Devore and Lucier make some simple arguments for the Haar Wavelet, and in the treatment of wavelets in general. While they admit Haar Wavelets can lack smoothness, they state the Haar Wavelet demonstrate the key features of wavelet decomposition. Their argument:

1. Let t be the independent variable.
2. Let $H = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]} = \begin{cases} 1 & 0 \leq \text{t} < \frac{1}{2} \\ -1 & \frac{1}{2} \leq \text{t} < 1 \\ 0 & \text{otherwise} \end{cases}$
3. Let the translation and dialation function be

$$H_{j,k}(t) = 2^{\frac{k}{2}}(2^k t - t) = 2^{k/2}\begin{cases} 1 & j2^{-k} \leq \text{t} < (j+\frac{1}{2})2^{-k} \\ -1 & (j+\frac{1}{2})2^{-k} \leq \text{t} < (j+1)2^{-k} \\ 0 & \text{otherwise} \end{cases}$$

4. What about $< H_{j,k}, H_{j',k'} >$
   a. Case $H_{j,k}$ and $H_{j',k'}$ are disjoint $< H_{j,k}, H_{j',k'} >= 0$
   b. Otherwise: $< H_{j,k}, H_{j',k'} >= 0$
5. $\{H_{j,k} | j, k \in Z\}$, the Haar basis, is L_2 complete. Each $S = S^0 \subset L_2(R)$ has a unique representation for dyadic shift intervals $j2^{-k}$.
6. Also, $S^\infty = L_2(R)$ and $S^{-\infty} = \{0\}$.

There is an exercise in proving this argument.

The main point of both forms of H (H and $H(t-k)$) form an orthonormal basis for W, which denotes wavelet space.

The Haar Decomposition and Smoothness space tends to be less understood. The whole point to the Haar Decomposition section is whether a wavelet can be L_p complete for an arbitrary $p \in [1, \infty)$. The answer is that for $p \in [2, \infty)$, this is true. The L_1 case shows the Haar Wavelet growing indefinately in energy. However, there is a reference to special cases in "Hardy space." At this time, the term itself is being researched as an educational exercise.

One question, what is the relationship between smoothness of f and the size of the Haar coefficents dependent on?

1. For a fixed $k \in Z$ the Haar functions $(H_{j,k}), j \in Z$ are stable.
2. The Jackson inequality (How well functions from $Lip(\alpha, L_p(R))$ can be approximated.
3. The Bernstein Inequality (Inequalities for Polynomials)

Another useful exercise is the proof of the Jackson and Bernstein Inequalities.

The fast wavelet transform is so far the same, a convolution scheme. This convolution scheme tosses out half of the values to produce convergence.

The mulivariate (multi-dimensional) Haar Wavelet Functions have the two methods:

1. Sum of a wavelet Polynomial, which is the collection of functions $2^{kd/2}\psi_v(2^k \cdot -j), j \in Z^d, k \in Z, v \in V\backslash\{0\}$

2. The collection of all shifts of $\chi_{[0,1]^d}$, which is the tensor product of the univariate space of piecewise functions with integer breakpoints.

Belkin