Wavelet Matrix Multiplcation

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1 Wavelet Matrix Multiplication

One of the key points for wavelet matrix multiplication is the proof that $W(A) \times W(B) = W(A \times B)$ If this is the case, then it is obvious that $W(A) \times W(B) = W(C) = W(A \times B = C)$. So far the proof is still weak. The reason is that an example proof is useful for proving something to not be the case, rather than being the case. However, a simple example does show some intuitive steps that would be necessary for a proof.

1.1 A 2×2 example

1.1.1 Conventional Multiplication

Conventional multiplication is spelled out as

$c_{i,j} = \sum_k a_{i,k} b_{k,j}$

For a 2×2 matrix, there is the following:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{pmatrix}$$

1.1.2 Wavelet Transform of two $2\times$ 2 matrices

For a wavelet transform on both matrix A and B, the results are:

$$W(A) = \frac{1}{2} \begin{pmatrix} a_{1,1} + a_{2,1} + a_{1,2} + a_{2,2} & a_{1,1} + a_{2,1} - a_{1,2} - a_{2,2} \\ a_{1,1} - a_{2,1} + a_{1,2} - a_{2,2} & a_{1,1} - a_{2,1} - a_{1,2} + a_{2,2} \end{pmatrix}$$

$$W(B) = \frac{1}{2}\begin{pmatrix} b_{1,1}+b_{2,1}+b_{1,2}+b_{2,2} & b_{1,1}+b_{2,1}-b_{1,2}-b_{2,2} \\ b_{1,1}-b_{2,1}+b_{1,2}-b_{2,2} & b_{1,1}-b_{2,1}-b_{1,2}+b_{2,2} \end{pmatrix}$$

1.1.3 Product of A and B in wavelet space

The conventional product of A and B can be transformed into wavelet space. The results of this matrix transform is as follows:

$$W(A \times B) = W\begin{pmatrix} a_{1,1}b_{1,1}+a_{1,2}b_{2,1} & a_{1,1}b_{1,2}+a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1}+a_{2,2}b_{2,1} & a_{2,1}b_{1,2}+a_{2,2}b_{2,2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2})+(a_{2,1}b_{1} \\ (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2})-(a_{2,1}b_{1} \end{pmatrix}$$

$$W(A \times B) == \frac{1}{2}\begin{pmatrix} (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2}+a_{2,1}b_{1,1}+a_{2,2}b_{2,1}+a_{2,1}b_{1,2}+a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}+a_{1,2}b \\ (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2}-a_{2,1}b_{1,1}-a_{2,2}b_{2,1}-a_{2,1}b_{1,2}-a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}+a_{1,2}b \end{pmatrix}$$

1.1.4 What is $W(A) \times W(B)$

Straight forward multiplication of $W(A) \times W(B)$ works out as follows:

$$W(A) \times W(B) = \frac{1}{2}\begin{pmatrix} a_{1,1}+a_{2,1}+a_{1,2}+a_{2,2} & a_{1,1}+a_{2,1}-a_{1,2}-a_{2,2} \\ a_{1,1}-a_{2,1}+a_{1,2}-a_{2,2} & a_{1,1}-a_{2,1}-a_{1,2}+a_{2,2} \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} b_{1,1}+b_{2,1}+b_{1,2}+b_{2,2} & b_{1,1}+b_{2, \\ b_{1,1}-b_{2,1}+b_{1,2}-b_{2,2} & b_{1,1}-b_{2, \end{pmatrix}$$

Of course this is better simplified.

$$\frac{1}{2}\begin{pmatrix} (a_{1,1}b_{1,1}+a_{2,1}b_{1,1}+a_{1,2}b_{2,1}+a_{2,2}b_{2,1}+a_{1,1}b_{1,2}a_{2,1}b_{1,2}+a_{1,2}b_{2,2}+a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}+a_{2,1}b_{1,1}+a_{1,2}b_{2,1}+ \\ (a_{1,1}b_{1,1}-a_{2,1}b_{1,1}+a_{1,2}b_{2,1}-a_{2,2}b_{2,1}+a_{1,1}b_{1,2}-a_{2,1}b_{1,2}+a_{1,2}b_{2,2}-a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}-a_{2,1}b_{1,1}+a_{1,2}b_{2,1}- \end{pmatrix}$$

Compared with $W(C)$ computed the conventional way:

$$W(C) = \frac{1}{2}\begin{pmatrix} (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2}+a_{2,1}b_{1,1}+a_{2,2}b_{2,1}+a_{2,1}b_{1,2}+a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}- \\ (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}+a_{1,1}b_{1,2}+a_{1,2}b_{2,2}-a_{2,1}b_{1,1}-a_{2,2}b_{2,1}-a_{2,1}b_{1,2}-a_{2,2}b_{2,2}) & (a_{1,1}b_{1,1}+a_{1,2}b_{2,1}- \end{pmatrix}$$

Notice that $W(A) \times W(B) = W(A \times B)$, in the case of 2×2 matrices.

1.2 Proof of Wavelet Matrix Multiplication

A reminder from Dr. Sinzinger, wavelets and their inverses are linear operators. As he correctly reminded me any linear operator has the following properties:

1. $L(AB) = L(A)\dot{L}(B)$
2. ψ^{-1} is a linear operator
3. $\psi^{-1}(\psi(A)\dot{\psi}(B)) = \psi^{-1}(\psi(A))\dot{\psi}^{-1}(\psi(B))$
4. $\psi^{-1}(\psi(A)) = A$
5. $\psi^{-1}(\psi(B)) = B$
6. Therefore: $AB = \psi^{-1}(\psi(A)\dot{\psi}(B))$

Thus this is sufficient proof that wavelet matrix multiplication is sound.