Vysoké učení technické v Brně
Fakulta informačních technologií

projekt do predmetu Grafové algoritmy

Téma 40 - Rešerše - Max-Flow Min-Cut teorém

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Abstrakt SK

Táto práca sa zaoberá využitím Max-Flow Min-Cut teorému v riešení problému existenicie uspokojíveho toku vo neorientovanom grafe s viacerímy komoditami. Podrobne je analyzovaný dokument [1], ktorý vylepšuje hranice pomeru hodnôt minimálneho rezu a maximálneho toku na $O(log\ k^2)$ pre všeobecné grafy a na $O(log\ k)$ pre planárne. V dokumente je definovaný mutikomoditný problém, význam pomeru minimálneho rezu ku maximálnemu toku a bude uvedený k-aproximačný algoritmus k zisteniu tohoto pomeru. Takmer všetky podklady pre prácu sú prevzaté z dokumentu [1] a táto práca je mienená ako doplnkový materiál k štúdií tohoto dokumentu.

Abstrakt CZ

Tato prace se zaoberá využitím Max-Flow Min-Cut teoremu v řešení problému existenice uspokojíveho toku v neorientovanom grafe s vícero komoditami. Podrobně je analyzován dokument [1], ktorý zlepšuje hřanice pomeru hodnot minimálního řezu a maximálního toku na $O(log\ k^2)$ pro obecné grafy a na $O(log\ k)$ pre planárny. V dokumente je definován mutikomoditný problém, význam poměru minimálneho řezu ku maximálnímu toku a bude uvedený k-aproximačný algoritmus k zisteniu tohoto pomeru. Takmer všechny podklady pro prácu sú prevzatý z dokumentu [1] a tato práce je mínen jako doplnkovej materiál k studií tohoto dokumentu.

Abstract EN

This work regards use of Max-Flow Min-Cut theorem in a problem solving of existence of feasible flow in unoriented graphs with multiple comodities. In detail is analyzed document [1], what improves, by that time the best known, bounds of a ratio of minimal cut to maximal flow to $O(log\ k^2)$ for general graphs and to $O(log\ k)$ for planar. In this document is defined muticomodity flow problem, meaning of a ratio of minimal cut to maximal flow and will be presented k-approximation algorithm for finding this ratio. Almost all material for the work were taken from document [1] and this work is intended to by used as additional material for the studies of this document.

1 Set of informal definitions

- Cut Capacity denoted $u(U)$ - sum of the cut-edge capacities
- Flow value of a cut denoted $f(U)$ - sum of the cut-edge flow values
- Feasible flow problem - does exist feasible flow in a given network G such that it satisfies all demands and does not violate capacity constraints?
- Multicommodity (feasible) flow problem - feasible flow problem with multiple comodities whoose count will be denoted k.
- Concurrent flow problem - aka. max flow problem, determine maximum precentage z^* such that at least z^* percent of each demand is shipped without violating capacity constraints
- Equivalnet problem to the concurrent flow problem is determining minimum ration λ, such that 100% of each demand are shipped withou violating capacity constraints, on a network with edge capacities $\lambda * u(e)$. It is clear that $z * \lambda = 1$
- Congestion of network - minimal ratio λ such that 100% of each demands is shipped in network with edge e capacities $\lambda \;*\; u(e)$ aka. every edge has capacity at least λ times larger.
- Minimum cut ratio aka. min cut denoted S^*- minimal ratio of a cut capacity to cut demand(over all cuts in network).
- Minimum cut maximum flow ratio - minimal ratio of min cut S^* to it's maximal flow value z(so the feasible flow exist).
- Approximation algorithm - an algorithm that may not find optimal solution for an optimalization problem, eg to estimate some value val.
- K-aproximation algorithm - an algorithm that is guaranteed to find solution to an optimalization problem that is at least equal to k times larger the optimal value in case we search for minimal value of a problem and smaller in case we search for it's maximal value.

2 Definitions and notation

Notation is taken from the mentioned document, the only changes that were made to the notation of S, z that are reffered in the source document as S^*, z^*. The meaning of the star may be in the minimlization of the value S and maximalization of the value z.

- Flow f_i of comodity i from s_i to t_i is collection of paths P_i from s_i to t_i, which are associated with real values(the path flow value). Let $f(P)$ be nonnegative for every $P \in P_i$. Value of a flow for that comodity is then $\sum_{P \in P_i} f_i(P)$. which is the total flow delivered from s_i to t_i. Total flow delivered from s_i to t_i accross edge (u, v) is:

$$f(u, v) = \sum \{f_i(P) | P \in P_i \; and \; (u, v) \in P\} \tag{1}$$

- Feasible multicommodity flow f consits of f_i for all comodities, while it does not violate edge capacity constraints, thus $f(u, v) \leq u(u, v)$, while $f(u, v) = \sum_{i=1}^{k} f_i(u, v)$.
- Maximum (feasible) multicommodity flow is percentage z of all demands.
- Cut Capacity $u(U) = \sum_{u \in U, v \in \overline{U}} u(u, v)$

- Demand accross the cut $d(U) = \sum_{i:|\{s_i,t_i\}\cap U|=1} d_i$
- Minimum cut ratio $S = min_{U\subset V} \frac{u(U)}{d(U)}$
- Minimum cut Maximum flow ratio S/z

3 Results of previous works

Klein, Rao, Agrawal and Rave showed that if demands and capacities are integral the min-cut max-flow ration is upper-bounded by $O(log\ C\ log\ D)$ where C denotes sum of all capacities and D denotes sum of all demands. Tragouda has improved this ratio to $O(log\ n\ log\ D)$. Garg, Vazirani and Yannnakakis has improved it to $O(log\ k\ log\ D)$. Klein, Plotkin and Rao proved that for planar graphs the ration is bounded by $O(log\ D)$.

4 Improved bounds of a min-cut max-flow ratio proof

This proof makes an assumption about input network, that is the sum of all demands D is upper bounded by a small-degree polynomial in k.

The previous works had results like $S/z = Y * O(log\ D)$, while D is sum of all demands. The $O(log\ D)$ factor can be transformed to $O(log\ \hat{D})$ while $\hat{D} = \sum_{i=1}^{k} d_i/d_{min}$ without assumption that demands are integral. In the following text we will denote variables after the rounding with apostrophe.

Proof of the previous statement can be achieved by rounding all demands to some integral multiple of d_{min}. Rounding demands cannot increase z, because along with increasing demands the value of z in the network remains at most as large as it was before rounding. Because when only demands are increasing then there cannot be larger percentage of a demands satisfyed than it was before rounding.

That rounding also changes value of S in network. It is clear that $S \leq 2S'$, because in the equation $S = min_{U\subset V} u(U)/d(U)$, the value of $d(U)$ may increase at most by a factor 2.(e.g. $d_{min} = 0.5$ and $d_{max} = 0.50001$ then $d'_{max} = 1.0$).

In a rounded problem we can state that sum of all demands is at most $\hat{D} + k \leq 2\hat{D}$, while the inequality implies from a mentioned assumpotion that sum of all demands is upper-bounded by a small-degree polynomial in k.

With use of results of Klein, Rao, Agrawal and Ravi $S/z = O(\log\ C\ log\ D)$ and $S \leq u(U)/d(U)$we get the following inequalities(note that this step is unclear for author of this document, clearly the inequality holds with an assumption that $S/z = O(log\ k\ log\ D)$, but it is not clear how from $S/z = O(\log\ C\ log\ D)$ we get member based on $log\ k$):

$$S' \leq S \leq 2S' \leq O(z'\ log\ k\ log(2\hat{D})) \leq O(z\ log\ k\ log\ \hat{D}) \quad (2)$$

With use of an assumption mentioned that is D is upper-bounded some small polynomial of k we get $d_{max}/d_{min} = O(k)$. So $\hat{D} = O(d_{min} + ... + d_{min} * k) = O(k)$. So for general networks and for planar ones is proven that:

$$S/z = O(log^2\ k) \quad S/z = O(log\ k) \quad (3)$$

This proofs the main result that is $S/z = O(log\ k\ log\ \hat{D})$. In the case of planar networks there is only the $log\ k$ multiplier missing from the same inequalities that were mentioned above.
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5 K-aproximation algorithm for finding min-cut max-flow ratio proof

Now we take all demands and distribute them by they value into groups Q_i with the demand lying in some interval. Lets say that $Q_i = \{d|(4k)^{i-1}d_{min} \leq d \leq (4k)^i d_{min}\}$ and let z_i, λ_i and S_i denote known variables while the underscore i represents the group Q_i.

Note that $\lambda_i = 1/z_i$, $max_i\lambda_i \leq \lambda$ and $min_iS_i \geq S$. The first applies for every graph. The second implies from that in a subgraph of the group Q_i may only decrease the congestion. The last one implies from that in a subgraph of the group Q_i the demand may only decrease and capacity remains the same so S_i may only increase its value.

Next step is to prove that $S_i \leq 2.25S$.

Let the $U \subset V$ define the cut that achieves ratio S, i.e. the most congested cut in the whole network. Let $d_j(U)$ denote the demand in group j and let i be the maximum index such that $d_i(U)$ is non-zero. Total demand of cut is then:

$$d(U) = d_i(U) + d_{i-1} + \sum_{j<i-1} d_j(U) \tag{4}$$

Since $d_i \leq 4d_j$ following inequality holds:

$$d(U) \leq \frac{5}{4}d_i(U) + d_{i-1} \tag{5}$$

So the min cut ratio can be written as:

$$S = \frac{u(U)}{d(U)} \geq \frac{u(U)}{(5/4)d_i(U) + d_{i-1}(U)} \tag{6}$$

$$\frac{u(U)}{(5/4)d_i(U) + d_{i-1}(U)} = \frac{4u(U)}{5d_i(U) + 4d_{i-1}(U)} = const. \tag{7}$$

$$const. \geq \begin{matrix} \frac{4}{9}\frac{u(U)}{d_i(U)} & \textbf{if } d_i \leq d_{i-1} \\ \frac{4}{9}\frac{u(U)}{d_{i-1}(U)} & \textbf{else} \end{matrix} \geq \frac{4}{9}min(S_i, S_{i-1}) \tag{8}$$

$$2.25S \geq min_iS_i \tag{9}$$

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Next step of the proof is to show that time required to find approximate optimal flows for each of the subpproblems defined by the groups Q_i for $i = 1, 2, 3...$ is the same as approximately solving the original multicomodity flow problem.

Lets say that $\lambda \leq 4max_i\lambda_i$. Then we have to prove that $S/z = O(log^2k)$. From a mentioned assumption we get $z \geq (1/4)min_iz_i$. Since $\lambda_iz_i = 1$ and $max_i\lambda_i \leq \lambda$ then $min_iz_i \geq z$. When we use inequality that clearly holds $S \leq min_iS_i$ we get:

$$S \leq min_iS_i \leq O(log^2k) * min_iz_i \leq O(log^2k) * z. \tag{10}$$

Next step is to prove following theorem.

Consider two multicomodity flow problems over the same network, where the first problem has to ship commodities of group Q and second in $Q^{'}$, denote their feasible flows $f, f^{'}$. Then if $4 * Demand(Q^{'}) \geq Demand(Q)$, then there exists feasible flow that satisfies all demands in Q and at least half of demands in $Q^{'}$.

The idea of a proof is as follows, we will regard to an edge e with capacity $u(e)$ as a collection of $u(e)$ edge-disjoint paths in the network. So flows $f, f^{'}$ is the collection of the edge-disjonint paths. Note that we allow some commodity from $Q^{'}$ to use some edge capacity that is also used by a some commodity from Q. This sitiation will be called *collision*. An edge on a both commodities paths will be called *collision edge*. Note that in general network and with such problem transformation we have to expect that there are multiple collisions of a commodity in Q.

Then we delete some small number of paths from $Q^{'}$ so we can reroute the flow of commodities in Q. Elimination of collision will be performed one at a time so this will give us a pseudopolynomial algorithm. Note that the deletion may be done in a arbitraty order, while we keep informations about the path deletion order and path edges that remain in the modified path from f of a commodity j from problem commodity set Q.

For the case of single commodity j we iteratively delete all collisions, by deleting edges on the paths between first and last collision edge with some path $P_{collision}$ of flow $f^{'}$ of a commodity j from $Q^{'}$. Note there may be multiple collision paths from $Q^{'}$ with collision of path from Q. We have to consider them in some arbitrary but fixed order also.

Let's assume that there is at least one more collision edge on the path between starting and ending collision edge and lets denote the set of collision edges with a commodity j as F_j on a path $P_{collision}$. By deleting the part of the path we get two subpaths that will be reffered as P_1, P_2.

At this point the analyzed document states that since there is a finite number of collisions with paths from Q with paths from $Q^{'}$, and since we are having correct method to delete the collision with paths of comodities we may do it. But it also states that every modified flow path by deletion is first on at most two flow paths of a commodity j from $Q^{'}$. The meaning of the first collision path of some path P of commodity from Q with some path $P^{'}$ of commodity j from $Q^{'}$ is that the collision paths are at most three different paths for j. And there are at least three collision edges. So the first collision path is the that contians the first collision edge from P.

Since every modified flow path by deletion is first on at most two flow paths of a commodity j from $Q^{'}$ so the total flow carried by the flow paths of commodity j that were removed is at most twice the total flow of all the commodities in Q. From the assumption that Q' has demand at least four times the total demand of all commodities in Q we get that the amount of removed flow for a commodity j from $Q^{'}$ is not larger than $d_j/2$ and hence the remaining(unremoved) paths of j carry at least $d_j/2$ amount of flow. i.e. they satisfy at least half of a demand of a commodity j.

It remains to show how the rerouting of modified paths P, that are splitted to subpaths P_1, P_2, for commodity from Q may be performed. We use the fact that paths from $Q^{'}$ for commodity j that are have non-zero intersection with paths P_1, P_2 and contains the first collision edges are free to be utilized by a broken path P, since collision edges are first not only on path P, but also on paths $P_1^{'}, P_2^{'}$. Since $P_1^{'}, P_2^{'}$ are also edge disjoint we get the main result and that is:

Consider two multicomodity flow problems over the same network, where the first problem has to ship commodities of group Q and second in $Q^{'}$, denote their feasible flows $f, f^{'}$. Then if $4 * Demand(Q^{'}) \geq Demand(Q)$, then there exists feasible flow that satisfies all demands in Q and at least half of demands in $Q^{'}$

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Now it only remians to prove the stated theorem $\lambda \leq 4 * max_i \lambda_i$

To prove the prevoius, we use another partition method on the set of all comodities. Now we consider commodities from Q_i(Note that in this proof there is not defined $Q_i^{'}$ by now, but Q_i represent the set of commodities that are having demand within some interval based on i). Lets consider only even indices i. For every group Q_i we have multicommodity flow f_i satisfying the capacity constraints $\lambda u(u)$ on every edge e. The document states that there exists multicomodity flow f_{even} that satisfies the capacity constriants $\lambda u(e)$ on every edge e and satisfies at least half of the demand of every commodity in groups Q_i for even i.

The following proof is based on the results of the prevoius proof. At first we show that for two such groups Q_1, Q_2 the claim is true.

Next we prove that the claim holds even for larger number of groups Q_i. In this step we apply previously prooven lemma by the subtitution $Q = Q_2 \cup ... \cup Q_{i-2}$ and $Q^{'} = Q_i$. Inductively, assume that there exist flow f that satisfies at least half of demand of each commodity in Q. Application of the result of the lemma on flows f, f_i implies that there exists a flow that satisfies all the demands that were satisfied by f and at least half of each demand in $Q^{'}$.

With application of the simmilar construction on the sets Q_i for odd i we conclude that there exist flows f_{even}, f_{odd} such that together they satisfy at least half of each demand. Moreover both flows separately satisfy capacity constriants $\lambda * u(e)$. Therefore, there exists a flow that satisfies all of the demands and satisfies capacity constraints $4 * \lambda * u(e)$

This proves that $\lambda \leq 4 * max_i \lambda_i$.

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Conclusion EN

In this paper was presented work of Serge A. Plotkin and Éva Tardos taken from [1]. This document should be used for learning purposes for anyone having enthusiasm for the graph theory. Author of this paper, by creating it, improved his skills in understanding one of the hardest fields of the graph teory, the multicomodity flow problems which are considered to be NP-complete.

Závěr CZ

V tomto dokumente bol prezentována práce Sergeje A. Plotkina a Évy Tardosove převzatej z [1]. Tento dokument by měl sloužit k učení pro kohokoli, kto má entuziazmus pro teoriu grafů. Autor tohoto dokumentu, tým že ho vytvořil, si zlepšil schopnosti v chápani v jedné z nejobtížnejšých oblasti teorie grafů, problému hledání toků v grafoch s vícero komoditami, tyto problémy patří do třídy složitosti NP úplné.

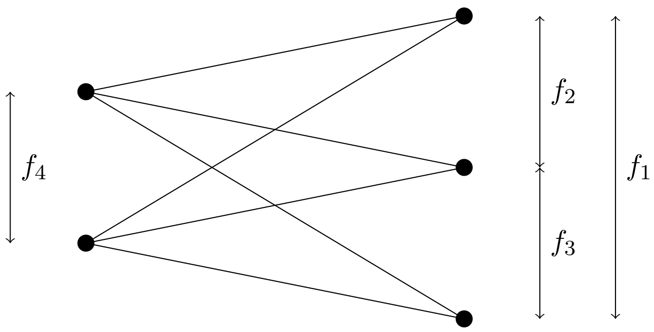
Záver SK

V tomto dokumente bola prezentovaná práca Sergeja A. Plotkina a Évy Tardosovej prevzatej z [1]. Tento dokument by mal slúžiť k učeniu pre kohokoľvek, kto má entuziazmus pre teóriu grafov. Autor tohoto dokumentu, tým že ho vytvoril, si zlepšil schopnosti v chápani v jednej z najťžšých oblastí teórie grafov, problému hľadania toku v grafoch s viacero komoditami, tieto problémy patria do triedy zložitosti NP úplné.

Literatúra

[1] Plotkin, S. A.; Tardos, E.: Improved Bounds on the Max-flow Min-cut Ratio for Multicommodity Flows. In *Proceedings of the Twenty-fifth Annual ACM Symposium on Theory of Computing*, STOC '93, New York, NY, USA: ACM, 1993, ISBN 0-89791-591-7, s. 691–697, doi:10.1145/167088.167263.
URL `http://doi.acm.org/10.1145/167088.167263`

A Example(s)



Obr. 1: $K_{2,3}$ with uniform demands and uniform capacaties does not have feasible maximum flow in integral space.