Q1

$$P: \quad f = n!$$
$$\{P\} \quad n, f := n+1, e \quad \{P\}$$

$$P \Longrightarrow P[n, f := n+1, e]$$

Assume antecedent and prove consequent.

$$\begin{array}{ll} & P[n, f := n+1, e] \\ = & \quad \langle \text{Definition of } P\text{; Textual Substitution} \rangle \\ & e = (n+1)! \\ = & \quad \langle (n+1)! = (n+1) \times n! \rangle \\ & e = (n+1) \times n! \\ = & \quad \langle \text{Assumption } P \rangle \\ & e = (n+1) \times f \end{array}$$

Q2.1

$$\{x = X \wedge y = Y\} \quad x, y :=? \quad \{x \leq y \ \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y)\}$$

Q2.2

$$\begin{array}{l} \{x = X \wedge y = Y\} \\ \textbf{if } x \leq y \textbf{ then skip else } x, y := y, x \\ \{x \leq y \ \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y)\} \end{array}$$

Q2.3

Prove

$$\{x = X \wedge y = Y \wedge x \leq y\} \textbf{ skip } \{x \leq y \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y)\}$$
$$\{x = X \wedge y = Y \wedge \neg(x \leq y)\} \ x, y := y, x \ \{x \leq y \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y)\}$$

$$x = X \wedge y = Y \wedge x \leq y \Longrightarrow x \leq y \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y)$$

$$\begin{array}{ll} & x = X \wedge y = Y \wedge x \leq y \\ \Longrightarrow & \quad \langle \text{Weakening/Strengthening} \rangle \\ & x \leq y \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y) \end{array}$$

$$\begin{array}{ll} & x = X \wedge y = Y \wedge \neg(x \leq y) \Longrightarrow (x \leq y \wedge (x = X \vee y = X) \wedge (x = Y \vee y = Y))[x, y := y, x] \\ = & \quad \langle \text{Textual Substitution} \rangle \\ & x = X \wedge y = Y \wedge \neg(x \leq y) \Longrightarrow y \leq x \wedge (y = X \vee x = X) \wedge (y = Y \vee x = Y) \end{array}$$

Assume antecedent and prove consequent.

$$
\begin{array}{ll}
 & y \le x \land (y = X \lor x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle \text{Definition of } \le \rangle \\
 & (y < x \lor y = x) \land (y = X \lor x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle \text{Assumption } x = X \land y = Y \land \neg(x \le y) \Longrightarrow y = x \equiv false \rangle \\
 & (y < x \lor false) \land (y = X \lor x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle \text{Identity of } \lor \rangle \\
 & y < x \land (y = X \lor x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle p < q \equiv \neg(p \ge q), \text{ with } p, q := y, x \rangle \\
 & \neg(y \ge x) \land (y = X \lor x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle \text{Distributivity of } \land \text{ over } \lor \rangle \\
 & (\neg(y \ge x) \land y = X \lor \neg(y \ge x) \land x = X) \land (y = Y \lor x = Y) \\
\Longleftrightarrow & \quad \langle \text{Distributivity of } \land \text{ over } \lor \rangle \\
 & ((\neg(y \ge x) \land y = X) \land (y = Y \lor x = Y)) \lor ((\neg(y \ge x) \land x = X) \land (y = Y \lor x = Y)) \\
\Longleftrightarrow & \quad \langle \text{Distributivity of } \lor \text{ over } \land \rangle \\
 & ((\neg(y \ge x) \land y = X) \land (y = Y \lor x = Y)) \lor \\
 & ((\neg(y \ge x) \land x = X \land y = Y) \lor (\neg(y \ge x) \land x = X \land x = Y)) \\
\Longleftrightarrow & \quad \langle \text{Assumption } x = X \land y = Y \land \neg(x \le y) \rangle \\
 & ((\neg(y \ge x) \land y = X) \land (y = Y \lor x = Y)) \lor (true \lor (\neg(y \ge x) \land x = X \land x = Y)) \\
\Longleftrightarrow & \quad \langle \text{Zero of } \lor \text{, twice} \rangle \\
 & true
\end{array}
$$

Q3

$$
\{0 \le n\} \quad m := ? \quad \begin{array}{l} \{(\text{odd}.m \Longrightarrow (\forall i \,|\, 0 \le i < n \land c[n-i-1] = b[i])) \lor \\ (\text{even}.m \Longrightarrow (\forall i \,|\, 0 \le i < n \land c[i] = 2 \times b[i]))\} \end{array}
$$

Q4.1

$$
\begin{array}{ll}
 & v \in \{b, c\} \\
= & \quad \langle \text{Set enumeration} \rangle \\
 & v \in \{x \,|\, x = b \lor x = c : x\} \\
= & \quad \langle x \in \{x \,|\, R\} \equiv R \rangle \\
 & v = b \lor v = c
\end{array}
$$

Q4.2

Proof via Mutual Implication

$$
\begin{array}{ll}
 & \{b, c\} = \{b, e\} \\
\iff & \quad \langle \text{Set enumeration} \rangle \\
 & \{x \mid x = b \vee x = c\} = \{x \mid x = b \vee x = e\} \\
\iff & \quad \langle \{x \mid Q\} = \{x \mid R\} \equiv (\forall x \mid\ : Q = R) \rangle \\
 & (\forall x \mid\ : x = b \vee x = c \equiv x = b \vee x = e) \\
\iff & \quad \langle \text{Distributivity of } \vee \text{ over } \equiv \rangle \\
 & (\forall x \mid\ : x = b \vee (x = c \equiv x = e)) \\
\iff & \quad \langle \text{Distributivity of } \vee \text{ over } \forall \rangle \\
 & x = b \vee (\forall x \mid\ : x = c \equiv x = e) \\
\iff & \quad \langle \text{Singleton membership} \rangle \\
 & x = b \vee (\forall x \mid\ : x \in \{c\} \equiv x \in \{e\}) \\
\iff & \quad \langle \text{Extensionality} \rangle \\
 & x = b \vee (\{c\} = \{e\}) \\
\implies & \quad \langle \text{Weakening/Strengthening} \rangle \\
 & \{c\} = \{e\} \\
\iff & \quad \langle \text{Set enumeration} \rangle \\
 & \{x \mid x = c\} = \{x \mid x = e\} \\
\iff & \quad \langle \text{One-point rule} \rangle \\
 & c = e
\end{array}
$$

$$
\begin{array}{ll}
 & c = e \\
\implies & \quad \langle \text{Leibniz, } \{b, z\}[z := p] \rangle \\
 & \{b, c\} = \{b, e\}
\end{array}
$$

Q5.1

$$
\begin{array}{ll}
 & (\forall M \mid M \subseteq X : (\sim M \cup M) \cup A = X) \\
\iff & \quad \langle \text{Excluded Middle} \rangle \\
 & (\forall M \mid M \subseteq X : X \cup A = X) \\
\iff & \quad \langle \text{Zero of } \cup \rangle \\
 & (\forall M \mid M \subseteq X : X = X) \\
\iff & \quad \langle \text{Metatheorem, } q \equiv q \equiv \mathit{true} \rangle \\
 & (\forall M \mid M \subseteq X : \mathit{true}) \\
\iff & \quad \langle (\forall x \mid R : \mathit{true}) \equiv \mathit{true} \rangle \\
 & \mathit{true}
\end{array}
$$

Q5.2

$$
\begin{array}{ll}
 & (\forall M \mid M \subseteq X : (\sim M \cap M) \cap A = \emptyset) \\
\iff & \quad \langle \text{Contradiction} \rangle \\
 & (\forall M \mid M \subseteq X : \emptyset \cap A = \emptyset) \\
\iff & \quad \langle \text{Zero of } \cap \rangle \\
 & (\forall M \mid M \subseteq X : \emptyset = \emptyset) \\
\iff & \quad \langle \text{Metatheorem, } q \equiv q \equiv \mathit{true} \rangle \\
 & (\forall M \mid M \subseteq X : \mathit{true}) \\
\iff & \quad \langle (\forall x \mid R : \mathit{true}) \equiv \mathit{true} \rangle \\
 & \mathit{true}
\end{array}
$$

Q6

$$
\begin{array}{ll}
 & \mathcal{P}\emptyset = \{\emptyset\} \\
\iff & \quad \langle \text{Extensionality} \rangle \\
 & (\forall v \mid\ : v \in \mathcal{P}\emptyset \equiv v \in \{\emptyset\}) \\
\iff & \quad \langle \text{Power set} \rangle \\
 & (\forall v \mid\ : v \subseteq \emptyset \equiv v \in \{\emptyset\}) \\
\iff & \quad \langle S \subseteq T \equiv S \cup T = T \rangle \\
 & (\forall v \mid\ : v \cup \emptyset = \emptyset \equiv v \in \{\emptyset\}) \\
\iff & \quad \langle \text{Identity of } \cup \rangle \\
 & (\forall v \mid\ : v = \emptyset \equiv v \in \{\emptyset\}) \\
\iff & \quad \langle \text{Singleton membership} \rangle \\
 & (\forall v \mid\ : v \in \emptyset \equiv v \in \{\emptyset\}) \\
\iff & \quad \langle \text{Identity of } \equiv \rangle \\
 & (\forall v \mid\ : \mathit{true}) \\
\iff & \quad \langle (\forall x \mid R : \mathit{true}) \equiv \mathit{true} \rangle \\
 & \mathit{true}
\end{array}
$$

Q7

$$
\begin{array}{ll}
& (\exists x \mid x \in S : x \notin T) \implies S \neq T \\
\iff & \quad \langle \text{Definition of Implication} \rangle \\
& \neg(\exists x \mid x \in S : x \notin T) \vee S \neq T \\
\iff & \quad \langle \text{Generalized De Morgan} \rangle \\
& (\forall x \mid x \in S : x \in T) \vee S \neq T \\
\iff & \quad \langle \text{Extensionality} \rangle \\
& S = T \vee S \neq T \\
\iff & \quad \langle \text{Excluded Middle} \rangle \\
& \mathit{true}
\end{array}
$$

Q8a

$$
\begin{aligned}
A \cup B - A \cap B &= \{\!| 121.\text{iMac}, 50.\text{desktop\_computer}, 8.\text{server} |\!\} - \{\!| 14.\text{iMac}, 6.\text{desktop\_computer}, 2.\text{server} |\!\} \\
&= \{\!| 107.\text{iMac}, 44.\text{desktop\_computer}, 6.\text{server} |\!\}
\end{aligned}
$$

Since both departments can share the same equipment, the total equipment required is the difference between the total needs and the amount of equipment that would be shared; department A requires more of every equipment so it is the universe.

Q8b

$$B - A = \emptyset$$

The question requires the difference between B and A, but since department A uses more of every equipment, there is no equipment that B uses that A does not use.

Q8c

$$A \cup B = \{\!| 121.\text{iMac}, 50.\text{desktop\_computer}, 8.\text{server} |\!\}$$

Since both departments require separate equipment, their needs for each equipment are summed.

Q9 Each category is a tuple that consists of one element from each of the two criteria, which implies $G \times W$.

$$
\begin{aligned}
\#(G \times W) &= \#G \bullet \#W \\
&= 3 \bullet 2 \\
&= 6
\end{aligned}
$$

Q10

Assume antecedent $A \subseteq U \wedge B \subseteq V$ and prove consequent.

$$
\begin{aligned}
& \langle x, y\rangle \in A \times B \\
\Longleftrightarrow \quad & \langle \text{Membership} \rangle \\
& x \in A \wedge y \in B \\
\Longrightarrow \quad & \langle \text{Assumption, } S \subseteq T \Longrightarrow x \in S \Longrightarrow x \in T \rangle \\
& x \in U \wedge y \in V \\
\Longleftrightarrow \quad & \langle \text{Membership} \rangle \\
& \langle x, y\rangle \in U \times V
\end{aligned}
$$

Q11

$$
\begin{aligned}
\text{Dom}.R &= \{x : \mathbb{N} \mid (\exists y \mid y \in \mathbb{N} : \langle x, y\rangle \in \{(x, y) \mid y < 200 \wedge y = x - 8)\}\} \\
&= [9, 207] \\
\text{Ran}.R &= \{y : \mathbb{N} \mid (\exists x \mid x \in \mathbb{N} : \langle x, y\rangle \in \{(x, y) \mid y < 200 \wedge y = x - 8)\}\} \\
&= [1, 199]
\end{aligned}
$$

Q12.1

$$
\begin{aligned}
& \langle a, c\rangle \in R \circ R^{-1} \\
\Longleftrightarrow \quad & \langle \text{Definition of } \circ \rangle \\
& (\exists b \mid b \in \mathbb{N} : \langle a, b\rangle \in R \wedge \langle b, c\rangle \in R^{-1}) \\
\Longleftrightarrow \quad & \langle \text{Definition of } R, R^{-1} \rangle \\
& (\exists b \mid b \in \mathbb{N} : \langle a, b\rangle \in \{(x, y) \mid y = 2x\} \wedge \langle b, c\rangle \in \{(y, x) \mid y = 2x\}) \\
\Longleftrightarrow \quad & \langle x \in \{x \mid R\} \equiv R \rangle \\
& (\exists b \mid b \in \mathbb{N} : b = 2a \wedge b = 2c) \\
\Longleftrightarrow \quad & \langle \text{Simplify} \rangle \\
& (\exists b \mid b \in \mathbb{N} : a = c)
\end{aligned}
$$

If the product of $R \circ R^{-1}$ results in only tuples $\langle a, c\rangle$ where $a = c$, then it is a subset of the identity relation and therefore deterministic.

Q12.2

$$\sim R = \{\langle x, y\rangle \mid y \neq 2x\}$$

Q12.3

From 12.1

$$R \circ R^{-1} = \{x \mid x \in \mathbb{N} : \langle x, x\rangle\}$$

Q12.4

$$
\begin{aligned}
& R \subseteq S \times T \Longleftrightarrow R^{-1} \subseteq T \times S \qquad \langle 14.19 \rangle \\
\Longrightarrow \quad & R \circ R^{-1} \subseteq S \times T \times S \\
\Longleftrightarrow \quad & R \circ R^{-1} \subseteq S \times S
\end{aligned}
$$

Q12.5

$$
\begin{aligned}
& R \subseteq S \times T \Longleftrightarrow R^{-1} \subseteq T \times S \qquad \langle 14.19 \rangle \\
\Longrightarrow \quad & R^{-1} \circ R \subseteq T \times S \times T \\
\Longleftrightarrow \quad & R^{-1} \circ R \subseteq T \times T
\end{aligned}
$$