The SVD

References can be found in the following textbooks:

- G. H. Golub and C. F. Van-Loan. *Matrix computations.* The Johns Hopkins University Press, third edition, 1996.
- L. N. Trefethen and D. Bau. *Numerical Linear Algebra.* SIAM, Philadelphia, PA, USA, 1997.

Source code can be found in the "numerical recipes" book. See http://www.nr.com.

Definitions

Any matrix $A \in \mathbb{R}^{m\times n}$ can be written as:

$$A = U\Sigma V^T, \quad U \in \mathbb{R}^{m\times m}, \ \Sigma \in \mathbb{R}^{m\times n}, \ V \in \mathbb{R}^{n\times n}$$

where U, V are orthogonal and Σ is diagonal. Furthermore, the diagonal elements of Σ are non-negative.

Thin SVD

Any matrix $A \in \mathbb{R}^{m\times k}$, $m \geq k$, can be written as:

$$A = U\Sigma V^T,$$

$$U = (u_1, \ldots, u_k) \in \mathbb{R}^{m\times k}, \quad V = (v_1, \ldots, v_k) \in \mathbb{R}^{k\times k}, \quad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^{k\times k}$$

where $U^T U = I$, $V^T V = I$, and $\sigma_1 \geq \ldots \geq \sigma_k \geq 0$.
The above matrix product can also be expressed as:

$$A = \sum_{j=1}^{k} \sigma_j u_j v_j^T$$

Example:

$$\begin{aligned}
A &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} && \text{thin matrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \text{SVD} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \text{thin SVD} \\
&= 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} && \text{expressed as outer product summation}
\end{aligned}$$