Section 1

p10 **Theorem 1.1**

Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
g. $c(d\mathbf{u}) = (cd)\mathbf{u}$
h. $1\mathbf{u} = \mathbf{u}$

p19 **Theorem 1.2**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutativity
b. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ Distributivity
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

p20 **Theorem 1.3**

Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

a. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
b. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

p22 **Theorem 1.4**

The Cauchy-Schwarz Inequality

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

p22 **Theorem 1.5**

The Triangle Inequality

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

p26 **Theorem 1.6**

Pythagoras' Theorem

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Section 2

p68	**Theorem 2.1**	Matrices A and B are row equivalent if and only if they can be reduced to the same row echelon form.
p72	**Theorem 2.2**	**The Rank Theorem** Let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then $\text{number of free variables} = n - \operatorname{rank}(A)$
p76	**Theorem 2.3**	If $[A \mid \mathbf{0}]$ is a homogeneous system of m linear equations with n variables, where $m < n$, then the system has infinitely many solutions.
p89	**Theorem 2.4**	A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.
p93	**Theorem 2.5**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.
p95	**Theorem 2.6**	Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_m]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A \mid \mathbf{0}]$ has a nontrivial solution.
p96	**Theorem 2.7**	Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (row) vectors in $\mathbb{R}^n$ and let A be the $m \times n$ matrix $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$ with these vectors as its rows. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\operatorname{rank}(A) < m$.
p97	**Theorem 2.8**	Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

Section 3

p144 **Theorem 3.1**

Let A be an $n \times m$ matrix, $\mathbf{e}_i$ a $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then

a. $\mathbf{e}_i A$ is the ith row of A and
b. $A\mathbf{e}_j$ is the jth column of A.

p154 **Theorem 3.2**

Algebraic Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be matrices of the same size and let c and d be scalars. Then

a. $A + B = B + A$ — Commutativity
b. $(A + B) + C = A + (B + C)$ — Associativity
c. $A + O = A$
d. $A + (-A) = O$
e. $c(A + B) = cA + cB$ — Distributivity
f. $(c + d)A = cA + dA$ — Distributivity
g. $c(dA) = (cd)A$
h. $1A = A$

p158 **Theorem 3.3**

Properties of Matrix Multiplication

Let A, B, and C be matrices (whose sizes are such that the indicated operations can be performed) and let k be a scalar. Then

a. $A(BC) = (AB)C$ — Associativity
b. $A(B + C) = AB + AC$ — Left distributivity
c. $(A + B)C = AC + BC$ — Right distributivity
d. $k(AB) = (kA)B = A(kB)$
e. $I_m A = A = AI_n$ if A is $m \times n$ — Multiplicative identity

p159 **Theorem 3.4**

Properties of the Transpose

Let A and B be matrices (whose sizes are such that the indicated operations can be performed) and let k be a scalar. Then

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. $(kA)^T = k(A^T)$
d. $(AB)^T = B^T A^T$
e. $(A^r)^T = (A^T)^r$ for all nonnegative integers r

p161 **Theorem 3.5**

a. If A is a square matrix, then $A + A^T$ is a symmetric matrix.
b. For any matrix A, AA^T and A^TA are symmetric matrices.

p164 **Theorem 3.6**

If A is an invertible matrix, then its inverse is unique.

p165 **Theorem 3.7**

If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^n$.

p165 **Theorem 3.8**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

p167 **Theorem 3.9**

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A is an invertible matrix and c is a nonzero scalar, then cA is an invertible matrix and

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

c. If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

d. If A is an invertible matrix, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

e. If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

p171 **Theorem 3.10**

Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

p172 **Theorem 3.11**

Each elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

p172 **Theorem 3.12**

The Fundamental Theorem of Invertible Matrices: Version 1

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.

p174 **Theorem 3.13**

Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

p175 **Theorem 3.14**

Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

p192 **Theorem 3.19**

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

p196 **Theorem 3.20**

Let B be any matrix that is row equivalent to a matrix A. Then $\text{row}(B) = \text{row}(A)$.

p196 **Theorem 3.21**

Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then N is a subspace of $\mathbb{R}^n$.

p197 **Theorem 3.22**

Let A be a matrix whose entries are real numbers. For any system of linear equations $A\mathbf{x} = \mathbf{b}$, exactly one of the following is true:

a. There is no solution.
b. There is a unique solution.
c. There are infinitely many solutions.

p202 **Theorem 3.23**

The Basis Theorem

Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

p204 **Theorem 3.24**

The row and column spaces of a matrix A have the same dimension.

p204 **Theorem 3.25**

For any matrix A,

$$\text{rank}(A^T) = \text{rank}(A)$$

p205 **Theorem 3.26**

The Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

p206 **Theorem 3.27**

The Fundamental Theorem of Invertible Matrices: Version 2

tional Portrait Gallery, London

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.

p207	**Theorem 3.28**	Let A be an $m \times n$ matrix. Then: a. $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$ b. The $n \times n$ matrix A^TA is invertible if and only if $\operatorname{rank}(A) = n$.
p208	**Theorem 3.29**	Let S be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for S. For every vector $\mathbf{v}$ in S, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$: $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$
p214	**Theorem 3.30**	Let A be an $m \times n$ matrix. Then the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x} \quad (\text{for } \mathbf{x} \text{ in } \mathbb{R}^n)$ is a linear transformation.
p216	**Theorem 3.31**	Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. More specifically, $T = T_A$, where A is the $m \times n$ matrix $A = [T(\mathbf{e}_1) \vdots T(\mathbf{e}_2) \vdots \cdots \vdots T(\mathbf{e}_n)]$
p220	**Theorem 3.32**	Let $T: \mathbb{R}^m \to \mathbb{R}^n$ and $S: \mathbb{R}^n \to \mathbb{R}^p$ be linear transformations. Then $S \circ T: \mathbb{R}^m \to \mathbb{R}^p$ is a linear transformation. Moreover, their standard matrices are related by $[S \circ T] = [S][T]$
p222	**Theorem 3.33**	Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then its standard matrix $[T]$ is an invertible matrix, and $[T^{-1}] = [T]^{-1}$

Section 4

p266 **Theorem 4.1** **The Laplace Expansion Theorem**

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\begin{aligned} \det A &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= \sum_{j=1}^{n} a_{ij}C_{ij} \end{aligned} \tag{5}$$

(which is the ***cofactor expansion along the*** i**th** ***row***) and also as

$$\begin{aligned} \det A &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ &= \sum_{i=1}^{n} a_{ij}C_{ij} \end{aligned} \tag{6}$$

(the ***cofactor expansion along the*** j**th** ***column***).

p269 **Theorem 4.2**

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

p269 **Theorem 4.3**

Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

p271 **Theorem 4.4**

Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.
b. If E results from multiplying one row of I_n by k, then $\det E = k$.
c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

p272 **Lemma 4.5**

Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

p272 **Theorem 4.6**

A square matrix A is invertible if and only if $\det A \neq 0$.

p272 **Theorem 4.7**

If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

p273	**Theorem 4.8**	If A and B are $n \times n$ matrices, then $$\det(AB) = (\det A)(\det B)$$
p274	**Theorem 4.9**	If A is invertible, then $$\det(A^{-1}) = \frac{1}{\det A}$$
p274	**Theorem 4.10**	For any square matrix A, $$\det A = \det A^T$$
p275	**Theorem 4.11**	**Cramer's Rule** Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by $$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A} \quad \text{for } i = 1, \ldots, n$$
p277	**Theorem 4.12**	Let A be an invertible $n \times n$ matrix. Then $$A^{-1} = \frac{1}{\det A}\operatorname{adj} A$$
p277	**Lemma 4.13**	Let A be an $n \times n$ matrix. Then $$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \det A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1} \quad (7)$$
p278	**Lemma 4.14**	Let A be an $n \times n$ matrix and let B be obtained by interchanging any two rows (columns) of A. Then $$\det B = -\det A$$
p295	**Theorem 4.15**	The eigenvalues of a triangular matrix are the entries on its main diagonal.
p295	**Theorem 4.16**	A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A.

p296 **Theorem 4.17** **The Fundamental Theorem of Invertible Matrices: Version 3**

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.

p296 **Theorem 4.18** Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a. For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.
b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
c. If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.

p297 **Theorem 4.19** Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors—say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

p297 **Theorem 4.20** Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

p302 **Theorem 4.21** Let A, B, and C be $n \times n$ matrices.

a. $A \sim A$
b. If $A \sim B$, then $B \sim A$.
c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

p302 **Theorem 4.22**

Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$
b. A is invertible if and only if B is invertible.
c. A and B have the same rank.
d. A and B have the same characteristic polynomial.
e. A and B have the same eigenvalues.

p304 **Theorem 4.23**

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

p306 **Theorem 4.24**

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all of the eigenspaces) is linearly independent.

p306 **Theorem 4.25**

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

p307 **Lemma 4.26**

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

p307 **Theorem 4.27**

The Diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.
b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.
c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

Section 5

p369 **Theorem 5.1**

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then these vectors are linearly independent.

p371 **Theorem 5.2**

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{w}$ be any vector in W. Then the unique scalars $c_1, \ldots, c_k$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

are given by

$$c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \quad \text{for } i = 1, \ldots, k$$

p373 **Theorem 5.3**

Let $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{w}$ be any vector in W. Then

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{w} \cdot \mathbf{q}_2)\mathbf{q}_2 + \cdots + (\mathbf{w} \cdot \mathbf{q}_k)\mathbf{q}_k$$

and this representation is unique.

p374 **Theorem 5.4**

The columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^TQ = I_n$.

p374 **Theorem 5.5**

A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

p375 **Theorem 5.6**

Let Q be an $n \times n$ matrix. The following statements are equivalent:

a. Q is orthogonal.
b. $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x}$ in $\mathbb{R}^n$.
c. $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

p376 **Theorem 5.7**

If Q is an orthogonal matrix, then its rows form an orthonormal set.

p376 **Theorem 5.8**

Let Q be an orthogonal matrix.

a. Q^{-1} is orthogonal.
b. $\det Q = \pm 1$
c. If λ is an eigenvalue of Q, then $|\lambda| = 1$.
d. If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is Q_1Q_2.

p379 **Theorem 5.9**

Let W be a subspace of $\mathbb{R}^n$.

a. $W^\perp$ is a subspace of $\mathbb{R}^n$.
b. $(W^\perp)^\perp = W$
c. $W \cap W^\perp = \{\mathbf{0}\}$
d. If $W = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

p379 **Theorem 5.10**

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{row}(A))^\perp = \text{null}(A) \quad \text{and} \quad (\text{col}(A))^\perp = \text{null}(A^T)$$

p384 **Theorem 5.11**

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Then there are unique vectors $\mathbf{w}$ in W and $\mathbf{w}^\perp$ in $W^\perp$ such that

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

p385 **Corollary 5.12**

If W is a subspace of $\mathbb{R}^n$, then

$$(W^\perp)^\perp = W$$

p386 **Theorem 5.13**

If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^\perp = n$$

p386 **Corollary 5.14**

The Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

p389 **Theorem 5.15**

The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= \text{span}(\mathbf{x}_1)\\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, & W_2 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2)\\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, & W_3 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\\
&\vdots\\
\mathbf{v}_k &= \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots\\
&\quad - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, & W_k &= \text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k)
\end{aligned}$$

Then for each $i = 1, \dots, k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W.

p401 **Theorem 5.17**

If A is orthogonally diagonalizable, then A is symmetric.

p401	**Theorem 5.18**	If A is a real symmetric matrix, then the eigenvalues of A are real.
p402	**Theorem 5.19**	If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.
p403	**Theorem 5.20**	**The Spectral Theorem** Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.
p411	**Theorem 5.23**	**The Principal Axes Theorem** Every quadratic form can be diagonalized. Specifically, if A is the $n \times n$ symmetric matrix associated with the quadratic form $\mathbf{x}^TA\mathbf{x}$ and if Q is an orthogonal matrix such that $Q^TAQ = D$ is a diagonal matrix, then the change of variable $\mathbf{x} = Q\mathbf{y}$ transforms the quadratic form $\mathbf{x}^TA\mathbf{x}$ into the quadratic form $\mathbf{y}^TD\mathbf{y}$, which has no cross-product terms. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and $\mathbf{y} = [y_1 \ \cdots \ y_n]^T$, then $$\mathbf{x}^TA\mathbf{x} = \mathbf{y}^TD\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$
p413	**Theorem 5.24**	Let A be an $n \times n$ symmetric matrix. The quadratic form $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$ is a. positive definite if and only if all of the eigenvalues of A are positive. b. positive semidefinite if and only if all of the eigenvalues of A are nonnegative. c. negative definite if and only if all of the eigenvalues of A are negative. d. negative semidefinite if and only if all of the eigenvalues of A are nonpositive. e. indefinite if and only if A has both positive and negative eigenvalues.
p414	**Theorem 5.25**	Let $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$ be a quadratic form with associated $n \times n$ symmetric matrix A. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the following are true, subject to the constraint $\lVert\mathbf{x}\rVert = 1$: a. $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$ b. The maximum value of $f(\mathbf{x})$ is λ_1, and it occurs when $\mathbf{x}$ is a unit eigenvector corresponding to λ_1. c. The minimum value of $f(\mathbf{x})$ is λ_n, and it occurs when $\mathbf{x}$ is a unit eigenvector corresponding to λ_n.