Answers

November 20, 2022

1 Part A

Question 1

$$y_t = \beta Y_t + \gamma X_t + u_t \tag{1}$$

$$Y_t = \delta_x X_t + \delta_Z Z_t + v_t \tag{2}$$

Substitute (2) into (1), we have

$$\begin{aligned} y_t &= \beta(\delta_x X_t + \delta_Z Z_t + v_t) + \gamma X_t + u_t \\ y_t &= (\beta + \delta_Z) Z_t + (\beta\delta_X + \gamma) X_t + +\beta v_t + u_t \\ y_t &= \pi_1 X_t + \pi_2 Z_t + +\beta v_t + u_t \end{aligned} \tag{3}$$

$$\pi_2 \equiv \beta\delta_Z$$

Hence, the the IV coefficient is the constant proportionality between reduced form coefficient and the first stage parameter.

Let

$$\epsilon_t = \beta v_t + u_t$$

Then it can be shown that

$$E(\epsilon_t) = 0; Var(\epsilon_t) = \beta^2\sigma_v^2 + \sigma_u^2 + 2\beta\sigma_{uv}$$

where

$$\sigma_V^2 \equiv Var(v_t); \sigma_u^2 \equiv Var(u_t); \sigma_{uv} \equiv Cov(u_t, v_t)$$

Question 2

Consider the following matrices:

$$X = \begin{pmatrix} X_1 & Z_1 & 0 & 0 \\ X_2 & Z_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ X_n & Z_n & 0 & 0 \\ 0 & 0 & X_1 & Z_1 \\ 0 & 0 & X_2 & Z_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & X_n & Z_n \end{pmatrix}$$

$$Y = (y_1, ..., y_n; Y_1, ..., Y_n)'$$

$$e = (u_1, ..., u_n; v_1, ..., v_n)'$$

$$\Pi = (\pi_1, \pi_2, \delta_x, \delta_z)'$$

Then the reduced-form equation can be written as

$$Y = X\Pi + e \tag{4}$$

Then the OLS estimator for Π is

$$\widehat{\Pi} = (X'X)^{-1}X'Y$$

By substituting Y into $\widehat{\Pi}$, we have

$$\widehat{\Pi} = (X'X)^{-1}X'(X\Pi + e) = \Pi + (X'X)^{-1}X'e$$

It can be shown that

$$E(\widehat{\Pi} - \Pi) = 0$$

$$E(\widehat{\Pi} - \Pi)(\widehat{\Pi} - \Pi)' = (X'X)^{-1}X'E(ee')X(X'X)^{-1} = (X'X)^{-1}X'\Omega X(X'X)^{-1}$$

where $\Omega = E(ee')$.

Question 3

The 2SLS estimator for β can be obtained by regression Y on $\widehat{Y}$ and X, where $\widehat{Y}$ is the predicted value from equation (2). Let $\tilde{\beta}$ be the 2SLS estimator for β, Then, the confidence set for

$$\tilde{\beta} \pm z_{0.5\alpha} se(\tilde{\beta}),$$

where z_τ is the τth percentile from the standard normal distribution.

Question 4

Let $\widehat{\beta} = \frac{\widehat{\pi}_2}{\widehat{\delta}_z}$ and the variance-covariance matrix of $(\widehat{\pi}_2, \widehat{\delta}_z)$ be denoted as $\widehat{\Sigma}$. Consider $H_0 : \pi_2 - \delta_0 \delta_z = 0$. The t-statistic is written as

$$t = \frac{\widehat{\pi}_2 - \delta_0 \widehat{\delta}_z)}{s_1 + \delta_0^2 s_2 - 2\delta_0 s_{12}}$$

where $vech(\widehat{\Sigma}) = (s_1, s_{12}, s_2)'$.

The confidence set for β based on Fieller method is written as

$$FCS(\delta; \alpha) = \{\delta_0 : (\widehat{\pi}_2 - \delta_0 \widehat{\delta}_z)^2 \leq z_{0.5\alpha}^2 (s_1 + \delta_0^2 s_2 - 2\delta_0 s_{12})\}.$$

This requires solving the following for δ_0:

$$A\delta_0^2 + 2B\delta_0 + C \leq 0,$$

where $A = \widehat{\delta}_z^2 - z_{0.5\alpha}^2 s_2$, $B = -\widehat{\pi}_2 \widehat{\delta}_z + z_{0.5\alpha}^2 s_{12}$, and $C = \widehat{\pi}_2^2 - z_{0.5\alpha}^2 s_1$. The roots should be

$$\delta_{01} = \frac{-B - \sqrt{\Delta}}{A}$$

$$\delta_{02} = \frac{-B + \sqrt{\Delta}}{A}$$

where $\Delta = B^2 - AC \neq 0$. Hence,

$$FCS(\delta; \alpha) = [\delta_{01}, \delta_{02}] \text{if} A > 0$$

$$FCS(\delta; \alpha) = [-\infty, \delta_{01}] \cup [\delta_{02}, \infty] \text{if} A < 0$$