Migration and inversion of seismic data

R. H. Stolt* and A. B. Weglein‡

ABSTRACT

Seismic migration and inversion describe a class of closely related processes sharing common objectives and underlying physical principles. These processes range in complexity from the simple NMO-stack to the complex, iterative, multidimensional, prestack, nonlinear inversion used in the elastic seismic case.

By making use of amplitudes versus offset, it is, in principle, possible to determine the three elastic parameters from compressional data. NMO-stack can be modified to solve for these parameters, as can prestack migration.

Linearized, wave-equation inversion does not inordinately increase the complexity of data processing. The principal part of a migration-inversion algorithm is the migration.

Practical difficulties are considerable, including both correctable and intrinsic limitations in data quality, limitations in current algorithms (which we hope are correctable), and correctable (or perhaps intrinsic) limitations in computer power.

INTRODUCTION

The words "migration" and "inversion" are, in fact, used to describe a number of related processes. To some people, seismic migration is strictly an imaging process. To others, seismic data already form an image which needs only to be massaged, or "migrated," into its proper location. To still others, migration is an inversion process which derives two-dimensional (or three-dimensional) maps of local reflectivity from seismic data.

Notions of seismic inversion are even more diverse, mainly because several inverse problems (and several approaches to solve them) can be formulated for seismic data. Traveltime information can be inverted for seismic velocities using "velocity analysis" or, for near-surface variations using "statics." A single stacked common-midpoint (CMP) trace forms an estimate of local reflectivity which, when combined with low-frequency velocity information (from well logs or velocity analysis), can be time-integrated to form an impedance esti-

mate. This process is sometimes referred to as inversion (Lindseth, 1979). More rigorous (and we hope, more realistic inversion) methods based on wave-equation analysis can also extract information from traveltimes and reflectivities.

The exact one-dimensional (1-D) acoustic inverse problem for plane waves at normal incidence has been vigorously attacked, solving, at least on paper, for absolute values of acoustic impedance (e.g., Goupillaud, 1961; Ware and Aki, 1969; Berryman and Greene, 1980; Howard, 1983). Plane waves at nonnormal incidence have also been addressed with individual solutions for velocity and density (e.g., Stolt and Jacobs, 1980; Coen, 1981 and 1982; Hooshyar and Razavy, 1983). The inverse acoustic linearized problem has been solved for point sources and receivers (Raz, 1981a, b). The inverse problem for elastic waves has also been approached, both by linearized approximation (Clayton, 1981) and through iterative modeling (McAulay, 1985). Some of the point-source methods (e.g., McAulay, 1985) use traveltime as well as reflectivity information. Others (e.g., Clayton, 1981) assume traveltimes are known a priori and analyze only reflectivity information. One-dimensional processing (based on the wave equation) of CMP gathers can, on the one hand, be viewed as a restriction of the general migration or inversion problem and, on the other hand, as an extension of conventional processing and single-trace inversion methods. The relevance of all these solutions to the real seismic problem has not always been clear. The real Earth is certainly not acoustic, nor exactly elastic, and it certainly is not exactly one-dimensional.

Considerable research has been concentrated on multidimensional wave-equation inversion, beginning with Cohen and Bleistein (1977). Mostly, the treatment has been in the linear approximation, ignoring transmission losses and multiple reflections, though approaches to full iterative solutions have appeared in the literature (Tarantola, 1984). Approaches to both many-offset (Clayton and Stolt, 1981; Raz, 1981a, b) and zero-offset (Bleistein and Cohen, 1979 and 1982) data have been considered.

Zero-offset inversion relates easily to poststack migration. Multidimensional migration and wave-equation inversion differ more in emphasis than in substance (migration is preoccupied with propagation, inversion with reflection or scattering), and approach essentially the same problem from

slightly different points of view. Papers describing the wave-equation migration/wave-equation inversion relationship include Phinney and Frazer (1978), Bleistein and Cohen (1982), Clayton and Stolt (1981), Weglein (1982a, b), Berkhout (1984), and Cheng and Coen (1984).

Many-offset inversion relates to prestack migration and combines elements of 1-D prestack inversion and poststack migration or inversion (Weglein and Wolf, 1985; Stolt and Benson, 1985). Indeed, wave-equation prestack migration can be considered a form of prestack inversion which solves for only one of the three elastic parameters.

A comprehensive treatment of the many forms of migration and inversion is far beyond the scope of this paper. We only suggest that wave-equation inversion provides a unifying conceptual framework in which other approaches may be understood and also provides the potential of moving substantially beyond present-day capabilities of seismic processing and analysis. Table 1 lists methods of seismic processing and analysis and the wave-equation inversion procedure to which each corresponds or from which it can be derived.

Here we first describe a wave-equation linearized inverse for the three elastic parameters in a vertically changing earth. We then discuss realistic objectives for the inversion process, address migration and multidimensional linearized inversion, and briefly discuss nonlinear methods.

THE ONE-DIMENSIONAL INVERSE PROBLEM

There is a close relationship between CMP NMO-stack processing and 1-D migration and inversion. CMP acquisition and processing rely on a ray-theoretical model, whereas migration and inversion tend to appeal directly to the wave equation. However, for frequencies and distances germane to seismic wave propagation, the geometrical (ray-theoretical) limit normally applies. This ensures at least a broad consistency between NMO-stack and conceptually deeper wave-equation inversion processes.

We can easily note the essential similarity between NMO-stack and a prestack Kirchhoff migration for a horizontally layered earth. It is also well-known that NMO-stack can be duplicated by an f-k dipless prestack migration (which is just NMO-stack in f-k space). Thus NMO-stack is, in its sphere, a legitimate process and is remarkably robust. In dealing with stacked data, even gross oversimplification often leads to useful results. For some applications, we can treat a stacked seismic trace as the result of a plane-wave experiment. This use (or abuse) of an obviously incorrect physical model is closely related to use of migration's exploding reflector model. It works well for primary reflections, provided locations are of more interest than amplitudes. Even amplitude problems can, to some extent, be patched up through minor modifications to the data. The final breakdown of the plane-wave model occurs when we try to incorporate multiple reflections. For that, the stacked seismic trace simply cannot be considered a plane wave.

Here, we lay out a simple physical model for primary reflections of compressional waves in an elastic, layered earth. We begin with the wave-theoretical model of seismic inversion and relate it to the conventional ray-theoretical model. The result is an inversion algorithm which amounts to a moveout correction and a weighted CMP stack. In principle, we can solve for each of the three elastic parameters (density, P veloc-

ity, and S velocity) by varying the weighting coefficients in the stack. Since this model ignores such niceties as multiple reflections and converted waves, it can only be viewed as incomplete. The relevance to exploration geophysics of an inverse to the 1-D elastic problem may be debatable, but it is not likely that the real Earth problem will be solved before this one.

The plane-wave reflection response

We begin with a simple stratified earth model characterized by the three elastic parameters:

$$\alpha = \text{compressional velocity},$$

$$\beta = \text{shear velocity, and}$$

$$\rho = \text{density}.$$

We pass a plane compressional wave down through this model earth (assuming perfect transmission), collect the total upgoing compressional primary reflection response, and return it to the surface. Although our ultimate interest is in localized sources and receivers, we begin with plane waves, confident (e.g., Treitel et al., 1982) that once we have one solution, we can find the other. It may seem a little odd (not to mention unnecessary) to include shear velocity in this model when our stated interest is in the compressional waves. In fact, the three elastic parameters constitute the minimum amount of realism necessary to describe the angular dependence of the reflection coefficients. The shear velocity is unnecessary to describe the transmission process, because we neglect compressional-to-shear conversion in the transmitted wave.

The appropriate downgoing wave is described by the WKBJ approximation (Morse and Feshbach, 1953). Consider a plane wave P of frequency ω and ray parameter

$$p = \sin\,\theta(z)/\alpha(z), \tag{1}$$

where $\theta(z)$ is the angle the wave makes with the vertical at depth z. In the WKBJ approximation, this wave has the form

$$P(p, z, \omega) = P(p, 0, \omega)\sqrt{\frac{\rho(z)\alpha(z)\cos\theta(0)}{\rho(0)\alpha(0)\cos\theta(z)}}$$

$$\times \exp\left[i\omega\int_0^z \frac{dz'}{\alpha(z')}\cos\theta(z')\right]. \tag{2}$$

The WKBJ approximation is the solution to the acoustic wave equation

$$\left(\rho\,\frac{d}{dz}\,\frac{1}{\rho}\,\frac{d}{dz} + \frac{\omega^2}{\alpha^2}\cos^2\theta\right)P = 0$$

in the limit of high frequencies. At lower frequencies, it neglects reflected energy, converted energy, and transmission losses. The z dependence of P is contained in both the rapidly varying phase term and the relatively slowly varying amplitude term. The more important phase term depends upon velocity α, but not density ρ. The amplitude term depends upon both. It does not compensate for transmission losses but, rather, conserves energy as acoustic properties change.

If we were to change variables from depth to traveltime via

$$\tau = \int_0^z \frac{dz'}{\alpha(z')}\cos\theta(z'), \tag{3}$$

and from P to ϕ, where

$$\phi = \eta P, \tag{4}$$

with

$$\eta = \sqrt{\frac{\cos\theta}{\rho\alpha}}, \tag{5}$$

then in the new coordinate system the WKBJ approximation would become the simple plane wave

$$\phi(p, \tau, \omega) = \phi(p, 0, \omega)e^{i\omega\tau}. \tag{6}$$

The wave equation in this coordinate system is (Jacobs and Stolt, 1980; Coen, 1981)

$$\left[\frac{\partial^2}{\partial\tau^2} + \omega^2 - V(\tau)\right]\phi(p, \tau, \omega) = 0, \tag{7}$$

where

$$V(\tau) = \eta^{-1}\partial^2\eta/\partial\tau^2. \tag{8}$$

In this coordinate system the WKBJ approximation amounts to neglect of the potential term $V(\tau)$. It is thus a first approximation in a Born-series expansion of the exact solution to the 1-D problem.

Having accepted the WKBJ approximation as an adequate description of transmission, we now need a reflection model.

Table 1. Aspects of the inverse seismic problem.

	Seismic processing and analysis methods	Wave equation inversion
One-dimensional	Traveltime velocity analysis	One-dimensional inversion with offset in the geometric acoustics limit (WKBJ)
	CMP stacked trace + Trace integration	Plane-wave, normal-incident, single-parameter linearized inversion
	Goupillaud equal traveltime inversion	Plane-wave, normal-incident, single-parameter exact inversion
	Picking + Reflectivity estimates + Amplitude/offset	One-dimensional multi-parameter (acoustic or elastic) linearized inversion with offset data)
	Picking + Reflectivity estimates + Amplitude/offset	Exact one-dimensional multiparameter inversion with offset data
Two- and three-dimensional	Post- and prestack migration	Post- and prestack linear single-parameter inversion
	Depth migration	Variable-background linearized inversion
	Elastic migration Three-component migration	Elastic linear inversion to find reflectivity
	Depth migration with in-built velocity analysis and multiple removal	Nonlinear (exact) constant-density acoustic inversion
		Iterative, nonlinear, variable-density acoustic and elastic inversion

We divide the model earth into a large number of layers, each thin enough that the elastic parameters can be considered constant within each layer and that they will exhibit only small changes from one layer to the next. (This may require a much finer discretization than the data we eventually want to describe. The layers are not meant to be thought of as a discretization interval for processing, but as a conceptual tool.)

Depth to the bottom of the ℓth layer is z_ℓ, and thickness of the layer will be $\Delta z_\ell = z_\ell - z_{\ell-1}$. Elastic properties in this layer are α_ℓ, β_ℓ, and ρ_ℓ. At the bottom of the ℓth layer, the incident wave is

$$P_i(p, z_\ell, \omega) = P_i(p, 0, \omega)\sqrt{\frac{\rho_\ell \alpha_\ell \cos \theta_0}{\rho_0 \alpha_0 \cos \theta_\ell}}$$

$$\times \exp\left[i\omega \sum_{\ell' \leqslant \ell} \Delta z_{\ell'} \cos \theta_{\ell'}/\alpha_{\ell'}\right]. \quad (9)$$

The compressional wave reflected from the layer bottom is, at z_ℓ, just the incident wave in equation (9) multiplied by a reflection coefficient:

$$P_{r\ell}(p, z_\ell, \omega) = P_i(p, z_\ell, \omega)R_\ell(\theta_\ell)\Delta z_\ell. \quad (10)$$

The reflection coefficient per layer R_ℓ depends upon the angle of incidence θ_ℓ of the plane wave in the ℓth layer. This angle obeys

$$\sin \theta_\ell = p\alpha_\ell. \quad (11)$$

For the reflection coefficient R_ℓ, we invoke Chapter 5 of Aki and Richards (1979). The exact Zoeppritz reflection coefficient is rather involved and unnecessarily complicated for small changes between adjacent layers. Consequently, we borrow an approximate expression [Aki and Richards, equation (5.44)]

$$R_\ell(\theta_\ell) \simeq \frac{1}{2}\left(1 - 4\frac{\beta_\ell^2}{\alpha_\ell^2}\sin^2\theta_\ell\right)\frac{\Delta\rho_\ell}{\rho_\ell\Delta z_\ell} + \frac{1}{2}\sec^2\theta_\ell\frac{\Delta\alpha_\ell}{\alpha_\ell\Delta z_\ell}$$

$$- 4\frac{\beta_\ell^2}{\alpha_\ell^2}\sin^2\theta_\ell\frac{\Delta\beta_\ell}{\beta_\ell\Delta z_\ell}. \quad (12)$$

The main benefit of choosing the expression (12) over an exact reflection coefficient is that expression (12) is linear in fractional changes of the elastic parameters. This simplifies the expression not only conceptually [just looking at expression (12), we easily see the effect of individual parameter changes], but computationally (the result is not sensitive to errors in absolute amplitude). Validity of expression (12) requires not only that changes at the layer boundary be small, but also that θ_ℓ not approach critical angle.

Given an expression for R_ℓ, it remains to sum contributions from all layers and propagate them to the surface. The result is

$$P_r(p, 0, \omega) = P_i(p, 0, \omega)\sum_\ell \Delta z_\ell R_\ell(\theta_\ell)$$

$$\times \exp\left[2i\omega \sum_{\ell' \leqslant \ell} \Delta z_{\ell'} \cos \theta_{\ell'}/\alpha_{\ell'}\right]. \quad (13)$$

When the largest layer thickness Δz_ℓ is forced to zero, equation (13) becomes an integral

$$P_r(p, 0, \omega) = P_i(p, 0, \omega)\int_0^\infty dz\, R(z, \theta)$$

$$\times \exp\left[2i\omega \int_0^z dz'\, \cos \theta(z')/\alpha(z')\right]. \quad (14)$$

In equation (14), θ, z, and p obey the relation $\sin \theta = p\alpha(z)$, and

$$R(z, \theta) = \frac{1}{2}\left[\frac{d\ln\rho}{dz} + \sec^2\theta\frac{d\ln\alpha}{dz}\right.$$

$$\left. - 4\frac{\beta^2}{\alpha^2}\sin^2\theta\frac{d\ln(\rho\beta^2)}{dz}\right] \quad (15)$$

is the reflection coefficient per unit depth.

Note that the linearized reflection coefficient in this expression has become, in a sense, "exact." That is, for a continuously varying elastic earth, equation (15) is as close to correct as any expression which neglects transmission losses, conversions, multiple reflections, and so on. Though "exact," equation (15) is not unique. In an inhomogeneous earth, the notion of a primary reflection gets a little fuzzy, and so does equation (15).

Equation (14) models the primary reflection response to an incoming plane wave of frequency ω and direction $\theta = \sin^{-1}[p\alpha(z)]$. The response is a sum over depth of phase-shifted, angle-dependent reflection coefficients. The basic form should neither surprise nor dissatisfy. Indeed, if we look at the special case of normal incidence ($p = \theta = 0$), then equation (14) reduces to

$$P_r(0, 0, \omega) = P_i(0, 0, \omega)\int_0^\infty dz\,\frac{d\ln(\rho\alpha)}{2\,dz}$$

$$\times \exp\left[2i\omega \int_0^z dz'/\alpha(z')\right]. \quad (16)$$

That is, the reflected wave is a sum over all depths of the logarithmic derivative of the acoustic impedance, retarded by two-way traveltimes. Equation (14) and its restriction equation (16) to normal incidence are reasonable forms for describing primary reflections, and especially the inverse problem. Although the dc values of the elastic parameters are filtered out by the z-derivative, the band-limited far-field nature of seismic data prevents their direct recovery from reflectivity information anyway. The low-frequency information regarding trends requires data from sources other than reflectivity. The forms of equations (14) and (16) are naturally suited for recovery of the high-frequency information carried by the data.

Relating plane waves to a point source

An impulsive point source generates a waveform which is expressed as a superposition of plane waves. The exact form depends on many experimental parameters and surface boundary conditions. For illustration, we stick with the simplest possible case. Instead of a reflecting surface at $z = 0$, we extend near-surface earth properties to infinite height. For an impulsive point source of stength A located at $z = 0$, the component of the ensuing waveform with frequency ω and ray parameter p will then be

$$P_i(p, 0, \omega) = \frac{iA\alpha_0}{2\omega \cos \theta_0}, \quad (17)$$

where θ_0 is θ at $z = 0$. Thus the reflected wave at $z = 0$ will be [combining equation (17) with equation (14)]

$$P_r(p, 0, \omega) = \frac{iA\alpha_0}{2\omega \cos \theta_0} \int_0^\infty dz\ R(z, \theta)$$

$$\times \exp\left[2i\omega \int_0^z dz'\ \cos \theta(z')/\alpha(z')\right]. \quad (18)$$

The reflected impulse response in p-τ coordinates

Evaluation of integral equation (18) would be slow (an N^2 operation) and inversion for R in terms of P very time-consuming. The simple solution is a change of variables from z to traveltime τ. With τ as defined in equation (3) ($d\tau = dz \cos \theta/\alpha$), equation (18) becomes

$$P_r(p, 0, \omega) = \frac{iA\alpha_0}{2\omega\sqrt{1 - p^2\alpha_0^2}} \int_0^\infty d\tau\ R(\tau, p)e^{2i\omega\tau}, \quad (19)$$

where R, expressed as a function of τ and p, is

$$R(\tau, p) = \frac{1}{2}\left[\frac{d \ln \rho}{d\tau} + \frac{1}{1 - p^2\alpha^2}\frac{d \ln \alpha}{d\tau}\right.$$

$$\left. - 4\frac{\beta^2}{\alpha^2}p^2\alpha^2\frac{d \ln \rho\beta^2}{d\tau}\right]. \quad (20)$$

$R(\tau, p)$ (reflection coefficient per unit time) is actually $R(z, \theta)$ times $dz/d\tau$. Because no reflections occur when $\tau < 0$, the integral in equation (19) is extended to $-\infty$. It is then recognizable as a Fourier transform from τ to ω of $R(\tau, p)$:

$$P_r(p, 0, \omega) = \frac{iA\alpha_0}{2\omega\sqrt{1 - p^2\alpha_0^2}}R(\omega, p). \quad (21)$$

Equation (21) is the second term (the incident wave P_i being the first) in an expansion of the exact solution to the point-source reflection problem. For our purposes we use equation (21).

To solve equation (21) for $R(\tau, p)$ we only have to take the inverse Fourier transform. Then

$$R(\tau, p) = \frac{2 \cos \theta_0}{\alpha_0 A}\frac{\partial}{\partial\tau}P_r(p, 0, \tau). \quad (22)$$

The relationship

$$\tau(z) = \int_0^z dz'[1 - p^2\alpha^2(z')]^{1/2}/\alpha(z') \quad (23)$$

between τ and z is different for each p. Consequently, an expression for $R(z, \theta)$ would be more useful than equation (22). We write

$$R(z, \theta) = \frac{2 \cos \theta(z) \cos \theta(0)}{\alpha(z)\alpha(0)A}\frac{\partial}{\partial\tau}P_r(p, 0, \tau). \quad (24)$$

Equation (24) is a simple relation between the angularly dependent reflection coefficient $R(z, \theta)$ and the reflection response to a point source in the p-τ domain. This relation can be used to work the forward problem (given R, calculate the data) or the inverse problem (given the data, calculate R).

To work the inverse problem requires three steps.

1. Put the data in the p-τ domain and differentiate

with respect to τ. For a full 3-D surface experiment, we could use the 3-D Fourier transform $(x, y, t) \rightarrow (k_x, k_y, \omega)$. For a 1-D earth, the lateral spatial frequency dependence must be only on $k = \sqrt{k_x^2 + k_y^2}$. Changing variables from k to $p = k/\omega$, we multiply by $-i\omega$ and inverse Fourier transform from ω to τ. Alternatively, a 2-D slant stack puts the data directly in the p-τ domain.

2. Use equation (3) or (23) to change variables from τ to z and multiply by $2 \cos \theta(z) \cos \theta(0)$ to form $R(z, \theta)$. The variable change is a form of moveout correction for each constant-p trace.

3. Fit $R(z, \theta)$ to equation (15) and determine the elastic parameters $d \ln \rho/dz$, $d \ln \alpha/dz$, and $d \ln (\rho\beta^2)/dz$. This step is easy to set up, but difficult to accomplish, as we discuss shortly.

Data collection in two dimensions

The plane-wave reflection problem from which the p-ω and p-τ results [equations (18) and (24)] were derived assumed a 3-D, (x, y, t), surface reflection experiment. For a 1-D earth, this is not required and leads to unnecessary complications. We replace equations (18) and (24) with equivalent 2-D expressions appropriate for point sources and receivers. The new expressions are called $2\frac{1}{2}$-D expressions to distinguish from a pure 2-D result which pertains to line sources and receivers. The way to accomplish this is to inverse-transform equation (18) back to the y domain and set $y = 0$. Data used are

$$D(k_x, k_y, \omega) = P_r(\sqrt{k_x^2 + k_y^2}/\omega, 0, \omega). \quad (25)$$

We inverse Fourier transform over k_y and set $y = 0$:

$$D(k_x, 0, \omega) = \frac{iA\alpha_0}{2\omega}\int d k_y \int dz\ e^{2i\omega\tau}R(z, \theta) \sec \theta_0. \quad (26)$$

The k_y integral can be performed by the method of stationary phase (Bleistein and Handelsman, 1975). The major contribution to this integral comes near the vicinity of the stationary point ($k_y = 0$) of the phase term exp ($2i\omega\tau$). The stationary phase approximation

$$\int d\mu\ e^{if(\mu)}g(\mu) = \frac{g(\mu_0)e^{if(\mu_0)}}{\sqrt{if''(\mu_0)}} \quad (27)$$

simplifies equation (26) to

$$D(k_x, 0, \omega) = \frac{A\alpha_0}{\sqrt{-2i\omega}\cos \theta_0}\int dz\ \frac{R(z, \theta)e^{2i\omega\tau}}{\left[\int_0^z dz'\ \alpha \sec \theta\right]^{1/2}}, \quad (28)$$

where all functions of θ are understood to be evaluated at the point $k_y = 0$, i.e., at

$$p = \frac{\sin \theta}{\alpha} = \frac{k_x}{\omega}. \quad (29)$$

Note that the stationary phase approximation relates the spatial point $y = 0$ to a wavenumber point $k_y = 0$, in apparent defiance of the uncertainty principle. In general, the stationary phase approximation replaces a wave-theoretic expression with its ray-theoretic counterpart.

The difference between equation (18) and equation (28), other than the restriction $k_y = 0$, is a 45 degree phase factor $\sqrt{-i\omega}$ and a divergence-correcting amplitude factor

$$\left[\int_0^z dz' \ \alpha(z') \ \sec \ \theta(z') \right]^{1/2}.$$

The combination of these two terms effectively converts a 2-D line source into a 3-D point source. The benefit of the $2\frac{1}{2}$-D expression is that only data at $y = 0$, need be collected, and only a 2-D Fourier transform (or a 1-D slant stack) is necessary to put the data in the form of equation (28).

A $2\frac{1}{2}$-D p-τ expression analogous to equation (24) is also possible. Define a filtered data set

$$Q(p, \omega) = \sqrt{-i\omega} D(k_x, 0, \omega). \tag{30}$$

Then, instead of equation (24), the relation between reflectivity and data is

$$R(z, \theta) = \frac{2 \cos \theta \cos \theta_0}{\alpha \ \alpha_0} \left[\int_0^z dz' \ \alpha(z') \ \sec \ \theta(z') \right]^{1/2} Q(p, \tau). \tag{31}$$

Computation of Q requires only a single (filtered) slant stack, after which calculation of the three elastic parameters proceeds in a straightforward manner.

Formulation in space-time

Although the p-τ expression (31) looks straightforward to evaluate, there is a catch: a slant stack of Fourier transform will accurately compute $Q(p, \tau)$ only if the data cover all source-receiver separations x. In practice, data must be collected over a finite number of offsets, and this limited aperture will, if ignored, distort Q.

Instead of some iterative scheme to solve this problem, it is easiest to deal with the spatial sampling problem in real space. Moreover, returning to space-time produces a natural extension of the concept of NMO-stack which accommodates a reflection coefficient with angular variation.

Starting with equation (28), we transform from k_x back to x, to make another stationary phase approximation:

$$D(x, 0, \omega) = \frac{A\alpha_0}{\sqrt{-2i\omega}} \int \frac{dz \ dk_x}{\cos \theta_0} \frac{e^{i(2\omega\tau + k_x x)} R(z, \theta)}{\left[\int_0^z dz' \ \alpha(z') \ \sec \ \theta(z') \right]^{1/2}}. \tag{32}$$

Although not explicitly expressed, remember that τ depends upon k_x via its defining equation (23) and the relation (29). At the stationary point of the k_x integral, p must be such that

$$x = 2 \int_0^z dz' \ \tan \ \theta(z') = 2p \int_0^z \frac{dz' \ \alpha(z')}{\sqrt{1 - p^2 \alpha^2(z')}}. \tag{33}$$

The stationary phase approximation to equation (32) is

$$D(x, 0, \omega) = \frac{A\alpha_0}{2 \cos \theta_0} \int dz \ \frac{R(z, \theta)}{h(z, \theta)}$$

$$\times \exp \left[2i\omega \int_0^z dz'/(\alpha \ \cos \ \theta) \right], \tag{34}$$

where

$$h(z, \theta) = \left(\int_0^z dz' \ \alpha \ \sec \ \theta \right)^{1/2} \left(\int_0^z dz' \ \alpha \ \sec^3 \ \theta \right)^{1/2}. \tag{35}$$

The next step is to return $D(x, 0, \omega)$ to the time domain with an inverse Fourier transform. This collapses the z-integral, leaving a simple relation between D and R, namely,

$$D(x, 0, t) = \frac{A\alpha_0 \alpha}{\cos \theta \cos \theta_0} \frac{R(z, 0)}{4h(z, 0)} R(z, \theta), \tag{36}$$

subject to the constraint

$$t = 2 \int_0^z \frac{dz'}{\alpha(z') \cos \theta(z')} = 2 \int_0^z \frac{dz'}{\alpha(z')\sqrt{1 - p^2\alpha^2(z')}}. \tag{37}$$

Equation (37) and the stationary phase condition in equation (33) determine z and θ (or p) as a function of x and t. These relations are recognizable as the curved-ray moveout curves for a layered earth. When θ is small, these relations can be replaced by simpler, straight-ray approximations. First define

$$t_0 = 2 \int_0^z \frac{dz'}{\alpha(z')}$$

and

$$\alpha_{rms} = \left[\frac{2}{t_0} \int_0^z dz' \ \alpha(z') \right]^{1/2}.$$

Then, expanding the square root in equation (33), obtain

$$x \simeq 2p \int_0^z dz' \ \alpha(z') = p\alpha_{rms}^2 t_0. \tag{38}$$

Similarly, expanding equation (37) and using equation (38) to replace p with x,

$$t \simeq 2 \int_0^z \frac{dz'}{\alpha(z')} + p^2 \int_0^z dz' \ \alpha(z') = t_0 + \frac{x^2}{2\alpha_{rms}^2 t_0}. \tag{39}$$

Whether we use the curved-ray moveout equations or the rms approximations, equation (36) is a remarkably simple relation between the reflection coefficients $R(z, \theta)$ and moveout-corrected/divergence-corrected seismic data. The surprising thing is not that this simple relation exists (except for the divergence term, it is almost obvious), but rather that such a simple, satisfying formula should result from wave-equation analysis.

Equation (36) is a powerful, accurate formula for modeling or inversion of 1-D earth data, subject to the constraints on the plane-wave linearized formula (14) wherein we consider only primary compressional wave reflections, assume perfect transmission, stay clear of critical angle, and agree to discretize finely enough that changes between adjacent intervals are small. Attenuation and anisotropy are likewise missing, and only the simplest of boundary conditions have been considered. The stationary phase approximations should not cause undue concern; realistic seismic data are thoroughly far-field, and the stationary phase approximation is a very good one (Bleistein and Cohen, 1982).

Recovery of elastic parameter changes from equation (36) follows a predictable course. First correct the data for moveout with a coordinate change $t \rightarrow z$ (or, equivalently, $t \rightarrow t_0$). Then correct the amplitude in the data with the divergence factors in equation (36) to obtain an estimate of $R(z, \theta)$. Finally, obtain a fit of equation (15) to $R(z, \theta)$ to obtain the individual elastic parameters. Each of the three logarithmic derivatives ends up as a weighted sum over offset of the data correct-

ed for moveout. The weight factors, which are different for each parameter so as to satisfy equation (15), represent the only substantial change from conventional moveout correction and stack.

REALISTIC LINEAR INVERSION GOALS

Because there normally are more than three offsets in the data, the relation in equation (15) provides more equations than unknowns. The inverse problem is therefore over-determined; nevertheless, a good estimate of all three parameters is very difficult to obtain. A large range of angles is necessary to distinguish between the three parameters. This need is made clear if we express equation (15) in powers of $p\alpha = \sin\theta$:

$$R(z, \theta) = \frac{1}{2}\left[\frac{d\ln\rho\alpha}{dz} + p^2\alpha^2\left(\frac{d\ln\alpha}{dz} - 4\frac{\beta^2}{\alpha^2}\frac{d\ln\rho\beta^2}{dz}\right)\right.$$
$$\left. + \frac{p^4\alpha^4}{1 - p^2\alpha^2}\frac{d\ln\alpha}{dz}\right]. \tag{40}$$

At normal incidence, only the first term (acoustic impedance) contributes. At 30 degrees, the coefficient of the last term is only 8 percent of that of the first term. The last term does not become as large as the first term until $\theta = 52$ degrees. This angle is not often achieved at depths of exploration interest, even if we could retain confidence in the linearized form at that angle.

The inverse to equation (40) will be, in most practical situations, ill-conditioned. The ability to extract reliably all three parameters from contemporary seismic data is likely to be more the exception than the rule. Even before noting that several factors external to a reflector can alter amplitudes as $\sin^4\theta$, we realize we are in trouble. The three-parameter linearized inverse seems to be in a squeeze; to differentiate between the parameters, a large range of offset angles is required. On the other hand, if we look at angles which are too large, the linearized model may break down. In a practical situation, it may be possible to separate the three parameters only at moderate depths where the angular aperture is large.

A way to increase the reliability of an inverse to equation (40) is through use of a priori information to constrain the result. If we could reduce the effective number of parameters from three to two, we would be much better off. There are many ways to accomplish this reduction (each involving the risk of imposing a solution which may not be true), but all are beyond the scope of this paper. We do point out that neglecting changes in shear modulus is not the answer. Note from equation (40) that shear modulus plays an important role in the angular dependence of the compressional reflection coefficient and, indeed, may often dominate it. Only when confining attention to near-normal incidence can shear modulus be safely neglected.

Another pitfall comes from the limitations of the linearized approximation. Although this approximation may be "exact" in a continuous, smoothly variable medium, changes which are large over the sampling interval—and presumably rapid compared to the frequencies in the data—violate the validity conditions. Validity conditions are essentially that the fractional change in each elastic property must remain small even when multiplied by tan θ. Weglein et al. (1985) pointed out that for a discrete change in one physical parameter, a lin-

earized inversion formula always predicts changes in the other parameters also. These false predictions, or artifacts, also noted in Hanson (1984), can be significant.

These artifacts can, to some extent, be dealt with at a cost of slightly greater algorithmic complexity. The confidence in an inversion can be improved by using angles only within the confidence interval of the data. For example, assume that several incidence angles need to be specified to invert for elastic properties. The range of reasonable angles should be chosen consistent with the minimum and maximum offset of the experiment and the depth of the particular reflector. Postcritical data must also be avoided. Additionally, Weglein et al. (1985) presented a method for distinguishing artifacts from true parameter changes based on f-k filtering. In space-time, the simplest measure is to confine attention to incidence angles of less than about 45 degrees (Hanson, 1984). If all else fails, we can try to invert the full Zoeppritz reflection coefficient; however, not just relative but absolute sizes of parameters then come into play.

When talking of discretization problems, we acknowledge that not just the linearization approximation, but the whole physical model, can be placed in jeopardy. Changes which are too fast for the frequencies in the data to "see" directly can affect perceived physical properties in many ways, introducing time delays, attenuation, and anisotropy. In extreme cases, such effects can perturb or even invalidate inversion results.

If data quality does not justify multiparameter inversion, it still may be possible to solve for a single parameter. By confining attention to near-normal incidence, acoustic impedance becomes a legitimate target. Most useful would be the absolute value of this property, though that is an elusive target. Even with low-frequency information supplied (or inferred), integration of the logarithmic derivative is treacherous. Data noise and transmission effects accumulate and multiples in the data bias the result (a spurious spike in the reflectivity series, when integrated, alters the background level of the parameters). Data normalization remains a problem. We expect better results over short intervals. Under good conditions, it is possible to map the relative sizes of impedance changes accurately. Even for poor data, we expect to document the existence and possibly the sign of an impedance change (Weglein and Gray, 1983).

To summarize, multiparameter inversion methods are used to determine the configuration of physical properties. Depending upon the data, there are various levels at which data can be inverted. The first, least ambitious level is to determine, at a given subsurface location (in depth or time), whether some property (maybe acoustic impedance) has changed. At the next level, we also determine the sign or phase of this change. At a higher level, we acquire amplitude information. Slightly higher, we recognize the "signature" (e.g., an unmistakable increase in amplitude with offset, see Ostrander, 1984) of some target geologic entity. At a still higher level, we distinguish more than one physical property and determine the sign of each change. At the highest level, absolutely true values of all relevant physical properties are determined. Of course, the highest level is desired when we can get it. However, even if the highest level is unobtainable, useful information may still be available with the sign of a parameter change being easier to predict than its amplitude.

In practice, the power to resolve individual parameters must diminish with depth. At deeper levels, raypaths approach

normal incidence, and only the ability to see acoustic impedance is left.

THE MULTIDIMENSIONAL INVERSE PROBLEM

The last decade has seen an explosion in migration and inversion technology. Since the early work of Claerbout demonstrated that wave-equation techniques were really applicable to seismic processing, a host of different processes have been presented. For a while, it seemed that competing technologies were developing—differential versus integral versus transform methods, then time migration versus depth migration versus reverse time migration, and finally, migration versus inversion. The contemporary view is that all these approaches share a common theoretical base. The Earth being complex and variable, it is necessary and desirable to keep an arsenal of specialized tools which are similar, but each one is customized to a particular application.

The major difference between single-parameter wave-equation inversion and wave-equation migration has turned out to be semantic; with its origins in scattering theory and applied mathematics, seismic inversion encourages a slightly different vocabulary than the more operationally based migration.

This is not to say that inversion concepts have not added to migration theory. Seismic inversion has provided a means to advance past recovery of reflectivity information to recovery of the physical changes which produce reflections. This, in turn, tied migration more firmly to 1-D data processing concepts from which the relationship between reflectivity and underlying physical changes has been better understood. Inversion concepts also have provided a framework for dealing with experimental limitations in real data sets. For example, migration usually deals with the problem of a source wavelet by ignoring it. Set up as an inverse scattering problem, the effect of the source wavelet can be incorporated, and the changes to the wavelet through the various processing steps can be observed and calculated. Likewise, effects of finite source-receiver aperture can be built into the problem.

Evolution of migration and inversion

Seismic migration has been around a long time. Deterministic migration methods are probably as old as the earliest seismic data. The advent of digital computers encouraged the growth of statistical migration methods—in particular, the "arc swinging" algorithms, whose modern counterpart is Kirchhoff migration. The Claerbout concept of a moving coordinate system reduced the cost and increased the fidelity of seismic migration to the point where it could be considered part of a standard processing sequence. Soon differential, integral, and f-k migration schemes abounded.

Cohen and Bleistein (1977) formulated the inverse seismic problem in terms of perturbation theory, presenting an apparent alternative to seismic migration. In this method, deviations from background values are solved by linearizing the inverse problem. The linearized inverse method developed by Bleistein and Cohen has concentrated on inversion of zero-offset seismic data by means of the scalar wave equation. This one-parameter inversion is now known to be closely related to poststack migration, both in theory and in practice. A one-parameter inversion is exactly right for zero-offset data. For

this case, there is enough information to recover only one earth parameter. [Only one parameter contributes; equation (40) shows that only acoustic impedance contributes to normal incidence seismic data.]

For prestack inversion, the scalar wave equation does not suffice. Though the original work of Cohen and Bleistein discussed both scalar and elastic fields, seismic inversion methods were essentially scalar until Raz (1981a, b) introduced an inversion scheme for acoustic (variable velocity and density) data. A nice feature of Raz's 1-D result (Raz, 1981a) was a space-time formula used to solve for density and velocity given data at two offsets. A second Raz result (Raz, 1981b) was a multidimensional linear solution for density and velocity perturbations. Clayton and Stolt (1981) also worked with the acoustic model, producing an f-k space result valid for variable background velocity and density. Raz (1982) published variable-velocity acoustic formulas which, with some restrictions, solve for absolute values of earth parameters. The importance of work with the acoustic wave equation should not be understated, particularly regarding advances in understanding of the multiparameter inverse problem. However, the acoustic wave equation does not quantitatively model reflections of compressional waves in the real Earth. As seen in equation (40), shear modulus changes make a major, perhaps dominant, contribution to the reflectivity away from normal incidence.

The elastic problem is more difficult to work with than the acoustic problem, mainly because it tends to lead toward lengthy equations. However, several researchers have described procedures, including Marfurt (1978), Clayton (1981), and recently Chen and Xie (1985) and Tarantola (1984).

The linear perturbation model

The first step in wrestling with the inverse seismic problem is to develop an adquate model of the reflection experiment and the reflection process. For a 1-D earth, this is made simpler by a change of coordinates from depth to traveltime [equation (3)]. This change may not be possible or helpful when velocities vary laterally (Hagin and Gray, 1984). This does not mean that the multidimensional problem is intractable. A great many tools are still available. The linearized perturbation method is a simple, effective approach to the multidimensional problem. In this method, we begin with a known or predetermined background model, close enough to true Earth properties that we can consider them to be small deviations or perturbations of the model. Typically, the background model is expected to describe wave propagation in the earth with reasonable accuracy, neglecting the generation of reflected waves. The perturbations from the model then generate the reflection data.

To illustrate the concepts, we use the acoustic wave equation. Although admittedly inadequate, the acoustic wave equation leads to results which are simple to describe and, better yet, simple to generalize. For a point source at $\mathbf{x}_s$, the total response P at $\mathbf{x}$, including reflections, obeys the acoustic wave equation

$$\left[\nabla \cdot \frac{1}{\rho(\mathbf{x})} \nabla + \frac{\omega^2}{\rho(\mathbf{x})\alpha^2(\mathbf{x})} \right] P(\mathbf{x} \mid \mathbf{x}_s; \omega) = -S(\omega)\delta(\mathbf{x} - \mathbf{x}_s). \quad (41)$$

Stolt and Weglein

The source term $S(\omega)$ represents the Fourier transform of a source wavelet.

The approximate (unperturbed) background wave field G_0, due to an impulsive point source satisfies

$$\left(\nabla \cdot \frac{1}{\rho_0}\nabla + \frac{\omega^2}{\rho_0 \alpha_0^2}\right)G_0(\mathbf{x}\,|\,\mathbf{x}_s;\,\omega) = -\delta(\mathbf{x} - \mathbf{x}_s). \qquad (42)$$

If G_0 is to track the propagation of a wave accurately, α_0 should be close to the true velocity α. On the other hand, if ρ_0 and α_0 vary too rapidly, G_0 will contain reflection energy. This may be all right, especially if we are using a successive approximation scheme. Ordinarily, it is desirable for G_0 to be reflection-free, if for no other reason than we are unlikely to know α and ρ sufficiently well a priori to reproduce accurately the reflections. The ideal choice for α_0 and ρ_0 would be local averages of the true values, too slowly varying to produce significant reflection energy for frequencies where $S(\omega)$ is non-zero. Although we may fall short of this ideal in practice, we proceed on the assumption that α_0 and ρ_0 are indeed local averages close to the true Earth parameters.

We designate the difference between the two wave operators in equations (41) and (42) as a potential V:

$$V(\mathbf{x},\,\omega) = \nabla \cdot \left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\nabla + \omega^2\left(\frac{1}{\rho\alpha^2} - \frac{1}{\rho_0\alpha_0^2}\right)$$

$$= \nabla \cdot \frac{a_2\nabla}{\rho} + \omega^2\left(\frac{a_1}{\rho_0\alpha_0^2}\right), \qquad (43)$$

where

$$a_2(\mathbf{x}) = \frac{\rho_0}{\rho} - 1,$$

and

$$a_1(\mathbf{x}) = \frac{\rho_0\alpha_0^2}{\rho\alpha^2} - 1.$$

This potential then contains the unknowns for which we would like to solve.

The difference D between the exact and modeled response obeys a simple identity

$$D(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega) = P(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega) - S(\omega)G_0(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega)$$

$$= \int d\mathbf{x}\,G_0(\mathbf{x}_g\,|\,\mathbf{x};\,\omega)V(\mathbf{x},\,\omega)P(\mathbf{x}\,|\,\mathbf{x}_s;\,\omega), \qquad (44)$$

which can be verified by applying the unperturbed wave operator to both sides of the equation. Equation (44) is nonlinear in the potential because P (which depends upon the potential) appears inside the integral along with the potential. We linearize it simply by approximating P by SG_0 in the right-hand side of equation (44):

$$D(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega) \simeq S(\omega)\int d\mathbf{x}\,G_0(\mathbf{x}_g\,|\,\mathbf{x};\,\omega)V(\mathbf{x},\,\omega)G_0(\mathbf{x}\,|\,\mathbf{x}_s;\,\omega). \qquad (45)$$

If G_0 is a good model of the transmitted wave minus reflections, then the interpretation of equation (45) is clear: The response P at $\mathbf{x}_g$ due to a source at $\mathbf{x}_s$ is the sum of a direct arrival SG_0 and an integral over all possible reflection points. The integrand is the product of a transmitted wave from $\mathbf{x}_s$ to the reflection point $\mathbf{x}$, times a reflection term $V(\mathbf{x},\,\omega)$, times a transmitted wave from $\mathbf{x}$ to $\mathbf{x}_g$. Should G_0 include reflection energy, then the equation still holds, but the interpretation does not. We state that the difference between the actual data

and the modeled data should approximately obey the linearized integral equation (45).

The form of equation (45) is reminiscent of the 1-D equation (14). However, there are two differences. The first is that the transmission term in equation (14) incorporates the correct velocity, whereas G_0 is guided by an approximation α_0. The second is that $V(\mathbf{x},\,\omega)$, though clearly related to reflectivity, is not exactly a reflection coefficient. The first difference is both good and bad. The good part is that it allows G_0 to be constructed without an exact knowledge of density and velocity. (If we had that knowledge, the inverse problem would be solved already.) The bad part is that, to the extent that α_0 differs from α, reflection points will be mislocated upon inversion. [This is also true for the 1-D case. The reflection coefficient $R(\tau,\,p)$ can be solved using equation (22) from the reflection data, even if velocity as a function of depth is unknown. Conversion from traveltime to correct depth, however, is possible only with the correct velocity.] A discussion of the second difference, the relation between V and R, follows.

WKBJ approximation in three dimensions

Some of the complexity of equation (45) can be removed by invoking the WKBJ approximation. Suppose that G_0 closely approximates a delta function wavefront so that in the frequency domain,

$$G_0(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega) \simeq g_0(\mathbf{x}_g\,|\,\mathbf{x}_s)\exp\left[i\omega\tau(\mathbf{x}_g\,|\,\mathbf{x}_s)\right]. \qquad (46)$$

Here, g_0 is the amplitude of the wavefront, which is assumed to be slowly varying and τ is the traveltime from source to receiver, which varies rapidly. Indeed,

$$|\nabla\tau| = 1/\alpha_0, \qquad (47)$$

while the direction of $\nabla\tau$ defines the local direction of the raypath from source to receiver. The amplitude g_0 is determinable from the first transport equation (Morse and Feshbach, 1953) for the WKBJ expansion.

Using the WKBJ approximation for G_0, equation (45) reduces to

$$D(\mathbf{x}_g\,|\,\mathbf{x}_s;\,\omega) = \omega^2 S(\omega)\int d\mathbf{x}\,\frac{g_0(\mathbf{x}_g\,|\,\mathbf{x})g_0(\mathbf{x}\,|\,\mathbf{x}_s)}{\rho_0\alpha_0^2}\,v(\mathbf{x},\,\theta)$$

$$\times \exp\left\{i\omega[\tau(\mathbf{x}_g\,|\,\mathbf{x}) + \tau(\mathbf{x}\,|\,\mathbf{x}_s)]\right\}, \qquad (48)$$

where $v(\mathbf{x},\,\theta)$ is a simplified potential function denuded of frequency and differential operator dependence:

$$v(\mathbf{x},\,\theta) = \left(\frac{\rho_0\alpha_0^2}{\rho\alpha^2} - 1\right) + \cos\,(2\theta)\left(\frac{\rho_0}{\rho} - 1\right). \qquad (49)$$

The angle 2θ is the angle between incident and reflected rays at the reflection point $\mathbf{x}$. Although not explicit in this formula, v depends upon source and receiver coordinates x_s and x_g through the angle of incidence θ. The simplification in equation (49) of the potential function is made possible by the WKBJ approximation, which is essentially the geometrical acoustics approximation. The impulse response G_0 has a well-defined direction at every point, so the derivatives contained in V are easily evaluated. The potential $v(\mathbf{x},\,\theta)$ is seen to be the sum of two terms, one involving density $\Delta\rho/\rho$ and the other bulk modulus $\Delta(\rho\alpha^2)/\rho\alpha^2$. As such, it is hardly identical to the 1-D reflection coefficient equation (15). However, if $\rho_0 \simeq \rho$ and $\alpha_0 \simeq \alpha$, then

$$v(\mathbf{x}, \theta) \simeq \ln \frac{\rho_0 \alpha_0^2}{\rho \alpha^2} + \cos (2\theta) \ln \frac{\rho_0}{\rho}, \qquad (50)$$

and better yet,[1]

$$-\frac{1}{4} \sec^2 \theta \frac{\partial}{\partial z} v(\mathbf{x}) \simeq \frac{1}{2} \left[\frac{\partial \ln \rho}{\partial z} + \sec^2 \theta \frac{\partial \ln \alpha}{\partial z} \right]$$

$$\equiv R(\mathbf{x}, \theta). \qquad (51)$$

The right-hand side of equation (51) is recognizable as the first two terms of the reflection coefficient equation (15). We therefore identify $R(\mathbf{x}, \theta)$ as a reflection coefficient[2] and $v(\mathbf{x}, \theta)$ as an integrated angle-filtered reflection coefficient. The last term in (15) does not appear in (51) because shear modulus changes are invisible to the acoustic wave equation. A way to view the shear modulus term is as a mysterious leakage of energy to and from an unseen other mode of propagation. As such, we are free to add it to equation (51). This step can be justified by recourse to the elastic wave equation (Clayton, 1981).

$2\frac{1}{2}$-D formulation

If $v(\mathbf{x}, \theta)$ does not change significantly in the y direction, equation (48) for D can be further simplified. We confine data values to $y_g = y_s = 0$.

The only significant y dependence in the right-hand side of equation (48) is in the traveltimes $\tau(\mathbf{x}_g | \mathbf{x})$ and $\tau(\mathbf{x} | \mathbf{x}_s)$. These are clearly minimized at $y = 0$. The g_0 terms can be replaced by their values at $y = 0$, leaving

$$D(\mathbf{x}_g | \mathbf{x}_s; \omega) = \omega^2 S(\omega) \int dx$$

$$\times \int dz\ g_0(\mathbf{x}_g | \mathbf{x}, z) g_0(\mathbf{x}, z | \mathbf{x}_s) \frac{v(x, z, \theta)}{\rho_0 \alpha_0^2}$$

$$\times \int dy\ \exp\left[i\omega(\tau(\mathbf{x}_g | \mathbf{x}) + \tau(\mathbf{x} | \mathbf{x}_s))\right]. \qquad (52)$$

Writing traveltime in the vicinity of $y = 0$ as

$$\tau(\mathbf{x}_g | \mathbf{x}) + \tau(\mathbf{x} | \mathbf{x}_s) \simeq \tau_0 + \tau_{yy} y^2/2, \qquad (53)$$

the y integral in equation (52) can be evaluated by stationary phase, leaving

$$D(\mathbf{x}_g | \mathbf{x}_s; \omega) = \frac{\omega^2 S(\omega)}{\sqrt{i\omega}} \int dx \int dz$$

$$\times \frac{g_0(\mathbf{x}_g | \mathbf{x}, z) g_0(\mathbf{x}, z | \mathbf{x}_s)}{\rho_0 \alpha_0^2 \sqrt{\tau_{yy}}} v(x, z, \theta) e^{i\omega\tau_0}.$$

$$(54)$$

Equation (54) differs little from a true 2-D formula. We observe the 45 degree phase term $\sqrt{i\omega}$ and a divergence correction term $\sqrt{\tau_{yy}}$, which together convert a 3-D experiment along the line $y = 0$ into an equivalent 2-D form. Provided τ_{yy} can be estimated (which, in the worst case, can be accomplished by ray shooting), the $2\frac{1}{2}$-D form presents little additional complication. Bleistein (1985) presented a formula for calculating τ_{yy} from in-plane raypath information. The result is $\tau_{yy} = 2/\sigma$, where σ is a ray parameter related to arc length s on the ray by $d\sigma/ds = \alpha(x, z)$.

Multidimensional linearized inversion

A prescription for inversion of the multidimensional formula in equation (48) or (54) is given in Clayton and Stolt (1981). The prescription involves a form of migration. We first downward-continue to depth z the reflection data D, both source and receiver coordinates. This, in principle, is done by application of the divergence theorem to both source and receiver coordinates along the surface $z = 0$. The amplitude of the downward continuation must be carefully computed and adjusted for divergence effects. The zero-time component of the downward-continued data is extracted by summing over frequencies, effecting a migration of the data. By retaining migrated data for $x_g \neq x_s$, information regarding angles of incidence is also retained.[3] To decode the angle of incidence information, we Fourier-transform the migrated data over all spatial coordinates. The result is proportional to the Fourier transform of the reflection coefficient in equation (51).

For the special case of a depth-dependent background velocity, we write an explicit $2\frac{1}{2}$-D formula (see the Appendix)

$$R(k_g - k_s, k_z, \theta)$$

$$= \frac{k_z}{4i} \sec^2 \theta \int dz\ e^{ik_z z}$$

$$\times \int \frac{d\omega\ q(z)}{\sqrt{i\omega}\ \rho_0 \alpha_0} \left(\int_0^z dz' \frac{\alpha_0 q}{q_g q_s} \right)^{1/2}$$

$$\times \left[\frac{q_g(0)q_s(0)}{q_g(z)q_s(z)} \right]^{1/2} \exp\left[-i\omega \int_0^z dz'\ q/\alpha \right] D(k_g | k_s; \omega).$$

$$(55)$$

In this expression,

$$q_g(z) = \sqrt{1 - k_g^2 \alpha_0^2/\omega^2}, \qquad (56)$$

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[1]To proceed to equation (51) from equation (50), we assumed that α_0 and ρ_0 are slowly varying, i.e., have negligible derivatives and do not produce noticeable reflections. The term $\cos (2\theta)$ which depends on the background velocity is likewise assumed to have negligible z-derivative. For background velocities which do produce reflections, equation (51) becomes the difference between modeled and actual reflection coefficients. For a slowly varying background, we emphasize what may be obvious: Although the potential technically depends on the choice of background parameters, it is actually, extremely insensitive to them. The logarithmic approximation in equation (51) is really an improvement to the form of equation (49), as noted in the section on 1-D.
[2]Strictly, the quantity R defined in equation (51) is a true reflection coefficient only when local velocity structure α_0 does not vary laterally. In general, the true reflection coefficient would be a derivative taken normal to local structure. Here we chose, somewhat arbitrarily, to work with the quantity R as defined in equation (51) and to call R, somewhat disingenuously, a reflection coefficient.

[3]For data at $x_g \neq x_s$ to be nonzero as $t \rightarrow 0$ may appear to violate causality. In fact, it is true that the data must go to zero (within the limits of source-receiver size and bandwidth) as x_g moves away from x_s. There is still the question, however, of how the data approach zero, and it turns out (because of the angular dependence of reflectivity) that the limit is sensitive to the direction of approach to a reflection point. The migrated data are not a simple delta function of $x_g - x_s$, though they are sharply peaked. A Fourier transform over offset and depth reveals that the migrated data are not constant in offset wavenumber (which would correspond to a delta function in offset) but, rather, they form a slowly varying function of offset wavenumber divided by depth wavenumber, [see equation (59)] i.e., a slowly varying function of angle of incidence.

$$q_s(z) = \sqrt{1 - k_s^2 \alpha_0^2/\omega^2},$$ (57)

and

$$q(z) = q_g(z) + q_s(z).$$ (58)

$R(k_m, k_z, \theta)$ is the double Fourier transform of the reflection coefficient $R(x, z, \theta)$ at the angle of incidence θ. θ is determined in this formula as the ratio of offset spatial frequency to depth spatial frequency:

$$\tan \theta = (k_g + k_s)/k_z.$$ (59)

Equation (55) is an extension [provided we use equation (15) for R] of the 1-D inversion formulas developed in the previous section. The prescription is to (1) Fourier transform the data from x_g, x_s, and t to k_g, k_s, and ω; (2) downward continue to depth z with a phase shift and the amplitude correction given inside the ω integral of equation (55); (3) sum over ω to effect a migration; (4) when the migration is complete for all z, Fourier transform from z to k_z to obtain the perturbation; (5) multiply by $k_z \sec^2 \theta/4i$ to convert the perturbation to a reflection coefficient; and (6) inverse Fourier transform over $k_m = k_g - k_s$ and k_z to yield a spatial map of the local reflectivity as a function of incidence angle.

Although explicit, equation (55) does look unwieldy. Given some further approximations (e.g., straight raypaths), some of the complexity can be removed. For example, if background velocity is assumed constant, then both the ω and z integrals are trivially evaluated, leaving

$$\tilde{R}(k_g - k_s, k_z, \theta) = \frac{1}{\rho_0} \sqrt{\frac{4ik_z q_g q_s[1 + (k_g + k_s)^2/k_z^2]}{1 + (k_g - k_s)^2/k_z^2}}$$

$$D(k_g \mid k_s; \omega),$$ (60)

where $\tilde{R}$ is a divergence-adjusted version of R:

$$\tilde{R}(x, z, \theta) = R(x, z, \theta)/\sqrt{z},$$ (61)

and ω and k_z are related as

$$k_z = \frac{\omega}{\alpha_0}(q_g + q_s) = \sqrt{\frac{\omega^2}{\alpha_0^2} - k_g^2} + \sqrt{\frac{\omega^2}{\alpha_0^2} - k_s^2}.$$ (62)

We showed above that the extension of the NMO-stack concept to accommodate an angular dependent reflection coefficient allowed the inversion for the three elastic parameters in a vertically varying medium. The elastic inversion then requires three different weighted sums over offset of moveout and divergence-corrected data. The analogous multidimensional situation results in different specific weighted averages of $D(k_g, k_s, \omega)$ (or its downward continued generalization) providing either a migration or the acoustic or elastic components of reflectivity. If the background is constant (and we assume line sources) the elastic generalization of the Clayton and Stolt (1981) result leads to (see, e.g., Clayton, 1981),

$$D(k_g, k_s, \omega) = \sum_{i=1}^{3} A_i(k_g, k_s, \omega) a_i(k_g - k_s, k_z),$$ (63)

where

$$A_1 = -\frac{\rho_0}{4q_s q_g},$$

$$A_2 = -\frac{\rho_0}{4q_s q_g}(q_g q_s - k_g k_s \alpha_0^2/\omega^2),$$

$$A_3 = -\frac{\rho_0}{4q_s q_g}\left(-\frac{2\beta_0^2}{\omega^2}[k_g q_s + k_s q_g]^2\right),$$

and the a_i are the Fourier transforms of

$$a_1 = \frac{\rho_0 \alpha_0^2}{\rho \alpha^2} - 1,$$

$$a_2 = \frac{\rho_0}{\rho} - 1,$$

$$a_3 = \frac{\rho_0 \beta_0^2}{\rho \beta^2} - 1.$$

The coefficients A_1 and A_2 are identical to those in Clayton and Stolt (1981). They appear slightly different because q_g and q_s are defined slightly differently there. The coefficient A_3, describing leakage of compressional energy into a shear mode, does not appear in Clayton and Stolt. From equation (63) the a_i are reconstructed [see e.g. Clayton and Stolt (1981)] as weighted averages of $D(k_g, k_s, \omega)$. Equation (63) can be rewritten as

$$D(k_g, k_s, \omega) = \frac{\rho_0}{4q_g q_s}\left(a_1 + \cos 2\theta a_2 - \frac{2\beta_0^2}{\alpha_0^2} \sin^2 2\theta a_3\right),$$

where θ is the half angle between the incident and reflected pressure wave

$$\theta = \frac{1}{2}[\sin^{-1}(k_g \alpha_0/\omega) + \sin^{-1}(k_s \alpha_0/\omega)].$$

The elastic generalization of equation (49) is

$$v(\mathbf{x}, \theta) = a_1(\mathbf{x}) + \cos 2\theta a_2(\mathbf{x}) - \frac{2\beta_0^2}{\alpha_0^2} \sin^2 2\theta a_3(\mathbf{x}),$$

and using equation (15) the relationship between the elastic $v(\mathbf{x}, \theta)$ and $R(\mathbf{x}, \theta)$ is also given by equation (51). Equation (63) is almost the same as equation (60). They differ because equation (60) incorporates the $2\frac{1}{2}$D correction [equation (A-9) of Clayton and Stolt (1981)].

For the more general case, Cohen and Hagin (1985) promised to generalize the zero-offset, space-time inversion formula to prestack data.

Whichever form we choose for a migration and inversion operator, a few conclusions can be made.

1. Inverting for the elastic parameters does not greatly complicate the migration process. We expect the expense of a migration and inversion algorithm to be dominated by the migration process.

2. The limitations in resolving all the elastic parameters noted for 1-D inversion still exist. The f-k formulas in equations (55) and (60) obscure the effects of finite source-receiver aperture but do not eliminate them. In fact, the resolving power of migration and inversion must diminish with depth, leaving ultimately only migration.

NONLINEAR METHODS

There are a small number of exact, nonlinear, multidimensional methods for solving acoustic and elastic inverse problems. For excellent reviews of 1-D methods, see Newton (1966) or Burridge (1980). For the multidimensional problem, there are several approaches which have been taken. One is to

invert the Born series iteratively. This line of research was originated by Jost and Kohn (1951), Kay and Moses (1955), Prosser (1969), and Razavy (1975), and was introduced to geophysics by Weglein et al. (1981) and Stolt and Jacobs (1980). The convergence properties of this series solution were studied by Prosser. Only "small" reflection data would allow the series to converge. Weglein (1982b) generalized some earlier Razavy (1975) work to invert a "distorted" Born series for an arbitrary background, thus allowing reflections and multiples in the background model. The goal was to allow accommodation of "larger" reflection data.

Another approach to solving the multidimensional inverse problem derives from the Gelfand-Levitan/Marchenko integral equation methods originally designed for the Schrödinger equation (Agronovich and Marchenko, 1963; Faddeev, 1963). In one dimension, a Liouville transformation exists to map elastic and acoustic problems to a Schrödinger equation and hence allows the machinery designed for the latter to be used. An interesting paper by Berryman and Greene (1980) showed that a discretized form of these integral equation methods leads to the original Goupillaud (1961) equal-traveltime inverse solution. However, even though multidimensional Schrödinger equations had been inverted (in theory) in two and three dimensions in Newton (1981) and Cheney (1984), no corresponding Liouville transformation was found to allow a multidimensional acoustic or elastic equation to be mapped into a multidimensional Schrödinger equation. Recently, Coen et al. (1984) presented a method for inverting a 2-D, variable-density acoustic equation by collecting point source data. Transforming the data over the dimension of which the medium is independent allows a 2-D Schrödinger equation to evolve. The Cheney 2-D Marchenko equation is used to solve the problem. Transmission as well as reflection data would be required.

Several authors have recently approached the multidimensional linear and nonlinear inverse problem from the generalized linear inverse point of view. Among these are Chen and Xie (1985), Berkhout (1984), Keys and Weglein (1983), and Tarantola (1984). The method basically consists of approaching the Fredholm I integral equations, which result from the linear Lippmann-Schwinger equation, by a generalized linear inverse method. These methods impose the constraints to determine a model which (1) is close to the reference model and (2) has the predicted data residuals close to the actual residuals. The residuals are the difference between the data and the background model-generated data. This method determines a "solution" whether or not the original integral equation possessed a single solution, many solutions, or no solutions. Confidence in these methods was discussed by several authors—e.g., Treitel and Lines (1985). The question of the convergence of the iterative solution (and the convergence to the "answer") of the generalized linear inverse approach to the nonlinear problem has yet to be addressed adequately.

Common to all the nonlinear approaches is a demand for massive amounts of computer power. Given the size of the seismic problem (a good-size 3-D survey should be looking at 10^8 to 10^9 subsurface points) and the fact that any nonlinear method must employ point-to-point correlations over substantial distances (roughly squaring the size of the problem), we begin to realize that today's "supercomputers" represent but a small step in the right direction toward the facilities necessary for a complete solution to the seismic problem.

CONCLUSION

Seismic processing has made significant advances during the last decade. Imaging techniques have matured and merged with seismic inversion to form potentially powerful tools for unraveling geologic structure and lithology under ideal conditions. A general principle for inversion is that the more ambitious the goal, the more hazardous and sensitive the procedure. However, the potential exists for predicting the three elastic parameters from compressional data. Practical difficulties are many. They include limitations in data quality (some removable with extra effort and some not) and in available computer resources.

ACKNOWLEDGMENTS

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APPENDIX

DERIVATION OF THE INVERSE FORMULA FOR A STRATIFIED BACKGROUND

There are a few steps between the general $2\frac{1}{2}$-D modeling formula in equation (54) and the stratified inverse in equation (55). The first step is to write a formula for the stratified 2-D Green's function G_0. From Clayton and Stolt (1981)

$$G_0(x, z \mid x', z'; \omega)$$

$$= \frac{-1}{2i\omega} \int dk_g \, e^{ik_g(x-x')} \left[\frac{\rho_0(z)\rho_0(z')\alpha_0(z)\alpha_0(z')}{q_g(z)q_g(z')} \right]^{1/2}$$

$$\times \exp\left[-i\omega \int_z^{z'} dz' \, q_g(z')/\alpha_0(z') \right] \tag{A-1}$$

with q_g as defined in equation (56) of this paper. With this formula, the $2\frac{1}{2}$-D modeling equation (54) is expressed in k-space as

$$D(k_g \mid k_s; \omega)$$

$$= \frac{\sqrt{i\omega}}{4} \int dz$$

$$\times \frac{v(k_g - k_s, z, \theta)\rho_0(z) \exp\left[-i\omega \int_0^z dz' \, q(z')/\alpha_0(z') \right]}{\left[q_g(0)q_s(0)q_g(z)q_s(z) \int_0^z dz' \, \alpha_0(z')q(z')/q_g(z')q_s(z') \right]^{1/2}}. \tag{A-2}$$

In this expression, q_g, q_s, and q are as defined in equations (56), (57), and (58). The incidence angle θ is given by

$$\tan \theta = \frac{(k_g + k_s)\alpha_0(z)}{\omega q(z)}. \tag{A-3}$$

As such, it is an implicit function of k_g, k_s, z, and ω.

The integral in the denominator of equation (A-2) compensates for $2\frac{1}{2}$-D divergence. Together with the phase term $\sqrt{i\omega}$, it comprises the term $\sqrt{\tau_{yy}}$ in equation (64).

The expression (A-2) is a forward modeling formula which translates the potential v into observable data. To derive v from D, we must effect a true amplitude migration M of the data. Write M as an integral over frequency of downward-

continued, normalized data:

$$M(k_g \mid k_s; z_0) \equiv \int d\omega \, \exp\left[i\omega \int_0^{z_0} dz' \, q(z')/\alpha_0(z') \right]$$

$$\times \, F(k_g, k_s, z_0, \omega) D(k_g \mid k_x; \omega) \qquad \text{(A-4)}$$

where the multiplier F will be chosen to normalize the $2\frac{1}{2}$-D migration properly.

With expression (A-2) for D, equation (A-4) is expandable to

$$M(k_g \mid k_s; z_0)$$

$$= \int d\omega \, \sqrt{\frac{i\omega}{4}} \, F(k_g, k_s, z_0, \omega)$$

$$\times \int \frac{dz \, \rho_0(z) v(k_g - k_s, z, \theta) \exp\left[-i\omega \int_{z_0}^{z} dz' \, q(z')/\alpha_0(z') \right]}{\left\{ q_g(0) q_s(0) q_g(z) q_s(z) \int_0^z dz' \, \alpha_0(z') q(z')/[q_g(z') q_s(z')]^{1/2} \right\}}.$$

$$\text{(A-5)}$$

For the ω integral, the rapid ω dependence is concentrated in the exponential term. Since this term oscillates rapidly unless $z \simeq z_0$, we conclude that the z integral contributes only for z near z_0. In this neighborhood, we can treat the background velocity as locally constant. The exponential term reduces to $\exp\left[-i\omega(z - z_0) q(z_0)/\alpha_0(z_0) \right]$. The angle of incidence can also be evaluated at z_0, $\tan\theta = (k_g + k_s)\alpha_0(z_0)/[\omega q(z_0)]$, as can all the other z dependence within the z integral except for the possibly rapidly varying potential $v(z)$. Thus

$$M(k_g \mid k_s; z_0)$$

$$\simeq \int d\omega \, \frac{\sqrt{i\omega}}{4}$$

$$\times \frac{\rho_0(z_0) F(k_g, k_s, z_0, \omega)}{\left[q_g(0) q_x(0) q_g(z_0) q_s(z_0) \int_0^{z_0} dz' \, \alpha_0(z') q(z')/(q_g(z') q_s(z')) \right]^{1/2}}$$

$$\times \int dz \, e^{-i\omega(z - z_0) q(z_0)/\alpha_0(z_0)} v(k_g - k_s, z, \theta). \qquad \text{(A-6)}$$

Equation (A-6) can be further simplified by a change of variables from ω to $k_z = \omega q(z_0)/\alpha(z_0)$. This simplifies the exponential and also removes the dependence of incidence angle θ on z_0. Since

$$d\omega/dk_z = \alpha_0(z_0) q_g(z_0) q_s(z_0)/q(z_0), \qquad \text{(A-7)}$$

we have

$$M(k_g \mid k_x; z_0) \simeq \int dk_z \, \frac{\sqrt{i\omega(k_z)}}{4}$$

$$\times \frac{\rho_0(z_0)\alpha_0(z_0) F(k_g, k_s, z_0, \omega(k_z))}{q(z_0)\left\{ \int_0^{z_0} dz' \, \alpha_0(z') q(z')/[q_g(z') q_s(z')] \right\}^{1/2}}$$

$$\times \left[\frac{q_g(z_0) q_s(z_0)}{q_g(0) q_s(0)} \right]^{1/2}$$

$$\times \int dz \, e^{-ik_z(z - z_0)} v(k_g - k_s, z, \theta). \qquad \text{(A-8)}$$

We now make F the reciprocal of all the unwanted terms in the k_z integral. Define

$$F(k_g, k_s, z_0, \omega) = \frac{4}{\sqrt{i\omega}} \, \frac{q(z_0)}{\rho_0(z_0)\alpha_0(z_0)}$$

$$\times \left\{ \int_0^{z_0} dz' \, \alpha_0(z') q(z')/[q_g(z') q_s(z')] \right\}^{1/2}$$

$$\times \left[\frac{q_g(0) q_s(0)}{q_g(z_0) q_s(z_0)} \right]^{1/2}. \qquad \text{(A-9)}$$

Then equation (A-8) reduces to

$$M(k_g \mid k_s; z_0) = \int dk_z \, e^{ik_z z_0} \int dz \, e^{-ik_z z} v(k_g - k_s, z, \theta). \qquad \text{(A-10)}$$

The z integral is just a Fourier transform of v from z-space to k_z-space:

$$M(k_g \mid k_s; z_0) = \int dk_z \, e^{ik_z z_0} v(k_g - k_s, k_z, \theta). \qquad \text{(A-11)}$$

If not for the k_z dependence of θ [in terms of k_z, we have $\tan\theta = (k_g + k_s)/k_z$], the k_z integral would be simply the inverse Fourier transform from k_z back to z_0. Even so, it is still true that the forward transform of M from z_0 to k_z equals v:

$$\int dz_0 \, M(k_g, k_s; z_0) e^{-ik_z z_0} = v(k_g - k_s, k_z, \theta). \qquad \text{(A-12)}$$

We note that by varying $k_g + k_s$, all values of θ can be computed for a fixed $k_g - k_s$ and k_z. Consequently, θ can be considered an independent parameter in equation (A-12).

By combining equations (A-4), (A-9), and (A-12), we almost have equation (55). The missing ingredient is equation (51) which relates the potential v to a reflection coefficient R. Using the relation

$$R(k_x, k_z, \theta) = \frac{-ik_z}{4} \sec^2 \theta v(k_x, k_z, \theta), \qquad \text{(A-13)}$$

equation (55) follows.