A. Singular Value Decomposition

Let $\mathbf{X}$ be an arbitrary $n \times m$ matrix[7] and $\mathbf{X}^T\mathbf{X}$ be a rank r, square, symmetric $n \times n$ matrix. In a seemingly unmotivated fashion, let us define all of the quantities of interest.

- $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_r\}$ is the set of *orthonormal* $m \times 1$ eigenvectors with associated eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ for the symmetric matrix $\mathbf{X}^T\mathbf{X}$.

$$(\mathbf{X}^T\mathbf{X})\hat{\mathbf{v}}_i = \lambda_i \hat{\mathbf{v}}_i$$

- $\sigma_i \equiv \sqrt{\lambda_i}$ are positive real and termed the *singular values*.

- $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \ldots, \hat{\mathbf{u}}_r\}$ is the set of *orthonormal* $n \times 1$ vectors defined by $\hat{\mathbf{u}}_\mathbf{i} \equiv \frac{\mathbf{1}}{\sigma_\mathbf{i}}\mathbf{X}\hat{\mathbf{v}}_\mathbf{i}$.

We obtain the final definition by referring to Theorem 5 of Appendix A. The final definition includes two new and unexpected properties.

- $\hat{\mathbf{u}}_\mathbf{i} \cdot \hat{\mathbf{u}}_\mathbf{j} = \delta_{ij}$

- $\|\mathbf{X}\hat{\mathbf{v}}_\mathbf{i}\| = \sigma_i$

These properties are both proven in Theorem 5. We now have all of the pieces to construct the decomposition. The