Critic of Spherical Wavelets

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May 17, 2003

Abstract Ideas The Power of the Wavelet

- Their power lies in the fact that they only require a small number of coefficents to represent general functions and large data sets accurately.
- Scalar functions defined on the sphere using biorthogonal wavelets
- Custom properties
- lifting scheme
- Reparameterizing directions over the set of visible surfaces
- Mapping directions to the unit square providing the avenue for construction of spherical wavelet.

The bases claim to allow fully adaptive subdivisions.

Examples

- Topographical Data
- Bi-direction reflection distribution functions
- Illumination techniques
- "The offer both theoretical characterization of smoothness, insights into the structure of functions and operators, and practical numerical tools which lead to faster computational algortihms."
- The challenge is to construct wavelets in general domains as they appear in graphics applications.

Reference to Lounsbery et al (LDW) for

- representing surface themselves
- representing functions defined on a surface.

Reference: Dahlke et al. intro to spherical wavelet (first) used a tensor product basis where one factor is an exponential spline.

Practical Applications:

1. Computer Graphics
    (a) Manipulations of spherical objects
    (b) display of spherical objects
    (c) remote sensing imagery

(d) simulation and modeling of bidirectional reflection distribution functions.

(e) illumination algorithms

(f) direction information processing (env maps and sphere views)

Outline of paper (in general)

1. Brief review of applications (computer graphics and others).

2. Wavelets on the sphere

3. Basic machinery of lifting and the fast wavelet x-form

4. Implementation

5. Simulations

Representing Functions on a sphere

- Dalton and geodesic sphere

- Hierarchical and non-hierarchical forms

- Sparseness is key

Example: Bidirectional reflectance distribution functions and radiance

- BRDF : $f_r(\vec{w_i}, \vec{x}, \vec{w_o})$ describes the relationship between

  - $\vec{w_i}$ incoming radiance

  - $\vec{x}$ point

- $\vec{w}_0$ outgoing radiance

- Spherical Harmonics needed to describe BRDF (involves the Fourier domain).
- A FFT has not been extended to the sphere domain
- One method derived from Monte Carlo simulations of micro-geometry.

Example: Radiance $L(\vec{x}, \vec{w})$ function.

- Defined over all surfaces and directions
- Global support introduces errors and ringing

Trouble for the spherical wavelet

- Suffers from distortions
- Contaminated wavelets from parameterization

Wavelets on the sphere (2nd generation wavelets)

- Note local support for compact support
- Smoothness (decay toward high-frequencies)

Base Philosophy: Build wavelets with all desirable properties (localization, fast-transfer) adapted to much more general settings than the real line.

One difference with classical wavelets are not the same throughout, but change locally to reflect the changing nature of the surface and its measure.

"In the setting of second generation wavelets, translation and dialation can no longer be used, and the Fourier transform becomes worthless as a construction tool."

Multi-resolution: Given A function $L_2 = L_2(S^2, d\omega)$. Call functions of finite energy defined over S^2.

Multi-resolution is defined as a sequence of closed subspaces $V_j \subset L_2$ with $j \geq 0$ such that

- $V_j \subset V_{j+1}$
- $\cup_{j \geq 0} V_j$ is dense in L_2

$K(j)$ is a general index set.

Note that every scaling function $\phi_{j,k}$, there exists coefficents $\{h_{i,j,k}\}$ such that

$$\phi_{j,k} = \sum_l h_{j,k,l} \phi_{j+1,l}$$

- $j \geq 0$
- $k \in K(j)$
- $l \in K(j+1)$
- $h_{j,k,l} = h_{l-2k}$ in the classic wavelet sense

Also, there is a concept of nested spaces, V_j, with bases of $\tilde{\psi}_{j,k}$ (dual-scan basis functions) which are bi-orthogonal to the scaling functions.

If the scaling functions and dual scaling functions coincide, then the scaling function forms an orthogonal basis.

If the multi-resolution and dual-multi-resolution analysis coincide, (not necessarily the scaling and dual scaling functions) then the scaling functions are semi-orthogonal.

Swelding's opinion on orthogonality

- Global support
- Assumption that coincidence does not occur
- General case is for bi-orthogonal cases

Wavelet Constructions: A basis space w_j that encodes the difference between two successive levels representation. Conditions for the wavelet basis space are as follows.

Wavelets: Multi-resolution wavelet construction

1. $V_j \oplus W_j$
2. The set of basis functions $\{\psi - j, m | j \geq, m \in M(j)\}$ such that $M(j) \subset X(j+1)$
   (a) is a Riesz basis for $L_2(S^2)$
   (b) is a Riesz basis of W_j
3. $\psi_{j,m}$ defines a spherical wavelet basis
4. $\psi_{j,m} = \sum_l g_{j,m,l} \phi_{j+1,l}$
5. Wavelets $\psi_{j,m}$ have $\bar{N}$ vanishing moments s.t.
   (a) for $0 \leq i \leq \bar{N} \exists < \psi j, m, P_i >= 0$
   (b) P_i are polynomials and $\bar{N}$ is independent of it.

6. P_i are defined as the restriction to the sphere.

7. There are basis and dual basis functions ($\tilde{\psi}_{j,m}$) which are bi-orthogonal.

8. The following equalities hold:

   (a) $< \tilde{\psi}_{j,m}, \phi_{j,k} >=< \tilde{\phi}_{j,k}, \psi_{j,m} >= 0$

   (b) $f = \sum_{j,m} < \tilde{\psi}_{j,m}, f > \psi_{i,m} = \sum_{j,m} \gamma_{j,m} \psi_{j,m}$

A Reisz base has some Hilbert space is a countable suset $\{f_k\}$ so that every element f of the space can be written uniquely as $f_i = \sum_k c_k f_k$ and positive constants A and B exist with

$A||f||^2 \leq \sum_k |c_k|^2 \leq B||f||^2$

The conditions for constructing a wavelet scaling function are simply function of coarser scaling functions.

$$\psi_{j+1,l} = \sum_k \tilde{h} \phi_{j,k} = \sum_m \bar{g}_{j,m,l} \psi_{j,m}$$

such that

- $k \in K(j)$
- $l \in K(j+1)$
- $m \in M(j)$

Proposition: Given a set of scaling function coefficients of function f.

$\{\lambda_{n,k} =< f, \emptyset > |k \in K(x)\}$

such that n is some finest resolution level.

Results: The fast wavelet transform recursively calculates the coarser approximation to the underlying functions.

- $\{\gamma_{j,n} | 0 \leq j < n; m \in M(j)\}$
- $\{\lambda_{0,k} | k \in (\emptyset)\}$

One step in the FWT computes a coarser level from a finer level ($j+1 \rightarrow j$)

- $\lambda_{j,k} = \sum_l \tilde{h}_{j+1,l} \lambda_{j+1,l}$
- $\gamma_{j,m} = \sum_l \tilde{g}_{j,m,l} \lambda_{j+1,l}$

A single step in the individual transform that computes the finer level from the coarser level ($j+1 \leftarrow j$).

$\lambda_{j+1,l} = \sum_k h_{j,k,l} \lambda_{j,k} + \sum_m g_{j,m,l} \gamma_{j,m}$