Consider the marginal prior density function

$$\begin{aligned}
\pi(\theta_1, \theta_2) &= \pi(\theta_1) \cdot \pi(\theta_2) \\
&= \text{normal}(\theta_1 \mid 0, \tau_1) \cdot \text{normal}(\theta_2 \mid 0, \tau_2) \\
&\propto \exp\left( -\frac{1}{2} \left[ \frac{\theta_1^2}{\tau_1^2} + \frac{\theta_2^2}{\tau_2^2} \right] \right),
\end{aligned}$$

and a probability density function that encodes a desired pushforward behavior on $\phi = \theta_1 + \theta_2$,

$$\pi(\phi) = \text{normal}(\phi \mid \mu, \tau_+) \propto \exp\left( -\frac{1}{2} \left( \frac{\phi - \mu}{\tau_+} \right)^2 \right).$$

Naively we might assume that multiplying these two probability density function together,

$$\begin{aligned}
\rho(\theta_1, \theta_2) &= \pi(\theta_1, \theta_2) \cdot \pi(\theta_1 + \theta_2) \\
&\propto \exp\left( -\frac{1}{2} \left[ \frac{\theta_1^2}{\tau_1^2} + \frac{\theta_2^2}{\tau_2^2} + \left( \frac{\theta_1 + \theta_2 - \mu}{\tau_+} \right)^2 \right] \right),
\end{aligned}$$

yields a prior density function consistent with the both the marginal and pushforward behavior. Unfortunately this is not the case.

To see why let's reparameterize to

$$\begin{aligned}
\eta_1 &= \theta_1 \\
\eta_2 &= \theta_1 + \theta_2.
\end{aligned}$$

The Jacobian determinant of this transformation is just

$$|J|(\eta_1, \eta_2) = \begin{vmatrix} \frac{\partial \eta_1}{\partial \theta_1} & \frac{\partial \eta_1}{\partial \theta_2} \\ \frac{\partial \eta_1}{\partial \theta_1} & \frac{\partial \eta_2}{\partial \theta_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

so that

$$\begin{aligned}
\rho(\eta_1, \eta_2) &= \rho\left(\theta_1(\eta_1, \eta_2), \theta_2(\eta_1, \eta_2)\right) \frac{1}{|J|(\eta_1, \eta_2)} \\
&= \rho\left(\eta_1, \eta_2 - \eta_1\right) \\
&\propto \exp\left( -\frac{1}{2} \left[ \frac{\eta_1^2}{\tau_1^2} + \left( \frac{\eta_2 - \eta_1}{\tau_2} \right)^2 + \left( \frac{\eta_2 - \mu}{\tau_+} \right)^2 \right] \right) \\
&\propto \exp\left( -\frac{1}{2} f(\eta_1, \eta_2) \right).
\end{aligned}$$

Expanding each term in square brackets and gathering like terms gives

$$\begin{aligned}
f(\eta_1, \eta_2) &= \frac{\eta_1^2}{\tau_1^2} + \frac{\eta_2^2}{\tau_2^2} - 2\frac{\eta_1 \cdot \eta_2}{\tau_2^2} + \frac{\eta_1^2}{\tau_2^2} + \frac{\eta_2^2}{\tau_+^2} - 2\frac{\mu \cdot \eta_2}{\tau_+^2} + \text{const} \\
&= \left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}\right)\eta_1^2 + \left(\frac{1}{\tau_2^2} + \frac{1}{\tau_+^2}\right)\eta_2^2 - 2\left(\frac{\eta_1}{\tau_2^2} + \frac{\mu}{\tau_+^2}\right)\eta_2 + \text{const} \\
&= \left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right)\eta_1^2 + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right)\eta_2^2 - 2\left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 \cdot \tau_+^2}\right)\eta_2 + \text{const.}
\end{aligned}$$

Completing the square for $\eta_2$ gives

$$\begin{aligned}
f(\eta_1, \eta_2) = & \left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right) \eta_1^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\tau_2^2 \cdot \tau_+^2}{\tau_2^2 + \tau_+^2}\right) \cdot \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 \cdot \tau_+^2}\right)\right]^2 \\
& - \left(\frac{\tau_2^2 \cdot \tau_+^2}{\tau_2^2 + \tau_+^2}\right) \cdot \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 \cdot \tau_+^2}\right)^2 \\
& + \text{const} \\
= & \left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right) \eta_1^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& - \left(\frac{\tau_2^2 \cdot \tau_+^2}{\tau_2^2 + \tau_+^2}\right) \cdot \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 \cdot \tau_+^2}\right)^2 \\
& + \text{const} \\
= & \left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right) \eta_1^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& - \left(\frac{\tau_2^2 \cdot \tau_+^2}{\tau_2^2 + \tau_+^2}\right) \cdot \frac{\eta_1^2 \cdot \tau_+^4 + 2\,\eta_1 \cdot \mu \cdot \tau_+^2 \cdot \tau_2^2}{\tau_2^4 \cdot \tau_+^4} \\
& + \text{const} \\
= & \left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right) \eta_1^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& - \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \frac{\eta_1^2 \cdot \tau_+^4 + 2\,\eta_1 \cdot \mu \cdot \tau_+^2 \cdot \tau_2^2}{\tau_2^2 \cdot \tau_+^2} \\
& + \text{const}
\end{aligned}$$

Gathering terms,

$$
\begin{aligned}
f(\eta_1, \eta_2) = \quad & \left[\left(\frac{\tau_1^2 + \tau_2^2}{\tau_1^2 \cdot \tau_2^2}\right) - \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \frac{\tau_+^2}{\tau_2^2}\right] \eta_1^2 \\
& - 2 \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \mu \cdot \eta_1 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& + \text{const} \\
= \quad & \left[\frac{(\tau_1^2 + \tau_2^2) \cdot (\tau_2^2 + \tau_+^2) - \tau_1^2 \cdot \tau_+^2}{\tau_1^2 \cdot \tau_2^2 \cdot (\tau_2^2 + \tau_+^2)}\right] \eta_1^2 \\
& - 2 \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \mu \cdot \eta_1 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& + \text{const} \\
= \quad & \left[\frac{\tau_2^2 \cdot (\tau_1^2 + \tau_2^2 + \tau_+^2)}{\tau_1^2 \cdot \tau_2^2 \cdot (\tau_2^2 + \tau_+^2)}\right] \eta_1^2 \\
& - 2 \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \mu \cdot \eta_1 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& + \text{const} \\
= \quad & \left[\frac{\tau_1^2 + \tau_2^2 + \tau_+^2}{\tau_1^2 \cdot (\tau_2^2 + \tau_+^2)}\right] \eta_1^2 \\
& - 2 \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \mu \cdot \eta_1 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& + \text{const.}
\end{aligned}
$$

Now we can complete the square for $\eta_1$ to give

$$\begin{aligned}
f(\eta_1, \eta_2) = & \quad \left[\frac{\tau_1^2 + \tau_2^2 + \tau_+^2}{\tau_1^2 \cdot (\tau_2^2 + \tau_+^2)}\right] \cdot \left[\eta_1 - \frac{\tau_1^2 \cdot (\tau_2^2 + \tau_+^2)}{\tau_1^2 + \tau_2^2 + \tau_+^2} \left(\frac{1}{\tau_2^2 + \tau_+^2}\right) \cdot \mu\right]^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \left(\frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right)\right]^2 \\
& + \text{const} \\
= & \quad \left[\frac{\tau_1^2 + \tau_2^2 + \tau_+^2}{\tau_1^2 \cdot (\tau_2^2 + \tau_+^2)}\right] \cdot \left[\eta_1 - \frac{\tau_1^2 \cdot \mu}{\tau_1^2 + \tau_2^2 + \tau_+^2}\right]^2 \\
& + \left(\frac{\tau_2^2 + \tau_+^2}{\tau_2^2 \cdot \tau_+^2}\right) \left[\eta_2 - \frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}\right]^2 \\
& + \text{const} \\
& = g(\eta_1) + h(\eta_1, \eta_2) + \text{const.}
\end{aligned}$$

Exponentiating these quadratics will yield normal density functions.

We can now substitute this back into the probability density function to give

$$\begin{aligned}
\rho(\eta_1, \eta_2) & \propto \exp\left(-\frac{1}{2} f(\eta_1, \eta_2)\right) \\
& \propto \exp\left(-\frac{1}{2} h(\eta_1, \eta_2\right) \cdot \exp\left(-\frac{1}{2} g(\eta_1)\right) \\
& = \rho(\eta_2 \mid \eta_1) \cdot \rho(\eta_1) \\
& = \text{ normal}\left(\eta_2 \mid \frac{\eta_1 \cdot \tau_+^2 + \mu \cdot \tau_2^2}{\tau_2^2 + \tau_+^2}, \frac{\tau_2 \cdot \tau_+}{\sqrt{\tau_2^2 + \tau_+^2}}\right) \\
& \quad \cdot \text{normal}\left(\eta_1 \mid \frac{\tau_1^2 \cdot \mu}{\tau_1^2 + \tau_2^2 + \tau_+^2}, \tau_1 \cdot \sqrt{\frac{\tau_2^2 + \tau_+^2}{\tau_1^2 + \tau_2^2 + \tau_+^2}}\right)
\end{aligned}$$

Because of this conditional decomposition we can immediately integrate out $\eta_2$ which gives the marginal density function for $\eta_1 = \theta_1$,

$$
\begin{aligned}
\rho(\eta_1) &= \int \mathrm{d}\eta_2 \, \rho(\eta_1, \eta_2) \\
&= \int \mathrm{d}\eta_2 \, \rho(\eta_2 \mid \eta_1) \cdot \rho(\eta_1) \\
&= \left[ \int \mathrm{d}\eta_2 \, \rho(\eta_2 \mid \eta_1) \right] \cdot \rho(\eta_1) \\
&= \rho(\eta_1) \\
&= \mathrm{normal}\left( \eta_1 \mid \frac{\tau_1^2 \cdot \mu}{\tau_1^2 + \tau_2^2 + \tau_+^2}, \tau_1 \cdot \sqrt{\frac{\tau_2^2 + \tau_+^2}{\tau_1^2 + \tau_2^2 + \tau_+^2}} \right),
\end{aligned}
$$

which is quite different from the original marginal of $\mathrm{normal}(\eta_1 \mid 0, \tau_1)$! The naive multiplication has both translated and scaled the original prior model.