HW 4

Thursday, October 16, 2025 17:32

Problem 1)

Find QR factorization of $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$Q = [\vec{e}_1, \dots, \vec{e}_k]$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2)$$

$$= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1$$

$$= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \emptyset = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{bmatrix}}$$

Problem 2)

Find the best fit line of $(0,0), (1,2), (2,1), (3,6)$

with weights $2/10, 3/10, 3/10, 2/10$:

$y = mx + b$

$$\underset{\mathbb{A}}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}} \underset{\vec{u}}{\begin{bmatrix} m \\ b \end{bmatrix}} = \underset{\vec{y}}{\begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}}$$

$$\mathbb{C} = \begin{bmatrix} 2/10 & 0 & 0 & 0 \\ 0 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 0 \\ 0 & 0 & 0 & 2/10 \end{bmatrix}$$

$$\mathbb{A}^T\mathbb{C}\mathbb{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2/10 & 0 & 0 & 0 \\ 0 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 0 \\ 0 & 0 & 0 & 2/10 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3/10 & 6/10 & 6/10 \\ 2/10 & 3/10 & 3/10 & 2/10 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 33/10 & 15/10 \\ 15/10 & 10/10 \end{bmatrix}$$

$$(\mathbb{A}^T\mathbb{C}\mathbb{A})^{-1} = \frac{100}{105} \begin{bmatrix} 10/10 & -15/10 \\ -15/10 & 33/10 \end{bmatrix}$$

$$\vec{u} = (\mathbb{A}^T\mathbb{C}\mathbb{A})^{-1}\mathbb{A}^T\mathbb{C}\vec{y}$$

$$= \frac{100}{105} \begin{bmatrix} 10/10 & -15/10 \\ -15/10 & 33/10 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2/10 & 0 & 0 & 0 \\ 0 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 0 \\ 0 & 0 & 0 & 2/10 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$= \frac{100}{105} \begin{bmatrix} -15/10 & -5/10 & 5/10 & 15/10 \\ 33/10 & 18/10 & 3/10 & -12/10 \end{bmatrix} \begin{bmatrix} 2/10 & 0 & 0 & 0 \\ 0 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 0 \\ 0 & 0 & 0 & 2/10 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$= \frac{\cancel{100}}{105} \begin{bmatrix} -30/\cancel{100} & -15/\cancel{100} & 15/\cancel{100} & 30/\cancel{100} \\ 66/\cancel{100} & 54/\cancel{100} & 9/\cancel{100} & -24/\cancel{100} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$105\left[\begin{matrix} 66/\cancel{100} & 54/\cancel{100} & 9/\cancel{100} & -24/\cancel{100}\end{matrix}\right]\left[\begin{matrix}1\\6\end{matrix}\right]$$

$$= \frac{1}{105}\left[\begin{matrix}165\\-27\end{matrix}\right]$$

$$m = \frac{165}{105} = \frac{11}{7}$$

$$b = -\frac{27}{105} = -\frac{9}{35}$$

$$\boxed{y = \frac{11}{7}x - \frac{9}{35}}$$

Problem 3)

$$u_t = Du_{xx}$$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Using backwards Euler, $u_t = \dfrac{u(t+\Delta t)-u(t)}{\Delta t}$

Using 2nd-order centered difference, $u_{xx} = \dfrac{u(x+\Delta x)-2u(x)+2u(x-\Delta x)}{\Delta x^2}$

$$\frac{u(t+\Delta t)-u(t)}{\Delta t} = D\,\frac{u(x+\Delta x)-2u(x)+u(x-\Delta x)}{\Delta x^2}$$

$$u(x,t)\left(-\frac{1}{\Delta t}+\frac{2D}{\Delta x^2}\right) - \frac{D}{\Delta x^2}u(x+\Delta x) - \frac{D}{\Delta x^2}u(x-\Delta x) = -\frac{1}{\Delta t}u(t+\Delta t)$$

Let $u(x,t) = Ge^{ikx}$:

$$u(x,t)\left(-\frac{1}{\Delta t}+\frac{2D}{\Delta x^2}\right) - \frac{D}{\Delta x^2}\left(Ge^{ik(x+\Delta x)} + Ge^{ik(x-\Delta x)}\right) = -\frac{1}{\Delta t}u(t+\Delta t)$$

$$u(x,t)\left(-\frac{1}{\Delta t}+\frac{2D}{\Delta x^2}\right) - \frac{D}{\Delta x^2}Ge^{ikx}\left(2G\cos(k\Delta x)\right) = -\frac{1}{\Delta t}u(t+\Delta t)$$

$$u(t+\Delta t) = \Delta t\, u(x,t)\left(\frac{1}{\Delta t} - \frac{2D}{\Delta x^2} + \frac{D}{\Delta x^2}2G\cos(k\Delta x)\right)$$

Growth term $G = \dfrac{u(t+\Delta t)}{u(t)}$

$$G = 1 + 2D\frac{\Delta t}{\Delta x^2}\left(-1 + G\cos(k\Delta x)\right)$$

$$G\left(1 - 2D\frac{\Delta t}{\Delta x^2}\cos(k\Delta x)\right) = 1 - 2D\frac{\Delta t}{\Delta x^2}$$

$$\boxed{G = \frac{1-2D\frac{\Delta t}{\Delta x^2}}{1-2D\frac{\Delta t}{\Delta x^2}\cos(k\Delta x)}}$$

Problem 4)

$$f(x) = 4+8x^2-x^4$$

- $\dfrac{\partial f(x)}{\partial x} = 16x - 4x^3$

$f(x) = 4 + 8x^2 - x^4$

- $\frac{\partial f(x)}{\partial x} = 16x - 4x^3$

  $\frac{\partial^2 f(x)}{\partial x^2} = 16 - 12x^2$

- y-intercept: $(0, 4)$

  $16x - 4x^3 = 0$

  $(4 - x^2)x = 0$

  Extrema: $x = -2, 0, 2$

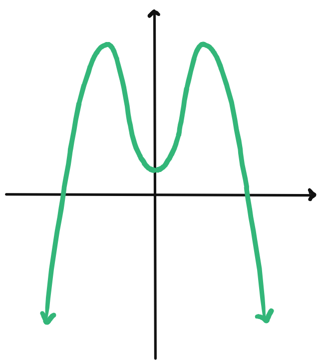
  Maxima: $(-2, 20), (2, 20)$

  Minimum: $(0, 4)$

  $16 - 12x^2 = 0$

  $x^2 = \frac{4}{3}$

  POIs: $\left(-\sqrt{\frac{4}{3}}, \frac{116}{9}\right), \left(\sqrt{\frac{4}{3}}, \frac{116}{9}\right)$

- 

  The function is even

- ```
  Root of f(x)=4+8x^2-x^4 using Newton's method:
  (2.9107225401716925, -0.0015183266996672273)
  Error: 0.0015183266996672273
  Number of iterations: 2
  ```

Problem 5)

$f(x) = x^3 - 3x - 3$

- $\frac{\partial f(x)}{\partial x} = 3x^2 - 3$

  $\frac{\partial^2 f(x)}{\partial x^2} = 6x$

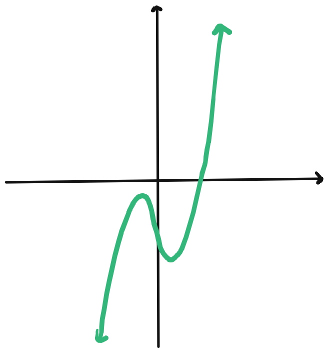
  $3x^2 - 3 = 0$
  ```

Extrema: $x = -1, 1$

Maximum: $(-1, -1)$

Minimum: $(1, -5)$

POI: $(0, -3)$

- 

This function is neither odd nor even

- ```
  Root of f(x)=x^3-3x-3 using Newton's method:
  (2.103835978835979, 0.0003348227593420461)
  Error: 0.0003348227593420461
  Number of iterations: 2
  ```

Problem (e)

- ```
  Root of f(x)=x^2-3 using bisection method:
  (1.4422607421875, 6.971588118176442e-05)
  Error: 6.971588118176442e-05
  Number of iterations: 13
  ```