A 2×2 example of wavelet multiplication

Daniel Beatty

February 17, 2004

Given Function

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Desired result

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}^2 = \begin{matrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{matrix}$$

Hypothesis:

$$\psi^{-1}((\psi A)^2) = A^2$$

$$\psi_c\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{matrix} \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{1}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{1}{2} - \frac{1}{2}\right) \\ \frac{1}{\sqrt{2}}(0 + \frac{1}{2}) & \frac{1}{\sqrt{2}}(0 - \frac{1}{2}) \end{matrix} = \begin{matrix} \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \end{matrix}$$

$$\psi_r\begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} \end{pmatrix} = \begin{matrix} \frac{1}{\sqrt{2}}(\frac{1}{2}\sqrt{2} + (\frac{1}{4}\sqrt{2})) & \frac{1}{\sqrt{2}}(0 + (-\frac{1}{4}\sqrt{2})) \\ \frac{1}{\sqrt{2}}(\frac{1}{2}\sqrt{2} - (\frac{1}{4}\sqrt{2})) & \frac{1}{\sqrt{2}}(0 - (-\frac{1}{4}\sqrt{2})) \end{matrix} = \begin{matrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$$

$$\psi_r\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{2}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 \end{pmatrix} = \begin{matrix} \frac{1}{4}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & 0 \end{matrix}$$

$$\psi_c\begin{pmatrix} \frac{1}{4}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\frac{1}{4}\sqrt{2} + \frac{1}{2}\sqrt{2}) & \frac{1}{\sqrt{2}}(\frac{1}{4}\sqrt{2} - \frac{1}{2}\sqrt{2}) \\ \frac{1}{\sqrt{2}}(\frac{1}{4}\sqrt{2} - 0) & \frac{1}{\sqrt{2}}(\frac{1}{4}\sqrt{2} - 0) \end{pmatrix} = \begin{matrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$$

$$\psi\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{matrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$$

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}^2 = \begin{matrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{matrix}$$

$$\psi_c^{-1}\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} = \begin{matrix} (\frac{1}{\sqrt{2}})((\frac{1}{2}) - (-\frac{1}{4})) & (\frac{1}{\sqrt{2}})((\frac{1}{2}) + (-\frac{1}{4})) \\ (\frac{1}{\sqrt{2}})(\frac{1}{4} - (0)) & (\frac{1}{\sqrt{2}})(\frac{1}{4} - (0)) \end{matrix} = \begin{matrix} \frac{3}{8}\sqrt{2} & \frac{1}{8}\sqrt{2} \\ \frac{1}{8}\sqrt{2} & \frac{1}{8}\sqrt{2} \end{matrix}$$

$$\psi_r^{-1}\left(\begin{matrix} \frac{3}{8}\sqrt{2} & \frac{1}{8}\sqrt{2} \\ \frac{1}{8}\sqrt{2} & \frac{1}{8}\sqrt{2} \end{matrix}\right) = \begin{matrix} (\frac{1}{\sqrt{2}})(\frac{3}{8}\sqrt{2} - (\frac{1}{8}\sqrt{2})) & (\frac{1}{\sqrt{2}})(\frac{1}{8}\sqrt{2} - (\frac{1}{8}\sqrt{2})) \\ (\frac{1}{\sqrt{2}})(\frac{3}{8}\sqrt{2} + (\frac{1}{8}\sqrt{2})) & (\frac{1}{\sqrt{2}})(\frac{1}{8}\sqrt{2} + (\frac{1}{8}\sqrt{2})) \end{matrix} = \begin{matrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{matrix}$$

$$\psi^{-1}\left(\begin{matrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{matrix}\right) = \begin{matrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{matrix}$$

Symbolic Form

Hypothesis:

$$\psi^{-1}((\psi A)^2) = A^2$$

$$\left(\begin{matrix} a & b \\ c & d \end{matrix}\right)^2 = \begin{matrix} a^2 + bc & ab + bd \\ ca + dc & bc + d^2 \end{matrix}$$

$$\psi_r\left(\begin{matrix} a & b \\ c & d \end{matrix}\right) = \begin{matrix} \frac{1}{\sqrt{2}}(a+b) & \frac{1}{\sqrt{2}}(a-b) \\ \frac{1}{\sqrt{2}}(c-d) & \frac{1}{\sqrt{2}}(c+d) \end{matrix} = \begin{matrix} \frac{1}{2}\sqrt{2}\,(a+b) & \frac{1}{2}\sqrt{2}\,(a-b) \\ \frac{1}{2}\sqrt{2}\,(c-d) & \frac{1}{2}\sqrt{2}\,(c+d) \end{matrix}$$

$$\psi_c\left(\begin{matrix} \frac{1}{2}\sqrt{2}\,(a+b) & \frac{1}{2}\sqrt{2}\,(a-b) \\ \frac{1}{2}\sqrt{2}\,(c-d) & \frac{1}{2}\sqrt{2}\,(c+d) \end{matrix}\right) = \begin{matrix} (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a+b) - \frac{1}{2}\sqrt{2}\,(c-d)) & (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a-b) - (\frac{1}{2}\sqrt{2}\,(c+d))) \\ (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a+b) + \frac{1}{2}\sqrt{2}\,(c-d)) & (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a-b) + (\frac{1}{2}\sqrt{2}\,(c+d))) \end{matrix}$$

$$\left(\begin{matrix} (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a+b) - \frac{1}{2}\sqrt{2}\,(c-d)) & (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a-b) - (\frac{1}{2}\sqrt{2}\,(c+d))) \\ (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a+b) + \frac{1}{2}\sqrt{2}\,(c-d)) & (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2}\,(a-b) + (\frac{1}{2}\sqrt{2}\,(c+d))) \end{matrix}\right)^2 = \begin{matrix} \frac{1}{2}\left(\frac{1}{2}\sqrt{2}\,(a+b) - \frac{1}{2}\right. \\ \frac{1}{2}\left(\frac{1}{2}\sqrt{2}\,(a+b) + \frac{1}{2}\sqrt{2}\,(c-d)\right)\left(\frac{1}{2}\right. \end{matrix}$$

$$= \begin{matrix} \frac{1}{2}a^2 + \frac{1}{2}ab - \frac{1}{2}ca - bc + \frac{1}{2}bd - \frac{1}{2}dc + \frac{1}{2}d^2 & \frac{1}{2}a^2 - \frac{1}{2}ca - \frac{1}{2}bd - \frac{1}{2}d^2 - \frac{1}{2}ab - \frac{1}{2}dc \\ \frac{1}{2}a^2 + \frac{1}{2}ab + \frac{1}{2}dc - \frac{1}{2}d^2 + \frac{1}{2}ca + \frac{1}{2}bd & \frac{1}{2}a^2 - bc + \frac{1}{2}d^2 - \frac{1}{2}ab + \frac{1}{2}ca - \frac{1}{2}bd + \frac{1}{2}dc \end{matrix}$$

$$\psi_c^{-1}\left(\begin{matrix} \frac{1}{2}a^2 + \frac{1}{2}ab - \frac{1}{2}ca - bc + \frac{1}{2}bd - \frac{1}{2}dc + \frac{1}{2}d^2 & \frac{1}{2}a^2 - \frac{1}{2}ca - \frac{1}{2}bd - \frac{1}{2}d^2 - \frac{1}{2}ab - \frac{1}{2}dc \\ \frac{1}{2}a^2 + \frac{1}{2}ab + \frac{1}{2}dc - \frac{1}{2}d^2 + \frac{1}{2}ca + \frac{1}{2}bd & \frac{1}{2}a^2 - bc + \frac{1}{2}d^2 - \frac{1}{2}ab + \frac{1}{2}ca - \frac{1}{2}bd + \frac{1}{2}dc \end{matrix}\right) =$$

$$\begin{matrix} (\frac{1}{\sqrt{2}})((\frac{1}{2}a^2 + \frac{1}{2}ab - \frac{1}{2}ca - bc + \frac{1}{2}bd - \frac{1}{2}dc + \frac{1}{2}d^2) - (\frac{1}{2}a^2 + \frac{1}{2}ab + \frac{1}{2}dc - \frac{1}{2}d^2 + \frac{1}{2}ca + \frac{1}{2}bd)) & (\sqrt{\frac{1}{2}})((\frac{1}{2}a^2 - \frac{1}{2}ca - \frac{1}{2} \\ (\frac{1}{\sqrt{2}})((\frac{1}{2}a^2 + \frac{1}{2}ab - \frac{1}{2}ca - bc + \frac{1}{2}bd - \frac{1}{2}dc + \frac{1}{2}d^2) + (\frac{1}{2}a^2 + \frac{1}{2}ab + \frac{1}{2}dc - \frac{1}{2}d^2 + \frac{1}{2}ca + \frac{1}{2}bd)) & (\sqrt{\frac{1}{2}})((\frac{1}{2}a^2 - \frac{1}{2}ca - \frac{1}{2} \end{matrix}$$

$$= \begin{matrix} -\frac{1}{2}\sqrt{2}\left(ca + bc + dc - d^2\right) & -\frac{1}{2}\sqrt{2}\left(ca + d^2 + dc - bc\right) \\ \frac{1}{2}\sqrt{2}\left(a^2 + ab - bc + bd\right) & \frac{1}{2}\sqrt{2}\left(a^2 - bd - ab - bc\right) \end{matrix}$$

$$\psi_r^{-1}\left(\begin{matrix} -\frac{1}{2}\sqrt{2}\left(ca + bc + dc - d^2\right) & -\frac{1}{2}\sqrt{2}\left(ca + d^2 + dc - bc\right) \\ \frac{1}{2}\sqrt{2}\left(a^2 + ab - bc + bd\right) & \frac{1}{2}\sqrt{2}\left(a^2 - bd - ab - bc\right) \end{matrix}\right) =$$

$$\begin{matrix} (\sqrt{\frac{1}{2}})(-\frac{1}{2}\sqrt{2}\left(ca + bc + dc - d^2\right) - (-\frac{1}{2}\sqrt{2}\left(ca + d^2 + dc - bc\right))) & (\sqrt{\frac{1}{2}})(-\frac{1}{2}\sqrt{2}\left(ca + bc + dc - d^2\right) + (-\frac{1}{2}\sqrt{2}\left(ca + \right. \\ (\sqrt{\frac{1}{2}})(\frac{1}{2}\sqrt{2}\left(a^2 + ab - bc + bd\right) - (\frac{1}{2}\sqrt{2}\left(a^2 - bd - ab - bc\right))) & (\sqrt{\frac{1}{2}})(\frac{1}{2}\sqrt{2}\left(a^2 + ab - bc + bd\right) - (\frac{1}{2}\sqrt{2}\left(a^2 - b\right. \end{matrix}$$

$$= \begin{matrix} -bc + d^2 & -c\,(a+d) \\ b\,(a+d) & b\,(a+d) \end{matrix}$$

$$\begin{matrix} -bc+d^2 & -c\,(a+d) \\ b\,(a+d) & b\,(a+d) \end{matrix} \neq \begin{matrix} a^2+bc & ab+bd \\ ca+dc & bc+d^2 \end{matrix}$$

1 Empirical Results

Each empirical result is from either a series of multiplications preconditioned either with wavelet packets or wavelet pyramids type wavelet transforms.

Sample One: Sandra 512×512 by wavelet transform packet

Number of Resolution	Original Energy $E(O)$	Estimate Energy $E(\widetilde{O})$	$(E(O) - E(\widetilde{O}))$	Fidelity $\sqrt{E((O) - (\widetilde{O}))}$
1	40043.8	40043.8	$-9.08501 \cdot 10^{-16}$	$2.16231 \cdot 10^{-13}$
2	40043.8	40043.8	$3.7666 \cdot 10-9$	0.0231922
3	40043.8	40043.8	$2.12923 \cdot 10^{-8}$	0.0331245
4	40043.8	40043.7	$1.15056 \cdot 10^{-7}$	0.046215
5	40043.8	40043.8	$-4.45904 \cdot 10^{-8}$	0.0563862
6	40043.8	40043.8	$4.85688 \cdot 10^{-8}$	0.0629457
7	40043.8	40043.8	$7.51241 \cdot 10^{-8}$	0.0666921

Sample Two: Sandra 512×512 by wavelet transform pyramid

Number of Resolution	Original Energy $E(O)$	Estimate Energy $E(\widetilde{O})$	$(E(O) - E(\widetilde{O}))$	Fidelity $\sqrt{E((O) - (\widetilde{O}))}$
1	40043.8	40043.8	$-9.08501 \cdot 10^{-16}$	$2.16231 \cdot 10^{-13}$
2	40043.8	40039.8	$9.93622\ cdot10^{-5}$	2.27862
3	40043.8	40039.2	0.000113292	4.04497
4	40043.8	40045.9	$-5.33258e-05$	6.7722
5	40043.8	40051.3	-0.000187741	11.1503

Sample Three: Molly Holly 256×256 by wavelet transform pyramid

Number of Resolution	Original Energy $E(O)$	Estimate Energy $E(\widetilde{O})$	$(E(O) - E(\widetilde{O}))$	Fidelity $\sqrt{E((O) - (\widetilde{O}))}$
1	582.864	582.864	$-9.75244 \cdot 10^{-16}$	$2.0396 \cdot 10^{-14}$
2	582.864	583.05	-0.000319839	1.19806
3	582.864	582.613	0.000430391	2.18712
4	582.864	579.473	0.00581664	3.19317
5	582.864	576.72	0.0105413	4.82652

Sample Four: Waterfall 512×512 by wavelet transform pyramid

Number of Resolution	Original Energy $E(O)$	Estimate Energy $E(\widetilde{O})$	$(E(O) - E(\widetilde{O}))$	Fidelity $\sqrt{E((O) - (\widetilde{O}))}$
1	6630.76	6630.76	$-9.60141e - 16$	$9.37836e - 14$
2	6630.76	6629.92	0.000127083	1.20049
3	6630.76	6626.6	0.000627646	2.37529
4	6630.76	6619.14	0.00175173	3.9194
5	6630.76	6595.14	0.00537249	6.63615
6	6630.76	6579.62	0.00771315	9.48774
7	6630.76	6567.17	0.00959077	12.1211

2 Issue of which MRE works and why

Theorem: $\psi(A) \cdot \psi(B) = \psi(A \cdot B)$

Fact : $\psi_{WPx}(A) \neq \psi^x(A)$ such that $\psi_{WPx}(A)$ is the x resolution of wavelet transform packets except for $x = 1$.

Fact: $\psi_{Wx}(A) \neq \psi^x(A)$ such that $\psi_{Wx}(A)$ is the x resolution of wavelet transform pyramids except for $x = 1$.

Proposed Hypothesis:

1. $\psi(\psi(A)) \cdot \psi(\psi(B)) = \psi(\psi(A \cdot B))$
2. $M : \psi(\psi(A)) \rightarrow \psi_{WP2}(A)$ such that ψ_{WP2} is a wavelet transform packet of 2 resolutions for all A capable of MRE or 2 resolutions.
3. $M_W : \psi_{WP2}(A) \rightarrow \psi(\psi(A))$ such that ψ_{WP2} is a wavelet transform packet of 2 resolutions for all A capable of MRE or 2 resolutions.