Problem Set 3

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4.2

V $= span(1, x, x^2)$ is a subspace of the inner product space $L^2([0, 1]; \mathbb{R})$. From the exercise in week 2, we know that D is an upper triangular matrix; thus, all the eigenvalues are 0 and the algebraic multiplicity is 3. The geometric multiplicity is 1, because there is one eigenvector for D.

4.4

(i)

By Exercise 4.3, we know that $p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - (a + d)\lambda + (ad - bc)$ The solutions to this equation are thus

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

Since A^H is Hermetian, $b = \bar{c}$, and the discriminant under the squareroot can be reduced to $\sqrt{(a + d)^2 - 4(ad - bc)} = a^2 + 2ad + d^2 - 4ad + 4bc = (a - d)^2 + 4bc$. Thus, $(a - d)^2$ and $4bc$ are both positive, the roots are real, and thus the eigenvalues of an Hermetian 2 x 2 matrix must be real.

(ii)

In a skew-Hermitian 2 x 2 matrix, $b = -\bar{c}$, which means that the roots are not real, and thus the eigenvalues must all me imarginary.

4.6

Proposition 4.1.22 states that the diagonal entries of an upper-triangular (or a lower-triangular) matrix are its eigenvalues.

L A be an upper triangular matrix. The determinant of this upper triangular matrix is represented by: $\det(A) = \prod_{i=1}^{n} a_i$.

To find the eigenvalues, solve the characteristic polynomial, $\det(\lambda I - A) = 0$.

Thus, we get $\det(\lambda I - A) = \prod_{i=1}^{n}(\lambda_i - a_{ii}) = 0 = (\lambda_1 - a_{11}) \cdot (\lambda_2 - a_{22}) \cdots (\lambda_n - a_{nn}) = 0$

Thus, the eigenvalues are the diagonal elements of matrix A.

4.8

V is the span of the set $S = \{sin(x), cos(x), sin(2x), cos(2x)\}$ in the vector space $C^\infty(\mathbb{R}; \mathbb{R})$.

(i)

We saw in a Chapter 3 exercise that this set is orthonormal. Thus, each vector in the set is independent and the vectors span. Thus, S is a basis for V.

(ii)

If D is the derivative operator, then

$$Dsin(x) = cos(x)$$

$$Dcos(x) = -sin(x)$$

$$Dsin(2x) = 2cos(2x)$$

$$Dcos(2x) = -2sin(2x)$$

Thus, D is represented by the following:

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii)

Two compementary D-invarient subspaces in V are $span(\{sin(x), cos(x)\})$ and $span(\{sin(x2), cos(2x)\})$.

4.13

$$A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

The roots of the characteristic polynomial are 0.4 and 1, thus these are the eigenvalues. From these eigenvalues, we get the eigenvectors $[2, 1]^T$ and $[1, -1]^T$. Thus,

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Plugging this in, we can confirm that this does indeed give us a diagonal matrix $P^(- 1)AP$:

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

4.15

If A is a semisimple matrix, then it is diagonalizable to matrix D, thus there exists P such that $P^{-1}AP = D$. Thus, $A^k = PD^kA^{-1}$. Given the f(x) in Theorem 4.3.12, $f(A) = a_0I + a_1A + ... + a_nA^n = P(a_0I + a_1D + ... + a_nD^n)P^{-1}$. Thus, the theorem is proven.

4.16

Recall

$$A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

(i)

Using Proposition 4.3.10, which states that if $A, B \in M_n(\mathbb{F})$ are similar, $A^k = P^{-1}B^kP$ for all $k \in \mathbb{N}$, and the results from 4.13, we see that

$$B^k = \begin{bmatrix} 1^k & 0 \\ 0 & 0.4^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix}$$

(ii)

Yes, the answer depends on the choice of norm.

(iii)

The eigenevalues are $f(1) = 3 + 5 + 1 = 9$ and $f(0.4) = 3 + 5 * 0.4 + 0.4^3 = 5.064$.

4.18

If λ is an eigenvalue of the $A \in M_n(\mathbb{F})$, then there exists a nonzero row vector $(\mathbf{x})^\mathbf{T}$ such that $(\mathbf{x})^\mathbf{T}\mathbf{A} = \lambda(\mathbf{x})^\mathbf{T}$.

Proof: A and A^T have the same characteristic polynomial, thus λ is an eigenvalue of A^T. Thus, $A^T(\mathbf{x}) = \lambda(\mathbf{x})$. Transposing this, we get $(\mathbf{x})^\mathbf{T}\mathbf{A} = \lambda(\mathbf{x})^\mathbf{T}$.

4.20

Lemma 4.4.2 states that if A is Hermitian and orthonormally similar to B, then B is also Hermitian.

Proof: from Def 4.4.1, we know $B = U^HAU$. Thus, $B = U^HAU = U^HA^HU = (U^HAU)^H = B^H$.

4.24

Since matrix A is Hermitian, then we have

$$\langle x, Ax\rangle = x^HAx = x^HA^Hx = \langle Ax, x\rangle = \overline{\langle x, Ax\rangle}$$

Thus, this must be real, and the numerator of the Rayleigh quotient must be real, and thus the Rayleigh quotient must take on only real values. We can show the opposite for skew-Hermitian matrices.

4.25

(i)

Since we know that $\langle x_j, x_j\rangle = 1$, we find that $(x_1x_1^H + ... + x_nx_n^H)x_j = x_jx_j^Hx_j = Ix_j$, thus, $I = x_1x_1^H + ... + x_nx_n^H$.

4.27

All positive definite matricies have the property $\langle x^H, Ax\rangle = x^HAx > 0 \forall x \neq 0$. Take e_i as the standard basis vector. We get $e_i^HAe_i > 0$. Thus, all the diagonal entries of A are real and positive.

4.33

We know by Exercise 4.31 that $||A||_2 = \sigma_1$. We know that $sup_{||x||_2=1,||y||_2=1}|y^H Ax| \leq sup_{||x||_2=1}||\sum X||_2 \leq \sigma 1$.

We also know $sup_{||x||_2=1,||y||_2=1}|y^H Ax| \geq |y^H Ax| = \sigma_1$.

Thus, $sup_{||x||_2=1,||y||_2=1}|y^H Ax| = \sigma_1$. By 4.31, we know $\sigma_1 = ||A||_2$, so the proof is done.

4.36

An example of a 2 x 2matrix whose determinant is nonzero and whose singular values are not equal to any of its eigenvalues is as follows:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Determinant: $det(A) = 0 - 1(2) = -2$

Singular values: 1, 2

Eigenvalues: $\pm\sqrt{2}$