ProbSet1 - Math

Problem Set 1

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Exercise 1.3

G_1: not an algebra or a sigma algebra because it is not closed under complements and finite unions.

G_2: algebra, but not a sigma-algebra because the unions are not countable.

G_3: algebra and sigma-algebra.

Exercise 1.7

$\{\emptyset, X\}$: A σ-algebra must, by definition, include $\emptyset$ and be closed under complements, thus, the smallest set must contain $\emptyset$ and its complement, X. This is thus the smalled σ-algebra.

$P(X)$: this is the largest set such that all elements in the set A are in P(X), thus, it must be the largest σ-algebra because it is closed under countable unions.

Exercise 1.10

Let $\{S_\alpha\}$ be a family of σ-algebras on X. Then $\cap_\alpha S_\alpha$ is also a σ-algebra.

To be a σ-algebra, $\cap_\alpha S_\alpha$ must contain the empty set, $\{\emptyset\}$. It does, because each S_α is a σ-algebra, so each contains the empty set, so the intersection of them contains the empty set.

$\cap_\alpha S_\alpha$ must also be closed under complements and finite unions. Suppose $A \in \cap_\alpha S_\alpha$. Then $A \in S_\alpha$ for some α. We know $A^c \in \cap_\alpha S_\alpha$ because each S_α is a σ-algebra. Thus, it is closed under complements and finite unions.

$\cap_\alpha S_\alpha$ must also be closed under countable unions. If we choose arbitrary sets $A_1, A_2, ... \in S_\alpha$, we know that each of these sets is in a σ-algebra and $\cup_{i=1}^{\infty} A_i \in \cap_\alpha \mathcal{S}_\alpha$. Thus, it is closed under countable unions.

By the definition of σ-algebra (showing the three properties above), $\cap_\alpha S_\alpha$ is also a σ-algebra.

Exercise 1.17

Let (X, S, μ) be a measure space. Prove the following:

μ is monotone: if $A, B \in S$, $A \subset B$, then $\mu(A) \leq \mu(B)$.

Let $A, B \in S$, and $A \subset B$. We notice A and B are disjoint, that is, $B = (B \cap A^c) \cup A$, where $(B \cap A^c) \cap A = \emptyset$. By definition, $\mu(B) = \mu(B \cap A^c) + \mu(A) \geq 0$. Since $\mu(B \cap A^c) \geq 0$, we know $\mu(A) \leq \mu(B)$. Thus, μ is monotone.

μ is countably subadditive: if $\{A_i\}_{i=1}^{\infty} \in A$, then $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty}(\mu(A_i))$.

Let $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - (A_1 \cap A_2) \ldots$ Then, $\cup_n A_n = \cup_n B_n$. From monotoncity, we know $\mu(A_n) \leq \mu(B_n)$. Thus, $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty}(\mu(A_i))$.

Exercise 1.18

Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda : \mathcal{S} \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure $(X, \mathcal{S})$.}

Let $(X, \mathcal{S}, \mu)$ be a measure space and let $B \in \mathcal{S}$. If $\lambda(A) : \mathcal{S} \to [0, \infty]$ where $\lambda(A) = \mu(A \cap B)$, then $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$, which satisfies the first property of a measure. Then, $\lambda(\cup_{i=1}^{\infty} A_i) = \mu((\cup_{i=1}^{\infty} A_i) \cap B) = \mu(\cup_{i=1}^{\infty}(A_i \cup B)) = \sum_{i=1}^{\infty} \mu(A_i \cup B) = \sum_{i=1}^{\infty} \lambda(A_i)$ because A is disjoint and thus $(A_i \cup B)$ is disjoint. Thus, the second property of a measure is satisfied, and $\lambda : \mathcal{S} \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure $(X, \mathcal{S})$.

Exercise 1.20

Let μ be a measure on $(X, \mathcal{S})$ and $(A_1 \subset A_2 \subset A_3 \subset \ldots, A_i \in \mathcal{S}, \mu(A_1) < \infty)$. Because this sequence of sets is decreasing, we know $A_1 - A_i$ increases as i increases, and $(A_i) < \infty$ for each $i \in \mathbb{N}$.

Then, $\lim_{i\to\infty}(A_1 - A_i) = A_1 - \lim_{i\to\infty} A_i = A_1 - A$. Then $\mu(\cap_{i=1}^{\infty} A_i) = \mu[A_1 - \cup_{i=1}^{\infty}(A_1 - A_i)] = \mu(A_1) - \mu(\cup_{i=1}^{\infty}(A_1 - A_i)) = \mu(A_1) - \lim_{i\to\infty}\mu(A_1 - A_i) = \mu(A_1) - \lim_{i\to\infty}[\mu(A_1) - \mu(A_i)] = \lim_{i\to\infty}\mu(A_i)$. So $\mu(A) = \lim_{n\to\infty}\mu(A_i)$.

Exercise 2.10

Explain why $(*)$ in the preceding theorem could be replaced by

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

.

Since μ^* is an outer measure, it is countably subadditive. Thus, $\mu^*(\cup_{i=1}^{\infty}) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Exercise 2.14

Why is it true that the Borel-algebra $\mathcal{B}(\mathbb{R})$ is a subset of $\mathcal{M}$? Hint: Caratheodory does most the work - you only need to show that $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$.

$\mathcal{O}$ is all open sets on $\mathbb{R}$ and $\sqsubseteq$ be the premeasure on $\mathbb{R}$. Let μ^* denote the outer measure generated by $\sqsubseteq$, and$\mathcal{M}$ denote the σ-algebra from the Caratheodory construction. Thus, $\sigma(\mathcal{O})$ is contained in the σ-algebra from the Caratheodory construction and thus $\sigma(\mathcal{O}) \subset \mathcal{M}$.

Exercise 3.1

Prove that every countable subset of the real line has Lebesgue measure 0.

Define $X = \{x_1, x_2, ...\}$ for all $x_i \in \mathbb{R}$. Then a cover of the set can be defined as $\{x - \epsilon, x + \epsilon\}$ for $\epsilon > 0$. Thus, $\lambda^*(\{x\}) = 0$ for all $x \in \mathbb{R}$, and since it is countable, $\lambda^*(\{X\}) \leq \sum_{i=1}^{\infty} \lambda^*(\{X_n\}) = 0$. Thus, the outer measure is restricted to $\mathcal{M}$ must have a measure 0 (this is the Lebesgue measure).

Exercise 3.4

Explain why the set (*) could be replaced by any of the following:

$$\{x \in X : f(x) \leq a\}$$

$$\{x \in X : f(x) > a\}$$

$$\{x \in X : f(x) \geq a\}$$

If $\{x \in X : f(x) < a\}$ is measurable ($\in \mathcal{M}$). Since $\mathcal{M}$ is a σ-algebra, the the complement $\{x \in X : f(x) < a\}^C = \{x \in X : f(x) \geq a\}$ is also in $\mathcal{M}$. Thus, $\{x \in X : f(x) = a\}$ is also in $\mathcal{M}$ and $\{x \in X : f(x) \leq a\}$ is also in $\mathcal{M}$. Finally, $\{x \in X : f(x) \leq a\}^C = \{x \in X : f(x) > a\}$ is also in $\mathcal{M}$.

Exercise 3.7

Explain why 2. and 4. imply 1.

If f and g are both continuous, measurable functions, and there exists and continuous mapping $f + g$ and $f * g$ that are both continuous functions, they are measurable functions by property 1, and thus property 4 holds. If a maximum exists for (f, g) then it is the supremum, and if a minimum exists for (f, g) it is the infimum, therefore by property 1, property 2 (that $max(f, g)$ and $min(f, g)$ are measurable) implies property 2.

Exercise 3.14

Prove (4): if f is bounded, the convergence in (1) is uniform.

For any $f : x \to \mathbb{R}$, assume by (1) that f $\exists\{s_n\}$ and is bounded. So $|f| < N$ for some N for all $x \in X$. Then $x \in E_i^N$ for some i. Let $M \in \mathbb{N}$ be a bound for f where $M \geq N$ satisfying $\frac{1}{2^M} < \epsilon$.

Thus, for all x and $n \geq M$, $|s_n(x) - f(x)| < \epsilon$. Thus, it follows that convergence is uniform.

Exercise 4.13

If f is measurable, $||f|| < \mathcal{M}$ on $E \subset \mathcal{M}$ and $\mu(E) < \infty$, then $f \in \mathcal{L}^1(\mu, E)$.

Since we know f is measurable and $||f|| = f^+ + f^-$ is bounded by Remark 4.10. Thus, f is integrable with respect to μ, in other words $f \in \mathcal{L}^1(\mu, A)$.

Exercise 4.14

If $f \in \mathcal{L}^1(\mu, E)$ then f is finite almost everywhere on E.

Since $f \in \mathcal{L}^1(\mu, E)$, it is measurable. Thus, we define $E_n = \{x \in E, f(x) \geq n\}$ such that $\int_E f d\mu < \int_{E_n} n d\mu = n\mu(E_n) < \infty$. Thus, f is finite almost everywhere on E.

Exercise 4.15

$f, g \in \mathcal{L}^1(\mu, E), f \leq g$ on $E \Rightarrow \int_E f d\mu \leq \int_E g d\mu$

$f, g \in \mathcal{L}^1(\mu, E)$ implies $f, g \geq 0$ are measurable on E and $E \in \mathcal{M}$. Since we also know $f \leq g$, we can apply Proposition 4.6 and get that $\int_E f d\mu \leq \int_E g d\mu$.

Exercise 4.16

If $f \in \mathcal{L}^1(\mu, E)$, $A \in \mathcal{M}$, $A \subset E \Rightarrow f \in \mathcal{L}^1(\mu, A)$

If $A \in \mathcal{M}$ and $A \subset E$, since f is measurable, we know $\int_A f = \int_A f^+ + \int_A f^-$. We also know that $||f|| = f^+ + f^- \Rightarrow f \in \mathcal{L}^1(\mu, E)$.

Exercise 4.21

Prove the corollary.

$$\begin{gathered}
A = (A - B) \cup B \\
\Rightarrow \mu(A) = \mu(A - B) + \mu(B) \\
\Rightarrow \mu(A) + \mu(B) = 0 \\
\Rightarrow \int_A f = \mu(A) - \mu(B) = \int_B f \\
\Rightarrow \int_A f - \mu(A) = \int_B f + \mu(B) \\
\Rightarrow \int_A f \leq \int_B f
\end{gathered}$$

by additive separability.