# Critic on Galerkin-Wavelet Solution to Strum-Louisville

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Example ODE Strum-Louisville

$Lu(t) = -\frac{d}{dt}(a(t)\frac{du}{dt})b(t)u(t) = f(t)\ \forall t \in [0,1]$

Dirichlet Conditions of $u(0) = u(1) = 0$

Properties:

1. "a" is a continuous species
2. L may be variable coefficient differential operator.
3. L is uniformly elliptic
4. Finite Constant $C_1$ , $C_2$ and $C_3$ exists such that $0 \leq C_1 \leq a(t) \leq C_2$ and $0 \leq b(t) \leq C_3$.

For Galerkin method we suppose that $\{v_j\}_j$ is a complete orthonormal system for $L^2[(0,1)]$ and that every $v_j$ is $C^2$ on $[(0,1)]$ and satisfies $v_j(0) = v_j(1) = 0$

- There is a finite set $\Lambda$ of indices j
- Subspace $S =$ span $\{v_j : j \in \Lambda\}$

The goal is to find the approximation to the solution u of $Lu(t)$ in the form $u_s = \sum_{k\in\Lambda} x_k v_k \in S$ such that $x_k$ is a scalar and is chosen such that $u_s$ behaves as a true solution on S.

## 1 Galerkin Method

Definition: The Galerkin method supposes that a complete orthonormal system $\{v_j\}_j$ is defined on $L^2([0,1])$ and every $v_j$ is $C^2$ on [0,1]. The boundary conditions of the $v_j$ is typically defined as well. The solution approximation is then defined on the span of this orthonormal system. Example:

$u_s = \sum_{k\in\Lambda}(x_k v_k) \in S$ such that S is a span of $v_j : j \in \Lambda$, and $x_j$ is a scalar. The catch is that $u_s$ should behave as true solution a system of linear equations. The linear equations are is the implicit set of equations for solving a PDE or ODE.

Frazier takes this one step further to show a parallel from Galerkin to a conventional implicit form. First he shows:

- $\langle Lu_s, v_j\rangle = \langle f, v_j\rangle\ \forall j \in \Lambda$ such that $\langle f, g\rangle = \int_0^1 f(t)g(\bar{t})dt$
- Furthermore: $\langle L(\sum_{k\in\Lambda} x_k v_k), v_j\rangle = \langle f, v_j\rangle\ \forall j \in \Lambda$ leading to
- $\sum_{k\in\Lambda}\langle Lv_k, v_j\rangle x_k = \langle f, v_j\rangle\ \forall j \in \Lambda$

The final connection is that each element of a matrix A defined $A = (a_{j,k})_{j,k\in\Lambda}$ is a scalar defined by $\langle Lv_k, v_j\rangle$.

- x is a vector $(x_k)_{k\in\Lambda}$
- y is a vector $(y_k)_{k\in\Lambda}$
- A is a matrix with rows and columns indexed by $\Lambda$ such that $A = (a_{j,k})_{j,k\in\Lambda} = \langle Lv_k, v_j\rangle$

With Galerkin, for all subsets in $\Lambda$ we obtain an approximation $u_s \in S \to u$. This is done by solving $Ax = y$ and using x to determine $u_s$.

Wavelets are chosen as a basis to ensure that the condition number is near unity and the matrix A becomes sparse.

1. Frazier modified wavelet system for $L^2(R)$ which is also complete orthonormal system $\{\psi_{j,k}\}$ such that $(j,k) \in \Gamma$ and $\Gamma$ is a certain subset of $Z \times Z$ that we do not specify.
2. Scale of $\psi_{j,k}$ is $2^{-j}$, and concentration is at or near $2^{-j}k$ and 0 elsewhere.
3. Near boundary points are substantially modified. $\forall(j,k) \in \Lambda$, $\psi_{j,k} \in C^2$ and satisfies:
   - Boundary conditions: $\psi_{j,k}(0) = \psi_{j,k}(1) = 0$
   - Key Estimate $g = \sum_{jk} c_{jk}\psi_{jk}$
   - Norm Equivalence: $C_4\sum_{ik} 2^{2j}|c_{j,k}|^2 \leq \int_0^1 |g'(t)|^2 dt \leq C_5\sum_{jk} 2^2|c_i j|^2$
4. Standard Scale and Translation Rule $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$
5. Chain Rule Applies to the Standard Scale and Translation rule: $\psi'_{j,k}(t) = 2^j 2^{j/2}\psi'(2^j - k) = 2^j(\psi')_{j,k}$
6. Modest convincing of derivative identity of $\psi = \psi'$ and Norm Equivalence.
7. Equivalent wavelet form

- $u_s = \sum_{k\in\Lambda} x_k v_k \in S$
- $u_s = \sum_{(j,k)\in\Lambda} x_k \psi_{j,k} \in S$
- $\sum_{(j,k)\in\Lambda} \langle L\psi_{j,k}, \psi_{l,m}\rangle = \langle f, \psi_{l,m}\rangle \ \forall (l,m) \in \Lambda$
- $A = [a_{l,m;j,k}]_{(l,m)(j,k)\in\Lambda} = \langle L\psi_{j,k}, \psi_{l,m}\rangle$

Let L be uniformly elliptic Strum-Liouville operator (i.e. an operator as defined in equation "a" satisfying relation "b". Suppose $g \in L^2([0,1])$ is $C^2$ on $[0,1]$ and satisfies $g(0) = g(1) = 0$, then $C_1 \int_0^1 |g'(t)|^2 df \leq \langle Lg, g\rangle \leq (C_2 + C_3) \int_0^1 |g'(t)|^2 dt$ where $C_1$ , $C_2$ and $C_3$ are the constants in the relation "b".

"a" $Lu(t) = \frac{d}{dt}(a(t)\frac{df}{dt}) + b(t)u(t) = f(t)$ s.t. $u(0) = u(1) = 0$

"b" $0 \leq C_1 \leq a(t) \leq C_2$ and $0 \leq b(t) \leq C_2$

"c" $M = D^{-1}AD^{-1}$ such that $D = [d_{lm,j,k}]_{(l,m)(j,k)\in\Delta}$ and

$d_{l,m,j,k} =$

"d" $C_4 \sum_{ik} 2^{2j} |c_{j,k}|^2 \leq \int_0^1 |g'(t)|^2 dt \leq C_5 \sum_{jk} 2^2 |c_i j|^2$

Let L be a uniformly elliptic Strum-Liouville operator. Let $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ be a complete orthonormal system for $L^2([0,1])$ such that each $\psi_{j,k}$ is $C^2$, satisfies $\psi_{j,k}(0) = \psi_{j,k}(1) = 0$, and such that the norm equivalence holds. Let $\Lambda$ be a finte subset of $\Gamma$. Let M be the matrix defined "c". Then the condition number of M satisfies:

$C_\#(M) \leq \frac{(C_2+C_3)C_5}{C_1C_4}$

for any finite-set $\Lambda$ , where $C_1$ , $C_2$, and $C_3$ are the constants in relation "b" and $C_4$ and $C_5$ are the constants in relation "d".

# References


[1] Michael W. Frazier *An Introduction to Wavelets to Wavelets Through Linear Algebra* copyright 1999 by the Springer-Verlag New York, Inc