ME40064: Systems Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 6

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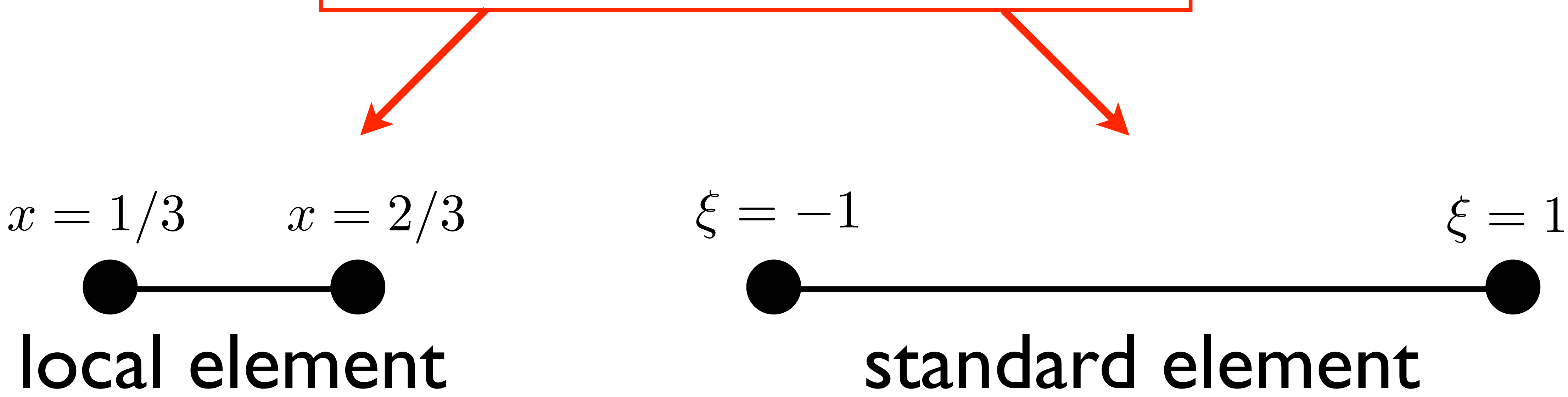
LECTURE 6
FEM: Local Element Matrix

- Understand how the local element matrix arises from basis function representation
- Ability to construct the local element matrix for diffusion operator

RECAP
Want To Perform Integration In An Element

$$\boxed{\int_{1/3}^{2/3} v f\, dx = \int_{-1}^{1} v f J\, d\xi}$$



Jacobian is the transformation between coordinates of local and standard elements $J = \left| \frac{dx}{d\xi} \right|$

LOCAL ELEMENT MATRIX
Start With LHS Of Equation

$$\boxed{\int_{-1}^{1} D \frac{\partial c}{\partial x} \frac{\partial v}{\partial x} J d\xi}$$

$$J = \left| \frac{dx}{d\xi} \right|$$

Remembering that:

$$c = c_0 \psi_0(\xi) + c_1 \psi_1(\xi)$$

$$x = x_0 \psi_0(\xi) + x_1 \psi_1(\xi)$$

$$v = \psi_0, \psi_1$$

LOCAL ELEMENT MATRIX
A More Compact Notation

The sum of basis functions:

$$c = c_0\psi_0(\xi) + c_1\psi_1(\xi)$$

$$x = x_0\psi_0(\xi) + x_1\psi_1(\xi)$$

Can be written in the form:

$$c = c_n\psi_n, \quad v = \psi_m$$

where a repeated subscript implies a summation.

LOCAL ELEMENT MATRIX
Expanding The Derivatives

Evaluate the following derivatives using the chain rule

$$\frac{dc}{dx} = c_0 \frac{d\psi_0}{d\xi}\frac{d\xi}{dx} + c_1 \frac{d\psi_1}{d\xi}\frac{d\xi}{dx}$$

Or in the more compact notation

$$\frac{dc}{dx} = c_n \frac{d\psi_n}{d\xi}\frac{d\xi}{dx}$$

$$\frac{dv}{dx} = \frac{d\psi_m}{d\xi}\frac{d\xi}{dx}$$

Noting that c_n are independent of x

LOCAL ELEMENT MATRIX
Evaluating The Terms

The integral is now:

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_m}{d\xi} \frac{d\xi}{dx} J d\xi$$

Note that for n,m = 0 & 1, there are four combinations of this equation

Now these terms will be evaluated for the first element of the mesh

$$\frac{dx}{d\xi} = \frac{x_1 - x_0}{2} \qquad J = \left| \frac{dx}{d\xi} \right| = \left| \frac{x_1 - x_0}{2} \right|$$

LOCAL ELEMENT MATRIX
Evaluating The Terms

For the element: $x_0 = 0, \quad x_1 = 1/3$

$$\frac{d\xi}{dx} = \frac{2}{(1/3 - 0)} = 6 \qquad J = \left|\frac{1/3 - 0}{2}\right| = \frac{1}{6}$$

Also need to evaluate: $\dfrac{d\psi_n}{d\xi}$

$$\psi_0 = \frac{1 - \xi}{2}, \qquad \frac{d\psi_0}{d\xi} = -\frac{1}{2}$$

$$\psi_1 = \frac{1 + \xi}{2}, \qquad \frac{d\psi_1}{d\xi} = \frac{1}{2}$$

LOCAL ELEMENT MATRIX
Evaluating The Terms

Multiplying by test function v, is multiplying by two functions, giving two equations:

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} J d\xi$$

LOCAL ELEMENT MATRIX

Matrices Make Their Entrance

Recap: repeated subscript implies summation

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

Recap: repeated subscript implies summation

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

Therefore these equations are actually:

$$c_0 \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi +$$

$$c_1 \int_{-1}^{1} D \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

And:

$$c_0 \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} J d\xi +$$

$$c_1 \int_{-1}^{1} D \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} J d\xi$$

If c is written as a vector, these integrals can be multiply the vector, if written in a matrix

$$\begin{bmatrix} Int_{00} & Int_{01} \\ Int_{10} & Int_{11} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

LOCAL ELEMENT MATRIX
Evaluating The Integrals

$$\begin{bmatrix} Int_{00} & Int_{01} \\ Int_{10} & Int_{11} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

That is the matrix entries are:

$$Int_{00} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

$$Int_{01} = Int_{10} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} J d\xi$$

$$Int_{11} = \int_{-1}^{1} D \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} \frac{d\psi_1}{d\xi} \frac{d\xi}{dx} J d\xi$$

We can perform these integrals manually:

$$Int_{00} = \int_{-1}^{1} D. - \frac{1}{2}.6. - \frac{1}{2}.6.\frac{1}{6} d\xi$$

$$= \int_{-1}^{1} \frac{6D}{4} d\xi = \left[\frac{3D}{2}(1 - (-1)) \right] = 3D$$

$$Int_{01} = Int_{10} = \int_{-1}^{1} D - \frac{1}{2}.6.\frac{1}{2}.6.\frac{1}{6} d\xi$$

$$= \int_{-1}^{1} -\frac{6D}{4} d\xi = \left[-\frac{3D}{2}(1 - (-1)) \right] = -3D$$

Finally:

$$Int_{11} = \int_{-1}^{1} D.\frac{1}{2}.6.\frac{1}{2}.6.\frac{1}{6}d\xi$$

$$= \int_{-1}^{1} \frac{6D}{4}d\xi = \left[\frac{3D}{2}(1-(-1))\right] = 3D$$

The Final Local Element Matrix

Putting these values in, the integral has become the following local element matrix:

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Note that this matrix is symmetric