}}k_1^B\frac{\langle mn\rangle}{\langle m1\rangle}-2\kappa(m,n-1)\cdot K_n\right)$$
$$=\sum_{m=3}^{n-1}\lambda_m\left(-2\kappa(1,n-1)\cdot K_n+\kappa_{\dot{A}B}(1,m-1)k_n^{\dot{A}}k_m^B\frac{\langle n1\rangle}{\langle m1\rangle}\right)$$
$$=-2K_n\cdot\kappa(1,n-1)c_{n-1}+\langle n1\rangle\sum_{m=3}^{n-1}\frac{k_{m\dot{C}}k_{1D}\kappa^{\dot{C}D}(1,m)\kappa_{\dot{A}B}(1,m)k_n^{\dot{A}}k_m^B}{\kappa(1,m-1)^2\kappa(1,m)^2}$$
$$=-2K_n\cdot\kappa(1,n-1)c_{n-1}+X. \tag{5.70}$$

At this point the first term in eq. (5.70) gives indeed the second term in eq. (5.68). The rest $R$ should give the term $\kappa(1,n)^2\lambda_n$. We find

$$R=\frac{K_{n\dot{C}D}k_2^{\dot{C}}k_1^D}{\langle 12\rangle^*}-X$$
$$=-\langle n1\rangle\Bigg\{-\frac{\langle n2\rangle^*}{\langle 12\rangle^*}+k_{1D}k_{n\dot{A}}$$
$$\times\sum_{m=3}^{n-1}\frac{2\kappa(1,m-1)\cdot K_m\kappa^{\dot{A}D}(1,m-1)-K_m^{\dot{C}D}\kappa_{\dot{C}B}(1,m-1)\kappa^{\dot{A}B}(1,m-1)}{\kappa(1,m-1)^2\kappa(1,m)^2}\Bigg\} \tag{5.71}$$

$$=-\langle n1\rangle\Bigg\{-\frac{\langle n2\rangle^*}{\langle 12\rangle^*}+k_{1D}k_{n\dot{A}}$$
$$\times\sum_{m=3}^{n-1}\left[\kappa^{\dot{A}D}(1,m-1)\left(\frac{1}{\kappa(1,m-1)^2}-\frac{1}{\kappa(1,m)^2}\right)-\frac{K_m^{\dot{A}D}}{\kappa(1,m)^2}\right]\Bigg\}$$
$$=-\langle n1\rangle k_{1D}k_{n\dot{A}}\left\{-\frac{\kappa^{\dot{A}D}(1,2)}{\kappa(1,2)^2}+\sum_{m=3}^{n-1}\left[\frac{\kappa^{\dot{A}D}(1,m-1)}{\kappa(1,m-1)^2}-\frac{\kappa^{\dot{A}D}(1,m)}{\kappa(1,m)^2}\right]\right\} \tag{5.72}$$
$$=\langle n1\rangle\frac{k_{1D}k_{n\dot{A}}\kappa^{\dot{A}D}(1,n)}{\kappa(1,n-1)^2}, \tag{5.73}$$

where use has been made of eq. (A.14) for obtaining eqs. (5.71) and (5.72). The final result (5.73) is indeed $\kappa(1,n)^2\lambda_n$, as can be seen from eq. (5.64). Thus the general form (5.62) for $J(1-2+\cdots l+)$ is valid. Complex conjugation of the currents (5.49) and (5.64) gives $J(1-2-\ldots n-)$ and $J(1+2-\ldots n-)$.

### 5.3. THE QUARK CURRENT AND ITS RECURSION RELATION

Firstly we rewrite the recursion relation (5.47), which is valid for the special gauge choice (5.42)–(5.44). Since it follows in general from eq. (A.13) that

$$\begin{aligned}
&\kappa_{\dot{A}C}(1,n)J_{\dot{B}D}(1,\ldots,m)J^{\dot{B}C}(m+1,\ldots,n)\\
&\quad=\{\kappa(1,n),J(m+1,\ldots,n)\}J_{\dot{A}D}(1,\ldots,m)\\
&\qquad-\{\kappa(m+1,n),J(1,\ldots,m)\}J_{\dot{A}D}(m+1,\ldots,n)\\
&\qquad+J_{\dot{A}C}(m+1,\ldots,n)J^{\dot{B}C}(1,\ldots,m)\kappa_{\dot{B}D}(1,n),
\end{aligned}\tag{5.74}$$

we obtain from eq. (5.47) another form of the recursion relation

$$J_{\dot{A}D}(1,\ldots,n)=-\frac{\kappa_{\dot{A}C}(1,n)}{\kappa(1,n)^2}\sum_{m=1}^{n-1}J_{\dot{B}D}(1,\ldots,m)J^{\dot{B}C}(m+1,\ldots,n).\tag{5.75}$$

For this, use has been made of current conservation and the special form (5.46), which leads to the vanishing of the last term in eq. (5.47). In terms of $X_{\dot{A}}(1,\ldots,n)$ one has

$$X_{\dot{A}}(1,\ldots,n)=-\frac{\kappa_{\dot{A}C}(1,n)}{\kappa(1,n)^2}\sum_{m=1}^{n-1}X_{\dot{B}}(1,\ldots,m)J^{\dot{B}C}(m+1,\ldots,n).\tag{5.76}$$

For a quark with positive helicity the recursion relation (3.9) is translated into the Weyl-van der Waerden spinor formalism in the same way as was done for the photon current in subsect. 5.1

$$J_{\dot{A}}(Q+;1,\ldots,n)=-\frac{[Q+\kappa(1,n)]_{\dot{A}C}}{[Q+\kappa(1,n)]^2}\sum_{m=0}^{n-1}J_{\dot{B}}(Q+;1,\ldots,m)J^{\dot{B}C}(m+1,\ldots,n),\tag{5.77}$$

with

$$J_{\dot{B}}(Q+)=-iq_{\dot{B}}.\tag{5.78}$$

We now see that the recursion relations for $X_{\dot{A}}(K_1,\ldots,K_n)$ and $J_{\dot{A}}(K_1;K_2,\ldots,K_n)$

are the same. The starting point for $X_{\dot{B}}(1+2+\cdots(n+1)+)$ is

$$X_{\dot{B}}(1+) = -\sqrt{2}\,\frac{k_{1\dot{B}}}{\langle 1+\rangle}\,. \tag{5.79}$$

Thus we find

$$J_{\dot{A}}(Q+;1+2+\cdots n+) = i\sqrt{\tfrac{1}{2}}\,\langle q+\rangle X_{\dot{A}}(Q+1+2+\cdots n+) \tag{5.80}$$

$$= i(\sqrt{2})^n \langle q+\rangle \frac{(Q+\kappa(1,n))_{\dot{A}C}b_+^C}{\langle +q\rangle\langle q1\rangle\langle 12\rangle\cdots\langle n+\rangle}$$

$$= -i(\sqrt{2})^n \frac{(Q+\kappa(1,n))_{\dot{A}C}b_+^C}{\langle q1\rangle\langle 12\rangle\cdots\langle n+\rangle}\,. \tag{5.81}$$

From the expression for $X_{\dot{A}}(Q+2+\cdots m+1-(m+1)+\cdots n+1)$, which one could evaluate in the gauge (5.43)–(5.44), we have similarly

$$J_{\dot{A}}(Q+;2+\cdots m+1-(m+1)+\cdots n+)$$

$$= i\sqrt{\tfrac{1}{2}}\,\langle q1\rangle X_{\dot{A}}(Q+2+\cdots m+1-(m+1)+\cdots n+)\,. \tag{5.82}$$

## 6. The amplitudes for specific helicity configurations

Since we have solved the recursion relation for currents in cases of specific helicity configurations we can calculate amplitudes for these situations as well. We do this for $n$-gluon scattering with and without the production of other particles. The additional particles are a quark-antiquark pair alone or in combination with a vector boson W, Z or $\gamma^*$.

The results prove certain conjectures in the literature to be correct.

### 6.1. SCATTERING OF $n$ GLUONS

From the currents we make $\mathscr{C}$-functions and from them the helicity amplitudes according to eqs. (4.1) and (4.2). With the explicit expression for $J^{\dot{A}B}(2+\cdots n+)$ in eq. (5.49) we have

$$\mathscr{C}(1\pm 2+\cdots n+) = \tfrac{1}{2}\kappa(2,n)^2 e^{\pm}_{1\dot{A}B} J^{\dot{A}B}(2+\cdots n+)\Big|_{\kappa(1,n)=0}$$

$$= \tfrac{1}{2}(\sqrt{2})^{n-1}\kappa(2,n)^2 e^{\pm}_{1\dot{A}B}\frac{\kappa_C^{\dot{A}}(2,n)b_+^C b_+^B}{\langle\langle +2,n+\rangle\rangle}\Bigg|_{\kappa(1,n)=0}$$

$$= 0\,. \tag{6.1}$$

The vanishing of this $\mathscr{C}$-function is due to the overall momentum conservation, which leads to a vanishing $\kappa(2, n)^2$. With the cyclic symmetry of the $\mathscr{C}$-function, also the $\mathscr{C}$-function with one negative helicity in an arbitrary position vanishes. The helicity amplitude then vanishes as well

$$\mathscr{M}(1 \pm 2+ \cdots n+) = 0. \tag{6.2}$$

The first non-trivial helicity amplitude is $\mathscr{M}(1-2-3+\cdots n+)$, for which we have to know $\mathscr{C}(1-3+\cdots m+2-(m+1)+\cdots n+)$. With the current $J(2-3+\cdots n+)$ we only obtain a $\mathscr{C}$-function with adjacent negative helicity gluons, arising from the $\kappa(2,n)^{-2}$ term in eq. (5.62):

$$\begin{aligned}
\mathscr{C}(1-2-3+\cdots n+) &= \tfrac{1}{2}\kappa(2,n)^2 e^{-}_{1\dot{A}B} J^{\dot{A}B}(2-3+\cdots n+)\Big|_{\kappa(1,n)=0} \\
&= \frac{k_{3\dot{A}}k_{1B}}{\langle 13\rangle^*} J^{\dot{A}B}(3+\cdots n+) \frac{\langle n2\rangle k_{2C}k_{n\dot{D}}\kappa^{\dot{D}C}(2,n)}{\kappa(2,n-1)^2}\Bigg|_{\kappa(1,n)=0} \\
&= (\sqrt{2})^{n-2} \frac{k_{3\dot{A}}k_{1B}K^{\dot{A}}_{1E}k^{B}_{2}k^{E}_{2}\langle n2\rangle k_{2C}k_{n\dot{D}}K^{\dot{D}C}_{1}}{\langle 13\rangle^*\langle\langle 23, n2\rangle\rangle\langle n1\rangle\langle n1\rangle^*}\Bigg|_{\kappa(1,n)=0} \\
&= \frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle\langle 12, n1\rangle\rangle}.
\end{aligned} \tag{6.3}$$

For gluon 1 and 2 polarization vectors (5.43) are used with $k_{3\dot{A}}$ as gauge spinor. From the subcyclic and cyclic identity we find

$$\begin{aligned}
&\mathscr{C}(1-3+2-4+\cdots n+) \\
&\quad = -\mathscr{C}(1-2-3+4+\cdots n+) \\
&\qquad -\mathscr{C}(1-2-4+3+\cdots n+) - \cdots - \mathscr{C}(1-2-4+\cdots n+3+) \\
&\quad = -\mathscr{C}(2-3+4+\cdots n+1-) - \mathscr{C}(2-4+3+\cdots n+1-) \\
&\qquad - \cdots - \mathscr{C}(2-4+\cdots n+3+1-).
\end{aligned} \tag{6.4}$$

From eq. (A.13) one derives

$$[\langle ab\rangle\langle bc\rangle\langle cd\rangle]^{-1} + [\langle ac\rangle\langle cb\rangle\langle bd\rangle]^{-1} = \frac{\langle ad\rangle}{\langle ab\rangle\langle bd\rangle}[\langle ac\rangle\langle cd\rangle]^{-1}, \tag{6.5}$$

and in general

$$[\langle ab\rangle\langle bc\rangle\langle cd\rangle\cdots\langle yz\rangle]^{-1}+[\langle ac\rangle\langle cb\rangle\langle bd\rangle\cdots\langle yz\rangle]^{-1}$$
$$+\cdots+[\langle ac\rangle\langle cd\rangle\cdots\langle yb\rangle\langle bz\rangle]^{-1}$$
$$=\frac{\langle az\rangle}{\langle ab\rangle\langle bz\rangle}[\langle ac\rangle\langle cd\rangle\cdots\langle yz\rangle]^{-1}. \tag{6.6}$$

With these relations we obtain

$$\mathscr{C}(1-3+2-4+\cdots n+)=\frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 45\rangle\cdots\langle n1\rangle}. \tag{6.7}$$

Using again the subcyclic, cyclic properties and eq. (6.7) we arrive at

$$\mathscr{C}(1-3+4+2-5+\cdots n+)+\mathscr{C}(1-4+3+2-5+\cdots n+)$$
$$=\frac{(\sqrt{2})^n}{2}\left(\frac{\langle 12\rangle}{\langle 24\rangle\langle 41\rangle}\right)\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 25\rangle\langle 56\rangle\cdots\langle n1\rangle}$$
$$=\frac{(\sqrt{2})^n}{2}\left(\frac{1}{\langle 13\rangle\langle 34\rangle\langle 42\rangle}+\frac{1}{\langle 14\rangle\langle 43\rangle\langle 32\rangle}\right)\frac{\langle 12\rangle^4}{\langle 25\rangle\langle 56\rangle\cdots\langle n1\rangle}. \tag{6.8}$$

Since we know from the recursion relations what kind of pole terms belong to a $\mathscr{C}$-function we see that the first and second term on the left hand side correspond to the first and second term on the right hand side. Repeating these arguments we find in the general case (details can be found in appendix C)

$$\mathscr{C}(1-3+4+\cdots m+2-(m+1)+\cdots n+)$$
$$=\frac{(\sqrt{2})^n}{2}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle\cdots\langle m2\rangle\langle 2(m+1)\rangle\cdots\langle n1\rangle}, \tag{6.9}$$

which gives the amplitude

$$\mathscr{M}(1-2-3+\cdots n+)=i(\sqrt{2})^n g^{n-2}\langle 12\rangle^4\sum_{P(1,\ldots,n-1)}\frac{(a_1\ldots a_n)}{\langle\langle 12,n1\rangle\rangle}. \tag{6.10}$$

The contribution to the cross section arises from

$$|\mathcal{M}(1-2-3+\cdots n+)|^2$$

$$= g^{2n-4}2^{4-n}N^{n-2}(N^2-1)$$

$$\times (K_1\cdot K_2)^4\left[\sum_{P(1,\dots,n-1)}\frac{1}{(K_1\cdot K_2)(K_2\cdot K_3)\dots(K_n\cdot K_1)}+\mathcal{O}\left(\frac{1}{N^2}\right)\right]. \quad (6.11)$$

This now proves the conjecture of ref. [6].

### 6.2. THE PROCESS PRODUCING $q\bar{q}$ AND $n$ GLUONS

In the amplitude for the production of $q\bar{q}$ and $k$ gluons we need (cf. eq. (4.11))

$$\mathcal{D}(Q+;1,\dots,k;P-) = iJ_{\dot{A}}(Q+;1,\dots,k)[Q+\kappa(1,k)]^{\dot{A}B}p_B, \quad (6.12)$$

with

$$Q+P+\kappa(1,k)=0. \quad (6.13)$$

For the special case, where all gluons have the same helicity we insert eq. (5.81) for $J_{\dot{A}}$ and find a vanishing $\mathcal{D}$. Consider the case where one of the gluons has a negative helicity, we use again the polarization vectors

$$e^+_{\dot{A}B}(i) = -\sqrt{2}\,\frac{k_{i\dot{A}}b_{+B}}{\langle i+\rangle}, \quad (6.14)$$

$$e^-_{\dot{A}B}(j) = -\sqrt{2}\,\frac{b_{-\dot{A}}k_{jB}}{\langle j-\rangle^*}. \quad (6.15)$$

Inserting the recursion relation (5.77), using eq. (5.46) and

$$J_{\dot{A}}(Q+;1,\dots,m) = cX_{\dot{A}}(Q+,1,\dots,m) \quad (6.16)$$

with some factor $c$ we find

$$\mathcal{D}(Q+;1,\dots,k;P-) = -i\sum_{m=0}^{k-1}J_{\dot{B}}(Q+;1,\dots,m)J^{\dot{B}C}(m+1,\dots,k)p_C$$

$$= -ic\langle p+\rangle\sum_{m=0}^{k-1}X_{\dot{B}}(Q+,1,\dots,m)X^{\dot{B}}(m+1,\dots,k). \quad (6.17)$$

On the other hand, the $n$-gluon $\mathscr{C}$-function is also related to the quantities $X_{\dot{A}}$

$$\mathscr{C}(1,\dots,n-)=\tfrac{1}{2}[\kappa(1,n-1)]^2 J_{\dot{A}D}(1,2,\dots,n-1)e_-^{\dot{A}D}(n) \tag{6.18}$$

with

$$\kappa(1,n)=0. \tag{6.19}$$

Using eqs. (5.75) or (5.76) and (6.15) we have

$$\begin{aligned}\mathscr{C}(1,2,\dots,n-)&=\tfrac{1}{2}K_{n\dot{A}C}b_{+D}e_-^{\dot{A}D}(n)b_+^C\sum_{m=1}^{n-2}X_{\dot{B}}(1,\dots,m)X^{\dot{B}}(m+1,\dots,n-1)\\ &=-\sqrt{\tfrac{1}{2}}\langle n+\rangle\langle +n\rangle\sum_{m=1}^{n-2}X_{\dot{B}}(1,\dots,m)X^{\dot{B}}(m+1,\dots,n-1).\end{aligned} \tag{6.20}$$

Since both eqs. (6.17) and (6.20) contain a similar sum of terms, we find

$$\begin{aligned}&\mathscr{D}(Q+;2+,\dots,m+,1-,(m+1)+,\dots,n;P-)\\ &=i\sqrt{2}\,\frac{c\langle p+\rangle}{\langle p+\rangle\langle +p\rangle}\mathscr{C}(Q+,2+,\dots,m+,1-,(m+1)+,\dots,n,P-)\\ &=\frac{\langle q1\rangle}{\langle p1\rangle}\mathscr{C}(Q+,2+,\dots,m+,1-,(m+1)+,\dots,n+,P-),\end{aligned} \tag{6.21}$$

where the factor $c$ is taken from eq. (5.82) in which case $b_+=k_1$. With the help of the explicit form of the $\mathscr{C}$-function (6.9) we now have

$$\begin{aligned}&\mathscr{D}(Q+;2+,\dots,m+,1-,(m+1)+,\dots,n+;P-)\\ &=(\sqrt{2})^n\frac{\langle p1\rangle^3\langle q1\rangle}{\langle pq\rangle\langle q2\rangle\langle 23\rangle\cdots\langle m1\rangle\langle 1,m+1\rangle\cdots\langle np\rangle},\end{aligned} \tag{6.22}$$

and therefore

$$\begin{aligned}&\mathscr{M}(Q+;1-,2+,\dots,n+;P-)\\ &=-i(\sqrt{2})^n g^n\frac{\langle p1\rangle^3\langle q1\rangle}{\langle pq\rangle}\sum_{P(1,\dots,n)}(a_1\dots a_n)_{ij}\frac{1}{\langle q1\rangle\langle 12\rangle\cdots\langle np\rangle}.\end{aligned} \tag{6.23}$$

This proves the conjecture of ref. [9]. For the square of this helicity amplitude we find after summing over the colours

$$|\mathcal{M}(Q+;1-,2+,\ldots,n+;P-)|^2 = 2^{2-n}g^{2n}N^{n-1}(N^2-1)\frac{(P\cdot K_1)^3(Q\cdot K_1)}{(P\cdot Q)}$$

$$\times \sum_{P(1,\ldots,n)} \frac{1}{(Q\cdot K_1)(K_1\cdot K_2)\ldots(K_n\cdot P)}. \tag{6.24}$$

The case, where the helicities of the quarks are opposite from those in eq. (6.23) is obtained by an interchange of p and q (modulo a phase factor), as can be seen by $C$-conjugation. When one wants the helicities of all particles in (6.23) to have the opposite value one should take the complex conjugate of all spinorial inner products $\langle ij \rangle$.

### 6.3. THE PROCESS PRODUCING $q\bar{q}$, $n$ GLUONS AND A VECTOR BOSON

In this section we show how processes with vector boson production can be incorporated, giving again explicit expressions for definite helicity combinations. Specifically we consider

$$e^+(P_+) + e^-(P_-) \to \gamma^* \to \bar{q}(Q_+) + q(Q_-) + g_1 + \cdots + g_n \tag{6.25}$$

and

$$Z(P) \to \bar{q} + q + g_1 + \cdots + g_n. \tag{6.26}$$

Similar results hold when a W instead of a Z is participating. Denoting the outgoing quark and antiquark momenta by $Q$ and $P$ we need the expression $\mathcal{S}_\mu$ of eq. (4.15) with

$$\Gamma_\mu = ie\left[a\gamma_\mu\frac{1-\gamma_5}{2} + b\gamma_\mu\frac{1+\gamma_5}{2}\right], \tag{6.27}$$

where in the case of reaction (6.25) for u, c, t quarks

$$a = b = -\tfrac{2}{3}, \tag{6.28}$$

whereas for d, s, b quarks

$$a = b = \tfrac{1}{3}, \tag{6.29}$$

and for reaction (6.26)

$$a = c_V + c_A, \tag{6.30}$$

$$b = c_V - c_A. \tag{6.31}$$

The quantities $c_V$ and $c_A$ have in the case of u, c, t quarks the form

$$c_V = \frac{1 - \frac{8}{3}s_W^2}{4s_W c_W}, \qquad c_A = \frac{1}{4s_W c_W}, \tag{6.32}$$

and for d, s, b quarks

$$c_V = \frac{\frac{4}{3}s_W^2 - 1}{4s_W c_W}, \qquad c_A = \frac{-1}{4s_W c_W}, \tag{6.33}$$

with

$$s_W = \sin\theta_W, \qquad c_W = \cos\theta_W. \tag{6.34}$$

For the production of a W instead of a Z we have $b = 0$ and a value $a_W$ for $a$ depending on the specific quark-antiquark pair.

First we consider the helicity combination, where only the part of $\Gamma_\mu$ which is proportional to $b$ can contribute. We find as in eqs. (5.14) and (5.15)

$$\begin{aligned}
&\mathscr{S}_\mu(Q_-+;1,2,\dots,n;Q_+-)\\
&= ieb\sum_{m=0}^{n} J(Q_-+;1,\dots,m)\gamma_\mu\frac{1+\gamma_5}{2}J(m+1,\dots,n;Q_+-) &(6.35)\\
&= -eb\sigma_\mu^{\dot{A}B}\sum_{m=0}^{n} J_{\dot{A}}(Q_-+;1,\dots,m)J_B(m+1,\dots,n;Q_+-). &(6.36)
\end{aligned}$$

For the gluons we choose the positive helicities with the polarization as in eq. (5.42) with $b_+ = q_+$ such that we have (cf. eqs. (5.33) and (5.81)) only one term

$$\begin{aligned}
\mathscr{S}_\mu(Q_-+;1,2,\dots,n;Q_+-) &= -eb\sigma_\mu^{\dot{A}B}J_{\dot{A}}(Q_-+;1,\dots,n)q_{+B}\\
&= ieb(\sqrt{2})^n\sigma_\mu^{\dot{A}B}\frac{[Q_-+\kappa(1,n)]_{\dot{A}C}q_+^C q_{+B}}{\langle q_-1\rangle\langle 12\rangle\cdots\langle nq_+\rangle}. &(6.37)
\end{aligned}$$

In a similar fashion one obtains

$$\mathscr{S}_\mu(Q_--;1,2,\dots,n;Q_++) = -iea(\sqrt{2})^n\sigma_\mu^{\dot{A}B}\frac{[Q_++\kappa(1,n)]_{\dot{A}C}q_-^C q_{-B}}{\langle q_-1\rangle\langle 12\rangle\cdots\langle nq_+\rangle}. \tag{6.38}$$

To this helicity contribution only the coefficient $a$ in eq. (6.27) contributes. The quantity $\mathscr{S}_\mu$ has to be contracted with $V^\mu$ which for reaction (6.25) assumes the forms

$$V_\mu(P_++, P_--) = e\frac{p_{+D}\sigma_\mu^{\dot{E}D}p_{-\dot{E}}}{\langle p_+p_-\rangle\langle p_+p_-\rangle^*}, \tag{6.39}$$

$$V_\mu(P_+-, P_-+) = e\frac{p_{+\dot{E}}\sigma_\mu^{\dot{E}D}p_{-D}}{\langle p_+p_-\rangle\langle p_+p_-\rangle^*}. \tag{6.40}$$

For the matrix element for a specific helicity combination one now finds

$$\begin{aligned}
&\mathscr{M}(P_+ +, P_- -, Q_- +, Q_+ -; 1+, \dots, n+) \\
&= g^n \sum_{P(1,\dots,n)} (a_1 \dots a_n)_{ij} V^\mu \mathscr{S}_\mu \\
&= ie^2 b(\sqrt{2})^n g^n \frac{p_{+D} p_{-\dot{E}} \sigma^{\mu \dot{E} D} \sigma_\mu^{\dot{A}B} (P_+ + P_-)_{\dot{A}C} q_+^C q_{+B}}{\langle p_+ p_- \rangle \langle p_+ p_- \rangle^*} \\
&\quad \times \sum_{P(1,\dots,n)} (a_1 \dots a_n)_{ij} \frac{1}{\langle q_- 1 \rangle \langle 12 \rangle \cdots \langle n q_+ \rangle} \\
&= -2ie^2 b(\sqrt{2})^n g^n \frac{\langle p_+ q_+ \rangle^2}{\langle p_+ p_- \rangle} \sum_{P(1,\dots,n)} \frac{(a_1 \dots a_n)_{ij}}{\langle q_- 1 \rangle \langle 12 \rangle \cdots \langle n q_+ \rangle} \qquad (6.41)
\end{aligned}$$

and

$$\begin{aligned}
&|\mathscr{M}(P_+ +, P_- -, Q_- +, Q_+ -; 1+, \dots, n+)|^2 \\
&= 2^{2-n} g^{2n} e^4 b^2 N^{n-1} (N^2 - 1) \\
&\quad \times \frac{(P_+ \cdot Q_+)^2}{(P_+ \cdot P_-)} \left( \sum_{P(1,\dots,n)} \frac{1}{(Q_- \cdot K_1)(K_1 \cdot K_2) \dots (K_n \cdot Q_+)} + \mathcal{O}\left(\frac{1}{N^2}\right) \right). \qquad (6.42)
\end{aligned}$$

Similar expressions arise for other helicity combinations of the fermions. This formula also applies to

$$e^-(P_- -) + q(Q_+ +) \to e^-(P_+ -) + q(Q_- +) + g(K_1 +) + \cdots + g(K_n +). \qquad (6.43)$$

In a similar fashion one could consider the process where a Z or W is produced by a lepton pair. Expressions like (6.39) and (6.40) should be used. Instead of this we sum over all polarization states of the Z, which means

$$\sum_{\text{pol.}} V_\mu V_\nu^* = -g_{\mu\nu} + \frac{P_\mu P_\nu}{m_Z^2}. \qquad (6.44)$$

Since $P^\mu \mathscr{S}_\mu = 0$, we have in the case of eq. (6.37)

$$\sum_{\text{pol.}} |\mathscr{S}_\mu V^\mu|^2 = -g_{\mu\nu} \mathscr{S}^\mu \mathscr{S}^{\nu *} = e^2 b^2 2^2 \frac{(P \cdot Q_+)^2}{(Q_- \cdot K_1)(K_1 \cdot K_2) \dots (K_n \cdot Q_+)}. \qquad (6.45)$$

For reaction (6.26) we have

$$\sum_{\text{pol. Z}} |\mathscr{M}(Q_-+, Q_+-; P; 1+, \dots, n+)|^2$$
$$= e^2 g^{2n} 2^{2-n} N^{n-1}(N^2-1)$$
$$\times b^2 (P\cdot Q_+)^2 \sum_{P(1,\dots,n)} \frac{1}{(Q_-\cdot K_1)(K_1\cdot K_2)\dots(K_n\cdot Q_+)}, \qquad (6.46)$$

$$\sum_{\text{pol. Z}} |\mathscr{M}(Q_--, Q_++; P; 1+, \dots, n+)|^2$$
$$= e^2 g^{2n} 2^{2-n} N^{n-1}(N^2-1)$$
$$\times a^2 (P\cdot Q_-)^2 \sum_{P(1,\dots,n)} \frac{1}{(Q_-\cdot K_1)(K_1\cdot K_2)\dots(K_n\cdot Q_+)}. \qquad (6.47)$$

For $Q_-$ and $Q_+$ we can take incoming momenta in eqs. (6.46) and (6.47) and for $P$ an outgoing momentum. Then (6.46) refers to a $-,+$ helicity combination for $Q_-, Q_+$ and (6.47) has $+,-$ helicity combination for the incoming quarks. In this way we describe quark-pair annihilation into a Z and $n$ gluons. When the Z is replaced by a W eq. (6.46) vanishes and in eq. (6.47) one should use for the parameter $a$ the appropriate $a_{\mathrm{W}}$.

## 7. Soft and collinear gluons

In this section we study how the gluon current behaves when a gluon becomes soft or when two gluons become collinear. In both cases a factorization occurs. For a current with a few gluons this can be shown explicitly. The recursion relation is used to extend these properties to an arbitrary number of gluons. Once one knows the behaviour of the currents in these specific kinematical situations, the properties of the $\mathscr{C}$-functions can be derived. Also the implications for the processes with a quark pair and a vector boson easily follow.

### 7.1. ONE SOFT QUARK OR GLUON

Consider a gluon current, where we take the limit $K_2 \to 0$. The leading terms in this limit arise from the propagators $[\kappa(1,2)]^{-2}$ and $[\kappa(2,3)]^{-2}$. Other propagators don't become singular. Let us assume, and we will show this later on, that we have in the soft limit for gluon 2 a factorization of the following currents

$$J(12) = s_{1\underline{2}} J(1), \qquad (7.1)$$

$$J(23) = s_{\underline{2}3} J(3). \qquad (7.2)$$

The coefficients $s_{ij}$ which contain the singular behaviour are to be determined later on. Property (7.2) now holds for an arbitrary number of gluons

$$J(23\ldots m) = s_{2\underline{3}} J(34\ldots m). \tag{7.3}$$

The recursion relation (2.19) shows the validity of (7.3) for $m = n$, when we assume (7.3) to be correct for $m < n$. In the limit $K_2 \to 0$ we have

$$\begin{aligned} J(23\ldots n) = {} & \frac{s_{2\underline{3}}}{[\kappa(3,n)]^2} \left( \sum_{m=3}^{n-1} [J(3\ldots m), J(m+1\ldots n)] \right. \\ & \left. + \sum_{m=3}^{n-2} \sum_{k=m+1}^{n-1} \{J(3\ldots m), J(m+1\ldots k), J(k+1\ldots n)\} \right) \\ = {} & s_{2\underline{3}} J(3\ldots n). \end{aligned} \tag{7.4}$$

Use has been made of the fact that the $m = 2$ term in the sum does not contribute in the soft limit. In a similar fashion one shows that

$$J(123\ldots n) = (s_{1\underline{2}} + s_{\underline{2}3}) J(134\ldots n) = s_2(1,3) J(134\ldots n). \tag{7.5}$$

What remains to be demonstrated is the correctness of eqs. (7.1) and (7.2). In the soft $K_2$ limit one has

$$J(12) = \frac{1}{[\kappa(1,2)]^2} \left( J(1) \cdot J(2) K_1 - 2K_1 \cdot J(2) J(1) \right), \tag{7.6}$$

$$J(23) = \frac{1}{[\kappa(2,3)]^2} \left( -J(2) \cdot J(3) K_3 + 2K_3 \cdot J(2) J(3) \right). \tag{7.7}$$

The expressions for $J(12)$ and $J(23)$ are proportional to polarization vectors of gluon 1 and 3, the gauge choice is different due to the terms proportional to $K_1$ and $K_3$. However such a change of gauge leads to a completely equivalent gluon current as has been shown in sect. 2. The proportionality factors are

$$s_{1\underline{2}} = -\frac{K_1 \cdot J(2)}{K_1 \cdot K_2}, \qquad s_{\underline{2}3} = \frac{K_3 \cdot J(2)}{K_3 \cdot K_2}. \tag{7.8}$$

$$s_2(1,3) = \left( \frac{K_3}{K_3 \cdot K_2} - \frac{K_1}{K_1 \cdot K_2} \right) \cdot J(2). \tag{7.9}$$

Specifying the helicity and using a gauge spinor as in eq. (5.42) one has

$$s_{1\underline{2}}^{+} = \sqrt{2}\,\frac{\langle 1+\rangle}{\langle 12\rangle\langle 2+\rangle}, \qquad s_{\underline{2}3}^{+} = -\sqrt{2}\,\frac{\langle 3+\rangle}{\langle 32\rangle\langle 2+\rangle},$$

$$s_{1\underline{2}}^{-} = (s_{1\underline{2}}^{+})^{*}, \qquad s_{\underline{2}3}^{-} = (s_{\underline{2}3}^{+})^{*},$$

$$s_2^{+}(1,3) = \sqrt{2}\,\frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle}, \qquad s_2^{-}(1,3) = (s_2^{+}(1,3))^{*}. \tag{7.10}$$

Note that in the last line the dependence on the gauge spinor drops out. For a $\mathscr{C}$-function the soft gluon limit $K_m \to 0$ follows from eq. (4.2)

$$\mathscr{C}(1,2,\ldots,m,\ldots,n) = s_m(m-1,m+1)\mathscr{C}(1,\ldots,m-1,m+1,\ldots,n). \tag{7.11}$$

This factorization has also been suggested in ref. [9] but no proof was given.

We now turn to the soft gluon behaviour of the spinorial current (3.9). For $K_1 \to 0$ one finds

$$J(Q;1) = -\frac{Q\cdot J(1)}{Q\cdot K_1}J(Q) = s_{q\underline{1}}J(Q), \tag{7.12}$$

$$J(Q;12) = (s_{q\underline{1}} + s_{\underline{1}2})J(Q;2) = s_1(Q,K_2)J(Q;2). \tag{7.13}$$

Using the recursion relation one obtains:
for $K_1 \to 0$

$$J(Q;12\ldots n) = s_1(Q,K_2)J(Q;2\ldots n), \tag{7.14}$$

for $K_m \to 0$

$$J(Q;1,\ldots,m,\ldots,n) = s_m(K_{m-1},K_{m+1})J(Q;1,\ldots,m-1,m+1,\ldots,n) \tag{7.15}$$

and similarly from eq. (3.13) for $K_n \to 0$

$$J(1\ldots,n;P) = s_n(K_{n-1},P)J(1\ldots,n-1;P). \tag{7.16}$$

For the quantities $\mathscr{D}$ and $\mathscr{S}_\mu$ which determines the amplitudes one finds in a soft gluon limit $K_m \to 0$

$$\mathscr{D}(1;2,\ldots,m,\ldots,n-1;n) = s_m(m-1,m+1)\mathscr{D}(1;2,\ldots,m-1,m+1,\ldots,n-1;n), \tag{7.17}$$

$$\mathscr{S}_\mu(1;2,\ldots,m,\ldots,n-1;n) = s_m(m-1,m+1)\mathscr{S}_\mu(1;2,\ldots,m-1,m+1,\ldots,n-1;n), \tag{7.18}$$

where $2 \leqslant m \leqslant n-1$.

In general multiple soft gluon emission is more involved than a repetition of eqs. (7.11), (7.17) and (7.18). This is due to the occurrence of propagators $[\kappa(m,l)]^{-2}$ which also become singular when $K_m, K_{m+1},\ldots, K_{l-1}$ become soft.

In the processes where quarks participate one can also consider the limit $Q \to 0$. For the spinorial current (5.77) one then has

$$J_{\dot{A}}(Q+;1) = iq_{\dot{C}}e^{\dot{C}B}(1)\frac{K_{1\dot{A}B}}{2Q\cdot K_1} = i\frac{q_{\dot{C}}k_{1B}e^{\dot{C}B}(1)}{2Q\cdot K_1}k_{1\dot{A}} = -\frac{q_{\dot{C}}k_{1B}e^{\dot{C}B}(1)}{2Q\cdot K_1}J_{\dot{A}}(K_1+). \tag{7.19}$$

The proportionality factor vanishes when the gluon has a negative helicity. Inserting eq. (5.42) we have for $Q \to 0$

$$J_{\dot{A}}(Q+;1+) = \frac{\sqrt{2}}{\langle qk_1\rangle}J_{\dot{A}}(K_1+), \tag{7.20}$$

and in general from (5.77)

$$J_{\dot{A}}(Q+;1+,2,\ldots,n) = \frac{\sqrt{2}}{\langle qk_1\rangle}J_{\dot{A}}(K_1+;2,\ldots,n). \tag{7.21}$$

Again, for $\mathscr{S}_\mu(Q+;1+,2,\ldots,n;P-)$ the same factorization occurs with $\mathscr{S}_\mu(K_1+;2,\ldots,n;P-)$. For $\mathscr{D}(Q+;1+,2,\ldots,n;P^-)$ one gets for $Q \to 0$ two singular terms. One arising from eq. (7.21) and another which originates from

$$\kappa(1,n)^{-2} = (2Q\cdot P)^{-1}. \tag{7.22}$$

Therefore we consider for $Q \to 0$ the following term in $\mathscr{D}$:

$$\begin{aligned}\mathscr{D}(Q+;1,\ldots,n;P-) &= -J_+(Q)\not{J}(1,\ldots,n)J_-(P)\\ &= -\frac{J_+(Q)\gamma_\nu J_-(P)}{2Q\cdot P}\kappa(1,n)^2 J^\nu(1,\ldots,n)\\ &= -\frac{\sqrt{2}}{\langle qp\rangle}\mathscr{C}(P-,1,\ldots,n).\end{aligned} \tag{7.23}$$

In conclusion for $Q \to 0$ the function $\mathscr{D}$ is a sum of two terms

$$\mathscr{D}(Q+;1+,2,\ldots,n;P_-) = \sqrt{2}\left(\frac{1}{\langle q1\rangle}\mathscr{D}(1+;2,\ldots,n;P-) - \frac{1}{\langle qp\rangle}\mathscr{C}(P-,1+,2,\ldots,n)\right). \tag{7.24}$$

7.2. TWO COLLINEAR PARTONS

When gluons 1 and 2 become collinear all terms containing $P^{-2}=[\kappa(1,2)]^{-2}$ in the gluon current become singular. Keeping those terms in the recursion relation means the following approximation to the current:

$$J(123)=\frac{1}{[P+K_3]^2}[J(12),J(3)], \tag{7.25}$$

$$J(1234)=\frac{1}{[P+\kappa(3,4)]^2}([J(12),J(34)]+[J(123),J(4)]+\{J(12),J(3),J(4)\}) \tag{7.26}$$

Suppose that $J(12)$ is a sum of 2 currents each belonging to a gluon with momentum $P$, but with different helicities. Then we have

$$J(1,2)=c_+J(P+)+c_-J(P-), \tag{7.27}$$

$$J(1,2,3)=c_+J(P+,3)+c_-J(P-,3), \tag{7.28}$$

$$J(1,2,3,4)=c_+J(P+,3,4)+c_-J(P-,3,4), \tag{7.29}$$

and by means of the recursion relation (2.19)

$$J(1,2,\dots,n)=c_+J(P+,3,4,\dots,n)+c_-J(P-,3,4,\dots,n). \tag{7.30}$$

We now want to establish that in the limit of two collinear gluons we indeed have property (7.27). Using polarization vectors like eq. (5.42) with gauge spinor $b$ for all positive helicity states and the complex conjugate for the negative helicity states we can derive explicit expressions for $J(12)$ in the collinear case. For the derivation of eq. (7.27) we add a gauge term $\sim P^\rho$ to $J(12)$ in the following way*

$$\begin{aligned} J^\rho(12) &= J_\nu(12)\left[g^{\nu\rho}-\frac{P^\nu B^\rho+B^\nu P^\rho}{P\cdot B}\right] \\ &= -J_\nu(12)\sigma^{\rho\dot{A}B}\sigma^{\nu\dot{C}D}\left[\frac{P_{\dot{A}D}b_B b_{\dot{C}}+P_{\dot{C}B}b_D b_{\dot{A}}}{4P\cdot B}\right], \end{aligned} \tag{7.31a}$$

or

$$J_{\dot{A}B}(12)=-J^{\dot{C}D}(12)\left[\frac{P_{\dot{A}D}b_B b_{\dot{C}}+P_{\dot{C}B}b_D b_{\dot{A}}}{2P\cdot B}\right]. \tag{7.31b}$$

For $B_\mu$ any null vector can be chosen, in particular the null vector connected with the gauge spinor $b$. The two terms in eq. (7.31b) will give $c_+$ and $c_-$ respectively.

In the collinear case, we introduce

$$\begin{aligned} K_1 &= zP, & k_1 &= \sqrt{z}\,p, \\ K_2 &= (1-z)P, & k_2 &= \sqrt{1-z}\,p. \end{aligned} \tag{7.32}$$

* In this way the residue of the $P^2$-pole decomposes in the collinear limit as a sum over helicity vectors $e^-_{p\nu}e^{+\rho}_p+e^+_{p\nu}e^{-\rho}_p$.

Explicit expressions for the helicity combinations are

$$J_{\dot{A}B}(1+,2+) = -\frac{\sqrt{2}}{\langle 12\rangle}\left[\frac{\langle 2b\rangle}{\langle 1b\rangle}J^{\dot{C}D}(2+) + \frac{\langle 1b\rangle}{\langle 2b\rangle}J^{\dot{C}D}(1+)\right] \times\left[\frac{P_{\dot{A}D}b_B b_{\dot{C}} + P_{\dot{C}B}b_D b_{\dot{A}}}{2P\cdot B}\right]$$
$$= \frac{2}{\langle 12\rangle}\left[\frac{\langle b2\rangle^*}{\langle 1b\rangle} + \frac{\langle b1\rangle^*}{\langle 2b\rangle}\right]\frac{\langle pb\rangle p_{\dot{A}} b_B}{\langle bp\rangle\langle bp\rangle^*}$$
$$= \frac{\sqrt{2}}{\langle 12\rangle\sqrt{z(1-z)}}J_{\dot{A}B}(P+), \qquad (7.33)$$

$$J_{\dot{A}B}(1+,2-) = -\frac{\sqrt{2}}{\langle 12\rangle\langle 12\rangle^*}\left[\frac{P_{\dot{A}D}b_B b_{\dot{C}} + P_{\dot{C}B}b_D b_{\dot{A}}}{2P\cdot B}\right]$$
$$\times\left[\frac{\langle 12\rangle^*\langle 2b\rangle}{\langle 1b\rangle}J^{\dot{C}D}(2-) + \frac{\langle 12\rangle\langle 1b\rangle^*}{\langle 2b\rangle^*}J^{\dot{C}D}(1+) + \frac{\sqrt{2}}{4}J_{\dot{E}F}(2+)J^{\dot{E}F}(1-)(K_1-K_2)^{\dot{C}D}\right]$$
$$= \frac{\sqrt{2}\,z^2}{\langle 12\rangle^*\sqrt{z(1-z)}}J_{\dot{A}B}(P+) + \frac{\sqrt{2}\,(1-z)^2}{\langle 12\rangle\sqrt{z(1-z)}}J_{\dot{A}B}(P-), \qquad (7.34)$$

which give

$$c_+(1+,2+) = \frac{\sqrt{2}}{\langle 12\rangle\sqrt{z(1-z)}}, \qquad c_-(1+,2+) = 0,$$
$$c_-(1-,2-) = c_+(1+,2+)^*, \qquad c_-(1-,2-) = 0,$$
$$c_+(1+,2-) = \frac{\sqrt{2}}{\langle 12\rangle^*}\sqrt{\frac{z^3}{1-z}}, \qquad c_-(1+,2-) = \frac{\sqrt{2}}{\langle 12\rangle}\sqrt{\frac{(1-z)^3}{z}},$$
$$c_-(1-,2+) = c_+(1+,2-)^*, \qquad c_+(1-,2+) = c_-(1+,2-)^*. \qquad (7.35)$$

For the $\mathscr{C}$-functions one finds from (7.30) in the collinear limit of gluon 1 and 2:

$$\mathscr{C}(1,2,3,n) = c_+(1,2)\mathscr{C}(P+,3,4,\ldots,n) + c_-(1,2)\mathscr{C}(P-,3,4,\ldots,n). \qquad (7.36)$$

This type of factorization also holds for $\mathscr{D}(Q;1,2,\ldots,n;P)$ and $\mathscr{S}_\mu(Q;1,\ldots,n;P)$. Again, eq. (7.36) has been stated without proof in ref. [9].

To be complete we also consider the case where quarks occur in the collinear parton pair.

When a quark becomes collinear with a gluon, one derives the factorization in a similar fashion as above. For $P = Q + K_1$, $P^2 \to 0$ and $Q = (1-z)P$, we find

$$J(Q\lambda;1\mu,2,\ldots,n) = d_\lambda(\lambda,\mu)J(P\lambda;2,\ldots,n), \qquad (7.37)$$

with

$$d_+(+,+) = \frac{-\sqrt{2}}{\langle qk_1\rangle\sqrt{z}}, \qquad d_+(+,-) = \frac{-\sqrt{2}}{\langle qk_1\rangle^*}\frac{1-z}{\sqrt{z}},$$

$$d_-(-,+) = \frac{-\sqrt{2}}{\langle qk_1\rangle}\frac{1-z}{\sqrt{z}}, \qquad d_-(-,-) = \frac{-\sqrt{2}}{\langle qk_1\rangle^*\sqrt{z}}. \tag{7.38}$$

For an outgoing antiquark with momentum $Q$ one has for $P = Q + K_n$, $P^2 \to 0$

$$J(1,2,\ldots,n\mu;Q\lambda) = -d_\lambda(\lambda,\mu)J(1,\ldots,n-1;P\lambda), \tag{7.39}$$

where $k_1$ is replaced by $k_n$ in the quantity $d_\lambda(\lambda,\mu)$.

From the behaviour of the currents in eqs. (7.37) and (7.39) a similar factorization follows for the functions $\mathscr{D}(Q_-;1,\ldots,n;Q_+)$ and $\mathscr{S}_\mu(Q_-;1,\ldots,n;Q_+)$, where $Q_-$ and $Q_+$ are the momenta of the outgoing quark and antiquark respectively.

Finally, also the quark and antiquark can become collinear. Considering the function $\mathscr{D}$ we get a singularity from $P^{-2} = (Q_+ + Q_-)^{-2} = [\kappa(1,n)]^{-2}$. The term in $\mathscr{D}$ containing this singularity is

$$\mathscr{D}(Q_-;1,\ldots,n;Q_+) = -J(Q_-)\not{J}(1,\ldots,n)J(Q_+). \tag{7.40}$$

In the collinear limit $J(Q_-)\gamma_\mu J(Q_+)$ becomes proportional to the sum of polarization vectors $e_\mu^\pm$ of a gluon with momentum $P$. Therefore we have

$$\frac{-1}{2Q_+\cdot Q_-}J(Q_-\lambda)\gamma_\rho J(Q_+\mu) = h_+(\lambda,\mu)e_\rho^+ + h_-(\lambda,\mu)e_\rho^- \tag{7.41}$$

and consequently for $P^2 \to 0$

$$\mathscr{D}(Q_-\lambda;1,\ldots,n;Q_+\mu) = h_+(\lambda,\mu)\mathscr{C}(P+,1,\ldots,n) + h_-(\lambda,\mu)\mathscr{C}(P-,1,\ldots,n). \tag{7.42}$$

The function $h$ can be most easily determined by considering

$$\begin{aligned}
&-\bar{u}_\lambda(Q_-)\gamma_\nu v_\mu(Q_+)\frac{g^{\nu\rho}}{2Q_+\cdot Q_-} \\
&= \bar{u}_\lambda(Q_-)\gamma_\nu v_\mu(Q_+)\frac{1}{2Q_+\cdot Q_-}\left[-g^{\nu\rho} + \frac{(Q_+ + Q_-)^\nu B^\rho + B^\nu(Q_+ + Q_-)^\rho}{(Q_+ + Q_-)\cdot B}\right] \\
&= \bar{u}_\lambda(Q_-)\gamma_\nu v_\mu(Q+)\frac{1}{4Q_+\cdot Q_-}\sigma^{\rho\dot{A}B}\sigma^{\nu\dot{C}D} \\
&\quad\times\left[\frac{(Q_+ + Q_-)_{\dot{A}D}b_B b_{\dot{C}} + (Q_+ + Q_-)_{\dot{C}B}b_D b_{\dot{A}}}{\{Q_+ + Q_-, B\}}\right],
\end{aligned} \tag{7.43}$$

where terms proportional to $P_\nu$, $P_\rho$ have been neglected on the basis of current conservation. Introducing

$$Q_- = zP, \qquad Q_+ = (1-z)P, \tag{7.44}$$

one finds

$$h_-(+,-) = \frac{\sqrt{2}(1-z)}{\langle q_+ q_- \rangle}, \qquad h_+(+,-) = \frac{-\sqrt{2}z}{\langle q_+ q_- \rangle^*},$$
$$h_-(-,+) = \frac{-\sqrt{2}z}{\langle q_+ q_- \rangle}, \qquad h_+(-,+) = \frac{\sqrt{2}(1-z)}{\langle q_+ q_- \rangle^*}. \tag{7.45}$$

## 8. Conclusions

Parton processes with $n$ gluons are of particular relevance for present and future accelerators. Six-parton processes have been calculated with a lot of inventivity (ref. [2] and references quoted there). A more systematic approach was introduced with the use of $\mathscr{C}$-functions (eq. (1.1)) and the related $\mathscr{D}$- and $\mathscr{S}_\mu$-functions (eqs. (4.10) and (4.14)). However in addition to this way of treating colour and in addition to the spinor calculus one wants a recursive technique which makes it possible to increase the number of gluons while still using previous results for fewer gluons. The method in this paper makes this feasible. A numerical evaluation of the seven-gluon process is underway. Once one knows the gluon current processes can be calculating involving quarks and vector bosons. Although we consider here only one quark-pair, the evaluation of processes with more quark-pairs also becomes simpler when one has the gluon current.

Besides the possibility of a systematic calculation of multi-parton processes the recursion relations are a useful tool to prove properties of $\mathscr{C}$-functions. Thus we proved quite general properties like cyclic and subcyclic relations, gauge invariance, reflection symmetry, soft and collinear factorization.

Moreover, a number of interesting hitherto unproven statements and conjectures on the specific expressions for $\mathscr{C}$- and $\mathscr{D}$-functions in situations where most of the helicities are equal [6,9] could also be proved.

Discussions with Drs. R. Kleiss, H. Kuijf and W.L. van Neerven are gratefully acknowledged.

## Appendix A

CONVENTIONS

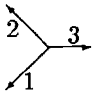
The colour matrices in the fundamental representations are normalized, such that

$$\mathrm{Tr}(T_a T_b) = (ab) = \tfrac{1}{2}\delta_{ab}, \tag{A.1}$$

$$[T_a, T_b] = i f_{abc} T_c, \tag{A.2}$$

$$\sum_a (T_a)_{ij} (T_a)_{kl} = \tfrac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}. \tag{A.3}$$

The 3-vertex, with 3-outgoing gluons, characterized by $K_i$, $\alpha_i$ and $a_i$ is

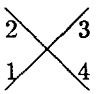



$$= -g f^{a_1 a_2 a_3} V_{\alpha_1 \alpha_2 \alpha_3}(K_1, K_2, K_3), \tag{A.4}$$

with

$$V_{\alpha_1 \alpha_2 \alpha_3}(K_1, K_2, K_3) = (K_1 - K_2)_{\alpha_3} g_{\alpha_1 \alpha_2} + (K_2 - K_3)_{\alpha_1} g_{\alpha_2 \alpha_3} + (K_3 - K_1)_{\alpha_2} g_{\alpha_3 \alpha_1}. \tag{A.5}$$

The 4-vertex is given by




$$= -i g^2 \sum_{C(1,2,3)} f^{a_1 a_2 y} f^{y a_3 a_4} K(\alpha_1, \alpha_2; \alpha_3, \alpha_4), \tag{A.6}$$

where $\Sigma_{C(1,2,3)}$ denotes a cyclic sum over 1, 2, 3 and

$$K(\alpha_1, \alpha_2; \alpha_3, \alpha_4) = g_{\alpha_1 \alpha_3} g_{\alpha_2 \alpha_4} - g_{\alpha_1 \alpha_4} g_{\alpha_2 \alpha_3}. \tag{A.7}$$

Useful relations are

$$i f^{a_1 a_2 a_3} = 2(a_1 a_2 a_3) - 2(a_2 a_1 a_3), \tag{A.8}$$

$$i^2 f^{y x_1 x_2} f^{y x_3 x} (\Omega_1 x_1)(\Omega_2 x_2)(\Omega_3 x_3) = \tfrac{1}{4}\{(\Omega_1 \Omega_2 \Omega_3 x) - (\Omega_2 \Omega_1 \Omega_3 x) + (\Omega_3 \Omega_2 \Omega_1 x) - (\Omega_3 \Omega_1 \Omega_2 x)\}, \tag{A.9}$$

where a summation has been carried out over $x_1$, $x_2$, $x_3$, and $y$. The quantities $\Omega_i$ are strings of $T$ matrices.

Furthermore, when $J_{\alpha_1}(1)$ and $J_{\alpha_2}(2)$ are conserved

$$K_i \cdot J(i) = 0, \tag{A.10}$$

we have

$$(K_1 + K_2)^{\alpha_3} J^{\alpha_1}(1) J^{\alpha_2}(2) V_{\alpha_1 \alpha_2 \alpha_3}(K_1, K_2, -(K_1 + K_2)) = (K_1^2 - K_2^2) J(1) \cdot J(2). \tag{A.11}$$

For the Weyl-van der Waerden spinors we use the conventions of ref. [3]. Often used relations are

$$K_{\dot A B} = \sigma^{\mu}_{\dot A B} K_{\mu}, \tag{A.12}$$

$$\psi_1^A \langle \psi_2 \psi_3 \rangle + \psi_2^A \langle \psi_3 \psi_1 \rangle + \psi_3^A \langle \psi_1 \psi_2 \rangle = 0, \tag{A.13}$$

$$\left(\sigma^{\mu}_{\dot A B} \sigma^{\nu \dot A C} + \sigma^{\nu}_{\dot A B} \sigma^{\mu \dot A C}\right) P_{\mu} Q_{\nu} = 2 P \cdot Q \delta_B{}^C. \tag{A.14}$$

## Appendix B

PROOF OF SOME PROPERTIES OF THE GLUON CURRENTS

The three properties (2.24)–(2.26) will be proven in this appendix. In the first and last case we use induction. The $n$-gluon current is given by the recursion relation (2.19), which contains currents with $l$ gluons, where $l<n$, thereby obeying the properties considered.

*B.1. The reflection property*

$$J(1,\dots,n)=\frac{1}{[\kappa(1,n)]^2}\left(\sum_{m=1}^{n-1}[J(1,\dots,m),J(m+1,\dots,n)]\right.$$
$$\left.+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\{J(1,\dots,m),J(m+1,\dots,k),J(k+1,\dots,n)\}\right) \tag{B.1}$$

$$=\frac{1}{[\kappa(1,n)]^2}\left(\sum_{m=1}^{n-1}-(-1)^{(m-1)+(n-(m+1))}\right.$$
$$\times[J(n,\dots,m+1),J(m,\dots,1)]$$
$$+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}(-1)^{(m-1)+(k-m-1)+(n-k-1)}$$
$$\left.\times\{J(n,\dots,k+1),J(k,\dots,m+1),J(m,\dots,1)\}\right)$$
$$=(-1)^{n-1}J(n,n-1,\dots,1). \tag{B.2}$$

*B.2. The cyclic sum property.* We have to show that a cyclic sum over $J(1,\dots,n)$ vanishes. This is equivalent with a cyclic sum over the r.h.s. of eq. (B.1). We rewrite this sum as a sum over terms which are seen to cancel explicitly:

$$\sum_{C(1,\dots,n)}J(1,2,\dots,n)$$
$$=\frac{1}{[\kappa(1,n)]^2}\sum_{C(1,\dots,n)}\left[\frac{1}{2}\sum_{m=1}^{n-1}\big([J(1,\dots,m),J(m+1,\dots,n)]\right.$$
$$+[J(m+1,\dots,n),J(1,\dots,m)]\big)$$
$$+\frac{1}{3}\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\big(\{J(1,\dots,m),J(m+1,\dots,k),J(k+1,\dots,n)\}$$
$$+\{J(k+1,\dots,n),J(1,\dots,m),J(m+1,\dots,k)\}$$
$$\left.+\{J(m+1,\dots,k),J(k+1,\dots,n),J(1,\dots,m)\}\big)\right]. \tag{B.3}$$

The terms containing $[J_1, J_2]$ cancel due to the antisymmetry of the bracket, whereas the terms containing $\{J_1, J_2, J_3\}$ cancel due to the fact that this bracket has the property that its cyclic sum vanishes:

$$\{J_1, J_2, J_3\} + \{J_2, J_3, J_1\} + \{J_3, J_1, J_2\} = 0. \tag{B.4}$$

*B.3. Current conservation.* When we contract (B.1) with $\kappa(1, n)$ we get terms of the following type

$$\begin{aligned}
&\kappa(1, n) \cdot [J(1,\dots, m), J(m+1,\dots, n)] \\
&\quad = ([\kappa(1, m)]^2 - [\kappa(m+1, n)]^2) J(1,\dots, m) \cdot J(m+1,\dots, n),
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
&\kappa(1, n) \cdot \{J(1,\dots, m), J(m+1,\dots, k) J(k+1,\dots, n)\} \\
&\quad = J(1,\dots, m) \cdot [J(m+1,\dots, k), J(k+1,\dots, n)] \\
&\qquad - J(k+1,\dots, n) \cdot [J(1,\dots, m), J(m+1,\dots, k)].
\end{aligned} \tag{B.6}$$

The second type of expression, inserted in the summation of eq. (B.1) leads to

$$\begin{aligned}
&\sum_{m=1}^{n-2} \sum_{k=m+1}^{n-1} J(1,\dots, m) \cdot [J(m+1,\dots, k), J(k+1,\dots, n)] \\
&\quad - \sum_{m=1}^{n-2} \sum_{k=m+1}^{n-1} J(k+1,\dots, n) \cdot [J(1,\dots, m), J(m+1,\dots, k)] \\
&\quad = \sum_{m=1}^{n-1} J(1,\dots, m) \cdot J(m+1,\dots, n) [\kappa(m+1, n)]^2 \\
&\quad - \sum_{k=1}^{n-1} J(1,\dots, k) \cdot J(k+1,\dots, n) [\kappa(1, k)]^2.
\end{aligned} \tag{B.7}$$

This sum cancels the sum of the terms of type (B.5) in eq. (B.1), thus proving current conservation.

## Appendix C

### FORM OF THE $\mathscr{C}$-FUNCTIONS FOR SPECIFIC HELICITIES

In this appendix we show that

$$\begin{aligned}
&\mathscr{C}(1-, 3+, \dots, k+, 2-, (k+1)+, \dots, n+) \\
&\quad = \frac{(\sqrt{2})^n}{2} \frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 34 \rangle \cdots \langle k2 \rangle \langle 2, k+1 \rangle \cdots \langle n-1, n \rangle \langle n1 \rangle}.
\end{aligned} \tag{C.1}$$

For $k = 3$ relation (C.1) is valid, as was shown in sect. 6. Suppose it to hold for all $k < m$, then for $k = m$ we find by using the cyclic and subcyclic identities that

$$\begin{aligned}
&\mathscr{C}(1-,3+,4+,\dots,m+,2-,(m+1)+,\dots,n+)\\
&\quad+\mathscr{C}(1-,4+,3+,\dots,m+,2-,(m+1)+,\dots,n+)+\cdots\\
&\quad\cdots+\mathscr{C}(1-,4+,\dots,m+,3+,2-,(m+1)+,\dots,n+)\\
&\quad=\tfrac{1}{2}(\sqrt{2})^n\langle 12\rangle^5\mathscr{N},
\end{aligned}\tag{C.2}$$

with

$$\mathscr{N}=(\langle 23\rangle\langle 31\rangle\langle 14\rangle\cdots\langle m2\rangle\langle 2,m+1\rangle\cdots\langle n1\rangle)^{-1}. \tag{C.3}$$

In deriving the r.h.s. of eq. (C.2) the remaining terms in the subcyclic identity have been summed with eq. (6.6). Their explicit form is given by eq. (C.1). From the recursion relation it can be seen which pole terms can occur in a $\mathscr{C}$-function and which ones can't. Terms like $\langle ij\rangle^{-1}$ are only allowed when gluons $i$ and $j$ are adjacent. Therefore eq. (C.2) leads to

$$\mathscr{C}(1-,3+,4+,\dots,m+,2-(m+1)+,\dots,n+)=\tfrac{1}{2}(\sqrt{2})^n\alpha_4\langle 23\rangle\langle 14\rangle\mathscr{N}, \tag{C.4}$$

$$\begin{aligned}
&\mathscr{C}(1-,4+,\dots,(i-1)+,3+,i+,\dots,m+,2-,(m+1)+,\dots,n+)\\
&\quad=\tfrac{1}{2}(\sqrt{2})^n\alpha_i\langle 23\rangle\langle 31\rangle\langle i-1,i\rangle\mathscr{N},
\end{aligned}\tag{C.5}$$

with $5 \leqslant i \leqslant m-1$, and

$$\mathscr{C}(1-,4+,\dots,m+,3+,2-,(m+1)+,\dots,n+)=\tfrac{1}{2}(\sqrt{2})^n\alpha_2\langle 31\rangle\langle m2\rangle\mathscr{N}, \tag{C.6}$$

where $\alpha_2, \alpha_4, \dots, \alpha_{m-1}$ have to be determined. Substituting the expressions (C.4)–(C.6) into eq. (C.2) gives

$$\langle 12\rangle^5=\langle 31\rangle\langle m2\rangle\alpha_2+\langle 23\rangle\langle 14\rangle\alpha_4+\langle 23\rangle\langle 31\rangle\sum_{i=5}^{m-1}\langle i-1,i\rangle\alpha_i. \tag{C.7}$$

Taking the special case $K_2 \to K_1$ we find

$$\sum_{i=5}^{m-1} \langle i-1, i\rangle \alpha_i = \frac{\langle 14\rangle \alpha_4 + \langle 1m\rangle \alpha_2}{\langle 13\rangle}, \tag{C.8}$$

which we can substitute in eq. (C.7), giving

$$\alpha_2 = \frac{\langle 12\rangle^4}{\langle 3m\rangle}. \tag{C.9}$$

Therefore we have determined

$$\mathscr{C}(1-,4+,5+,\ldots,m+,3+,2-,(m+1)+,\ldots,n+)$$
$$= \frac{(\sqrt{2})^n}{2} \frac{\langle 12\rangle^4}{\langle 14\rangle\langle 45\rangle \cdots \langle m3\rangle\langle 32\rangle\langle 2, m+1\rangle \cdots \langle n-1, n\rangle\langle n1\rangle}, \tag{C.10}$$

which is the same as eq. (C.1), thus proving the statment for $k = m$.

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