Computational Notes

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1 Elliptic Partial Differential Equations

Example equation $\nabla[g(x,y)\nabla\phi(x,y)] + h(x,y)\phi(x,y) = f(x,y)\ \forall(x,y) \in R$.

There is also the Poisson form:

$\nabla^2\phi(x,y) = \frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} = f(x,y)$

In working with these PDE forms the boundaries are typically spelled out in rectangular form for the left, right, top and bottom boundaries as follows:

- left: $\phi(0,y) = 0$
- right: $\frac{\partial\phi}{\partial x}(a,y) = 0$
- bottom: $\phi(x,0) = 0$
- top: $\frac{\partial\phi}{\partial y}(a,y) = 0$

Central Difference formulas are to derive the finite-difference equation for an interior grid (i,j).

Poison's Equation becomes the following after applying the Central Difference formulas:

$\frac{\phi_{i-1,j}-2\phi_{i,j}+\phi_{i+1,j}}{\Delta x^2} + \frac{\phi_{i-1,j}-2\phi_{i,j}+\phi_{i+1,j}}{\Delta y^2} = f_{i,j}$

Boundaries are also spelled out in discrete form.

Also, for the specified boundaries, the derivation of those points are unnecessary. Also those specified points are used in deriving the points near the edges.

There are two given approaches to implement boundary conditions that are spelled out for the right and top boundaries. One method yields the right and top boundaries as being equal to second to the last element. The other uses the last two elements to spell out the last element. (such as on page 803 and 804).

There are two further methods for solving the elliptical partial differential equations, Robbins and cases of irregular boundaries.

2 PDE Intro Generically

In general two sources agree that mathematically there are three basic types of PDE formulas: hyperbolic, parabolic, and elliptic. The classifications are based on the curves which propagate through the function. For these types there are prototype functions.

- Hyperbolic: $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$
- Parabolic: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(D \frac{\partial u}{\partial x})$
- Elliptical: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \rho(x, y)$

These equations are given such that v and D are both constants, and ρ is a given function. The elliptical formula is Poisson's Equation and in the special case of $\rho = 0$ is Laplace's Equation. The hyperbolic and parabolic are source definitions for initial value/ Cauchy problems. Some cases for parabolic and hyperbolic functions use time-evolution methods (special case of shooting method) such that the initial values propagate to subsequent values as time progresses. Boundary value problems require the area of consideration to be specified. Bottom line, the goal of PDE's is to fill in the blanks by some model / converging function.

So what is the issue in these cases? BVP tend come down a series of simultaneous algebraic equations. For example the elliptical version comes to:

$\frac{u_{j+1,l} - 2u_{j,l} + u_{j-1,l}}{\Delta x^2} + \frac{u_{j,l+1} - 2u_{j,l} + u_{j,l-1}}{\Delta x^2} = \rho_{j,l}$

These methods eventually bring up an equation which is meant satisfy the linear set of equations.

$A \cdot u = b$

A is typically sparse, and allows methods such as relaxation methods to split the matrix. Upon splitting, the goal is to guess the value of u and iterative compute out the correct solution for u. The idea for wavelets in PDE's would most likely be both to precondition the matrix, and make it as sparse as possible.

Methods for consideration are:

- The Incomplete Cholesky Conjugate Gradient Method (ICCG)

- Strongly Implicit Procedure
- Analyze Factorize Operate Approach
- Multi-Grid Approaches
- Monte-Carlo Methods
- Spectral Methods
- Variational Methods

3 ODE

3.1 Euler's Method

$x_i = x_0 i\dot{h}$

$y_{i+1} = y_i + hy_i + \frac{h^2}{2}y_i^{''} + ... + \frac{h^2}{2}y_i^n$

Approximately: $y_{i+1} = y_i + hy_i + \frac{h^2}{2}y_i^{''}(\zeta)$

$y_i = y(x_i)$

$y_{i+1} = y(x_{i+1})$

$y_i^{'} = \frac{dy}{dx}(x_i) = f(x_i, y_i)$

This works in case that some initial value is also known and steady intervals are defined.

This can be enhanced by selecting an interval small enough to reduce truncated error to its smallest value.

Integration can also be used since

$\int_{x_i}^{x_{i+1}} f(x,y)dx \approx f(x_i, y_i)(x_{i+1}, x_i)$

A system of differential equations can be use at specific points $y_{i,i+1}y_{j,i} + hf_j(x_i, y_1, ..., y_n, i)$

Euler's Method can be made to use higher order terms

$y(x_0) = y_0$

$y_i + 1 = y_i + hf(x_i, y_i) + \frac{h^2}{2!}f(x_i, y_i) + ... + \frac{h^n}{n!}f(x_i, y_i)$

$y^{''} = f^{''}(x,y) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}f(x,y)$

3.2 Heun's Method

1. Start with initial condition $y(x_0) = y_0$.
2. $\forall i \in [0, N) \bigcup Z$ determine the slope at the beginning step, $f(x_i, y_i)$
3. Find the slope at the end step as $f(x_{i+1}, y_{i+1})$.
4. Determine the value of y at x_{i+1} as $y_i + 1 = y_i + h[\frac{f(x_i,y_i)+f(x_{i+1},y_{i+1})+f(x_{i+1},y_{i+1})}{2}]$
5. Find corrector iteratively such that $|\frac{y_{i+1}^{k+1}-y_{i+1}^{k}}{y_{i+1}^{k}}| \leq \epsilon\; y_i{+}1 = y_i{+}h[\frac{f(x_i,y_i)+f(x_{i+1},y_{i+1})+f(x_{i+1},y_{i+1})}{2}]$

3.3 Runge-Kutta Method

"Runge-Kutta methods require only one initial point to start the procedure ... and requires several evaluations of $f(x, y)$ for each step of integration.

Runge-Kutta Method: $y_{i+1} = y_i + h\alpha(x, i, y_i, h)$ such that

- $\alpha(x_i, y_i, h)$ is the increment function (average slope)
- $\alpha(x_i, y_i, h) = c_1k_1 + c_2k_2 + ... + c_nk_n$
- n is the order of Runge-Kutta method
- $c_1, ..., c_n$ are constants
- $k_1, k_2, ..., k_n$ are recurrence relations defined as
  - $k_1 = f(x_i, y_i)$
  - $k_2 = f(x_i + p_2h, y_i + a_21hk_i)$
  - $k_3 = f(x_i + p_3h, y_i + a_31hk_i + a_32hk_2)$
  - $k_n = f(x_i + p_3h, y_i + a_{n,1}hk_i + ... + a_{n,n-1}hk_2)$
- Rewrite $y_{i+1} = y_i + h\sum_{j=1}^{n} c_jk_j$
- $k_j = f(x_i + p_jh, y_i + \sum^{j-1} l = 1a_{jl}k_l)$

The mechanisms allow a 1st, 2nd, 3rd, and nth order Runge-Kutta function to be defined.

Note that the recurrence relations must be computed each time and constants must be selected for the appropriate order.

"Usually a self-starting method that has the same order of accuracy as that of the multi-step method is used in the first few steps to generate the solution needed for starting the multistep method."

Adam's Method

3.4 Boundary Value Problems

The problem specifications are for two or more end points and 2nd or higher order in the solution.

An example:

$\frac{d^2y(x)}{dx} = f(x, y, \frac{dy}{dx})$; $\forall x \in [a, b]$

$y(x = a) = \alpha$ $y(x = b) = \beta$

Shooting Method (pg 727) General Procedure

1. Guess the unspecified initial conditions of the differential equation and prepare it for solution based on initial value problem methods.
2. Derive the variational equations denoting the sensitivity of the dependent variable based on the guess initial conditions.
3. Integrate the differential equation and the variation equations along the x direction as a set of simultaneous initial value equations.
4. Use the results of Step 3 to correct the guessed initial conditions
5. Repeat steps 2 through 4 until the specified boundary conditions are satisfied.

Development of the procedure $\frac{d^2(x)}{dx^2} = f(x, y, \frac{dy}{dx})$; $\forall x \in [a, b]$

$y(x = a) = y_a$

$\frac{dy}{dx}(x = b) = y_b$

Next define variables

1. $y_1(x) = y(x)$
2. $y_2(x) = \frac{dy_1}{dx}(x) = \frac{dy}{dx}(x)$

4 PDE Classic

The general mathematical definition of the partial differential equation is as follows:

The quasi - Linear $A\frac{\partial^2\theta}{\partial x^2} + B\frac{\partial^2\theta}{\partial x\partial y} + C\frac{\partial^2\theta}{\partial y^2} + D\frac{\partial^2\theta}{\partial x} + E\frac{\partial^2\theta}{\partial y^2} + F\theta = G(x, y, \theta, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y})$

The Linear Form: $A\frac{\partial^2\theta}{\partial x^2} + B\frac{\partial^2\theta}{\partial x\partial y} + C\frac{\partial^2\theta}{\partial y^2} + D\frac{\partial^2\theta}{\partial x} + E\frac{\partial^2\theta}{\partial y^2} + F\theta = G(x, y, \theta, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y})$

If $H \equiv 0$ then it is linear and homogeneous.

4.1 General Classifications

There are three general classifications of PDE and one case that is mention only but a few times in mathematical literature. The four classifications are with their practical characteristic :

- Elliptical (Equilibrium States)
- Parabolic (Diffusion states)
- Hyperbolic (oscillating or vibrating states)
- Ultra-Hyperbolic

4.2 Initial Value and Boundary Value Conditions

Both: Initial and Boundary conditions are specified such that a unique solution to the partial differential equation.

- Initial Value Problems have an open region
- Boundary Value Problems are specified at all boundaries with respect to all independent variables

There are three types of general boundary conditions which can be specified for a partial differential equations:

1. Dirichlet Condition (Dependent Value)
2. Neumann Condition (Gradient of Dependent variable)
3. Robbins (mixed)

4.3 Elliptical Differential Equations

One example type that shows up for Elliptical PDE problems is heat transfer, and typically Poison's formula is used as the model in the 2-D world.

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Also, it is typical that polygon is chosen to define the boundary conditions. Typically, the polygon is defined by either constants either in the first derivative or normal space. An example would be a rectangle specified in the first derivative for the right and top edges and constants for the lower and left edges. Also, the central difference formula is used to setup a linear set equations to be solved. [1]. The trick is to set up the matrix, and then solve it by any means (classic or wavelet based).

4.4 Parabolic Differential Equations

With Parabolic Differential Equations, an explicit method is introduced. The central difference method is applied to part of a grid with accuracy of $O(x^2)$ and $O(t^2)$. When substituted into the simplest parabolic form; the new form is

$\phi_{i,j+1} = \phi_{i,j+1} + \frac{2\alpha^2 \Delta t}{\Delta x^2}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j})$

This is useful computation. The next step is computed from the current and previous steps. However, this explicit form is unstable, and tends to yield results inconsistent with the boundaries. In order for this form to be stable :

$\phi_{i,j+1} = p\phi_{i+1,j} + q\phi_{i,j} + r\phi i - 1, j$ such that $p, q, r > 0$ and $p + q + r \leq 1$.

Limits on the explicit Method and alternatives by implicit methods are as follows:

1. Imposed limits on Δx and Δt. Dependences in explicit methods are directly limited to 3 values of the many values which it theoretically should.
2. The implicit method is approximately the second derivative $\frac{\partial^2\theta}{\partial x^2}|_{i,j}$ "by the finite difference formula involving θ at an advanced time (t_{j+1})" [1]. A mid-point is computed using a central - difference formula
   $\frac{\partial\theta}{\partial t}|_{i,j+1/2} = \frac{\theta_{i,j+1}\theta_{i,j}}{\Delta t}$
3. The 2nd p.d. applied with central-difference formula. There is catch with a weighting parameters.
4. Variable Weighted Implicit Formula can be used with the following conditions:
   - weighting factor θ
   - more than one unknown variable at the time step $j + 1$

4.5 Method of Lines

The main idea is to convert the PDE into a system of first order ODE(s) by approximating only the spatial derivative, $\frac{\partial^2\theta}{\partial x^2}(x, t)$ using a finite difference formula. There are special conditions, and they are as follows:

- Dirichlet Condition: $\theta_0 = \theta(x_0) = u_0$ such that u_0 is a constant for $t > 0$
- Neumann Condition: $\frac{\partial \theta_0}{\partial x} = \frac{\partial \theta}{\partial x}(x = 0) = v_0$ such that v_0 is a constant for $t > 0$

Note: For cases of more than one spacial dimension, more dimensions are needed to solve the system.

Again, this is solved by the Central Difference Formula.

5 Diffusion Problems

6 Classic Methods: Bessel and Legendre

The basic point of both Bessel and Legendre is to assign a solution of polynomials to a differential equation problem. Each have a particular partial and ordinary differential equations, for which they solve. In most texts, these methods are provided as an analytic method for solving the differential equation in question. These functions are orthogonal either in the range of -1 to 1 (Legendre) or 0 to 1 (Bessel).

There are several cousin polynomials to these general cases such as Leguerre, Hermite, and the Strum Liouville system for the Legnedre polynomials. In case of the Bessel functions, Hankel, Kei-Ker, Ber-Bei, and modified Bessel have alternate forms to the general Bessel function method. Again, most text such Spiegle and Farlow treat the analytical version.

?What about the numerical version?

7 Navier-Stokes and Wavelets

Reference page 145:

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