Bil345_ HW1

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1) Seven Sided Dices

a. $\max(x,y,z) < 5$

Assume k is the number in top face of a dice. k can be 1,2,3,or 4 to be max<5 The number of sample points for event A is $4 \cdot 4 \cdot 4 = 64$ by multiplication rule.

Also, the number of elements in the sample space is $7 \cdot 7 \cdot 7 = 343$ by multiplication rule

Therefore,

$$P[A] = \frac{|A|}{|S|} = \boxed{\frac{64}{343}}$$

b. $\min(x,y,z) > 3$ It is similar with question 1.a k can be 4,5,6,7 to be min > 3.

The number of sample points of event B $4 \cdot 4 \cdot 4 = 64$ by independency and multiplication rule. From previous question, the size of sample space is 343.

Therefore;

$$P[B] = \frac{|B|}{|S|} = \boxed{\frac{64}{343}}$$

c. $P(B|A) = \dfrac{P(A\cap B)}{\underbrace{P(A)}_{\text{from question 1.a}}}$

To find $P(A\cap B)$; we know that event A and event B happen in same time.

Values of top faces of all dices must be 4 because the minimum of them must be more than 3 and the maximum of them must be less than 5. Only one case which every dice can have the value is 4.

$$P(A\cap B) = \frac{|A\cap B|}{|\text{Sample Space}|} = \frac{1}{343}$$

Therefore;

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{343}}{\frac{64}{343}} = \boxed{\frac{1}{64}}$$

d. Complement of event C $= C' =$ the sum of the numbers being less than or equal to 5.

Let's count the number of elements in C'.

Sum $= 3$; $\dfrac{3!}{3!}$; $(x=1, y=1, z=1)$

Sum $= 4$; $\left(\dfrac{3!}{2!}\right)$; $(x=1, y=1, z=2), (x=1, y=2, z=1), (x=2, y=1, z=1)$

Sum $= 5$; $\left(\dfrac{3!}{2!}\right)$; $(x=1, y=1, z=3), (x=1, y=3, z=1), (x=3, y=1, z=1)$

$\left(\dfrac{3!}{2!}\right)$; $(x=1, y=2, z=2), (x=2, y=1, z=2), (x=2, y=2, z=1)$

$|C'| = 1+3+6 = 10$ $\quad P(C') = \dfrac{10}{343}$

$$P(C) = 1 - P(C') = 1 - \frac{10}{343} = \boxed{\frac{333}{343}}$$

e. $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = ?$

max of them 1

Event D has only one case which is $(x=1, y=1, z=1)$.

So, $P(D) = \dfrac{1}{343}$.

Event C and event D are disjoint events, because

$C \cap D = \emptyset$ so that, $|C \cap D| = 0$, $P(C \cap D) = 0$ ↳ impossible event

Therefore; $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{0}{\frac{1}{343}} = \boxed{0}$

2. Circuit System

Let event X means "component X works" for X=A,B,C,D.

a. The question asks that;

$P(A) \cap [P(B \cap C) \cup P(D)] = ?$

independency

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) . [P(B \cap C) + P(D) - P(B \cap C \cap D)] = ?$

$P(A) . [P(B) . P(C) + P(D) - P(B) . P(C) . P(D)] = ?$

$0{,}8 . [0{,}4 . 0{,}7 + 0{,}5 - 0{,}4 . 0{,}7 . 0{,}5] = 0{,}512$

Let event W be "the system works".

b. Question $P(A|W') = ?$

$P(A|W') = \dfrac{P(A \cap W')}{P(W')} = \dfrac{P(A \text{ works but the system not})}{1 - P(W)}$

↳ from previous question

$$P(A\cap W') = P(\underbrace{A}_{\text{A çalışıyor olmalı}} \cap \underbrace{(B\cap C)'}_{\text{B ve C aynı anda çalışırsa sistem de çalışır.}} \cap \underbrace{D'}_{\text{D çalışırsa sistem de çalışır.}}) = P(A)\cdot\left[1-P(B\cap C)\right]\cdot P(D')$$

$$= P(A)\left[1-P(B)\cdot P(C)\right]\cdot\left[1-P(D)\right]$$

$$= 0{,}8\cdot\left[1-0{,}4\cdot 0{,}7\right]\cdot 0{,}5 = 0{,}288$$

Therefore, $\dfrac{P(A\cap W')}{1-\underbrace{P(W)}_{0{,}512}} = \dfrac{0{,}288}{1-0{,}512} = \boxed{0{,}59}$

c $P(C\mid W) = ? = \dfrac{P(C\cap W)}{P(W)}$

$$P(C\cap W) = P(\underbrace{A}_{\text{sistemin çalışması için A çalışmak zorunda.}}\cap C\cap \underbrace{(B\cup D)}_{\text{sistemin çalışıyor olması için ikisinden biri çalışıyor olmalı.}}) = P(A)\cdot P(C)\cdot\left[P(B)+P(D)-P(B\cap D)\right]$$

$$= 0{,}8\cdot 0{,}7\cdot\left[0{,}4+0{,}5-0{,}2\right]$$

$$= 0{,}392$$

Therefore, $\dfrac{P(C\cap W)}{P(W)} = \dfrac{0{,}392}{0{,}512} = \boxed{0{,}765}$

d. $P(D'\mid W) = ? = \dfrac{P(D'\cap W)}{P(W)}$ ⇝ from question 2.a

$$P(D'\cap W) = P(\underbrace{A\cap(B\cap C)}_{\text{D çalışmaz iken sistemin çalışması için hepsi çalışıyor olmalı}}\cap D') = P(A)\cdot P(B)\cdot P(C)\cdot\left[1-P(D)\right]$$

$$= 0{,}8\cdot 0{,}4\cdot 0{,}7\cdot 0{,}5$$

$$= 0{,}112$$

Therefore, $\dfrac{P(D'\cap W)}{P(W)} = \dfrac{0{,}112}{0{,}512} = \boxed{0{,}218}$

3. Software Practice

$P(P_{k+1} \mid P_k) = 0{,}7$ $P(P_{k+1} \mid S_k) = 0{,}9$

$P(S_{k+1} \mid P_k) = 0{,}3$ $P(S_{k+1} \mid S_k) = 0{,}1$

$* \quad P_K + S_K = 1$

$$P(P_K) = P(P_K \cap E) = P(P_K \cap (P_{K-1} \cup S_{K-1}))$$

$$= P(P_K \cap P_{K-1}) \cup P(P_K \cap S_{K-1})$$

$$= \underbrace{P(P_K \mid P_{K-1})}_{0{,}7} . P(P_{K-1}) + \underbrace{P(P_K \mid S_{K-1})}_{0{,}9} . P(S_{K-1}) \quad * \text{Bayes Rule}$$

$$= 0{,}7 . P(P_{K-1}) + 0{,}9\, P(S_{K-1})$$

$$P(S_K) = P(S_K \cap E) = P(S_K \cap (P_{K-1} \cup S_{K-1}))$$

$$= P(S_K \cap P_{K-1}) + P(S_K \cap S_{K-1})$$

$$= P(S_K \mid P_{K-1})\, P(P_{K-1}) + P(S_K \mid S_{K-1}) . P(S_{K-1})$$

$$= 0{,}3\, P(P_{K-1}) + 0{,}1 . P(S_{K-1})$$

→ yararlanarak soruları gözeceğim.

a. $P(P_3) = 0{,}7 . P(P_2) + 0{,}9\, P(S_2)$

$$= 0{,}7 . \left[0{,}7\, P(P_1) + 0{,}9\, P(S_1)\right] + 0{,}9 \left[0{,}3 . P(P_1) + 0{,}1\, P(S_1)\right]$$

$$= 0{,}49 . P(P_1) + 0{,}63 . P(S_1) + 0{,}27 . P(P_1) + 0{,}09 . P(S_1)$$

$$= 0{,}76 . P(P_1) + 0{,}72 . P(S_1)$$

$$= 0{,}76 . \left[0{,}7 . \underbrace{P(P_0)}_{1} + \cancel{0{,}9\, P(S_0)}_{0}\right] + 0{,}72 . \left[0{,}3\, \underbrace{P(P_0)}_{1} + \cancel{0{,}1\, P(S_0)}_{0}\right]$$

$P(P_0) = 1$, $P(S_0) = 0$ } by default

$$= 0{,}76 \; 0{,}7 + 0{,}72 . 0{,}3 = \boxed{0{,}748}$$

b. $P(S_4) = 0,3 . \boxed{\overset{\text{a şıkkının cevabı.}}{P(P_3)}} + 0,1\, \underbrace{P(S_3)}_{1-P(P_3)}$

$= 0,3 . 0,748 + 0,1 . (1 - 0,748) = \boxed{0,2496}$

c. * conditional probability ; given that $P(S_1) = 1 \iff P(P_1) = 0$

$P(P_3 \mid S_1) = 0,7 . P(P_2) + 0,9 . P(S_2)$

$= 0,7 . \left[\cancel{0,7 . \underbrace{P(P_1)}_{0}} + 0,9 . \underbrace{P(S_1)}_{1} \right] + 0,9 . \left[\cancel{0,3 . \underbrace{P(P_1)}_{0}} + 0,1 . \underbrace{P(S_1)}_{1} \right]$

$= 0,7 . 0,9 + 0,9 . 0,1 = \boxed{0,72}$

d $P(S_1 \mid P_3) = \dfrac{P(S_1 \cap P_3)}{P(P_3)} = \dfrac{P(P_3 \mid S_1) \quad P(S_1)}{P(P_3)}$

from question c

from question a

$P(S_1) = 0,3 . \underbrace{P(P_0)}_{1} + 0,1 . \underbrace{P(S_0)}_{0} = 0,3$

Therefore ; $\dfrac{P(P_3 \mid S_1) . P(S_1)}{P(P_3)} = \dfrac{0,72 . 0,3}{0,748} = \boxed{0,288}$

4. Shopping Center

- It is disjoint because the sets of the values of p and q are countable inf. Σ !

a. $g(p) = \sum_{q=0}^{\infty} \frac{1}{2^{p+q+2}} = \frac{1}{2^{p+2}} \sum_{q=0}^{\infty} \frac{1}{2^q} = \frac{1}{2^{p+2}} \cdot \lim_{r\to\infty} \sum_{q=0}^{r} \frac{1}{2^q}$

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$$= \frac{1}{2^{p+2}} \cdot \lim_{r\to\infty} \frac{1 - \cancel{\left(\frac{1}{2}\right)^{r-1}}^{\,0}}{\frac{1}{2}} \rightsquigarrow 2$$

$$= \frac{1}{2^{p+2}} \cdot 2 = 2^{1-p-2} = 2^{-(p+1)}$$

$$\boxed{g(p) = 2^{-(p+1)}}$$

b. The random variables P and Q are said to be statically independent if and only if

$f(p,q) = g(p)\, h(q)$ for all (x,y) within $\{Z^+ \cup \{0\}\}$.

from previous question

marginal distributions

$$h(q) = \sum_{p=0}^{\infty} \frac{1}{2^{p+q+2}} = \frac{1}{2^{q+2}} \sum_{p=0}^{\infty} \frac{1}{2^p} = 2^{-(q+1)}$$

geometrik seri
2

$$f(p,q) = \frac{1}{2^{(p+q+2)}} \overset{?}{=} g(p) \cdot h(q) = 2^{-(p+1)} \cdot 2^{-(q+1)}$$

✓

$$= 2^{-(p+q+2)} = \frac{1}{2^{(p+q+2)}}$$

Therefore, P and Q are independent.

c. $P(P+Q<6 \mid P>2) = ? = \dfrac{P(\overbrace{(P+Q<6)\cap(P>2)}^{A})}{P(P>2)}$

$(p,q) = \left\{ \begin{array}{l} (3,2),\ (3,1),\ (3,0), \\ (4,1),\ (4,0),\ (5,0) \end{array} \right\}$ $\Rightarrow$ $P(A) = P(p=3, q=2) + P(p=3, q=1) + P(q=3, q=0)$

$+ P(p=4, q=1) + P(p=4, q=0) + P(p=5, q=0)$

$p+q<6$ and $p>2$

$P(P>2) = 1 - P(P\leq 2)$

$= 1 - \{ g(P=0) + g(P=1) + g(P=2) \}$

$= 1 - \{ 2^{-1-0} + 2^{-1-1} + 2^{-1-2} \}$

$= \dfrac{1}{8}$

$= \dfrac{1}{2^{5+2}} + \dfrac{1}{2^{4+2}}(2) + \dfrac{1}{2^{3+2}}(2^2) +$

$\dfrac{1}{2^{5+2}} + \dfrac{1}{2^{4+2}}(2) + \dfrac{1}{2^{5+2}}$

$= \dfrac{11}{2^7}$

$\Rightarrow \dfrac{\frac{11}{2^7}}{\frac{1}{2^3}} = \dfrac{11}{16} = \boxed{0{,}68}$

d. $E[Q] = \sum_{q=0}^{\infty} q \underbrace{\left(\sum_{p=0}^{\infty} f(p,q)\right)}_{h(q)} = ?$

from question b

$= \sum_{q=0}^{\infty} q \cdot 2^{-(q+1)} = \dfrac{1}{4} \cdot \underbrace{\sum_{q=0}^{\infty} q \cdot \left(\dfrac{1}{2}\right)^{q-1}} = \dfrac{1}{4} \cdot \dfrac{1}{\left(1-\frac{1}{2}\right)^2} = 1$

$0 + 1 + 2x^1 + 3x^2 + \ldots + \ldots + q \cdot x^{q-1} + \ldots = ? = \dfrac{1}{(1-x)^2}$

türevi alınır

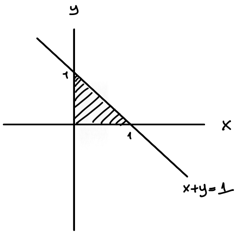
$x^0 + x^1 + x^2 + x^3 + \ldots + \ldots + x^q + \ldots =$ geometrik seri $= \dfrac{1}{1-x}$

5 Lifetime of Components at a Computer System

lifetime ⇝ continuous ($\int$)*

a. 1. $f(x,y) \geq 0$ for all (x,y)

2. $\int_x \int_y f(x,y)\,dy\,dx = 1$ should be true.



$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

$$\int_0^1 \int_0^{1-x} c\,x\,y^2\,dy\,dx = \int_0^1 \left(\frac{c\,x\,y^3}{3} \Bigg|_0^{1-x} \right) dx$$

$$= \int_0^1 \frac{c\,x\,(1-x)^3}{3}\,dx = \frac{c}{3} \cdot \int_0^1 (x - 3x^2 + 3x^3 - x^4)\,dx$$

$(1-x)^3 = 1 - 3x + 3x^2 - x^3$

$$= \frac{c}{3} \left(\frac{x^2}{2} - \frac{3 \cdot x^3}{3} + \frac{3 \cdot x^4}{4} - \frac{x^5}{5} \right) \Bigg|_0^1$$

$$= \frac{c}{3} \left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right] = \frac{c}{3} \cdot \frac{1}{20} = 1$$

$$\boxed{c = 60}$$

b. $$F(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^x \int_0^{1-x} f(x,y)\,dy\,dx = \int_0^x \int_0^{1-x} 60\,x\,y^2\,dy\,dx$$

$$= \int_0^x \left[20\,x\,y^3 \Big|_0^{1-x} \right] dx = \int_0^x 20\,x\,(1-x)^3\,dx = 20 \cdot \int_0^x (x - 3x^2 + 3x^3 - x^4)\,dx$$

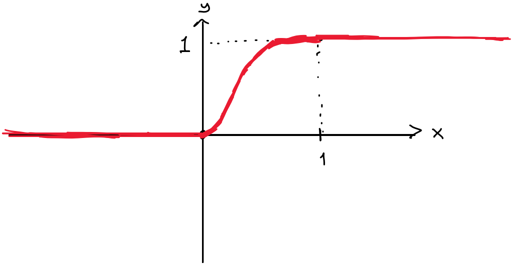
$$= 20 \left[\frac{x^2}{2} - \frac{3\,x^3}{3} + \frac{3 \cdot x^4}{4} - \frac{x^5}{5} \right] \Bigg|_0^x$$

$$= 10\,x^2 - 20x^3 + 15x^4 - 4x^5$$

$$= x^2 (10 - 20x + 15x^2 - 4x^3)$$

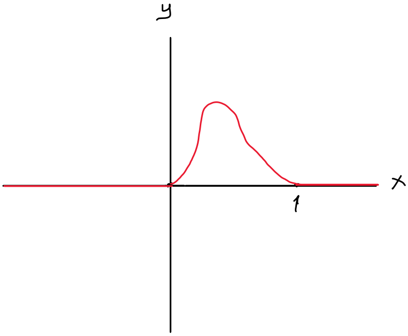
$$F(x) = \begin{cases} 0 & x < 0 \\ x^2(10 - 20x + 15x^2 - 4x^3) & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

x'in sınırlarına uygun şekilde yazılmalı



c. $\int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{1-x} 60xy^2\,dy = \frac{60xy^3}{3}\Big|_{0}^{1-x} = 20.x.(1-x)^3$

$$\text{PDF}(X) = \begin{cases} 20.x.(1-x)^3 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$



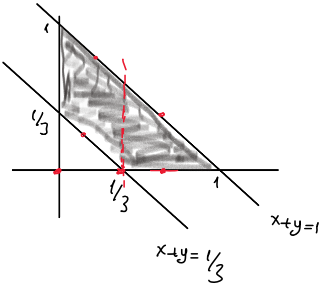
d. $P(Y \leq 1/3 \mid X = 0,4) = ?$

$\int_{-\infty}^{1/3} f(y|0.4)\,dy = \int_{-\infty}^{1/3} \frac{f(0.4,\,y)}{g(0.4)}\,dy = ?$

from previous questions
PDF(X) = g(x)

$\int_{0}^{1/3} \frac{\cancel{60}^{3} . \cancel{0,4} . y^2}{\cancel{20} . \cancel{0,4} . (0,6)^3}\,dy = \frac{\cancel{3}}{(0,6)^3} . \frac{y^3}{\cancel{3}}\Big|_{0}^{1/3} = \left(\frac{10}{18}\right)^3 = \boxed{0,171}$

e. $P\left(X+Y \geq \frac{1}{3}\right) = ?$ $x+y \leq 1$!



$0 \leq x \leq \frac{1}{3} \Rightarrow \frac{1}{3}-x \leq y \leq 1-x$

$\frac{1}{3} \leq x \leq 1 \Rightarrow 0 \leq y \leq 1-x$

$$\int_0^{\frac{1}{3}}\int_{\frac{1}{3}-x}^{1-x} 60xy^2\,dy\,dx + \int_{1/3}^{1}\int_0^{1-x} 60xy^2\,dy\,dx = \int_0^{1/3} 20xy^3\Big|_{\frac{1}{3}-x}^{1-x} dx + \int_{1/3}^{1} 20xy^3\Big|_0^{1-x} dx$$

$$= \int_0^{1/3} 20x\Big[\underbrace{(1-x)^3 - \left(\tfrac{1}{3}-x\right)^3}_{(1-3x+3x^2-x^3)-\left(\frac{1}{27}-\frac{1}{3}x+x^2-x^3\right)}\Big]dx + \int_{1/3}^{1} 20x(1-x)^3\,dx$$

$$= \int_0^{\frac{1}{3}} 20x\left[\frac{26}{27} - \frac{8}{3}x + 2x^2\right]dx + \int_{1/3}^{1} 20x\left(1-3x+3x^2-x^3\right)dx$$

$$= 20\cdot\left[\frac{\cancel{26}^{\,13}}{27}\cdot\frac{x^2}{\cancel{2}} - \frac{8}{3}\cdot\frac{x^3}{3} + \cancel{2}\cdot\frac{x^4}{\cancel{4}\,2}\Bigg|_0^{1/3} + \frac{x^2}{2} - \cancel{3}\frac{x^3}{\cancel{3}} + 3\frac{x^4}{4} - \frac{x^5}{5}\Bigg|_{1/3}^{1}\right]$$

$$= 20\cdot(0{,}0490534) = \boxed{0{,}98}$$

f. $E[X] = \int_{-\infty}^{\infty} x \cdot \underbrace{PDF(x)}_{\text{from question c}} \cdot dx = \int_{0}^{1} x \left(20x\, \underbrace{(1-x)^3}_{1-3x+3x^2-x^3}\right) dx$

$$= 20 \cdot \int_{0}^{1} \left(x^2 - 3x^3 + 3x^4 - x^5\right) dx = 20 \cdot \left[\frac{x^3}{3} - \frac{3 \cdot x^4}{4} + \frac{3 \cdot x^5}{5} - \frac{x^6}{6} \right] \Bigg|_{0}^{1}$$

$$= 20 \cdot \left[\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right] = 20 \cdot \frac{1}{60} = \boxed{\frac{1}{3}}$$
