# 2. Graph Terminology

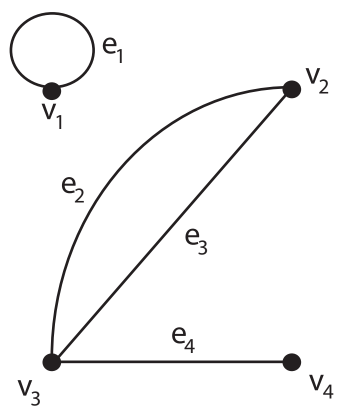
## 2.1. Undirected Graphs.

DEFINITIONS 2.1.1. *Suppose $G = (V, E)$ is an undirected graph.*

(1) *Two vertices $u, v \in V$ are* **adjacent** *or* **neighbors** *if there is an edge $e$ between $u$ and $v$.*
   - *The edge $e$* **connects** *$u$ and $v$.*
   - *The vertices $u$ and $v$ are* **endpoints** *of $e$.*

(2) *The* **degree** *of a vertex $v$, denoted $deg(v)$, is the number of edges for which it is an endpoint. A loop contributes twice in an undirected graph.*
   - *If $deg(v) = 0$, then $v$ is called* **isolated***.*
   - *If $deg(v) = 1$, then $v$ is called* **pendant***.*

EXAMPLE 2.1.1. *$V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$.*




(1) *$v_2$ and $v_3$ are adjacent.*
(2) *$deg(v_1) = 2$*
(3) *$deg(v_2) = 2$*
(4) *$deg(v_3) = 3$*
(5) *$deg(v_4) = 1$*

Discussion

Notice that in computing the degree of a vertex in an undirected graph a loop contributes two to the degree. In this example, none of the vertices is isolated, but $v_4$ is pendant. In particular, the vertex $v_1$ is not isolated since its degree is 2.

### 2.2. The Handshaking Theorem.

THEOREM 2.2.1. **(The Handshaking Theorem)** *Let $G = (V, E)$ be an undirected graph. Then*

$$2|E| = \sum_{v \in V} deg(v)$$

PROOF. Each edge contributes twice to the sum of the degrees of all vertices. □

Discussion

Theorem 2.2.1 is one of the most basic and useful combinatorial formulas associated to a graph. It lets us conclude some facts about the numbers of vertices and the possible degrees of the vertices. Notice the immediate corollary.

COROLLARY 2.2.1.1. *The sum of the degrees of the vertices in any graph must be an even number.*

In other words, it is impossible to create a graph so that the sum of the degrees of its vertices is odd (try it!).

### 2.3. Example 2.3.1.

EXAMPLE 2.3.1. *Suppose a graph has 5 vertices. Can each vertex have degree 3? degree 4?*

(1) *The sum of the degrees of the vertices would be* $3 \cdot 5$ *if the graph has 5 vertices of degree 3. This is an odd number, though, so this is not possible by the handshaking Theorem.*
(2) *The sum of the degrees of the vertices if there are 5 vertices with degree 4 is 20. Since this is even it is possible for this to equal* $2|E|$.

Discussion

If the sum of the degrees of the vertices is an even number then the handshaking theorem is not contradicted. In fact, you can create a graph with any even degree you want if multiple edges are permitted. However, if you add more restrictions it may not be possible. Here are two typical questions the handshaking theorem may help you answer.

EXERCISE 2.3.1. *Is it possible to have a graph $S$ with 5 vertices, each with degree 4, and 8 edges?*

EXERCISE 2.3.2. *A graph with 21 edges has 7 vertices of degree 1, three of degree 2, seven of degree 3, and the rest of degree 4. How many vertices does it have?*

THEOREM 2.3.1. *Every graph has an even number of vertices of odd degree.*

PROOF. Let $V_o$ be the set of vertices of odd degree, and let $V_e$ be the set of vertices of even degree. Since $V = V_o \cup V_e$ and $V_o \cap V_e = \emptyset$, the handshaking theorem gives us

$$2|E| = \sum_{v \in V} deg(v) = \sum_{v \in V_o} deg(v) + \sum_{v \in V_e} deg(v)$$

or

$$\sum_{v \in V_o} deg(v) = 2|E| - \sum_{v \in V_e} deg(v).$$

Since the sum of any number of even integers is again an even integer, the right-hand-side of this equations is an even integer. So the left-hand-side, which is the sum of a collection of odd integers, must also be even. The only way this can happen, however, is for there to be an even number of odd integers in the collection. That is, the number of vertices in $V_o$ must be even. □

Theorem 2.3.1 goes a bit further than our initial corollary of the handshaking theorem. If you have difficulty with the last sentence of the proof, consider the following facts:

- odd + odd = even
- odd + even = odd
- even + even = even

If we add up an odd number of odd numbers the previous facts will imply we get an odd number. Thus to get an even number out of $\sum_{v \in V_o} deg(v)$ there must be an even number of vertices in $V_o$ (the set of vertices of odd degree).

While there must be an even number of vertices of odd degree, there is no restrictions on the parity (even or odd) of the number of vertices of even degree.

This theorem makes it easy to see, for example, that it is not possible to have a graph with 3 vertices each of degree 1 and no other vertices of odd degree.

### 2.4. Directed Graphs.

DEFINITIONS 2.4.1. *Let $G = (V, E)$ be a directed graph.*

*(1) Let $(u, v)$ be an edge in $G$. Then $u$ is an* **initial vertex** *and is* **adjacent to** *$v$. The vertex $v$ is a* **terminal vertex** *and is* **adjacent from** *$u$.*
*(2) The* **in-degree** *of a vertex $v$, denoted $deg^{-}(v)$ is the number of edges which terminate at $v$.*

(3) *Similarly, the* **out-degree** *of $v$, denoted $deg^+(v)$, is the number of edges which initiate at $v$.*

### 2.5. The Handshaking Theorem for Directed Graphs.

THEOREM 2.5.1. *For any directed graph $G = (V, E)$,*

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

Discussion

When considering directed graphs we differentiate between the number of edges going into a vertex verses the number of edges coming out from the vertex. These numbers are given by the in-degree and the out-degree.

Notice that each edge contributes one to the in-degree of some vertex and one to the out-degree of some vertex. This is essentially the proof of Theorem 2.5.1.

EXERCISE 2.5.1. *Prove Theorem 2.5.1.*

### 2.6. Underlying Undirected Graph.

DEFINITION 2.6.1. *If we take a directed graph and remove the arrows indicating the direction we get the* **underlying undirected graph**.

Discussion

There are applications in which you may start with a directed graph, but then later find it useful to consider the corresponding undirected graph obtained by removing the direction of the edges.

Notice that if you take a vertex, $v$, in a directed graph and add its in-degree and out-degree, you get the degree of this vertex in the underlying undirected graph.

### 2.7. New Graphs from Old.

DEFINITIONS 2.7.1. *1. $(W, F)$ is a* **subgraph** *of $G = (V, E)$ if*

$$W \subseteq V \text{ and } F \subseteq E.$$

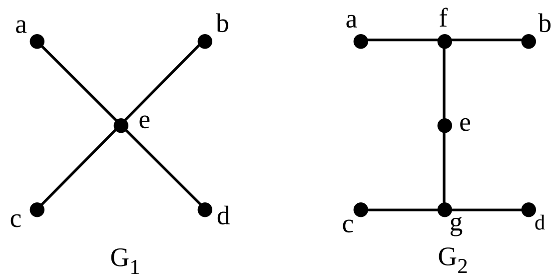
*2. Given graphs $G_1$ and $G_2$, their union*

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

3. *Given graphs $G_1$ and $G_2$, their* **join**, *denoted by $G_1 * G_2$, is the graph consisting of the union $G_1 \cup G_2$ together with all possible edges connecting a vertex of $G_1$ that is not in $G_2$ to a vertex of $G_2$ that is not in $G_1$.*

EXAMPLE 2.7.1. *Suppose $G$ has vertex set $V = \{a, b\}$ and one edge $e = \{a, b\}$ connecting $a$ and $b$, and $H$ has a single vertex $c$ and no edges. Then $G * H$ has vertex set $\{a, b, c\}$ and edges $\{a, b\}$, $\{a, c\}$, $\{b, c\}$.*

EXERCISE 2.7.1. *Find the union and join of the graphs $G_1$ and $G_2$ below.*




EXERCISE 2.7.2. *Prove that the union of two simple graphs is a simple graph.*

EXERCISE 2.7.3. *Prove that the join of two simple graphs is a simple graph.*

### 2.8. Complete Graphs.

DEFINITION 2.8.1. *The* **complete graph** *with $n$ vertices, denoted $K_n$, is the simple graph with exactly one edge between each pair of distinct vertices.*

Discussion

There are a certain types of simple graphs that are important enough that they are given special names. The first of these are the **complete graphs**. These are the simple graphs that have the maximal number of edges for the given set of vertices. For example, if we were using graphs to represent a local area network, a complete graph would represent the maximum redundancy possible. In other words, each pair of computers would be directly connected. It is easy to see that any two complete graphs with $n$ vertices are isomorphic, so that the symbol $K_n$ is ambiguously used to denote any such graph.

Complete graphs also arise when considering the question as to whether a graph $G$ is **planar**, that is, whether $G$ can be drawn in a plane without having any two edges intersect. The complete graphs $K_1$, $K_2$, $K_3$, and $K_4$ are planar graphs, but $K_n$ is not planar if $n \geq 5$. Draw $K_4$ without making the edges intersect, then try to draw $K_5$ without creating an unwanted intersection between edges. (Notice that $K_{n+1}$ can be created from $K_n$ by adding one new vertex and an edge from the new vertex to each vertex of $K_n$.)

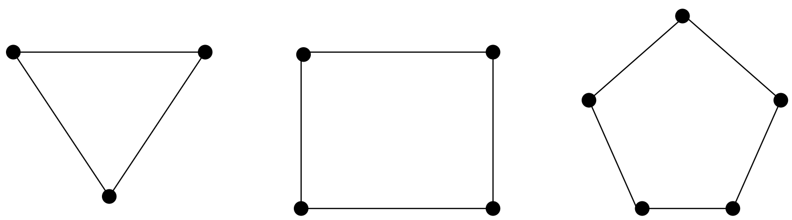
EXERCISE 2.8.1. *Prove that the complete graph $K_n$, $n \geq 1$, is the join $K_{n-1} * G$, where $G$ is a graph with one vertex and no edges.*

### 2.9. Cycles.

DEFINITION 2.9.1. *A* **cycle** *with $n$ vertices $\{v_1, v_2, ..., v_n\}$, denoted by $C_n$, is a simple graph with edges of the form $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.*

Discussion

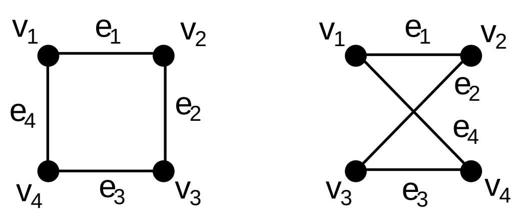
Notice that a cycle must have at least 3 vertices. Here are examples of the first three possibilities:



$C_3$ $C_4$ $C_5$

Local area networks that are configured this way are often called **ring** networks.

Notice that the following two graphs are isomorphic. Pay close attention to the labels.




The point of the last illustration, is that sometimes you have to redraw the graph to see the ring shape. It also demonstrates that a graph may be planar even though this fact may not be evident from a given representation.

### 2.10. Wheels.

DEFINITION 2.10.1. *A* **wheel** *is a join $C_n * G$, where $C_n$ is a cycle and $G$ is a graph with one vertex and no edges. The wheel with $n + 1$ vertices is denoted $W_n$.*

Discussion

Notice that a wheel is obtained by starting with a cycle and then adding one new vertex and an edge from that vertex to each vertex of the cycle. Be careful! The index on the notation $W_n$ is actually one less that the number of vertices. The $n$ stands for the number of vertices in the cycle that we used to get the wheel.

### 2.11. $n$-Cubes.

DEFINITION 2.11.1. *The $n$-**cube**, denoted $Q_n$, is the graph with $2^n$ vertices represented by the vertices and edges of an $n$-dimensional cube.*

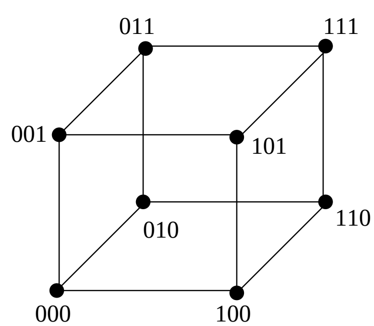
These graphs can be constructed recursively as follows:

1. Initial Condition. A 0-cube is a graph with one vertex and no edges.
2. Recursion. Let $Q_n^1$ and $Q_n^2$ be two disjoint $n$-cubes, $n \geq 0$, and let $f\colon Q_n^1 \to Q_n^2$ be an isomorphism. $Q_{n+1}$ is the graph consisting of the union $Q_n^1 \cup Q_n^2$, together with all edges $\{v, f(v)\}$ joining a vertex $v$ in $Q_n^1$ with its corresponding vertex $f(v)$ in $Q_n^2$.

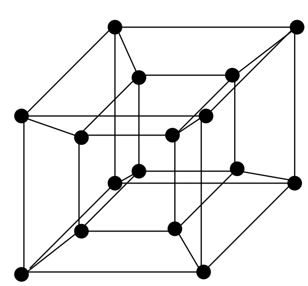
$Q_n$ can also be represented as the graph whose vertices are the bit strings of length $n$, having an edge between each pair of vertices that differ by one bit.

Discussion

The $n$-Cube is a common way to connect processors in parallel machines. Here are the cubes $Q_3$ and $Q_4$.




3-cube



4-cube

EXERCISE 2.11.1. *Find all the subgraphs of $Q_1$, and $Q_2$.*

EXERCISE 2.11.2. *Label the vertices of $Q_4$ appropriately, using bit strings of length four.*

EXERCISE 2.11.3. *Use your labeling of the vertices of $Q_4$ from Exercise 2.11.2 to identify two disjoint $Q_3$'s, and an isomorphism between them, from which $Q_4$ would be obtained in the recursive description above.*

EXERCISE 2.11.4. *Prove that $Q_{n+1} \subseteq Q_n^1 * Q_n^2$, where $Q_n^1$ and $Q_n^2$ are disjoint $n$-cubes, $n \geq 0$.*

EXERCISE 2.11.5. *Prove that the 2-cube is not (isomorphic to) the join of two 1-cubes.*

EXERCISE 2.11.6. *Draw the graph $Q_5$. [Hint: Abandon trying to use a "cube" shape. Put* 00000 *on the top of the page and* 11111 *on the bottom and look for an organized manner to put the rest in between.]*

## 2.12. Bipartite Graphs.

DEFINITION 2.12.1. *A simple graph $G$ is* **bipartite** *if $V$ can be partitioned into two nonempty disjoint subsets $V_1$ and $V_2$ such that every edge connects a vertex in $V_1$ and a vertex in $V_2$.*

DEFINITION 2.12.2. *A bipartite graph is* **complete** *if $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that there is an edge from every vertex $V_1$ to every vertex in $V_2$.*

*$K_{m,n}$ denotes the complete bipartite graph when $m = |V_1|$ and $n = |V_2|$.*

Discussion

The definition of a bipartite graph is not always consistent about the necessary size of $|V_1|$ and $|V_2|$. We will assume $V_1$ and $V_2$ must have at least one element each, so we will not consider the graph consisting of a single vertex bipartite.

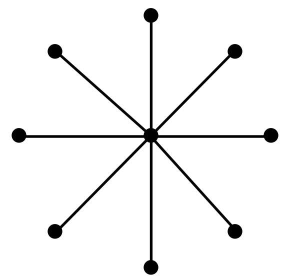
Note: There are no edges connecting vertices in $V_1$ in a bipartite graph. Neither are there edges connecting vertices in $V_2$.

EXERCISE 2.12.1. *How many edges are there in the graph $K_{m,n}$?*

EXERCISE 2.12.2. *Prove that a complete bipartite graph $K_{m,n}$ is the join $G_m * G_n$ of graphs $G_m$ and $G_n$, where $G_m$ has $m$ vertices and no edges, and $G_n$ has $n$ vertices and no edges.*

## 2.13. Examples.

EXAMPLE 2.13.1. *A* **star network** *is a $K_{1,n}$ bipartite graph.*

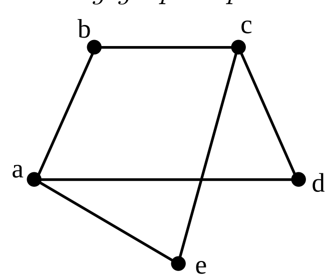


$K_{1,8}$

EXAMPLE 2.13.2. *$C_k$, for k even, is a bipartite graph: Label the vertices $\{v_1, v_2, ..., v_k\}$ cyclicly around $C_k$, and put the vertices with odd subscripts in $V_1$ and the vertices with even subscripts in $V_2$.*

(1) Suppose $V_1$ is a set of airlines and $V_2$ is a set of airports. Then the graph with vertex set $V = V_1 \cup V_2$, where there is an edge between a vertex of $V_1$ and a vertex of $V_2$ if the given airline serves the given airport, is bipartite. If every airline in $V_1$ serves every airport in $V_2$, then the graph would be a **complete bipartite graph**.
(2) Supplier, warehouse transportation models where an edge represents that a given supplier sends inventory to a given warehouse are bipartite.

EXERCISE 2.13.1. *Is the following graph bipartite?*




EXERCISE 2.13.2. *Prove that $Q_n$ is bipartite. [Hint: You don't need mathematical induction; use the bit string model for the vertex set.]*

Bipartite graphs also arise when considering the question as to whether a graph is planar. It is easy to see that the graphs $K_{1,n}$ and $K_{2,n}$ are planar for every $n \geq 1$. The graphs $K_{m,n}$, however, are not planar if both $m$ and $n$ are greater than 2. In particular, $K_{3,3}$ is not planar. (Try it!) A theorem, which we shall not prove, states that *every* nonplanar graph contains (in some sense) a subgraph (see Slide 15) isomorphic to $K_5$ or a subgraph isomorphic to $K_{3,3}$.

REMARK 2.13.1. *The important properties of a graph do not depend on how it is drawn. To see that two graphs, whose vertices have the same labels, are isomorphic, check that vertices are connected by an edge in one graph if and only if they are also connected by an edge in the other graph.*