Heat Equation Finite Element Solution

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Part 1: Derivation of the Weak Form

$u_t - u_{xx} = f(x,t)$, $x \in (0,1)$, $t \in (0,1)$

$f(x,t) = (\pi^2 - 1)e^{-t}\sin(\pi x)$

$$\int_0^1 \frac{\partial}{\partial t}u(x,t)\,v(x)\,\partial x - \int_0^1 \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}v(x,t)\,\partial x = \int_0^1 f(x,t)\,v(x)\,\partial x$$

$$\int_0^1 \frac{\partial}{\partial t}u(x,t)\,v(x)\,\partial x - \int_0^1 \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}v(x,t)\,\partial x = \int_0^1 (\pi^2 - 1)e^{-t}\sin(\pi x)\,v(x)\,\partial x$$

- Forward Euler: $\dfrac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$

$$\frac{1}{\Delta t}\int_0^1 u(x,t+\Delta t)v(x) - \frac{1}{\Delta t}\int_0^1 u(x,t+\Delta t)v(x) - u(x,t)\int_0^1 \frac{\partial}{\partial x}v(x)\frac{\partial}{\partial x}v(x)\,\partial x = \int_0^1 f(x,t)\,v(x)\,\partial x$$

$v(x)$ $v(x)$

$$\frac{1}{\Delta t}\vec{u}^{(n+1)}\underbrace{\int_0^1 V_j(x)V_i(x)}_{\mathbb{M}} - \frac{1}{\Delta t}\vec{u}^{(n)}\underbrace{\int_0^1 V_j(x)V_i(x)}_{\mathbb{M}} - \vec{u}^{(n)}\underbrace{\int_0^1 \frac{\partial}{\partial x}V_j(x)\frac{\partial}{\partial x}V_i(x)\,\partial x}_{\mathbb{K}} = \underbrace{\int_0^1 f(x,t)\,V_i(x)\,\partial x}_{\mathbb{F}}$$

$$\frac{1}{\Delta t}\vec{u}^{(n+1)}\underbrace{\int_{-1}^1 \hat{\phi}_i(x)\hat{\phi}_j(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{M}} - \frac{1}{\Delta t}\vec{u}^{(n)}\underbrace{\int_{-1}^1 \hat{\phi}_i(x)\hat{\phi}_j(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{M}} - \vec{u}^{(n)}\underbrace{\int_{-1}^1 \left(\frac{\partial \hat{\phi}_i}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \hat{\phi}_j}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{K}} = \underbrace{\int_{-1}^1 \hat{f}(x,t)\hat{\phi}_i(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{F}}$$

$$\vec{u}^{(n+1)} = \Delta t\,\mathbb{M}^{-1}\left(\mathbb{F} + \frac{1}{\Delta t}\vec{u}^{(n)}\mathbb{M} + \vec{u}^{(n)}\mathbb{K}\right) = \vec{u}^{(n)} + \Delta t\,\mathbb{M}^{-1}\left(\mathbb{F} - \mathbb{K}\vec{u}^{(n)}\right)$$

- Backwards Euler: $\dfrac{u(x,t-\Delta t) - u(x,t)}{\Delta t}$

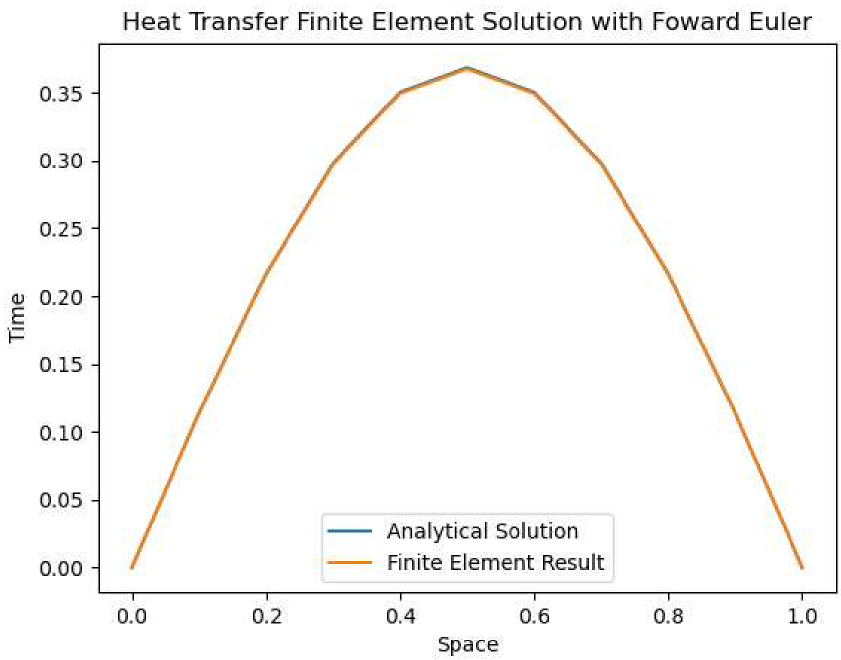
$$\frac{1}{\Delta t}\vec{u}^{(n-1)}\underbrace{\int_0^1 V_j(x)V_i(x)}_{\mathbb{M}} - \frac{1}{\Delta t}\vec{u}^{(n)}\underbrace{\int_0^1 V_j(x)V_i(x)}_{\mathbb{M}} - \vec{u}^{(n)}\underbrace{\int_0^1 \frac{\partial}{\partial x}V_j(x)\frac{\partial}{\partial x}V_i(x)\,\partial x}_{\mathbb{K}} = \underbrace{\int_0^1 f(x,t)\,V_i(x)\,\partial x}_{\mathbb{F}}$$

$$\frac{1}{\Delta t}\vec{u}^{(n-1)}\underbrace{\int_{-1}^1 \hat{\phi}_i(x)\hat{\phi}_j(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{M}} - \frac{1}{\Delta t}\vec{u}^{(n)}\underbrace{\int_{-1}^1 \hat{\phi}_i(x)\hat{\phi}_j(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{M}} - \vec{u}^{(n)}\underbrace{\int_{-1}^1 \left(\frac{\partial \hat{\phi}_i}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \hat{\phi}_j}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{K}} = \underbrace{\int_{-1}^1 \hat{f}(x,t)\hat{\phi}_i(x)\frac{\partial x}{\partial \xi}\partial \xi}_{\mathbb{F}}$$

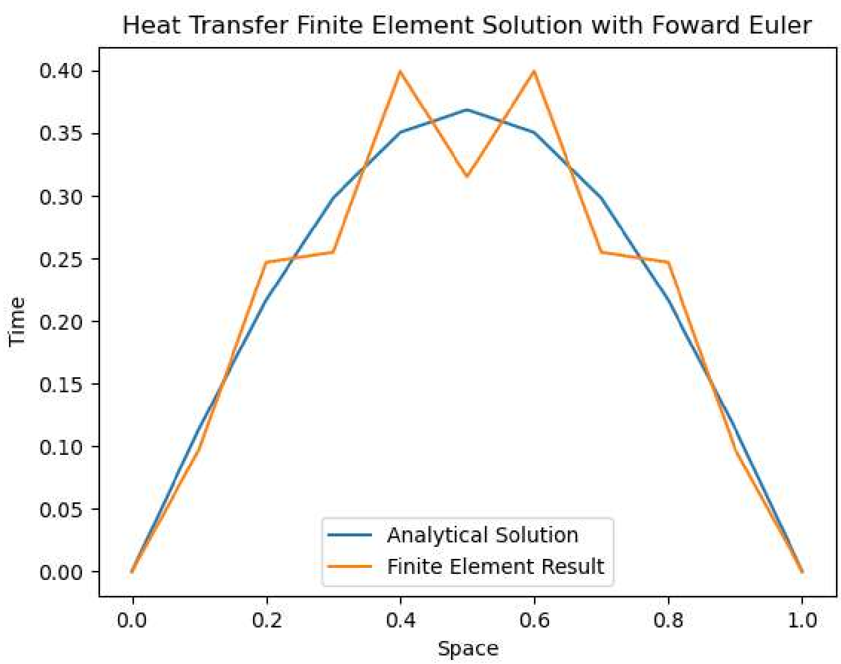
$$\vec{u}^{(n)} = \left(\frac{1}{\Delta t}\mathbb{M} + \mathbb{K}\right)^{-1}\left(\frac{1}{\Delta t}\mathbb{M}\vec{u}^{(n-1)} + \mathbb{F}\right)$$

Part 2: Forward Euler

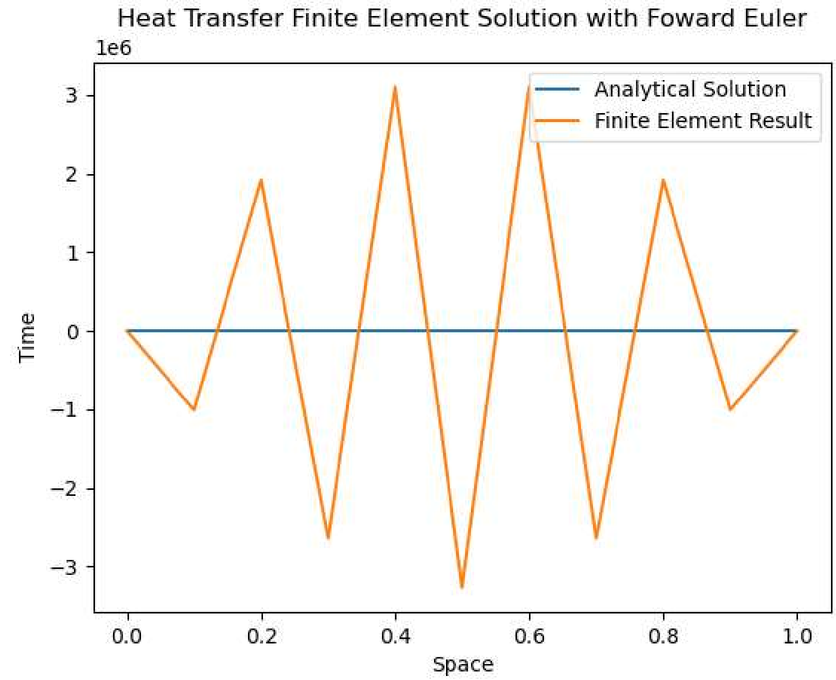
Using $\Delta t = \frac{1}{551}$ and $N = 11$:



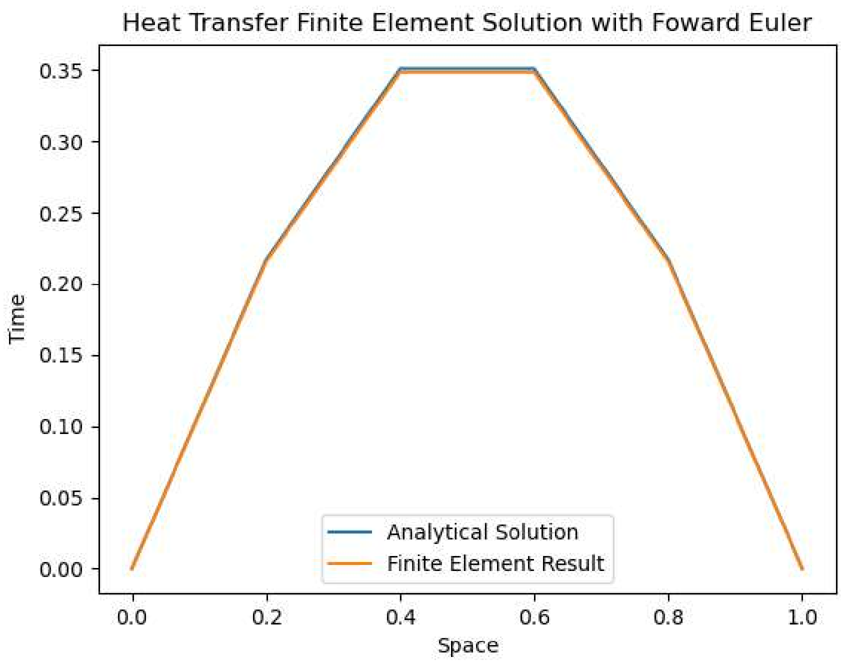
Instability began to occur at $\Delta t = \frac{1}{541}$.



This continued, as the solution continued to assume the form shown at $\Delta t = \frac{1}{531}$.

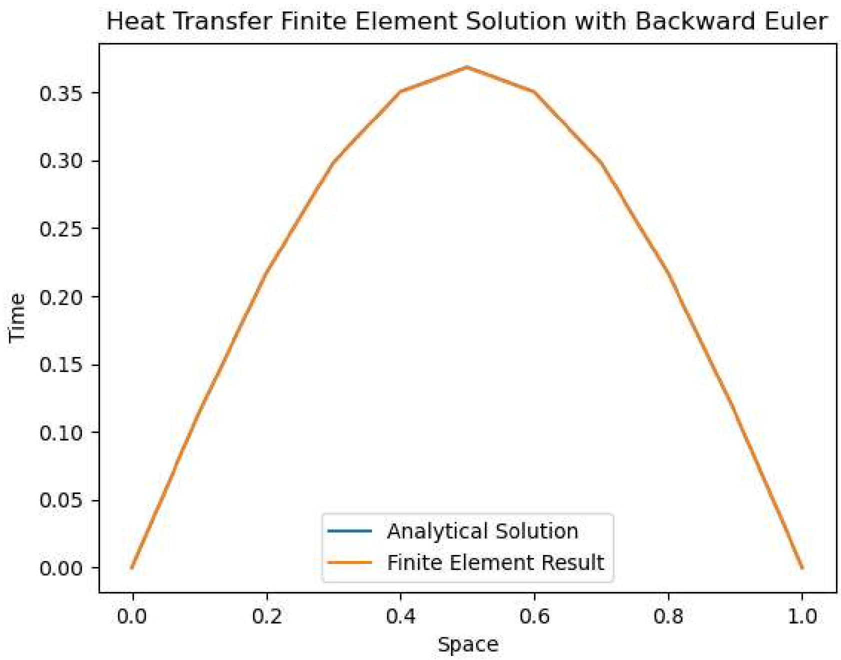


However, with fewer nodes, the stability improved, albeit with less precise results. This is the solution with $N = 6$ and $\Delta t = \frac{1}{301}$.

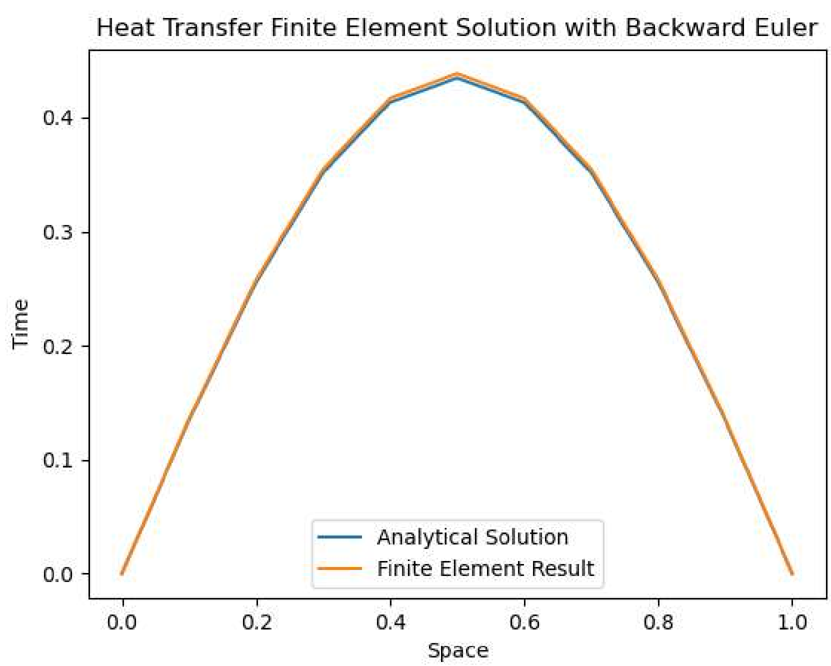


Part 3: Backward Euler

Using $\Delta t = \frac{1}{551}$ and $N = 11$:



Even at $\Delta t = \frac{1}{6}$ and $N = 11$, where the timestep size was much larger than the spatial step size of 0.1, the backward Euler method shows remarkable stability.



The stability is due to the backward Euler method being an implicit method. The matrix multiplied with u^n to obtain u^{n+1} is guaranteed to have eigenvalues less than or equal to 1, which in practice means it cannot add onto the solution in a way that causes it to blow out of proportion.