Brief Article

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1 Wavelet Matrix Multiplication

Wavelet based matrix multiplication is sound only as long as the wavelet operator is an linear orthonormal operator operator and produces MRE sections which cancel in the inverse wavelet transform. In this chapter, the matrix multiplication with the Haar Transform is first shown. Empirical examples of MRE with wavelet transform pyramids and wavelet packet transform to illustrate a flaw with performing Wavelet Based Matrix Multiplication with those schemes. Next a formal proof that one particular MRE scheme is valid. Last, empirical example of the sound form MRE Wavelet Based Matrix Multiplication.

1.1 Matrix Multiply with the Haar Transform

Given
Vectors: f, g N $\times$ N matrices: A, B

Required

$$\psi(A)\psi(B) = \psi(AB)$$

Proof

There are two formulae required for transforming either A or B into the wavelet domain.

$$\psi(A) = \psi_{1R}(\psi_{1C}(A)) = \psi_{1C}(\psi_{1R}(A))$$

$$\psi(B) = \psi_{1R}(\psi_{1C}(B)) = \psi_{1C}(\psi_{1R}(B))$$

Let $C' = \psi(A)\psi(B)$ then the above formulae define C' as follows:

$$C_{i,j} = \langle \psi_{1C}(\psi_{1R}(A))_{ri}, \psi_{1R}(\psi_{1C}(B))_{ci} \rangle$$

$$C_{i,j} = \langle \psi_{1R}(\psi_{1C}(A))_{ri}, \psi_{1C}(\psi_{1R}(B))_{ci} \rangle$$

$$\left\langle A^C_{ri}, B^R_{cj} \right\rangle = \sum_{k=0}^{row/2-1} \frac{1}{2}((A_{ri})_k+(A_{ri})_{k+1})(B_{cj})_k+(B_{cj})_{k+1})+\sum_{k=0}^{row/2-1} \frac{1}{2}((A_{ri})_k-(A_{ri})_{k+1})(B_{cj})_k-(B_{cj})_{k+1})$$

so that

$$\psi_{1C}(A) = \begin{cases} \frac{A_{i,j}+A_{i,j+1}}{\sqrt{2}} & j < col \\ \frac{A_{i,j}-A_{i,j+1}}{\sqrt{2}} & j \geq col \end{cases}$$

$$\psi_{1R}(A) = \begin{cases} \frac{A_{i,j}+A_{i+1,j}}{\sqrt{2}} & i < row \\ \frac{A_{i,j}-A_{i+1,j}}{\sqrt{2}} & i \geq row \end{cases}$$

In this case, there are four cases to show for:

1. $i < row$ and $j < col$
2. $i \geq row$ and $j < col$
3. $i < row$ and $j \geq col$
4. $i \geq row$ and $j \geq col$

Case 1

$$C'_{ij} = \left\langle \frac{A^R_{ri,j} + A^R_{ri+1}}{\sqrt{2}}, \frac{B^C_{cj} + B^C_{cj+1}}{\sqrt{2}} \right\rangle$$

$$C'_{i,j} = \frac{1}{2}(\left\langle A^R_{ri}, B^C_{cj} \right\rangle + \left\langle A^R_{ri+1}, B^C_{cj} \right\rangle + \left\langle A^R_{ri}, B^C_{cj+1} \right\rangle + \left\langle A^R_{ri+1}, B^C_{cj+1} \right\rangle)$$

$$\psi(AB) = \psi(C)$$

$$(C) = \psi \left\langle A_{ri}, B_{cj} \right\rangle$$

$$\psi_{1C}(C)_{i,j} = \frac{1}{\sqrt{2}} \langle A_{ri}, B_{cj} \rangle + \langle A_{ri+1}, B_{cj} \rangle$$
$$\psi_{1C}(C)_{i,j+1} = \frac{1}{\sqrt{2}} \langle A_{ri}, B_{cj+1} \rangle + \langle A_{ri+1}, B_{cj+1} \rangle$$

$$\psi(C) = \frac{1}{\sqrt{2}}(\psi_{1C}(C)_{i,j} + \psi_{1C}(C)_{i,j+1})$$

$$\psi(C) = \frac{1}{2}(\langle A_{ri}, B_{cj} \rangle + \langle A_{ri+1}, B_{cj} \rangle + \langle A_{ri}, B_{cj+1} \rangle + \langle A_{ri+1}, B_{cj+1} \rangle)$$

Case 2

$$C'_{ij} = \left\langle \frac{A^R_{ri,j} - A^R_{ri+1}}{\sqrt{2}}, \frac{B^C_{cj} + B^C_{cj+1}}{\sqrt{2}} \right\rangle$$

$$C'_{i,j} = \frac{1}{2}(\langle A^R_{ri}, B^C_{cj} \rangle - \langle A^R_{ri+1}, B^C_{cj} \rangle + \langle A^R_{ri}, B^C_{cj+1} \rangle - \langle A^R_{ri+1}, B^C_{cj+1} \rangle)$$

$$\psi(AB) = \psi(C)$$
$$(C) = \psi \langle A_{ri}, B_{cj} \rangle$$
$$\psi_{1C}(C)_{i,j} = \frac{1}{\sqrt{2}} \langle A_{ri}, B_{cj} \rangle - \langle A_{ri+1}, B_{cj} \rangle$$
$$\psi_{1C}(C)_{i,j+1} = \frac{1}{\sqrt{2}} \langle A_{ri}, B_{cj+1} \rangle - \langle A_{ri+1}, B_{cj+1} \rangle$$

$$\psi(C) = \frac{1}{\sqrt{2}}(\psi_{1C}(C)_{i,j} + \psi_{1C}(C)_{i,j+1})$$

$$\psi(C) = \frac{1}{2}(\langle A_{ri}, B_{cj} \rangle - \langle A_{ri+1}, B_{cj} \rangle + \langle A_{ri}, B_{cj+1} \rangle - \langle A_{ri+1}, B_{cj+1} \rangle)$$

Case 3

$$C'_{ij} = \left\langle \frac{A^R_{ri,j} + A^R_{ri+1}}{\sqrt{2}}, \frac{B^C_{cj} - B^C_{cj+1}}{\sqrt{2}} \right\rangle$$

$$C'_{i,j} = \frac{1}{2}(\langle A^R_{ri}, B^C_{cj} \rangle + \langle A^R_{ri+1}, B^C_{cj} \rangle - \langle A^R_{ri}, B^C_{cj+1} \rangle - \langle A^R_{ri+1}, B^C_{cj+1} \rangle)$$

$$\psi(AB) = \psi(C)$$

$$(C) = \psi \left\langle A_{ri}, B_{cj} \right\rangle$$

$$\psi_{1C}(C)_{i,j} = \frac{1}{\sqrt{2}} \left\langle A_{ri}, B_{cj} \right\rangle + \left\langle A_{ri+1}, B_{cj} \right\rangle$$

$$\psi_{1C}(C)_{i,j+1} = \frac{1}{\sqrt{2}} \left\langle A_{ri}, B_{cj+1} \right\rangle + \left\langle A_{ri+1}, B_{cj+1} \right\rangle$$

$$\psi(C) = \frac{1}{\sqrt{2}}(\psi_{1C}(C)_{i,j} - \psi_{1C}(C)_{i,j+1})$$

$$\psi(C) = \frac{1}{2}(\left\langle A_{ri}, B_{cj} \right\rangle + \left\langle A_{ri+1}, B_{cj} \right\rangle - \left\langle A_{ri}, B_{cj+1} \right\rangle - \left\langle A_{ri+1}, B_{cj+1} \right\rangle)$$

Case 4

$$C'_{ij} = \left\langle \frac{A^R_{ri,j} - A^R_{ri+1}}{\sqrt{2}}, \frac{B^C_{cj} - B^C_{cj+1}}{\sqrt{2}} \right\rangle$$

$$C'_{i,j} = \frac{1}{2}(\left\langle A^R_{ri}, B^C_{cj} \right\rangle - \left\langle A^R_{ri+1}, B^C_{cj} \right\rangle - \left\langle A^R_{ri}, B^C_{cj+1} \right\rangle + \left\langle A^R_{ri+1}, B^C_{cj+1} \right\rangle)$$

$$\psi(AB) = \psi(C)$$

$$(C) = \psi \left\langle A_{ri}, B_{cj} \right\rangle$$

$$\psi_{1C}(C)_{i,j} = \frac{1}{\sqrt{2}} \left\langle A_{ri}, B_{cj} \right\rangle - \left\langle A_{ri+1}, B_{cj} \right\rangle$$

$$\psi_{1C}(C)_{i,j+1} = \frac{1}{\sqrt{2}} \left\langle A_{ri}, B_{cj+1} \right\rangle - \left\langle A_{ri+1}, B_{cj+1} \right\rangle$$

$$\psi(C) = \frac{1}{\sqrt{2}}(\psi_{1C}(C)_{i,j} - \psi_{1C}(C)_{i,j+1})$$

$$\psi(C) = \frac{1}{2}(\left\langle A_{ri}, B_{cj} \right\rangle - \left\langle A_{ri+1}, B_{cj} \right\rangle - \left\langle A_{ri}, B_{cj+1} \right\rangle + \left\langle A_{ri+1}, B_{cj+1} \right\rangle)$$

1.2 A 2×2 example

The feasibility of multiplication in the wavelet domain is demonstrated directly using a 2×2 matrix. The coefficients of the matrices are multiplied both according to normal matrix multiplication and the modified wavelet multiplication operator. In the end the resulting coefficients are seen to be the same.

1.2.1 Conventional Multiplication

Conventional multiplication is spelled out as

$$c_{i,j} = \sum_k a_{i,k} b_{k,j}.$$

For a 2×2 matrix, this can be presented by

$$\begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} = \begin{pmatrix} a_1^1 b_1^1 + a_1^2 b_2^1 & a_1^1 b_1^2 + a_1^2 b_2^2 \\ a_2^1 b_1^1 + a_2^2 b_2^1 & a_2^1 b_1^2 + a_2^2 b_2^2 \end{pmatrix}$$

1.2.2 Wavelet Transforms of the Matrices

For a wavelet transform, the result on matrix A is

$$W(A) = \frac{1}{2} \begin{pmatrix} (a_1^1 + a_2^1 + a_1^2 + a_2^2) & (a_1^1 + a_2^1 - a_1^2 - a_2^2) \\ (a_1^1 - a_2^1 + a_1^2 - a_2^2) & (a_1^1 - a_2^1 - a_1^2 + a_2^2) \end{pmatrix}, \tag{1}$$

and for matrix B it is

$$W(B) = \frac{1}{2} \begin{pmatrix} (b_1^1 + b_2^1 + b_1^2 + b_2^2) & (b_1^1 + b_2^1 - b_1^2 - b_2^2) \\ (b_1^1 - b_2^1 + b_1^2 - b_2^2) & (b_1^1 - b_2^1 - b_1^2 + b_2^2) \end{pmatrix}. \tag{2}$$

1.2.3 Product of A and B in wavelet space

The conventional product of A and B can be transformed into wavelet space. The product of the two matrices is

$$A \cdot B = \begin{pmatrix} (a_1^1 b_1^1 + a_1^2 b_2^1) & (a_1^1 b_1^2 + a_1^2 b_2^2) \\ (a_2^1 b_1^1 + a_2^2 b_2^1) & (a_2^1 b_1^2 + a_2^2 b_2^2) \end{pmatrix}.$$

The wavelet transform of this matrix is represented by

$$W(A \cdot B) = \frac{1}{2} \begin{pmatrix} \psi(A) & \psi(V) \\ \psi(H) & \psi(D) \end{pmatrix}$$

where

$$\begin{aligned}
\psi(A) &= (a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2) + (a_2^1b_1^1 + a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2)\\
\psi(V) &= (a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2) + (a_2^1b_1^1 + a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2)\\
\psi(H) &= (a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2) - (a_2^1b_1^1 + a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2)\\
\psi(D) &= (a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2) - (a_2^1b_1^1 + a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\psi(A) &= a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2 + a_2^1b_1^1 + a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2\\
\psi(V) &= a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2 + a_2^1b_1^1 + a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2\\
\psi(H) &= a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2 - a_2^1b_1^1 - a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2\\
\psi(D) &= a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2 - a_2^1b_1^1 - a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2
\end{aligned}$$

1.2.4 The product of the waveletized matrices

Straight forward multiplication of $W(A) \cdot W(B)$ represented by equations 1 and 2 works out as follows:

$$W(A) \cdot W(B) = \frac{1}{4}\begin{pmatrix} W_A & W_V \\ W_H & W_D \end{pmatrix}$$

where

$$\begin{aligned}
W_A &= (a_1^1 + a_2^1 + a_1^2 + a_2^2)(b_1^1 + b_2^1 + b_1^2 + b_2^2) + (a_1^1 + a_2^1 - a_1^2 - a_2^2)(b_1^1 - b_2^1 + b_1^2 - b_2^2)\\
W_V &= (a_1^1 + a_2^1 + a_1^2 + a_2^2)(b_1^1 + b_2^1 - b_1^2 - b_2^2) + (a_1^1 + a_2^1 - a_1^2 - a_2^2)(b_1^1 - b_2^1 - b_1^2 + b_2^2)\\
W_H &= (a_1^1 - a_2^1 + a_1^2 - a_2^2)(b_1^1 + b_2^1 + b_1^2 + b_2^2) + (a_1^1 - a_2^1 - a_1^2 + a_2^2)(b_1^1 - b_2^1 + b_1^2 - b_2^2)\\
W_D &= (a_1^1 - a_2^1 + a_1^2 - a_2^2)(b_1^1 + b_2^1 - b_1^2 - b_2^2) + (a_1^1 - a_2^1 - a_1^2 + a_2^2)(b_1^1 - b_2^1 - b_1^2 + b_2^2)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
W_A &= a_1^1b_1^1 + a_2^1b_1^1 + a_1^2b_2^1 + a_2^2b_2^1 + a_1^1b_1^2 + a_2^1b_1^2 + a_1^2b_2^2 + a_2^2b_2^2\\
W_V &= a_1^1b_1^1 + a_2^1b_1^1 + a_1^2b_2^1 + a_2^2b_2^1 - a_1^1b_1^2 - a_2^1b_1^2 - a_1^2b_2^2 - a_2^2b_2^2\\
W_H &= a_1^1b_1^1 - a_2^1b_1^1 + a_1^2b_2^1 - a_2^2b_2^1 + a_1^1b_1^2 - a_2^1b_1^2 + a_1^2b_2^2 - a_2^2b_2^2\\
W_D &= a_1^1b_1^1 - a_2^1b_1^1 + a_1^2b_2^1 - a_2^2b_2^1 - a_1^1b_1^2 + a_2^1b_1^2 - a_1^2b_2^2 + a_2^2b_2^2
\end{aligned}$$

This can then be compared to the coefficients of $W(A \cdot B)$ which were

$$\begin{aligned}
\psi(A) &= a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2 + a_2^1b_1^1 + a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2\\
\psi(V) &= a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2 + a_2^1b_1^1 + a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2\\
\psi(H) &= a_1^1b_1^1 + a_1^2b_2^1 + a_1^1b_1^2 + a_1^2b_2^2 - a_2^1b_1^1 - a_2^2b_2^1 - a_2^1b_1^2 - a_2^2b_2^2\\
\psi(D) &= a_1^1b_1^1 + a_1^2b_2^1 - a_1^1b_1^2 - a_1^2b_2^2 - a_2^1b_1^1 - a_2^2b_2^1 + a_2^1b_1^2 + a_2^2b_2^2
\end{aligned}$$

Notice that $W(A) \cdot W(B) = W(A \cdot B)$, in the case of 2×2 matrices.

2 Multi Resolution Expansion Proof

In a previous section, wavelet based matrix multiplication was proven such that

$$\psi(A) \cdot \psi(B) = \psi(A \cdot B)$$

In this section the question of whether or not there is MRE form which is sound for matrix multiplication. There are few facts that are relevant and were made obvious in the empirical analysis section of this chapter.

- $\psi_{WPx}(A) \neq \psi^x(A)$ such that $\psi_{WPx}(A)$ is the x resolution of wavelet transform packets except for $x = 1$.
- $\psi_{Wx}(A) \neq \psi^x(A)$ such that $\psi_{Wx}(A)$ is the x resolution of wavelet transform pyramids except for $x = 1$.

The next hypothesis to be answered is can the wavelet transform be applied more than once to condition a matrix for matrix multiplication? This question is answered by the following lemma.

$$\psi^2(A) \cdot \psi^2(B) = \psi^2(A \cdot B)$$

Proof:
The theorem

$$\psi(A) \cdot \psi(B) = \psi(A \cdot B)$$

is proven as fact.

$$\psi^2(A) = \psi(\psi(A))$$
$$\psi^2(A) = \cdot\psi^2(B) = \psi(\psi(A)) \cdot \psi(\psi(B))$$
$$\psi(\psi(A)) \cdot \psi(\psi(B)) = \psi(\psi(A) \cdot \psi(B))$$
$$\psi(\psi(A) \cdot \psi(B)) = \psi(\psi(A \cdot (B))$$
$$\psi(\psi(A \cdot (B)) = \psi^2(A \cdot B)$$

Therefore:

$$\psi^2(A) \cdot \psi^2(B) = \psi^2(A \cdot B)$$

Proposed Hypothesis:

1. $\exists M : \psi(\psi(A)) \rightarrow \psi_{WP2}(A)$ such that ψ_{WP2} is a wavelet transform packet of 2 resolutions for all A capable of MRE or 2 resolutions.
2. $\exists M_W : \psi_{WP2}(A) \rightarrow \psi(\psi(A))$ such that ψ_{WP2} is a wavelet transform packet of 2 resolutions for all A capable of MRE or 2 resolutions.

$$\psi(S) = \begin{matrix} A_1 & V_1 \\ H_1 & D_1 \end{matrix}$$

$$\psi(\psi(S)) = \begin{matrix} \phi_c(\phi_r(A_1)) & \phi_c(\phi_r(V_1)) & \phi_c(\psi_r(A_1)) & \phi_c(\psi_r(V_1)) \\ \phi_c(\phi_r(H_1)) & \phi_c(\phi_r(D_1)) & \phi_c(\psi_r(H_1)) & \phi_c(\psi_r(D_1)) \\ \psi_c(\phi_r(A_1)) & \psi_c(\phi_r(V_1)) & \psi_c(\psi_r(A_1)) & \psi_c(\psi_r(V_1 \\ \psi_c(\phi_r(H_1)) & \psi_c(\phi_r(D_1)) & \psi_c(\psi_r(H_1)) & \psi_c(\psi_r(D_1)) \end{matrix}$$

$$\psi(\psi(S)) = \begin{matrix} AA_1 & AV_1 & VA_1 & VV_1 \\ AH_1 & AD_1 & VH_1 & VD_1 \\ HA_1 & HV_1 & DA_1 & DV_1 \\ HH_1 & HD_1 & DH_1 & DD_1 \end{matrix}$$

$$\psi^2_{PA}(S) = \begin{matrix} AA_1 & VA_1 & AV_1 & VV_1 \\ AH_1 & DA_1 & HV_1 & DV_1 \\ HA_1 & VH_1 & AD_1 & VD_1 \\ HH_1 & DH_1 & HD_1 & DD_1 \end{matrix}$$

The mapping here can be confusing due to the language. To describe this mapping, a quad tree structure is used. The identification of the tree elements is by the operations on matrix S, and operations on results of operations from S. This is the same as in the previous matrix representation. The values in elements are positions in the matrix for which a sub-matrix of the matrix represents the elements operational identity.

Given a standard wavelet transform packet tree, the mapping is developed level by level. At the second the level, the position of these sub-matrices in the combined matrix is given relative to the standard and ψ^2's tree.

$(\psi^2_{PA})_A(x) \rightarrow \psi^2(x_1)_A$

$(\psi^2_{PA})_V(x) \rightarrow \psi^2(x_1)_V$

$(\psi^2_{PA})_H(x) \rightarrow \psi^2(x_1)_H$

$(\psi^2_{PA})_D(x) \rightarrow \psi^2(x_1)_D$

In three resolutions:

$$
\begin{array}{cccccccc}
A_3A_2A_1 & A_3A_2V_1 & A_3V_2A_1 & A_3V_2V_1 & V_3A_2A_1 & V_3A_2V_1 & V_3V_2A_1 & V_3V_2V_1
\end{array}
$$

$$
\psi^3(S) = \psi(\psi(\psi(S))) = \begin{array}{cccccccc}
A_3A_2H_1 & A_3A_2D_1 & A_3V_2H_1 & A_3V_2D_1 & V_3A_2H_1 & V_3A_2D_1 & V_3V_2H_1 & V_3V_2D_1 \\
A_3H_2A_1 & A_3H_2V_1 & A_3D_2A_1 & A_3D_2V_1 & D_3H_2A_1 & D_3H_2V_1 & D_3D_2A_1 & D_3D_2V_1 \\
A_3H_2H_1 & A_3H_2D_1 & A_3D_2H_1 & A_3D_2D_1 & V_3H_2H_1 & V_3H_2D_1 & V_3D_2H_1 & V_3D_2D_1 \\
H_3A_2A_1 & H_3A_2V_1 & H_3V_2A_1 & H_3V_2V_1 & D_3A_2A_1 & D_3A_2V_1 & D_3V_2A_1 & D_3V_2V_1 \\
H_3A_2H_1 & H_3A_2D_1 & H_3V_2H_1 & H_3V_2D_1 & D_3A_2H_1 & D_3A_2D_1 & D_3V_2H_1 & D_3V_2D_1 \\
H_3H_2A_1 & H_3H_2V_1 & H_3D_2A_1 & H_3D_2V_1 & D_3H_2A_1 & D_3H_2V_1 & D_3D_2A_1 & D_3D_2V_1 \\
H_3H_2H_1 & H_3H_2D_1 & H_3D_2H_1 & H_3D_2D_1 & D_3H_2H_1 & D_3H_2D_1 & D_3D_2H_1 & D_3D_2D_1
\end{array}
$$