ON THE RELATIONSHIP BETWEEN THE KARHUNEN-LOEVE TRANSFORM AND THE PROLATE SPHEROIDAL WAVE FUNCTIONS

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ABSTRACT

We find a close relationship between the discrete Karhunen-Loeve transform (KLT) and the discrete prolate spheroidal wave functions (DPSWF). We show that the DPSWF form a natural basis for an expansion of the eigenfunctions of the KLT in the frequency domain, and then determine more general conditions that any set of functions must obey to be a valid basis. We also present approximate solutions for small, medium, and large filter orders. The medium order solution suggests that the principal eigenfunction is, to a high degree of approximation, the principal DPSWF modulated so that its center frequency coincides with the peak of maximum energy in the signal spectrum. We then use this result to propose a new basis.

1. INTRODUCTION

In Thompson's [6] seminal work on non-parametric spectral estimation using multi-taper windows, he first suggested that the discrete prolate spheroidal wave functions (DPSWF) may be a suitable basis in which to expand the eigenfunctions of the discrete Karhunen-Loeve transform (KLT) in the frequency domain. We intend to strengthen the connection. Our motivation is a desire to understand the frequency domain behavior of the solutions to the KLT. We seek to show that the DPSWF are a natural basis for expressing the eigenfunctions of the KLT. By natural, we mean that the expansion can be found without assuming it, and also that the expansion is sparse so that the eigenfunctions can be expanded in terms of only a few DPSWF. We begin by reviewing the KLT.

2. KARHUNEN-LOEVE TRANSFORM

Let $x(n)$ be a zero-mean, wide-sense stationary signal with autocorrelation function $r(m) = E[x(n)x(n+m)]$ and power spectrum $S(f)$, related through the Wiener-Khintchin relation

$$r(n) = \int_{-1/2}^{1/2} S(f) e^{i2\pi nf} df . \tag{1}$$

Furthermore, let the signal be filtered by a finite impulse response (FIR) filter with weights $\boldsymbol{w} = [w(0) \ldots w(N-1)]^{\dagger}$ and frequency response $\Phi(f)$. The discrete Karhunen-Loeve transform (KLT) arises from asking the following question: what is the optimal filter that maximizes the output power

$$P = \boldsymbol{w}^{\dagger}\boldsymbol{R}\boldsymbol{w} = \int_{-1/2}^{1/2} |\Phi(f)|^2 S(f) df \tag{2}$$

subject to the constraint that the filter have unity L2 norm

$$\boldsymbol{w}^{\dagger}\boldsymbol{w} = \int_{-1/2}^{1/2} |\Phi(f)|^2 df = 1 . \tag{3}$$

Here, $\boldsymbol{R}$ is the $N \times N$ symmetric positive-definite Toeplitz autocorrelation matrix with components $R_{ij} = r(i-j)$.

2.1 Time domain

In the time domain, it is easy to show that maximizing (2) subject to (3) results in the discrete KLT:

$$\boldsymbol{R}\boldsymbol{w} = \lambda \boldsymbol{w} . \tag{4}$$

Thus, the optimal matched filter is an eigenvector of the autocorrelation matrix, and is sometimes called an eigenfilter. There will be N solutions to (4), however only the one corresponding to the largest eigenvalue also maximizes (2). The other solutions may prove useful if we seek a second matched filter with an output uncorrelated with the first one, and so on. The eigenfilters are orthogonal, alternately symmetric or anti-symmetric, and the two that correspond to the largest and smallest eigenvalues have all their zeros on the unit circle [3].

2.2 Frequency domain

In the frequency domain, maximizing (2) subject to (3) can be done by inspection and results in

$$|\Phi(f)|^2 = \delta(f - f_{\max}) , \quad f_{\max} = \max_f [S(f)] \tag{5}$$

a Dirac delta function centered at the peak value of the signal spectrum. However, since $\boldsymbol{w}$ is index limited in the time domain, it cannot have infinite energy concentration in the frequency domain because of the time-bandwidth product rule. This means that the constraint that $\boldsymbol{w}$ is index limited is *not* embodied in the frequency domain parts of (2) and (3), and thus (5) cannot be a valid solution. Thomson [6] recognized that the correct way to obtain the frequency-domain eigensystem equation is by direct transformation of (4). By defining the discrete Fourier transform (DFT) as

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$$\Phi(f) = \varepsilon \sum_{n=0}^{N-1} w(n) e^{i2\pi f[n-(N-1)/2]} \qquad (6)$$

the frequency response of the filter can be guaranteed real, for convenience, by defining the constant ε to be either 1 or *i*, depending on whether $\boldsymbol{w}$ is symmetric or anti-symmetric, respectively. Then, taking the DFT of both sides of (4) and substituting (1) there results the frequency domain eigensystem equation:

$$\int_{-1/2}^{1/2} \frac{\sin[N\pi(f-f')]}{\sin[\pi(f-f')]} S(f')\Phi(f')df' = \lambda\Phi(f) . \qquad (7)$$

Equation (7) is a homogenous Fredholm equation of the second kind with a Dirichlet kernel and weighting function $S(f)$. It will have N solutions, $\Phi_k(f)$ k=1...N, referred to as eigenfunctions, however only the one corresponding to the largest eigenvalue also maximizes (2). The eigenfunctions are either symmetric or anti-symmetric about the origin and they are orthogonal with or without the spectrum as weighting function (Thomson [6]):

$$\int_{-1/2}^{1/2} \Phi_k(f)\Phi_l(f)df = \frac{1}{\lambda_k}\int_{-1/2}^{1/2} \Phi_k(f)\Phi_l(f)S(f)df = \delta_{kl} . \qquad (8)$$

When $S(f)$ is a bandlimited process, such that $S(f)$=1 for $|f| \le W$ and zero elsewhere, the resulting equation defines the discrete prolate spheroidal wave functions (DPSWF) (Slepian [5])

$$\int_{-W}^{W} \frac{\sin[N\pi(f-f')]}{\sin[\pi(f-f')]} U_k(f', N, W)df' = \Lambda_k(N, W)U_k(f, N, W) \qquad (9)$$

where the functional dependence of the DPSWF and their eigenvalues on the filter order N and half-bandwidth W has been emphasized. The DPSWF are doubly orthogonal, as in (8), except the second integral is over {-W,W} due to the bandlimited nature of $S(f)$. The eigenvalues Λ_k represent the fraction of energy within the band and the first 2NW eigenvalues are close to one, with the remaining eigenvalues approaching zero. The time domain sequences corresponding to the DPSWF are called the discrete prolate spheroidal sequences (DPSS). Of all index limited sequences of length N, the DPSS have the greatest concentration of energy in the band $|f| \le W$, subject to orthogonality constraints.

3. EXACT SOLUTIONS

3.1 Separable Kernel Approach

Thomson [6] first suggested on heuristic grounds that the DPSWF may be a suitable basis in which to expand the eigenfunctions. We now show that such an expansion arises naturally in the process of solving (9), without ever explicitly presuming the expansion.

A closed form algebraic solution of a homogenous Fredholm equation of the second kind can be found whenever the kernel is separable or degenerate, meaning expressible as a sum of products of the form

$$\frac{\sin[N\pi(f-f')]}{\sin[\pi(f-f')]} = \sum_{n=0}^{N-1} Q_n(f) \cdot Q_n(f') \qquad (10)$$

where $Q_n(f)$ is an unspecified family of functions. Substituting (10) into the frequency domain eigensystem equation (7) leads to

$$\sum_{n=0}^{N-1} Q_n(f) \int_{-1/2}^{1/2} Q_n(f')S(f')\Phi(f')df' = \lambda\Phi(f) . \qquad (11)$$

Noting that the integral is just a constant

$$c(n) \equiv \int_{-1/2}^{1/2} Q_n(f')S(f')\Phi(f')df' \qquad (12)$$

it becomes clear that (11) is just an expansion in Hilbert space of the eigenfunctions in terms of the basis functions $Q_n(f)$:

$$\sum_{n=0}^{N-1} c(n)Q_n(f) = \lambda\Phi(f) . \qquad (13)$$

We can solve for the expansion coefficients by multiplying both sides of (13) by $Q_m(f) \cdot S(f)$ and integrating:

$$\sum_{n=0}^{N-1} c(n) \int_{-1/2}^{1/2} Q_m(f)Q_n(f)S(f)df = \lambda c(m) . \qquad (14)$$

Regarding the expansion coefficients as the components of a vector $\boldsymbol{c}$, they are themselves the solution of an eigensystem equation

$$\boldsymbol{Ac} = \lambda\boldsymbol{c} \qquad (15)$$

of an $N \times N$ matrix $\boldsymbol{A}$ whose elements are given by

$$A_{mn} \equiv \int_{-1/2}^{1/2} Q_m(f)Q_n(f)S(f)df . \qquad (16)$$

Again, there will be N solutions to (15), each corresponding to a solution to (7).

There are several ways to separate the Dirichlet kernel. One is through the use of a Fourier basis:

$$\frac{\sin[N\pi(f-f')]}{\sin[\pi(f-f')]} = \sum_{n=0}^{N-1} e^{i\pi(N-1-2n)f} e^{-i\pi(N-1-2n)f'} . \qquad (17)$$

Use of (17) with the previous development leads to $\boldsymbol{A} \Rightarrow \boldsymbol{R}$ and thus back to the *time domain* eigensystem equation (4). However, the Dirichlet kernel is also separable in terms of the DPSWF

$$\frac{\sin[N\pi(f-f')]}{\sin[\pi(f-f')]} = \sum_{n=0}^{N-1} U_n(f-f_0, N, W)U_n(f'-f_0, N, W) \qquad (18)$$

where f_0 is an arbitrary frequency offset. Use of (18) in conjunction with the previous development leads naturally to an expansion of the k^{th} eigenfunction in terms of the DPSWF

$$\Phi_k(f) = \frac{1}{\lambda_k}\sum_{n=0}^{N-1} \boldsymbol{c}_k(n)U_n(f-f_0, N, W) \qquad k = 0 \ldots N-1 \qquad (19)$$

where $\boldsymbol{c}_k(n)$ is the n^{th} component of the k^{th} eigenvector solution of $\boldsymbol{A}\boldsymbol{c}_k = \lambda_k \boldsymbol{c}_k$, with $\boldsymbol{A}$ having elements

$$A_{mn} = \int_{-1/2}^{1/2} U_m(f-f_0, N, W) U_n(f-f_0, N, W) S(f) df . \qquad (20)$$

Thus, an expansion of the eigenfunctions in terms of the DPSWF falls out naturally as a result of the separability of the Dirichlet kernel. The problem with the expansion in (19), however, is that while $\Phi_k(f)$ is (anti-)symmetric about the origin, $U_n(f\text{-}f_0)$ is not, and thus may not be a sparse representation.

3.2 Expansion Approach

We now summarize Thomson's approach and assume an expansion in terms of an arbitrary but complete basis

$$\Phi(f) = \sum_{n=0}^{N-1} c(n) Q_n(f) . \qquad (21)$$

Substituting (21) into (7) and multiplying both sides by $Q_m(f)$ and integrating there results

$$\sum_{n=0}^{N-1} c(n) \int_{-1/2}^{1/2} Q_n(f') S(f') df' \int D(f-f') Q_m(f) df = \lambda \sum_{n=0}^{N-1} c(n) \int Q_m(f) Q_n(f) df \qquad (22)$$

where $D(f-f')$ denotes the Dirichlet kernel. For the basis to be "natural", we desire that the double integral be replaced with a single integral so that the equation be numerically tractable. First, note that the integration of $Q_m(f)$ with the Dirichlet kernel in (22) is a convolution, since $D(f-f')$ is symmetric, if the limits of integration are {-½,½}, and thus equivalent to multiplication by a boxcar window of length N in the time domain. However, if $Q_m(f)$ already corresponds to an index limited sequence of length N, the windowing has no effect and the convolution simply acts as the identity operator and returns $Q_m(f)$. Then the left-hand side of (22) is the same as the left-hand side of (14), and we arrive at the generalized eigensystem equation $\boldsymbol{A}\boldsymbol{c} = \lambda \boldsymbol{B}\boldsymbol{c}$, where the elements of $\boldsymbol{A}$ are given by (16) and $\boldsymbol{B}$ by the inner product $\langle Q_m(f), Q_n(f) \rangle$ (an integral over {-½,½}).

Another possibility for simplifying (22) is for $Q_m(f)$ to be related to the DPSWF. If the limits of integration are chosen properly, then we can use the defining equation for the DPSWF (9) to reduce the integration of $Q_m(f)$ with the Dirichlet kernel to a multiplication of the DPSWF by its eigenvalue. Thomson's approach was to divide the digital frequency axis into bands of equal width and use the first few principal DPSWF frequency shifted to the center of each band. For our approach, we first seek insight from approximate solutions.

4. APPROXIMATE SOLUTIONS

4.1 Large Order Solution

As $N \to \infty$, the Dirichlet function becomes a Dirac delta function and (7) becomes

$$\Phi(f) S(f) = \lambda \Phi(f) \qquad (23)$$

the solution of which is

$$\Phi_k(f) = \delta(f - f_k) , \quad \lambda_k = S(f_k) . \qquad (24)$$

Thus, the eigenfilters form a fixed Fourier basis, and the eigenvalues are the power spectrum. Each eigenvalue is doubly degenerate, associated with two real sinusoidal eigenvectors separated in phase by 90 degrees. The principal eigenvalue, λ_0, is the maximum value of the power spectrum: $\lambda_0 = \max_f [S(f)]$.

4.2 Medium Order Solution

The results from (5) suggest that, even when the filter order is not infinite, the principal eigenfunction will have maximum energy concentration within a frequency band compatible with an index-limited sequence. The index limited sequence with the largest concentration of energy in the frequency domain is, in fact, the DPSWF with the largest eigenvalue. However, instead of finding the maximum value of the spectrum, as in the large order solution, the medium order solution involves finding the peak with the largest total energy compatible with the bandwidth of the DPSWF. This frequency shifting of the DPSWF must be done so as to preserve the symmetry of the principal eigenfunction. Therefore, we propose that *the solution to the maximization of (2) is, to a high degree of approximation, a modulated DPSWF*:

$$\Phi_0(f) \approx U_0(f - f_0, N, W_0) + U_0(f + f_0, N, W_0) . \qquad (25)$$

The optimal values $\hat{f}_0$ and $\hat{W}_0$ are those that maximize the energy

$$E_{max} \approx \max_{f_0, W_0} \int_{-1/2}^{1/2} |\Phi_0(f)|^2 S(f) df . \qquad (26)$$

Since the proposed eigenfunction is symmetric, if the spectrum $S(f)$ is also symmetric about its local maximums, then $\hat{f}_0$ will coincide with one of these local maximums.

To test this supposition we created a signal with the power spectrum shown in Fig. 1a, for which the power spectrum of the principal eigenfilter for a filter order of 20 is shown in Fig. 1b. We then found the principal DPSWF for the same filter order and searched among all possible modulations and half-bandwidths to find the optimal values according to (26). The optimal DPSWF power spectrum is shown in Fig 1c, and is nearly identical to that of the principal eigenfilter. Figs. 2a and 2b show the total output power (2) as a function of the modulation frequency and half-bandwidth of the DPSWF, respectively. Note that the optimal DPSWF is relatively insensitive to the half-bandwidth parameter, W, and peaks at a value close to the optimal time-bandwidth product, $2WN = 2 \times 0.02 \times 20 = 0.8$, which implies that the value of W can be chosen based on the dimensionality N of the tapped delay line, irrespective of the particular signal spectrum. Note also that the optimal matched energy is very sensitive to the frequency offset, a desirable property.

It is possible, however, to conceive of situations where this approximation may not hold. For example, a spectrum that is flat everywhere, except for isolated zeros, will not have a well-defined peak with maximum energy. However, for the vast majority of spectra, the optimal filter will be well approximated by a modulated DPSWF.

4.3 Small Order Solution

Casdagli et al. [2] showed that when the autocorrelation matrix is viewed as a sampled version of the continuous time autocorrelation function, the time domain eigenvectors can be approximated by the discrete Legendre polynomials when the window width, $N \cdot \tau_s$, is small, where τ_s is the sampling period. For the purely discrete case, as N becomes small, the time-bandwidth product law dictates that the bandwidth of the eigenfilters will become large. The eigenfilters then form a fixed basis in the sense that they become increasingly insensitive to the specific spectral shape of the signal. It is interesting to note that in the limit as the bandwidth approaches the extremes $W \to 0$ and $W \to ½$, the DPSWF become the Legendre polynomials.

5. A NEW BASIS

Our goal is to choose a basis that will result in a sparse representation of the eigenfunctions in the expansion (21). Based on the approximate solutions, it seems natural to choose a DPSWF basis modulated so that the center frequency of the principal DPSWF coincides with the largest peak in the signal spectrum:

$$Q_n(f) = [U_n(f - \hat{f}_0, N, W) + U_n(f + \hat{f}_0, N, W)] . \qquad (27)$$

The half-bandwidth can be chosen on the basis of the time-bandwidth product rule such that $2NW$=1. Using (27) in (22) and choosing the limits of integration to be $\{f_0 - W, f_0 + W\}$ *and* $\{-f_0 - W, -f_0 + W\}$, it can be shown that the integration with the Dirichlet kernel in (22) can be approximated as

$$\int D(f - f')Q_m(f)df \approx \Lambda_m(N, W)Q_m(f) . \qquad (28)$$

Equation (22) then becomes the generalized eigenvalue equation

$$\boldsymbol{Ac} = \lambda \boldsymbol{Bc} \qquad (29)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $N \times N$ matrices with elements given by

$$A_{mn} \equiv \Lambda_m(N, W) \int_{-1/2}^{1/2} Q_m(f)Q_n(f)S(f)df \qquad (30)$$

$$B_{mn} \equiv \int_{\hat{f}_0 - W}^{\hat{f}_0 + W} Q_m(f)Q_n(f)df + \int_{-\hat{f}_0 - W}^{-\hat{f}_0 + W} Q_m(f)Q_n(f)df \qquad (31)$$

6. CONCLUSION

We've shown that the DPSWF form a natural basis for the frequency domain eigenfunctions of the KLT and that the DPSWF modulated to the peak of maximum energy in the signal spectrum may be the most sparse representation, at least for the principal eigenfunction. While this may seem a mathematical curiosity, we believe it has practical applications. In particular, for on-line tracking of the optimal matched filter, it may be possible to adapt a single parameter, the modulation frequency of the DPSWF, f_0, instead of the N filter weights required of present on-line methods [4]. We shall pursue this in future research as well as show that the proposed basis is in fact sparse for all the eigenfunctions.

7. REFERENCES

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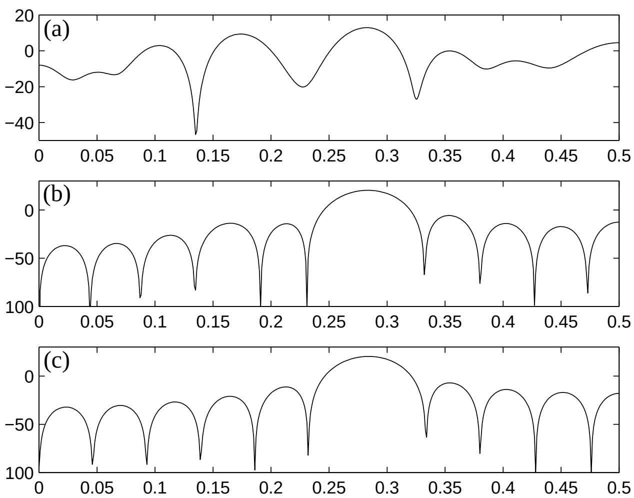

Figure 1. Power spectrums (dB) vs. frequency (digital): (a) original signal; (b) 1st eigenfilter; (c) optimal modulated 1st DPSWF.

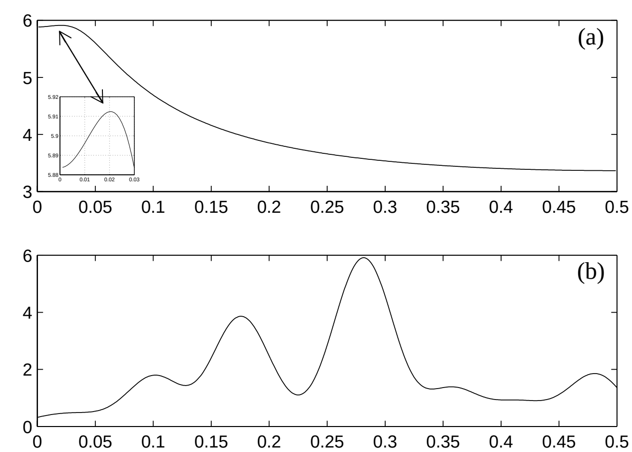

Figure 2. (a) Output power vs. f_0 for optimal half-bandwidth $\hat{W}_0$; (b) filter energy vs. W for optimal frequency offset $\hat{f}_0$.