Massive MIMO Networks: Spectral, Energy, and Hardware Efficiency

Exercises

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This document contains a series of exercises on some of the topics covered in:

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The exercises have been designed to test the understanding of basic theorems, concepts, and results of Massive MIMO. In most cases, the proofs are rather short if you know the necessary theory.

This document might be updated with corrections and additional exercises. The latest version is available at: `https://massivemimobook.com`

Exercises for Section 1: Introduction and Motivation

1. Consider an infinitely large cellular network with equal-sized hexagonal cells. The distance between neighboring BSs is 200 m. The communication bandwidth is 20 MHz and the SE per cell is 3 bit/s/Hz/cell.
   (a) What is the area throughput of this network?
   (b) If the SE is increased to 30 bit/s/Hz/cell, which distance between the neighboring BSs will result in the same area throughput as in (a)?

2. Consider a discrete memoryless channel with input $x \in \mathbb{C}$ and output $y \in \mathbb{C}$ given by
$$y = (h_{\mathrm{LoS}} + h_{\mathrm{NLoS}})\, x + n \tag{1}$$
   where $n \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ is independent circularly symmetric complex Gaussian noise. The channel response consists of two parts: h_{LoS} is deterministic and known at the receiver, $h_{\mathrm{NLoS}} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ is random and its realization is unknown to the receiver (and the transmitter). Use Corollary 1.3 to compute a lower bound on the capacity of this channel, when the input is power-limited as $\mathbb{E}\{|x|^2\} \leq p$.

3. The SE expression $\mathsf{SE}_0^{\mathrm{LoS}}$ in Eq. (1.17) is derived under the assumption that signals $s_0, s_1 \sim \mathcal{N}_{\mathbb{C}}(0, p)$ in the two cells have the same power p.
   (a) Generalize the expression of $\mathsf{SE}_0^{\mathrm{LoS}}$ for the case when $s_0 \sim \mathcal{N}_{\mathbb{C}}(0, p_0)$ and $s_1 \sim \mathcal{N}_{\mathbb{C}}(0, p_1)$.
   (b) Compute the corresponding SE expression $\mathsf{SE}_1^{\mathrm{LoS}}$ for the UE in cell 1.
   (c) What happens to the SEs if $p_0 \to \infty$ for a fixed value of p_1? What happens to the SEs if $p_1 \to \infty$ for a fixed value of p_0?
   (d) Suppose an infinite amount of power is available, so that p_0 and p_1 can take any values. Propose an alternative communication approach that gives infinite SE to both UEs.

4. The channel responses from two single-antenna UEs to an M-antenna BS are denoted as $\mathbf{h}_1 \in \mathbb{C}^M$ and $\mathbf{h}_2 \in \mathbb{C}^M$, respectively. Consider LoS propagation and assume the UEs are respectively located in the angle directions φ_1 and φ_2, as seen from the BS.
   (a) Assume the BS is equipped with a uniform linear array (ULA) with $d_{\mathrm{H}} = 1/2$. Derive the channel responses for the two UEs, under the assumptions that the first element in each vector is 1.
   (b) Compute $|\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2$ and simplify the expression. Let $\varphi_1 = \pi/6$ and plot $|\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2$ for $\varphi_2 \in [0, 2\pi)$. Explain the results.

(c) Assume the BS is equipped with a uniform circular array (UCA) where the distance between adjacent antennas is 1/2 wavelength. Derive the channel responses for the two UEs, under the assumptions that the first element in each vector is 1.

(d) Plot $|\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2$ for $\varphi_1 = \pi/6$ and $\varphi_2 \in [0, 2\pi)$. Compare the results with (b).

Hint: Eq. (1.23) may be useful.

5. Consider the received signal in Eq. (1.25) for a two-UE scenario. Suppose the channels are arbitrary deterministic vectors $\mathbf{h}_0^0$ and $\mathbf{h}_1^0$.

   (a) Compute an SE expression SE_0 for UE in cell 0. The expression should apply for an arbitrary choice of combining vector $\mathbf{v}_0$.

   (b) Under which conditions on $\mathbf{h}_0^0$ and $\mathbf{h}_1^0$ will MR combining, as defined in Eq. (1.26), maximize the SE from (a)?

   (c) Suppose $\mathbf{h}_0^0$ and $\mathbf{h}_1^0$ are LoS channels, modeled as in Eq. (1.23). Give an interpretation of the conditions from (b), first for finite M and then as $M \to \infty$.

6. Consider the lower bound on the UL SE in Eq. (1.32). Suppose we have a bandwidth of 100 MHz and want to achieve an information rate of at least 500 Mbit/s per UE.

   (a) If $M = 1$ and $\bar{\beta} = -20$ dB, which SNRs can deliver the required information rate?

   (b) If $M = 2$ and $\bar{\beta} = -20$ dB, which SNRs can deliver the required information rate?

   (c) If $M = 10$ and $\bar{\beta} = -20$ dB, which SNRs can deliver the required information rate?

7. M-MMSE combining in Eq. (1.42) is claimed to minimize the MSE $\mathbb{E}\{|s_{0k} - \mathbf{v}_{0k}^{\mathrm{H}}\mathbf{y}_0|^2\}$ between the desired signal s_{0k} and the receive combined signal $\mathbf{v}_{0k}^{\mathrm{H}}\mathbf{y}_0$ in Eq. (1.40).

   (a) Prove that M-MMSE indeed minimizes $\mathbb{E}\{|s_{0k} - \mathbf{v}_{0k}^{\mathrm{H}}\mathbf{y}_0|^2\}$, when the channels are deterministic and known.

   (b) What is the minimum value of the MSE?

   Hint: Completing the squares might be easier than taking derivatives.

8. A lower bound on the UL sum SE is given in Eq. (1.44).

   (a) What is the SE per UE, using this bound?

   (b) What happens to the SE per UE when both M and K grow without bound, such that the ratio $M/K \to c$ for some constant c? Consider the following three cases: $c = 0$, $c = +\infty$, and c is a finite non-zero constant.

9. Lemma 1.8 contains DL sum SE expressions with MR precoding. In the paragraph after the lemma it is claimed that "the UL simulations in Figure 1.16–1.17 are representative for the DL performance as well". Generate a DL counterpart to Figure 1.16a, using MR precoding, and discuss to what extent the claim is correct.

Exercises for Section 2: Massive MIMO Networks

1. Let $\mathbf{h}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M)$ and $\mathbf{h}_{jk}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M)$ be two independent random variables.

   (a) Compute $\mathbb{E}\{\mathbf{h}_{li}^j\}$, $\mathbb{E}\{\|\mathbf{h}_{li}^j\|^2\}$, and $\mathbb{E}\{\|\mathbf{h}_{li}^j\|^4\}$.

   (b) What is the mean value and variance of $\frac{\|\mathbf{h}_{jk}^j\|^2}{\mathbb{E}\{\|\mathbf{h}_{jk}^j\|^2\}}$?

   (c) What is the mean value and variance of $\frac{(\mathbf{h}_{li}^j)^{\mathrm{H}}\mathbf{h}_{jk}^j}{M}$?

2. According to Definition 2.4, asymptotic channel hardening occurs if $\frac{\|\mathbf{h}_{jk}^j\|^2}{\mathbb{E}\{\|\mathbf{h}_{jk}^j\|^2\}} \to 1$ as $M_j \to \infty$, where the convergence is almost sure. A mathematically weaker, but practically similar, definition would involve convergence in mean square. A sequence $X_1, X_2, \ldots$ of random variables converges in mean square to X if $\mathbb{E}\{|X_n - X|^2\} \to 0$ as $n \to \infty$.

   (a) Define asymptotic channel hardening using convergence in mean square. Obtain a simplified condition for asymptotic channel hardening with correlated Rayleigh fading channels.

   (b) Prove whether or not asymptotic channel hardening (with convergence in mean square) holds under Assumption 1 in Section 4.3. Are any further conditions on the correlation matrix required?

3. According to Definition 2.5, asymptotic favorable propagation occurs if $\frac{(\mathbf{h}_{li}^j)^{\mathrm{H}}\mathbf{h}_{jk}^j}{\sqrt{\mathbb{E}\{\|\mathbf{h}_{li}^j\|^2\}\mathbb{E}\{\|\mathbf{h}_{jk}^j\|^2\}}} \to 0$ as $M_j \to \infty$, where the convergence is almost sure. A mathematically weaker, but practically similar, definition would involve convergence in mean square. A sequence $X_1, X_2, \ldots$ of random variables converges in mean square to X if $\mathbb{E}\{|X_n - X|^2\} \to 0$ as $n \to \infty$.

   (a) Define asymptotic favorable propagation using convergence in mean square. Obtain a simplified condition for asymptotic favorable propagation with correlated Rayleigh fading channels.

   (b) Prove whether or not asymptotic favorable propagation (with convergence in mean square) holds under Assumption 1 in Section 4.3. Are any further conditions on the correlation matrices required?

4. For the local scattering model, Eq. (2.23) gives a general expression for a correlation matrix $\mathbf{R}$.

   (a) Suppose the angle $\bar{\varphi}$ is uniformly distributed between $-\pi$ and $+\pi$. Show that the off-diagonal elements of $\mathbf{R}$ are non-zero in this case.

   (b) Give an example of an angular PDF $f(\bar{\varphi})$ for which Eq. (2.23) becomes a scaled identity matrix.

   (c) Discuss the practical implications. How reasonable is it to observe i.i.d. Rayleigh fading channels?

Exercises for Section 3: Channel Estimation

1. Consider the received pilot signal $\mathbf{y}_{jjk}^p$ in Eq. (3.5) and assume Rician fading channels: $\mathbf{h}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\bar{\mathbf{h}}_{li}^j, \mathbf{R}_{li}^j)$, where $\bar{\mathbf{h}}_{li}^j \in \mathbb{C}^{M_j}$ is the mean value and $\mathbf{R}_{li}^j \in \mathbb{C}^{M_j \times M_j}$ is the covariance matrix.

   (a) What is the correlation matrix of $\mathbf{h}_{li}^j$?

   (b) Derive the MMSE estimator of $\mathbf{h}_{jk}^j$ based on the received pilot signal in Eq. (3.5).

   (c) Compare the MSE with that of the MMSE estimator in Theorem 3.1, for Rayleigh fading channels with the same covariance matrices. What are the differences?

2. Consider the received pilot signal

$$\mathbf{y}^p = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{n}$$

   where $\mathbf{h}_1 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_1)$, $\mathbf{h}_2 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_2)$, and $\mathbf{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M)$ are independent. Suppose we estimate $\mathbf{h}_1$ as $\hat{\mathbf{h}}_1 = \mathbf{A}\mathbf{y}^p$ for some deterministic matrix $\mathbf{A}$.

   (a) Compute $\mathbb{E}\{\hat{\mathbf{h}}_1^{\mathrm{H}}\mathbf{h}_2\}$ and $\mathbb{E}\{\mathbf{h}_2\hat{\mathbf{h}}_1^{\mathrm{H}}\}$.

   (b) Suppose $\mathbf{R}_2 = \mathrm{diag}(\beta_2^{(1)}\mathbf{I}_{M/2}, \beta_2^{(2)}\mathbf{I}_{M/2})$. Determine a non-zero $\mathbf{A}$ that makes $\mathbb{E}\{\hat{\mathbf{h}}_1^{\mathrm{H}}\mathbf{h}_2\}$ from (a) zero.

   (c) Are $\hat{\mathbf{h}}_1$ and $\mathbf{h}_2$ uncorrelated when using $\mathbf{A}$ and $\mathbf{R}_2$ from (b)?

   (d) Suppose $\mathbf{R}_1 = \mathrm{diag}(\beta_1^{(1)}\mathbf{I}_{M/2}, \beta_1^{(2)}\mathbf{I}_{M/2})$. Under what condition is $\mathbb{E}\{\hat{\mathbf{h}}_1^{\mathrm{H}}\mathbf{h}_1\}$ non-zero when using $\mathbf{A}$ and $\mathbf{R}_2$ from (b)?

3. Consider the received pilot signal

$$\mathbf{Y}^p = \mathbf{h}_1 \boldsymbol{\phi}_1^{\mathrm{T}} + \mathbf{h}_2 \boldsymbol{\phi}_2^{\mathrm{T}} + \mathbf{N}$$

where $\mathbf{h}_1 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_1)$ and $\mathbf{h}_2 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_2)$ are independent channels and $\mathbf{N}$ has i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ elements. The pilot sequences $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ satisfy $\|\boldsymbol{\phi}_1\|^2 = \|\boldsymbol{\phi}_2\|^2 = \tau_p$, but are neither orthogonal nor parallel vectors; that is, $0 < |\boldsymbol{\phi}_1^{\mathrm{H}} \boldsymbol{\phi}_2| < \tau_p$.

   (a) Suppose we first compute $\mathbf{y}^p = \mathbf{Y}^p \boldsymbol{\phi}_1^{\star}$. Compute the MMSE estimator of $\mathbf{h}_1$ given the observation $\mathbf{y}^p$. Derive a closed-form expression for the MSE.

   (b) Suppose we compute the MMSE estimator of $\mathbf{h}_1$ given the original received signal $\mathbf{Y}^p$. Derive a closed-form expression for the MSE.

   (c) Are the results in (a) and (b) equal? If not, which of the two approaches gives the smallest MSE?

   (d) What will change if $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are orthogonal ($\boldsymbol{\phi}_1^{\mathrm{H}} \boldsymbol{\phi}_2 = 0$) or parallel ($\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2$)?

   Hint: Lemma B.17 and Corollary B.18 will be useful. If $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors, then* $\mathbf{a}\mathbf{b}^{\mathrm{T}} = (\mathbf{b} \otimes \mathbf{I})\mathbf{a}$.

4. Consider the EW-MMSE estimator in Corollary 3.4.

   (a) What is the distribution of the estimate?

   (b) What is the distribution of the estimation error?

   (c) Compute the correlation matrix between the estimate and estimation error.

5. Consider the LS estimator in Eq. (3.35).

   (a) What is the distribution of the estimate?

   (b) What is the distribution of the estimation error?

   (c) Compute the correlation matrix between the estimate and estimation error.

Exercises for Section 4: Spectral Efficiency

1. Consider the UL channel capacity bound in Theorem 4.1 for a network with only one cell and one UE.

   (a) Derive the receive combining vector maximizes the capacity bound.

   (b) Under what conditions is the receive combining vector in (a) equivalent to MRC?

   Hint: The matrix inversion lemma might be useful.

2. Consider the case of Rician fading channels $\mathbf{h}_{li}^{j} \sim \mathcal{N}_{\mathbb{C}}(\bar{\mathbf{h}}_{li}^{j}, \mathbf{R}_{li}^{j})$, where $\bar{\mathbf{h}}_{li}^{j} \in \mathbb{C}^{M_j}$ is the mean value and $\mathbf{R}_{li}^{j} \in \mathbb{C}^{M_j \times M_j}$ is the covariance matrix. Suppose there are no UL pilots.

   (a) Explain why $\bar{\mathbf{h}}_{jk}^{j}$ is the MMSE estimate of $\mathbf{h}_{jk}^{j}$ in this case.

   (b) Suppose MR combining is applied using the MMSE estimate from (a). Derive a closed-form expression for the UL channel capacity bound in Theorem 4.4.

   (c) Is there any coherent interference in this case?

3. Suppose ZF combining is used and consider the special case of spatially uncorrelated fading (i.e., $\mathbf{R}_{li}^{j} = \beta_{li}^{j} \mathbf{I}_{M_j}$ for $j = 1, \ldots, L$, $l = 1, \ldots, L$ and $i = 1, \ldots, K_l$).

   (a) Derive a closed-form expression for the UL capacity bound in Theorem 4.4, assuming a pilot reuse factor $f = 1$, such that every pilot is reused in every cell.

   (b) Compare the expression with the corresponding expression for MR in Eq. (4.19). Explain the similarities and differences. Identify (extreme) conditions on M_j and K_j such that either MR or ZF are preferable.

   Hint: Properties of Wishart matrices are useful when computing the expectations.

4. The ZF processing scheme defined in Eq. (4.10) is designed to only cancel interference between the UEs in the same cell. When there is strong inter-cell interference, it can be desirable to also cancel interference from UEs in other cells.
   (a) Construct a multi-cell ZF receive combining scheme for BS j that cancels interference from all UEs in the entire network, assuming correlated Rayleigh fading channels with MMSE channel estimation.
   (b) What is the minimum number of antennas that BS j needs in order to apply the combining scheme developed in (a)?
   (c) If we consider uncorrelated Rayleigh fading channels, then some of the MMSE channel estimates that BS j obtains might be parallel vectors. Define a multi-cell ZF receive combining scheme for this special case. What is the minimum number of BS antennas in this case?

5. Consider a single-cell setup with two UEs having the channels $\mathbf{h}_1 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_1)$ and $\mathbf{h}_2 \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R}_2)$, respectively. These UEs share a pilot sequence such that the processed received pilot signal is

$$\mathbf{y}^p = \tau_p \mathbf{h}_1 + \tau_p \mathbf{h}_2 + \mathbf{n}^p$$

   where $\mathbf{n}^p \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \tau_p \mathbf{I}_M)$ is independent noise. In the UL data transmission, the received signal is

$$\mathbf{y} = \mathbf{h}_1 s_1 + \mathbf{h}_2 s_2 + \mathbf{n}$$

   where $s_1 \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ and $s_2 \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ are the independent data signals and $\mathbf{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M)$ is the receiver noise.
   (a) Suppose $\mathbf{v}_1 = \mathbf{y}^p$ is used as receive combining vector for user 1. Compute a UL SE for user 1 using Theorem 4.4.
   (b) Suppose $\mathbf{v}_1 = \mathbf{A}_1 \mathbf{y}^p$ is used as receive combining vector for user 1, where $\mathbf{A}_1$ is an arbitary deterministic matrix. Compute an SE for user 1 using Theorem 4.4.
   (c) Identify the coherent and non-coherent interference terms in the expressions from (a) and (b), assuming that $\mathrm{tr}(\mathbf{R}_i)$ is proportional to M and $\|\mathbf{R}_i\|_2$ is constant, for $i = 1, 2$.
   (d) Consider the desired signal term and coherent interference term from (b). Assume

$$\mathbf{R}_1 = \mathrm{diag}(\beta_1^{(1)} \mathbf{I}_{M/2}, \beta_1^{(2)} \mathbf{I}_{M/2}), \quad \mathbf{R}_2 = \mathrm{diag}(\beta_2^{(1)} \mathbf{I}_{M/2}, \beta_2^{(2)} \mathbf{I}_{M/2}).$$

   Find $\mathbf{A}_1$ such that the coherent interference term vanishes but the desired signal term is non-zero. Is in any additional condition on $\mathbf{R}_1$ and $\mathbf{R}_2$ required for this to hold?

6. Assume UE k in cell j knowns all the channels and precoding vectors perfectly.
   (a) Compute a lower bound on the DL ergodic channel capacity using Corollary 1.3.
   (b) What happens with the DL ergodic capacity bound in Theorem 4.9 as $\tau_c \to \infty$, assuming that $\tau_d = \tau_c - \tau_p$ and τ_p is constant.
   (c) Compare the bounds in (a) and (b) in the asymptotic regime where $\tau_c \to \infty$. Which bound provides the largest values?

7. Consider a single-cell single-user scenario where the UE has perfect CSI, but the BS lacks CSI. The received signal during DL data transmission is

$$y = \mathbf{h}^{\mathrm{H}} \mathbf{w} \varsigma + n$$

   where $\mathbf{h} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{R})$ is the fading channel, $\varsigma \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ is the data signal, and $n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ is noise.
   (a) Compute an expression for the DL ergodic channel capacity using Corollary 1.2, assuming that $\mathbf{w}$ is an arbitrary deterministic vector.
   (b) Compute a closed-form expression of the bound in (a).
   (c) Find a unit-norm precoding vector that maximizes the expressions from (a) and (b).
   (d) Is the answer from (c) unique? What happens in the special case of $\mathbf{R} = \beta \mathbf{I}_M$?