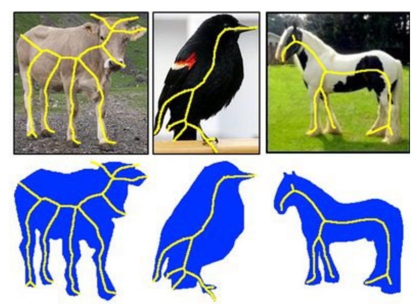
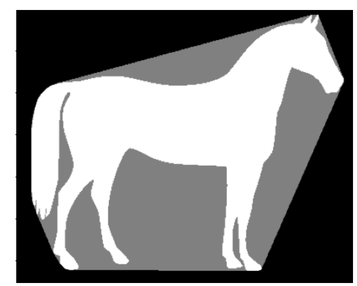
Chapter 9. Morphological Image Processing

Introduction

- Morphology
  - A branch of biology concerned with the form and structure of animals and plants
- Mathematical morphology
  - A tool for extracting image components useful for the representation and description of image shape including
    - Boundaries
    - Skeletons
    - convex hulls
  - Can also be used for
    - Filtering
    - Thinning
    - Pruning





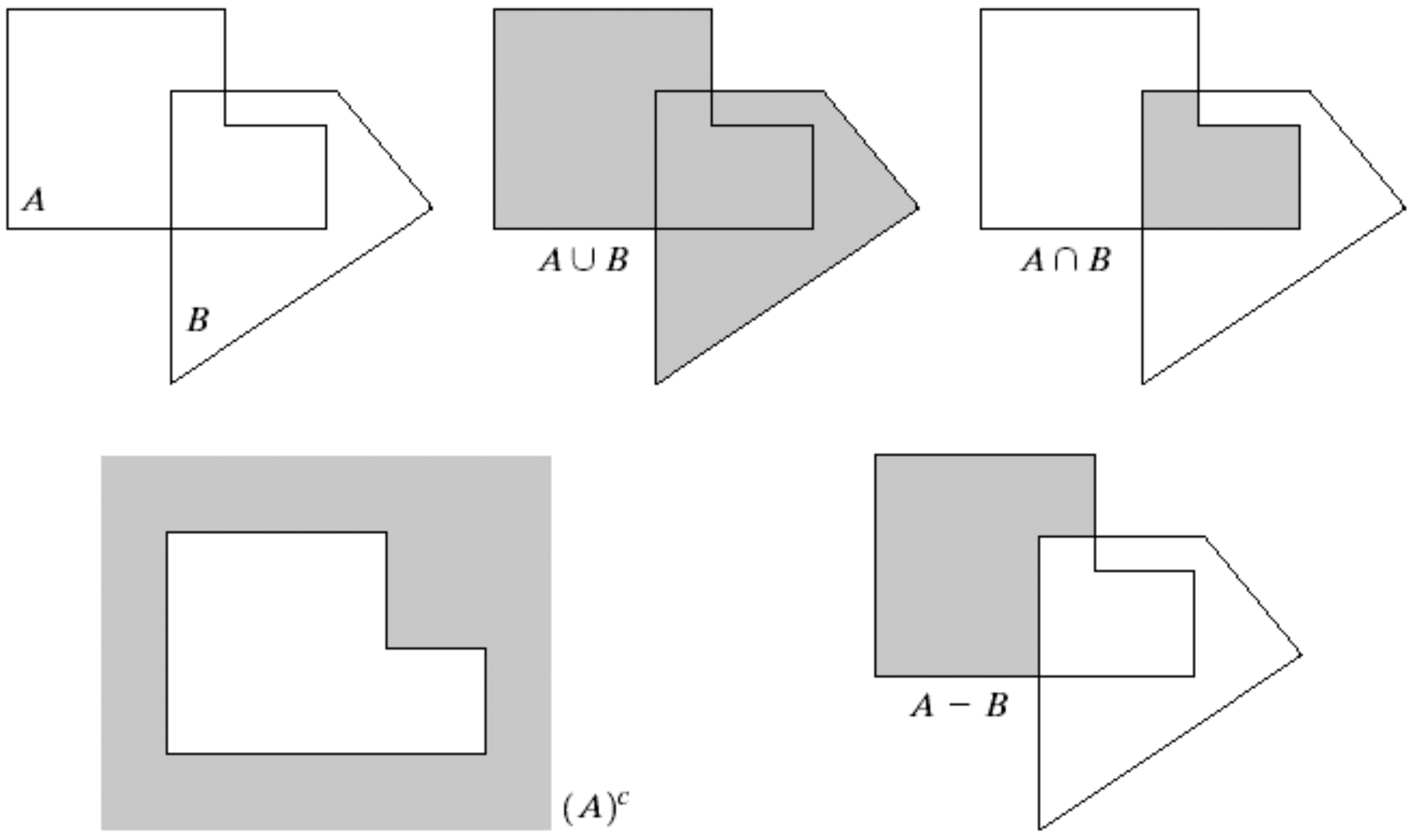
Preliminary

- Set theory: language of mathematical morphology
- Sets represent objects in an image
  - For example, the set of all black pixels in a binary image
- For binary images, sets are members of the 2-D integer space Z^2
  - Each element of the set is a tuple (2-D vector) whose coordinates are the (x,y) coordinates of a black (or white depending on convention) pixel in the image
- Gray-scale digital images are represented as sets in Z^3
  - Coordinates and gray-scale value
- Higher dimensional sets could represent attributes such as color, time varying components, etc.

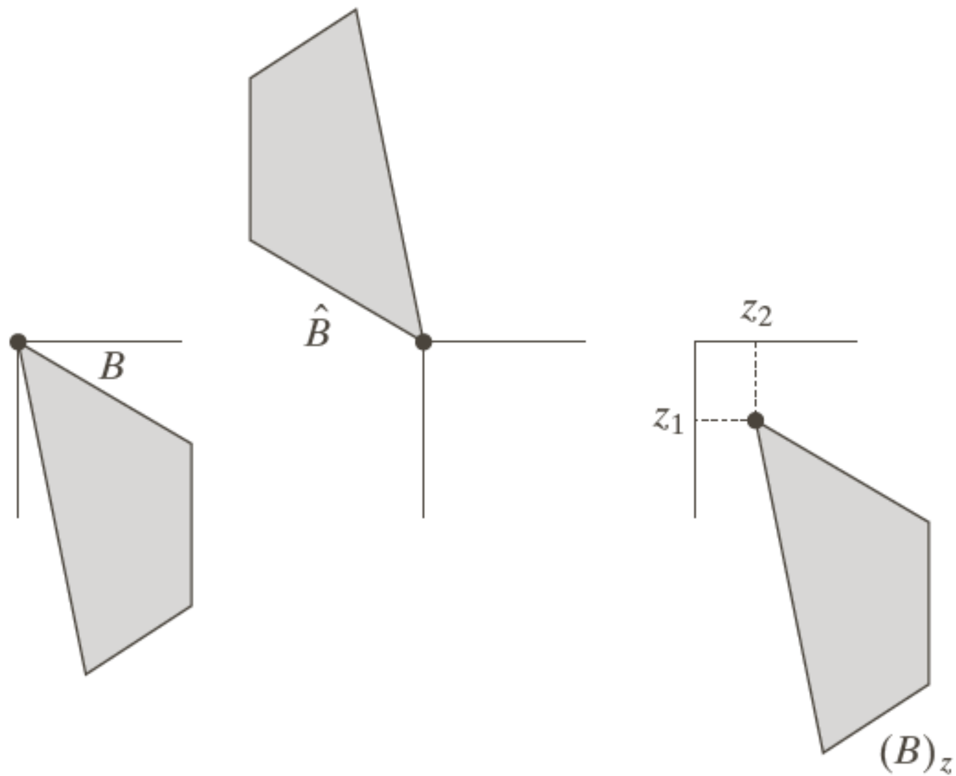
Definitions and Notations

- SET (Ω)
  - A collection of objects (elements)
- membership ($\in$)
  - If ω is an element (member) of a set Ω, then $\omega \in \Omega$
- Subset ($\subset$)
  - Let A, B are two sets. If for every a $\in$ A, we also have a $\in$ B, then set A is a *subset* of B; that is, $A \subset B$
  - If $A \subset B$ and $B \subset A$, then A = B.
- Empty set ($\varnothing$)
- Complement set
  - If $A \subset \Omega$, then its complement set $A^c = \{\omega | \ \omega \in \Omega$, and □ $A\}$
- Union ($\cup$)
  - $A \cup B = \{\omega | \ \omega \in A$ **or** $\omega \in B\}$
- Intersection ($\cap$)
  - $A \cap B = \{\omega | \ \omega \in A$ **and** $\omega \in B\}$
- Set difference ($-$)
  - $B - A = B \cap A^c$
  - Note: $B - A \neq A - B$
- Disjoint sets
  - A and B are disjoint (mutually exclusive) if $A \cap B = \varnothing$

Set Relations



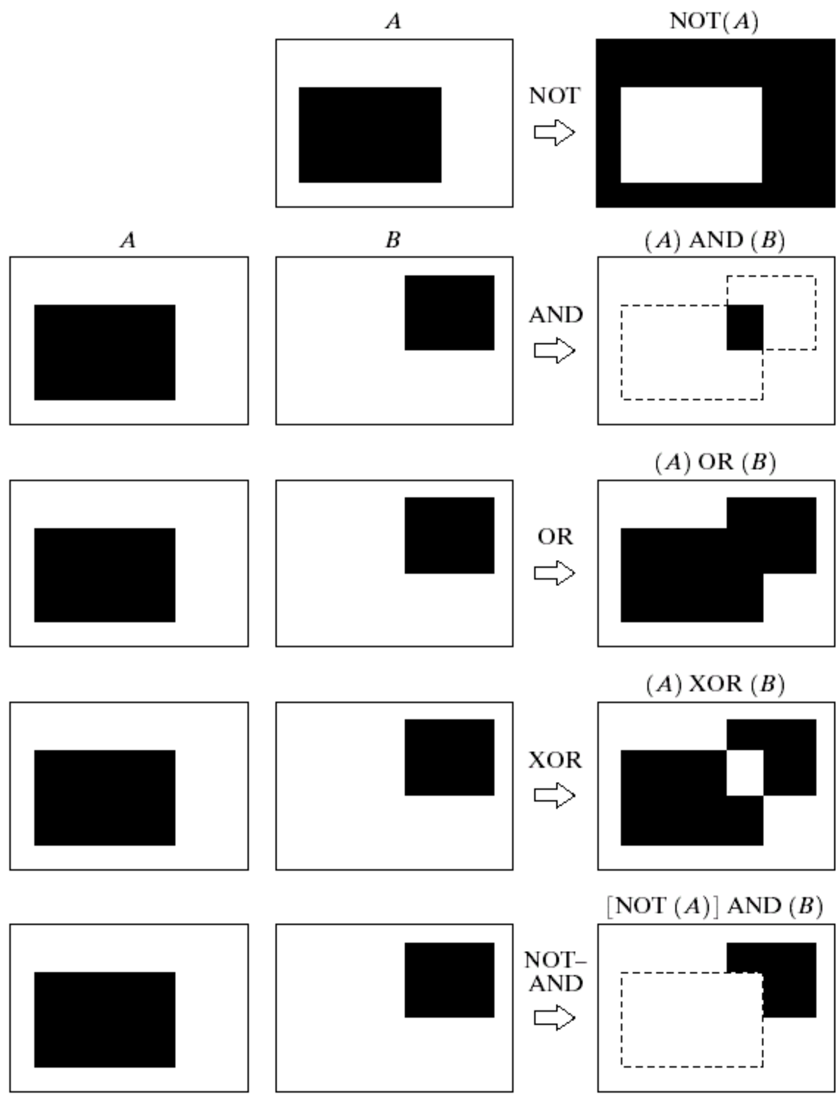
Translation and Reflection



- Reflection $\hat{B} = \{w \mid w = -b, \text{for } b \in B\}$

- Translation $(B)_Z = \{c \mid c = b + z, \text{for } b \in B\}$

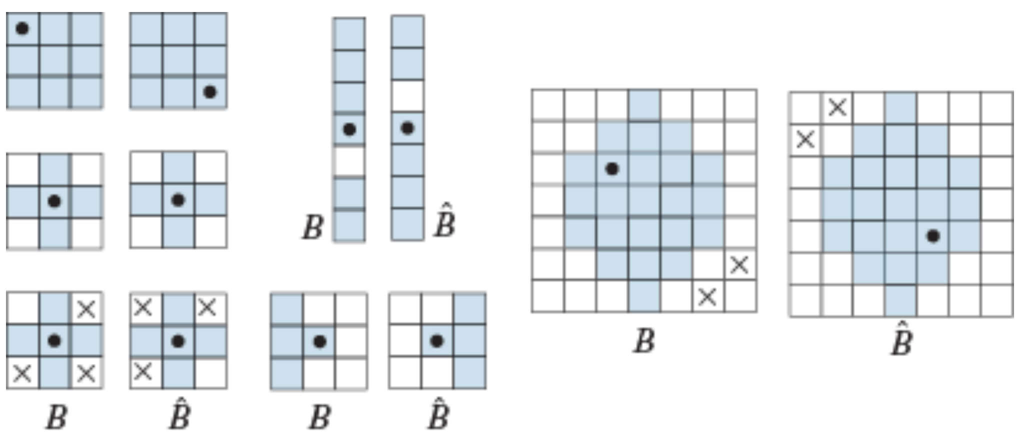
Logical Operations between Binary Images



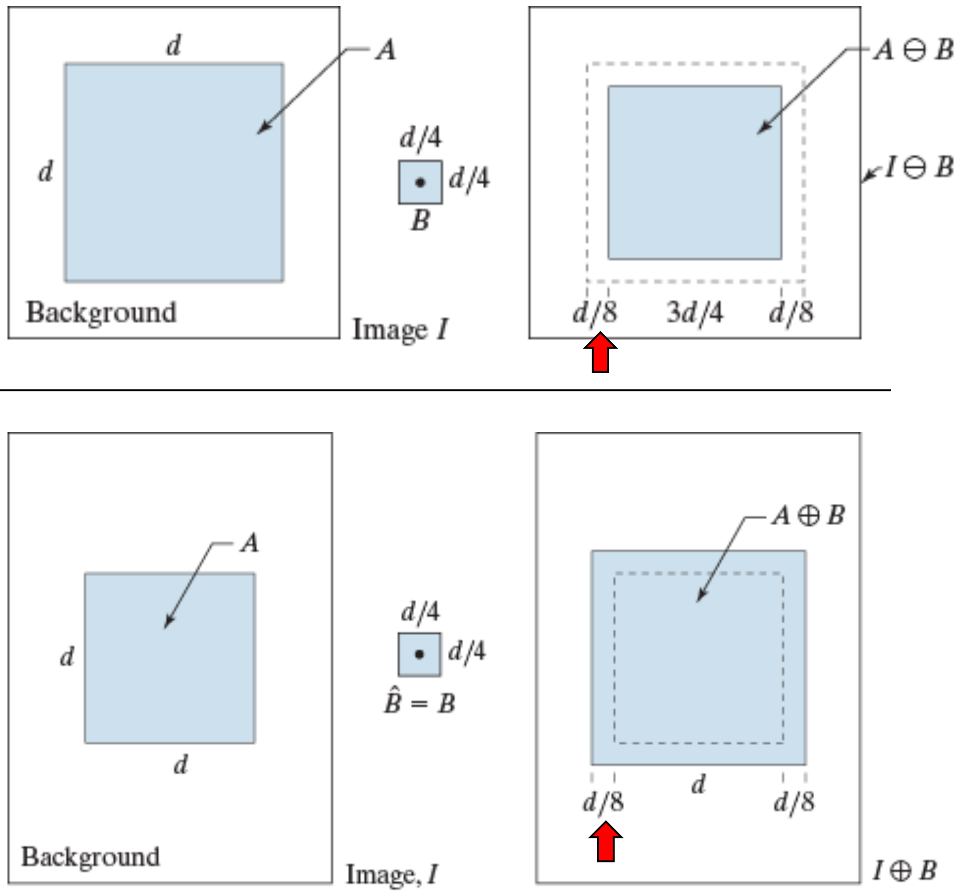
Structuring Element

- An SE is a small set (or subimage) used to “probe” an area of interest for certain properties
  - May be of arbitrary shape and size
  - In practice, an SE is generally a regular geometric shape (square, rectangle, diamond, etc.)
  - Generally padded to a rectangular array for image processing

FIGURE 9.2 Structuring elements and their reflections about the origin (the ×'s are don't care elements, and the dots denote the origin). Reflection is rotation by 180° of an SE about its origin.



Erosion and Dilation



A: Object (foreground)
B: structure element

- Dilation expands an image, and erosion shrinks it.
- Many morphological algorithms in this chapter are based on these 2 primitives.
- Erosion

$$A \ominus B = \{z \mid (B)_z \subseteq A\}$$

- Dilation

$$A \oplus B = \left\{ z \,\middle|\, (\hat{B})_z \cap A \neq \varnothing \right\}$$

$$= \left\{ z \,\middle|\, \left[(\hat{B})_z \cap A \right] \subseteq A \right\}$$

- Duality Relation

$$\left(A \ominus B \right)^c = A^c \oplus \hat{B}$$

$$\left(A \oplus B \right)^c = A^c \ominus \hat{B}$$

Duality

- Erosion and dilation are duals of each other with respect to set complementation and reflection

$$
\begin{aligned}
(A \ominus B)^c &= \left\{ z \mid (B)_Z \subseteq A \right\}^c \\
&= \left\{ z \mid (B)_Z \cap A^c = \varnothing \right\}^c \\
&= \left\{ z \mid (B)_Z \cap A^c \neq \varnothing \right\} \\
&= A^c \oplus \hat{B}
\end{aligned}
$$

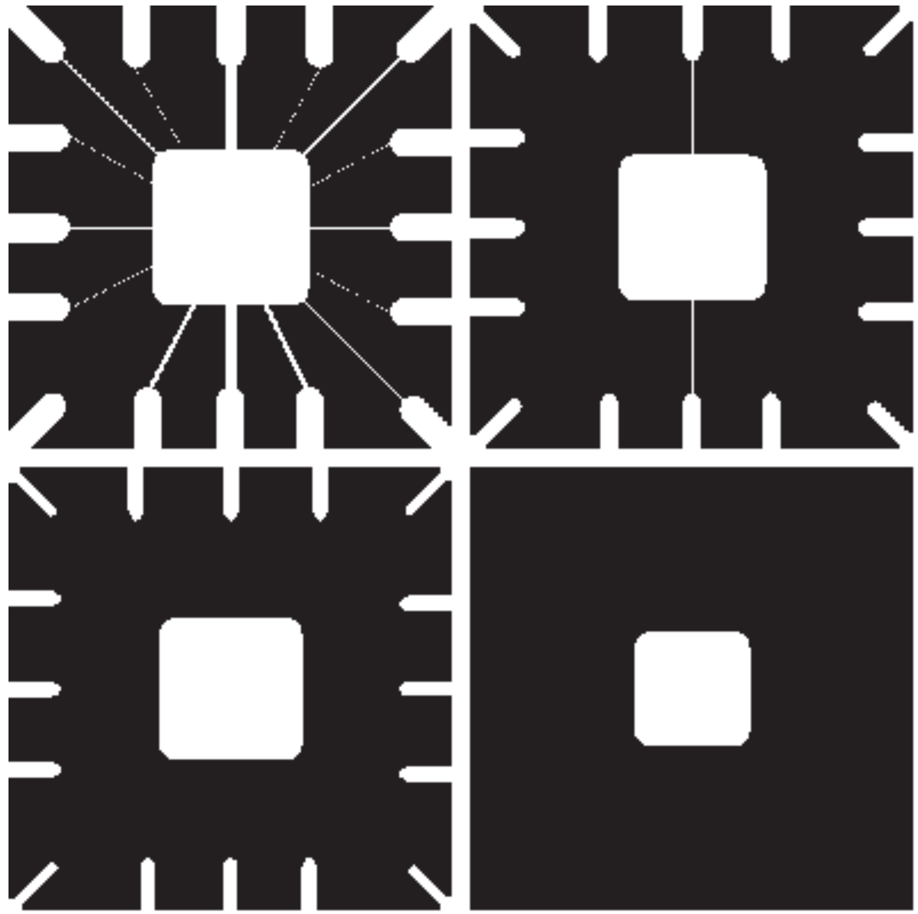
$$
\begin{aligned}
(A \oplus B)^c &= \left\{ z \mid \left(\hat{B}\right)_Z \cap A \neq \varnothing \right\}^c \\
&= \left\{ z \mid \left(\hat{B}\right)_Z \cap A^c = \varnothing \right\} \\
&= A^c \ominus \hat{B}
\end{aligned}
$$

Application of Erosion

- ❑ Using erosion to remove image components

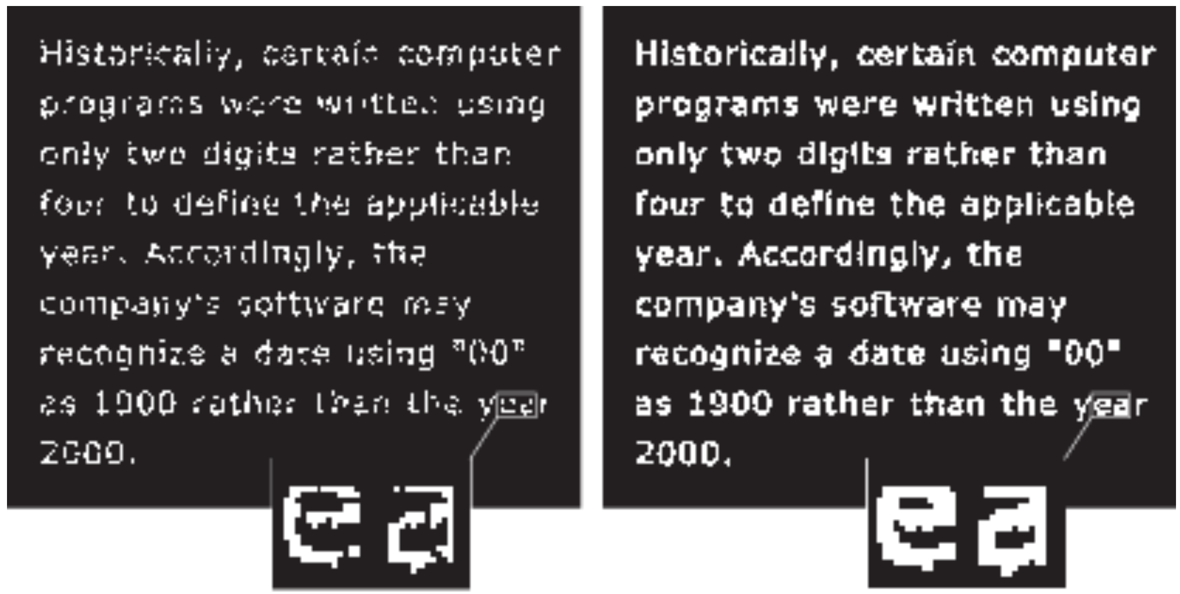
a	b
c	d

FIGURE 9.5 Using erosion to remove image components. (a) A 486×486 binary image of a wire-bond mask in which foreground pixels are shown in white. (b)–(d) Image eroded using square structuring elements of sizes 11×11, 15×15, and 45×45 elements, respectively, all valued 1.



Application of Dilation

- ❑ Using dilation to repair broken characters in an image



1	1	1
1	1	1
1	1	1

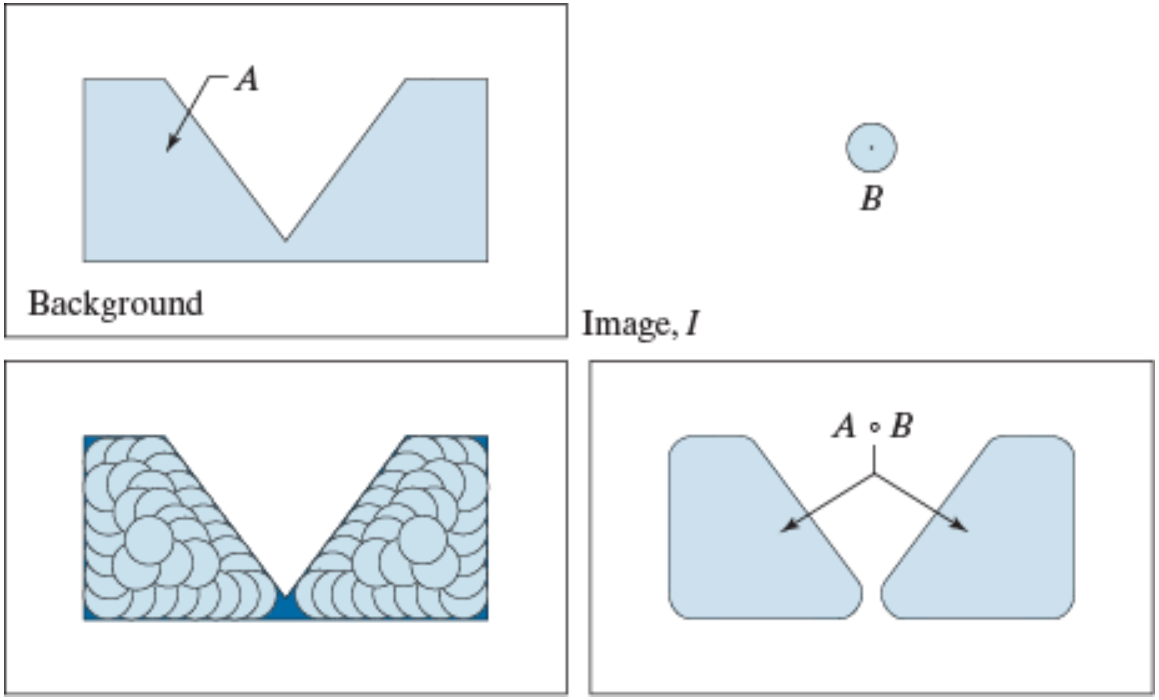
a b c

FIGURE 9.7
(a) Low-resolution text showing broken characters (see magnified view).
(b) Structuring element.
(c) Dilation of (a) by (b). Broken segments were joined.

Opening

- Opening generally smoothens the contour of an object, breaks narrow isthmuses, and eliminates thin protrusions

$$A \circ B = (A \ominus B) \oplus B$$



a b
c d

FIGURE 9.8 (a) Image I, composed of set (object) A and background. (b) Structuring element, B. (c) Translations of B while being contained in A. (A is shown dark for clarity.) (d) Opening of A by B.

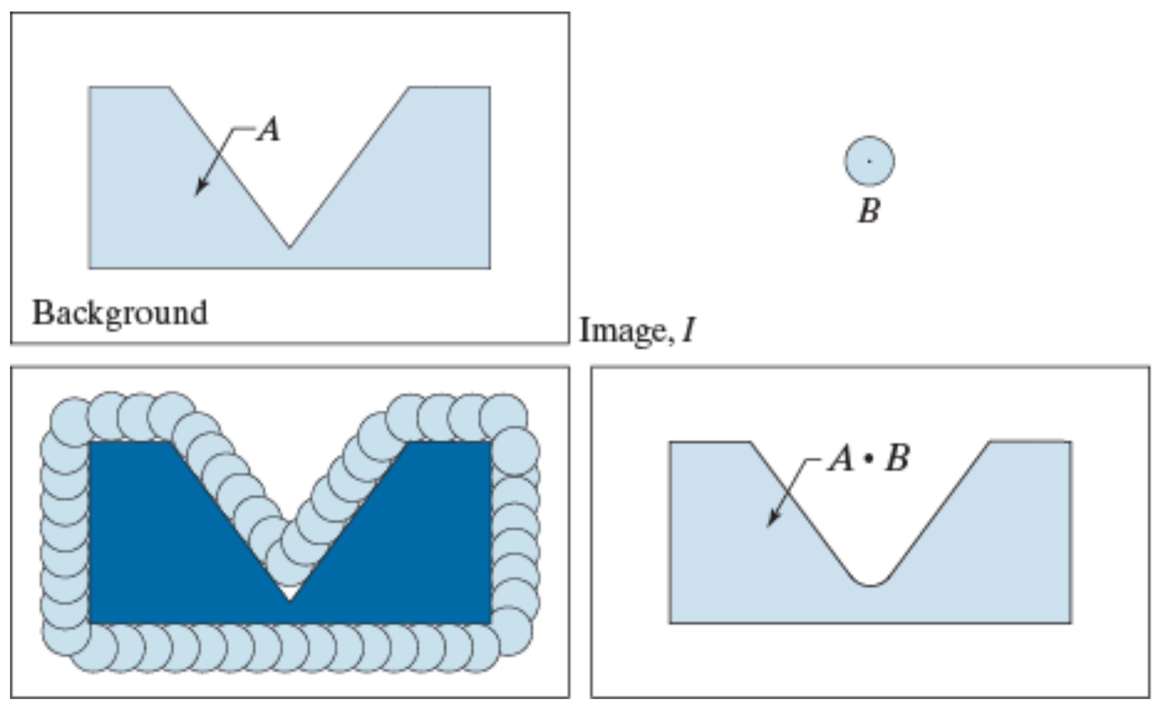
Closing

- Closing also tends to smooth sections of contours, but it generally fuses narrow breaks and long thin gulfs, eliminates small holes, and fills gaps in the contour

$$A \bullet B = (A \oplus B) \ominus B$$

a b
c d

FIGURE 9.9
(a) Image I, composed of set (object) A, and background.
(b) Structuring element B.
(c) Translations of B such that B does not overlap any part of A. (A is shown dark for clarity.)
(d) Closing of A by B.



Geometric Interpretation

- The opening of *A* by *B* is the union of all the translations of *B* so that *B* fits entirely in *A*

$$A \circ B = \bigcup \left\{ (B)_Z \mid (B)_Z \subseteq A \right\}$$

- The closing of *A* by *B* is the complement of the union of all the translations of *B* that do not overlap *A*.

$$A \bullet B = \left[\bigcup \left\{ (B)_Z \mid (B)_Z \cap A = \varnothing \right\} \right]^c$$

Closing has a similar geometric interpretation, except that now we translate B outside A.

- Opening and closing are duals of each other with respect to set complementation and reflection

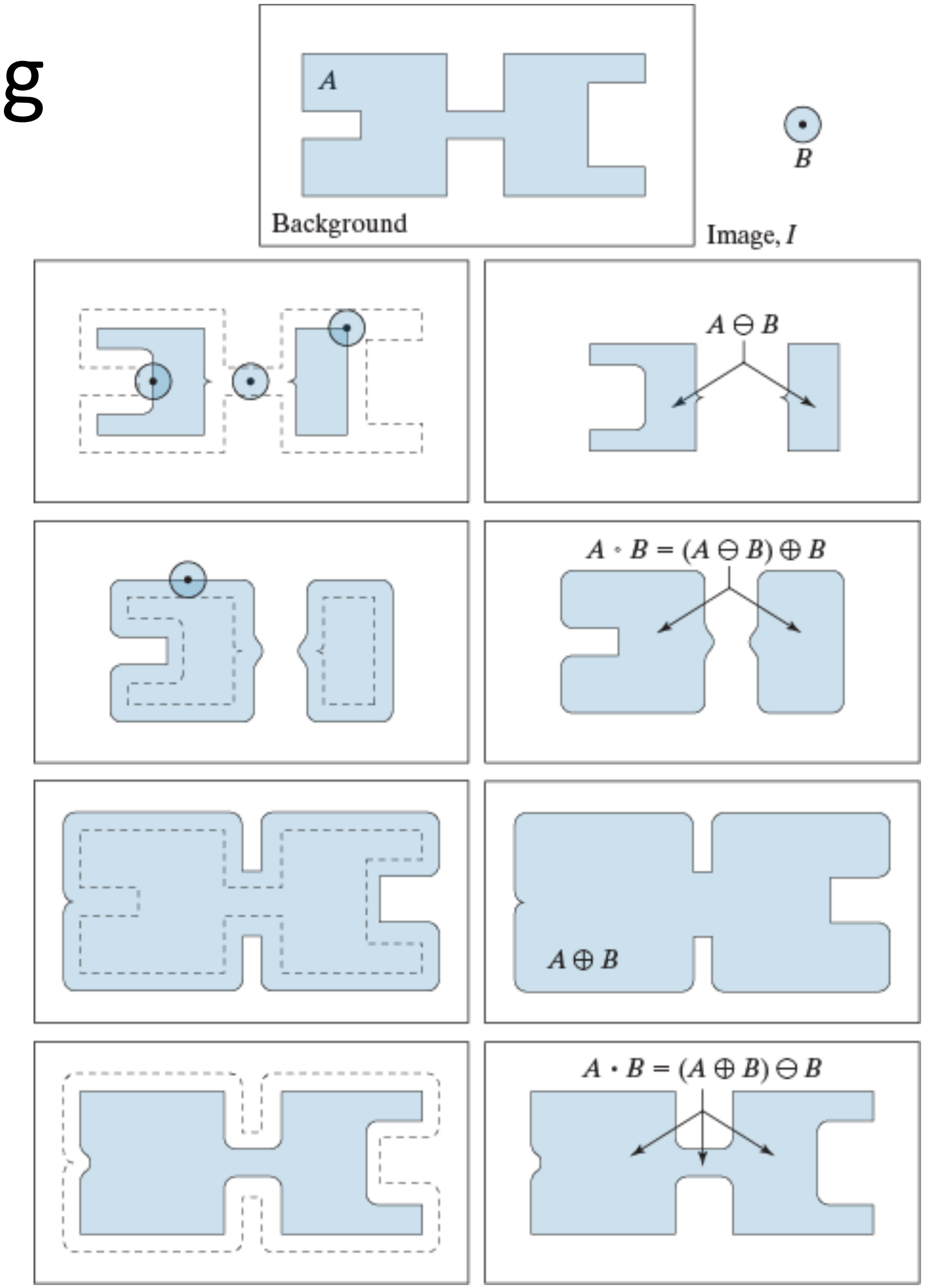
$$\left(A \circ B \right)^c = (A^c \bullet \hat{B})$$

$$\left(A \bullet B \right)^c = (A^c \circ \hat{B})$$

Opening and Closing

a
b c
d e
f g
h i

FIGURE 9.10 Morphological opening and closing. (a) Image I, composed of a set (object) A and background; a solid, circular structuring element is shown also. (The dot is the origin.) (b) Structuring element in various positions. (c)-(i) The morphological operations used to obtain the opening and closing.



Properties of Opening and Closing

- Properties of Opening

(a) $A \circ B$ is a subset (subimage) of A

(b) if C is a subset of D, then $C \circ B$ is a subset of $D \circ B$

(c) $(A \circ B) \circ B = A \circ B$

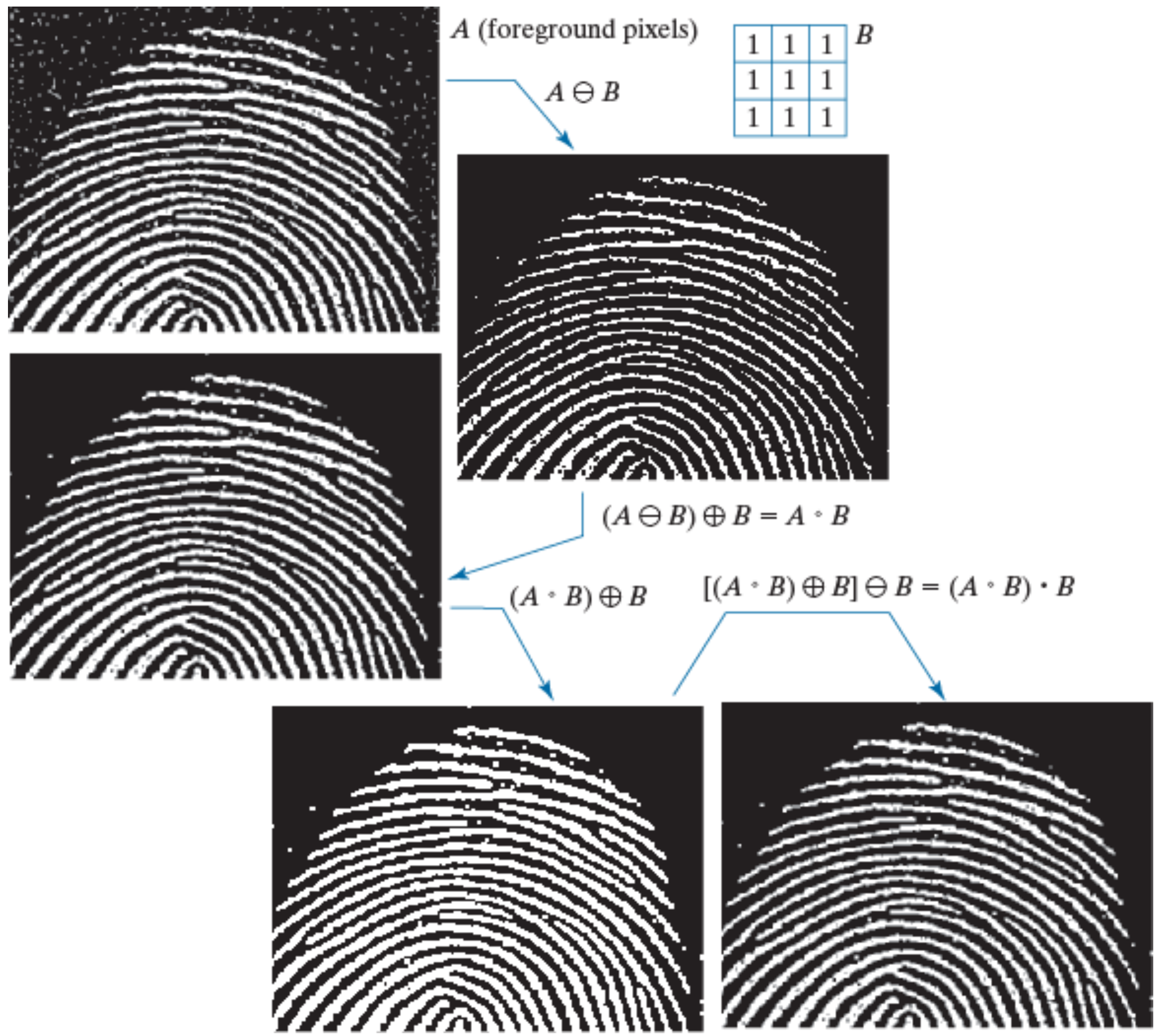
- Properties of Closing

(a) A is subset (subimage) of $A \bullet B$

(b) If C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$

(c) $(A \bullet B) \bullet B = A \bullet B$

Filtering Using Opening and Closing



a b
d c
e f

FIGURE 9.11
(a) Noisy image. (b) Structuring element. (c) Eroded image. (d) Dilation of the erosion (opening of A). (e) Dilation of the opening. (f) Closing of the opening. (Original image courtesy of the National Institute of Standards and Technology.)

Hit-or-Miss Transform for Shape Detection

- HMT utilizes two structuring elements B_1 and B_2
  - B_1 for detecting shapes in the foreground
  - B_2 for detecting shapes in the background
- Let $B=(B_1, B_2)$ and A denote the foreground, then HMT of image I by B is defined as

$$A \circledast B = \left\{ z \,\middle|\, (B_1)_z \subseteq A \text{ and } (B_2)_z \subseteq A^c \right\}$$
$$= \left(A \ominus B_1 \right) \cap \left(A^c \ominus B_2 \right)$$

- Interpretation

  The HMT is the set of translations, z, of structure elements B_1 and B_2 such that B_1 finds a match in the foreground while B_2 finds a match in the background

Illustration of Hit-or-Miss Transform

- Find the origin of object (set) *D* in image

a	b
c	d
e	f

FIGURE 9.12
(a) Image consisting of a foreground (1's) equal to the union, A, of set of objects, and a background of 0's.
(b) Image with its foreground defined as A^c.
(c) Structuring elements designed to detect object D.
(d) Erosion of A by B_1.
(e) Erosion of A^c by B_2.
(f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origin of their respective components. Each dot is a single pixel.

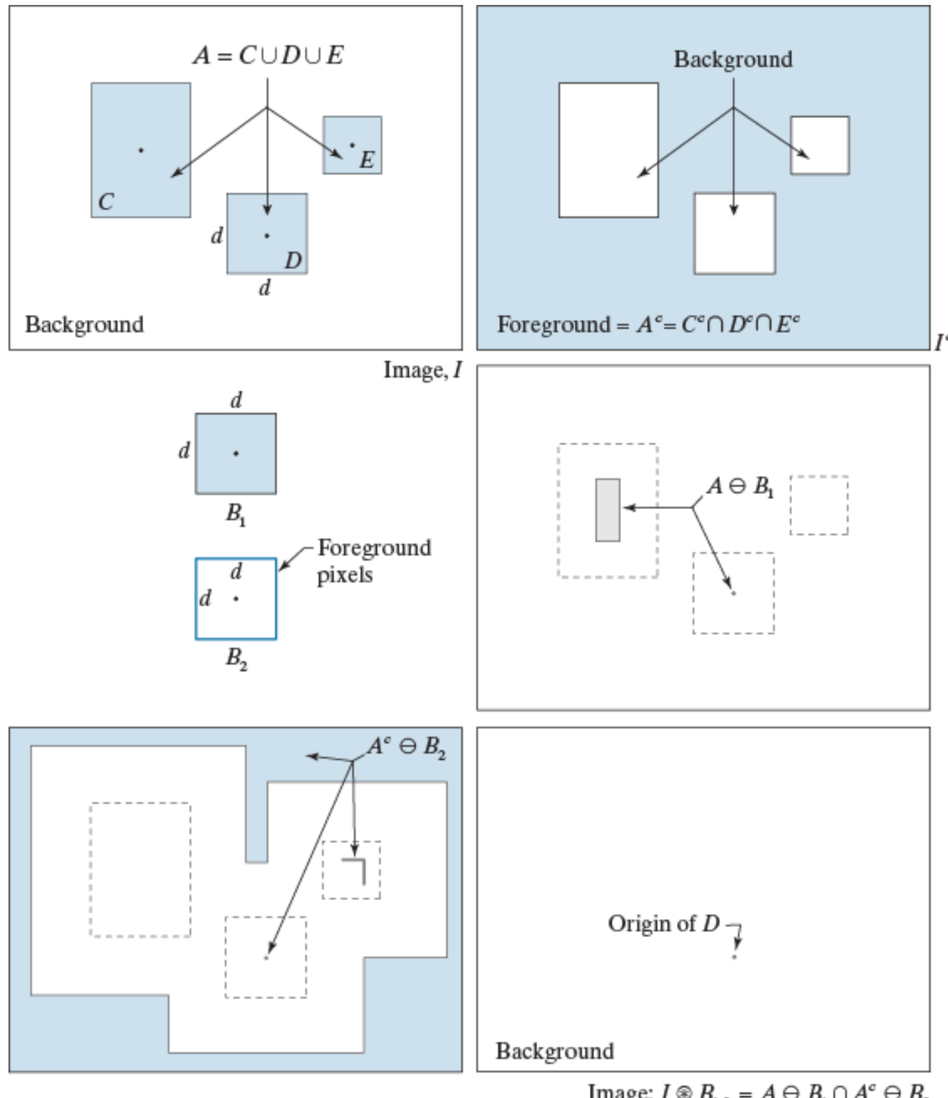


Illustration of Hit-or-Miss Transform

- Using a single SE to process both FG and BG simultaneously

$$A \circledast B = \left\{ z \,\middle|\, (B)_z \subseteq I \right\} \qquad (9\text{-}17)$$

B: the shape to be detected + a border of one-pixel width

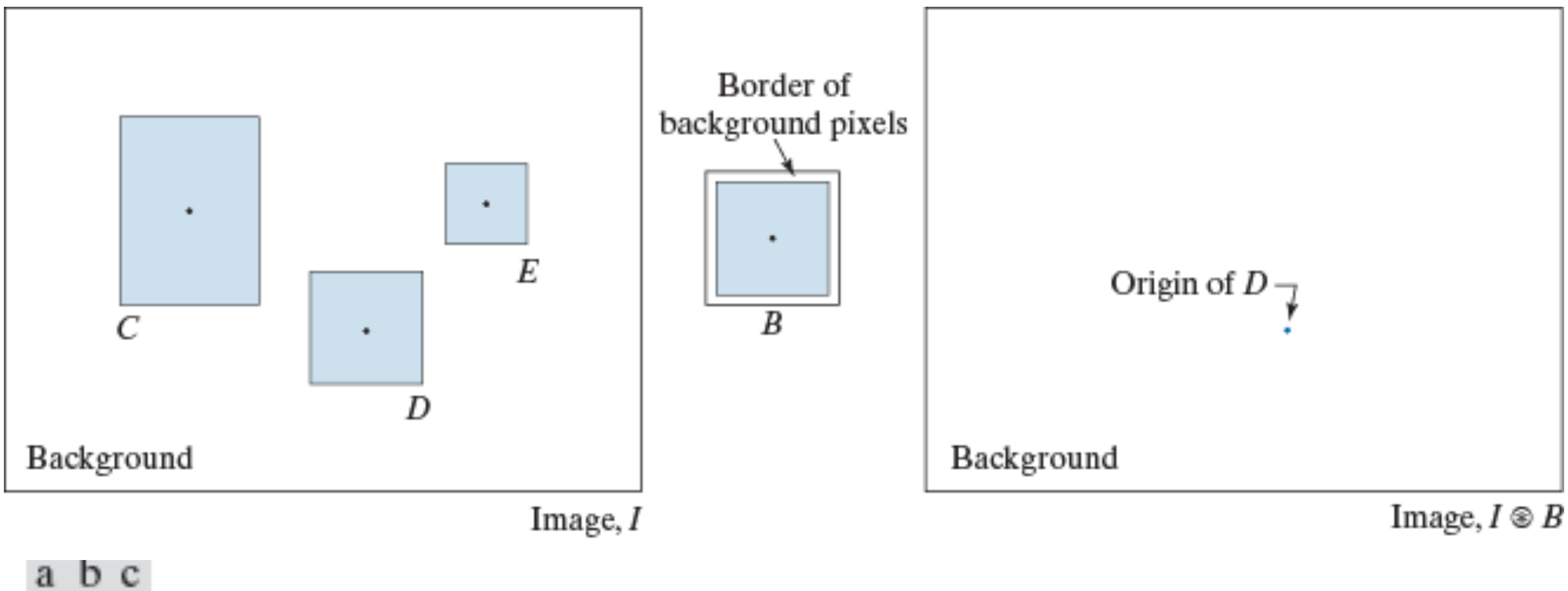
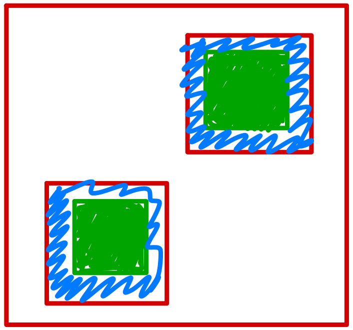


FIGURE 9.13 Same solution as in Fig. 9.12, but using Eq. (9-17) with a single structuring element.



Boundary Extraction

$$\beta(A) = A - \left(A \ominus B \right) \tag{9-18}$$

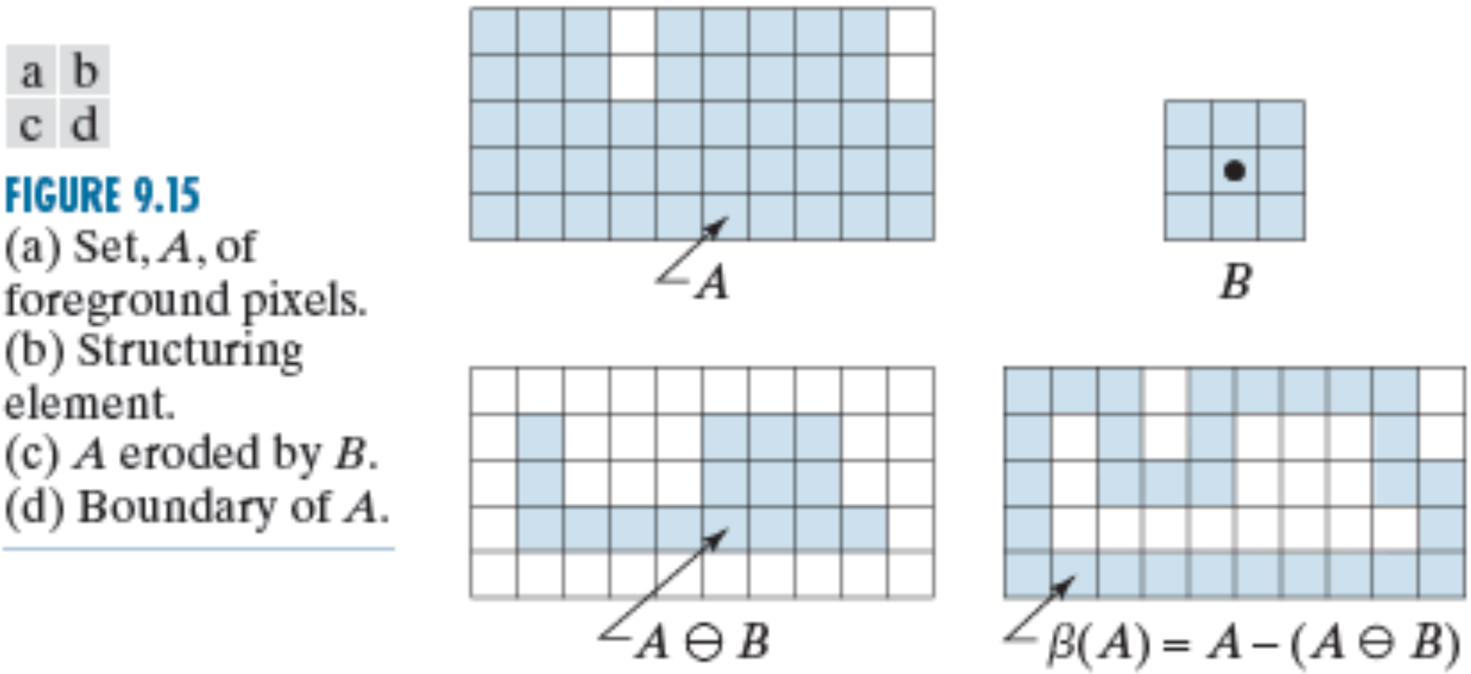
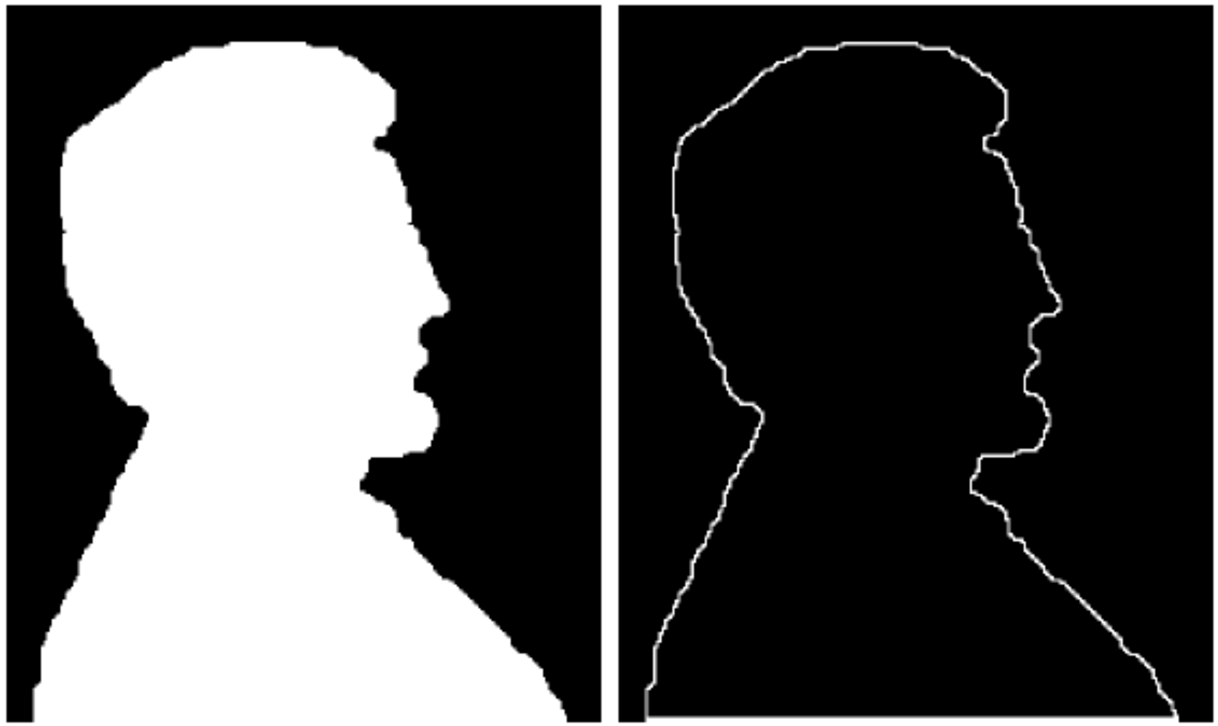


FIGURE 9.15
(a) Set, A, of foreground pixels.
(b) Structuring element.
(c) A eroded by B.
(d) Boundary of A.

Boundary Extraction (2)

a b

FIGURE 9.16
(a) A binary image.
(b) Result of using Eq. (9-18) with the structuring element in Fig. 9.15(b).



Hole Filling

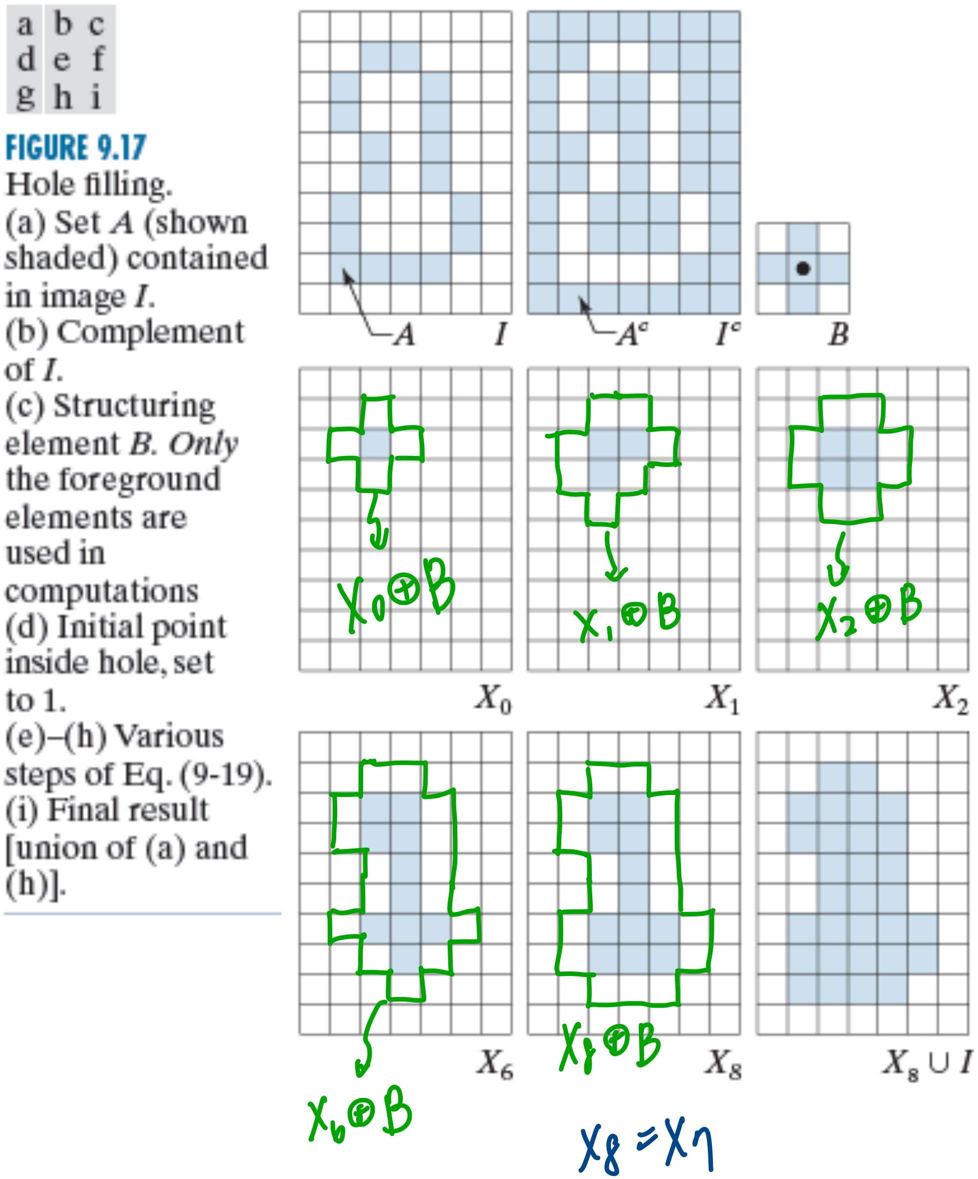


FIGURE 9.17 Hole filling. (a) Set A (shown shaded) contained in image I. (b) Complement of I. (c) Structuring element B. *Only* the foreground elements are used in computations (d) Initial point inside hole, set to 1. (e)–(h) Various steps of Eq. (9-19). (i) Final result [union of (a) and (h)].

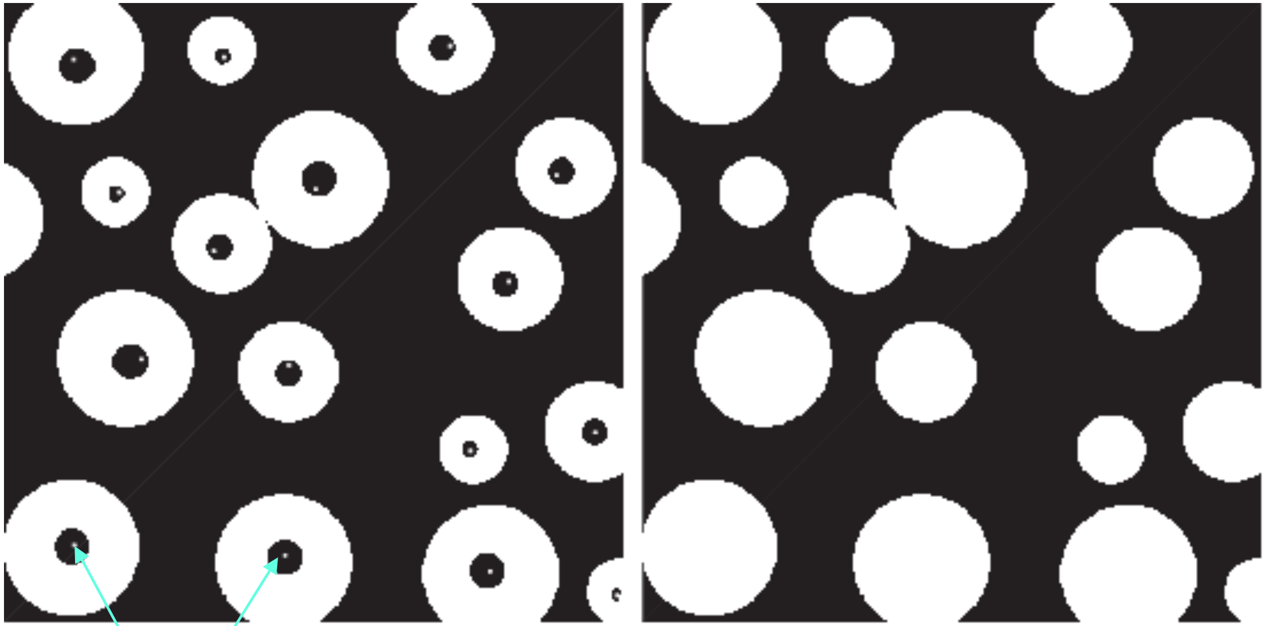
Given a point of a hole, do

1. Form an array X_0 of 0's
2. Set the given pixel to 1 in X_0
3. $X_k = (X_{k-1} \oplus B) \cap I^c, k = 1, 2, 3, \ldots$
4. Stop when $X_k = X_{k-1}$
5. Output $X_k \cup I$

Hole Filling

a b

FIGURE 9.18
(a) Binary image. The white dots inside the regions (shown enlarged for clarity) are the starting points for the hole-filling algorithm.
(b) Result of filling all holes.



Starting points

Extraction of Connected Components

- Let A be a set containing one or more connected components and B be the structuring element
- Form an array X_0 (of the same size as the array containing A) with elements all 0's except at the location known to correspond to a point in a connected component of A, which is set to 1.
- Perform

$$X_k = (X_{k-1} \oplus B) \cap A, \;\; k = 1, 2, 3, \ldots$$

until $X_k = X_{k-1}$. Then $Y = X_k$.

Extraction of Connected Components (2)

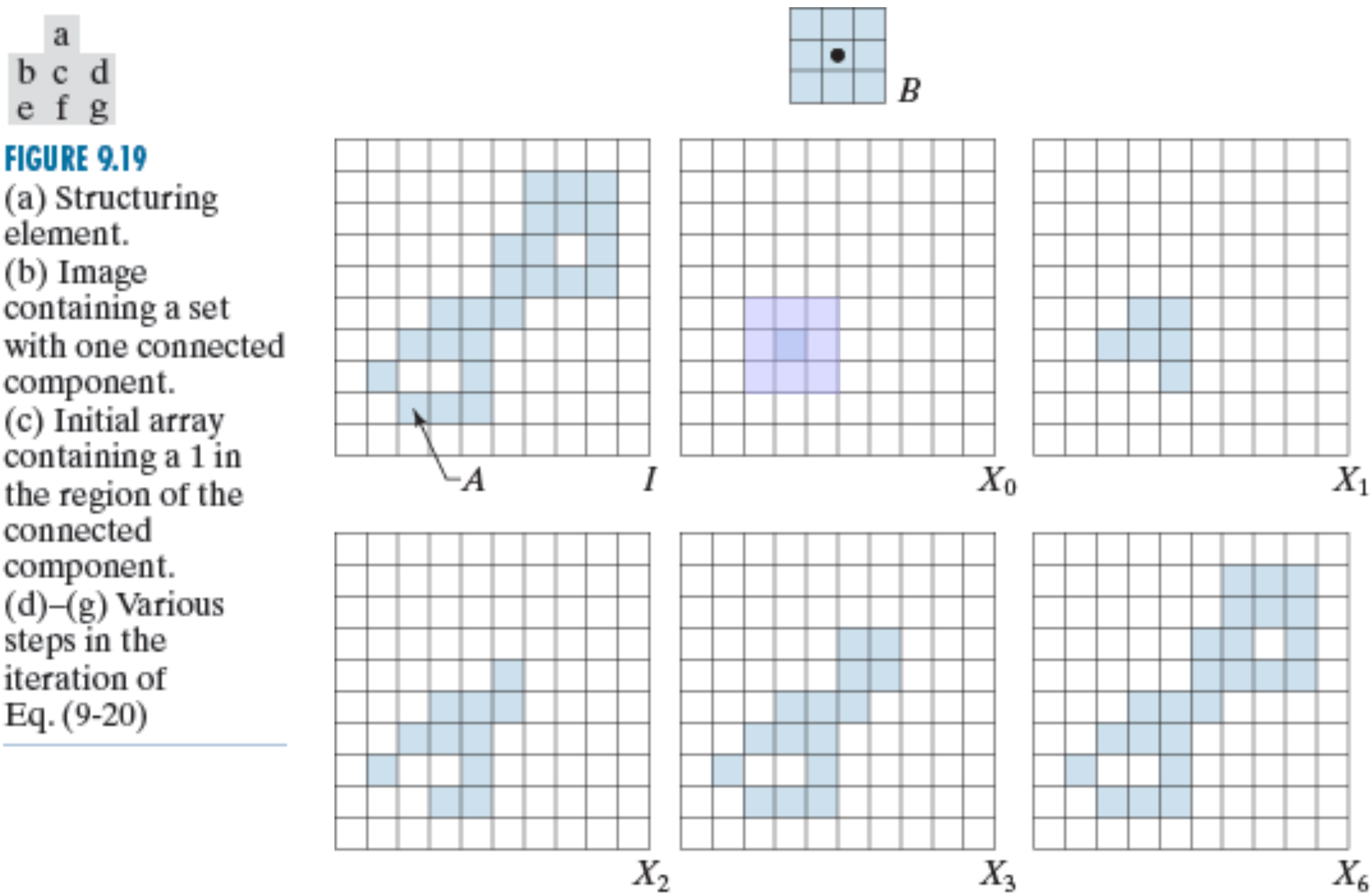


FIGURE 9.19
(a) Structuring element.
(b) Image containing a set with one connected component.
(c) Initial array containing a 1 in the region of the connected component.
(d)–(g) Various steps in the iteration of Eq. (9-20)

Convex Hull

- The convex hull H of an arbitrary set A is the smallest convex set containing A

Let B^i, $i = 1,\ 2,\ 3,\ 4$, represent four structuring elements.

The procedure consists of implementing the equation:

$$X_k^i = (X_{k-1} \circledast B^i) \cup A$$
$$i = 1, 2, 3, 4 \ \text{ and } \ k = 1, 2, 3, \ldots$$

with $X_0^i = A$.

When the procedure converges, or $X_k^i = X_{k-1}^i$, let $D^i = X_k^i$,

the convex hull of A is

$$C(A) = \bigcup_{i=1}^{4} D^i$$

Convex Hull (2)

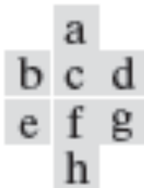
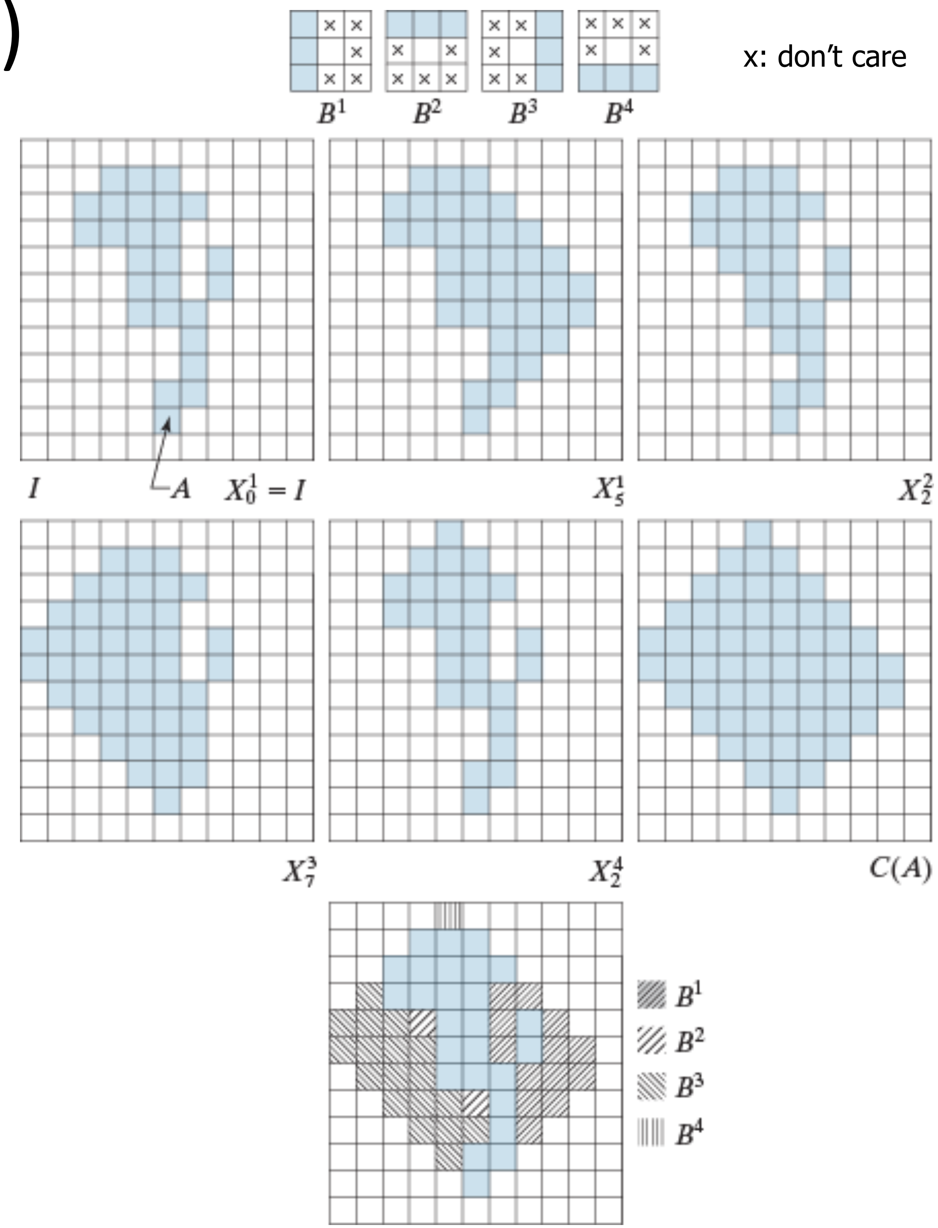
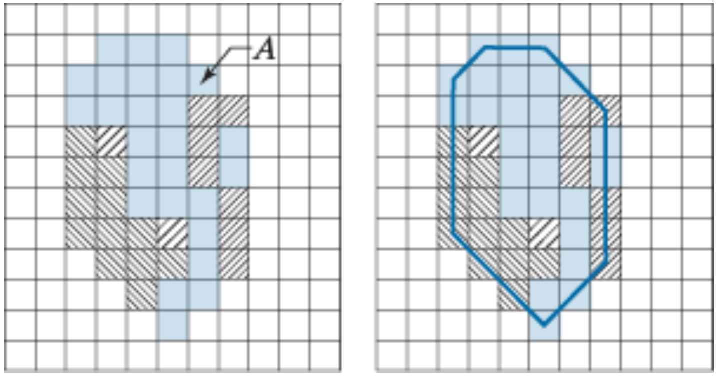


FIGURE 9.21
(a) Structuring elements.
(b) Set A.
(c)–(f) Results of convergence with the structuring elements shown in (a).
(g) Convex hull.
(h) Convex hull showing the contribution of each structuring element.



Convex Hull (3)

- Shortcoming of the algorithm: the convex hull can grow beyond the min dimensions required to guarantee convexity.
- A simple approach to reduce this effect: Place limits so that the convex hull does not go beyond the vertical and horizontal dimensions of set A. But this is only a simple fix and may not yield the closest approximation.
- We may increase the accuracy by including structuring elements in additional directions, such as the diagonals.

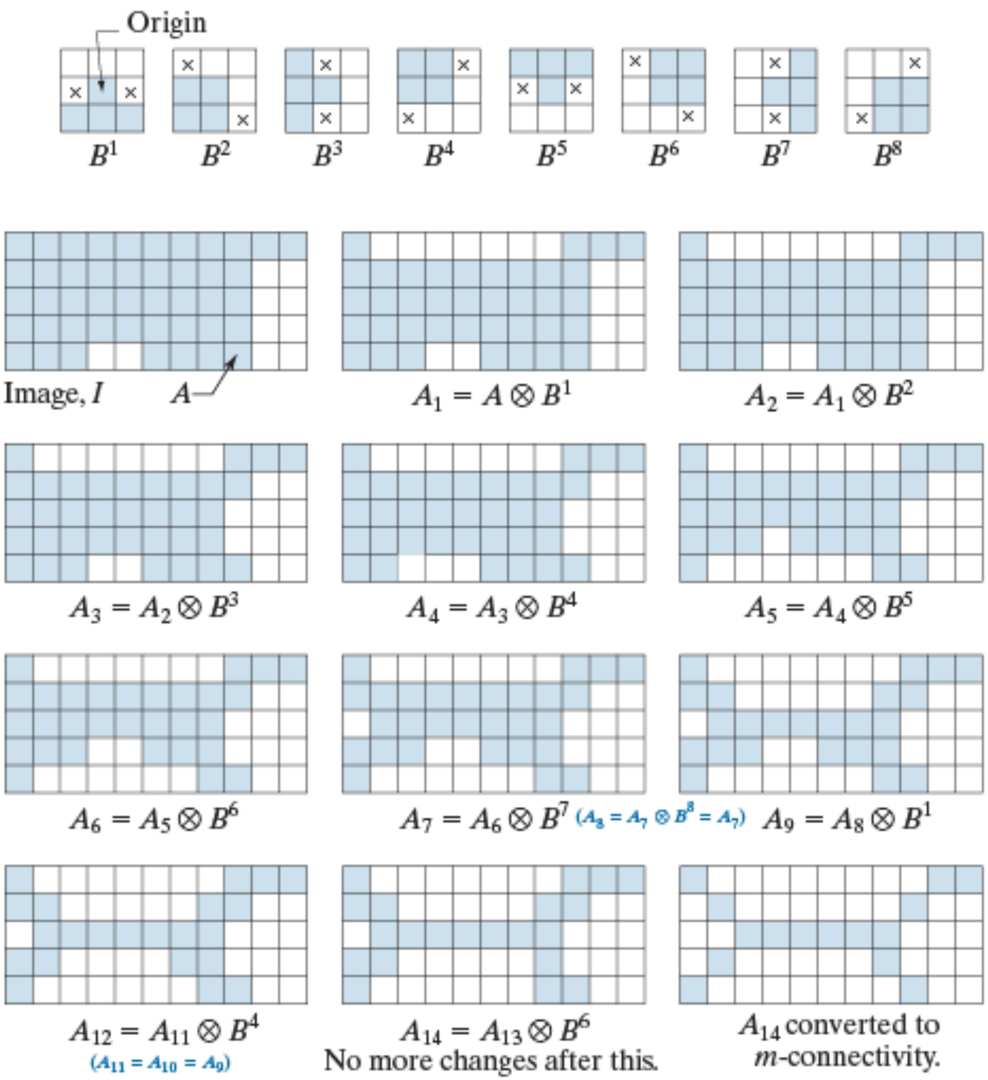


a b

FIGURE 9.22
(a) Result of limiting growth of the convex hull algorithm.
(b) Straight lines connecting the boundary points show that the new set is convex also.

✡ WHY not ⊛ erosion ?

Thinning



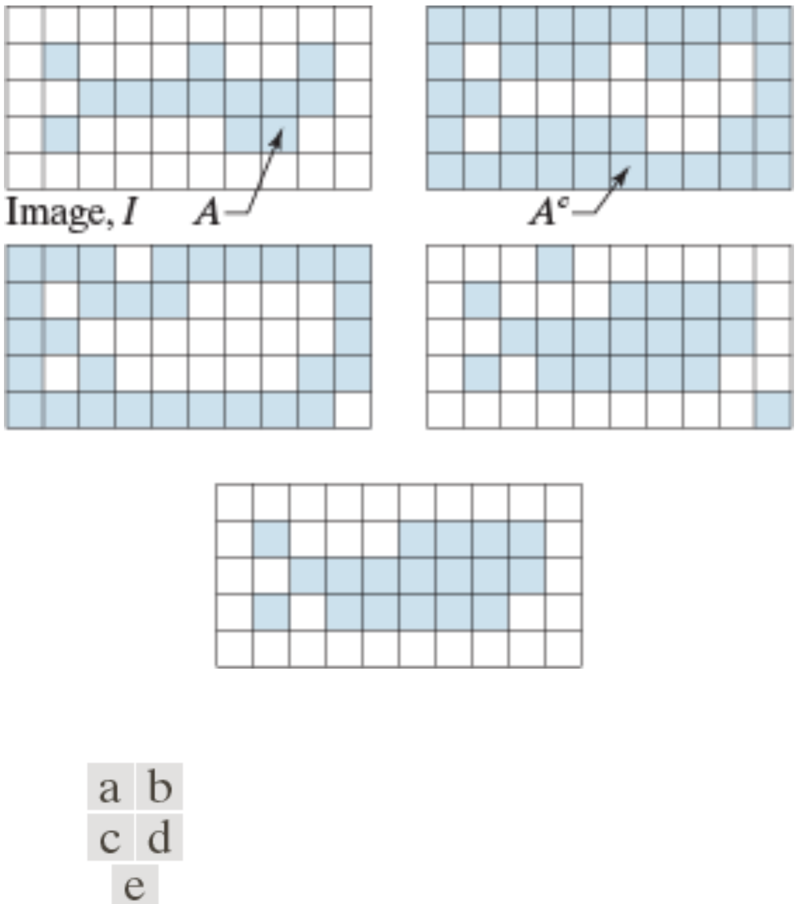
$$A \otimes B = A - (A \circledast B)$$
$$= A \cap (A \circledast B)^c$$
$$\{B\} = \{B^1, B^2, \ldots, B^n\}$$
$$A \otimes \{B\} = \left(\left(\ldots\left(\left(A \otimes B^1\right) \otimes B^2\right)\ldots\right) \otimes B^n\right)$$

	a	
b	c	d
e	f	g
h	i	j
k	l	m

Figure 9.23

(a) Typically, a sequence of rotated structuring elements are used for thinning. (b) Set A. (c) Result of thinning with first element. (d)–(j) results of thinning with the next 7 elements. (k)–(m) There was no change from A_7 to A_8, A_9 to A_{10}, and A_{10} to A_{11}. The result is converted to m-connectivity.

Thickening



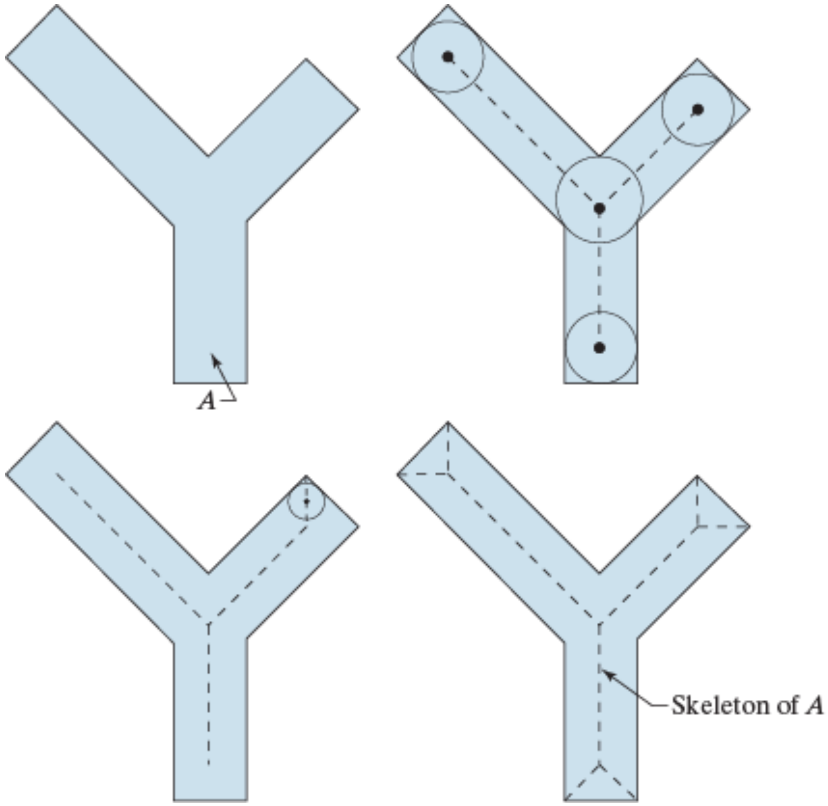
$$A \odot B = A \cup (A \circledast B)$$

$$A \odot \{B\} = \left((\ldots((A \odot B^1) \odot B^2)\ldots) \odot B^n\right)$$

- Thickening is the dual of thinning. So the structuring elements have the same form as those in Figure 9.23(a), but with all 1's and 0's interchanged.
- Usually, thickening a set *A* is obtained by thinning *A*^{c} and then taking the complement of the result.
- Finally, a pruning process is applied as a post step to remove disconnected points.

FIGURE 9.24 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

Skeleton



a b
c d

FIGURE 9.25
(a) Set A.
(b) Various positions of maximum disks whose centers partially define the skeleton of A.
(c) Another maximum disk, whose center defines a different segment of the skeleton of A.
(d) Complete skeleton (dashed).

A skeleton $S(A)$ of a set A has the following properties:

a) If z is a point of $S(A)$ and $(D)_z$ is the largest disk centered at z and contained in A, one cannot find a larger disk containing $(D)_z$ and simultaneously included in A. Such $(D)_z$ is called a maximum disk.
b) If $(D)_z$ is a maximum disk, it touches the boundary of A at two or more different places.

Skeleton (2)

The skeleton of A is obtained by

$$S(A) = \bigcup_{k=0}^{K} S_k(A) \qquad \text{(9-28)}$$

with

$$S_k(A) \triangleq (A \ominus kB) - (A \ominus kB) \circ B \qquad \text{(9-29)}$$

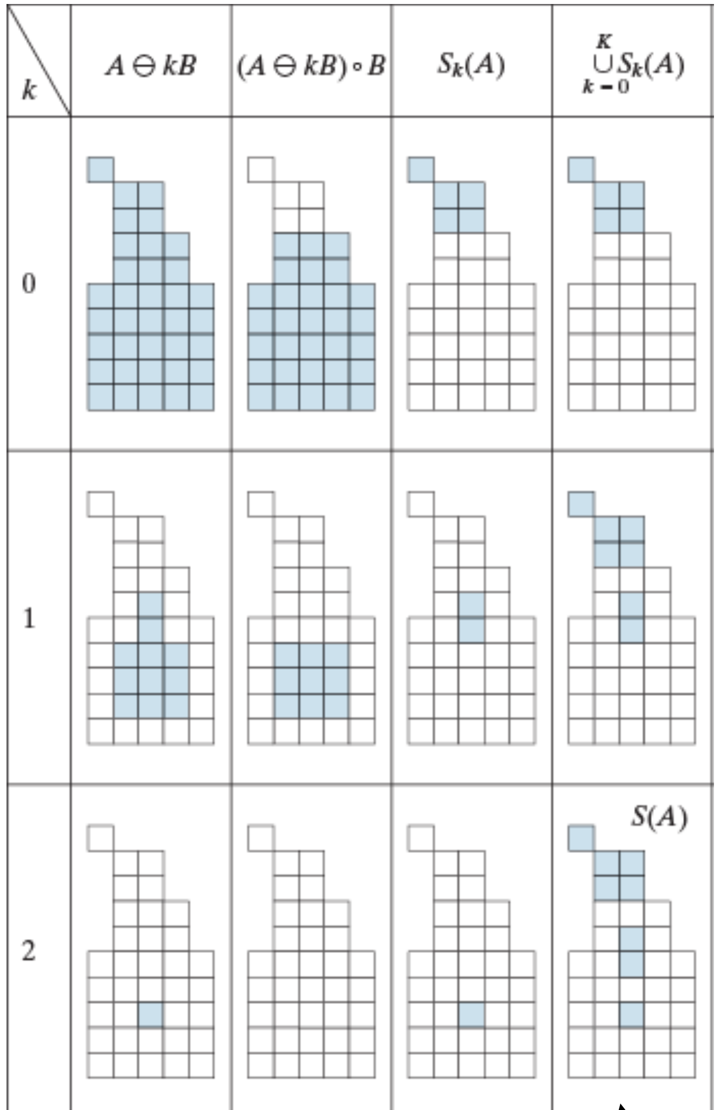
where B is the structuring element, and $(A \ominus kB)$ denotes k successive erosions of A by B; that is,

$$(A \ominus kB) \triangleq \left((\ldots((A \ominus B) \ominus B) \ominus \ldots) \ominus B\right) \qquad \text{(9-30)}$$

K is the last iterative step before A erodes to an empty set,

$$K = \max\{k \mid A \ominus kB \neq \phi\} \qquad \text{(9-31)}$$

Skeleton (3)

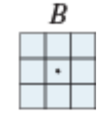


skeleton

FIGURE 9.26 Implementation of Eqs. (9-28) through (9-33). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

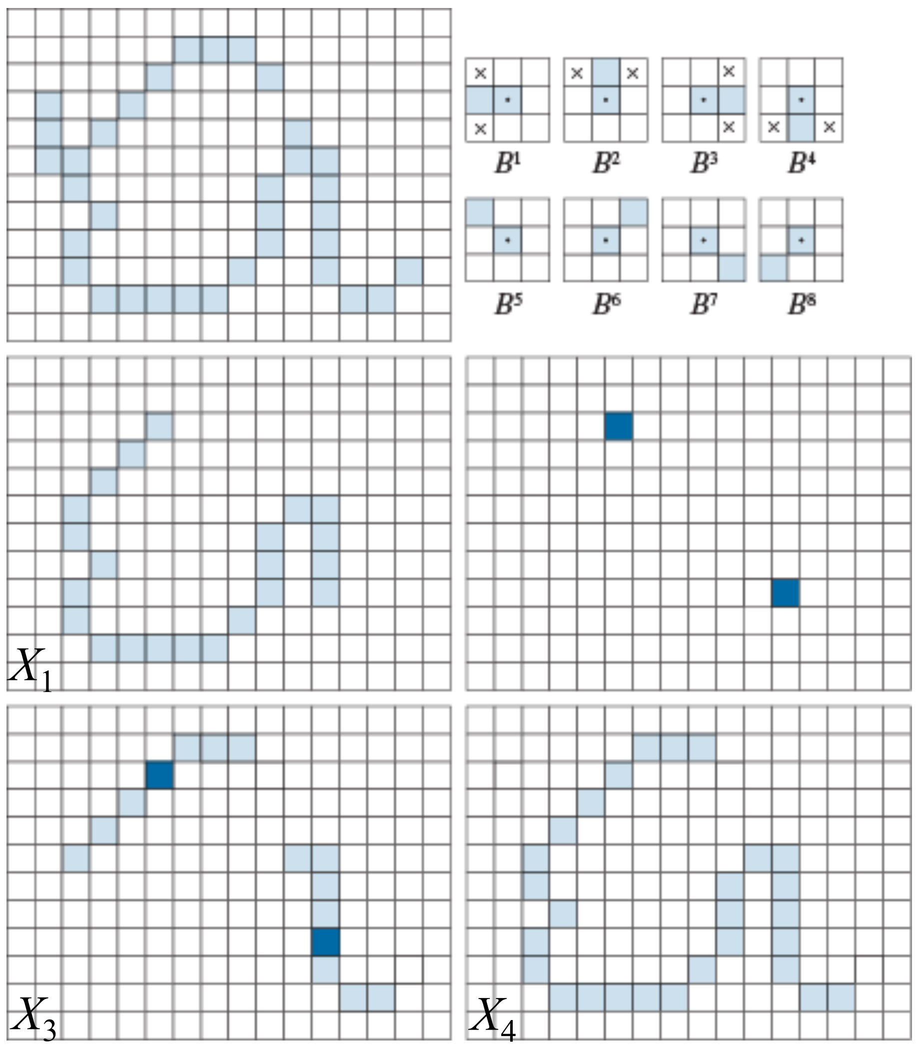
K=2

B



Pruning

- Pruning is an essential post step to clean up the parasitic components created by thinning and skeletonizing



a b
c d
e f

FIGURE 9.27
(a) Set A of foreground pixels (shaded).
(b) SEs used for deleting end points.
(c) Result of three cycles of thinning.
(d) End points of (c).
(e) Dilation of end points conditioned on (a).
(f) Pruned image.

1. Thinning

$$X_1 = A \otimes \{B\}$$

2. Endpoint extraction

$$X_2 = \bigcup_{k=1}^{8} \left(X_1 \circledast B^k \right)$$

3. Dilation of endpoints

$$X_3 = \left(X_2 \oplus H \right) \cap A$$

$$H: 3 \times 3 \text{ SE of 1's}$$

4. Take union

$$X_4 = X_1 \cup X_3$$

Suppressing a spur branch by successively eliminating its endpoints.

Gray-Scale Morphology

- The following basic operations will be extended to gray
- scale images:
  - Dilation
  - Erosion
  - Opening
  - Closing
- These operations will be used to develop morphological algorithms such as
  - Boundary extraction
  - Region partitioning
  - Smoothing
  - Sharpening

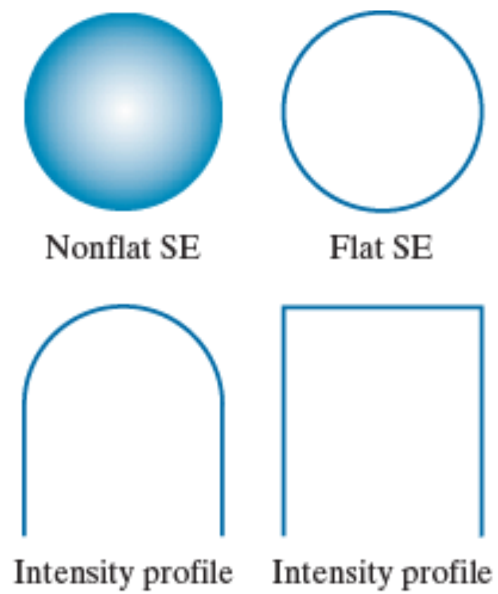
Gray-Scale Morphology

$f(x, y)$: gray-scale image

$b(x, y)$: structuring element

a b
c d

FIGURE 9.36 Nonflat and flat structuring elements, and corresponding horizontal intensity profiles through their centers. All examples in this section are based on flat SEs.



Erosion and Dilation with Flat SE

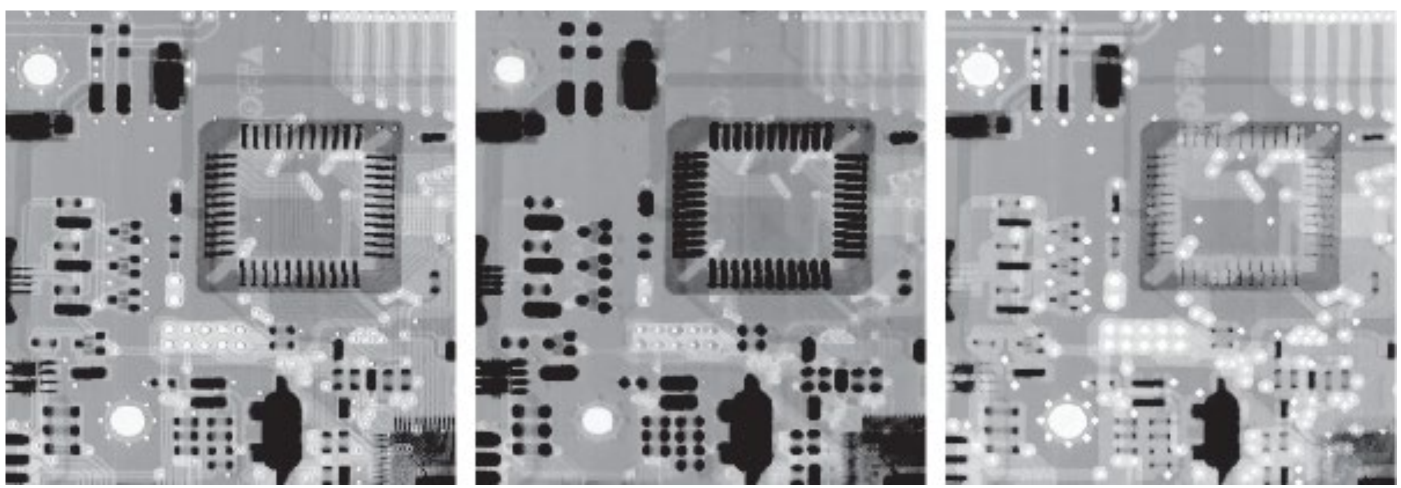
$$\left[f \ominus b\right](x,y) = \min_{(s,t)\in b}\left\{f(x+s, y+t)\right\}$$

$$\left[f \oplus b\right](x,y) = \max_{(s,t)\in b}\left\{f(x-s, y-t)\right\}$$

An example:

a b c

FIGURE 9.37 (a) Gray-scale X-ray image of size 448×425 pixels. (b) Erosion using a flat disk SE with a radius of 2 pixels. (c) Dilation using the same SE. (Original image courtesy of Lixi, Inc.)



original image erosion dilation

Erosion with a Nonflat SE

$$\left[f \ominus b_N\right](x, y) = \min_{(s,t)\in b}\left\{f(x+s, y+t) - b_N(s,t)\right\}$$

- *f* and *b* are functions, not sets.

  Gray scale: $(s+x),(t+y) \in D_f;(x,y) \in D_b$

  Binary: b has to be completely contained by A.

  analogous

- The general effect of performing erosion on a gray scale image
  - The values of *b* > 0
  - The output image tends to be <u>darker</u>
  - Bright details that are smaller in area than the structuring element are <u>reduced</u>, with the degree of reduction determined by the gray level values surrounding the bright details, and by the shape and values of b.

Dilation with a Nonflat SE

$$\left[f \oplus b_N \right](x,y) = \max_{(s,t)\in b} \left\{ f(x-s, y-t) + b_N(s,t) \right\}$$

- *f* and *b* are functions, not sets.

  Gray scale: $(s-x),(t-y) \in D_f; (x,y) \in D_b$

  Binary: the 2 sets have to overlap by at least 1 element

  analogous

- The general effect of performing dilation on a gray scale image
  - The values of *b* > 0
  - The output image tends to be <u>brighter</u>.
  - Dark details are either <u>reduced</u> or <u>eliminated</u>, depending on how their values and shapes relate to *b* used for dilation.

Duality: Erosion and Dilation

$$\left[f \ominus b \right]^c (x, y) = \left(f^c \oplus \hat{b} \right)(x, y)$$

where $f^c = -f(x, y)$ and $\hat{b} = b(-x, -y)$

$$\left[f \ominus b \right]^c = \left(f^c \oplus \hat{b} \right)$$

Similary,

$$\left(f \oplus b \right)^c = (f^c \ominus \hat{b})$$

Gray-Scale Opening and Closing

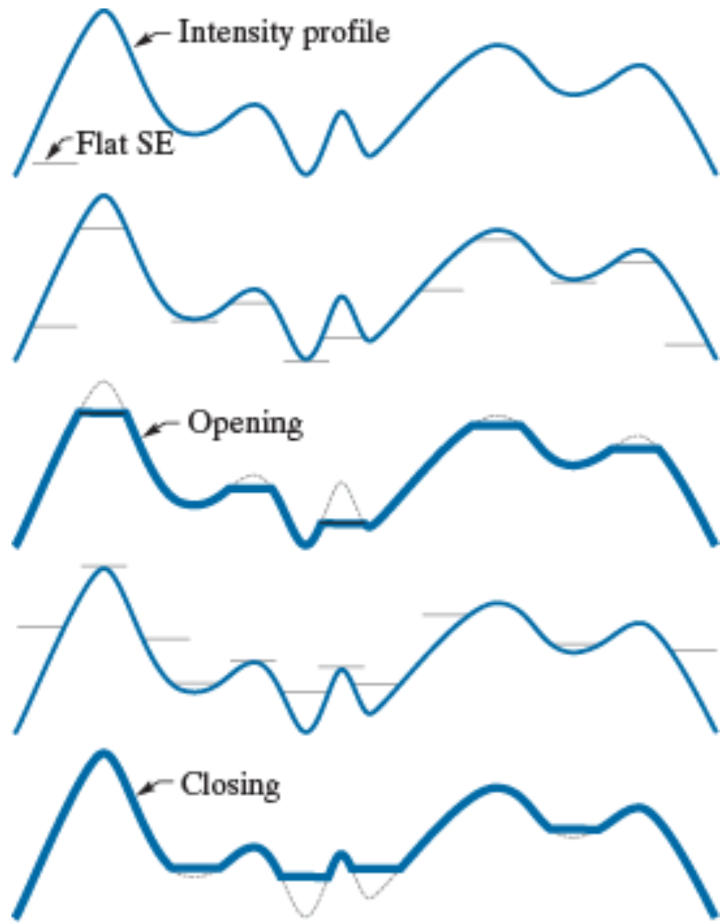
$$f \circ b = \left(f \ominus b \right) \oplus b$$

$$f \bullet b = \left(f \oplus b \right) \ominus b$$

$$\left(f \bullet b \right)^c = f^c \circ \hat{b} = -f \circ \hat{b}$$

$$\left(f \circ b \right)^c = f^c \bullet \hat{b} = -f \bullet \hat{b}$$

Gray-Scale Opening and Closing in 1D



a
b
c
d
e

FIGURE 9.38
Grayscale opening and closing in one dimension. (a) Original 1-D signal. (b) Flat structuring element pushed up underneath the signal. (c) Opening. (d) Flat structuring element pushed down along the top of the signal. (e) Closing.

Properties of Gray-Scale Opening

$(a)\quad f \circ b \dashv f$

$(b)\quad \text{if } f_1 \dashv f_2, \text{ then } \left(f_1 \circ b\right) \dashv \left(f_2 \circ b\right)$

$(c)\quad \left(f \circ b\right) \circ b = f \circ b$

where the notation $e \dashv r$ denotes that the domain of e is a subset of the domain of r and that $e(x, y) \leq r(x, y)$.

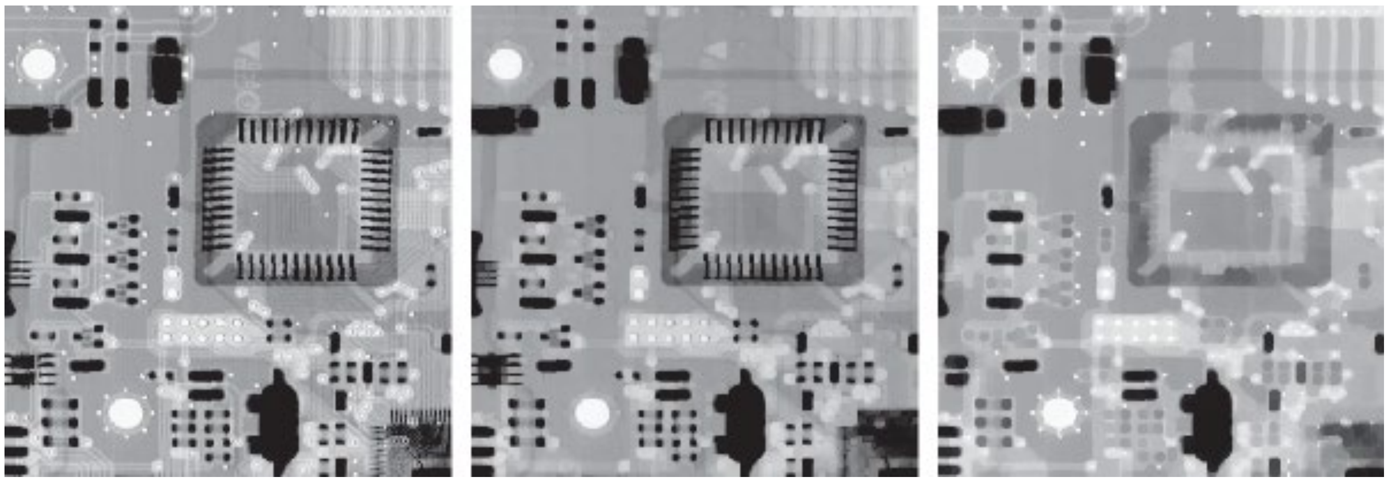
Properties of Gray-scale Closing

$$(a)\quad f \dashv f \bullet b$$

$$(b)\quad \text{if } f_1 \dashv f_2,\ \text{then } \left(f_1 \bullet b\right) \dashv \left(f_2 \bullet b\right)$$

$$(c)\quad \left(f \bullet b\right) \bullet b = f \bullet b$$

Gray-Scale Opening and Closing



a b c

FIGURE 9.39 (a) A grayscale X-ray image of size 448×425 pixels. (b) Opening using a disk SE with a radius of 3 pixels. (c) Closing using an SE of radius 5.

opening

closing

All bright features are attenuated, but the effect on dark featu4es and background are negligible.

The bright details and background are relatively unaffected, but the dark features are attenuated.