# Section 1

p12 **Definition** A vector $\mathbf{v}$ is a ***linear combination*** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. The scalars $c_1, c_2, \ldots, c_k$ are called the ***coefficients*** of the linear combination.

p18 **Definition** If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the ***dot product*** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

p20 **Definition** The ***length*** (or ***norm***) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

p23 **Definition** The ***distance*** $\mathrm{d}(\mathbf{u}, \mathbf{v})$ between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

p24 **Definition** For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

p26 **Definition** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are ***orthogonal*** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

p27 **Definition** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, then the ***projection of*** $\mathbf{v}$ ***onto*** $\mathbf{u}$ is the vector $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ defined by

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$

p41

## Table 1.2 Equations of Lines in $\mathbb{R}^2$

| Normal Form | General Form | Vector Form | Parametric Form |
|---|---|---|---|
| $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ | $ax + by = c$ | $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ | $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$ |

## Table 1.3 Lines and Planes in $\mathbb{R}^3$

| | Normal Form | General Form | Vector Form | Parametric Form |
|---|---|---|---|---|
| Lines | $\begin{cases} \mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}_1 \\ \mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{p}_2 \end{cases}$ | $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$ | $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ | $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$ |
| Planes | $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ | $ax + by + cz = d$ | $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ | $\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$ |

In the case where the line $\ell$ is in $\mathbb{R}^2$ and its equation has the general form $ax + by = c$, the distance $d(B, \ell)$ from $B = (x_0, y_0)$ is given by the formula

p43

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} \tag{3}$$

p44

$$d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \tag{4}$$

# Section 2

p58 **Definition** A ***linear equation*** in the $n$ variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the ***coefficients*** $a_1, a_2, \ldots, a_n$ and the ***constant term*** $b$ are constants.

p65 **Definition** A matrix is in ***row echelon form*** if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom.
2. In each nonzero row, the first nonzero entry (called the ***leading entry***) is in a column to the left of any leading entries below it.

p66 **Definition** The following ***elementary row operations*** can be performed on a matrix:

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

p68 **Definition** Matrices $A$ and $B$ are ***row equivalent*** if there is a sequence of elementary row operations that converts $A$ into $B$.

p73 **Definition** A matrix is in ***reduced row echelon form*** if it satisfies the following properties:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a ***leading 1***).
3. Each column containing a leading 1 has zeros everywhere else.

p76 **Definition** A system of linear equations is called ***homogeneous*** if the constant term in each equation is zero.

p90 **Definition** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the ***span*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\mathrm{span}(S)$. If $\mathrm{span}(S) = \mathbb{R}^n$, then $S$ is called a ***spanning set*** for $\mathbb{R}^n$.

p93 **Definition** A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_k$, *at least one of which is not zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is called ***linearly independent.***

# Section 3

p138 **Definition** A ***matrix*** is a rectangular array of numbers called the ***entries,*** or ***elements,*** of the matrix.

p141 **Definition** If $A$ is an $m \times n$ matrix and $B$ is an $n \times r$ matrix, then the ***product*** $C = AB$ is an $m \times r$ matrix. The $(i, j)$ entry of the product is computed as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

p149 If $A$ is a square matrix and $r$ and $s$ are nonnegative integers, then

1. $A^rA^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

p151 **Definition** The ***transpose*** of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ obtained by interchanging the rows and columns of $A$. That is, the $i$th column of $A^T$ is the $i$th row of $A$ for all $i$.

p151 **Definition** A square matrix $A$ is ***symmetric*** if $A^T = A$—that is, if $A$ is equal to its own transpose.

p152 A square matrix $A$ is symmetric if and only if $A_{ij} = A_{ji}$ for all $i$ and $j$.

p163 **Definition** If $A$ is an $n \times n$ matrix, an ***inverse*** of $A$ is an $n \times n$ matrix $A'$ with the property that

$$AA' = I \quad \text{and} \quad A'A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an $A'$ exists, then $A$ is called ***invertible.***

p169 The inverse of a product of invertible matrices is the product of their inverses in the reverse order.

p169 If $A$ is an invertible matrix and $n$ is a positive integer, then $A^{-n}$ is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}$$

p170 **Definition** An ***elementary matrix*** is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

p192 **Definition** A ***subspace*** of $\mathbb{R}^n$ is any collection $S$ of vectors in $\mathbb{R}^n$ such that:

1. The zero vector $\mathbf{0}$ is in $S$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in $S$, then $\mathbf{u} + \mathbf{v}$ is in $S$. ($S$ is ***closed under addition.***)
3. If $\mathbf{u}$ is in $S$ and $c$ is a scalar, then $c\mathbf{u}$ is in $S$. ($S$ is ***closed under scalar multiplication.***)

p195 **Definition** Let $A$ be an $m \times n$ matrix.

1. The ***row space*** of $A$ is the subspace $\text{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of $A$.
2. The ***column space*** of $A$ is the subspace $\text{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of $A$.

p197 **Definition** Let $A$ be an $m \times n$ matrix. The ***null space*** of $A$ is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. It is denoted by $\text{null}(A)$.

p198 **Definition** A ***basis*** for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors in $S$ that

1. spans $S$ and
2. is linearly independent.

p202

1. Find the reduced row echelon form $R$ of $A$.
2. Use the nonzero row vectors of $R$ (containing the leading 1s) to form a basis for $\text{row}(A)$.
3. Use the column vectors of $A$ that correspond to the columns of $R$ containing the leading 1s (the pivot columns) to form a basis for $\text{col}(A)$.

p203 **Definition** If $S$ is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for $S$ is called the ***dimension*** of $S$, denoted $\dim S$.

p204 **Definition** The ***rank*** of a matrix $A$ is the dimension of its row and column spaces and is denoted by $\text{rank}(A)$.

p204 **Definition** The ***nullity*** of a matrix $A$ is the dimension of its null space and is denoted by $\text{nullity}(A)$.

p208 **Definition** Let $S$ be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for $S$. Let $\mathbf{v}$ be a vector in $S$, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. Then $c_1, c_2, \ldots, c_k$ are called the ***coordinates of v with respect to*** $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the ***coordinate vector of v with respect to*** $\mathcal{B}$.

p213 **Definition** A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a ***linear transformation*** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$ and all scalars $c$.

p221 **Definition** Let $S$ and $T$ be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then $S$ and $T$ are ***inverse transformations*** if $S \circ T = I_n$ and $T \circ S = I_n$.

# Section 4

p254 **Definition** Let $A$ be an $n \times n$ matrix. A scalar $\lambda$ is called an ***eigenvalue*** of $A$ if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an ***eigenvector*** of $A$ corresponding to $\lambda$.

p256 **Definition** Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. The collection of all eigenvectors corresponding to $\lambda$, together with the zero vector, is called the ***eigenspace*** of $\lambda$ and is denoted by $E_\lambda$.

p264 **Definition** Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the ***determinant*** of $A$ is the scalar

$$\det A = |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1)$$

p265 **Definition** Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the ***determinant*** of $A$ is the scalar

$$\begin{aligned} \det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned} \quad (3)$$

p292 The eigenvalues of a square matrix $A$ are precisely the solutions $\lambda$ of the equation

$$\det(A - \lambda I) = 0$$

p292 Let $A$ be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of $A$.
2. Find the eigenvalues of $A$ by solving the characteristic equation $\det(A - \lambda I) = 0$ for $\lambda$.
3. For each eigenvalue $\lambda$, find the null space of the matrix $A - \lambda I$. This is the eigenspace $E_\lambda$, the nonzero vectors of which are the eigenvectors of $A$ corresponding to $\lambda$.
4. Find a basis for each eigenspace.

p301 **Definition** Let $A$ and $B$ be $n \times n$ matrices. We say that $A$ ***is similar to*** $B$ if there is an invertible $n \times n$ matrix $P$ such that $P^{-1}AP = B$. If $A$ is similar to $B$, we write $A \sim B$.

p303 **Definition** An $n \times n$ matrix $A$ is ***diagonalizable*** if there is a diagonal matrix $D$ such that $A$ is similar to $D$—that is, if there is an invertible $n \times n$ matrix $P$ such that $P^{-1}AP = D$.

# Section 5

Hence, the projection of the vector **v** onto this line is just $P\mathbf{v}$.

p366 **Problem 1** Show that $P$ can be written in the equivalent form

$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

p369 **Definition** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal—that is, if

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{whenever} \quad i \neq j \quad \text{for } i, j = 1, 2, \ldots, k$$

p370 **Definition** A set of vectors in $\mathbb{R}^n$ is an ***orthonormal set*** if it is an orthogonal set of unit vectors. An ***orthonormal basis*** for a subspace $W$ of $\mathbb{R}^n$ is a basis of $W$ that is an orthonormal set.

p372 **Definition** An ***orthogonal basis*** for a subspace $W$ of $\mathbb{R}^n$ is a basis of $W$ that is an orthogonal set.

p378 **Definition** Let $W$ be a subspace of $\mathbb{R}^n$. We say that a vector **v** in $\mathbb{R}^n$ is ***orthogonal to*** $W$ if **v** is orthogonal to every vector in $W$. The set of all vectors that are orthogonal to $W$ is called the ***orthogonal complement of*** $W$, denoted $W^\perp$. That is,

$$W^\perp = \{\mathbf{v} \text{ in } \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \text{ in } W\}$$

p382 **Definition** Let $W$ be a subspace of $\mathbb{R}^n$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for $W$. For any vector **v** in $\mathbb{R}^n$, the ***orthogonal projection of* v *onto* $W$** is defined as

$$\operatorname{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k$$

The ***component of* v *orthogonal to* $W$** is the vector

$$\operatorname{perp}_W(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v})$$

p400 **Definition** A square matrix $A$ is ***orthogonally diagonalizable*** if there exists an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $Q^TAQ = D$.

p409 **Definition** A ***quadratic form*** in $n$ variables is a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form

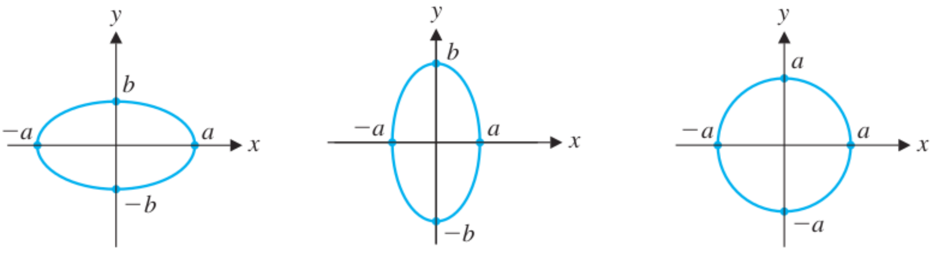
$$f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$$

where $A$ is a symmetric $n \times n$ matrix and **x** is in $\mathbb{R}^n$. We refer to $A$ as the ***matrix associated with*** $f$.

p412 **Definition** A quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is classified as one of the following:

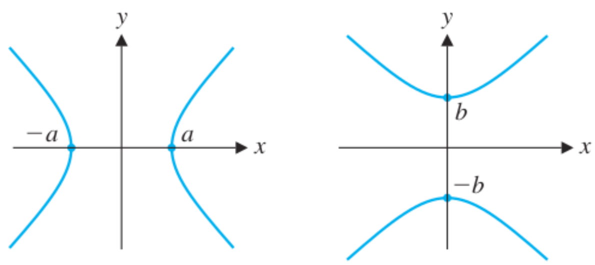
1. ***positive definite*** if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$
2. ***positive semidefinite*** if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$
3. ***negative definite*** if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$
4. ***negative semidefinite*** if $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$
5. ***indefinite*** if $f(\mathbf{x})$ takes on both positive and negative values

p416 Ellipse or Circle: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $a, b > 0$



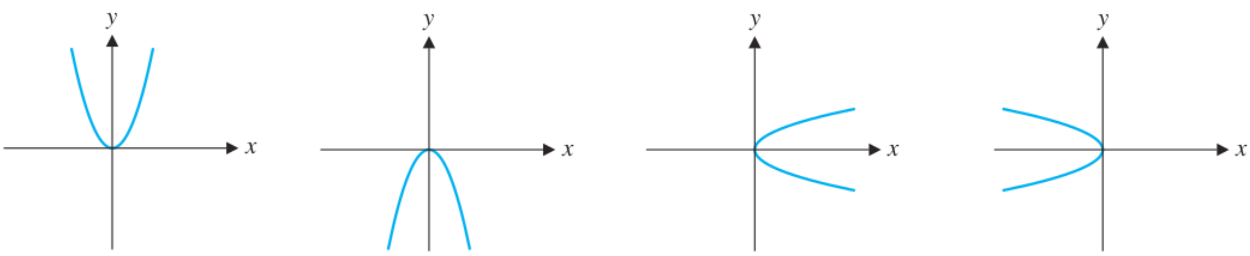

$a > b$ $a < b$ $a = b$

Hyperbola




$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a, b > 0$ $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1,\ a, b > 0$

Parabola




$y = ax^2,\ a > 0$ $y = ax^2,\ a < 0$ $x = ay^2,\ a > 0$ $x = ay^2,\ a < 0$