7.6. NORMAL POPULATIONS WITH UNEQUAL VARIANCE–COVARIANCE MATRICES

When the populations are normal but the variance–covariance matrices are not equal, the allocation problem is more difficult. In the case of equal variance–covariance matrices, the logarithm of the ratio of the distribution functions was simplified by the cancellation of a quadratic term. However, in the unequal variance–covariance case, that term cannot be removed and the result is called a "quadratic rule." When $\Sigma_1 \neq \Sigma_2$,

$$
\begin{aligned}
\log(f_1/f_2) &= \tfrac{1}{2}\log(|\Sigma_2|/|\Sigma_1|) - \tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_1)^{\mathrm{T}}\Sigma_1^{-1}(\mathbf{x} - \boldsymbol{\mu}_1) \\
&\quad + \tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_2)^{\mathrm{T}}\Sigma_2^{-1}(\mathbf{x} - \boldsymbol{\mu}_2) \\
&= C_0 - \tfrac{1}{2}[\mathbf{x}^{\mathrm{T}}(\Sigma_1^{-1} - \Sigma_2^{-1})\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}(\Sigma_1^{-1}\boldsymbol{\mu}_1 - \Sigma_2^{-1}\boldsymbol{\mu}_2)] \qquad (7.28)
\end{aligned}
$$

where $C_0 = \frac{1}{2}\ln(|\Sigma_1|/|\Sigma_2|) - \frac{1}{2}(\boldsymbol{\mu}_1^{\mathrm{T}}\Sigma_1^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2^{\mathrm{T}}\Sigma_2^{-1}\boldsymbol{\mu}_2)$. This is a *quadratic* rule because the middle term of Eq. (7.28) is a quadratic form in the features.