

PRIMES IN ARITHMETIC PROGRESSIONS TO LARGE MODULI AND SIEGEL ZEROES

THOMAS WRIGHT

ABSTRACT. Let χ be a Dirichlet character mod D with $L(s, \chi)$ its associated L -function, and let $\psi(x, q, a)$ be Chebyshev's prime-counting function for primes congruent to a modulo q . We show that under the assumption of an exceptional character χ with $L(1, \chi) = o((\log D)^{-5})$, for any $q < x^{\frac{2}{3}-\varepsilon}$, the asymptotic

$$\psi(x, q, a) = \frac{\psi(x)}{\phi(q)} \left(1 - \chi\left(\frac{aD}{(D, q)}\right) + o(1) \right)$$

holds for almost all a with $(a, q) = 1$. We also find that for any fixed a , the above holds for almost all $q < x^{\frac{2}{3}-\varepsilon}$ with $(a, q) = 1$. Previous prime equidistribution results under the assumption of Siegel zeroes (by Friedlander-Iwaniec and the current author) have found that the above asymptotic holds either for all a and q or on average over a range of q (i.e. for the Elliott-Halberstam conjecture), but only under the assumption that $q < x^\theta$ where $\theta = \frac{30}{59}$ or $\frac{16}{31}$, respectively.

1. INTRODUCTION

We recall first the definition of the Dirichlet L -function:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Here, χ is a Dirichlet character modulo an integer $q > 2$. We will assume that χ is non-principal, and hence the above sum is convergent for $\operatorname{Re}(s) > 0$.

The study of primes in arithmetic progressions is closely related to the study of when $L(s, \chi)$ equals zero. Dirichlet's work found a zero-free region around $s = 1$, while larger zero-free regions would allow for better error terms for this theorem. Indeed, one of the most famous conjectures in mathematics is the belief that all of these zeroes are, in fact, on the half-line:

Generalized Riemann Hypothesis. *For a Dirichlet character χ , let $L(s, \chi) = 0$ for $s = \sigma + it$ with $\sigma > 0$. Then $\sigma = \frac{1}{2}$.*

Of course, we are nowhere close to proving this. In the case where the zero is real, the best effective and ineffective bounds come from Landau [6] and Siegel [12], respectively:

Theorem 1.1 (Landau, 1918). *There exists an effectively computable positive constant C such that for any q and any character χ mod q , if $L(s, \chi) = 0$ and s is real, then*

$$s < 1 - \frac{C}{q^{\frac{1}{2}} \log^2 q}.$$

Theorem 1.2 (Siegel, 1935). *For any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that if $L(s, \chi) = 0$ and s is real then*

$$s < 1 - C(\varepsilon)q^{-\varepsilon}.$$

However, most zeroes are far closer to the half-line than these bounds indicate. In fact, it is known (see [4], [6], [13]) that for any q , every zero of $L(s, \chi)$ except at most one will obey a much smaller bound:

Theorem 1.3. *There is an effectively computable positive constant C such that*

$$\prod_{\chi \bmod q} L(s, \chi) = 0$$

has at most one solution on the region

$$\sigma \geq 1 - \frac{C}{\log q(2 + |t|)}.$$

If such a zero exists, s must be real, and the character for which $L(s, \chi) = 0$ must be a non-principal real character.

A zero of this type, if it is to exist, is called a *Siegel zero* or an *exceptional zero*, and the associated character is called an exceptional character. We note that the definition given here (or, indeed, in the literature in general) for a Siegel zero is not particularly rigorous, since this definition depends on the choice of the constant C .

2. SIEGEL ZEROES

Interestingly, the existence of Siegel zeroes would lead to some surprisingly nice properties among the primes. Most notably, the existence of Siegel zeroes would allow us to prove (among other things) the twin prime conjecture [5], small gaps between general m -tuples of primes [15], the existence of large intervals where the Goldbach conjecture is true [8], a hybrid Chowla and Hardy-Littlewood conjecture [14], and results about primes in arithmetic progressions that would allow the modulus q to be greater than \sqrt{x} [3], [16]. It is this last result that is of interest in the present paper.

In the definitions below, we will assume that $(a, q) = 1$. We recall that Chebyshev's functions are given by

$$\begin{aligned} \psi(x) &= \sum_{n \leq x} \Lambda(n), \\ \psi(x, q, a) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n), \end{aligned}$$

where Λ is the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

In 2003, Friedlander and Iwaniec [3] proved the following:

Theorem 2.1 (Friedlander-Iwaniec, 2003). *Let χ be a real character mod D . Let $x > D^r$ with $r = 554,401$, let $q = x^\theta$ with $\theta < \frac{233}{462}$, and let $(a, q) = 1$. Then*

$$(1) \quad \psi(x, q, a) = \frac{\psi(x)}{\phi(q)} \left(1 - \chi \left(\frac{aD}{(D, q)} \right) + O \left(L(1, \chi) (\log x)^{r^r} \right) \right).$$

While this theorem gives us an understanding of the distribution of primes in arithmetic progressions beyond the so-called $x^{\frac{1}{2}}$ barrier, it requires a rather extreme Siegel zero. Indeed, this theorem is only non-trivial if

$$L(1, \chi) = o\left((\log D)^{-554,401^{554,401}}\right).$$

In a recent work [16], the current author relaxed the requirements on θ and $L(1, \chi)$ to $\theta < \frac{30}{59} - \varepsilon$ and

$$L(1, \chi) = o\left((\log D)^{-7}\right),$$

in addition to proving that (1) holds for almost all $q \sim x^\theta$ with $\theta < \frac{16}{31} - \varepsilon$.

It was noted in [3] that if the methods of that paper have an obvious technical obstruction at $\theta = \frac{2}{3}$, which we will discuss below. In light of this, it seems that $\theta = \frac{2}{3} - \varepsilon$ might be the best possible result using these methods. In this paper, we show that in several contexts, we can indeed reach this bound for θ .

Since we generally only require $L(1, \chi)$ to be smaller than \log to a power, we define the following. For some large power of A (say, $A = 10,000$), write

$$\mathcal{L}(\chi) = \max\{L(1, \chi), \log^{-A} x\}.$$

Throughout this paper, we will generally assume that any $\varepsilon < \frac{1}{500}$ and $\alpha < \frac{1}{500}$, as this will be sufficiently small for our purposes.

We will prove two results. The first is a fairly sharp version of the Brun-Titchmarsh theorem for $q < x^{\frac{2}{3}-\varepsilon}$, which can be restated as a lower bound on the number of primes congruent to $a \bmod q$ for every $q < x^{\frac{2}{3}-\varepsilon}$ and almost all a with $(a, q) = 1$:

Theorem 2.2. *Let x and D be such that $\log D = o(\log x)$, let $\sqrt{x} < q < D^{-1}x^{\frac{2}{3}-\varepsilon}$ for any $\varepsilon > 0$, and let $(a, q) = 1$. Then*

$$\psi(x, q, a) \leq \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right) + O(\mathcal{L}(\chi) \log^5 x)}{\phi(q)} \psi(x).$$

Moreover, for a given q as above, for any function $h = h(x) < 1$, the equation

$$\psi(x, q, a) \geq \frac{1 - h - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} \psi(x),$$

holds for all but

$$O\left(\frac{\phi(q)}{h} (\mathcal{L}(\chi) \log^5 x)\right)$$

values of a with $(a, q) = 1$.

This result is obviously non-trivial if $L(1, \chi) = o(\log^5 x)$.

For the other result, we recall that the Bombieri-Vinogradov theorem and the Elliott-Halberstam conjecture consider the question of which θ allow for the following inequality to hold for arbitrary A :

$$(2) \quad \sum_{q \leq x^\theta} \max_{(a, q)=1} \left| \psi(x, q, a) - \frac{\psi(x)}{\psi(a)} \right| \ll \frac{x}{\log^A x}.$$

It is known that (2) holds for $\theta < 1/2$, and the Elliott-Halberstam conjecture posits that this holds for all $\theta < 1 - \varepsilon$ for any $\varepsilon > 0$. Under the assumption of Siegel zeroes, it has been proven by the current author [16] that this holds for $\theta = \frac{16}{31} - \varepsilon$.

However, if one changes the inequality in (2) to consider the sum over a fixed congruence class, we can consider instead the weaker question of which θ allow for the following:

$$(3) \quad \sum_{\substack{q \sim x^\theta \\ (a,q)=1}} \left| \psi(x, q, a) - \frac{\psi(x)}{\phi(q)} \right| = o(x).$$

It is known that when considering (3) instead of (2), one can move slightly past $\theta = 1/2$ to $\theta = 1/2 + h(x)$ for any function h such that $h(x) = o(1)$ [1]. (One can also move even further past θ for both (2) and (3) if one is to consider specific well-chosen subsets of $q \sim Q$ - see e.g. [9] for a more thorough discussion of such results.)

Since (3) appears to be slightly more tractable than (2) in the unconditional setting, it would stand to reason that this would be true under the assumption of a Siegel zero as well. Here, we prove that this is indeed the case:

Theorem 2.3. *Let $a \in \mathbb{Z}$, let x and D be such that $\log D = o(\log x)$, and let Q be such that $Q < x^{\frac{2}{3}-\varepsilon}$. Then*

$$\sum_{\substack{q \sim Q \\ (a,q)=1}} \left| \psi(x, q, a) - \frac{\psi(x)}{\phi(q)} \right| \ll x \mathcal{L}(\chi) \log^5 x.$$

3. λ IN ARITHMETIC PROGRESSIONS

Let $*$ denote the Dirichlet convolution, let $\chi = \chi_D$ be an exceptional character mod D , and define

$$\begin{aligned} \lambda &= \chi * 1, \\ \lambda' &= \chi * \log, \\ \nu &= \mu * \mu\chi, \end{aligned}$$

as well as the following sums:

$$\begin{aligned} S'(x, q, a) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda'(n), \\ S(x, q, a) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda(n), \\ S'(x, q) &= \sum_{\substack{n \leq x \\ (n, q)=1}} \lambda'(n), \\ S(x, q) &= \sum_{\substack{n \leq x \\ (n, q)=1}} \lambda(n). \end{aligned}$$

In [3], the authors evaluate $\psi(x, q, a)$ by exploiting the fact that

$$\Lambda = \mu * \log = \mu * \log * (1 * \mu)\chi = \log * \chi * \mu * \chi\mu = \lambda' * \nu.$$

The authors first show that

$$\sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d \leq D^2}} \sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d \leq D^2}} \lambda'(m)\nu(d) = \frac{1 - \chi\left(\frac{aD}{(D,q)}\right)}{\phi(q)} \sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d \leq D^2}} \lambda'(m)\nu(d) + O\left(q^{\frac{1}{2}+\varepsilon} + \frac{x}{q} \log^3 x L(1, \chi)\right).$$

The rest of the paper then shows that

$$(4) \quad \sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d > D^2}} \lambda'(m)\nu(d) \ll \frac{x}{q} \log^A x L(1, \chi) + q^{\frac{115}{58} + \alpha}$$

for a large value of A and a small α . To do this, they use the fact that $|\nu(d)| \leq \lambda(d)$ and then apply a method of Landreau [7] to reduce the quaternary sum $\lambda' * \lambda$ to a ternary sum $\lambda * 1$, which allows them to apply results from ternary sums in arithmetic progressions. The work of [16] treats these ternary sums a bit more carefully and then also uses more recent Kloosterman sum results to improve the error term in (4).

In this paper, we will use the exceptional character χ to mimic the Möbius function μ more directly. More specifically, an exceptional character χ is a multiplicative function for which $\chi(p) = -1$ for “most” primes p in a large region beginning around D and ending at some large D_L , where D_L depends on both D and the location of the zero. So for a natural number n whose divisors are on this interval, it will usually be the case that $\chi(n) = \mu(n)$. So let

$$R = \max\{D^5, x e^{-(\log x)^{\frac{1}{2}}}\}.$$

For a multiplicative function like μ , Tao and Teravainen [14] coined the term “Siegel model” to describe the function μ_{Siegel} where $\mu_{\text{Siegel}} = \mu(p)$ if $p \leq R$ and $\mu_{\text{Siegel}} = \chi(p)$ if $p > R$. We simplify this idea to consider only R -rough numbers, as we define

$$\lambda'_R(n) = \lambda'(n) \mathbf{1}_{P(n) > R},$$

$$\lambda_R(n) = \lambda(n) \mathbf{1}_{P(n) > R},$$

where $P(n)$ denotes the smallest prime divisor of n . For ease of computation, we also define the square-free analogue to λ'_R and λ'_R :

$$\lambda'_W(n) = \lambda'_R(n) \mu(n)^2,$$

$$\lambda_W(n) = \lambda_R(n) \mu(n)^2,$$

which will be helpful since $\lambda(jk) = \lambda(j)\lambda(k)$ if jk is square-free.

Since $\chi(n) = \mu(n)$ should be true for most n in the support of λ'_W , it should usually be the case that $\lambda'_W(n) = \Lambda(n)$. We can then use our understanding of λ' in arithmetic progressions mod q to gain a similar understanding of λ'_W in arithmetic progressions. Since $\Lambda(n) \leq \lambda'(n)$ when n is not a prime power, we can then use this to give an upper bound for Chebyshev’s function $\psi(x, q, a)$:

$$(5) \quad \psi(x, q, a) \leq \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda'_W(n) = \frac{x(1 - \chi_D\left(\frac{aD}{(D,q)}\right) + o(1))}{\phi(q)}.$$

As this upper bound is actually fairly sharp, we know that for almost all a , the inequality on the left-hand side can actually be replaced with an equality, else one would end up with $\psi(x)$ having a main term smaller than x , which is impossible.

For the second main theorem, a key insight is that for any nonnegative function f ,

$$(6) \quad \sum_{q \sim Q} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) \leq \sum_{n \leq x} f(n) \tau(n-a),$$

where $\tau(n)$ is the divisor function. Landreau [7] introduced the idea of bounding a divisor function via inequalities of the form

$$\tau(n) \ll \sum_{\substack{d|n \\ d \leq n^{\frac{1}{r}}}} \tau(d)^{\beta}$$

for some β that depends on r , and Friedlander and Iwaniec [3] later applied this to results about Siegel zeroes. We will use the strongest available bound of this form, which comes from Munshi [10]:

Lemma 3.1 (Munshi, 2011). *For any natural number n and any real number $r > 2$,*

$$\tau(n) \ll \sum_{\substack{d|n \\ d \leq n^{\frac{1}{r}}}} \tau(d)^{\beta},$$

where

$$\beta = -\frac{\log r}{\log 2} + r \left(1 + \left(1 - \frac{1}{r} \right) \frac{\log \left(1 - \frac{1}{r} \right)}{\log 2} \right).$$

In particular, if $r \leq 4$ then $\beta < 1$.

Hence, we can bound (6) as

$$(7) \quad \ll \sum_{d \leq x^{\frac{1}{4}}} \tau(d) \sum_{\substack{n \leq x \\ n \equiv a \pmod{d}}} f(n).$$

We are now left to consider the inner sum over a smaller modulus than q , which makes it easier to show that the sum is small. In practice, we will be able to decompose Λ into a main term, which we can evaluate directly, and an error term, which is a sum that looks like (6).

The paper, then, will proceed in six parts. Sections 4 and 5 establish the behavior of λ' and λ , both modulo q and in general. Section 6 shows the functions λ' and λ'_R are similarly equidistributed in arithmetic progressions, while Section 7 shows that this distribution holds for λ'_W as well. Section 8 uses the relationship between λ'_W and Λ to find an upper bound for $\psi(x, q, a)$ and prove Theorem 2.2. Finally, Section 10 uses the trick in (6) to prove Theorem 2.3

We noted earlier that there is a technical obstruction at $\theta = 2/3$. The obstruction comes from the fact that these methods work by analyzing the behavior of a twisted k -fold divisor function in arithmetic progressions. In [3], Friedlander and Iwaniec removed one variable from the quaternary divisor function $\lambda * \lambda'$ so that they could consider instead a ternary divisor function $\lambda * 1$. Since the ternary divisor function allows one to find information modulo q for a $q > \sqrt{x}$, this allowed the authors to move q past the $x^{\frac{1}{2}}$ barrier to $\theta = \frac{233}{462}$. Our insight is that one can further reduce

this problem to one about a twisted binary divisor function, and since the binary divisor function is understood modulo q for $q \leq x^{\frac{2}{3}-\varepsilon}$, our result allows for q up to this bound. It is not clear that one could move beyond the $x^{\frac{2}{3}}$ -barrier, however, while still using the machinery of divisor functions.

4. BACKGROUND LEMMAS: DISTRIBUTION OF λ

In this section, we examine the distribution of λ and λ' in arithmetic progressions. For the latter, we simply restate Proposition 4.2 of [3]:

Lemma 4.1. *For any $\alpha > 0$,*

$$S'(x, q, a) = \frac{1 - \chi_D \left(\frac{aD}{(D, q)} \right)}{\phi(q)} S'(x, q) + O \left((Dq)^{\frac{1}{2} + \alpha} \right).$$

This is non-trivial if $Dq < x^{\frac{2}{3}-3\alpha}$.

Following largely the same framework as the [3] proof of the above, we can prove a similar theorem about λ :

Lemma 4.2. *For any $\alpha > 0$,*

$$S(x, q, a) = \frac{1 + \chi_D \left(\frac{aD}{(D, q)} \right)}{\phi(q)} S(x, q) + O \left((Dq)^{\frac{1}{2} + \alpha} \right).$$

This is non-trivial if $Dq < x^{\frac{2}{3}-3\alpha}$.

Proof. Let $f(u)$ be a smooth function supported on $(0, x+y)$ such that $f(u) = 1$ on $[1, x]$. In particular, we will set $y = q^{1+\alpha}$. Moreover, define

$$S_f(x, q, a) = \sum_{n \equiv a \pmod{q}} \lambda(n) f(n)$$

and

$$S_f(x, q) = \sum_{n \in \mathbb{N}} \lambda(n) f(n).$$

Note that by Shiu's theorem [11],

$$S_f(x, q, a) = S(x, q, a) + O \left(\sum_{\substack{x \leq n \leq x+y \\ n \equiv a \pmod{q}}} \tau(n) \right) = S(x, q, a) + O(q^{2\alpha}),$$

and similarly,

$$S_f(x, q) = S(x, q) + O(q^{1+2\alpha}).$$

For any character $\chi_q \bmod q$, write

$$Z(s, \chi_q) = \sum_{n \in \mathbb{N}} \lambda(n) \chi(n) n^{-s} = L(1, \chi_D \chi_q) L(1, \chi_q).$$

and define

$$\tilde{f}(s) = \int_0^\infty f(u) u^{s-1} du.$$

Letting $(1 + \varepsilon)$ denote the line at $\operatorname{Re}(s) = 1 + \varepsilon$, we can express $S_f(x, q, a)$ as a contour integral:

$$S_f(x, q, a) = \frac{1}{\phi(q)} \sum_{\chi_q \bmod q} \overline{\chi_q(a)} \frac{1}{2\pi i} \int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds.$$

Similarly, letting χ_0 denote the principal character mod q ,

$$S_f(x, q) = \frac{1}{2\pi i} \int_{(1+\varepsilon)} Z(s, \chi_0) \tilde{f}(s) ds,$$

Note that

$$S_f(x, q, a) = \frac{1}{\phi(q)} S_f(x, q) + \frac{1}{\phi(q)} \sum_{\chi_q \neq \chi_0} \overline{\chi_q(a)} \frac{1}{2\pi i} \int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds.$$

Moreover, if $D|q$ then there exists a character χ_q such that $\chi_q = \chi_D \chi_0$, and hence we have

$$S_f(x, q, a) = \frac{1 + \chi_D(a)}{\phi(q)} S_f(x, q) + \frac{1}{\phi(q)} \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} \overline{\chi_q(a)} \frac{1}{2\pi i} \int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds.$$

For the remaining sum of characters, we move the contour of integration for each of these integrals to $(-\varepsilon)$. Since all of the remaining $L(s, \chi)$ have non-principal χ , the L-functions are analytic and hence we have no poles. So for each such L and χ_q ,

$$\int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds = \int_{(-\varepsilon)} L(s, \chi_q) L(s, \chi_q \chi_D) \tilde{f}(s) ds.$$

We exploit the functional equation

$$L(s, \chi_q) = W(\chi_q) q^{\frac{1}{2}-s} X(s) \Gamma(1-s) L(1-s, \chi_q),$$

where $W(\chi) = \frac{G(\chi)}{\sqrt{q}}$ is the normalized Gauss sum and $|X(s)| = O(1)$ is dependent only on s . So

$$\begin{aligned} & \frac{1}{\phi(q)} \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} \int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds \\ &= \frac{1}{\phi(q)} \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} W(\chi_q) W(\chi_q \chi_D) \overline{\chi_q(a)} \\ & \quad \cdot \frac{1}{2\pi i} \int_{(-\varepsilon)} L(1-s, \chi_q) L(1-s, \chi_q \chi_D) \Gamma(1-s)^2 X(s)^2 q^{1-2s} D^{\frac{1}{2}-s} \tilde{f}(s) ds. \end{aligned}$$

We can write

$$L(1-s, \chi_q) L(1-s, \chi_q \chi_D) = \sum_{n \in \mathbb{N}} \frac{c(n) \chi_q(n)}{n^{1-s}},$$

where $|c(n)| \leq \tau(n)$. Isolating the terms in the integral and sum that have characters χ_q , we then consider

$$\begin{aligned} & \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} \overline{\chi_q(a)} W(\chi_q) W(\chi_q \chi_D) L(1-s, \chi_q) L(1-s, \chi_D) \\ &= \sum_{n \in \mathbb{N}} \frac{c(n)}{n^{1-s}} \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} \chi_q(n \bar{a}) W(\chi_q) W(\chi_q \chi_D). \end{aligned}$$

By Corollary 4.1 of [3] and the fact that $Re(s) = -\varepsilon$, this is

$$\ll \sum_{n \in \mathbb{N}} \frac{c(n)\sqrt{Dq}}{n^{1+\varepsilon}} \ll \sqrt{Dq}.$$

Moreover, integrating \tilde{f} by parts repeatedly gives

$$\tilde{f}(s) \ll \frac{1}{|s|} \min \left(1, \left(\frac{x}{y|s|} \right)^2 \right).$$

Thus,

$$\left| \frac{1}{\phi(q)} \sum_{\chi_q \neq \chi_0, \chi_0 \chi_D} \int_{(1+\varepsilon)} Z(s, \chi_q) \tilde{f}(s) ds \right| \ll \frac{(q)^{\frac{3}{2}+2\varepsilon} D^{\frac{1}{2}+\varepsilon}}{\phi(q)} \int_{(-\varepsilon)} \tilde{f}(s) ds \ll (Dq)^{\frac{1}{2}+3\varepsilon}.$$

Letting $\varepsilon = \frac{\alpha}{3}$ then completes the lemma. \square

5. BACKGROUND LEMMAS: MULTIPLICATIVITY AND DECOMPOSITION

Here, we also give two lemmas about the decomposition of λ and λ' that will help us exploit the multiplicativity of λ . The first one allows us to decompose λ' :

Lemma 5.1. *Let $(d, n) = 1$. Then*

$$\lambda'(dn) = \lambda(d)\lambda'(n) + \lambda'(d)\lambda(n).$$

Proof. We can write

$$\lambda'(dn) = \sum_{l|dn} \chi(l) \log \left(\frac{dn}{l} \right).$$

Since $(d, n) = 1$, we can split l uniquely into $l = d_1 n_1$ where $d_1 | d$ and $n_1 | n$. So

$$\begin{aligned} \lambda'(dn) &= \sum_{d_1 | d} \sum_{n_1 | n} \chi(d_1) \chi(n_1) \log \left(\frac{d}{d_1} \cdot \frac{n}{n_1} \right) \\ &= \sum_{d_1 | d} \sum_{n_1 | n} \left(\chi(d_1) \chi(n_1) \log \left(\frac{d}{d_1} \right) + \chi(d_1) \chi(n_1) \log \left(\frac{n}{n_1} \right) \right) \\ &= \lambda'(d)\lambda(n) + \lambda(d)\lambda'(n). \end{aligned}$$

\square

In order to transition from λ'_R to λ'_W as mentioned in the introduction, we will also need to be able to split $\lambda(n)$ into square-free and square parts. We give an inequality for this in the following lemma.

Lemma 5.2. *Write $n = st$ where s is a square and t is square-free. Then*

$$\lambda(n) \leq \lambda(s)\lambda(t),$$

Proof. We first recall that if χ is an exceptional character then, for any prime p , we have $\chi(p) = \pm 1$ or 0. If $\chi(p) = -1$ then $\lambda(p^{2k}) = 1$ and $\lambda(p) = 0$, and hence

$$\lambda(p^{2k+1}) = 0 = \lambda(p)\lambda(p^{2k}).$$

If $\chi(p) = 0$ then $\lambda(p^r) = 1$ for every $r \geq 1$, and hence

$$\lambda(p^{2k+1}) = 1 = \lambda(p)\lambda(p^{2k}).$$

If $\chi(p) = 1$, we have

$$\lambda(p^{2k+1}) = \tau(p^{2k+1}) = 2k + 2 \leq (2k + 1)(2) = \tau(p^{2k})\tau(p) = \lambda(p^{2k})\lambda(p).$$

So for a given natural number n , write

$$n = p_1^{2k_1+j_1} \cdots p_t^{2k_t+j_t}$$

where k_i is a whole number and j_i is 0 or 1. Since λ is multiplicative, we then have

$$\begin{aligned} \lambda(n) &= \lambda(p_1^{2k_1+j_1} \cdots p_t^{2k_t+j_t}) = \lambda(p_1^{2k_1+j_1}) \cdots \lambda(p_t^{2k_t+j_t}) \\ &\leq \lambda(p_1^{2k_1})\lambda(p_1^{j_1}) \cdots \lambda(p_t^{2k_t})\lambda(p_t^{j_t}) = \lambda(p_1^{2k_1} \cdots p_t^{2k_t})\lambda(p_1^{j_1} \cdots p_t^{j_t}) = \lambda(s)\lambda(t). \end{aligned}$$

□

We also note for later that

$$\sum_{n \leq x} \lambda(n) = xL(1, \chi) + O(D\sqrt{x}),$$

and

$$\sum_{D^2 \leq n \leq x} \frac{\lambda(n)}{n} = L(1, \chi) \log x,$$

as these are Lemma 5.1 and (5.9) of [3], respectively.

6. A ROUGH λ' FUNCTION

Recall the definitions of λ'_R and λ_R given in Section 3. Analogously to the definitions of $S'(x, q)$ and $S'(x, q, a)$, we define

$$\begin{aligned} S'_R(x, q) &= \sum_{\substack{n \leq x \\ (n, q) = 1}} \lambda_R(n), \\ S'_R(x, q, a) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda'_R(n). \end{aligned}$$

We prove that something like Lemma 4.1 holds for these variants as well.

Lemma 6.1. *For any $\alpha > 0$,*

$$S'_R(x, q, a) = \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} S'_R(x, q) + O\left((Dq)^{\frac{1}{2} + \alpha} + \frac{x \log^5 x \mathcal{L}(\chi)}{\phi(q)}\right).$$

Proof. Define

$$P(R) = \prod_{p \leq R} p.$$

Then

$$S'_R(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q} \\ (n, P(R)) = 1}} \lambda'(n)w(n) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda'(n) \sum_{\substack{d|n \\ d|P(R)}} \mu(d) = \sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d|P(z)}} \sum_{\substack{dm \leq x \\ dm \equiv a \pmod{q} \\ d|P(z)}} \mu(d)\lambda'(dm).$$

We split the sum over d into ranges. Choose some small $\eta > 0$. Then

$$S'_R(x, q, a) = \sum_{\substack{dm \leq x, d \leq x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \sum_{\substack{dm \leq x, d \leq x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \mu(d)\lambda'(dm) + \sum_{\substack{dm \leq x, d > x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \sum_{\substack{dm \leq x, d > x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \mu(d)\lambda'(dm).$$

For the second sum, we note that since d is R -smooth, if $d > x^\eta$ then d must have a (not necessarily unique) divisor $l|d$ such that $x^{\frac{\eta}{2}} \leq l \leq Rx^{\frac{\eta}{2}}$. Letting $d = lj$,

$$\begin{aligned} & \sum_{\substack{dm \leq x, d > x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \mu(d) \lambda'(dm) \\ & \ll \log x \sum_{\substack{x^{\frac{\eta}{2}} \leq l \leq Rx^{\frac{\eta}{2}} \\ l|P(R)}} \tau(l) \sum_{\substack{jm \leq \frac{x}{l} \\ jm \equiv al \pmod{q}}} \tau(jm) \\ & \ll \log x \sum_{l_1 \leq x^{\frac{\eta}{4}}} \sum_{\substack{\frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq \frac{Rx^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} \sum_{\substack{k \leq \frac{x}{l_1 l_2} \\ k \equiv al_1 l_2 \pmod{q}}} \tau_4(k), \end{aligned}$$

where $k = jm$ and $l = l_1 l_2$.

By Shiu's theorem, the inner sum can be bounded by

$$\sum_{\substack{k \leq \frac{x}{l_1 l_2} \\ k \equiv al_1 l_2 \pmod{q}}} \tau_4(k) \ll \frac{x \log^3 x}{l_1 l_2}.$$

For the sum over l_2 , we can split this sum into dyadic intervals, finding

$$\sum_{\substack{\frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq \frac{Rx^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} \frac{1}{l_2} \ll \sum_{r=0}^{\log_2 R} \sum_{\substack{2^r \frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq 2^{r+1} \frac{x^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} \frac{1}{l_2} \ll \sum_{r=0}^{\log_2 R} \frac{l_1}{2^r x^{\frac{\eta}{2}}} \sum_{\substack{2^r \frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq 2^{r+1} \frac{x^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} 1.$$

By the standard estimate for smooth numbers [2], we know that for

$$u = \frac{\log x}{\log y},$$

the number of y -smooth numbers less than x is $\ll x\rho(u) \ll xu^{-u}$, where ρ is the Dickman-de Bruijn function. So this sum over l_2 and r is

$$\ll \sum_{r=0}^{\log_2 R} e^{-\frac{\eta \log x}{4 \log R}} \ll e^{-\frac{\eta \log x}{4 \log R}} \log R.$$

So

$$\log x \sum_{l_1 \leq x^{\frac{\eta}{4}}} \sum_{\substack{\frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq \frac{Rx^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} \sum_{\substack{k \leq \frac{x}{l_1 l_2} \\ k \equiv al_1 l_2 \pmod{q}}} \tau_4(k) \ll \frac{x \log^4 x}{q} \sum_{l_1 \leq x^{\frac{\eta}{4}}} \frac{1}{l_1} \sum_{\substack{\frac{x^{\frac{\eta}{2}}}{l_1} \leq l_2 \leq \frac{Rx^{\frac{\eta}{2}}}{l_1} \\ l_2|P(R)}} \frac{1}{l_2} \ll \frac{x}{q} e^{-\frac{\eta \log x}{5 \log R}}.$$

Hence,

$$S'_R(x, q, a) = \sum_{\substack{dm \leq x, d \leq x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \mu(d) \lambda'(dm) + O\left(\frac{x}{q} e^{-\frac{\eta \log x}{5 \log R}}\right).$$

For the remaining sum, since λ' has the decomposition given in Lemma 5.1, we write $m = jk$, where $(k, d) = 1$ and $\text{rad}(j)|d$. Then

$$\sum_{\substack{dm \leq x, d \leq x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq \frac{x}{d} \\ \text{rad}(j)|d}} \sum_{\substack{k \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} [\mu(d)\lambda'(dj)\lambda(k) + \mu(d)\lambda(dj)\lambda'(k)].$$

We can again split the sum over j into $j \leq x^\eta$ and $j > x^\eta$. Note that for the sum over $j > x^\eta$, we have

$$\log x \sum_{\substack{x^\eta < j \leq \frac{x}{d} \\ \text{rad}(j)|d|P(z)}} \sum_{\substack{m \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} \tau(djk) \ll \frac{x}{dq} e^{-\frac{\eta \log x}{5 \log R}}$$

by the same reasoning as before. So

$$\begin{aligned} & \sum_{\substack{dm \leq x, d \leq x^\eta \\ dm \equiv a \pmod{q} \\ d|P(z)}} \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{m \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} [\mu(d)\lambda(dj)\lambda'(k) + \mu(d)\lambda'(dj)\lambda(k)] + O\left(\frac{x}{q} e^{-\frac{\eta \log x}{6 \log R}}\right). \\ &= \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{m \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} [\mu(d)\lambda(dj)\lambda'(k) + \mu(d)\lambda'(dj)\lambda(k)] + O\left(\frac{x}{q} e^{-\frac{\eta \log x}{6 \log R}}\right). \end{aligned}$$

Noting that $\frac{x}{dj} \geq x^{1-2\eta}$ for some small choice of η , we can apply Lemmas 4.1 and 4.2 to find for any $\varepsilon > 0$,

$$\begin{aligned} & \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{m \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} [\mu(d)\lambda(dj)\lambda'(k) + \mu(d)\lambda'(dj)\lambda(k)] \\ &= \frac{1 - \chi_D\left(\frac{aD}{(D,q)}\right)}{\phi(q)} \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{k \leq \frac{x}{dj} \\ (k,q)=1}} \mu(d)\lambda(dj)\lambda'(k) \\ &+ \frac{1 + \chi_D\left(\frac{aD}{(D,q)}\right)}{\phi(q)} \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{k \leq \frac{x}{dj} \\ (k,q)=1}} \mu(d)\lambda'(dj)\lambda(k) + O\left(q^{\frac{1}{2}+\varepsilon} x^{2\eta} + \frac{x}{q} e^{-\frac{\eta \log x}{6 \log R}}\right). \end{aligned}$$

For the latter expression

$$\begin{aligned} & \frac{1 + \chi_D\left(\frac{aD}{(D,q)}\right)}{\phi(q)} \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{k \leq \frac{x}{dj} \\ (k,q)=1}} \lambda'(dj)\lambda(k) \ll \frac{x \log x L(1, \chi)}{\phi(q)} \sum_{d \leq x^\eta} \sum_{j \leq x^\eta} \frac{\tau(dj)}{dj} \\ & \ll \frac{x \log x L(1, \chi)}{\phi(q)} \sum_{r \leq x^{2\eta}} \frac{\tau_4(r)}{r} \\ & \ll \frac{x \log^5 x L(1, \chi)}{\phi(q)}. \end{aligned}$$

So

$$\begin{aligned} & \sum_{\substack{d \leq x^\eta \\ d|P(z)}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d}} \sum_{\substack{m \leq \frac{x}{dj} \\ dkj \equiv a \pmod{q}}} [\mu(d)\lambda(dj)\lambda'(k) + \mu(d)\lambda'(dj)\lambda(k)] \\ &= \frac{1 - \chi_D\left(\frac{aD}{(D,q)}\right)}{\phi(q)} \sum_{\substack{d \leq x^\eta \\ d|P(z) \\ (d,q)=1}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d \\ (j,q)=1}} \sum_{\substack{k \leq \frac{x}{dj} \\ (k,q)=1}} \mu(d)\lambda(dj)\lambda'(k) + O\left(q^{\frac{1}{2}+\varepsilon}x^{2\eta} + \frac{x}{q}e^{-\frac{\eta \log x}{6 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)}\right). \end{aligned}$$

By essentially the same reasoning,

$$S'_R(x, q) = \sum_{\substack{d \leq x^\eta \\ d|P(z) \\ (d,q)=1}} \sum_{\substack{j \leq x^\eta \\ \text{rad}(j)|d \\ (j,q)=1}} \sum_{\substack{k \leq \frac{x}{dj} \\ (k,q)=1}} \mu(d)\lambda'(dj)\lambda'(k) + O\left(xe^{-\frac{\eta \log x}{6 \log R}} + x \log^5 x L(1, \chi)\right).$$

Taking $\eta = \frac{1}{4}\varepsilon$ and $\varepsilon = \frac{\alpha}{2}$, the lemma then follows. \square

7. A SQUARE-FREE λ' FUNCTION

Recall now the definitions of λ'_W and λ_W from Section 3. We next show that the difference between λ'_R and λ'_W is minimal.

Lemma 7.1. *Let $\sqrt{x} \leq q < D^{-1}x^{\frac{2}{3}-3\alpha}$. Then for any $\alpha > 0$,*

$$S'_W(x, q, a) = S'_R(x, q, a) + O\left(q^{\frac{1}{2}+\alpha} + \frac{x \log x}{qR^{1-2\alpha}}\right),$$

and

$$S'_W(x, q) = S'_R(x, q) + O\left(\frac{x}{R^{1-2\alpha}}\right).$$

Hence,

$$S'_W(x, q, a) = \frac{1 - \chi_D\left(\frac{aD}{(D,q)}\right)}{\phi(q)} S'_W(x, q) + O\left((Dq)^{\frac{1}{2}+\alpha} + \frac{x}{q}e^{-\frac{\varepsilon \log x}{24 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)}\right).$$

Proof. As in Lemma 5.2, we write $n = st$ where s is a square and t is square-free. Note that $|S'_W(x, q, a) - S'_R(x, q, a)|$ will be a sum comprised of numbers n that have a square factor of size at least R^2 . So for any $\alpha > 0$,

$$\begin{aligned} |S'_W(x, q, a) - S'_R(x, q, a)| &\leq \sum_{\substack{st \leq x \\ st \equiv a \pmod{q} \\ s > R^2}} \lambda'_R(st) \\ &\ll \log x \sum_{R^2 < s \leq \frac{x}{q^{1+\alpha}}} \tau(s) \sum_{\substack{t \leq \frac{x}{s} \\ t \equiv a\bar{s} \pmod{q}}} \tau(t) + x^{o(1)} \sum_{\frac{x}{q^{1+\alpha}} < s < q^{1+\alpha}} \sum_{\substack{t \leq \frac{x}{s} \\ t \equiv a\bar{s} \pmod{q}}} 1 \\ &\quad + x^{o(1)} \sum_{R < t < \frac{x}{q^{1+\alpha}}} \sum_{\substack{q^{1+\alpha} < s \leq \frac{x}{t} \\ s \equiv a\bar{t} \pmod{q}}} 1 \end{aligned}$$

Write $s = d^2$. For the first term, we can use Shiu's theorem, along with the fact that $\tau(s) \ll s^\alpha$ for any $\alpha > 0$:

$$(8) \quad \log x \sum_{R^2 < s \leq \frac{x}{q^{1+\alpha}}} \tau(s) \sum_{\substack{t \leq \frac{x}{s} \\ t \equiv a\bar{s} \pmod{q}}} \tau(t) \ll \frac{x \log x}{q} \sum_{R < d \leq \sqrt{\frac{x}{q^{1+\alpha}}}} \frac{d^{2\alpha}}{d^2} \ll \frac{x \log x}{q R^{1-2\alpha}}.$$

For the second term, we express the congruence condition via exponential sums. We change notation from t to v to indicate that we have dropped the condition on t being square-free, finding:

$$\begin{aligned} \sum_{\substack{\frac{x}{q^{1+\alpha}} < s < q^{1+\alpha} \\ s=d^2 \\ (s,q)=1}} \sum_{\substack{v \leq \frac{x}{s} \\ v \equiv a\bar{s} \pmod{q}}} 1 &= \frac{1}{q} \sum_{r=0}^{q-1} \sum_{\substack{\sqrt{\frac{x}{q^{1+\alpha}}} < d < q^{\frac{1}{2} + \frac{\alpha}{2}} \\ (d,q)=1}} \sum_{r \leq \frac{x}{d^2}} \exp\left(\frac{r(d^2 v - a)}{q}\right) \\ &\ll \sqrt{\frac{x}{q^{1-\alpha}}} + \frac{1}{q} \sum_{r=1}^{q-1} \sum_{\sqrt{\frac{x}{q^{1+\alpha}}} < d < q^{\frac{1}{2} + \frac{\alpha}{2}}} \frac{1}{\|d^2 r/q\|}, \end{aligned}$$

where the first term is the $r = 0$ term, and $\|\cdot\|$ indicates the distance to the closest integer. We change variables, letting $r' = d^2 r$ and finding

$$\begin{aligned} \frac{1}{q} \sum_{r=1}^{q-1} \sum_{\sqrt{\frac{x}{q^{1+\alpha}}} < d < q^{\frac{1}{2} + \frac{\alpha}{2}}} \frac{1}{\|d^2 r/q\|} &= \frac{1}{q} \sum_{\sqrt{\frac{x}{q^{1+\alpha}}} < d < q^{\frac{1}{2} + \frac{\alpha}{2}}} \sum_{r'=1}^{q-1} \frac{1}{\|r'/q\|} \\ &\ll \log x \sum_{\sqrt{\frac{x}{q^{1+\alpha}}} < d < q^{\frac{1}{2} + \frac{\alpha}{2}}} 1 \\ &\ll q^{\frac{1}{2} + \frac{3}{4}\alpha}. \end{aligned}$$

Since $q \geq \sqrt{x}$,

$$\sqrt{\frac{x}{q^{1-\alpha}}} \leq q^{\frac{1}{2} + \frac{3}{4}\alpha},$$

and hence the $1 \leq r \leq q-1$ -term dominates the $r = 0$ -term, meaning that

$$x^{o(1)} \sum_{\substack{\frac{x}{q^{1+\alpha}} < s < q^{1+\alpha} \\ s=d^2 \\ (s,q)=1}} \sum_{\substack{t \leq \frac{x}{s} \\ t \equiv a\bar{s} \pmod{q}}} 1 \ll q^{\frac{1}{2} + \alpha}.$$

Finally, with the third expression, note that the number of solutions to $d^2 \equiv b \pmod{q}$ is $\ll 2^{\omega(q)} = x^{o(1)}$ if $d < q$, where $\omega(q)$ denotes the number of unique prime divisors of q . So

$$x^{o(1)} \sum_{R < t < \frac{x}{q^{1+\alpha}}} \sum_{\substack{q^{\frac{1}{2} + \frac{\alpha}{2}} < d \leq \sqrt{\frac{x}{t}} \\ d^2 \equiv a\bar{t} \pmod{q}}} 1 \ll x^{o(1)} \sum_{R < t < \frac{x}{q^{1+\alpha}}} 1 \ll \frac{x^{1+o(1)}}{q^{1+\alpha}}.$$

This term is dominated by the term in (8).

Putting all three of these results together then proves the first equation in the lemma.

For the second equation in the lemma, we write again $d^2 = s$ and split the sum over s into $s \leq x^{\frac{3}{4}}$ and $s > x^{\frac{3}{4}}$, noting that $\lambda'_R(s) \leq \tau(s) \log s \ll s^\alpha$:

$$\begin{aligned} |S'_W(x, q) - S'_R(x, q)| &\ll \log x \sum_{R < d \leq x^{\frac{3}{8}}} \tau(d^2) \sum_{t \leq \frac{x}{d^2}} \tau(t) + x^{o(1)} \sum_{t < x^{\frac{1}{4}}} \sum_{d \leq \sqrt{\frac{x}{t}}} 1 \\ &\ll x \sum_{R < s \leq x^{\frac{3}{8}}} \frac{1}{d^{2-2\alpha}} + x^{\frac{1}{2}+o(1)} \sum_{t < x^{\frac{1}{4}}} \sqrt{\frac{1}{t}} \\ &\ll \frac{x}{R^{1-2\alpha}} + x^{\frac{5}{8}+o(1)}. \end{aligned}$$

The first summand is clearly larger than the second, completing the proof of the lemma. \square

8. PRIME SUPPORT

Next, we will show that the behavior of $S'_W(x, q)$ is largely the same as that of $\psi(x)$, while $S'_W(x, q, a)$ provides roughly the upper bound that one would expect for $\psi(x, q, a)$. This will allow us to prove Theorem 2.2.

Lemma 8.1.

$$S'_W(x, q) = x + O\left(R \log^2 x + x \log x L(1, \chi)\right).$$

Proof. For a given n , write $n = n_1 n_{-1}$, partitioned such that $p|n_j$ implies $\chi(p) = j$ for $\chi = \chi_D$. (If no $p|n$ is such that $\chi(p) = j$ then we write $n_j = 1$.) Note that any prime $p|n$ must be such that $\chi(p) \neq 0$, since $(n, q) = 1$. So by Lemma 5.1,

$$S'_W(x, q) = \sum_{n_1 n_{-1} \leq x} \lambda'_W(n_1 n_{-1}) = \sum_{n_1 n_{-1} \leq x} [\lambda'_W(n_1) \lambda_W(n_{-1}) + \lambda_W(n_1) \lambda'_W(n_{-1})].$$

Note that if $n_{-1} \neq 1$ then $\lambda_W(n_{-1}) = 0$, since either n_{-1} is square-free (and hence $\lambda(n_{-1}) = 0$) or else $\mu(n_{-1})^2 = 0$. Moreover, $\lambda'_W(n_{-1})$ is exactly the same as $\Lambda(n_{-1})$, since $\chi(d) = \mu(d)$ for any $d|n_{-1}$. So we break the sum into three cases: $n_{-1} = 1$, $n_1 = 1$, and the rest.

(9)

$$S'_W(x, q) = \sum_{R < n_1 \leq x} \lambda'_W(n_1) + \sum_{\substack{n_{-1} \leq x \\ p|n_{-1} \Rightarrow p > R}} \Lambda(n_{-1}) + \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R}} \lambda_W(n_1) \lambda'_W(n_{-1}).$$

For the first term, since $\lambda'_W(n_1) \leq \tau(n_1) \log x = \lambda(n_1) \log x$ for every n_1 ,

$$\sum_{R < n_1 \leq x} \lambda'_W(n_1) \ll \log x \sum_{R < n_1 \leq x} \lambda(n_1) \ll x \log x L(1, \chi).$$

For the middle term, we can apply the prime number theorem:

$$\sum_{\substack{n_{-1} \leq x \\ p|n_{-1} \Rightarrow p > R}} \Lambda(n_{-1}) = x - \sum_{\substack{R < m \leq x \\ p|m \Rightarrow \chi(p)=1}} \Lambda(m) + O\left(x e^{-(\log x)^{\frac{3}{5}-\epsilon}} + \sum_{\substack{p^k \leq x \\ p|q \text{ or } p < R}} \Lambda(p)\right).$$

For the sum over m , we again have

$$\sum_{\substack{R < m \leq x \\ p|m \Rightarrow \chi(p)=1, p > R}} \Lambda(m) \leq \log x \sum_{R < m \leq x} \lambda(m) \ll x \log x L(1, \chi).$$

For the sum over $p|q$, we see that $k \leq \log x$ and the number of $p|q$ is bounded by $\log x$, and hence

$$\sum_{\substack{p^k \leq x \\ p|q, p > R}} \Lambda(p) \ll \log^3 x.$$

For the sum over $p < R$, we have

$$\sum_{\substack{p^k \leq x \\ p < R}} \Lambda(p) \ll R \log^2 x.$$

Since $R > xe^{-(\log x)^{\frac{3}{5}-\varepsilon}}$ by assumption, we then have

$$\sum_{\substack{n_{-1} \leq x \\ p|n_{-1} \Rightarrow p > R}} \Lambda(n_{-1}) = x + O(R \log^2 x + x \log x L(1, \chi)).$$

Finally, for the last term of (9),

$$\begin{aligned} \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R}} \lambda_W(n_1) \Lambda(n_{-1}) &\leq \sum_{R < n_{-1} < 2\sqrt{x}} \Lambda(n_{-1}) \sum_{R < n_1 \leq \frac{x}{n_{-1}}} \lambda(n_1) + \sum_{R < n_1 \leq 2\sqrt{x}} \lambda(n_1) \sum_{R < n_{-1} < \frac{x}{n_1}} \Lambda(n_{-1}) \\ &\ll x L(1, \chi) \sum_{R < n_{-1} < \frac{x}{R}} \frac{\Lambda(n_{-1})}{n_{-1}} + x \sum_{R < n_1 \leq 2\sqrt{x}} \frac{\lambda(n_1)}{n_1} \\ &\ll x L(1, \chi) \log x. \end{aligned}$$

□

Lemma 8.2.

$$\psi(x, q, a) \leq S'_W(x, q, a) + O(R \log^2 x).$$

Proof. Following the decomposition in (9), we write

$$\begin{aligned} S'_W(x, q, a) &\geq \sum_{\substack{R < n_1 \leq x \\ n_1 \equiv a \pmod{q}}} \lambda'_W(n_1) + \sum_{\substack{n_{-1} \leq x \\ p|n_{-1} \Rightarrow p > R \\ n_{-1} \equiv a \pmod{q}}} \Lambda(n_{-1}) \\ &= \sum_{\substack{R < n_1 \leq x \\ p|n_1 \Rightarrow p > R \\ n_1 \equiv a \pmod{q}}} (\lambda'_W(n_1) - \Lambda(n_1)) + \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) + O\left(\sum_{\substack{p^k \leq x \\ p < R}} \Lambda(p) + \sum_{\substack{p^k \leq x \\ p|q}} \Lambda(p)\right) \\ &\geq \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) + O(R \log^2 x), \end{aligned}$$

since $\lambda'_W(n_1) - \Lambda(n_1) \geq 0$ when n_1 is R -rough.

□

9. MAIN THEOREMS: THEOREM 2.2

Finally, we can prove the first main theorem:

Theorem 9.1. *Let $\sqrt{x} < q < D^{-1}x^{\frac{2}{3}-3\alpha}$ for any $\alpha > 0$, let $(a, q) = 1$, and let $A > 0$. Then*

$$\psi(x, q, a) \leq \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} \psi(x) + O(E(x)),$$

where

$$E(x) = \frac{x}{q} e^{-\frac{\varepsilon \log x}{24 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)} + R \log^2 x.$$

Moreover, if q satisfies the bounds given above and $h = h(x)$ is a function with $h < 1$, then the equation

$$\psi(x, q, a) \geq \frac{1 - h + \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} \psi(x),$$

holds for all but

$$O\left(\frac{\phi(q)^2 E(x)}{hx}\right)$$

values of a with $(a, q) = 1$.

Proof. From Lemma 7.1,

$$\psi(x, q, a) \leq S'_W(x, q, a) + O(R \log^2 x).$$

From Lemmas 6.1, 8.1, and 8.2,

$$\begin{aligned} S'_W(x, q, a) &= S'_R(x, q, a) + O\left(q^{\frac{1}{2}+\alpha} + \frac{x \log x}{q R^{1-2\alpha}}\right) \\ &= \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} S'_R(x, q) + O\left((Dq)^{\frac{1}{2}+\alpha} + \frac{x}{q} e^{-\frac{\alpha \log x}{24 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)}\right) \\ &= \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} S'_W(x, q) + O\left(\frac{x}{q} e^{-\frac{\alpha \log x}{24 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)}\right) \\ &= \frac{x \left(1 - \chi_D\left(\frac{aD}{(D, q)}\right)\right)}{\phi(q)} + O\left(\frac{x}{q} e^{-\frac{\alpha \log x}{24 \log R}} + \frac{x \log^5 x L(1, \chi)}{\phi(q)} + R \log^2 x\right), \end{aligned}$$

where we have absorbed smaller terms into larger ones. Letting $E(x)$ be the error term then gives us

$$(10) \quad \psi(x, q, a) \leq \frac{x \left(1 - \chi_D\left(\frac{aD}{(D, q)}\right)\right)}{\phi(q)} + E(x),$$

proving the first half of the theorem.

For the second half of the theorem, we have trivially

$$(11) \quad \sum_{a \in \mathbb{Z}_q^*} \psi(x, q, a) = \psi(x) = x + O\left(x e^{-(\log x)^{\frac{3}{5}-\varepsilon}}\right),$$

and

$$\sum_{a \in \mathbb{Z}_q^*} \frac{1 - \chi_D\left(\frac{aD}{(D, q)}\right)}{\phi(q)} = 1.$$

So let \mathcal{C} be the set of a such that

$$\psi(x, q, a) \leq \frac{(1-h)x}{\phi(q)}$$

for some h . Bounding the error term in (11) with R ,

$$\begin{aligned} x + O(R) &= \sum_{a \in \mathbb{Z}_q^*} \psi(x, q, a) \\ &\leq (\phi(q) - |\mathcal{C}|) \left(\frac{x}{\phi(q)} + O(E(x)) \right) + |\mathcal{C}| \frac{(1-h)x}{\phi(q)} \\ &= x - \frac{hx|\mathcal{C}|}{\phi(q)} + O((\phi(q) - |\mathcal{C}|)E(x)). \end{aligned}$$

Collapsing down the inequalities, we have

$$x + O(R) \leq x - \frac{hx|\mathcal{C}|}{\phi(q)} + O((\phi(q) - |\mathcal{C}|)E(x)).$$

Since $R \ll E(x)$, this can be rewritten as

$$|\mathcal{C}| \ll \frac{\phi(q)}{hx} (\phi(q)E(x)).$$

This completes the theorem. \square

10. MAIN THEOREMS: THEOREM 2.3

Finally, we prove the following, from which one easily deduces Theorem 2.3:

Theorem 10.1. *Let $\sqrt{x} < Q < D^{-1}x^{\frac{2}{3}-\alpha}$ for any $\alpha > 0$. Then for any fixed integer $a \neq 0$,*

$$\sum_{\substack{q \sim Q \\ (a,q)=1}} \left| \psi(x, q, a) - \frac{\psi(x)}{\phi(q)} \right| \ll xe^{-\frac{\alpha \log x}{24 \log R}} + x \log^5 x L(1, \chi) + R \log^2 x.$$

Proof. We again recall the decomposition in (9):

$$(12) \quad S'_W(x, q, a) = \sum_{\substack{R < n_1 \leq x \\ n_1 \equiv a \pmod{q}}} \lambda'_W(n_1) + \sum_{\substack{n_{-1} \leq x \\ p|n_{-1} \Rightarrow p > R \\ n_{-1} \equiv a \pmod{q}}} \Lambda(n_{-1}) + \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{q}}} \lambda_W(n_1) \lambda'_W(n_{-1}).$$

We show first that when summed over $q \sim Q$, $S'_W(x, q, a)$ will generally be a good approximation for $\psi(x, q, a)$. To this end,

$$\begin{aligned} &\sum_{\substack{q \sim Q \\ (a,q)=1}} |S'_W(x, q, a) - \psi(x, q, a)| \\ &\leq \sum_{\substack{q \sim Q \\ (a,q)=1}} \left(\sum_{\substack{R < n_1 \leq x \\ n_1 \equiv a \pmod{q}}} (\lambda'_W(n_1) - \Lambda(n_1)) + \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{q}}} \lambda_W(n_1) \lambda'_W(n_{-1}) + O(R \log^2 x) \right) \end{aligned}$$

Recall that $\lambda'_W(n_1) - \Lambda(n_1) \geq \lambda(n_1) \log x$. So by Lemma 4.2,

$$\begin{aligned} \sum_{\substack{q \sim Q \\ (a,q)=1}} \sum_{\substack{R < n_1 \leq x \\ n_1 \equiv a \pmod{q}}} (\lambda'_W(n_1) - \Lambda(n_1)) &\ll \log x \sum_{\substack{q \sim Q \\ (a,q)=1}} \sum_{\substack{R < n_1 \leq x \\ n_1 \equiv a \pmod{q}}} \lambda(n_1) \\ &\ll \log x \sum_{\substack{q \sim Q \\ (a,q)=1}} \frac{1}{\phi(q)} \sum_{R < n_1 \leq x} \lambda(n_1) \\ &\ll xL(1, \chi) \log x. \end{aligned}$$

Hence

$$\sum_{\substack{q \sim Q \\ (a,q)=1}} |S'_W(x, q, a) - \psi(x, q, a)| \ll \log x \sum_{\substack{q \sim Q \\ (a,q)=1}} \left(\sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{q}}} \lambda_W(n_1) \right) + O(xL(1, \chi) \log x + R \log^2 x).$$

Now, if $n_1 n_{-1} \equiv a \pmod{q}$ then $q | n_1 n_{-1} - a$, and hence we can write

$$\sum_{\substack{q \sim Q \\ (a,q)=1}} \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{q}}} \lambda_W(n_1) \leq \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R}} \lambda_W(n_1) \tau(n_1 n_{-1} - a).$$

By Lemma 3.1, we can bound this with

$$\ll \sum_{d \leq x^{\frac{1}{3}-\alpha}} \tau(d) \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{d}}} \lambda_W(n_1).$$

Note that if $n_{-1} \geq x^{\frac{1}{3}} > d^{1+\alpha}$ then

$$\sum_{\substack{x^{\frac{1}{3}} \leq n_{-1} \leq \frac{x}{n_1} \\ n_{-1} \equiv a \overline{n_1} \pmod{d}}} 1 \ll \frac{1}{d} \sum_{n_{-1} \leq \frac{x}{n_1}} 1,$$

while if $n_{-1} < x^{\frac{1}{3}}$ then $n_1 \geq x^{\frac{2}{3}}$, and hence

$$\sum_{\substack{x^{\frac{2}{3}} \leq n_1 \leq \frac{x}{n_{-1}} \\ n_1 \equiv a \overline{n_{-1}} \pmod{d}}} \lambda_W(n_1) \ll \sum_{\substack{x^{\frac{2}{3}} \leq n_1 \leq \frac{x}{n_{-1}} \\ n_1 \equiv a \overline{n_{-1}} \pmod{d}}} \lambda(n_1) \ll \frac{1}{\phi(d)} \sum_{n_1 \leq \frac{x}{n_{-1}}} \lambda(n_1).$$

So

$$\begin{aligned}
& \sum_{d \leq x^{\frac{1}{3}-\alpha}} \tau(d) \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R \\ n_1 n_{-1} \equiv a \pmod{d}}} \lambda_W(n_1) \\
& \ll \sum_{d \leq x^{\frac{1}{3}-\alpha}} \frac{\tau(d)}{\phi(d)} \sum_{\substack{n_1 n_{-1} \leq x \\ n_1, n_{-1} > R}} \lambda(n_1) \\
& \ll x \sum_{d \leq x^{\frac{1}{3}-\alpha}} \frac{\tau(d)}{d} \left[L(1, \chi) \sum_{R < n_{-1} \leq 2x^{\frac{2}{3}}} \frac{1}{n_{-1}} + \sum_{R < n_{-1} \leq 2x^{\frac{1}{3}}} \frac{\lambda(n_1)}{n_1} \right] \\
& \ll x L(1, \chi) \log^3 x.
\end{aligned}$$

Hence

$$\sum_{\substack{q \sim Q \\ (a, q)=1}} |S'_W(x, q, a) - \psi(x, q, a)| \ll x L(1, \chi) \log^4 x + R \log^2 x.$$

From Lemma 7.1, if $Q < D^{-1} x^{\frac{2}{3}-3\alpha}$ then

$$\sum_{\substack{q \sim Q \\ (a, q)=1}} \left| S'_W(x, q, a) - \frac{1}{\phi(q)} S'_W(x, q) \right| \ll x e^{-\frac{\alpha \log x}{24 \log R}} + x \log^5 x L(1, \chi),$$

and we recall from Lemma 8.1 that

$$S'_W(x, q) = \psi(x) + O(R \log^2 x + x \log x L(1, \chi)).$$

Choosing the largest terms from these expressions, the theorem then follows. \square

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WOFFORD COLLEGE, 429 N. CHURCH ST., SPARTANBURG, SC 29302, USA