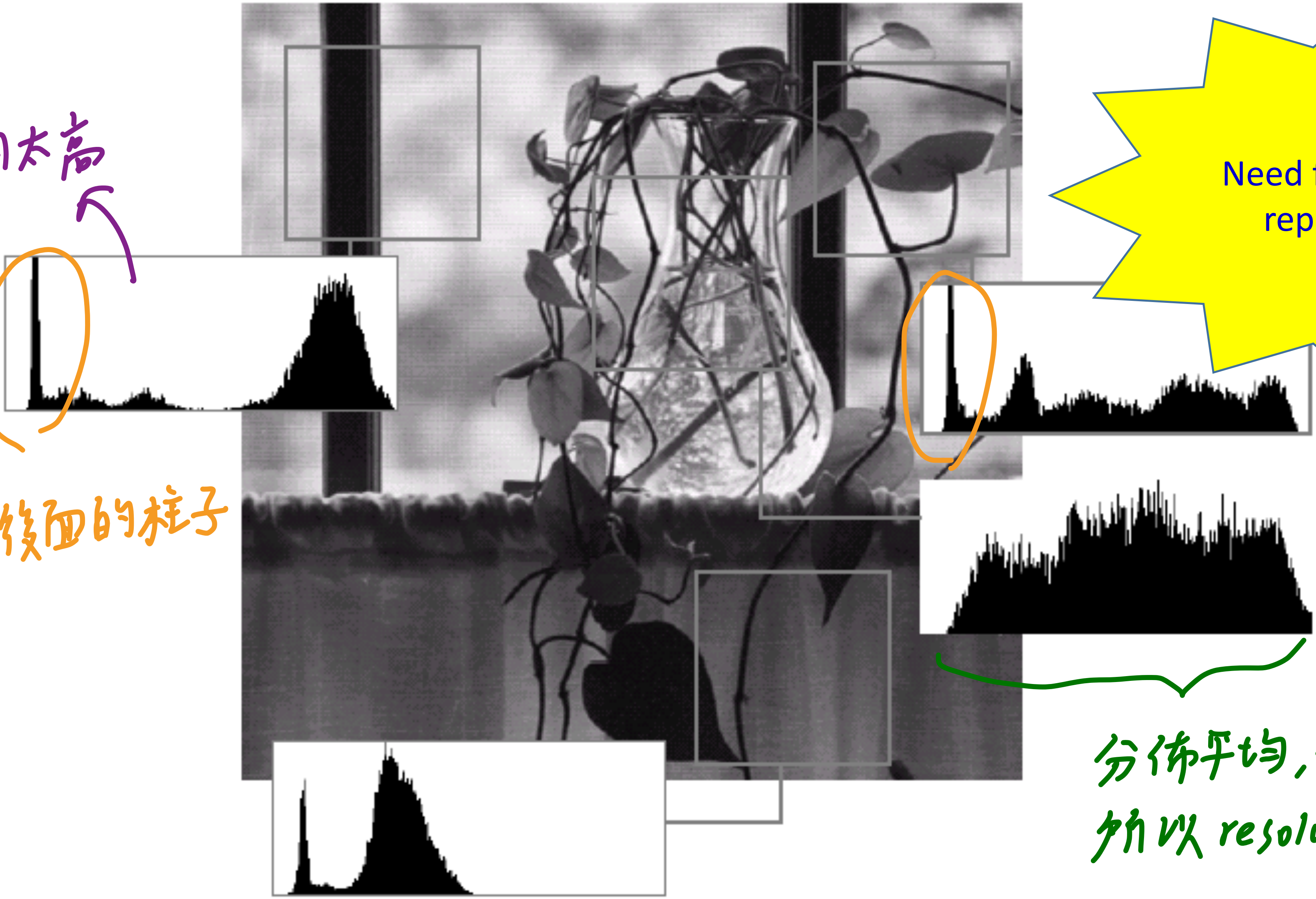
EE 5098 – Digital Image Processing

7. Wavelet Transform

Nonstationary Properties of Natural Images

detail不多,
∴ resolution不用太高



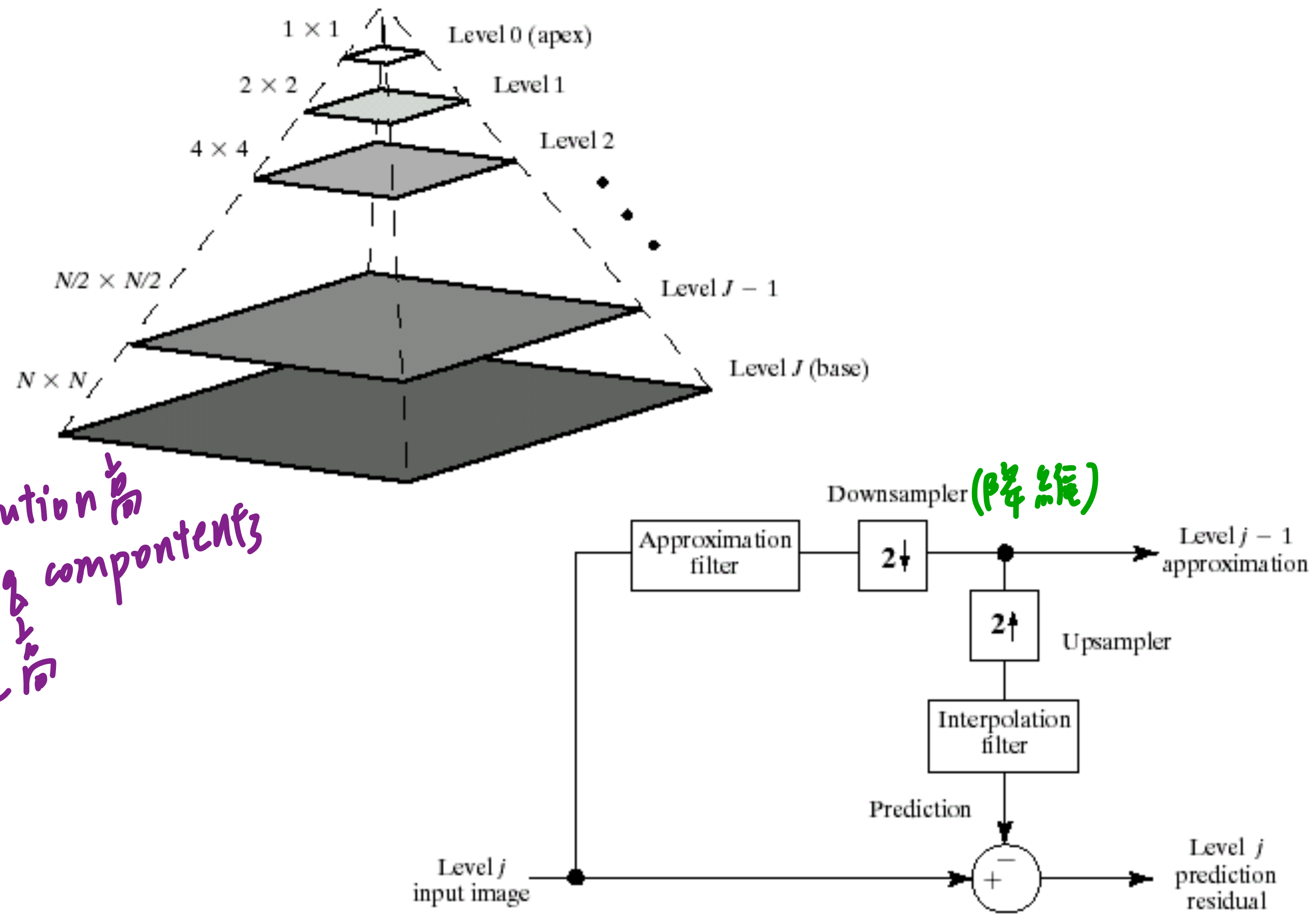
可能是後面的柱子

Need for multi-scale representation

分佈平均,代表detail多
所以resolution不能太低

Pyramidal Image Representation

resolution 低



a b

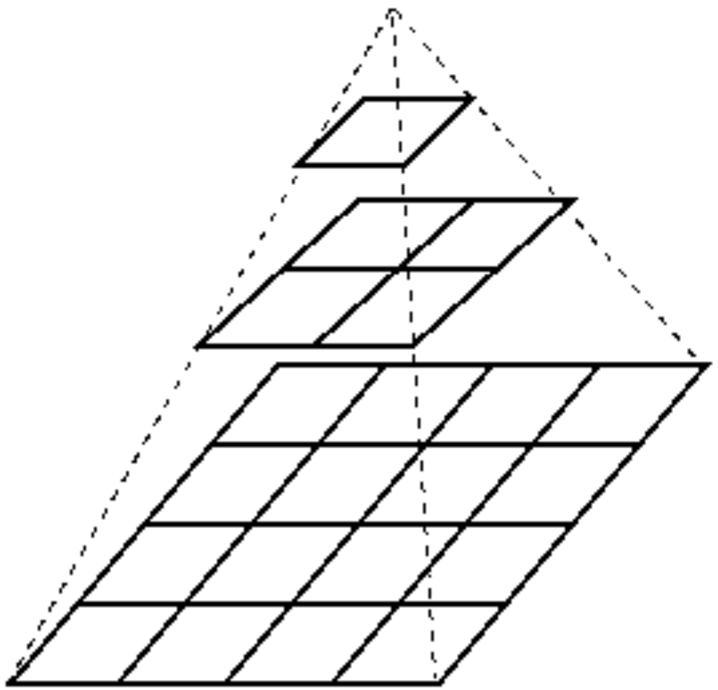
FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.

resolution 高
high freq compontents
的含量高

Gaussian Pyramid Representation

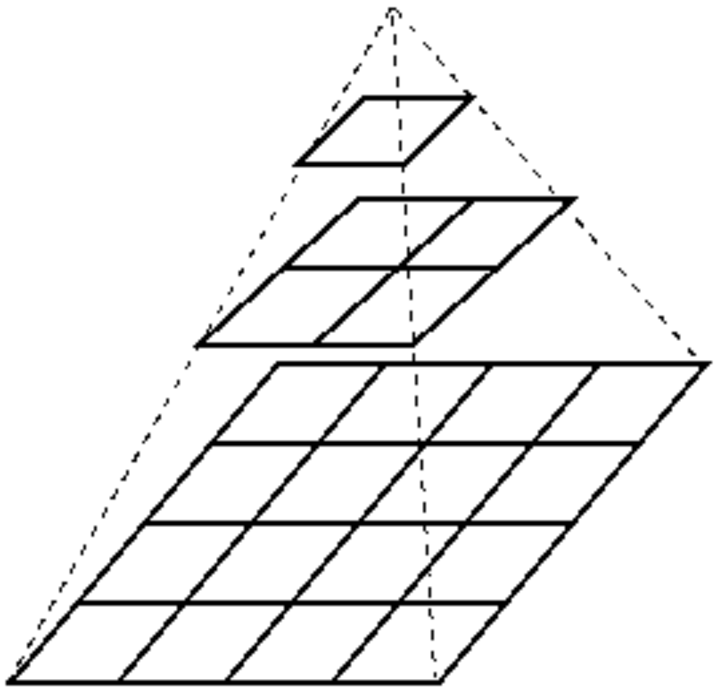
- Start with bottom layer
- Successively decimate by ½ in both dimensions using Gaussian anti-alias filtering

  大幅減少
- Since the frequency response of the Gaussian filter has some leakage beyond the frequency $\omega = \pi/2$, images in the upper levels contain aliasing.

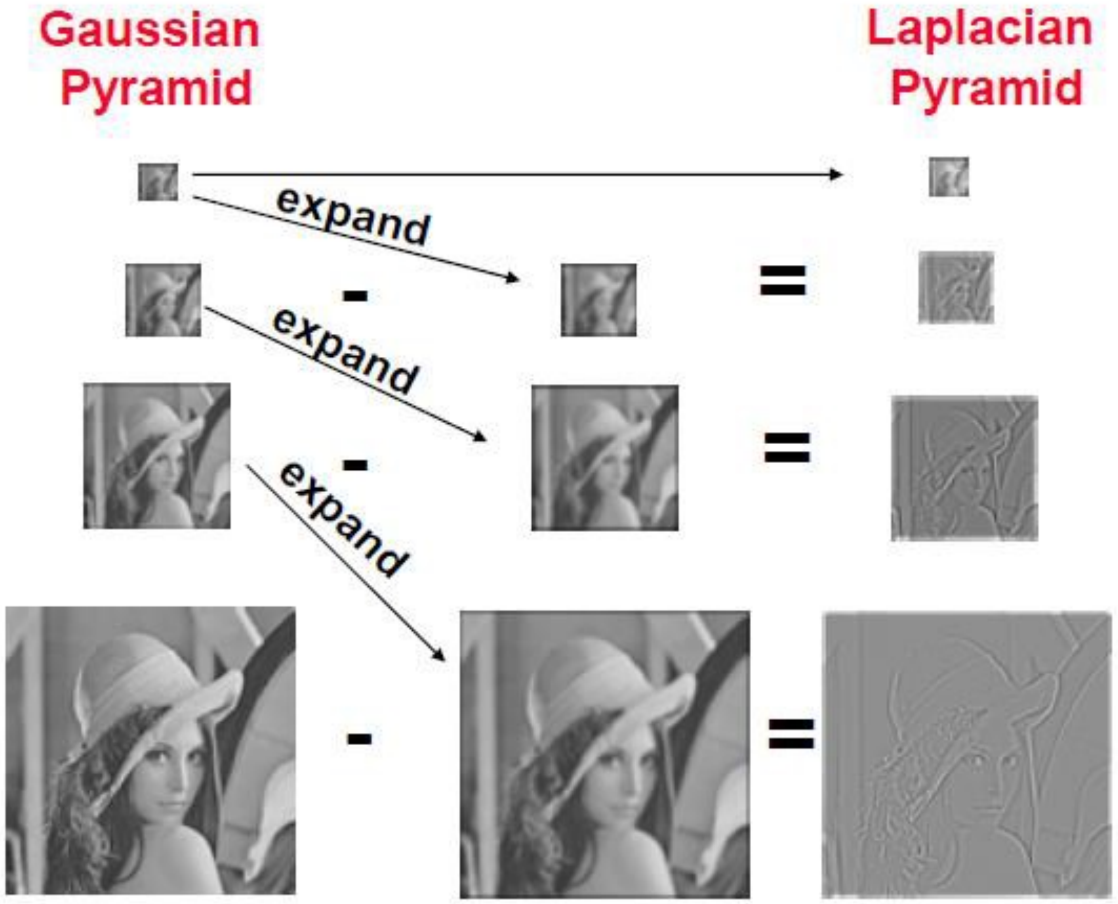


Laplacian Pyramid Representation

- Start with the thumbnail image at the top level
- Interpolate by 2 in both dimensions and subtract from the same level of the Gaussian pyramid.
- Keep difference image
- Repeat for all levels

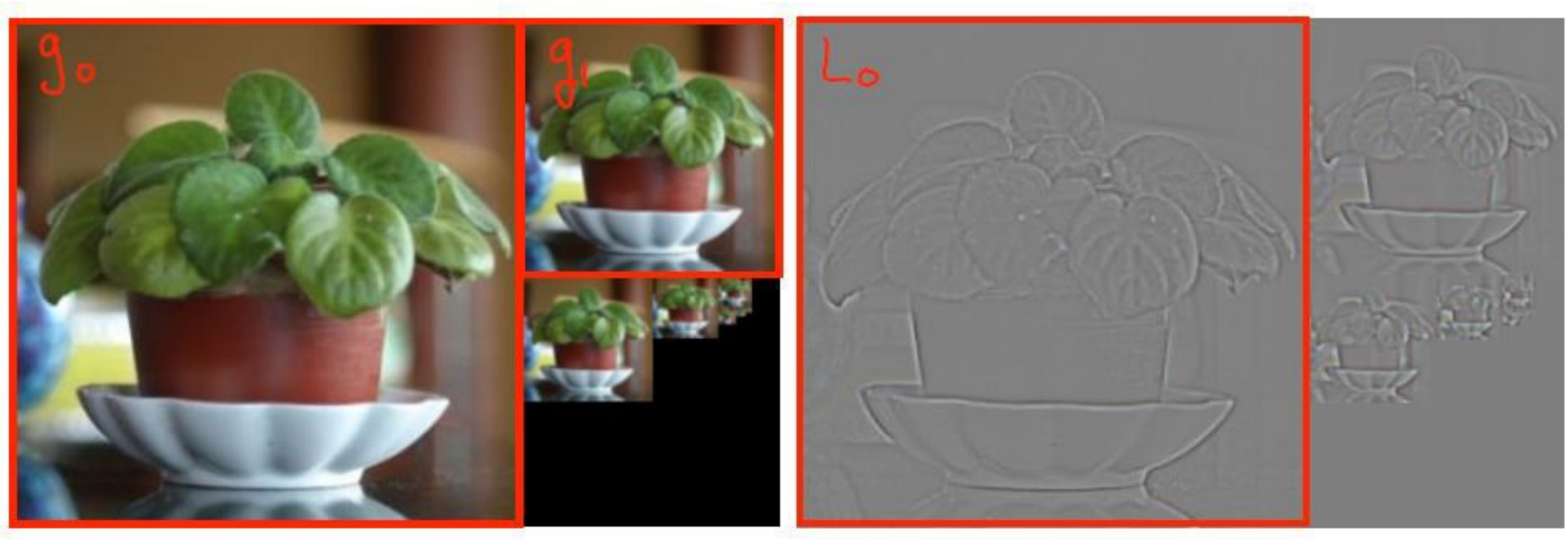


Thumbnail image + difference images

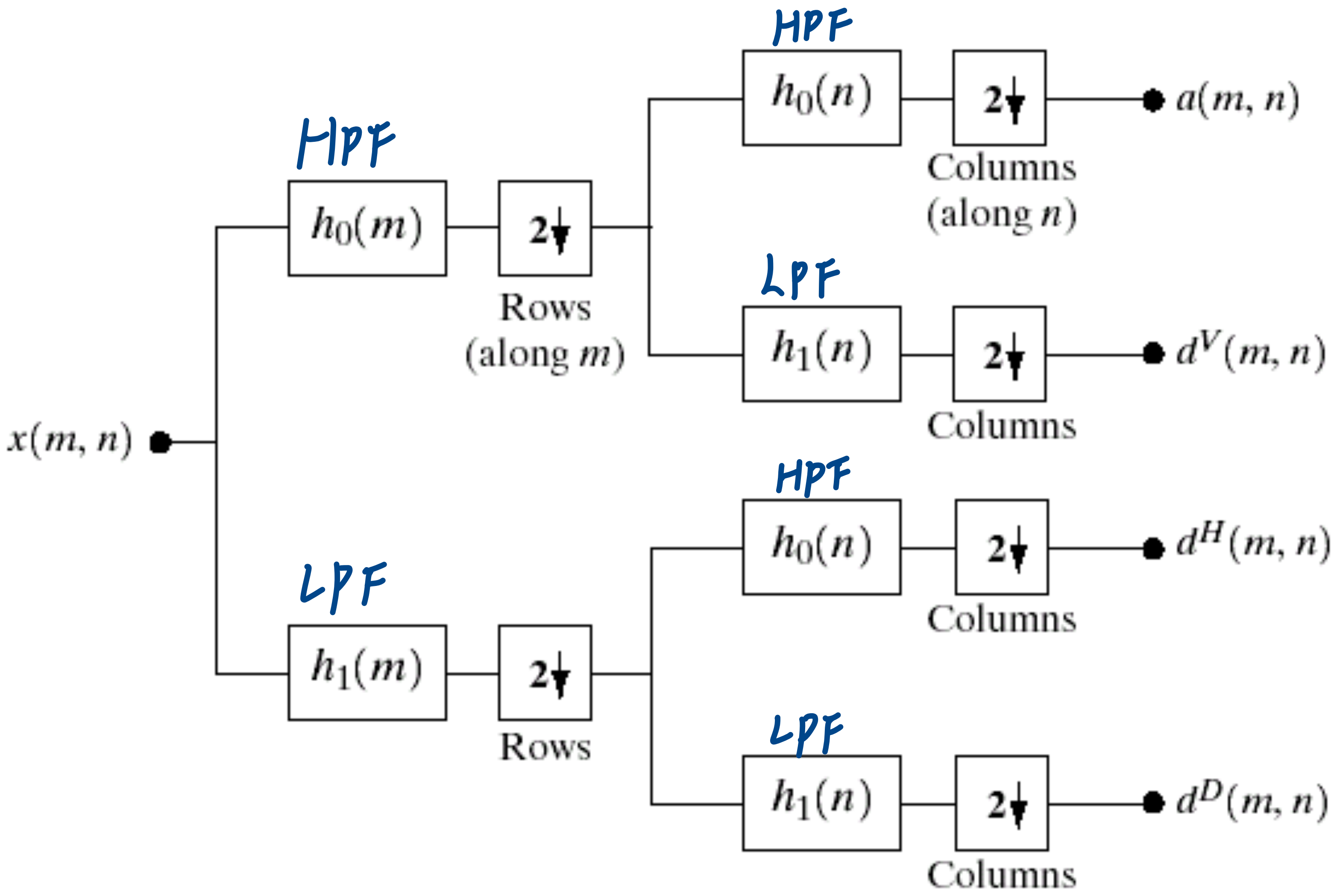


Laplacian Pyramid Representation

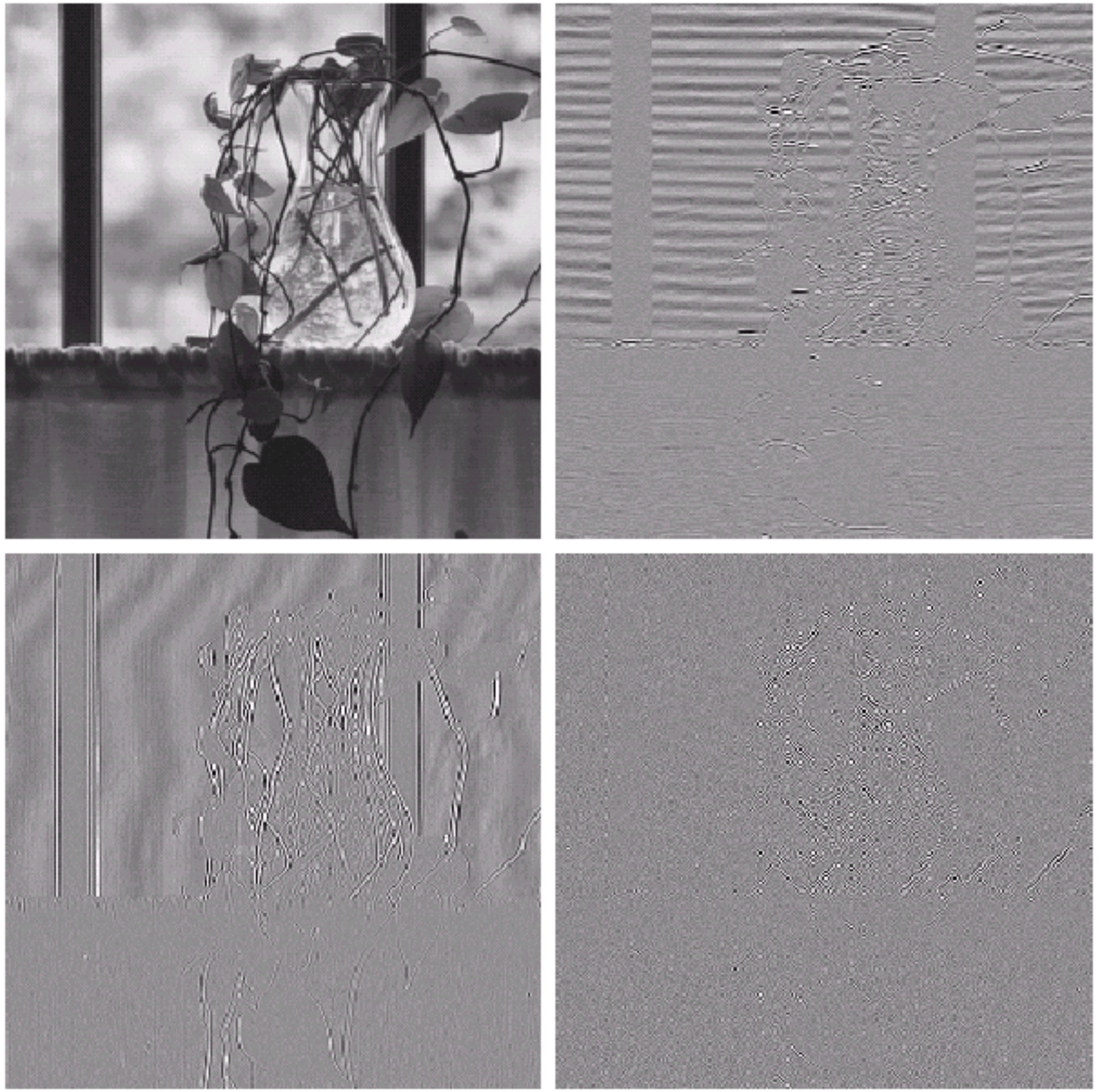
$L_0 = g_0 - \text{interp}(g_1)$
$L_1 = g_1 - \text{interp}(g_2)$



2D Subband Filter



Subband Decomposition Example

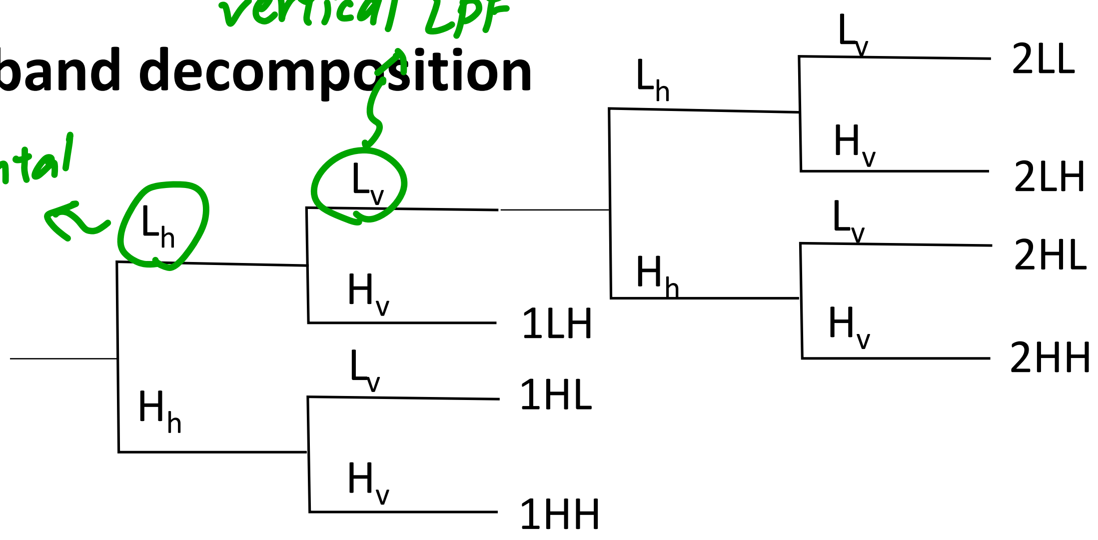


2D Wavelet Representation

vertical LPF

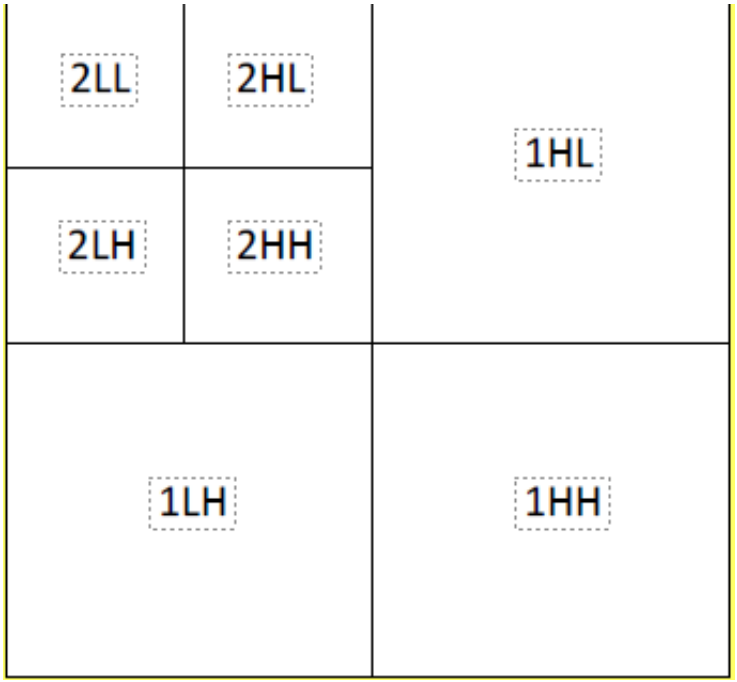
2 level, 7-band decomposition

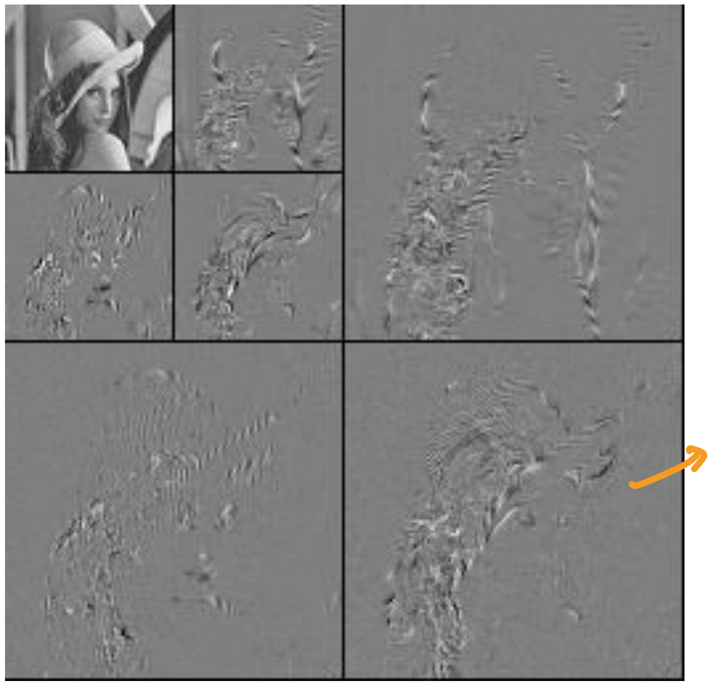
horizontal LPF



L_h

H_h





1HH
的correlation
和原圖滿高的

Wavelet Transform

- Wavelet representation provides a *multi- resolution* and *multi-frequency* description of an image signal with localization in space and frequency.
- Wavelet transform decomposes a non-stationary signal (like the real-world image and video) into a set of multi-scaled subbands in which each component becomes relatively more stationary and hence easier to code.

和 fourier transform 不同的地方

fft 無法保留 location information

e.g. 在 fft 看到某個東西，但你不會知道它在 spatial domain 的哪

原因：fourier 轉換是積分 p12

Motivations for Wavelet Transform

- The motivation for using wavelet is to provide a set of basis functions that decompose a signal in time over parameters in the frequency domain and the time domain simultaneously.
- While Fourier transform only pins down the frequency content of a signal, wavelets pin down the frequency content at different parts of the image.
- That is, the basis functions of the wavelet transform are localized in both time and frequency.
- Wavelet transform decomposes the input signal into components that are easier to deal with, have special interpretations, or can be thresholded for compression purposes.

Why Another Way for Signal Decomposition?

ex. [1,5,3,7,2] 取平均=3.6

但3.6無法得知

每個位置上的具体

數值.

- To answer the question, let's examine the Fourier transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt$$

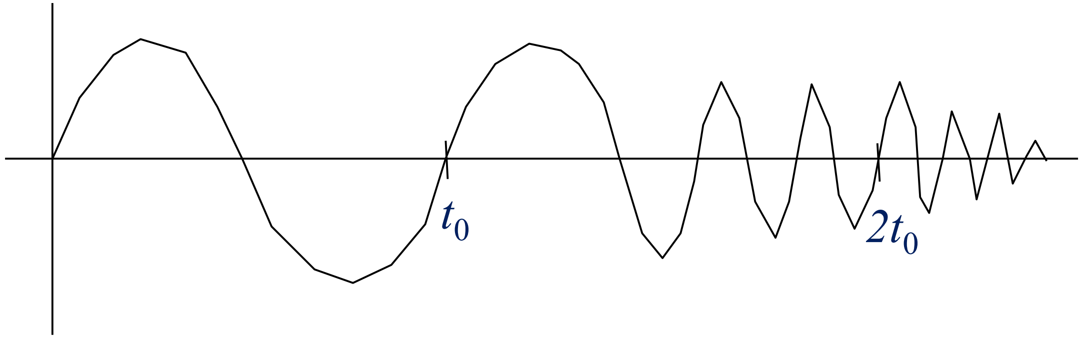
- Integration is an **average** operation. Thus, the analysis we obtain in some sense is an average analysis. → 完全失去 location info.

- Looking at a Fourier transform of a signal, we can say, for example, that there is a large component of frequency 20KHz in the signal, but we cannot tell when in time this component occurs.

- The converse is true for the time function f(t), which provides exact information about the value of the function at each instance of time but does not directly provide spectral information of the signal.

↳ 相反地，知道 location 但不知道 freq

- Take the following signal as an example,



- We would like to know not only the frequency components but when in time the particular frequency component occurred.
- One way to obtain this information is via the short-term Fourier transform (STFT). With STFT, we break the time signal into pieces of length T and apply Fourier analysis to each piece.
- This way, we can say that a component of 10 KHz occurred in the third piece. But this method generates distortion in the form of boundary effects.

- To reduce the boundary effects, we **window** each piece before we take the Fourier transform.

- If the window shape is given by $g(t)$, the STFT is given by

$$F(j\omega,\ \tau) = \int_{-\infty}^{\infty} f(t) g^*(t-\tau) e^{j\omega t} dt$$

- If $g(t)$ is a Gaussian, the STFT is called the Gobor transform.

- The problem with the STFT is the fixed window size. In order to obtain the low-pass component at the beginning of the function shown in the previous page, the window size should be at least t_0.

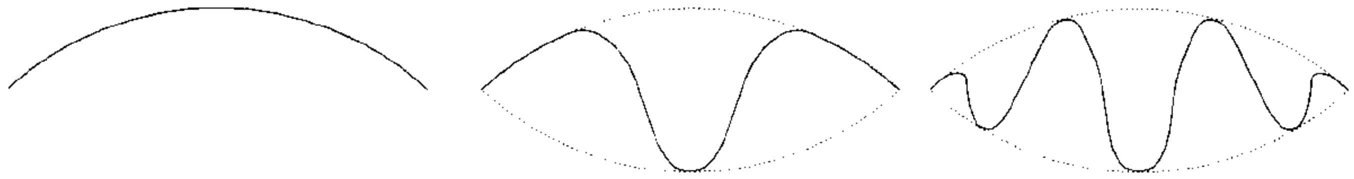
- However, a window of size t_0 or greater will not be able to accurately localize the high frequency spurt. Thus, if we want to have finer resolution in time, we end up with a lower resolution in frequency domain. How do we get around the problem?

- Consider the STFT in terms of basis expansion and just look at one interval for the moment:

$$F(m,\ 0) = \int_{-\infty}^{\infty} f(t) g^*(t) e^{jm\omega_0 t} dt$$

fixed window size
但 g(t) 不同

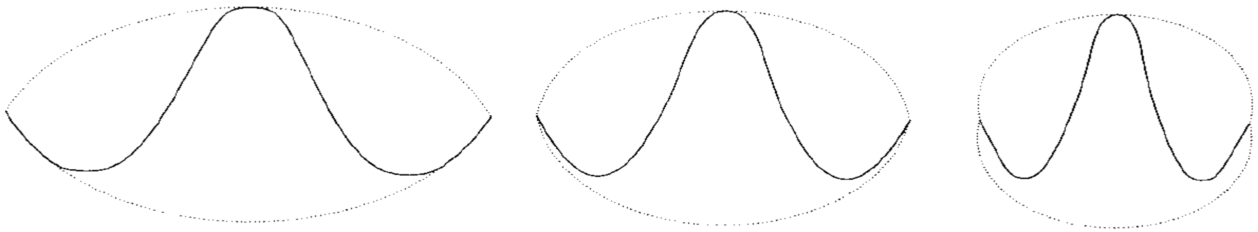
- The basis functions are $g(t),\ g(t)e^{j\omega_0 t},\ g(t)e^{j2\omega_0 t}$, etc



- We can see that we have a window with constant size, and within this window, we have sinusoids with an increasing number of cycles.

- What if we have a different set of basis functions in which the number of cycles of the sinusoids is constant, but the size of the window keeps changing?

→ 每個 window 的 freq 一樣，但 window size 不同



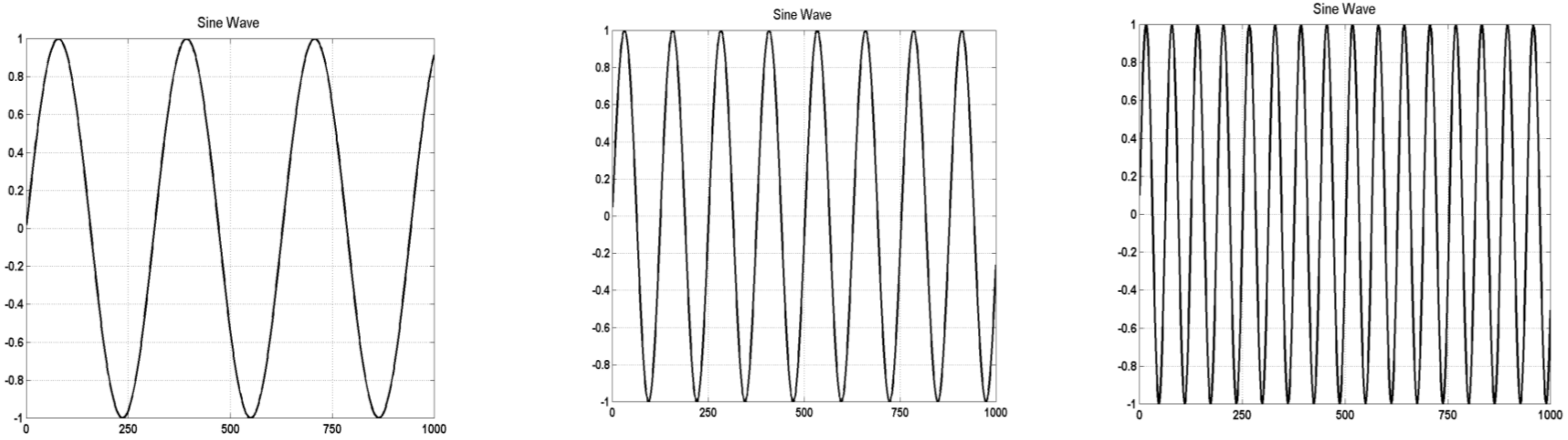
- The lower (higher) frequency functions cover a longer (shorter) time interval—exactly what we want.
- If we can write our function in terms of these functions and their translates, we have a representation that gives us time and frequency localization and can provide high frequency resolution at low frequencies (longer time window) and high time resolution at high frequency (shorter time window). → 分成很多小波，保留 location、freq
- This is essentially the basic idea behind wavelets.

小波

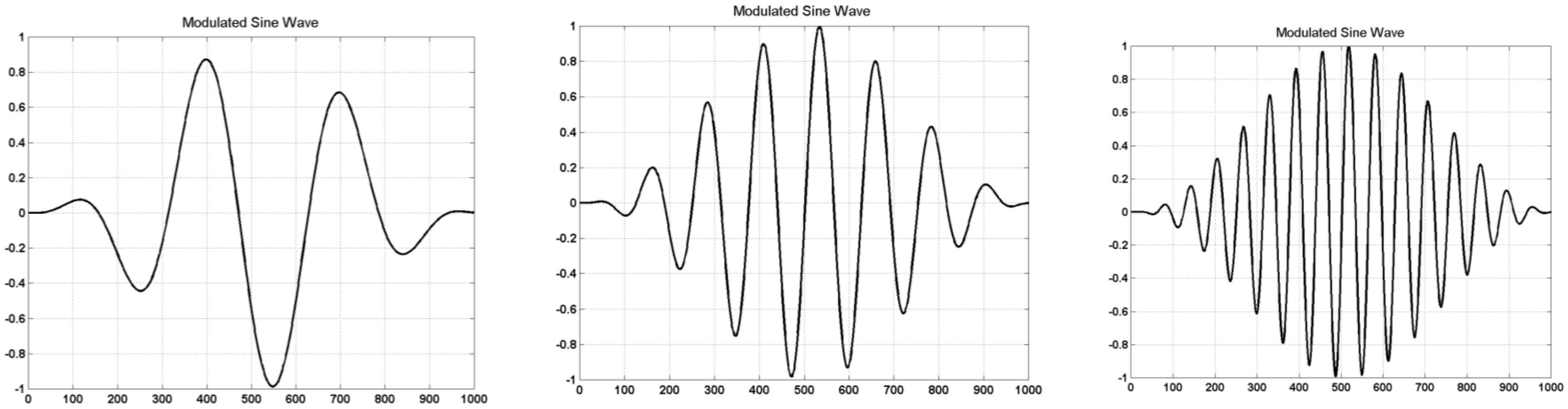
- In the above example, we started out with a single function and generated the other functions by changing the size (or scaling) of this signal function and translating it. This function is called the mother wavelet.
- The mother wavelet has zero mean—the admissibility condition for wavelets.

FT and Short-Term Fourier Transform

Fourier basis functions (infinite support)



Short-term Fourier basis functions (finite support)



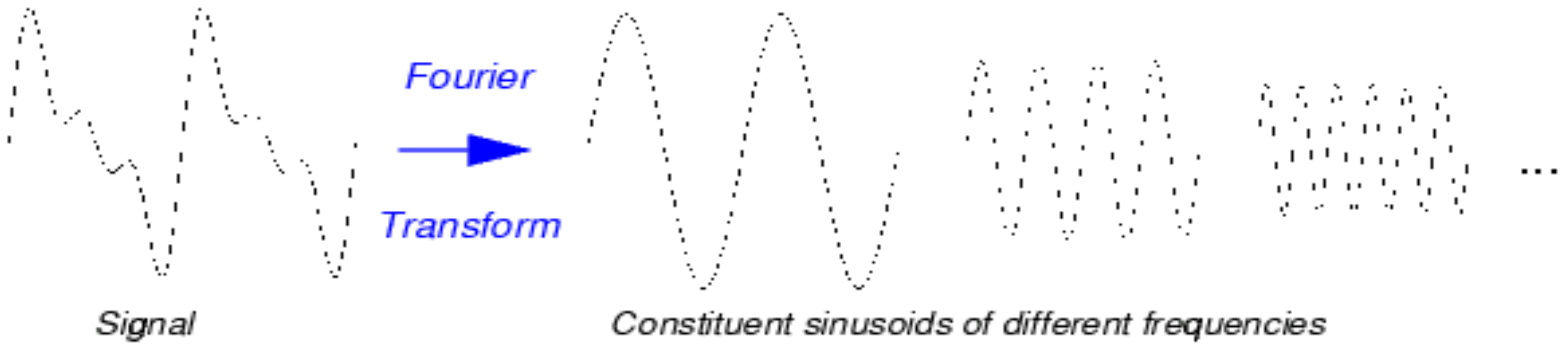
Fourier Transform vs. Wavelet Transform

<u>Fourier Transform</u>

- Basis functions: sinusoids
- Only offer frequency information
- Lose time (location) coordinate completely
- Analyze the **whole** signal
- Short pieces lose "frequency" meaning

<u>Wavelet Transform</u>

- Basis functions: small waves (wavelets)
- Localized time-frequency analysis
- Frequency + temporal information
- Short signal pieces also have significance
- Scale = Frequency band

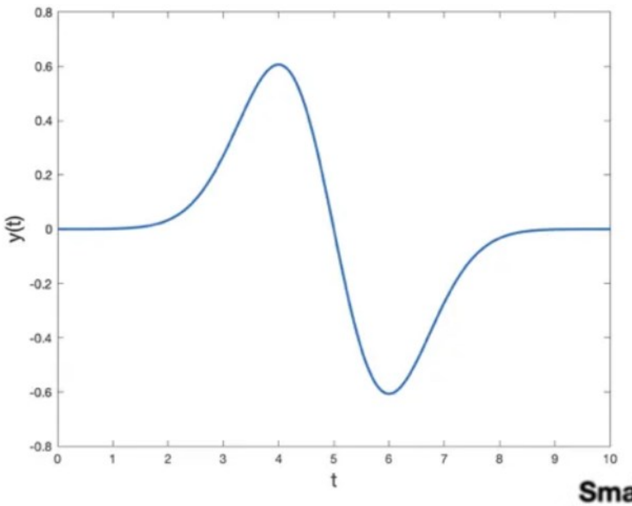


Fourier analysis doesn't work well on discontinuous, "bursty" data such as music, video, power, sesmic,…

Wavelets

Wave-like oscillation localized in space (time)

Example:



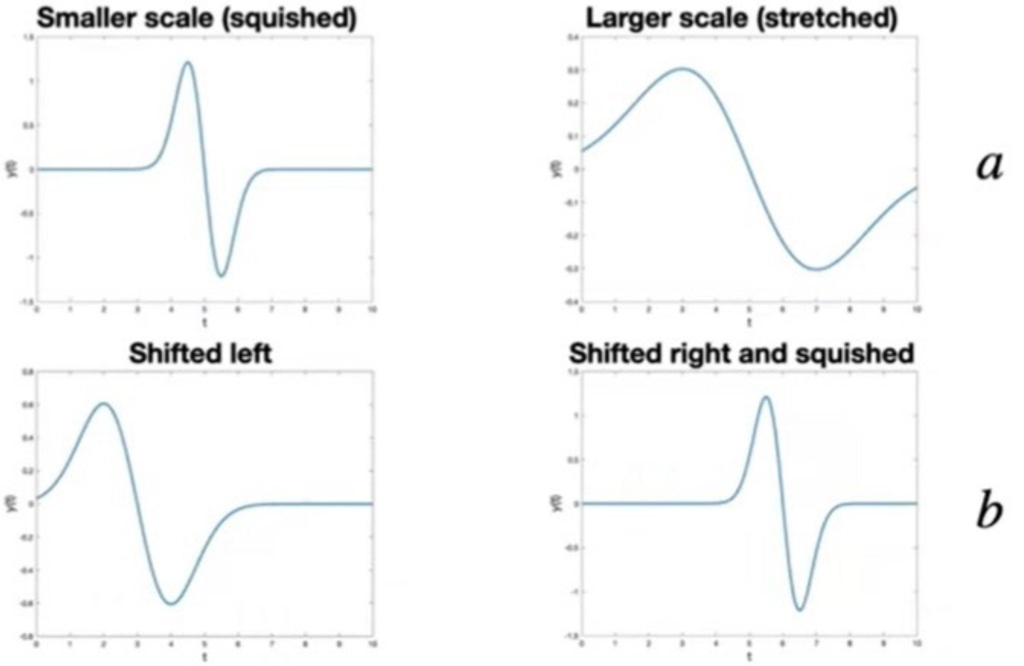
$$-(x-b)e^{\frac{-(x-b)^2/(2a^2)}{\sqrt{2\pi}a^3}}$$

First derivative of Gaussian Function

[Source: S. Talebi]

Two Properties

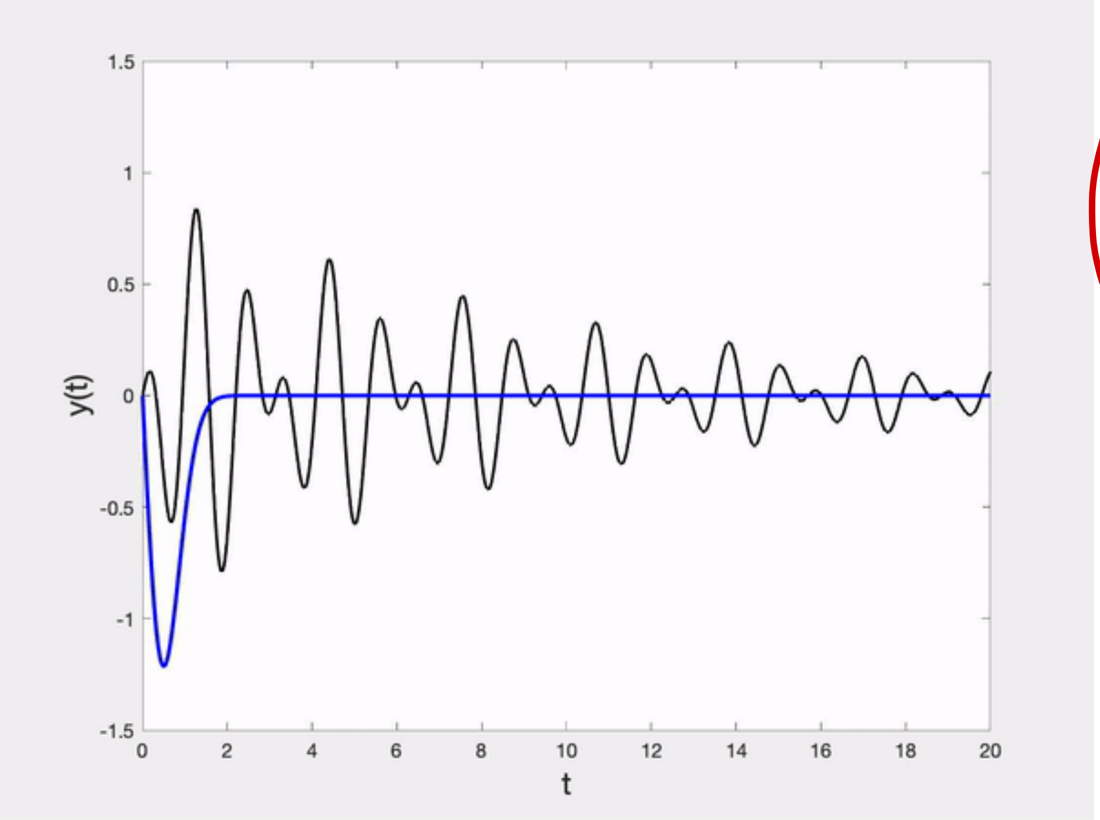
- **Scale** (dilation) – How squished or stretched wavelet is relative to frequency
- **Location** – where wavelet is positioned in time



P.S. Addison Wavelet transforms and the ECG: A review, Physiological Measurement, 26(5), p. R155, 2005.

Wavelet Transform

Decomposition of a signal using wavelets of varying scale and location



Basic idea: Compute how much of wavelet is in the signal for a particular scale and location.

That is, evaluate convolution of signal and wavelet at varying scales.

Why wavelets?

- Traditional FT gives global average over entire signal, thus may obscure local information
- WT can extract local spectral and temporal information simultaneously
- Variety of wavelets to choose from

Wavelet Transform

Two methods

Continuous Wavelet Transform (CWT)

Mother (basis) wavelets are defined everywhere

$$\psi = \psi(t)$$

Transform must be discretized for implementation

$$\boxed{T(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\, \psi^* \frac{(t-b)}{a} dt}$$

$T(a,b)$ may have redundant information since same feature may be *captured* by multiple scales

Discrete Wavelet Transform (DWT)

Mother (basis) wavelets are only defined on discrete grid

$$\psi = \psi_{m,n}(t)$$

Transform is already discrete!

$$\boxed{T_{m,n} = \int_{-\infty}^{\infty} x(t)\, \psi_{m,n}(t)\, dt}$$

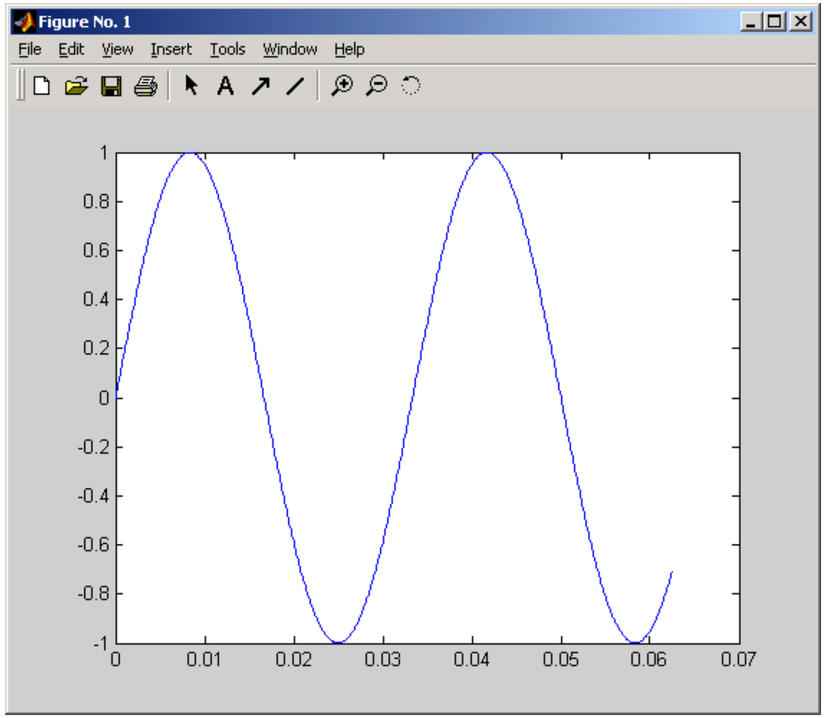
Coefficients ($T_{m,n}$) do not have redundant information since discretized wavelets can be defined to be orthonormal

$$\int_{-\infty}^{\infty} \psi_{m,n}(t)\psi_{m',n'}(t)\, dt = \begin{cases} 1 & \text{if} \quad m = m' \text{ and } n = n' \\ 0 & \text{otherwise.} \end{cases}$$

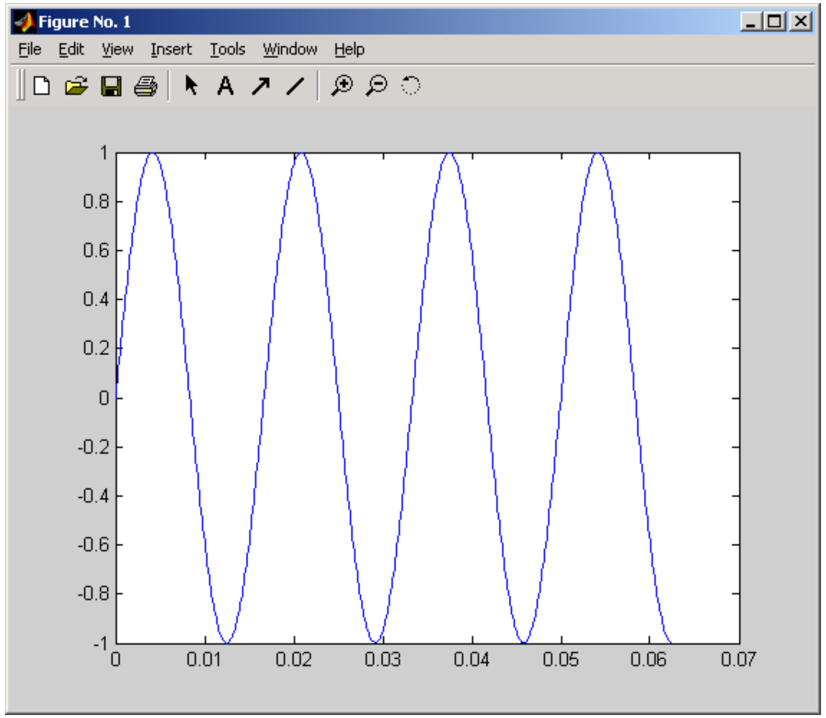
Scaling—Value of "Stretch"

- Scaling a wavelet simply means stretching (or compressing) it.

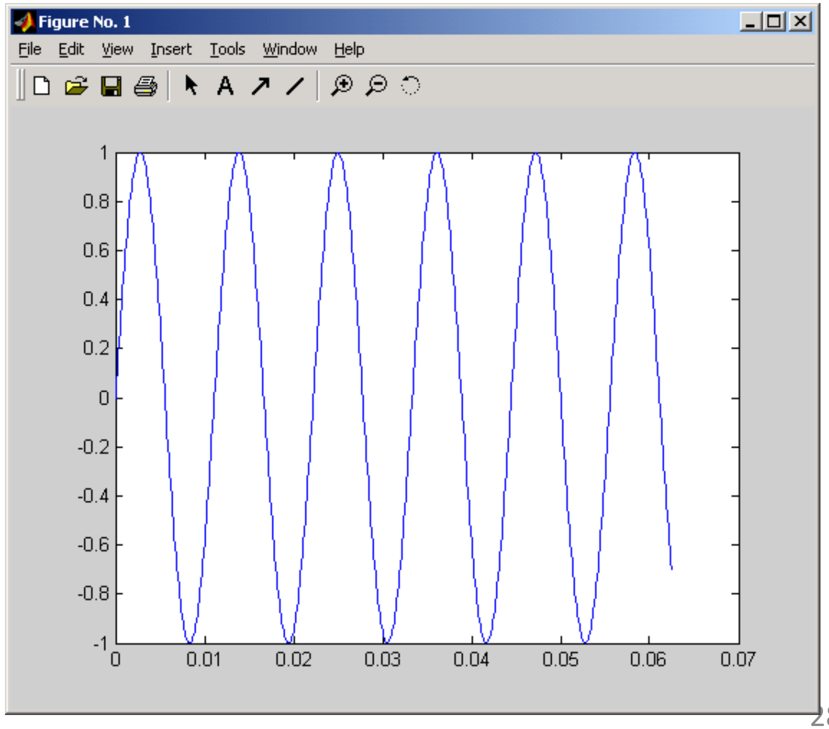
f(t) = sin(t)
scale factor1



f(t) = sin(2t)
scale factor 2



f(t) = sin(3t)
scale factor 3



More on Scaling

- It lets you either narrow down the frequency band of interest, or determine the frequency content in a narrower time interval

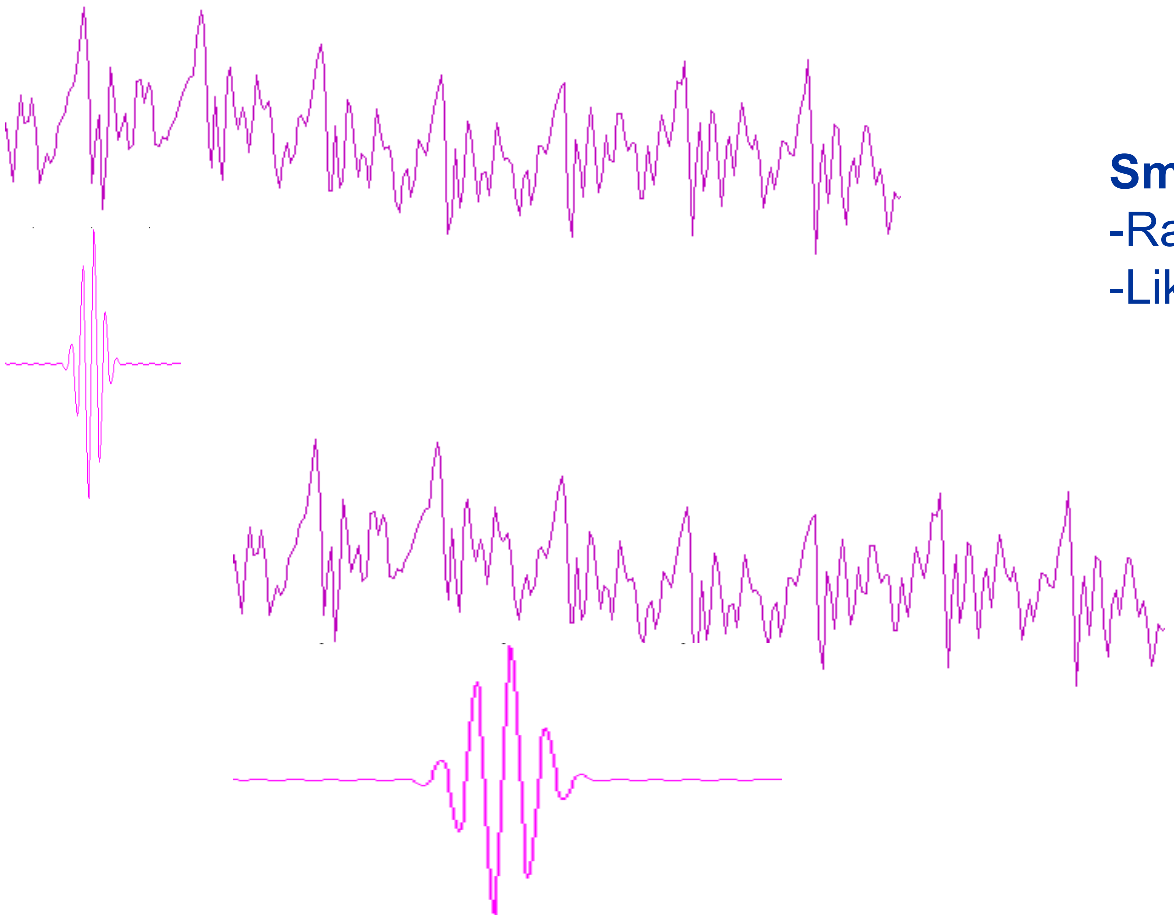
- Scaling ≈ frequency band

- Good for non-stationary data

- Low scale → Compressed wavelet → Rapidly changing details → High frequency

- High scale → Stretched wavelet → Slowly changing, coarse features → Low frequency

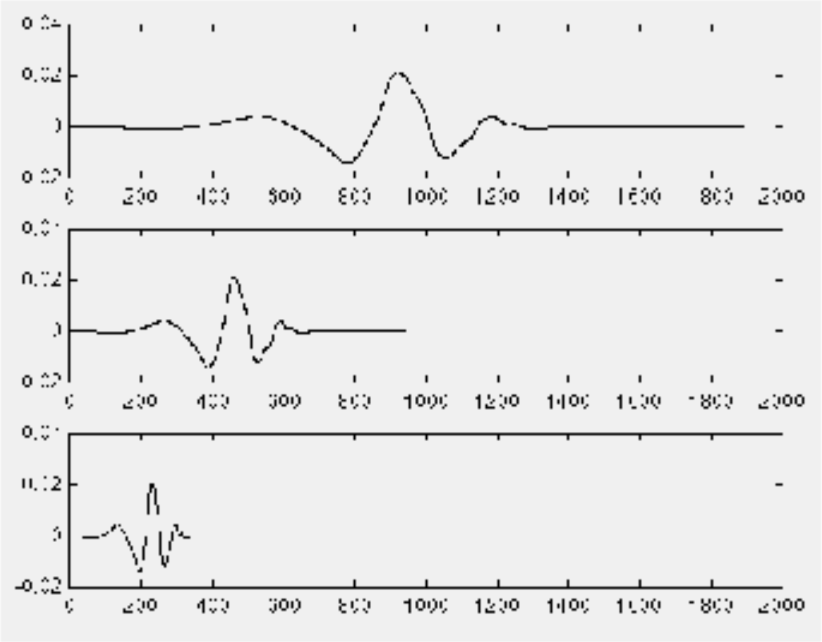
Scale Is Sort of Like Frequency



Small scale
-Rapidly changing details,
-Like high frequency

Large scale
-Slowly changing
 details
-Like low frequency

Scale Is Sort of Like Frequency



$$f(t) = \psi(t) \quad ; \quad a = 1$$

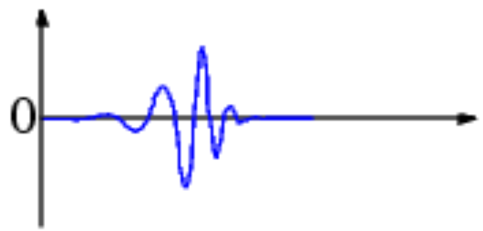
$$f(t) = \psi(2t) \; ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \; ; \quad a = \frac{1}{4}$$

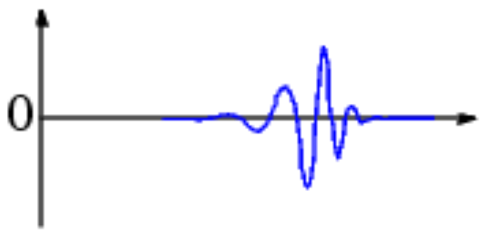
The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.

Shifting

Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function f(t)   by *k* is represented by f(t-k)

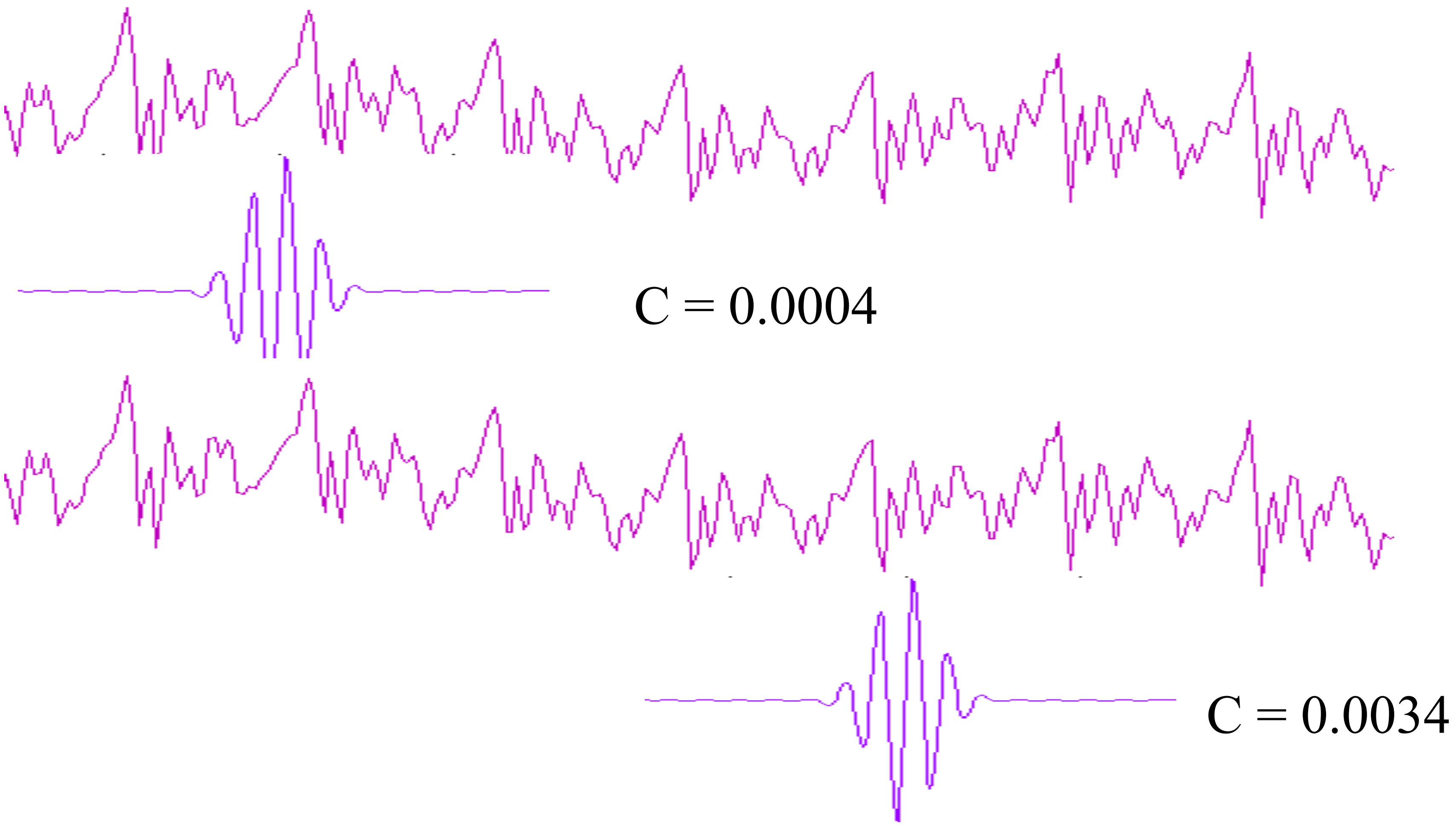


Wavelet function
$\psi(t)$



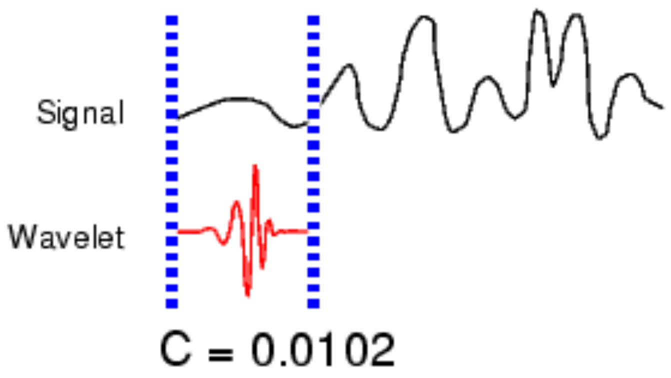
Shifted wavelet function
$\psi(t-k)$

Shifting



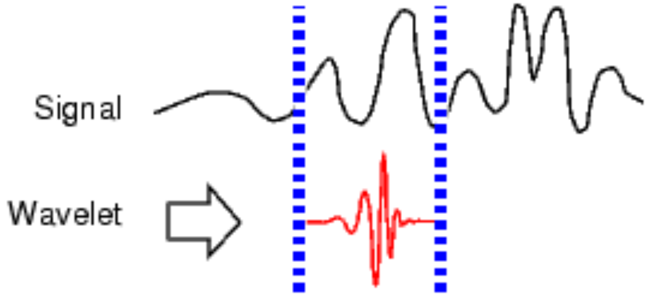
Five Steps of a Continuous Wavelet Transform

1. Take a wavelet and compare it to a section at the start of the original signal.
2. Calculate a correlation coefficient c

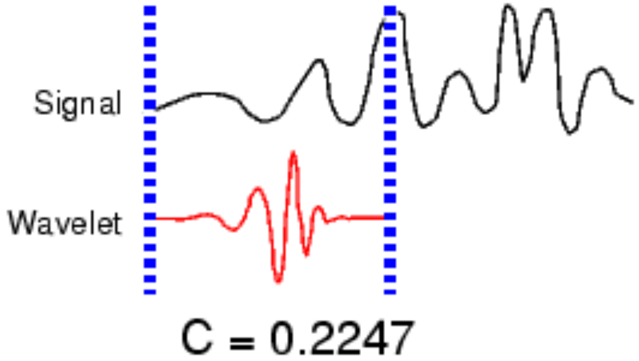


Five Steps of a Continuous Wavelet Transform

3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.

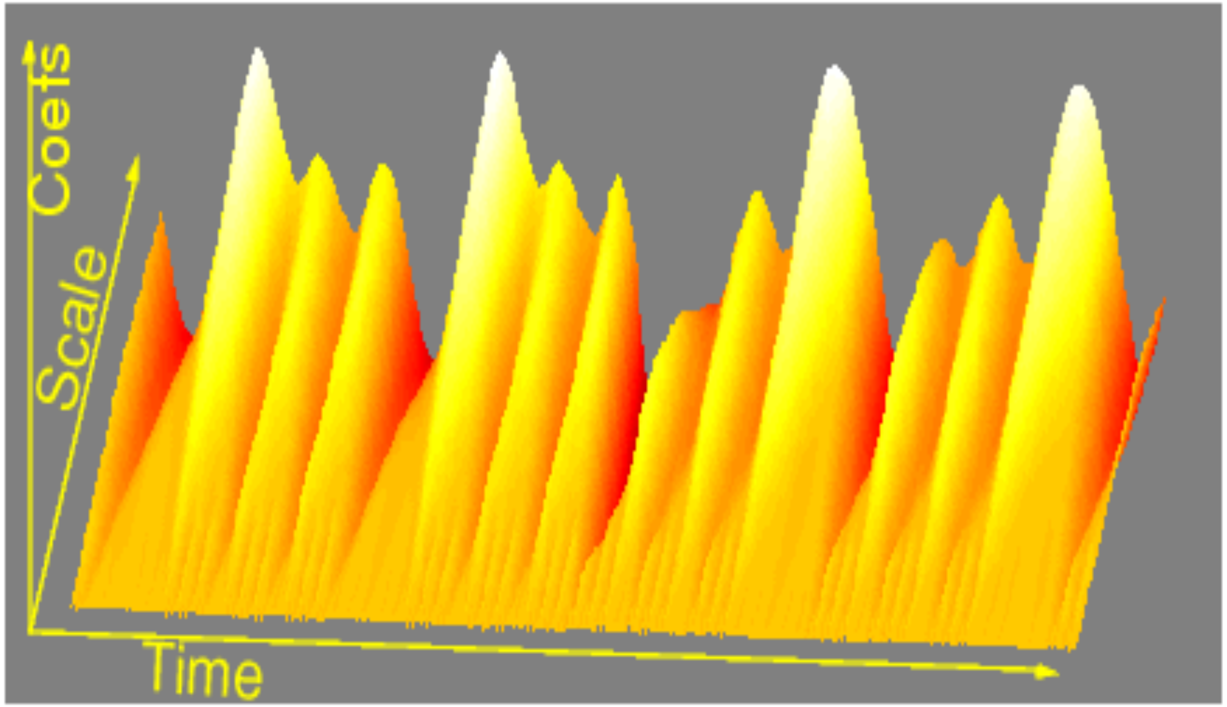


4. Scale (stretch) the wavelet and repeat steps 1 through 3.



5. Repeat steps 1 through 4 for all scales.

Coefficient Plots



Discrete Wavelet Transform

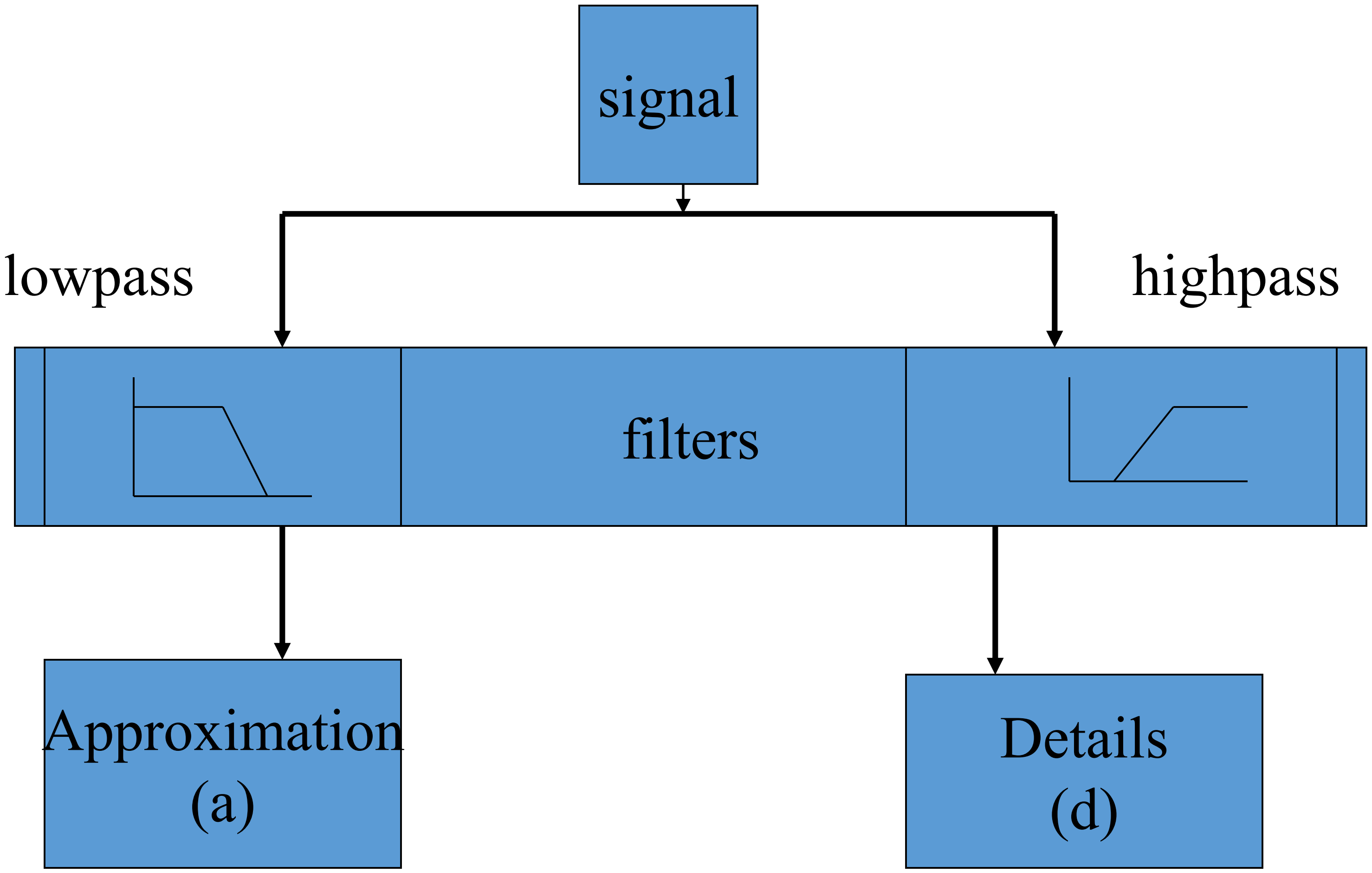
- “Subset” of scale and position based on power of two
  - rather than every “possible” set of scale and position in continuous wavelet transform

- Behaves like a filter bank: signal in, coefficients out

- Down-sampling necessary

  (twice as much data as original signal)

Discrete Wavelet Transform

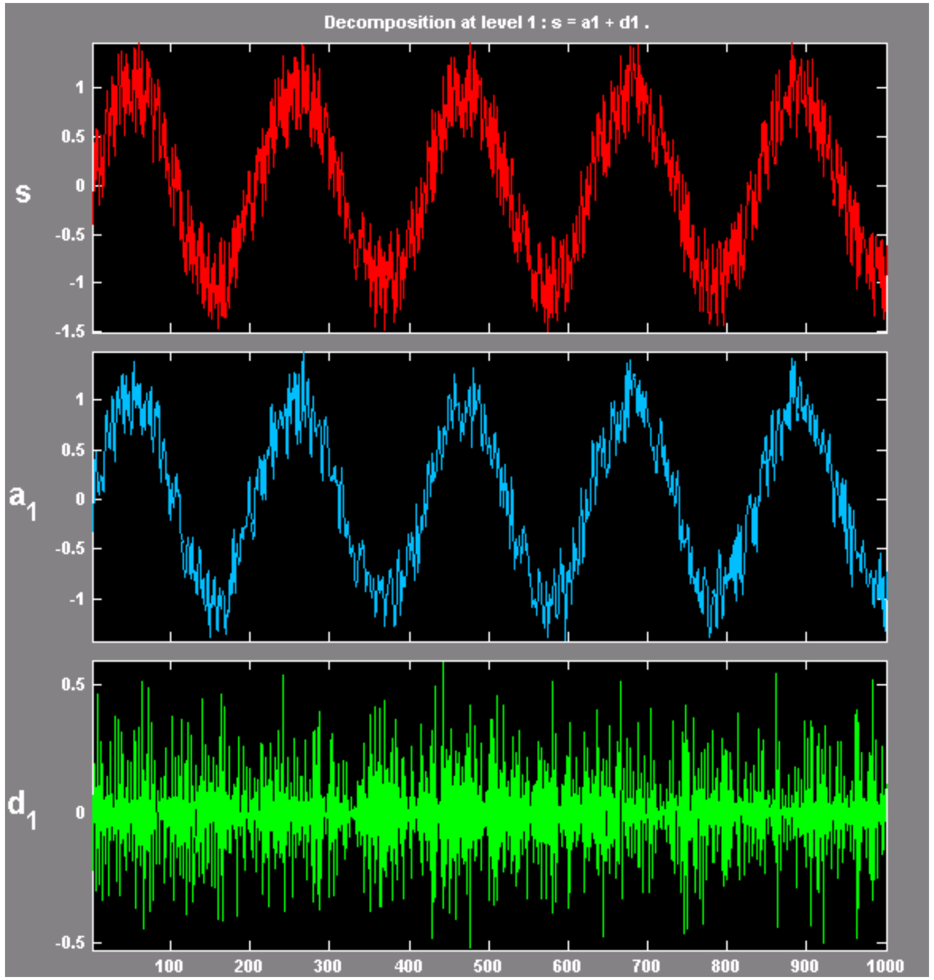


Results of Wavelet Transform — Approximation and Details

- Low frequency:
  - approximation (a)

- High frequency
  - details (d)

- "Decomposition" can be performed iteratively



Example of Multi-Level Decomposition

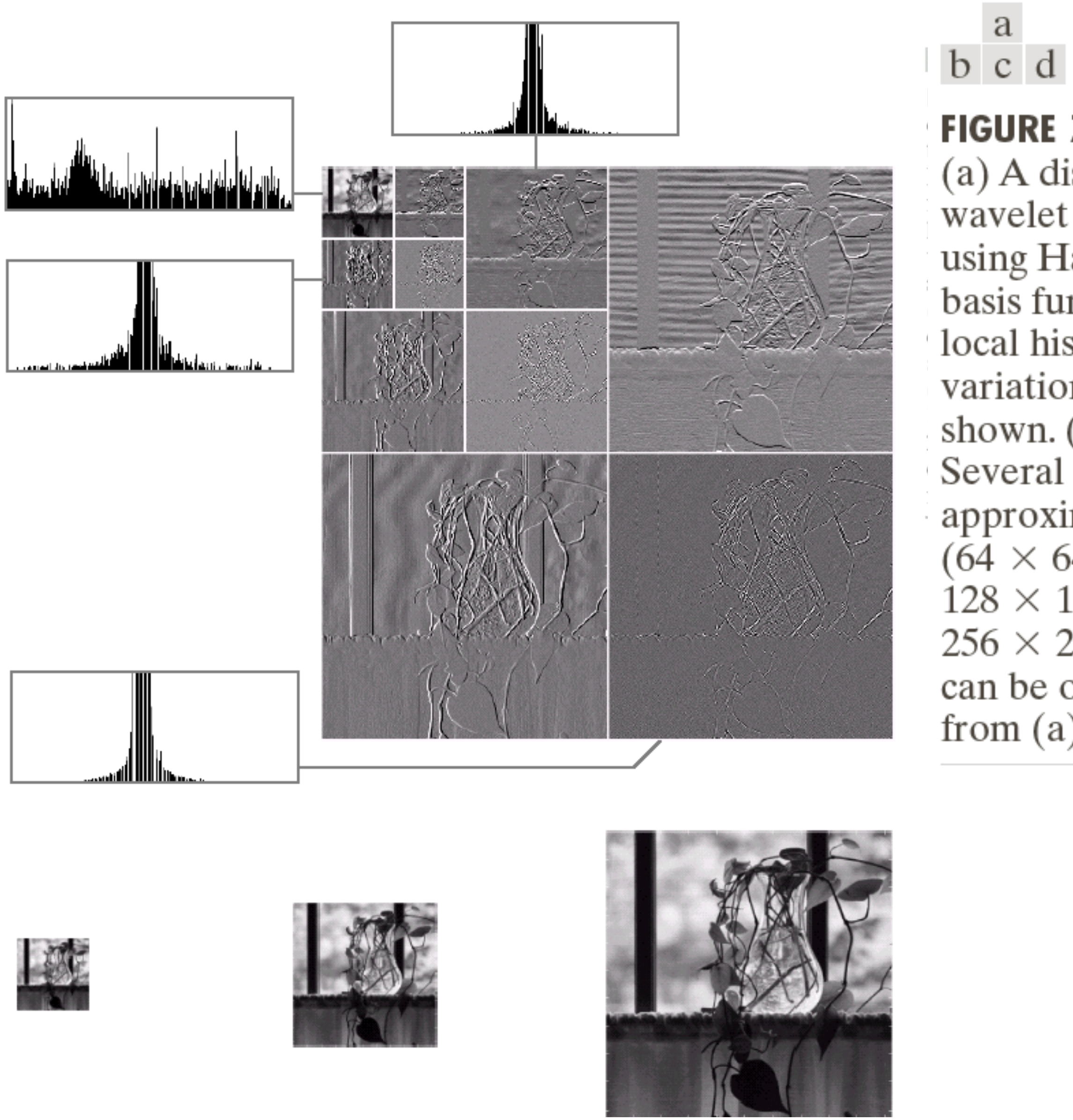


FIGURE 7.10
(a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).