ProbSet2

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Problem 3.1

(i)

$$\frac{1}{4}||x+y||^2 - ||x-y||^2 = \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle)$$

$$= \frac{1}{4}(((\langle x+y, x\rangle + \langle x+y, y\rangle) - (\langle x-y, x\rangle + \langle x-y, -y\rangle))$$

$$= \frac{1}{4}(((\langle x, x+y\rangle + \langle y, x+y\rangle) - (\langle x, x-y\rangle + \langle -y, x-y\rangle)$$

$$= \frac{1}{4}(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - (\langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle))$$

$$= \frac{1}{4}(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle)$$

$$= \frac{1}{4}(4\langle x, y\rangle)$$

$$\langle x, y\rangle$$

(ii)

$$\frac{1}{2}||x+y||^2 - ||x-y||^2 = \frac{1}{2}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle)$$

$$= \frac{1}{2}(((\langle x+y, x\rangle + \langle x+y, y\rangle) - (\langle x-y, x\rangle + \langle x-y, -y\rangle))$$

$$= \frac{1}{2}(((\langle x, x+y\rangle + \langle y, x+y\rangle) - (\langle x, x-y\rangle + \langle -y, x-y\rangle)$$

$$= \frac{1}{2}(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - (\langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle))$$

$$= \frac{1}{2}(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle)$$

$$= \frac{1}{2}(2\langle x, x\rangle + 2\langle y, y\rangle)$$

$$= \frac{1}{2}(2||x||^2 + 2||y||^2)$$

$$||x||^2 + ||y||^2$$

Problem 3.2

$$\frac{1}{4}(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2)$$
$$= \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i(\langle x-iy, x-iy\rangle - i(\langle x+iy, x+iy\rangle)$$
$$= \frac{1}{4}(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle - \langle x,-y\rangle - \langle -y,x\rangle - \langle -y,-y\rangle +$$
$$i\langle x,x\rangle + i\langle x,-iy\rangle + i\langle -iy,x\rangle + i\langle -iy,-iy\rangle - i\langle x,x\rangle - i\langle x,iy\rangle - i\langle iy,x\rangle - i\langle iy,iy\rangle)$$
$$= \langle x,y\rangle$$

Problem 3.3

(i)

$$< x, x^5 >= \int_0^1 x^6 dx = x^7/7|_0^1 = 1/7$$
$$||x|| = \int_0^1 x^2 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$$
$$||x^5|| = \int_0^1 x^{10} dx = \frac{x^1}{11}|_0^1 = \frac{1}{11}$$

Thus, we see that $\cos\theta = \frac{\frac{1}{7}}{\sqrt{\frac{1}{3}\frac{1}{11}}} = \sqrt{33}/7$, thus, $\theta \approx 34.5$.

(ii)

$$< x^2, x^4 >= \int_0^1 x^6 dx = x^7/7|_0^1 = 1/7$$
$$||x^2|| = \int_0^1 x^4 dx = \frac{x^5}{5}|_0^1 = \frac{1}{5}$$
$$||x^4|| = \int_0^1 x^8 dx = \frac{x^1}{9}|_0^1 = \frac{1}{9}$$

Thus, we see that $\cos\theta = \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}\frac{1}{9}}} = \sqrt{45}/7$, thus, $\theta \approx 17$.

Problem 3.8

(i)

Check if basis is normalized (norms = 1):

$$||cos(t)|| = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt = \frac{1}{\pi}\frac{cos(x)sin(x)+x}{2}|_{-\pi}^{\pi} = 1$$

Using the same method, $||sin(t)||$ is also 1.

$$||cos(2t)|| = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(2t)dt = \frac{1}{\pi}\frac{sin(4x)+4x}{8}|_{-\pi}^{\pi} = 1$$

Using the same method, $||sin(2t)||$ is also 1.

Check if orthoganal (inner products = 0):

$$\langle cos(t), sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(t)dt = \frac{sin^2(x)}{2}|_{-\pi}^{\pi} = sin^2(\pi) - sin^2(-\pi) = 0$$

Checking all the other inner products finds them all equal to 0, thus, S is an orthonormal set.

(ii)

$$||t|| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt} = \frac{1}{\sqrt{\pi}}\frac{t^3}{3}|_{-\pi}^{\pi} = \frac{1}{\sqrt{\pi}}(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}) = \frac{1}{\sqrt{\pi}}(\frac{2\pi^3}{3})$$

$$||t|| = \frac{2\pi^{5/2}}{3}$$

(iii)

$$proj_X(cos(3t)) = 0$$

because $\langle t, cos(3t)\rangle = 0 \forall x \in S$

(iv)

$$proj_X(t) = 0 + 2\pi sin(t) + 0 - \pi sin(2t) = 2\pi sin(t) - \pi sin(2t)$$

Problem 3.9

Proof : A rotation (2.17) $\mathbb{R}^2$ is an orthonormal transformation because

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^T \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} =$$

$$\begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & 0\\ 0 & \cos^2(\theta)+\sin^2(\theta)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Since $RR^T = I$, this proof is complete.

Problem 3.10

(i)

$\Rightarrow$: Let Q be an orthonormal operator on some field. Choose an arbitrary x,y. Then, $\langle x, y\rangle = \langle Q(x), Q(y)\rangle \forall x, w \in$ the field. This implies that $\langle Q(x), Q(y)\rangle = (Qx)^H Qy = x^H Q^H Qy = x^H y$. iff $Q^H Q = I$ if x and y were arbitrarily chosen. We also see that when $Q^H Q = QQ^H, \langle Q(x), Q(y)\rangle = (Qx)^H Qy = x^H Q^H Qy = x^H y = \langle x, y\rangle$.

(ii)

$||Qx|| = \sqrt{\langle Qx, Qx\rangle} = x^H Q^H Qx = \sqrt{\langle x, x\rangle} = ||x||$

(iii)

If Q is an orthonormal matrix, this means $QQ^H = Q^HQ = I \implies Q^H = Q^{-1}$. Thus, since $(Q^H)^H = Q$, $(Q^H)^HQ^H = Q^H(Q^H)^H = I$. For Q orthonormal, Q^H is orthonormal, and since $Q^{-1} = Q^H$, Q^{-1} is orthonormal.

(iv)

Columns of Q are orthonormal. Let c_i be the i^th column of Q. Since Q is orthonormal, we know that $(Q^HQ)_{ij} = c_i^Hc_j =< c_i, c_j >$ is 1 if $i = j$ and 0 if $i \neq j$. Thus, the columns of Q are orthonormal.

(v)

We can show that the converse is not true by example. For example, the determinate of

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

is $2(1/2) - 0 = 1$, but this matrix is not orthonormal.

(vi)

$$\begin{aligned} (Q_1Q_2)^HQ_1Q_2 &= Q_2^HQ_1^HQ_1Q_2 = Q_2^HQ_2 = I \\ Q_1Q_2(Q_1Q_2)^H &= Q_1Q_2Q_2^HQ_1^H = Q_1Q_1^H = I \end{aligned}$$

Therefore, the product Q_1Q_2 is orthonormal.

Problem 3.11

In the end, we would get zero vectors and basis vectors that would be the dimensions of X (the number of linearly independent vectors in X), and we would throw out the zero vectors to get our final orthonormal basis.

Problem 3.16

(For this problem, I received help from other students in the class.) ### (i) Let $A \in \mathbb{M}_{mxn}$ where $\text{rank}(A) = n \leq m$. Then there exist orthonormal $Q \in \mathbb{M}_{mxm}$ and upper triangular $R \in \mathbb{M}_{mxn}$ such that $A = QR$. Since $\tilde{Q} = -Q$ is still orthonormal $(-Q(-Q)^H = -Q(-Q^H) = QQ^H = I$ and similarly one shows $(-Q)^H(-Q) = I)$ and $\tilde{R} = -R$ is still upper triangular, $A = QR = \tilde{Q}\tilde{R}$. Therefore QR-decomposition is not unique.

(ii)

Suppose now that A is invertible and can be decomposed into two different QR decompositions: QR and $\tilde{Q}\tilde{R}$, where the diagonal entries of R and $\tilde{R}$ are strictly positive. Then both R and $\tilde{R}$ are invertible and we conclude that $\tilde{R}^{-1}R = Q^H\tilde{Q}$. Since R and $\tilde{R}$ are upper triangular, so is the LHS of the previous equation. On the other hand, since Q and $\tilde{Q}$ are orthonormal, so is the RHS. Therefore $\tilde{R}^{-1}R = I$ and, by unicity of the inverse, we conclude that $R = \tilde{R}$, and so $Q = \tilde{Q}$.

Problem 3.17

$$
\begin{aligned}
&(A^TA)x = A^Tb \\
\implies &(\hat{R}^T\hat{Q}^T\hat{Q}\hat{R})x = A^Tb \\
\implies &R^T\hat{R}x = A^Tb \\
\implies &\hat{R^T}\hat{R}x = (\hat{Q}\hat{R})^Tb \\
\implies &\hat{R^T}\hat{R}x = \hat{R^T}\hat{Q^T}b \\
\implies &\hat{R}x = \hat{Q^T}b
\end{aligned}
$$

Thus, we see that solving these systems is equivalent.

Problem 3.23

$$||x|| - ||y|| \leq ||x|| + ||y|| = ||x + y||$$

Thus, $||x|| - || - y|| \leq ||x + -y|| = ||x - y||$

We also see that $||y|| - ||x|| = ||y|| + || - x|| \leq ||y - x|| = ||x - y||$. Thus, $|||x|| - ||y||| \leq ||x - y|| \forall x, y \in V$.

Problem 3.24

(i)

$||f||_{L^1} = \int_a^b |f(t)|dt$

Consider $f, g \in C[a, b]$, with $f(t) \geq g(t)$ for all $t \in [a, b]$, then,

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt$$

Set $g(t) = 0$, and $|f(t)| \geq 0$. Then,

$$\int_a^b f(t)dt \geq \int_a^b 0dt = 0$$

Also, scale is preserved. Given $h \in \mathbb{F}$ and $f \in C[a, b]$,

$$||hf|| = \int_a^b |hf(t)|dt = \int_a^b |h||f(t)|dt = |h| \int_a^b |f(t)|dt = |h|||f||$$

Lastly, the triangle inequality holds ($||f + g|| \leq ||f|| + ||g||$).

Thus, $||f||_{L^1}$ is a norm on $C([a, b]; \mathbb{F})$.

(ii)

$||f||_{L^2}$

Similar proof to part (i) - show positivity, scale preservation, and triangle inequality.

(iii)

$||f||_{L^\infty}$

Similar proof to part (ii) - show positivity, scale preservation, and triangle inequality..

Problem 3.28

(i)

We know that

$$\sup_{x\neq 0}\frac{||Ax||_1}{||x||_1}\leq \sup_{x\neq 0}\frac{||Ax||_1}{||x||_1}\leq \sqrt{n}\sup_{x\neq 0}\frac{||Ax||_2}{||x||_2}$$

and

$$\sup_{x\neq 0}\frac{||Ax||_1}{||x||_1}\geq \sup_{x\neq 0}\frac{||Ax||_2}{||x||_1}\geq \frac{1}{\sqrt{n}}\sup_{x\neq 0}\frac{||Ax||_2}{||x||_2}$$

Thus, the two above imply that $\frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq \sqrt{n}||A||_2$.

(ii)

We know that

$$\sup_{x\neq 0}\frac{||Ax||_2}{||x||_2}\leq \sup_{x\neq 0}\frac{\sqrt{n}||Ax||_\infty}{||x||_\infty}$$

and

$$\sup_{x\neq 0}\frac{||Ax||_2}{||x||_2}\geq \sup_{x\neq 0}\frac{1}{\sqrt{n}}\frac{||Ax||_\infty}{||x||_\infty}$$

Thus, the two above imply that $\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{n}||A||_\infty$.

Problem 3.29

Any orthonormal matrix $Q \in M_n(\mathbb{F})$ has ||Q||=1.

$$||Qx|| = (< Qx, Qx >)^{1/2} = \left(< Q^H Qx, x >\right)^{1/2} = (< x, x >)^{1/2} = ||x||$$

Thus,

$$||Q|| = \sup_{x\neq 0}\frac{||Qx||}{||x||} = 1$$

Now let $R_x : \mathbb{M}_n(\mathbb{F}) \to \mathbb{F}^n$ be the linear transfromation $A \mapsto Ax$ for any $x \in \mathbb{F}^n$. Here, we notice

$$||R_x|| = \sup_{A\neq 0}\frac{||Ax||}{||A||} = \sup_{A\neq 0}\frac{||Ax||||x||}{||A||||x||} \leq \sup_{A\neq 0}\left(\frac{||Ax||||x||}{||Ax||}\right) = ||x||$$

Thus, we have proved equality.

Problem 3.30

Show positivity: $||A||$ is a norm on M_n, thus $||A|| \geq 0$. Therefore, if $||A||_s = ||SAS^{-1}||$, $||A||_s = ||SAS^{-1}||$ will also b ≥ 0.

Show scale preservation: $||\alpha A|| = ||\alpha SAS^{-1}|| = \alpha||A||_s$

Show triangle inequality: $||A + B||_s = ||S(A + B)S^{-1}|| \leq ||SAS^{-1}|| + ||SBS^{-1}|| = ||A||_s + ||B||_s/$

Problem 3.37

Find the unique $q \in V$ s.t. $L[p] = \langle q, p \rangle$

Basis $= \{1, x, x^2\}$ L evaluated on the basis vectors finds:

$$L(1) = 0$$

$$L(x) = 1$$

$$L(x^2) = 2$$

p can be written using this basis,

$$L(p) = L(a_1 1 + a_2 x + a_3 x^2) = a_1 L(1) + a_2 L(x) + a_3 L(x^2) = \langle (L(1) \cdot 1, L(x), L(x^2)), (a_1, a_2, a_3) \rangle$$

Getting the unique$q = (0, 1, 2)$.

Problem 3.38

Let Basis $= \{1, x, x^2\}$ We see that the matrix representation of D is:

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix representation of the adjoint of D is given by the Hermitian conjugate:

$$D^* = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

Problem 3.39

(i)

$\langle (S{+}T)v, w \rangle = \langle Sv, w \rangle {+} \langle Tv, w \rangle = \langle v, S^* w \rangle {+} \langle v, T^* w \rangle = \langle v, (S^* {+} T^*) w \rangle \langle \alpha T^* v, w \rangle = \alpha \langle Tv, w \rangle = \alpha \langle v, T^* w \rangle = \langle v, \overline{\alpha} T^* w \rangle$

Thus, we can see that $(S + T)^* = S^* + T^*$ and $(\alpha T)^* = \bar{\alpha} T^*$.

(ii)

$$\langle S^* w, v \rangle = \overline{\langle v, S^* w \rangle} = \overline{\langle Sv, w \rangle} = \langle w, Sv \rangle$$

Thus, we can see that $(S^*)^* = S$.

(iii)

$$\langle STv, w \rangle = \langle Tv, S^* w \rangle = \langle v, T^* S^* w \rangle = \langle v, (T^* S^*) w \rangle$$

Thus, we can see that $(ST)^* = T^* S^*$.

(iv)

From (iii), we see $T^*(T^*)^{-1} = (TT^{-1})^* = I^* = I \implies (T^*)^{-1} = (T^{-1})^*$

Problem 3.40

(i)

By definition of Frobenious inner product, $\langle B, AC\rangle_F = tr(B^H AC) = tr((A^H B)^H C) = \langle A^H B, C\rangle_F$

(ii)

$$\langle A_2, A_3A_1\rangle = \text{ tr}(A_2^H A_3A_1) = \text{ tr}(A_1A_2^H A_3) = \text{ tr}(A_2A_1^H A_3) = \langle A_2A_1^*, A_3\rangle$$

(iii)

Given $B, C \in \mathbb{M}_n(\mathbb{F})$, we have $< B, AC - CA >=< B, AC > - < B, CA >$. From (ii) applied to the second term we get $< B, CA >=< BA^*, C >$. Also,

$$< B, AC >= \text{tr}(B^H AC) = \text{tr}((A^H B)^H C) =< A^H B, C >=< A^* B, C >$$

Thus, $(T_A)^* = T_{A^*}$