Matrix Multiplication via Wavelets

by

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A Master's Thesis Defense

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Conference Room (206), Computer Science Building

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Some of the most important questions in computer science are how quickly can a computation be performed and how accurate are the results. This thesis addresses these issues using Wavelets for Matrix Multiplication. Matrix multiplication itself is one of the more fundamental operations known to linear algebra, and computational scientist have been trying to shave the efficency of matrix multiply from its defining algorithm's $O(N^3)$ to something closer to $O(N^2)$. Both sparse and non-sparse means have been tried in an effort to reduce computations.

This thesis utilizes the Haar Wavelet Transform for Matrix Multiplication to precondition dense matrices. Then they can be approximated with sparse matrices and quickly multiplied together. This thesis shows how this transformation can be performed, and how accurate it is, and how precise the result is even with the lower energy elements discarded.

Topics of Discussion

- Wavelet Overview
  - Basic Wavelet
  - Multi-Resolution
    - Multi-Resolution Analysis
    - Multi-Resolution Expansion
    - ψ^n Expansion
- Numerical Examples
- Matrix Multiply
  - Regular
  - Sparse
  - Wavelet Preconditioning
  - Formal Proof
  - Empirical Results

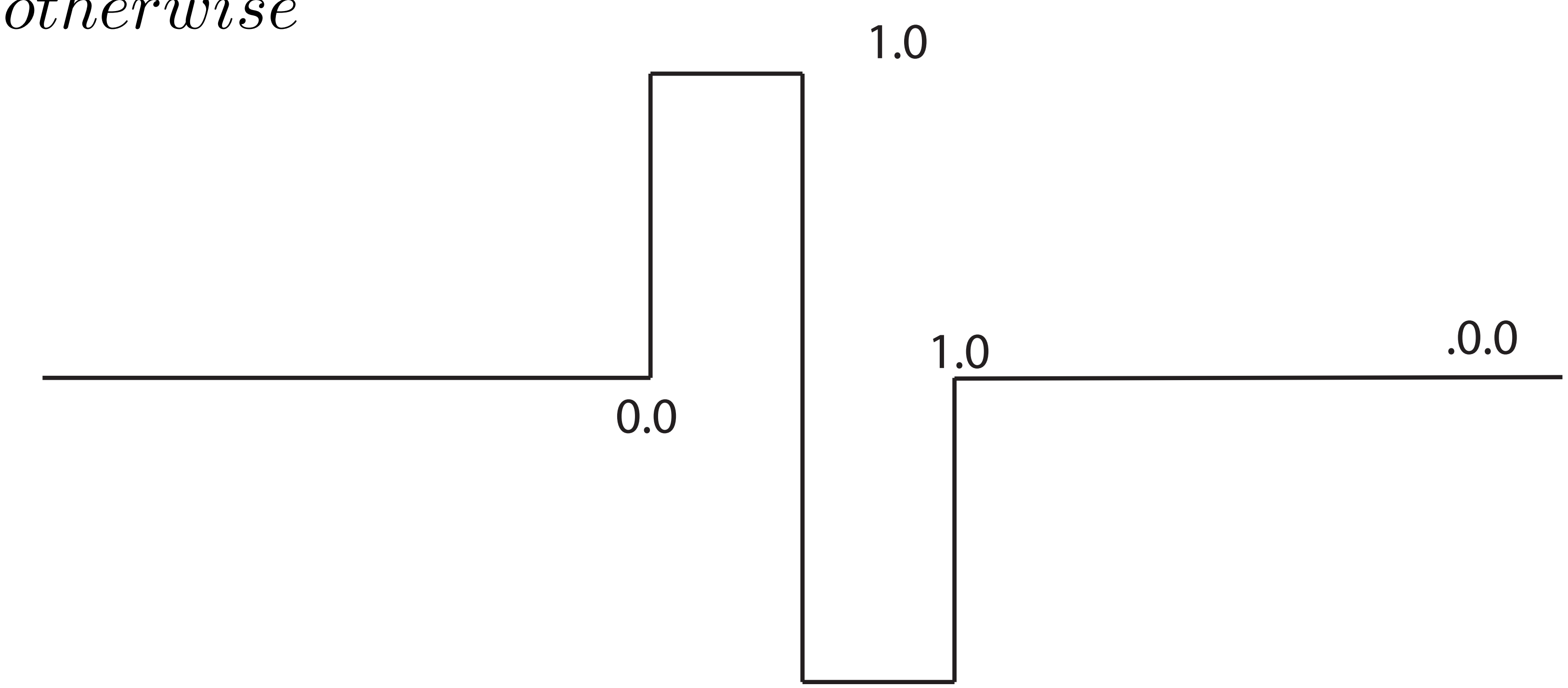
Wavelet Basis Function

The two mandatory properties of a wavelet basis function are:

- it must square integrable, and
- must have a zero average, i.e.:

$$\int_{-\infty}^{+\infty} \psi(x)\ dx = 0.$$

$$\psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & otherwise \end{cases} \quad (1)$$



Strict Definition of a Wavelet Basis

"A function $\psi \in L_2(R)$ is called an orthonormal wavelet if the family $\{\psi_{j,k}\}$ defined

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\forall j, k \in Z$$

is an orthonormal basis of $L_2(R)$ where $\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\delta_{k,m}, \forall j, k, l, m \in Z$ and every $f \in L_2(R)$ can be written as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x)$$

where the series convergences and is

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x)$$

in $L_2(R)$ such that

$$\lim_{M_1,M_2,N_1,N_2} ||f - \sum_{j=-M_2}^{N_2} \sum_{k=-M_1}^{N_1} c_{j,k}\psi_{j,k}|| = 0$$

The simplest example of orthonormal wavelets is the Haar Transform."

[10] C. Chui. *An Introduction to Wavelets.* Academic Press San Diego, 1992.

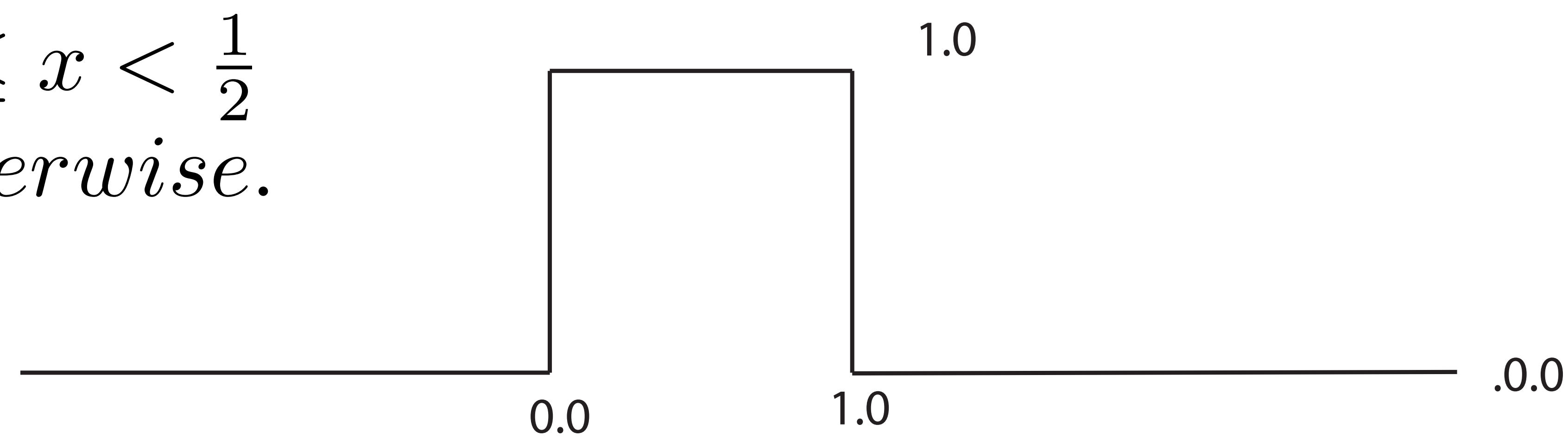
Wavelet Averaging Basis

An averaging basis function which must meet the following criteria

- be square integrable,
- satisfy the orthonormal translation-dilation property, and
- the wavelet basis function must be orthogonal with the average basis function.

Haar Averaging Basis

$$\phi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 0 & otherwise. \end{cases}$$



Introductory Wavelet Statement

- The convolution wavelet transform is a mapping from $L^2(\mathbb{R})$ defined in terms of the wavelet pair, the convolution operation on an $L^2(\mathbb{R})$ array, and an element selector.
- A wavelet pair is defined as wavelet basis function with a corresponding wavelet averaging basis function

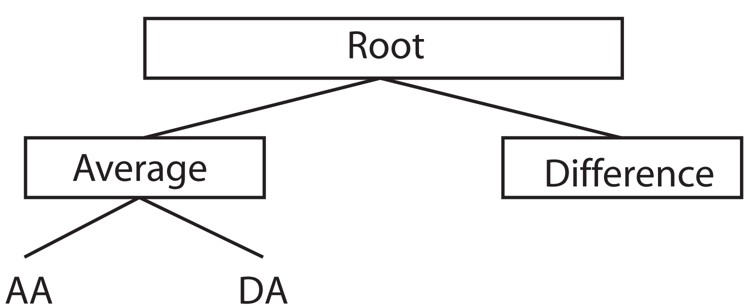
Convolution Wavelet Transform

The Convolution wavelet transform can be described using three steps

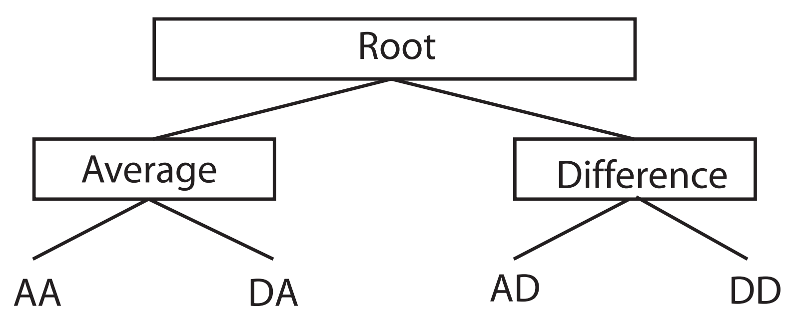
1. Convolve $A = S * \phi$ and $D = S * \psi$
2. Selectively Filter: Map $A \rightarrow A'$ and $D \rightarrow D'$
3. Concatenate: $W(S) = (A'|D')$

Multi-Resolution

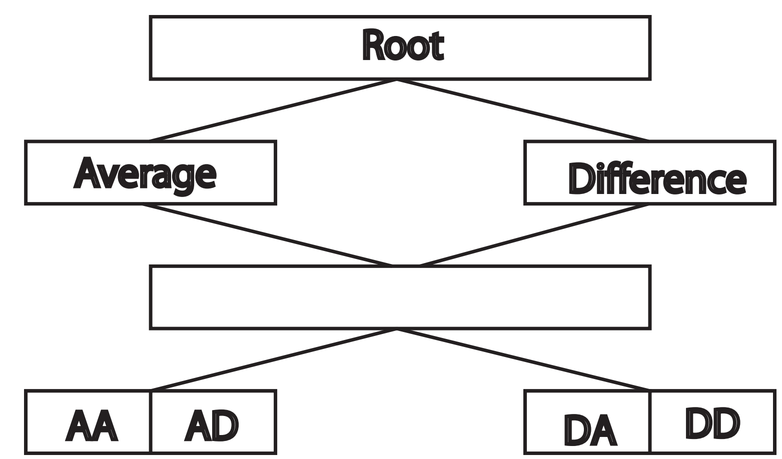
- A tree scheme is used in this thesis to represent the various forms of wavelet multi-resolution methods.
- Multi-Resolution as it applies to wavelets has three basic forms.



Multi-Resolution Analysis

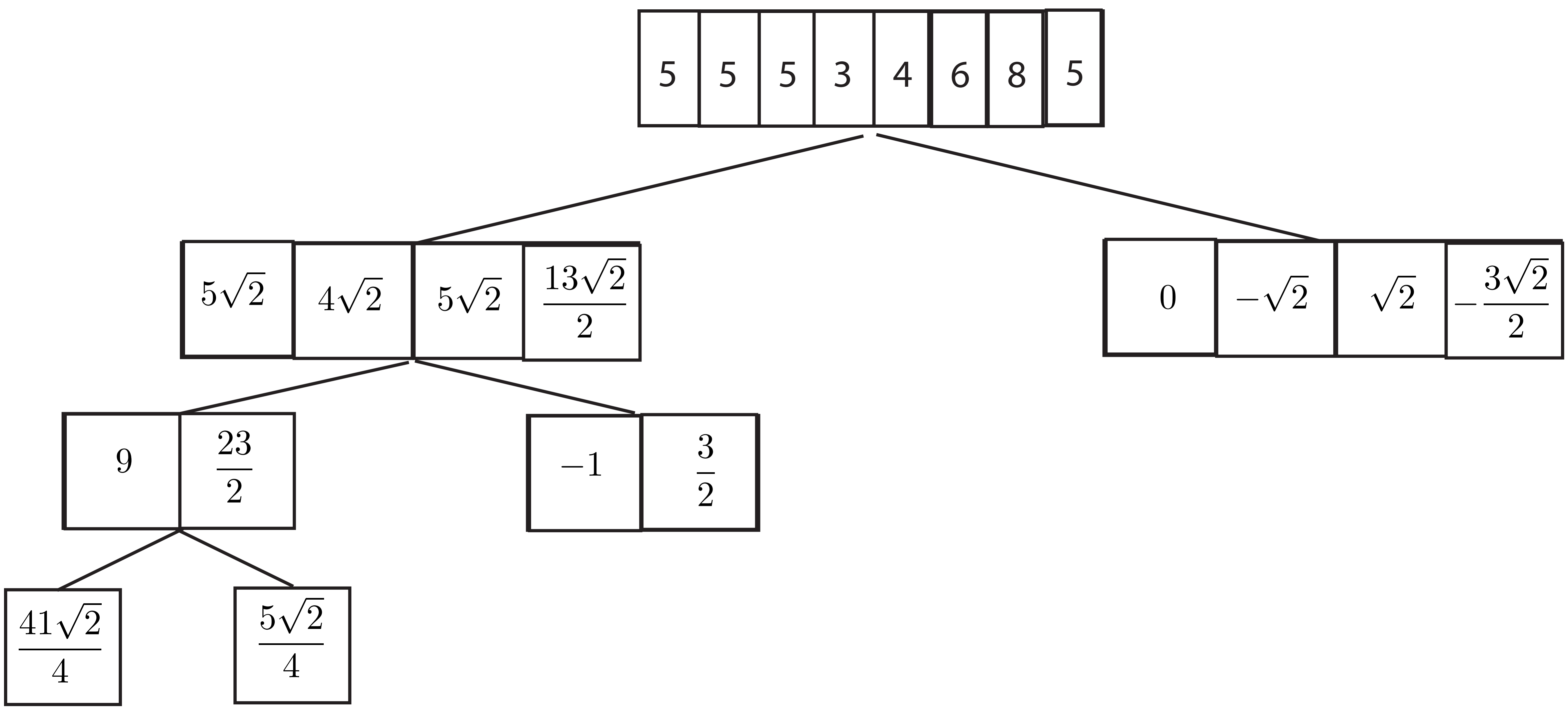


Multi-Resolution Expansion

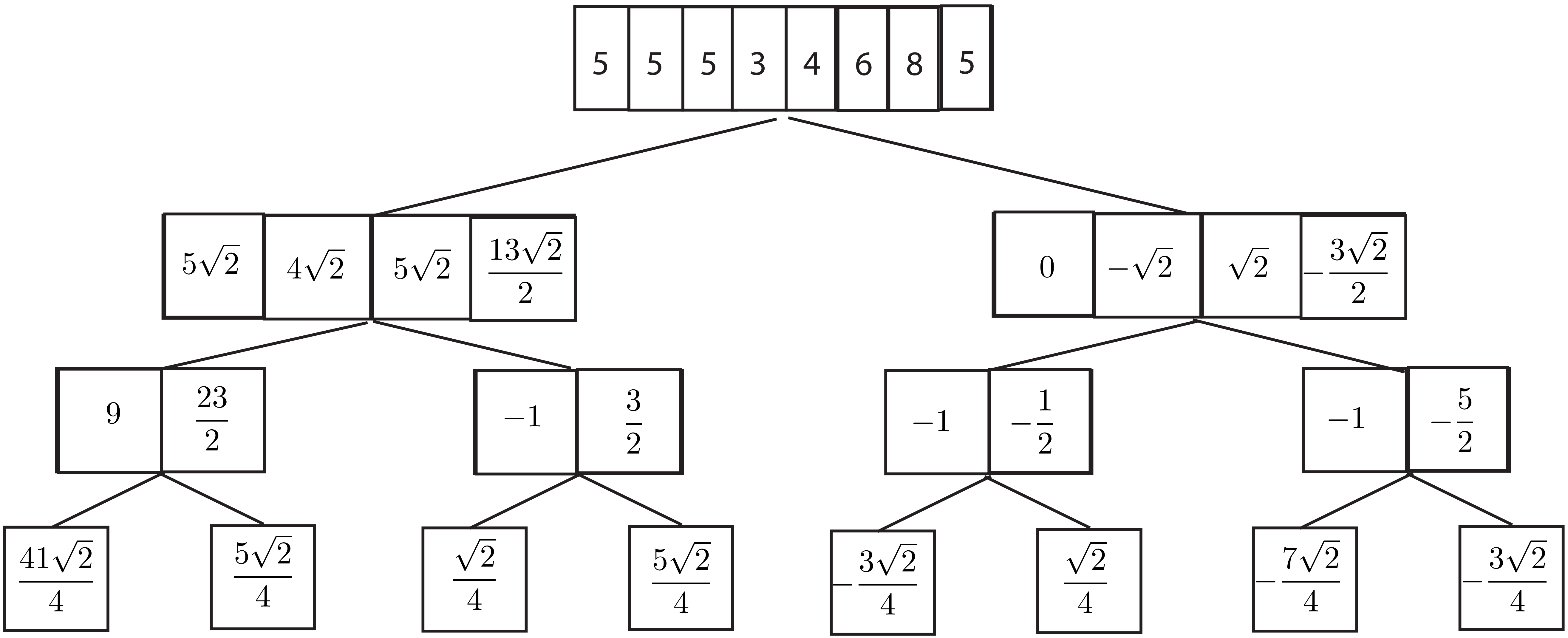


$\psi^{\mathbf{n}}$ **Expansion**

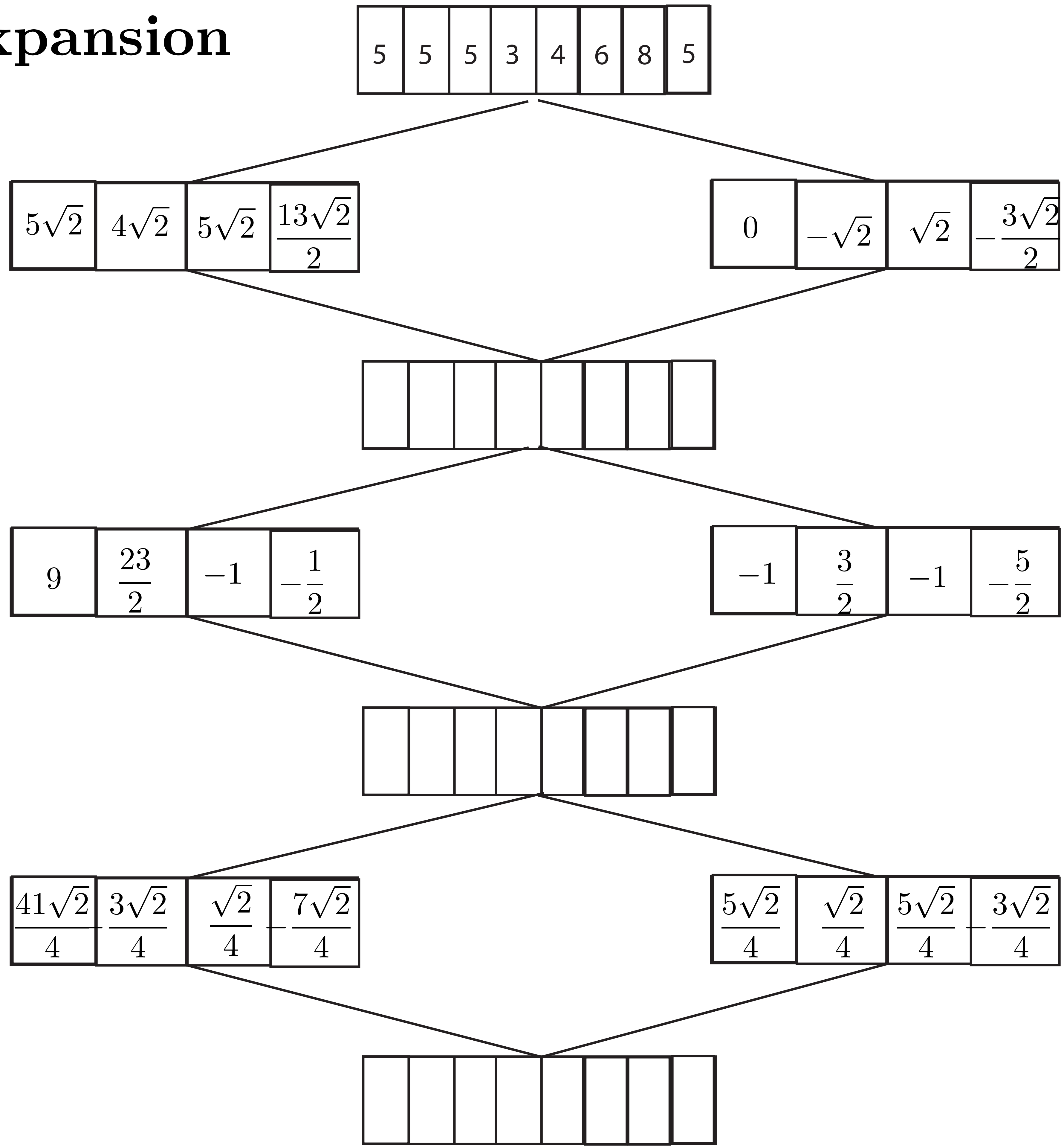
Numerical Example MRA



Numerical Example MRE



$\psi^{\mathbf{n}}$ Expansion



2-D Wavelet Transform

- The 2D transform has four components:
  - A: the average (average of the rows and columns)
  - V: vertical (average of the columns and difference of the rows)
  - H: horizontal (average of the rows and difference of the columns)
  - D: diagonal (difference of the rows and columns)
- Two basic methods of computing the 2-D CWT have been explored: the 1-D to 2-D method and the Vector-Matrix method

$$B \Rightarrow \begin{pmatrix} A & V \\ H & D \end{pmatrix}$$

1-D to 2-D Method

Algorithm 1 Wavelet Row Transform 1- D to 2 -D Method

Require: Wavelet Transform, , and Wavelet Pair (ψ and ϕ)
Require: Matrix, $S \in \mathbb{R}^{N \times M}$
  for $i = 0$ to $N - 1$ **do**
    $R = S.getRow(i)$
    $S.row(i) = [R * \phi, R * \psi]$
  end for

Vector-Matrix Method

Algorithm 1 Wavelet Transform: Vector - Matrix Method: Row Transform

Require: Wavelet Pair
Require: Temporary Vector S
Require: Matrix, $A \in \mathbb{R}^{M \times N}$
  for $i = 0$ to M **do**
    load S from A_i where A_i is the row vector at row i
    $S \xrightarrow{\psi_1} R$
    Load R into result matrix α at α_i
  end for
  Return α

Algorithm 2 Wavelet Transform: Vector - Matrix Method: Column Transform

Require: Wavelet Pair
Require: Temporary Vector S
Require: Matrix, $A \in \mathbb{R}^{M \times N}$
  for $j = 0$ to N **do**
    load S from A_j where A_i is the column vector at column j
    $S \xrightarrow{\psi_1} R$
    Load R into result matrix α at α_j
  end for
  Return α

2-D Wavelet Transform

Original

Transformed

Multi-Resolution

There exist Multi-Resolution for 2-D structures such as matrices. Those methods are:

- 2- D Multi-Resolution Analysis
- 2-D Multi-Resolution Expansion
- 2-D ψ^n expansion

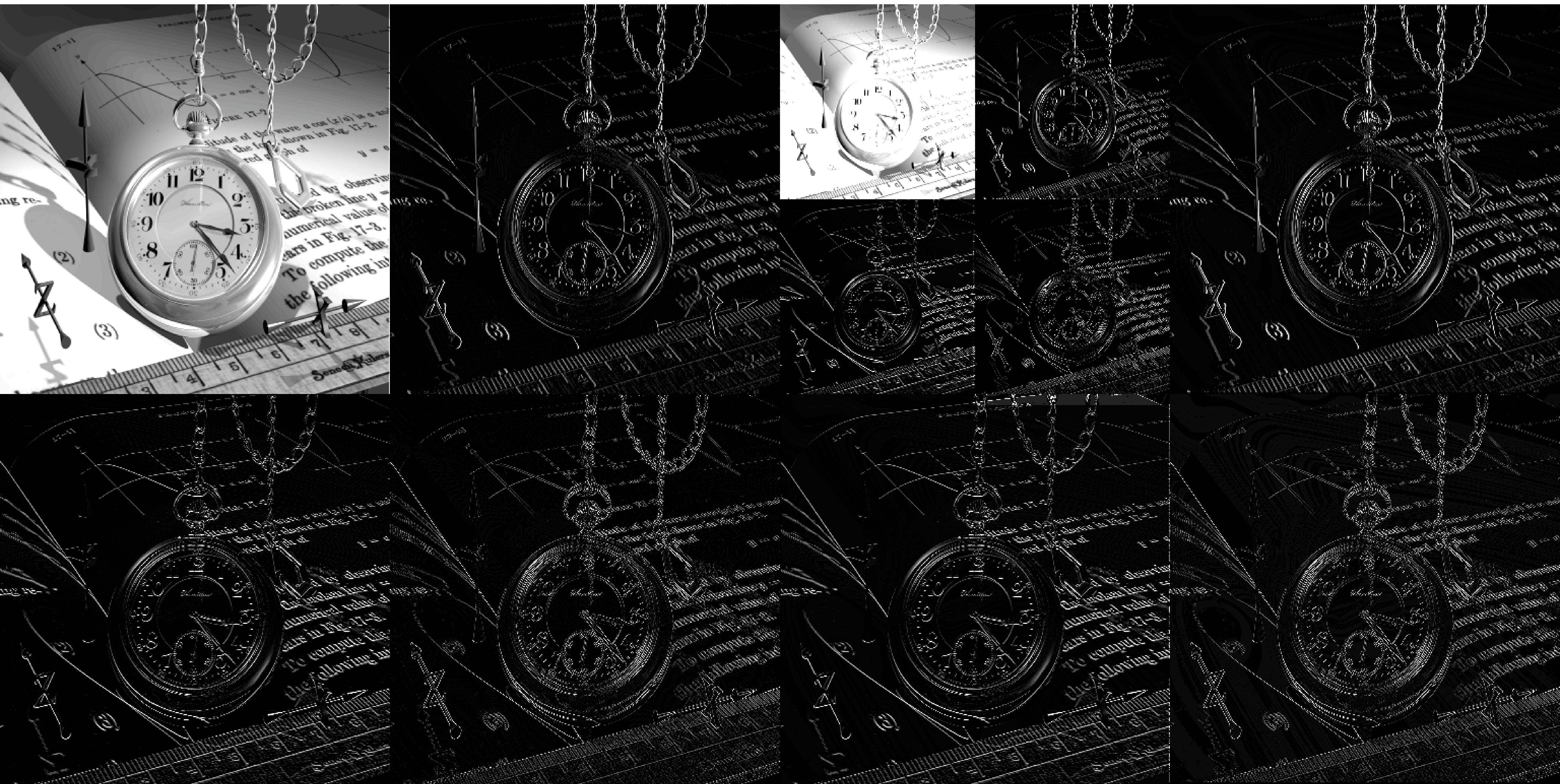
2D MRA

Algorithm 1 Wavelet Transform: Wavelet Pyramid Method: Driving Transform

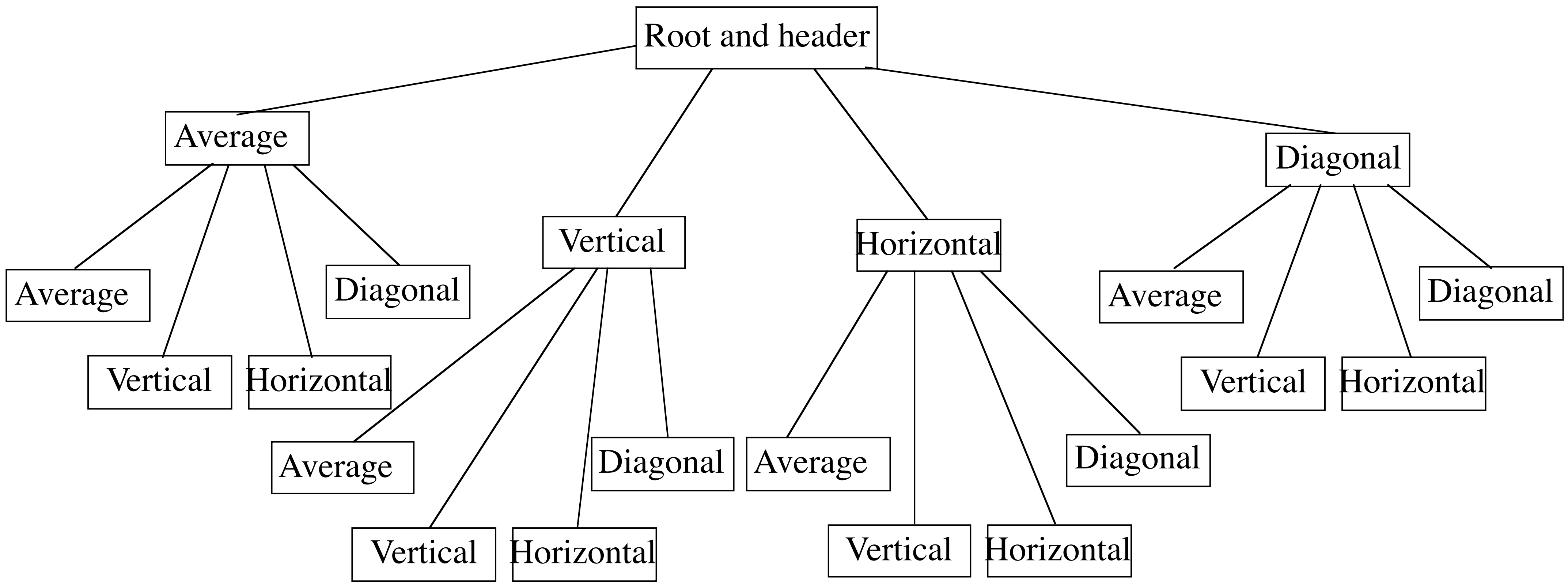
Require: Wavelet Pair hA and hD of length w_l
Require: Matrix A of size $M \times N$
Require: Number of resolutions, r

Initialize matrix α to $M \times N$ and set equal to A
Initialize matrix β to $M \times N$ and set equal to zero.
for $k = 0$ to r **do**
  $M' = \frac{M}{2^k}$
  $N' = \frac{N}{2^k}$
  call row transform for matrix α, with dimension limits M' , N' store in β
  call column transform for matrix β, with dimension limits M' , N' store in α
end for
Return α as result

2D MRA



2-D Multi-Resoltion Expansion



2-D MR Expansion

One Resolution

Two Resolutions

$\psi^{\mathbf{n}}$ Expansion

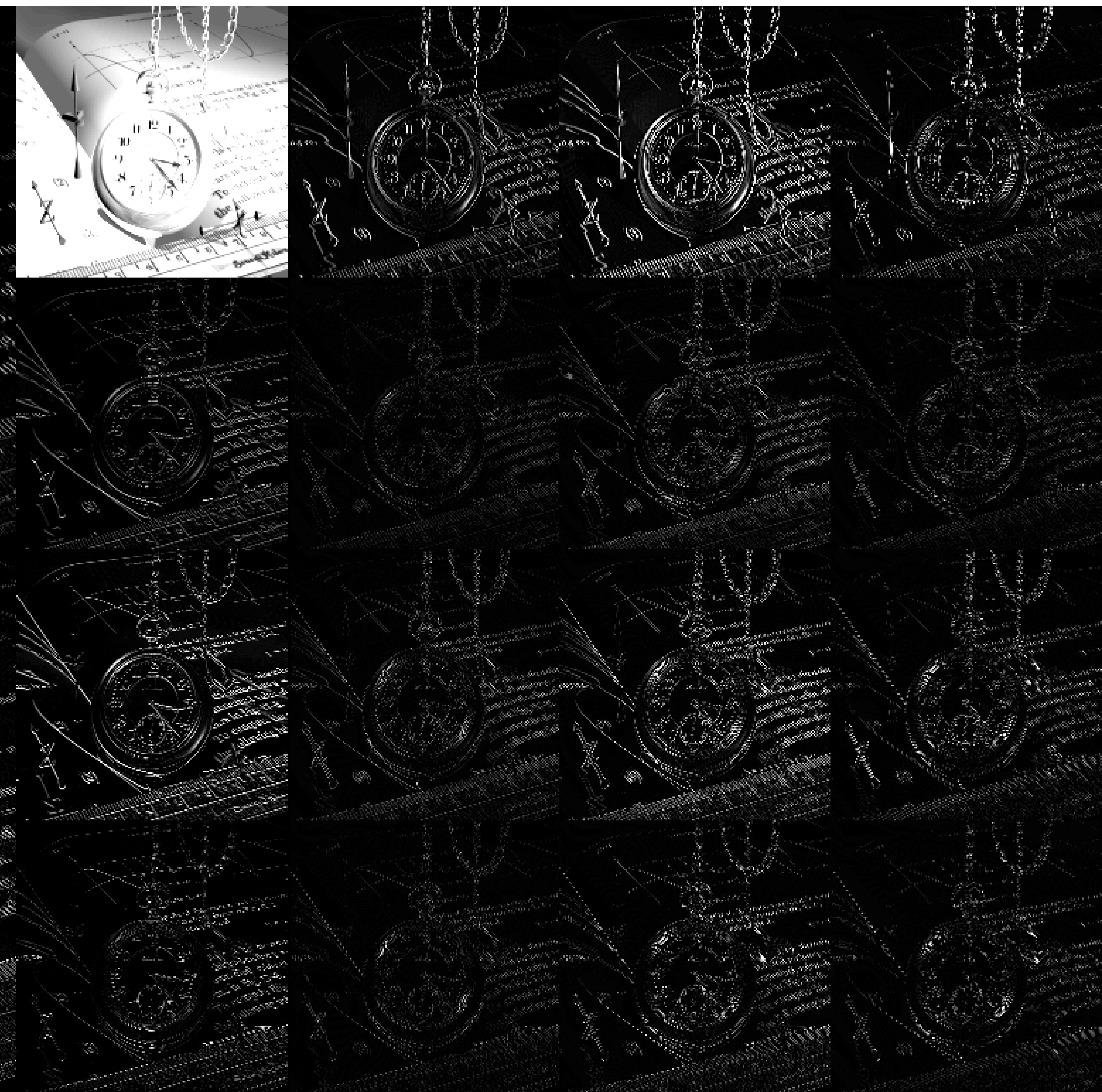
Algorithm 1 Wavelet Transform: MRE with queue controlled visits of the Quad Tree

Require: Wavelet Transform, MRE, and wavelet pair (ψ and ϕ)
Require: Matrix S
  Load S into temporary matrix T
  for $i = 1$ to n **do**
    $T \xrightarrow{\psi_c} X$
    $X \xrightarrow{\psi_r} T$
  end for
  return T as the transformed matrix

Psi N Expansion

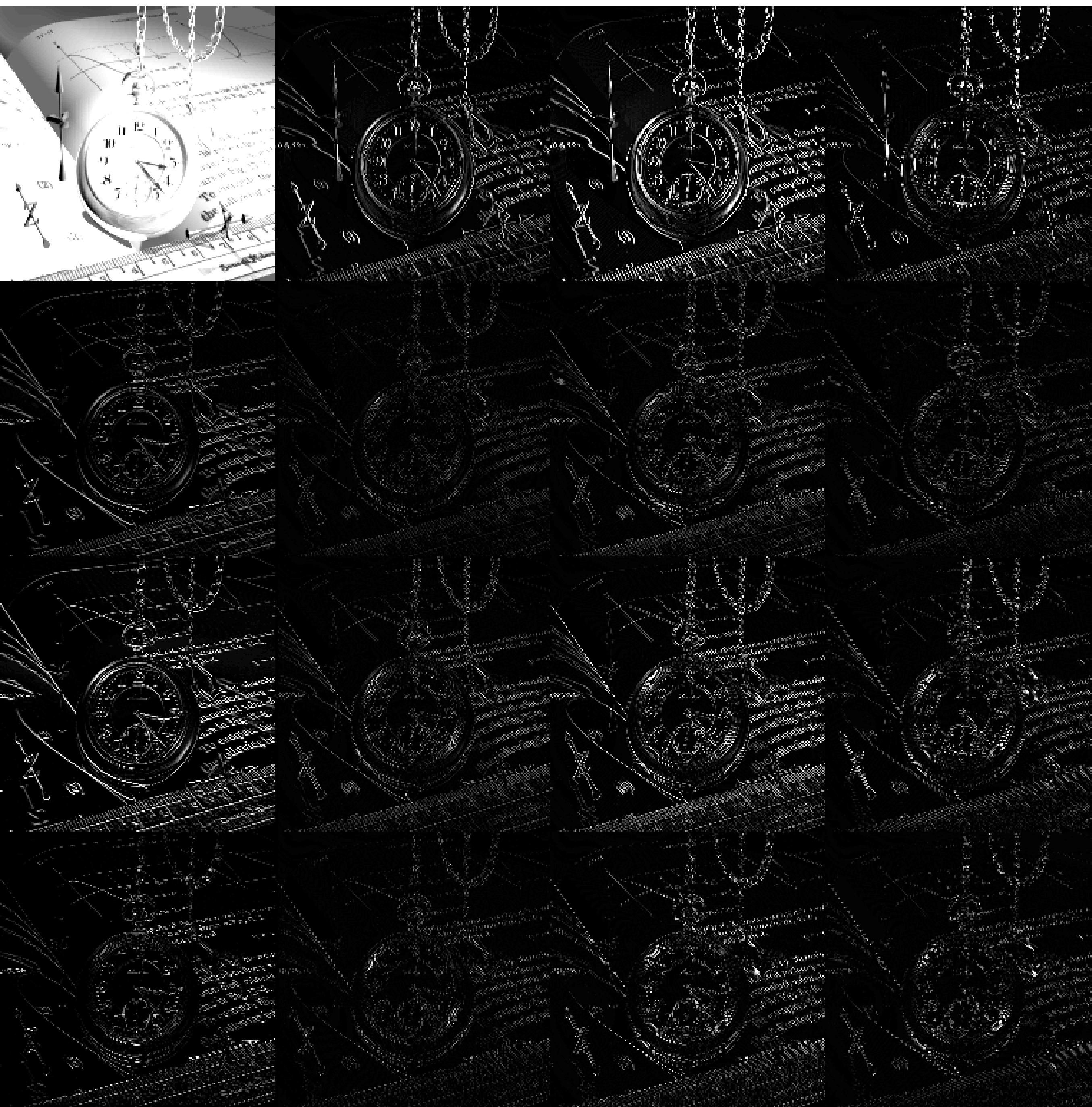
One Resolution

Two Resolutions



Comparison of Expansions

$\psi^{\mathbf{n}}$ **Expansion**



MR Expansion

Matrix Multiplication

- Wavelet Matrix Multiply
- Wavelet Matrix Multiply Sparse

Sparse Multiply

- One of the primary method performing matrix multiply is by using an array of vectors to represent the sparse matrix to multiply against any matrix which is similar to the “Chain-Vector Matrix Multiply”
- The vector contains the element and the position of itself in the matrix.
- There is a penalty if both matrices are in a Array of Vectors system for seaching. If just one, then the search for the matching element
- The speed up for this method is proportional how sparse the matrix is.

Algorithms for Matrix Multiplication

Algorithm 1 Haar Wavelet Multiplication

Require: Matrices A and B

$\hat{A} \leftarrow \psi^n(A)$

$\hat{B} \leftarrow \psi^n(B)$

$\hat{C} \leftarrow \hat{A} \cdot \hat{B}$

$C \leftarrow \psi^{-n}\left(\hat{C}\right)$

Algorithm 2 Sparse Haar Wavelet Multiplication

Require: Matrices A and B

$\hat{A} \leftarrow \psi^n(A)$

$\hat{B} \leftarrow \psi^n(B)$

$l_A \leftarrow \text{sparsify}\left(\hat{A}\right)$

$l_B \leftarrow \text{sparsify}\left(\hat{B}\right)$

$l_C \leftarrow \text{sparse multiply}\left(l_A, l_B\right)$

$\hat{C} \leftarrow \text{densify}\left(\hat{C}\right)$

$C \leftarrow \psi^{-n}\left(\hat{C}\right)$

Let there be sparsity

$$A_{i,j} = \begin{cases} 0 & |A_{i,j}| \leq \epsilon \\ A_{i,j} & otherwise \end{cases}$$

Results

- MRA Results
- MRE Results
- ψ^n Expansion Results
  - waterfall Sascha Ledinsky *h*ttp://oz.irtc.org/ftp/pub/stills/1998-10-31/waterfa1.txt Internet Raytracing Competition copyright October 31, 1998
  - Jaime Vives Piqueres *Always running, never the same...* http://oz.irtc.org/ftp/
  - Kevin Odhner *Pocket Watch on a Gold Chain*
  - fabien a. p. petitcolas Opera House of Lyon open domain
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  - Signal and Image Processing Institute at the University of Southern California *F-16* The copyright status of this images in unclear.
  - Signal and Image Processing Institute at the University of Southern California *Peppers* The copyright status of this images in unclear.
  - Signal and Image Processing Institute at the University of Southern California *Fishing Boat* The copyright status of this images in unclear.

MRA Analysis

Resolution	Original Energy	Estimate Energy	Fidelity
1	6630.76	6630.76	$9.37836 \cdot 10^{-14}$
2	6630.76	6629.92	1.20049
3	6630.76	6626.6	2.37529
4	6630.76	6619.14	3.9194
5	6630.76	6595.14	6.63615
6	6630.76	6579.62	9.48774
7	6630.76	6567.17	12.1211

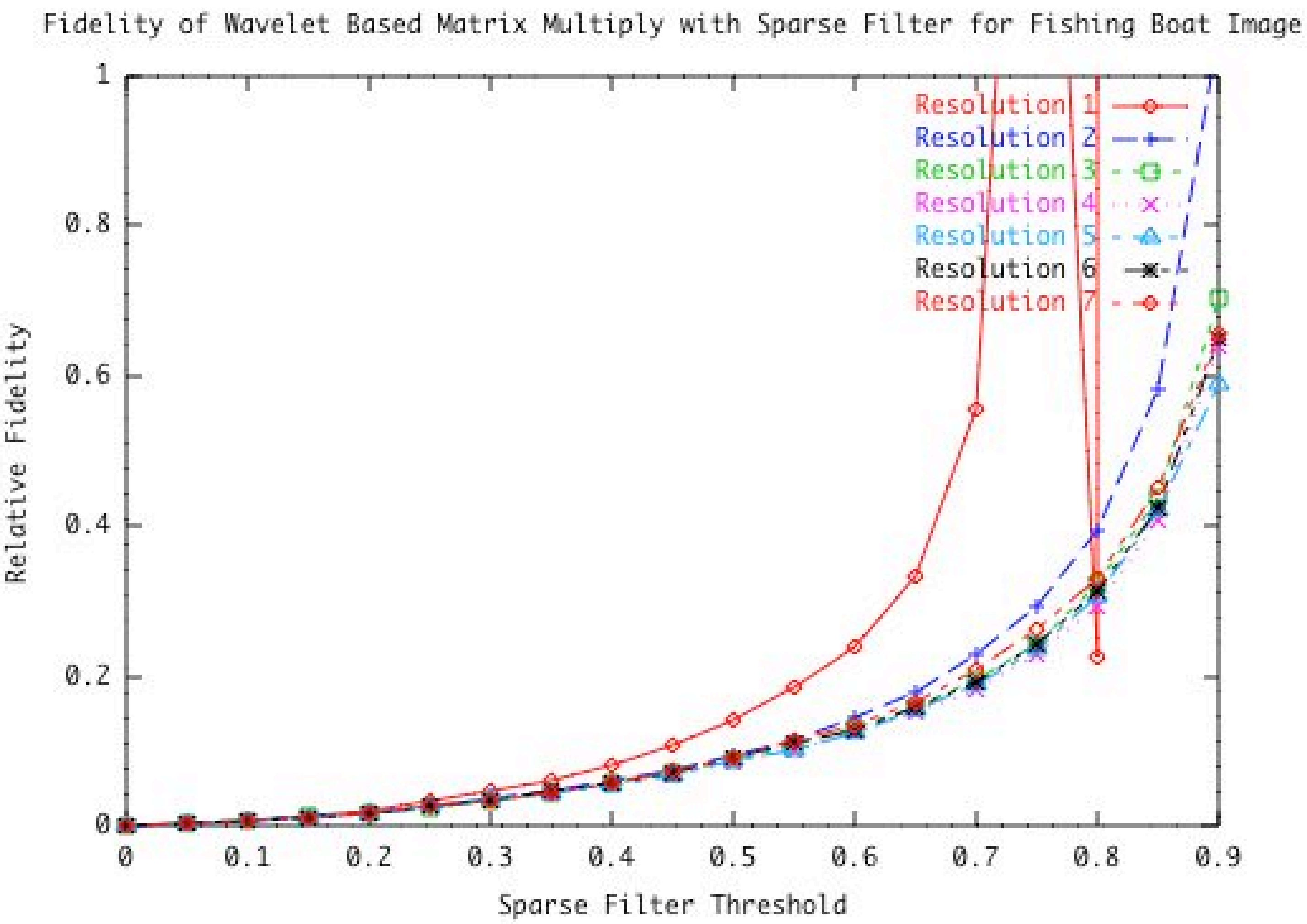
Table 1: Waterfall 512×512 by MRA.

MRE Results

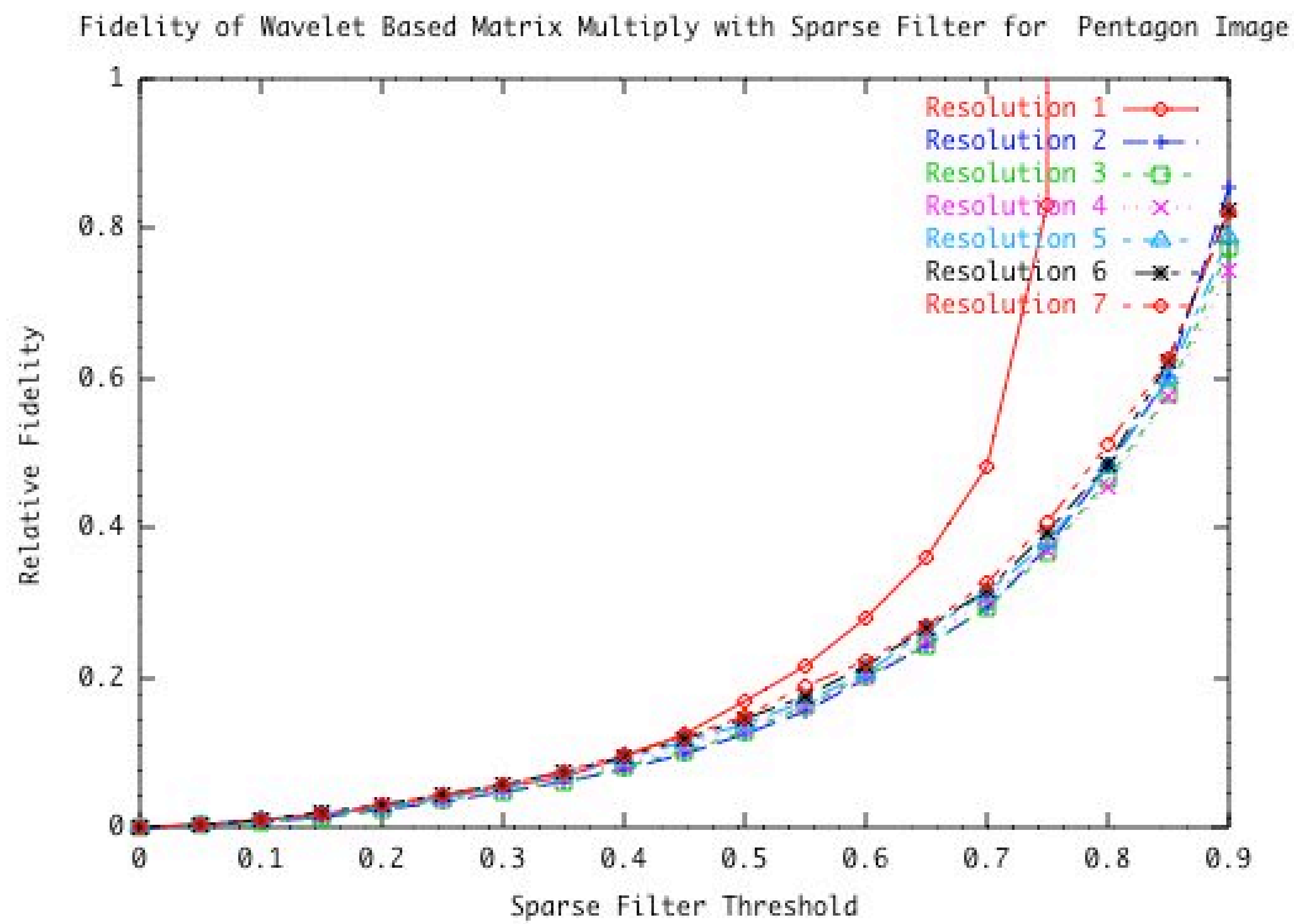
Resolution	Original Energy	Estimate Energy	Fidelity
1	12972.4	12972.4	$1.26464 \cdot 10^{-13}$
2	12972.4	12972.4	0.059622
3	12972.4	12972.4	0.072957
4	12972.4	12972.4	0.0849579
5	12972.4	12972.4	0.09721
6	12972.4	12972.4	0.104528
7	12972.4	12972.4	0.119789

Table 1: Peppers 512×512 by MRE.

Fishing Boat and its Results

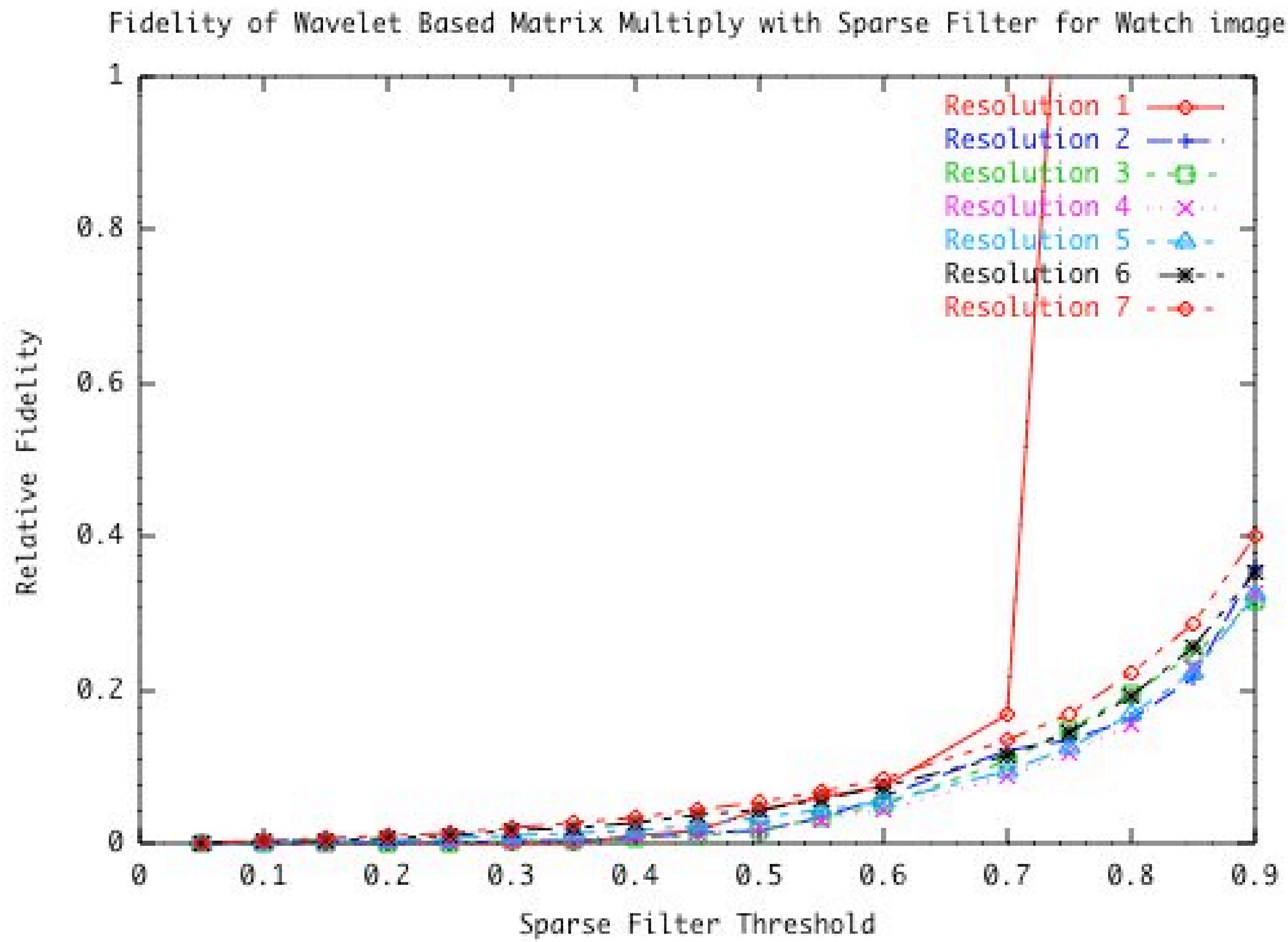


Pentagon



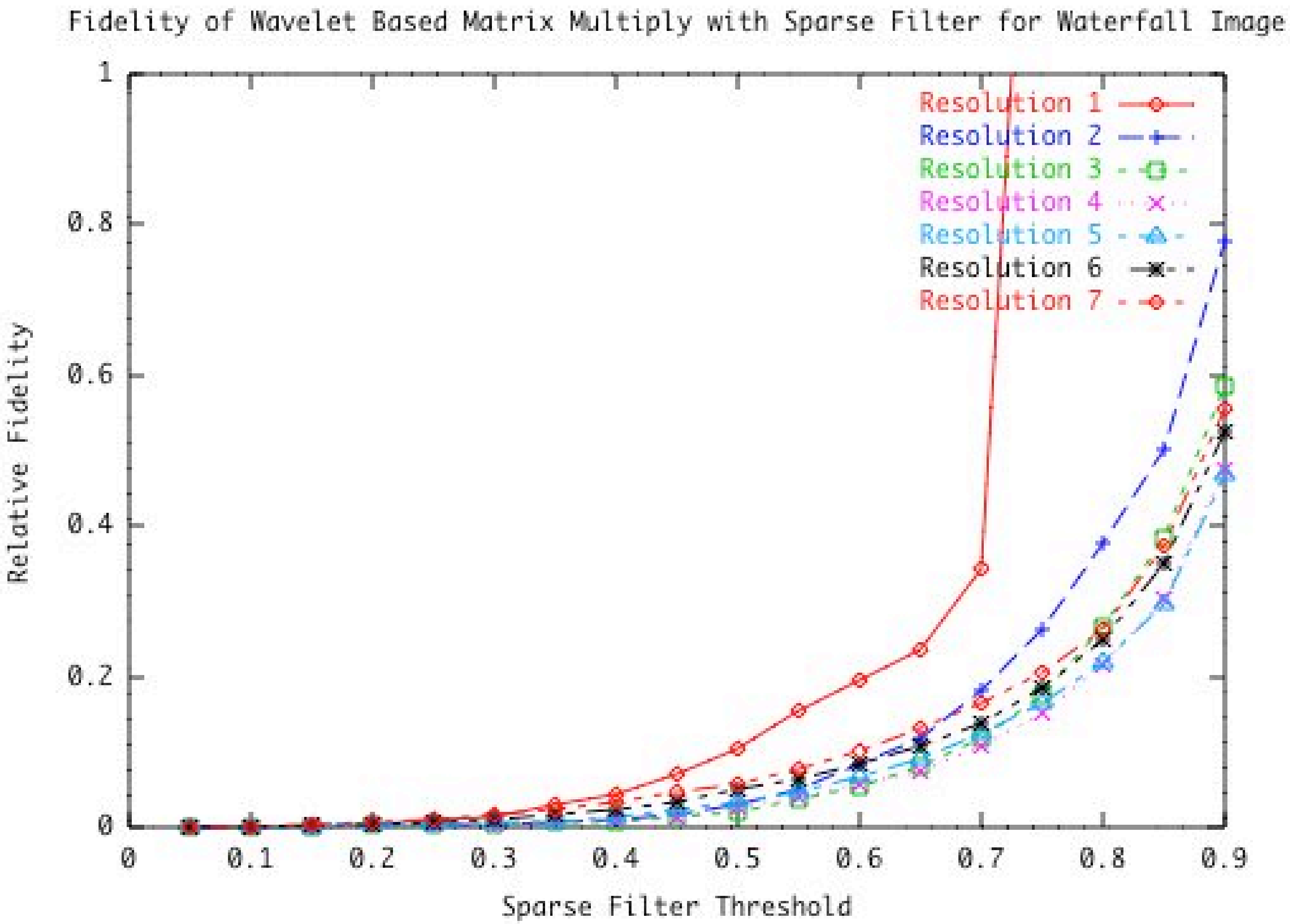
Fidelity of Wavelet Based Matrix Multiply with Sparse Filter for Pentagon Image
Resolution 1
Resolution 2
Resolution 3
Resolution 4
Resolution 5
Resolution 6
Resolution 7
Relative Fidelity
Sparse Filter Threshold
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7
0.9

Watch



Fidelity of Wavelet Based Matrix Multiply with Sparse Filter for Watch image
Relative Fidelity
Sparse Filter Threshold
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Resolution 1
Resolution 2
Resolution 3
Resolution 4
Resolution 5
Resolution 6
Resolution 7

Water Fall



Fidelity of Wavelet Based Matrix Multiply with Sparse Filter for Waterfall Image
Resolution 1
Resolution 2
Resolution 3
Resolution 4
Resolution 5
Resolution 6
Resolution 7
Relative Fidelity
Sparse Filter Threshold
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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[2] Fishing boat. Signal and Image Processing Institute at the University of Southern California. The copyright status of this image is unclear.

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Questions and Answers