Inverse

$$\partial \underline{\underline{I}}^E = \begin{bmatrix} s & u \\ u & t \end{bmatrix}$$

$$\underline{\underline{I}}^E = \begin{bmatrix} s/2 & u/2 \\ u/2 & t/2 \end{bmatrix}$$

Note:
Inverse of Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & - \\ -c & a \end{bmatrix}$$

$$\left(\underline{\underline{I}}^E\right)^{(-1)} = \begin{bmatrix} s/2 & u/2 \\ u/2 & t/2 \end{bmatrix}^{(-1)}$$

$$= \frac{1}{\underbrace{\left(\frac{s}{2}\right)\left(\frac{t}{2}\right)}_{4} - \underbrace{\left(\frac{u}{2}\right)\left(\frac{u}{2}\right)}_{4}} \begin{bmatrix} t/2 & -u/2 \\ -u/2 & s/2 \end{bmatrix}$$

$$= \frac{4}{st - u^2} \times \begin{bmatrix} t/2 & -u/2 \\ -u/2 & s/2 \end{bmatrix}$$

$$x = W \times \frac{\partial V^m}{\partial \tau^2} \qquad y = W \times \frac{\partial V^m}{\partial \tau^2}$$

$$\hookrightarrow \frac{\partial V^m}{\partial \tau^2} = \frac{x}{W} \qquad \hookrightarrow \frac{\partial V^m}{\partial \tau^2} = \frac{y}{W}$$

Substituting

$$Var(V^m) \approx \left(\frac{\partial V^m}{\partial(\omega^2, \tau^2)}\right)\left(\underline{\underline{I}}^E\right)^{(-1)}\left(\frac{\partial V^m}{\partial(\omega^2, \tau^2)}\right)$$

Note:
Write in matrix form (Expanding)

$$\frac{\partial V^m}{\partial(\omega^2, \tau^2)} = \begin{bmatrix} \partial V^m / \partial \omega^2 \\ \partial V^m / \partial \tau^2 \end{bmatrix} \begin{matrix} \to y \\ \to x \end{matrix}$$

Does it matter?

Substitute in and expand matrix form

$$= \begin{bmatrix} \partial V^m/\partial\omega^2 & \partial V^m/\partial\tau^2 \end{bmatrix} \frac{4}{(st-u^2)} \begin{bmatrix} t/2 & -u/2 \\ -u/2 & s/2 \end{bmatrix} \times \begin{bmatrix} \partial V^m/\partial\omega^2 \\ \partial V^m/\partial\tau^2 \end{bmatrix}$$

$$= \begin{bmatrix} y/W & x/W \end{bmatrix} \times \frac{4}{(st-u^2)} \times \begin{bmatrix} t/2 & -u/2 \\ -u/2 & s/2 \end{bmatrix} \times \begin{bmatrix} y/W \\ x/W \end{bmatrix}$$

multiply everything out

$$Var(V^m) = \frac{2(sy^2+tx^2-2uxy)}{W^2(st-u^2)}$$

Now Find ζ (degrees of freedom)

Substitute $\frac{1}{W}$ for $E(V^m)$ in the equation for degrees of freedom:

$$\zeta = \frac{2[E(V^m)]^2}{Var(V^m)} = \frac{2[\frac{1}{W}]^2}{Var(V^m)}$$

Substituting $Var(V^m)$

$$\zeta = \cancel{2}\left(\frac{1}{W}\right)^2 \left(\frac{W^2(st-u^2)}{\cancel{2}(sy^2+tx^2-2uxy)}\right)$$

$$\boxed{= \frac{st-u^2}{sy^2+tx^2-2uxy}}$$

$$w_j = \frac{k_j}{k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)}$$

Take partial derivative of above with respect to $\hat{\tau}^2$.

$$\frac{\partial w_j}{\partial \hat{\tau}^2} = \frac{\partial}{\partial \hat{\tau}^2}(k_j)\left(k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)\right)^{(-1)}$$

$$= (-1)(k_j)\left(k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)\right)^{-2} \times$$

$$(k_j + \varnothing + \varnothing + \varnothing)$$

$$= \frac{-(k_j)^2}{\left(k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)\right)^2} = \boxed{w_j^2}$$

Now take partial derivative of w_j with respect to $\hat{\omega}^2$

$$\frac{\partial w_j}{\partial \hat{\omega}^2} = (-1)(k_j)\left(k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)\right)^{-2} \times$$

$$(\varnothing + \varnothing + 1 + \varnothing)$$

$$= \frac{-(k_j)}{\left(k_j\hat{\tau}^2 + k_j\rho\sigma_j^2 + \hat{\omega}^2 + (1-\rho)(\sigma_j^2)\right)^2}\frac{(k_j)}{(k_j)} = \boxed{\frac{-w_j^2}{k_j}}$$

$\longrightarrow$

Since V^m is $\frac{1}{W}$ and $W = \sum_{j=1}^{J} w_j^{(-1)}$

$$x = W \frac{\partial V^m}{\partial \hat{\tau}^2} = W \times \frac{\partial}{\partial \hat{\tau}^2}\left(\sum_{j=1}^{J} \omega_j\right)^{-1} =$$

take partial derivative

$$= \cancel{W}\left(\frac{-1}{\cancel{\left(\sum_{j=1}^{J} \omega_j\right)^2}}\right) \times \sum_{j=1}^{J} \frac{\partial \omega_j}{\partial \hat{\tau}^2}$$

$$= \cancel{W}\left(\frac{\cancel{-1}}{\cancel{W^2}}\right) \times \sum_{j=1}^{J} \cancel{-}\,\omega_j^2$$

$$\boxed{= \frac{1}{W} \sum_{j=1}^{J} \omega_j^2}$$

$$y = W \frac{\partial V^m}{\partial \hat{\omega}^2}$$

$$= W \frac{\partial}{\partial \hat{\omega}^2}\left(\sum_{j=1}^{J} \omega_j\right)^{-1}$$

$$= \cancel{W}\left(\frac{\cancel{-1}}{\cancel{W^2}}\right) \times \sum_{j=1}^{J} \cancel{-}\frac{\omega_j^2}{k_j}$$

$$= \boxed{\frac{1}{W} \sum_{j=1}^{J} \frac{\omega_j^2}{k_j}}$$

$$\underline{\Phi}_j = (\tau^2 + \rho\sigma_j^2)\,\underline{1}_j\,\underline{1}_j' + (\omega^2 + (1-\rho)\sigma_j^2)\,\underline{I}_j$$

Let
$$\underline{\Phi} = \text{diag}(\underline{\Phi}_1, \underline{\Phi}_2, \ldots, \underline{\Phi}_J)$$

Q: Is this a diagnol matrix? Dimensions:

$J \times J$? or $k \times k$?

$$\underline{Q} = \underline{\Phi}^{-1} - \underline{\Phi}^{-1}\,\underline{1}\,(\underline{1}'\,\underline{\Phi}^{-1}\,\underline{1})^{-1}\,\underline{1}'\,\underline{\Phi}^{-1}$$

So, $\underline{\Phi}$ must be a square matrix to take inverse

Dimensions of Q?

$$s = tr\left(\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\tau}^2}\,\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\tau}^2}\right)$$

$$t = tr\left(\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\omega}^2}\,\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\omega}^2}\right)$$

$$u = tr\left(\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\tau}^2}\,\underline{Q}\,\frac{\partial\Phi}{\partial\hat{\omega}^2}\right)$$

$$\underline{Z} = \text{diag}(1_1, 1_2, \ldots 1_J)$$

What is dimensions of $\underline{Z}$?

Why isn't this an identity matrix too?

and
$$\frac{\partial\Phi}{\partial\hat{\tau}^2} = \underline{Z}\,\underline{Z}'$$

$$\frac{\partial\Phi}{\partial\hat{\omega}^2} = \underline{I} \quad \text{with dimension } k \times k.$$

so, $s = tr(\underline{Q}\,\underline{Z}\,\underline{Z}'\,\underline{Q}\,\underline{Z}\,\underline{Z}') = tr(\underline{Z}'\,\underline{Q}\,\underline{Z}\,\underline{Z}'\,\underline{Q}\,\underline{Z})$

Why is this commutative? Does that mean they are both diagnol matrices with same dimensions? How is $\underline{Z}$ different from $\underline{I}$?

$t = tr(\underline{Q}\,\underline{Q})$

$u = tr(\underline{Q}\,\underline{Z}\underline{Z}'\,Q) = tr(Z'QQZ)$