MARC HENRARD

Interest Rate Modelling in the Multi-curve Framework

Foundations, Evolution and Implementation

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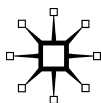
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Foundations, Evolution and Implementation

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Preface

The first lines of this book were written in 2006. At the time the term *multi-curve framework*, which is used for the book's title, had not been coined and my idea was only to write a couple of pages for a note. In the meantime, August 2007 changed the course of writing on interest rate curve modelling for ever.

> Festina lente.
>
> *Latin saying*

> Personal translation: Haste slowly.

Why did it take me so long?

> Chacun sa méthode... Moi, je travaille en dormant et la solution de tous les problèmes, je la trouve en rêvant. (Each his own method ... Myself, I work sleeping and the solution to all problems, I find it dreaming).
>
> *Drôle de drame (1937) – Marcel Carné*

This means a lot of nights spent working to dream up all these pages.

The starting point of the reflection was my quest to answer the question 'What is the present value of an FRA(Forward Rate Agreement)?' in a convincing way. I could not find a satisfactory answer in the literature. The answers I could find were either 'it is trivial', or a description of a replication argument for which it was not acceptable to discuss the numerous hidden hypotheses. Discussing the hypothesis was not politically correct as a scientist nor as a business executive. For the former, it questioned a foundation of quantitative finance, and thus the developments built on those foundations. For the latter, it was not seen positively by board members and supervisors as it casted doubts on accounting figures – maybe rightly so – and had legal implications.

The reason for my interest in discounting is that

> Gentlemen prefer bonds.
>
> *Andrew Mellon – 1855–1937*

Ironically, the first article to come out of those reflections, titled *Irony in derivatives discounting*, was published just one month before the now famous August 2007. The publication was not a prediction of what would happen in the derivative market just after and should not be seen as a premonition. Neither should it be seen as a cause of the crisis. It was nevertheless an indication of inconsistencies in

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1 | Introduction

1.1 9 August 2007

9 August 2007 is probably the day that the *multi-curve framework* became paramount, even if the term had not yet been coined at that time. What happened on that date can be found in history books. From an interest rate curve modelling perspective, it can be summarised by

> Houston, we have a problem!
>
> *Attributed to the crew of the US's Apollo 13 – 14 April 1970*

From the pre-crisis period to the time of finishing this book, the pricing of Ibor[1] and overnight-related derivatives has changed significantly. Before 2007, the standard textbook formulas,[2] using a unique multipurpose curve in each currency, were generally used in banks and software packages.

Nevertheless not everybody believed this was a perfect approach. Several banks, at least in their front-office, were using different pragmatic approaches incorporating cost of funding, risk-free rates different from Libor and similar issues. I have personally developed and used related techniques for pricing on a trading desk from 2003 onward. I would not, however, call the techniques used at that time a theoretically sound framework. It was more a set of ad hoc adjustments done for specific products with a narrow range of applicability. A theoretical foundation was missing and the coherency between instruments was not guaranteed.

For non-par swaps,[3] noticeable quote differences could be observed between banks with different credit qualities and quantitative development levels. It was a clear sign that adjustments to the textbook formulas were done in some places (and

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[1] We use the term Ibor as a generic term for all rates fixing with similar rules. In particular the term covers Libor (originally ten currencies and since May 2013 reduced to six currencies), Euribor (EUR), Tibor (JPY), Cibor (DKK), and so on. See Appendix B for the conventions in the main currencies and Quantitative-Research (2012c) for more currencies and details.

[2] Some of them are described in the first part of Chapter 2.

[3] The interbank swaps are mainly traded at par, with the noticeable exception of the asset swaps.

probably not in others). Since then, several stories have appeared in newspapers and journals describing how some more knowledgeable banks may have used that knowledge to make profits at the detriment of their counterparts and clients. It is difficult to know how much was systematic misinformation and how much was simply taking the risk of using techniques other than the consensus. It should serve the readers as a warning that the current multi-curve framework, as described in this book, is today's equivalent of the 2007 textbook formulas. The framework is still evolving. and the approach is not future proof. As this book is published, there are a lot of people researching and implementing similar subjects in academia and in the industry. In particular, Chapter 8 is a work in progress, both for its theoretical developments and its implementation in practice.

Since the development of the framework became public some people have claimed that the framework was used as early as the mid-1990s in some places. I have not seen any evidence of those claims and all the texts from that period that are now publicly surfacing fall in the *precursor* category, which is discussed in the next section. They described some incoherencies in the single-curve textbook formulas, and proposed some local adjustments, but failed to propose a global, theoretically sound and practically implementable alternative.

Certainly the framework was not developed by one person or in one place. It is the collective work of many people, coming to the problem from a theoretical or practical point of view. It is a child with a lot of fathers and mothers, a child that is still growing fast.

What is certain is that the framework became paramount even before most people knew it existed, and that happened in August 2007.

1.2 Foundations, evolution, and implementation

This book's subtitle is *Foundations, Evolution, and Implementation*. The subtitle acts as a quick summary of the book content.

The *foundations* are paramount. As the framework is different from the previous one-curve approach in fundamental aspects for index-linked financial instruments, one can not rely on a 'copy and paste' style approach. Any claim, even the most basic one, should be backed by clear definitions and clear proofs. Before building the skyscraper that is the pricing of an exotic derivative, one has not only to be sure that the foundations are sound but also indicate explicitly where they are. In some cases, mathematically sophisticated developments are made, only to notice later that their domain of applicability is empty. Their starting hypotheses, which seem acceptable in a one-curve world, lead to contradictions for the multi-curve framework. This is why the book systematically uses an 'axiomatic approach'. The fundamental

hypotheses are displayed in the text as quotes with a bold letter in front reminding us of the content of the hypothesis – such as **D** for the discounting hypothesis at the beginning of Section 2.2. These *axioms* or fundamental hypotheses are the foundations of the framework. No development in interest rate modelling can be done without referring to them or to their equivalent in a different framework.

The *evolution* is important to understand how this new framework started and where it is coming from. Several choices made in the new framework are recycling previous approaches with appropriate and justified twists. Why and where those twists were made is an important piece of information. Looking at the final product, it may seem to have been produced by black magic. It is not any more when seen through the eyes of Darwin, as the result of evolution.

In chronological order, the evolution came before the foundations. The theory is more an explanation of the practice obtained by tinkering with different approaches than a new, ground-breaking theory that started to be used from scratch. We refer the reader to the book *Antifragile* by Taleb (2012) for more on how practice often precedes theory. In this book, we reverse the chronological order and start with the foundations.

Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine : la loi de la probabilité.

Louis Bachelier, Théorie de la spéculation, 1900

Personal translation: If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

This quote, which is the final sentence of Bachelier's 1900 thesis, can be applied in some ways to this book. We do not claim that the theory described appeared in 2007 and started to be used by the market. It first started to be used, and then its use was codified. It was first used in a particular context, on interest rate derivatives desks, and then coherency requirements, analysed by the theory, lead to its general use in finance. The market, unwittingly, complies to a law and the framework tries to axiomatise it.

The *implementation* is the 'raison d'être' of the framework and of this book. The foundations make sense only if they are used in practice. Every piece of theory, every example in this book, was created because it is useful in practice and was implemented for a practical reason to serve a practitioner request. It is possible to implement all of it in a unique and coherent framework. Moreover, the framework is not a goal by itself but is the foundation and an initial building block of

interest rate modelling, including forex – as interest rates in several currencies – and inflation.

When you have read and fully implemented all the details of this book you will have completed only the first stage of your journey! Thankfully, you need not set out upon the project on your own. You can join an open community that has worked on it, through its foundations, evolution and implementation.

1.3 Standard textbook framework

The book by Hull (2006), one of the most popular introductory textbooks to derivative pricing, is used as an example of the way earlier literature treats the curves question. In the section on the *type of rates* in the *Interest rates* chapter (p. 76) the existence of both Libor and Libid is acknowledged. The latter is described in the above book as the rate at which a cash rich investor can invest in the interbank market. In 2006, those rates were relatively close to the OIS (Overnight Indexed Swap) rates.

When it comes to valuing the first derivatives (Section 4.7: Forward Rate Agreements) the explanation is 'the assumption underlying the contract is that the borrowing or lending would normally be done at Libor'. This is first a misleading statement on the instrument itself. The reality is that the contract *settlement amount* is computed by discounting with a *Libor fixing* rate between the end of the accrual period and its start. There is *no actual borrowing or lending* and there is *no assumption* in the FRA[4] contract, only a clear (contingent) settlement formula. For the valuation of the instrument before its fixing date, the approach described in the above book is to use the same Libor rate to discount the resulting quantity to the valuation date. But there is no justification in the text for choosing that particular rate from the different ones described in the previous sections. The choice of Libor for that purpose is a modelling choice and not a contractual obligation. This vagueness is certainly a witness of the consensus at that time: if the instrument is related to an Ibor rate, use similar rates for everything, even if there is no modelling or legal reason to do so. The hidden explanation is that it is easier and everybody is doing it.

Another standard textbook on swaps and curve construction is Sadr (2009). In the description of the pricing of a swap, the terms used to describe the rates linked to the Ibor leg are 'hypothetical loan', 'deposits', 'funding' and 'risk-free rates'. The text indicates that to obtain the standard one-curve formula, those different rates have to be the same. Using the equivalence between those rates the book proceeds through standard arbitrage-free arguments to obtain the standard swap pricing formula.

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[4] The FRA contract details are described in Appendix B.3.

The approach is described in the chapter called 'Swaps: It's still About Discounting', which for a 2009 book is maybe an unfortunate title. There is no analysis of what happens if the rates are different. The book also describes a method that separates the forward and discount curves, in a way similar to Fruchard et al. (1995) described below. This can be viewed as an example of pre-crisis consensus where Libor, risk-free and funding were somehow considered equivalent.

In his book on model risk, Morini (2011) has a good description of the multiple hidden hypotheses that were used to price simple instruments in that framework.

It is a valuable exercise to read the past approaches with the hindsight of the recent developments and to ask why it was done in that way. Every reader interested in model risk or model validation should do the exercise as a warning to any model, including the ones presented here.

More details on the standard textbook framework are provided at the start of Chapter 2. We use this introduction as a starting point for our *evolution* analysis and as a justification on the choice of fundamental hypotheses and definitions for the rest of the book.

1.4 The precursors

We now review some of the *precursors* literature. We describe why the questions asked and the features described in those articles were important, but also why we do not incorporate them in the *multi-curve framework* literature stricto sensu.

These precursors articles include Fruchard et al. (1995), Tuckman and Porfirio (2003) and Boenkost and Schmidt (2004).

Fruchard et al. (1995) describes the existence of a cross-currency basis and proposes an approach to the pricing of cross-currency swaps. One of the hypotheses (Hypothesis 4 of the paper) is that floating references are independent of their payment frequency, and implicitly that they can be obtained from the same pseudo-discount factor curve. The justification is: 'This is a generally accepted consequence of the risk neutral probability approach.' This approach is based on a market reference for liquidity, the USD Libor (Hypothesis 5). In that main currency, there is a unique curve for discounting and forward estimation. In other currencies there is one discounting and one forward curve. The discounting curves in the other currencies are deduced from the (interest rate swaps) IRS fixed rates and cross-currency basis spread. The paper explicitly excludes the possibility of basis between Ibor rates of different tenors. This basis is one of the features of the post August 2007 market. The approach is able to take into account one cross-currency basis between the base currency and each other currency. The explanation of the difference between currencies is based on an hypothetical difference of liquidity. It potentially explains a difference between currencies but not within a given currency. The formulas

to obtain the pricing are quite involved and use several layers of derivatives and integrals.

Tuckman and Porfirio (2003) also focuses on cross-currency swaps. The authors clearly point to incoherences in the then approach for the pricing of those instruments. The solution proposed uses two swaps of the same tenor, one of them not available in the market – a zero-coupon swap – but does not propose a theoretically satisfactory solution. It is an important hint, as it uses two different types of asset to build the curves, like the current multi-curve framework uses two types of instrument. Unfortunately, one of the two assets proposed by the article does not exist in the market and the solution can not be implemented in practice.

Boenkost and Schmidt (2004) also analyses cross-currency swaps starting from the visible incoherencies. They propose two different approaches. One of them is based on two curves in each currency. The two curves are not used for different financial purposes but one is used for each leg of a swap. The result is a different set of incoherencies, like a swap in its last period with zero netted cash-flows having a non-zero present value.

All the above articles asked very good questions, and...

> In mathematics, the art of posing a question is more important than the art of solving one.
>
> *Georg Cantor, 1867*

1.5 Early multi-curve framework literature

To my knowledge, the first article to propose a coherent valuation framework where the discounting is explicitly differentiated from the index forward estimation was *The irony in the derivatives discounting* (Henrard 2007). It was published in July 2007, just one month before the crisis started. To my knowledge there is no link between the article publication and the crisis.

The article focuses on discounting in interest rate derivatives. The starting observation is that different curves are used to value different instruments (OIS, IRS, cross-currency swaps), which is clearly incoherent and creates portfolio level arbitrages.

The approach proposed described only one Ibor curve; it is not an important feature of the approach but a description of the then reality. The approach can be used with several Ibor related curves without changes. The spread between the curves is described in a multiplicative way and is equivalent to the β we define in Chapter 2; our notation is adapted from that used in the paper. Only the case of a constant spread is discussed in the original paper; again, it is not an important feature of the approach but more a description of the then reality.

The first instrument for which the price is discussed is the FRA and the paper provides a pricing formula, including all the hypotheses required to obtain the result. The valuation of swaps in the framework and the impact on the value and risk for not-at-par instruments is discussed. The approach is extended in a coherent way to (Short Term Interest Rate) STIR futures, caps/floors and swaptions. The Ibor fixings are clearly dissociated from the discounting rate through the introduction of an exogenous spread. The spread can be obtained from the market value of different instruments; this is equivalent to the curve building process we propose in Chapter 5.

For practical examples, the article uses an OIS-based discounting curve (called Libid in the paper). The article was probably overly simplistic, as it used only a constant spread between the curves and did not use explicitly different curves for the different Ibor tenors. Nevertheless it was an early proposal leading to today's most commonly used approach.

The importance of the multi-curve framework, with each curve having a specific purpose, is attested by the numerous related literature that appeared in the following years. Throughout the book we have tried to cite as many sources as possible in the relevant sections. We want to give readers the opportunity to go back to the original sources. In this section we comment on a couple of the early papers which describe some particular aspects of the framework. In the literature the framework has received many names. Some of those names are: *two curves*, *multi-curve framework*, *derivative tenor curves*, *funding-Ibor*, *discounting-estimation*, *discounting-forecast*, *discounting-forward*, and *multi-curve market*. Through the book, we will use the name used in the title: *multi-curve framework*.

Ametrano and Bianchetti (2009) is the earliest paper describing the impact of the multi-curve framework on curve construction. Kijima et al. (2009) is important as the first paper describing the impact of collateral. Our approach to collateral, described in Chapter 8, is different but leads to the same fundamental result. Morini (2009) links the framework to the credit risk of the Libor basket obligors. Chibane et al. (2009) describes a multi-curve framework similar to the one described here and a bootstrapping approach to curve calibration. Bianchetti (2010) proposes a description of a multi-curve approach. Piterbarg (2010) is an early description of collateral when collateral rates and risk-free rates are equal. Moreni and Pallavicini (2010) propose a parsimonious simultaneous modelling of both discounting and forward curves. Mercurio (2009), Mercurio (2010a) and Mercurio (2010c) propose a comprehensive Libor Market Model approach for discounting and forward curves. Traven (2010) presents a simplified analysis of the framework based on a restricted set of instruments and on spread around a dominant curve. Bianchetti and Carlicchi (2011) analyses the impact of the framework on optional instruments like swaptions and cap/floor.

In Figure 1.1 the spreads between EUR swaps with different floating leg indexes are displayed in relation to the above literature for the period 2004–13. Note the
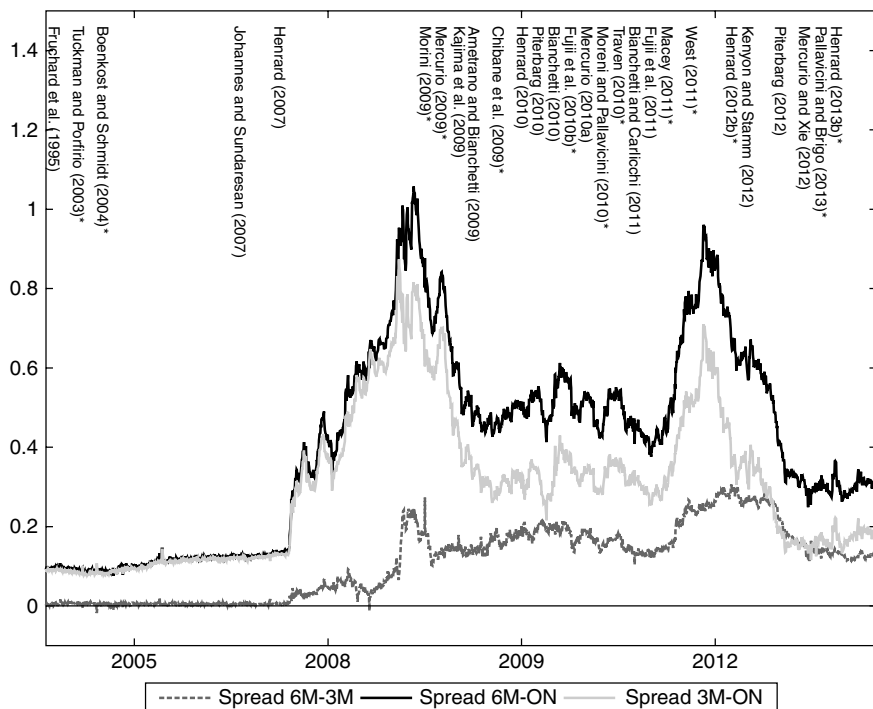
Figure 1.1 Relationship between swap spreads and multi-curves related literature. The spreads are between six months Euribor and three months Euribor and between six months Euribor and OIS, both for swaps with two years tenor.

burst of activity a couple of years after the start of the period with larger spreads. The dates on the graph are approximative in order to avoid text overlap. Literature with a * indicates working papers that have not been published in peer reviewed journals. Obviously the dates proposed are biased, as the publication in peer reviewed journals is strongly impacted by the referee process and the publication back-log. The chronology nevertheless clearly indicates a burst of activity soon after the actual widening of the different spreads, a period where the impact of the new approach was larger.

In the years following the spread widening, a large effort has been made by most banks and software developers to adjust their systems to the new reality. Several financial institutions had started these developments in one form or another before the actual crisis. Nevertheless, robust and general implementations as described here take time and they were not available immediately.

The foundations of the multi-curve framework presented in this book are mainly based on the approach described in Henrard (2010) and Henrard (2013a), the first version of which dates back to mid-2008. It is to my knowledge the first theoretically

sound publicly available description of what is now called the *multi-curve framework*. We changed some of the original definition to introduce a more general framework as described in Henrard (2012c).

In that new multi-curve framework context, all of the previous textbook approaches to derivative pricing, and in particular the valuation of vanilla products such as IRS, FRA and futures need to be reviewed carefully. Even for some of the vanilla instruments, strong modelling assumptions are required to obtain simple formulas.

The notions of *yield curves* and *discount factors* need to be handled carefully. The framework presented here is based on a unique curve used to discount all the cash-flows related to the derivatives, whatever the tenors they relate to, and other curves for index estimation. Even if those estimation curves are different from the discounting and old curves, we will in some circumstances use the same names of discount factor, yield curve and forward rate.

The choice of the discounting curve is by itself an open question. Different people will choose different curves. One possibility is to relate it to the rate that can be achieved for the residual cash; the name used then is the *funding* approach. Another possible starting point is that most of the interbank transactions are done on a collateral basis, and in many contracts the collateral renumeration is set to the same rate as the one used in OIS. In particular, Kijima et al. (2009) showed that in the presence of collateral, the derivative pricing is related to discounting at collateral rate. The exact meaning of the claim needs to be made precise; this will be done in Chapter 8.

In Chapter 2, we suppose that the curves are given. The actual curve calibration process, that is how the market information can be used to obtain those theoretical objects, is proposed in Chapter 5. The curves related to indexes are often called *estimation*, *projection*, *forecast* or *forward* curves. Here the plural is used as there is one curve for each index. In practice, the most popular tenors are one, three, six and twelve months and each currency has its favourite family of indexes – Libor, Euribor, CDOR and so on. In theory one should have one curve for each tenor (two, four, and so on months) and for each index family for each currency. For those curves we will also use the terms yield, discount factor and forward rate at the risk of slight abuse of terminology. The forward curves will never be used for discounting.[5] In the next section, we set the main hypothesis used in the framework. Then we price the simple derivatives (IRS, FRA, STIR futures). The term 'simple' has to be taken with a pinch of salt. In this framework, OISs and FRAs are really contingent claims and a curve hypothesis is needed to price them. Of course STIR futures require a so-called convexity adjustment and a model must be specified. In the subsequent chapter, we describe curve construction in detail. We not only

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[5] Except for FRAs where the discounting using the Ibor fixing rate is part of the contract specification.

describe the basic requirements for such a process but also describe some nice-to-haves that are useful tools to have in practice. To our knowledge this is the most generic curve calibration process described in the literature, from both a theoretical and an implementation point of view.

1.6 Collateral and funding

The description in the first chapters of the multi-curve framework is in a credit risk-free world without the market reality of the *collateral* and *funding* issues. Even if the central theme of the book is the multi-curve framework and not the collateral and funding issues, a book on the curve framework resulting from the crisis would be missing an important part of the reality if the crisis impact on collateral and funding was not discussed at all. Chapter 8 describes some of the issues related to those features. It provides results that were initially presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012). It also presents extensions of the above results. In particular we generalise the definition of collateral and propose results about collateral with assets themselves collateralised – called collateral square. Those results are original and were first published in Henrard (2013b). Our approach to proofs borrows heavily on Macey (2011) and Piterbarg (2012) even if we add the missing self-financing property to the original sketches.

1.7 How to read this book

The easiest way to read this book is to start at Chapter 1 and go to Chapter 8, with a stop in the Appendices when required – you are allowed to eat and sleep between chapters. This is certainly not the only way to read it. In the graph below I have represented the *dependency graph* between the different chapters of the book. The dotted lines for the Appendices indicate where they fit best, but some readers may want to skip them entirely. The links with bidirectional arrows indicate that in some way the two chapters depend on each other and are best read in parallel if the reader has that capacity.
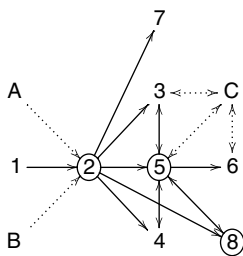
From the graph, the reader can see that, in the mind of the author, the most important chapters are Chapter 2 (Foundations), Chapter 5 (Curve calibration), and Chapter 8 (Collateral).

I tried to write this book in the way I would like to read it. The different *axioms* are clearly evidenced and for each result I have tried to be clear on which hypotheses are required. I have added as many references as possible. The reader should have the opportunity to go to the original literature and compare different approaches. I have also added a lot of cross references between sections, to allow the reader to move between sections with related subjects.

1.8   What is not in this book

A large part of the collateral and funding issues are not discussed in this book. For more in-depth analysis of these issues, we refer to the recent book Brigo et al. (2013). In particular we do not analyse non-perfect collateral. In general the credit risk issues, including Credit Value Adjustment (CVA), are not part of this book. We again refer to the above recent book for more on that subject.

The relation between the Ibor index level and the credit risk embedded in the basket is not discussed. A early account of it can be found in Morini (2009) and a more recent explanation in Crépey (2013) and Filipovic and Trolle (2013). In this book we do not try to explain from first principles the level of the spreads; we take them as an input to the framework.

The inflation derivatives can be viewed as a particular case of Ibor related derivatives in the sense that an index, different from the value of risk-free bonds, is used to reset floating payments. But the inflation market is sufficiently different, with its own idiosyncrasies, and would require an extra chapter or two, not a simple paragraph. We have not tried to cover the subject here.

This book discuss only derivatives, and not assets such as bonds or equities. The bond market would be a mixture of credit, discounting and maybe Ibor (for Floating Rate Notes). This is not covered in this book.

2 | The Multi-curve Framework Foundations

2.1 One-curve world

We start this chapter with a description of some of the one-curve framework pricing formulas. They are important from an historical point of view but also because they explain the evolution of the multi-curve framework, and where its formulas and definitions come from. Chapter 3 proposes different implementations of the framework described here.

Below, we start by describing briefly what an Ibor coupon is. This is the basic financial instrument from which the multi-curve framework is built. More details regarding the conventions related to different coupons and swaps are available in Appendix B. Here we only recall the details important for the understanding of the theory. Note also that we will refrain from using the term *Forward Rate Agreement* (FRA) for one period swaps or single payment Ibor floating coupons. The term FRA has clear market terms and conditions, which are described in Section B.3, including a special settlement procedure which makes them unsuitable as a starting instrument. Some documents in the literature use the term FRA in a loose way as a starting instrument to simplify the description. In my opinion this creates more confusion than simplification.

The general idea of the one-curve world is that all interest rate derivatives depend on only one curve, which is supposed to be at the same time the risk-free curve and the curve relevant for Ibor. The fixed future cash-flows are discounted with that curve. The discount factor at time t for a maturity u is denoted $P(t, u)$. The theoretical deposits underlying the Ibor indexes are priced using the same curve. At any fixing date, the deposit that pays the notional at the settlement date and receives the notional plus the Ibor interest at maturity is supposed to be a fair deposit; this is a deposit for which the total present value is zero. The total value includes the initial settlement of notional and the final payment of notional plus interest. For the purpose of computing the present value, the unique curve is used.

Let j designate an index, such as 'USD LIBOR three months'. We define a j-Ibor floating coupon as a financial instrument which pays at the end of the underlying

period the Ibor rate set at the start of the period. The details of the instrument are as follows. The rate is set or fixed at a date t_0 for the period $[u, v]$ $(t_0 \le u < v)$ of length j; at the end date v the amount paid in currency X is the Ibor fixing $I_X^j(t_0)$ multiplied by the conventional accrual factor δ for the period. The lag between t_0 and u is called the *spot lag* or *settlement lag*. The difference between u and v is the tenor of the index j. All periods and accrual factors should be calculated according to the day count, business day convention, calendar and end-of-month rule appropriate to the relevant Ibor or overnight rate. We describe these conventions for the main currencies in Appendix B. The graphical representation of the coupon, with 0 being today, is given below. The vertical undulating curves represent floating payments and the vertical plain lines represent fixed payments.
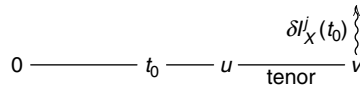
We would like to price this instrument using simple no arbitrage arguments. At this stage we suppose that we are in a credit risk-free world and that the Ibor fixing is equal to the risk-free rate to be paid over the period $[u, v]$ if a deposit is agreed in t_0.

There's no such thing as a free lunch.

Milton Friedman, 1975

To obtain the price we add synthetic payments and reception of the notional both at the start date and at the end date. Ignoring the settlement risk, paying and receiving the same amount at the same date has a zero value. Adding these cash-flows has not changed the total present value of our instrument. This is represented in the next graph.
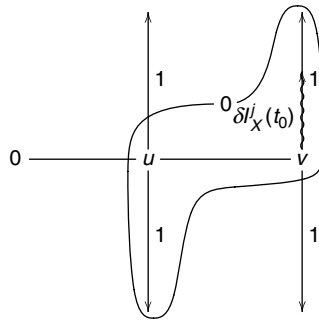
We link three of the five payments together: the payment of the initial notional, and the receipt of the final notional and of the interest $\delta I_X^j(t_0)$. The value of this bundle of three cash-flows is very easy to compute. The interest received is the fair

interest for a deposit over the period $[u, v]$ if decided in t_0, the notional flows are the standard settlement of the deposit at the start and repayment of the notional at maturity. Viewed from t_0, this instrument has, whatever the fair deposit rate is in t_0, a value of 0; there is no advantage or disadvantage to entering into such a transaction. What is the value today of a transaction that has in all states of the world a value of 0 in t_0? Obviously by no arbitrage argument the value is 0. The three cash-flows circled in the above graph have a total value of 0. We simply remove them and are left with the value described in the next graph.

Note that the fixing date t_0 no longer appears in the graph. This does not mean that any fixing date can be chosen. We were able to simplify the representation only because the coupon period $[u, v]$ is the standard period for the Ibor index fixed in t_0. Even if the date does not appear explicitly at this stage, it appears implicitly through the conventions of the underlying Ibor index and in the simplification done previously. This remark will be important when we deal with in-arrear or delayed payments.
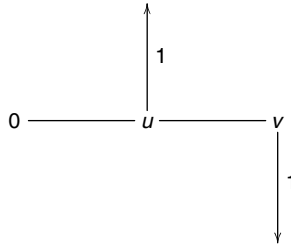
The total value of the remaining cash-flows, which is also the value of the coupon, can be written in 0 as

$$cpn = P(0, u) - P(0, v).$$

This description of the value of a Ibor coupon is often called the *cash-flow equivalent* valuation of the coupon. The floating coupon is replaced by two fixed cash-flows for valuation purposes. This description is valid only under the hypothesis that the Ibor rate is a risk-free rate corresponding to the curve P. The description will be modified later when we introduce actual Ibor indexes, which are not equivalent to risk-free deposits.

The above formula can be modified as

$$cpn = P(0, v) \left(\frac{P(0, u)}{P(0, v)} - 1 \right) = P(0, v) \delta \frac{1}{\delta} \left(\frac{P(0, u)}{P(0, v)} - 1 \right).$$

If we use the definition

$$F = \frac{1}{\delta} \left(\frac{P(0, u)}{P(0, v)} - 1 \right) \tag{2.1}$$

and call the number F the *estimated forward rate*, we have the formula

$$cpn = P(0, v)\delta F.$$

The value of the coupon is the discounted (from payment date to today) value of the estimated forward rate. This description of the value is called *discounting forward rates*. This is a second approach to Ibor coupon valuation that will also appear when we introduce the multi-curve framework in the next sections.

We also price a market FRA in this framework. An FRA is described by dates similar to a coupon (see Appendix B for details): a fixing date t_0 and a period $[u, v]$ with accrual factor δ. A FRA is also described by a reference rate K. The settlement is done in u with the receipt of the amount $\delta(I_X^j(t_0) - K)/(1 + \delta I_X^j(t_0))$ by the FRA buyer. If the rate $I_X^j(t_0)$ is the risk-free rate for the period $[u, v]$, the above payment is equal in present value to the (risk-free) amount $\delta(I_X^j(t_0) - K)$ in v. The value of the FRA is equivalent to the value of receiving the floating rate $I_X^j(t_0)$ and paying the fixed rate K (multiplied by the relevant accrual factor) in v. This last value is the same as that of a one period swap with fixed rate K.

Under all the simplifying assumptions used above, the value of an FRA is equivalent to the value of a one period swap. But this is not a financial description, it only shows that the two products have an equal present value under a specific strong modelling assumption; this is an equal theoretical result under that specific model.

In this sense, it can be claimed that an approach using the modified, simplified and non-market version of an FRA makes sense only if one does not work in a true multi-curve framework. The use of the simplified version is an indication that one works in a single-curve framework.

2.2 Discounting curve

The multi-curve framework starts with the description of the fundamental curve, which is the first of the multiple curves. The starting point is the discounting of known cash-flows. This is the first hypothesis on which the multi-curve framework is based. The cash-flows considered are credit risk-free cash-flows. The impact of collateral on the framework is discussed in Chapter 8.

D: The instrument paying one unit of currency X in u is an asset for each u and each currency X. Its value in t is denoted $P_X^D(t, u)$. The value is continuous in t.

Remember that time is money.

Benjamin Franklin – 1748

With these curves we are able to value fixed cash-flows. The discounting curves are functions $P_X^D : [0, T] \to (0, +\infty)$. To be able to model curves with the usual tools of quantitative finance one needs to fix an upper bound on the time frame on which the modelling is done. This is described in, for example, (Hunt and Kennedy, 2004, Section 7.4.4). This is why we impose an upper bound T. If the bound T is taken sufficiently large, it will have no impact in practice.

We do not restrict discount factors to be below 1 as we want to allow negative rates. Similarly we do not impose that P_X^D is monotonically decreasing in the second variable. In this sense the treatment presented here is more general than the one presented in (Andersen and Piterbarg, 2010, Section 6.1.1). We restrict only the discount factors to be positive. We want to work in an arbitrage-free world. As the payment at maturity u is $1 > 0$, its present value today t, represented by the discount factor $P_X^D(t, u)$, should be strictly positive.

2.3 Forward curves

Our goal is to price Ibor- and overnight-related derivatives, in particular IRSs and OISs. We need an hypothesis to say that these instruments exist in the framework we are describing.

In this section, we adopt the misuse of language of calling instruments *Ibor coupons* even if the underlying index is not strictly speaking an Ibor index, but an index playing the same role or an overnight index.

As the period addition, t + period j, is used often we adopt the notation $t + j$ for that date, without clarifying in which unit the j is; it is usually clear from the context.

Our existence hypothesis for the Ibor coupons reads as

I^CPN: The value of a j-Ibor floating coupon is an asset for each tenor j, each fixing date and each currency. Its value is a continuous function of time.

The subscript CPN stands for CouPoN. We will use the same notation convention for the different quantities related to the Ibor coupons.

This hypothesis is implicit in most of the literature. It is important to state it explicitly as this is not a consequence of the existence of the discounting curve. This hypothesis is one of the *foundations* described in the introduction.

Once we have assumed that the instrument is an asset, we can give its value a name. We do this indirectly through the curves F_X^j below. These curves are called *forward curves*, *projections curves* or *estimation curves* in the literature. We will use the term *forward curves* most of the time. At this stage we insist that the curves F_X^j are pure definition and the use of the word *curve* should be understood in the mathematical sense, not in the financial sense.

Definition 2.1 (Forward coupon rate). *The* forward curve $F_X^{\mathrm{CPN},j}$ *is the continuous function such that,*

$$P_X^D(t,v)\delta F_X^{\mathrm{CPN},j}(t,u,v) \tag{2.2}$$

is the price in t of the j-Ibor coupon with fixing date t_0, start date u and maturity date v ($t \le t_0 \le u = \mathrm{Spot}(t_0) < v$).

The reason for this definition is to keep the usual formulas involving forward rate computation[1]. The standard terms of *discounting curve* and *forward rate* will still be used. But at this stage we insist that they are only definitions and names. One should not attach too strong a financial intuition to those mathematical objects even if we use standard financial terms to designate them.

> La mathématique est l'art de donner le même nom à des choses différentes.
>
> *Henri Poincaré.*

Personal translation: Mathematics is the art of giving the same name to different items.

If the floating coupons have a price, the functions $F_X^{\mathrm{CPN},j}$ exist. At this stage the only link between the curves and market rates is that the Ibor rate fixing in t_0 for the period j, denoted $I_X^j(t_0)$, is

$$I_X^j(t_0) = F_X^{\mathrm{CPN},j}(t_0,u,v). \tag{2.3}$$

To obtain this equality the value time-continuity was used. Take a coupon with fixing in t_0. Its value at $t_0 - \epsilon$ is

$$P_X^D(t_0 - \epsilon, v)\delta F_X^{\mathrm{CPN},j}(t_0 - \epsilon, u, v).$$

After the fixing, the floating coupon becomes a fixed coupon and its value in $t_0 + \epsilon$ is

$$P_X^D(t_0 + \epsilon, v)\delta I_X^j(t_0).$$

The value is obtained simply by discounting the known amount from v. By taking the limit on both values when $\epsilon \overset{>}{\to} 0$, one gets the result for $I_X^j(t_0)$.

The text is written with the intuition that the coupon is related to I_X, which is an interest rate index over the period $[u,v]$. Fundamentally there are two types of assets: the risk-free bonds and the index paying coupons; nothing would be different in this chapter if I_X^j was not an interest rate but any other type of index, such

---

[1] We use the simple interest notation as this is the most used convention in the market. We could also have used something like $P_X^D(t,v)((1+F_X^{\mathrm{CPN},j}(t,u,v))^\delta - 1)$ if we were mainly working in the Brazilian market, where rates are quoted on a daily compounded basis.

as the amount of snowfall in a city or the golf ball production by a manufacturer. Obviously the products would be called *snow swap* or *golf ball swap* but the theory and mathematics would remain the same.

Note also that as the coupons are assets in our economy from hypothesis $\mathbf{I^{CPN}}$, using Definition 2.1, the forward rates $F_X^{\text{CPN},j}(.,u,v)$ are martingales in the $P_X^D(.,v)$-numeraire. This is simply coming from the fact that F_X^j is the value of an asset divided by the numeraire. This remark will be very important when we model the forward rates. Even if it seems trivial at this stage, we write that result as a theorem.

Theorem 2.1 (Martingale property of forward rates). *Under the hypotheses $\mathbf{D}$ and $\mathbf{I^{CPN}}$, the forward rates $F_X^{\text{CPN},j}(.,u,v)$ described in Definition 2.1 are martingales in the $P_X^D(.,v)$-numeraire.*

This result could have been used as the definition of forward rate. It would be possible to define the rate $F_X^{\text{CPN},j}(t,u,v)$ through the expectation

$$F_X^{\text{CPN},j}(t,u,v) = \text{E}^v\left[I_X^j(t_0) \,\middle|\, \mathcal{F}_t \right].$$

We prefer the definition through the present value as it defines the quantities used through financial realities and not to mathematical abstraction. We will use a different approach when the mathematical abstraction definition gives a more general approach to a problem, like in the case of collateral definition in Chapter 8.

The above theorem justifies the development of an Libor Market Model (LMM) on those quantities. The forward rates are martingales and can be modelled by drift-less stochastic equations. This will be discussed in Chapter 7.

Several implementations of these definitions are possible. They will be described in Chapter 3.

We should also add a couple of remarks on the dates. We use the notation $t+j$ as if the time displacement was a real addition. This is not the case. There is no inverse due to non-business dates, several dates u can lead to the same $u+j$. This is not an exceptional case; three days a week, in the standard following rule, have the date $u+j$ on the same Monday.

We also define the forward risk-free rate as follows.

Definition 2.2 (Forward risk-free rate). *The risk-free forward rate over the period $[u,v]$ is given at time t by*

$$F_X^D(t,u,v) = \frac{1}{\delta}\left(\frac{P_X^D(t,u)}{P_X^D(t,v)} - 1 \right). \tag{2.4}$$

This definition will be useful when we introduce spreads. Moreover some definitions and results are written in a more symmetric way with this notation.

2.4 Interest rate swap

With hypothesis $\mathbf{I}^{\mathbf{CPN}}$ and the related definition, the computation of the present value of vanilla IRSs is straightforward. The definition was selected for that reason. An IRS is described by a set of fixed coupons or cash-flows c_i at dates $\tilde{t}_i$ $(1 \leq i \leq \tilde{n})$. For those flows, the discounting curve is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ $(1 \leq i \leq n)$. The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS in $t < t_0$ is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P_X^D(t, t_i) \delta_i F_X^{\mathrm{CPN}, j}(t, t_{i-1}, t_i). \tag{2.5}$$

In the textbook one-curve pricing approach, the IRSs are usually priced through either the *discounting forward rate* approach or the *cash-flow equivalent* approach. The *discounting forward rate* approach is similar to the above formula.

The *cash-flow equivalent* approach consists in replacing, for valuation purposes, the (receiving) floating leg by receiving the notional at the period start and paying the notional at the period end. We would like to have a similar result in our new framework. To this end we have the following definition.

Definition 2.3 (Multiplicative coupon spread). *The* multiplicative spread *between a forward curve and the discounting curve is*

$$\beta_X^{\mathrm{CPN}, j}(t, u, v) = \left(1 + \delta F_X^{\mathrm{CPN}, j}(t, u, v)\right) \frac{P_X^D(t, v)}{P_X^D(t, u)} = \frac{1 + \delta F_X^{\mathrm{CPN}, j}(t, u, v)}{1 + \delta F_X^D(t, u, v)}. \tag{2.6}$$

This type of multiplicative spread is quite natural in the interest rate context. The factor β is the investment factor to be composed with the risk-free investment factor to obtain the Ibor investment factor. It could also be written as

$$\left(1 + \delta F_X^D(t, u, v)\right) \left(1 + \delta F_X^\beta\right) = \left(1 + \delta F_X^{\mathrm{CPN}, j}(t, u, v)\right)$$

for a clear definition of F_X^β.

Obviously the value of this variable is constant at 1 if $F_X^D = F_X^j$, in particular in the one-curve world. With the above definition, a j-Ibor coupon present value is

$$P_X^D(t, v) \delta F_X^{\mathrm{CPN}, j}(t, u, v) = P_X^D(t, v) \left(\beta_X^{\mathrm{CPN}, j}(t, v, u) \frac{P_X^D(t, u)}{P_X^D(t, v)} - 1\right)$$

$$= \beta_X^{\mathrm{CPN}, j}(t, u, v) P_X^D(t, u) - P_X^D(t, v).$$

The value is equal to the value of receiving β_X^j notional at the period start and paying the notional at the period end.

For the forward risk-free rate F_X^D defined above, the floating coupon price can also be written as

$$P_X^D(t,v)\delta\left(\beta_X^{\mathrm{CPN},j}(t,u,v)F_X^D(t,u,v) + \frac{1}{\delta}\left(\beta_X^{\mathrm{CPN},j}(t,u,v) - 1\right)\right)$$

which is the formula originally proposed in Henrard (2007).

The cash-flow equivalent description can be extended to the valuation of swaps. The present value of the swap from Equation (8.18) is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^D(t,\tilde{t}_i) - \sum_{i=1}^{n}\left(\beta_X^{\mathrm{CPN},j}(t,t_{i-1},t_i)P_X^D(t,t_{i-1}) - P^D(t,t_i)\right).$$

Grouping the terms with the same discount factor together and defining $\bar{n}$, $\bar{t}_i$ and d_i appropriately, one has

$$\sum_{i=0}^{\bar{n}} d_i(t)P_X^D(t,\bar{t}_i).$$

The quantities d_i depend on the curves and are not really constant. With some hypotheses that we will use later, the quantities will be constant and the situation will be in some sense equivalent to the one-curve world.

A consequence of hypothesis $\mathbf{I^{CPN}}$ and the definition of $\beta_X^{\mathrm{CPN},j}$ is that $\beta_X^{\mathrm{CPN},j}(.,t_{i-1},t_i)$ is a martingale in the $P_X^D(.,t_{i-1})$ numeraire. The Ibor coupon value is $\beta_X^{\mathrm{CPN},j}(t,t_{i-1},t_i)P_X^D(t,t_{i-1}) - P_X^D(t,t_i)$. The coupon is an asset due to hypothesis $\mathbf{I^{CPN}}$ and so its value divided by the numeraire $P_X^D(t,t_{i-1})$ is a martingale. The second term, the zero-coupon $P_X^D(t,t_i)$, is also an asset, hence its rebased value is also a martingale. The rebased first term is thus also a martingale and its value is $\beta_X^{\mathrm{CPN},j}P_X^D(t,t_{i-1})/P_X^D(t,t_{i-1}) = \beta_X^{\mathrm{CPN},j}(t)$. This proves that $\beta_X^{\mathrm{CPN},j}(.,t_{i-1},t_i)$ is a martingale under the $P^D(.,t_{i-1})$-measure.

On the other side, P_X^j cannot be an asset nor its rebased value a martingale. If it were, we would have for any numeraire N,

$$P_X^j(0,t) = N_0\,\mathrm{E}^N\left[N_t^{-1}P_X^j(t,t)\right]$$

$$= N_0\,\mathrm{E}^N\left[N_t^{-1}1\right]$$

$$= N_0\,\mathrm{E}^N\left[N_t^{-1}P_X^D(t,t)\right] = P_X^D(0,t)$$

that is, we would have two assets paying the same amount in t in all states of the universe and they would be equal at any previous date. The two curves P_X^D and P_X^j would be identical, which is a contradiction with the starting hypothesis that we have two different assets.

As with the one-curve framework, we can define a forward swap rate. This is the fixed rate for which the vanilla IRS price is 0:

$$S_t^j = \frac{\sum_{i=1}^n P_X^D(t,t_i)\delta_i F_X^{\text{CPN},j}(t,t_{i-1},t_i)}{\sum_{i=1}^{\tilde{n}} P_X^D(t,\tilde{t}_i)\tilde{\delta}_i}. \tag{2.7}$$

Note that the swap rates of multi-period swaps depend on both the forward curve and the discounting curve. In that aspect, the forward rates of Ibor coupons and the forward rates of multi-period swaps are fundamentally different.

The OIS swaps, paying a composition of overnight rates, have a different term sheet than the IRS described above. They are discussed in Section 2.7.

In practice: swap fixing

The description of swaps above is valid for swaps with one fixed leg and the other leg made of floating coupons only. In practice, this is not the case for most of the swaps on the trade date. The standard swaps are traded with an effective date equal to the spot date. The definition of spot is different for each currency and is described in Section B.8.

When the Ibor index fixing for the day has already taken place (after 11:00 am London time for Libor), the first coupon of the floating leg is not a floating coupon anymore but a fixed coupon. In that case, the value of the swap is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^D(t,\tilde{t}_i) - P_X^D(t,t_1)\delta_1 I_X^j(t^*) - \sum_{i=2}^n P_X^D(t,t_i)\delta_i F_X^{\text{CPN},j}(t,t_{i-1},t_i)$$

where t^* is the fixing time. In practice, it is important in the valuation, including when calibrating the curves, to clearly distinguish between the coupons that have already been fixed and the ones that have not.

More generally the swaps that have *aged* or *seasoned*, that is, swap after their trading date, will also have one or more of the floating coupons already fixed. Most of the time, one of the coupons is fixed. If the valuation time is between the fixing of one coupon, except the first one, and the payment date of the previous one, there will be two fixed coupons.

In practice: dates mismatch in swaps

In the description of j-Ibor floating coupons we indicated that the payment is done at the end of the fixing period, that is, we set our notation with the v used in P_X^D and the one used in $F_X^{\text{CPN},j}$ the same. As indicated in Appendix B.6, due to business day conventions, there can be a mismatch between those dates. The *payment date* (v in P_X^D) can be several days before the *fixing period end date* (v in $F_X^{\text{CPN},j}$). The difference between these two dates is often one or two days but can be up to six.

To avoid introducing cumbersome notation we will not make the distinction between these dates. In theory the mismatch should lead to convexity adjustment for those payments.

In Section 6.2 we analyse the impact of this convexity adjustment on pricing in the Gaussian HJM model. As the impact is negligible, we generally ignore it. In all the other computations we use the actual dates, computed using all the appropriate conventions, but ignore the related convexity adjustments.

2.5 Forward rate agreement

An FRA with *FRA discounting* settlement rule is an instrument linked to a period of length j, a fixing date t_0, an accrual factor δ and a fixed rate K. At the fixing date t_0, the Ibor rate $I_X^j(t_0)$ is recorded. The contractual payment in $u = \mathrm{Spot}(t_0)$ (the start date) received by the contract buyer is

$$\frac{\delta(I_X^j(t_0) - K)}{1 + \delta I_X^j(t_0)}.$$

More details about the FRA conventions are given in Appendix B.3.

The origin of the formula can be traced back to the computation of the value as the difference between the Ibor fixing and the fixed rate discounted at the Ibor rate from the end date to the settlement date. The rate is not paid at the end of its natural period but at the start, and is discounted with the fixing rate itself; we insist that the Ibor discounting used in the above formula is not a modelling choice but part of the contract term sheet. It is not directly a floating coupon or a one period swap as defined in the previous section. In that sense, the above definition of an FRA, in line with a market FRA term sheet, is different from those of Ametrano and Bianchetti (2009), Bianchetti (2010), Chibane et al. (2009) and Mercurio (2009).

We would like to value this instrument in a simple way in our economy. In a general contingent claim formula with a generic numeraire N and associated expected value $\mathrm{E}^N[]$, its value in 0 is

$$N_0 \, \mathrm{E}^N \left[N_u^{-1} \frac{\delta(I_X^j(t_0) - K)}{1 + \delta I_X^j(t_0)} \right].$$

Here we cannot use the usual trick of postponing the payment to $v = u + j$ by deciding to invest in t_0 from u to v at the rate $I_X^j(t_0)$ (multiplying by $1 + \delta I_X^j(t_0)$) and selecting $P_X^j(.,v)$ as the numeraire to simplify the formula. The reason is that the investment cannot be done at the Ibor rate but only at the risk-free rate.

Our goal at this stage is to obtain a relatively simple, coherent and practical approach to Ibor derivatives pricing. To achieve the simplicity, our next hypothesis is related to the spreads between the curves, as defined through the quantities $\beta_X^j(t)$. Other spread hypotheses will be analysed in later chapters.

S0$^{\text{CPN}}$: The multiplicative spreads $\beta_X^j(t, u, u+j)$, as defined in Equation (2.6), are constant through time: $\beta_X^j(t, u, u+j) = \beta_X^j(0, u, u+j)$ for all t and u.

The hypothesis **S0$^{\text{CPN}}$** can be viewed as the equivalent of the constant continuously compounded spread used in Henrard (2007). Here, the spread is not constant across maturities but deterministic and given by its initial values. The continuously compounded spread is $\ln(\beta_X^j(u, v))/(u - v)$. In that sense, the framework described here is a direct extension of the one developed in Henrard (2007) but adapted to the situation where the spreads are not equal for all maturities.

The hypothesis **S0$^{\text{CPN}}$** is equivalent to the hypothesis that β_X^j is deterministic. The equivalence can be obtained easily through a martingale argument. We know that $\beta_X^j(t, u, v)$ is a $P_X^D(., u)$ martingale. If $\beta_X^j(., u, v)$ is deterministic, $\beta_X^j(0, u, v) = \mathrm{E}^u\left[\beta_X^j(t, u, v)\right] = \beta_X^j(t, u, v)$. This proves that when β is deterministic, it is constant in t.

Theorem 2.2 *Under hypotheses D, I$^{\text{CPN}}$ and S0$^{\text{CPN}}$ the price in t of the FRA of length j with fixing date t_0, accrual factor δ and reference rate K is*

$$P_X^D(t, u)\frac{\delta(F_X^{\text{CPN},j}(t, u, v) - K)}{1 + \delta F_X^{\text{CPN},j}} = P_X^D(t, v)\frac{\delta(F_X^{\text{CPN},j}(t, u, v) - K)}{\beta_X^j(t, u, v)}$$

where $u = \mathrm{Spot}(t_0)$ and $v = \mathrm{Spot}(t_0) + j$.

Proof: We use the strategy to invest in t_0 the value from u to v. With the $P_X^D(., v)$ numeraire, the value is

$$P_X^D(t, v)\,\mathrm{E}^v\left[(1 + \delta F_X^D(t_0, u, v))\frac{\delta(I_X^j(t_0) - K)}{1 + \delta I_X^j(t_0)}\,\bigg|\,\mathcal{F}_t\right].$$

We replace $I_X^j(t_0)$ by $F_X^{\text{CPN},j}(t_0, u, v)$ and use the definition of $\beta_X^j(t)$ to obtain

$$P_X^D(t, v)\,\mathrm{E}^v\left[(1 + \delta F_X^D(t_0, u, v)) - (1 + \delta K)(\beta_X^j(t_0, u, v))^{-1}\,\bigg|\,\mathcal{F}_t\right].$$

We use the constancy of β_X^j and the martingale property of F_X^D to obtain the results.
□

The above result relies heavily on hypothesis **S0$^{\text{CPN}}$**. Even if the floating coupon value exists by **I$^{\text{CPN}}$** and the related definition, this is not enough to price the

FRA; an extra hypothesis is necessary to obtain a simple formula. Discounting the FRA with the Ibor rate between the start date and the end date creates an adjustment represented by the coefficient β_X^j. The above formula is used by most systems decoupling discounting and forward curves. The hypothesis **S0$^{\text{CPN}}$** justifying such a formula is seldom provided.

(Mercurio 2010a, Appendix A) analyses the impact of non-constant spread on the valuation of market FRAs. His analysis shows that the adjustment required when the spread is not constant depends on the spread size and its volatility. With realistic market data, he obtains adjustments of up to 0.5 bps for a ten year FRA. For FRAs below one year, as typically used in curve constructions, the adjustment he obtains is below 0.1 bps.

An FRA is said to be at-the-money when the rate K is such that the instrument value is 0. Using the above result, this is the case when

$$K = F_0^j = \frac{1}{\delta} \left(\beta_X^j(t_1, t_2) \frac{P^D(0, t_1)}{P^D(0, t_2)} - 1 \right).$$

This is essentially the FRA fair rate obtained in Mercurio (2009) with, in his notation,

$$\beta_X^j = \frac{1}{R + (1 - R)\, \mathrm{E}^{\cdot}\, [Q(t_1, t_2)]}.$$

The adjustment factor above is linked in Mercurio (2009) to credit-related parameters such as recovery rate and default probability. It is also linked to the quantity $K_L(t, t_1, t_2)$ defined in (Kijima et al., 2009, Equation (4.7)). In their article the spread is linked to some model parameters; in the approach described here the spread is fitted to the market curves.

Note also that to compute the FRA fair rate, it is enough to know the forward curve F_X^j while to compute its present value, one needs to know both the forward curve and the discounting curve.

2.6 STIR Futures

A general pricing formula for interest rate futures in the one-factor Gaussian HJM model in the one-curve framework was proposed in Henrard (2005). The formula extended a previous result proposed in Kirikos and Novak (1997). The pricing in a displaced diffusion LMM with skew is analysed in Jäckel and Kawai (2005). Piterbarg and Renedo (2004) analyse the pricing of futures in some general stochastic volatility model to study the impact of the smile; they also analyse the pricing of options on futures.

The Gaussian HJM formula was extended to the multi-curve framework in Henrard (2007). In this section, we describe the pricing of futures under the hypotheses

ICPN and **S0CPN** in a multi-factor Gaussian HJM multi-curve framework. The exact notation for the multi-factor Gaussian HJM framework used here is given in Appendix 9. The pricing of the futures in the LMM with stochastic basis is proposed in Mercurio (2010b). In Mercurio and Xie (2012), the pricing of futures for an additive stochastic basis spread is analysed. The pricing of futures in a framework with multiplicative stochastic basis is proposed in Chapter 7.

The future *fixing* or *last trading date* is denoted t_0. The fixing is on the Ibor rate between $u = \text{Spot}(t_0)$ and $v = u + j$. The fixing accrual factor for the period $[u, v]$ is δ. More details on the STIR futures conventions can be found in Appendix B.4.

The futures price is denoted Φ_t^j. On the fixing date, the relation between the futures price and the fixing rate is

$$\Phi_{t_0}^j = 1 - I_X^j(t_0).$$

The futures margining is done on the futures price (multiplied by the notional and the futures accrual factor).

When analysing futures prices, we will use the generic futures price process theorem (Hunt and Kennedy 2004, Theorem 12.6), which states that

$$\Phi_t^j = \text{E}^{\mathbb{N}}\left[\left.\Phi_{t_0}^j\right|\mathcal{F}_t\right]$$

where $\text{E}^{\mathbb{N}}[]$ is the cash account numeraire expectation. A generalisation of this theorem is proved in Chapter 8. The formula above can be obtained as a special case of formula (8.3) for a collateral rate $c_t = 0$. The link between pricing under collateral and futures is explained in that chapter.

Theorem 2.3 *Let $0 \leq t \leq t_0 \leq u < v$. In the multi-factor Gaussian HJM multi-curve framework under the hypotheses D, I^{CPN} and $S0^{CPN}$, the price in t of the futures fixing in t_0 for the period $[u, v]$ with fixing accrual factor δ is given by*

$$\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t, u, v))\gamma(t, t_0, u, v) \tag{2.8}$$

where the convexity adjustment factor $\gamma(t, t_0, u, v)$ is given in Appendix A.

Proof: The above number can be written as

$$1 - I_X^j(t_0) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\left(1 + \delta I_X^j(t_0)\right)$$

$$= 1 + \frac{1}{\delta} - \frac{1}{\delta}\left(1 + \delta F_X^{\text{CPN},j}(t_0, u, v)\right).$$

The investment factor $1 + \delta F_X^j(t_0, u, v)$ can be written as the product of $\beta_X^j(t_0)$ and a ratio of discount factors. Using Lemma A.1, the ratio of discount factors is

$$\frac{P^D(t_0, v)}{P^D(t_0, u)} = \frac{P^D(t, v)}{P^D(t, u)} \exp\left(-\alpha(t, t_0, u, v)X_{t, t_0} - \frac{1}{2}\alpha^2(t, t_0, u, v)\right)\gamma(t, t_0, u, v)$$

By hypothesis $\mathbf{S0^{CPN}}$, the coefficient $\beta_X^j(t_0)$ is constant. The expected value of the above exponential is 1 and the announced result follows. □

The pricing formula mixes the rate in the forward curve $F_X^{\mathrm{CPN},j}$ and the constant γ. The constant depends on the model parameters of the discounting curve P^D. The result incorporates the forward rate, the discounting curve model and the interaction between them. The result strongly depends on the interaction between the forward curve and the discounting curve given by $\mathbf{S0^{CPN}}$.

The convexity adjustment is the difference between the futures rate and the forward rate, that is, the convexity adjustment is given by

$$1 - \Phi_t^j - F_X^{\mathrm{CPN},j}(t,u,v) = \frac{1}{\delta}(1 + \delta F_X^{\mathrm{CPN},j}(t,u,v))(\gamma(t,t_0,u,v) - 1).$$

The adjustment is positive when $\gamma > 1$.

The quantity γ, given by Equation (A.3), is in most practical cases above 1. Recently the convexity adjustment has been negative in EUR as described in Carver (2013). Pozdnyakov and Steele (2009) describe technical conditions in which the adjustment in the one-curve world is positive. From the formula of γ, it will be the case if $v(s,u) \cdot v(s,v) > |v(s,u)|^2$ for all s ($t \le s \le t_0$). This can be interpreted geometrically as the projection of $v(s,v)$ on $v(s,u)$ is larger than $v(s,u)$, or, for a certain definition of 'volatility', the volatility of the longer term bond of maturity v is larger than the shorter bond of maturity u.

In Chapter 7 we will propose several other approaches to futures pricing when the spread between the curves, represented by β, is not constant but stochastic.

2.7 Overnight indexed swaps

The coupons of OISs differ from those of Ibor swaps. The coupons on an OIS are computed by compounding the overnight rates and are paid at the end of a given period (often three months or one year). Moreover, the payment is done with a lag (usually two days after the last fixing publication in EUR and USD). More details on the OIS conventions can be found in Appendix B.9.

In this section we analyse the impact of the compounding on the pricing. The impact of compounding in the multi-curve framework was originally analysed in Quantitative Research (2012d).

The impact of the delayed payment is ignored here. It was proved in Henrard (2004) that the adjustment is minimal and can be neglected in practice. We will come back to the feature and propose its analysis in Chapter 6.

The description of an overnight indexed coupon is as follows. The times associated are denoted t_i ($i = 0, \ldots, n$). They correspond to successive business days

in the relevant calendar. The fixing for the period $[t_{i-1}, t_i]$ is denoted $I_X^O(t_{i-1})$ $(i = 1, \ldots, n)$ and the accrual factor in the index convention is δ_i.

The overnight indexed coupon pays the amount

$$\left(\prod_{i=1}^{n} (1 + \delta_i I_X^O(t_{i-1})) \right) - 1$$

in t_p. In this section we suppose that $t_p = t_n$.

Let $P_X^O(s, t)$ denote the OIS forward curve in the multi-curve framework. The meaning of OIS forward curve is the one described in $\mathbf{I}^{\mathbf{CPN}}$ and its related definition: the pricing of a one period (from t_s to t_e, paying $\delta_e I_e^O$) overnight index coupon is given in s by

$$P_X^D(s, t_e) \delta F_X^O(s, t_s, t_e).$$

As before we define

$$\beta_X^O(s, t, t + 1d) = (1 + \delta F_X^O(s, t, t + 1d)) \frac{P_X^D(s, t + 1d)}{P_X^D(s, t)} \tag{2.9}$$

where $t + 1d$ has to be understood as t plus one business day.

We work under the constant spread hypothesis $\mathbf{S0}^{\mathbf{CPN}}$ with $j = O$. The present value of the coupon (in the $P_X^D(., t_n)$ numeraire) is given by

$$P_X^D(0, t_n) \, \mathrm{E}^n \left[\prod_{i=1}^{n} \left(1 + \delta_i F_X^O(t_{i-1}, t_{i-1}, t_i) \right) - 1 \right] \tag{2.10}$$

Obviously the part of the value that requires analysis is the product. Using the definition of β^O and the constance of $\beta_X^O(s, t, t + 1d)$ over time, we have

$$\mathrm{E}^n \left[\left(\prod_{i=1}^{n} (1 + \delta_i F_X^O(t_{i-1}, t_{i-1}, t_i)) \right) \right]$$

$$= \mathrm{E}^n \left[\left(\prod_{i=1}^{n} \beta_X^O(t_{i-1}, t_i) \frac{P_X^D(t_{i-1}, t_{i-1})}{P_X^D(t_{i-1}, t_i)} \right) \right]$$

$$= \mathrm{E}^n \left[\left(\prod_{i=1}^{n-1} \beta_X^O(t_{i-1}, t_i) \frac{P_X^D(t_{i-1}, t_{i-1})}{P_X^D(t_{i-1}, t_i)} \beta_X^O(t_{n-1}, t_n) \right. \right.$$

$$\left. \left. \times \mathrm{E}^n \left[\frac{P_X^D(t_{n-1}, t_{n-1})}{P_X^D(t_{n-1}, t_n)} \middle| \mathcal{F}_{t_{n-2}} \right] \right) \right]$$

$$= \mathrm{E}^n \left[\left(\prod_{i=1}^{n-1} \beta_X^O(t_{i-1}, t_i) \frac{P_X^D(t_{i-1}, t_{i-1})}{P_X^D(t_{i-1}, t_i)} \beta_X^O(t_{n-1}, t_n) \frac{P_X^D(t_{n-2}, t_{n-1})}{P_X^D(t_{n-2}, t_n)} \right) \right]$$

$$= \mathrm{E}^n \left[\left(\prod_{i=1}^{n-2} \beta_X^O(t_{i-1}, t_i) \frac{P_X^D(t_{i-1}, t_{i-1})}{P_X^D(t_{i-1}, t_i)} \beta_X^O(t_{n-2}, t_{n-1}) \right. \right.$$

$$\left. \left. \times \beta_X^O(t_{n-1}, t_n) \frac{P_X^D(t_{n-2}, t_{n-2})}{P_X^D(t_{n-2}, t_n)} \right) \right]$$

$$= \mathrm{E}^n \left[\left(\prod_{i=1}^{n-2} \beta_X^O(t_{i-1}, t_i) \frac{P_X^D(t_{i-1}, t_{i-1})}{P_X^D(t_{i-1}, t_i)} \beta_X^O(t_{n-2}, t_{n-1}) \right. \right.$$

$$\left. \left. \times \beta_X^O(t_{n-1}, t_n) \frac{P_X^D(t_{n-3}, t_{n-2})}{P_X^D(t_{n-3}, t_n)} \right) \right].$$

In the second equality we use the fact that $P_X^D(s, t)$ is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and that $\beta_X^O(t_{n-1}, t_n)$ is constant. In the third equality we use the martingale property of the asset $P_X^D(., x)$ in the $P_X^D(., t_n)$ numeraire. The fourth equality is simply the simplification of fractions. In the fifth equality we use the same technique as in the second and third.

The same procedure as the one in the second and third equalities can be repeated down for all factors in the product and we obtain

$$\mathrm{E}^n \left[\frac{P_X^D(t_0, t_0)}{P_X^D(t_0, t_n)} \left(\prod_{i=1}^{n} \beta_X^O(t_{i-1}, t_i) \right) \right].$$

After once more applying the martingale property, we obtain the following result.

Theorem 2.4 *Under the hypotheses **D**, **I**^{CPN} and **S0**^{CPN}, the price of the overnight indexed coupon described above is given by*

$$P_X^D(0, t_n) \left(\prod_{i=1}^{n} (1 + \delta_i F_X^0(0, t_{i-1}, t_i)) - 1 \right).$$

This means that with the hypothesis **S0**^{CPN}, the overnight indexed coupon can be priced as if the forward rate were composed and then discounted.

When the pseudo-discount factors description of the overnight forward curves is used, as described in Chapter 3, the value of the coupon can be simplified further.

The value can be computed directly as a ratio of discount factors, without requiring to compute the actual composition of the daily rates. In practice, when dealing with overnight forward curves, we use the pseudo-discount factor approach. This is faster due to the above simplification. Moreover as the index is very short term, there is virtually no arbitrary part in the pseudo-discount factor curve.

2.8 Forex and cross-currency swaps

Up to now, everything discussed is valid for any currency and we have reviewed only single-currency instruments. We have also supposed that the risk-free curves are given.

For cross-currency instruments like forex swaps and cross-currency swaps, cash-flows in different currencies are involved. Our convention is that the present value of each cash-flow is evaluated in its own currency; at this stage no conversion into another currency is done.

Forex spot, forward or swap are simply sets of fixed cash-flows in each currency. The cash-flows in one currency are valued using the relevant discounting curve and produce a result in that currency. The total present value is two amounts in two different currencies.

As described in Appendix B.14, the forex swaps are mainly interest rate products. The main market information they convey is the difference in interest rate for a given period between two currencies. As such, they create links between the curves in different currencies.

Through multi-currency instruments a link between the discounting curves in different currencies is imposed by the market. To keep the coherence of the framework, the curve in one currency follows from the choice in any other currency. The valuation of cross-currency swaps is done as for single-currency swaps. The present value of each coupon is computed (in its own currency) using the previous formulas, and they are added. The total present value is two amounts, one in each of the swap currencies. To convert the multi-currency amount to one currency, today's exchange rate is used.

The existence of par forex and cross-currency swaps, that is, with zero total present value, will impose a relationship between discounting curves in different currencies and exchange rates. Note that it is the total converted present value which is null, not each of the present values in each currency. Each leg of a forex swap transaction does not have a zero present value. Even if the forex swaps are mainly interest rate products, and this is why they are used in curve calibration, they create small currency exposures. The currency exposure exists from the trading moment and does not only appear due to market movements. The only case where there is no currency exposure is when the interest rate in the first currency of the swap is zero over the swap period. Paying one amount and receiving the same amount at

the end is a fair deal by itself, without introducing the second currency. This is a very rare occurrence.

The introduction of multi-currency instruments and the multi-currency coherence they impose also means that interest rates are becoming forex rate dependent; changing the exchange rate and keeping the market quotes of forex and cross-currency swaps will have an impact on the interest rate obtained. Some remarks on the forex sensitivity of interest rate curves is proposed in the curve calibration chapter.

3 | Variation on a Theme

Up to now, we have worked with generic forward curves $F_X^{\text{CPN},j}(t, u, v)$ and we have not discussed their implementation. If there were infinitely many market instruments, one for each starting date u – or even better, one for each starting time – the general curve would be a good enough description of the full economy. The market would provide the forward rate for each possible date and the curve description would merely be a data storage and not a data modelling tool.

In practice, there are a lot less market instruments on which we can build the forward curves; there are a lot less than one instrument for each time u. Often one has monthly or quarterly information on the short part of the curve, provided by FRA, futures or short term swaps, and annual information at best on the long part of the curve, provided by swaps or swap futures. One needs to resort to a modelling mechanism of some sort to find the intermediary value, often combined with interpolation. Chapter 4 discusses the subject of interpolation.

In the next two sections, we outline two ways to describe the curves; the two methods lead to different implementations. Each of them can be combined with different interpolation schemes. This dual approach was initially described in Henrard (2012c). A third approach is proposed in Fries (2013) based on coupons present values.

The first implementation uses pseudo-discount factors. This is certainly the most commonly used approach in practice; in a lot of literature and software packages it is even the definition of multi-curve framework. Starting from the description of the multi-curve framework we proposed in the previous chapter, it may seems strange to use such a convoluted approach to describe the curves. This is mainly due to the *evolution* of the framework from the one-curve framework. It allows us to have the same description of the curves for the discounting curve and the forward curves. As we will see in the next chapter, a lot of the curve interpolation literature focuses on the impact of interpolation in that specific framework.

The second implementation, which in many ways should really be the first, is to describe the forward curve directly by modelling the forward rates themselves. As we will see later in this chapter and in the next one, many of the drawbacks of simple

interpolation, such as linear interpolation, can be reduced by using the direct curve description in terms of forward rates. Some similarities between discounting and forward curves are lost. As the theoretical descriptions of the two types of curve are very different, having different implementations is not a problem. On the positive side, the intuition of the forward curves is more direct and not obtained through a ratio of discount factors.

In the last section of this chapter we describe an approach to the multi-curve framework different from the one described in the rest of the book. The section can be viewed as a side story, where the story of the main character's cousin is told. It can be skipped without losing essential information for the rest of this book. Nevertheless it is an interesting simplified version of the 'real world' where there is not only one set of instruments but several sets distinguished by their collateral rules. This explanation will really make sense only after Chapter 8. In the alternative framework, the forward curves are based on STIR futures instead of being based on Ibor coupons. This is certainly a meaningful approach if one has a portfolio composed mainly of futures with only a limited number of swaps.

3.1 Forward curves through pseudo-discount factors

The pseudo-discount factor forward curves are defined as follows.

Definition 3.1 (Coupon pseudo-discount factor curves). *The forward curve $P_X^{\text{CDF},j}$ (t,s) is the continuous function defined for $t \leq s$ such that $P_X^{\text{CDF},j}(t,t) = 1$, $P_X^{\text{CDF},j}(t,s)$ is an arbitrary strictly positive function for $t \leq s < \text{Spot}(t) + j$, and for $t_0 \geq t$, $u = \text{Spot}(t_0)$ and $v = u + j$ one has*

$$
F_X^{\text{CPN},j}(t,u,v) = \frac{1}{\delta} \left(\frac{P_X^{\text{CDF},j}(t,u)}{P_X^{\text{CDF},j}(t,v)} - 1 \right). \tag{3.1}
$$

The origin of the above definition can be traced back to the one-curve world, where the forward rate was written with a similar formula described by Equation (2.1). Nevertheless the substance of the two formulas is very different. The one-curve formula is a result obtained from different hypotheses and from no arbitrage condition. The above formula is merely a definition. The pseudo-discount factor should be viewed as the 'wrong number used in the wrong formula to obtain the correct result' type of approach. The formula is better understood through the *evolution* than through the *foundations*.

Definition 3.1, which refers to an arbitrary function, is itself arbitrary in more than one way. The definition fixes the first j-period as the arbitrary part; one could instead fix any other j period and deduce the rest of the curve from there.

In Baviera and Cassaro (2012) the authors propose to use the arbitrariness on the second period by imposing a specific interpolation in that period and building the first period in a way similar to our definition. One could even take an arbitrary decomposition of the j-period time interval into subintervals and distribute those subintervals arbitrarily on the real axis in such a way that, modulo the j-periods, they recompose the initial j-period. As another arbitrary choice, one could also change the value of $P_X^{\text{CDF},j}(t,t)$ to any value different from 1. As only the ratios between two values are used and never a value on its own, the choice of initial value has no impact on the end results. One could also impose an arbitrary value for $P_X^{\text{CDF},j}$ in $(t,\text{Spot}(t))$ instead of in (t,t). As the curve is used only for forward computation, it is used only with time $s > \text{Spot}(t)$; the value for shorter times is irrelevant.

This arbitrariness of the pseudo-discount factors makes it very difficult to design a model for those values. All the relevant models I'm aware of model the value $F_X^{\text{CPN},j}(t,u,v)$ and not the pseudo-discount factors $P_X^{\text{CDF},j}(t,u)$. This modelling problem with pseudo-discount factors is discussed further in Section 7.1.

Curve calibration

The curve calibration is the subject of Chapter 5. In this section we describe a very simple curve calibration to emphasise the difference between pseudo-discount factor curve description and the direct rate curve description of the next section.

A relatively standard way to calibrate the curves P_X^D and $P_X^{\text{CDF},j}$ is to select a set of market instruments for which the market quotes are known and an equal number of node points. An interpolation scheme is selected and the discount factors (or the associated rates) on the node points are calibrated to reproduce the market quotes. The market forward rates $F_X^{\text{CPN},j}(0,u,u+j)$ can be computed from the forward pseudo-discount factors curve using Formula (3.1). The pseudo-discount factors are calibrated using the formula and the instruments are priced from the pseudo-discount factors using the same formula. Everything is coherent and we could stop here. The big difference with a direct approach is the interpolation. Do you interpolate the pseudo-discount factors or the forward rates?

A typical forward rate curve using the pseudo-discount factors approach is displayed in black in Figure 3.1. The swap data used to calibrate the curve are those proposed in (Andersen and Piterbarg, 2010, Section 6.2) in a one-curve setting. The interpolation scheme is linear on (continuously compounded) rates. For our calibration, the swaps are fixed versus three months Ibor. We use a discounting curve with flat OIS market rates at 4%. The discounting curve is mostly irrelevant for the discussion in this section.

The familiar sawtooth pattern can be seen. There are two angles in the curve for each node point. One when the fixing period end date is on a node and one when the fixing period start date is on the node. One of the reasons for this unpleasant
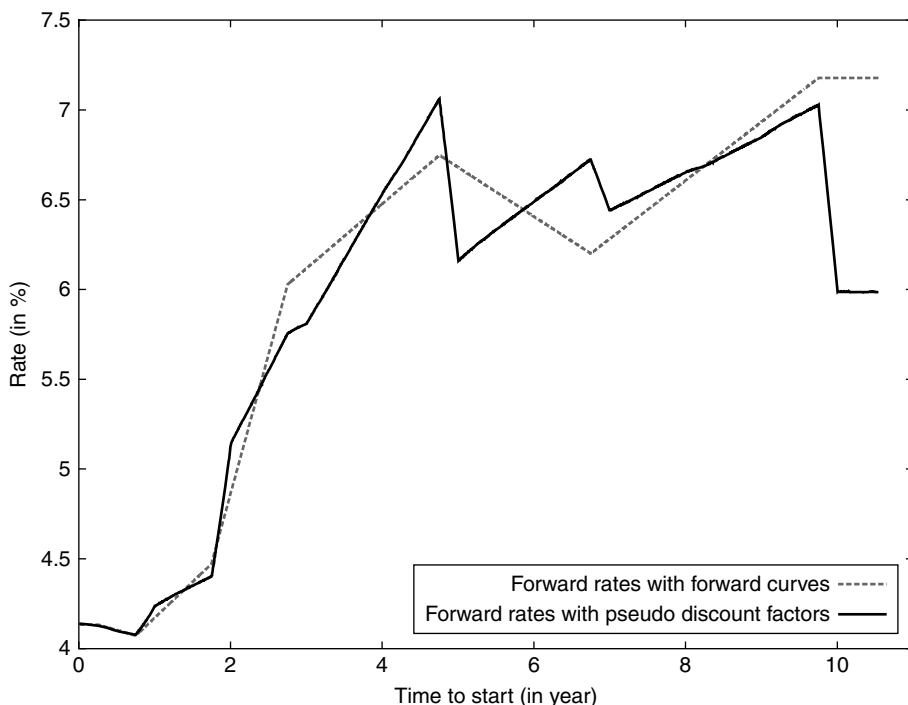
Figure 3.1 Forward Ibor three months rates computed using pseudo-discount factors in black and direct forward rate curve in dotted line. Both curves are calibrated using the same market data (see text).

shape is probably that we have an intuition on a market quantity – the forward rate – but model it indirectly through a ratio of discount factors where our intuition is diluted. The graph represents the forward rate, which is of interest to the trader and risk manager, but the data is stored and calibrated using a different mechanism.

3.2 Direct forward curves

In this approach we do not need an intermediary function. We work directly with the curve $F_X^{\text{CPN},j}(0,u,v)$, with $u > \text{Spot}(0)$ the variable of this one-dimensional function. This approach, which seems the natural approach in the multi-curve framework, has not been documented until recently and software implementations seems to mostly ignore it.

Note that in this framework, the *Ibor discounting* is impossible as there is no discount factor associated to the Ibor curves. There is no longer an arbitrary part

to the curve. The curve is defined unambiguously (as long as the corresponding market instruments exist) for all $u \geq \text{Spot}(0)$.

> Modeling the forward rate by constructing the discount or spot rates is similar to fixing a pair of eyeglasses while wearing gloves. [...] Instead, if one models the forward curve directly and embeds all of the desired properties into it, then by construction the spot and discount rates will have the characteristics that the modeler requires.
>
> *Jerassy-Etzion, Stripping the Yield Curve With*
> *Maximally Smooth Forward Curves, 2010*

Curve calibration

The advantage of this approach is that the market rates on which we have some intuition are modelled directly. In some senses, and borrowing a well known name, it could be called the *Libor Market Model* of forward curve description – not of curve dynamic as its namesake.

There is no longer a requirement for an arbitrary part like in Definition 3.1 of the pseudo-discount factor approach. The interpolation and constraints can be imposed directly on the market quantities. In Figure 3.1, the forward rate using the same data as the previous approach and the same linear interpolation scheme, but with the interpolation directly on the forward rate, is presented in dotted line.

It is to each market maker or risk manager to decide which one he prefers. With the reported data, the forward rates display less sawtooth effect with the direct forward rate approach. With some other market rates, the picture may be different.

In Table 3.1 we give the figures of the maximum variation of the three months rates three months apart. It is expected that the view of three months rates at some distant time in the future will not change too widely for relatively small differences in dates. Larger variations may indicate a problem with the way we estimate forward curves.

The three month period was chosen as this is the problematic period artificially introduced by the specific form of Definition 3.1. As can be seen from the numbers,

Table 3.1 The maximum changes of three months rates over a three month period – maximum increase and maximum decrease

Interpolation	Pseudo-discount factors		Direct forward	
	Increase	Decrease	Increase	Decrease
Linear	69.1	104.1	37.4	27.5
Natural cubic spline	36.7	15.4	38.5	17.5

Figures in basis points.
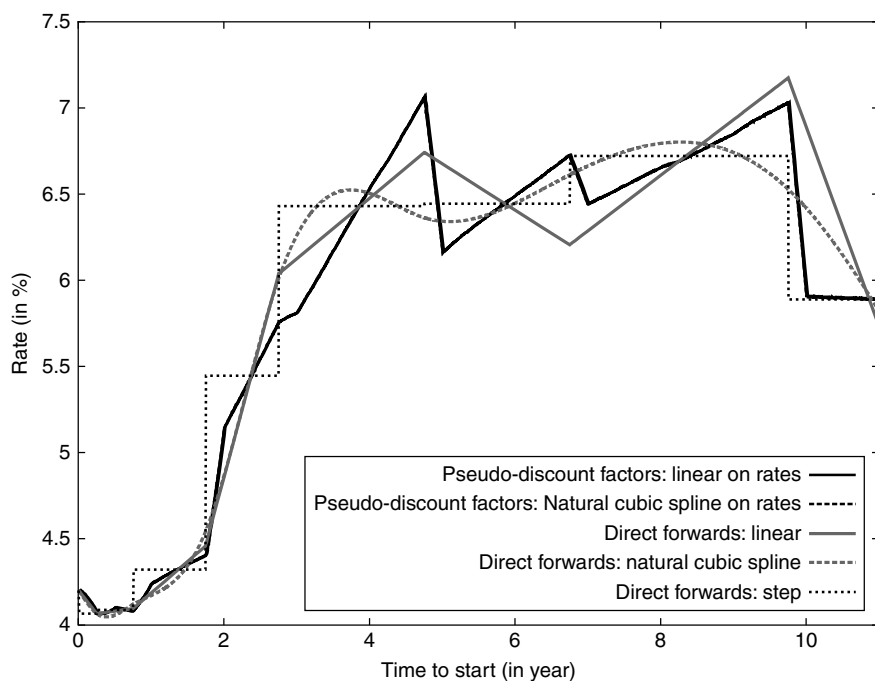
Figure 3.2 Forward Ibor three months rates computed using pseudo-discount factors (in black) and direct forward rate curve (in grey) with natural cubic spline interpolation. The dashed lines are using linear interpolation and the dotted line a step function.

the interpolation on pseudo-discount factors introduces a significantly larger variation. The figures using the linear interpolation on the direct forward rate are not very different from the cubic spline interpolation on the pseudo-discount factors.

In Figure 3.2, we did the same comparison as in the previous figure, this time using a smoother interpolation mechanism. The interpolation scheme is natural cubic spline.

For the direct rates, we have also added an interpolation through step function (dotted line). As the curve represents the forward rate and not the rate to maturity, the step function is not necessarily the worst choice. In all cases, it avoids unnecessary swings in the forward rate function.

The choice of the curve implementation and interpolation mechanism not only has an impact at the level of the forward rates but also on the implied hedging. One quality often requested from interpolation mechanisms is locality; this is for example the third desirable feature quoted in Hagan and West (2006). In Table 3.2 we give the sensitivity to the instrument quotes used to build the forward curve using the two implementations proposed above and two interpolation schemes. The instrument we want to hedge is a 1Yx5Y forward swap.

Table 3.2 Sensitivity to the change of each market quote input (rescale to a one basis point change)

Impl.	Int.	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	12Y
DF	Lin.	1	−2	−9,659	−17	−45	26,404	26,478	0	0
	Spl.	−386	1,151	−11,519	4,942	−9,727	33,231	28,468	−4,077	1,075
Direct	Lin.	−344	2,036	−15,066	14,803	−24,220	48,696	17,256	0	0
	Spl.	−7,639	19,156	−31,931	27,677	−29,480	44,723	21,918	−1,616	297

DF: Pseudo-discount factors implementation; Direct: Direct forward rate implementation. Lin: Linear interpolation; Spl.: Natural cubic spline.

From the localisation perspective, the best behaviour is probably the one of linear interpolation with pseudo-discount factors. The cubic spline create non-local interpolation both before the first payment and after the last payment. The linear interpolation on direct rate create non-local effects before the first payment but not after the maturity. The cubic spline on direct rates creates even larger non-local effects.

As seen through the forward rate graph and the hedging properties, there is no perfect solution to the curve representation problem. But certainly the direct interpolation on rate opens a new range of possibilities to explore before making an educated choice on which approach to use in practice.

3.3 Futures multi-curve framework

In this section, we describe an alternative framework. Unlike the main framework of this book it is not based on Ibor coupons, but on STIR futures.

The existence hypothesis in this framework replaces hypothesis $\mathbf{I^{CPN}}$. It links Ibor futures to martingale futures price processes in the sense of Hunt and Kennedy (2004, Section 12.4).

$\mathbf{I^{FUT}}$: The prices of the j-Ibor futures are martingale futures price processes for each fixing date.

From the existence hypotheses, we can define the futures rates $F_X^{\mathrm{FUT},j}$. The description and market conventions of STIR futures can be found in Appendix B.4. The futures price in t for a futures with settlement date u is denoted $\Phi_t^j(u)$.

Definition 3.2 (Futures rate). *The futures rate $F_X^{\mathrm{FUT},j}(t, u)$ is the number such that for $t_0 \geq t$, $u = \mathrm{Spot}(t_0)$ and $v = u + j$ we have*

$$\Phi_t^j(u) = 1 - F_X^{\mathrm{FUT},j}(t, u, v). \tag{3.2}$$

The futures rate is obtained directly from the futures price without convexity adjustment.

With that definition, the link between $I_X^j(t_0)$ and $F_X^{\text{FUT},j}(t,u)$ is

$$I_X^j(t_0) = F_X^{\text{FUT},j}(t_0, u, v).$$

On the fixing date, the futures rate is equal to the fixing rate. Obviously no convexity adjustment is required as there is no margining time left.

Like in the coupon framework, we define a spread as follows.

Definition 3.3 (Futures multiplicative spread). *The variable* $\beta_X^{\text{FUT},j}(t,u,v)$, *called the* (multiplicative) *spread, is defined as a ratio*

$$\beta_X^{\text{FUT},j}(t,u,v) = (1 + \delta F_X^{\text{FUT},j}(t,u,v)) \frac{P_X^D(t,v)}{P_X^D(t,u)}. \tag{3.3}$$

In the coupon framework an hypothesis often used is that the ratios $\beta_X^{\text{CPN},j}$ are constant through time (hypothesis **S0$^{\text{CPN}}$**). We propose to use a similar hypothesis here: a deterministic spread hypothesis.

SD$^{\text{FUT}}$: The multiplicative spreads $\beta_X^{\text{FUT},j}(t,u,v)$ are deterministic for all u.

For reasons explained below, one cannot have a constant spread hypothesis in this case, except with constant spread at 0.

An Ibor coupon pays the amount $\delta I_X^j(t_0)$ in v. Its today's value is given by the following theorem.

Theorem 3.1 (**Coupon value**). *In the futures multi-curve framework, under the hypotheses* **D**, **I^{FUT}** *and* **SD^{FUT}**, *the price of the Ibor coupon fixing in t_0 for the period* $[u,v]$ *is given by*

$$P_X^D(0,u)\beta_X^{\text{FUT},j}(t_0,u,v) - P^D(0,v).$$

Proof: The proof is immediate. It is enough to use the link between $I_X^j(t_0)$ and $F_X^{\text{FUT},j}$, use the definition of $\beta_X^{\text{FUT},j}$ and take the expectation with $P_X^D(.,v)$ as numeraire. The result follows from the martingale property of the rescaled asset $P_X^D(.,u)$ under the $P_X^D(.,v)$ numeraire. $\qquad\square$

The formula is very close to the one for the Ibor coupon in the coupon multi-curve framework. The difference is that here the $\beta^{\text{FUT},j}$ is taken in t_0, not in 0. We mentioned above that the quantity is not constant, so the two formulas are different. This is to be expected as the definitions of $F_X^{\text{CPN},j}$ and $F_X^{\text{FUT},j}$ are different.

Suppose we use a Gaussian HJM model on the discounting curve. The model is described in Appendix A. From Lemma A.1, it can be seen that $\gamma(.,0,v,u) P_X^D(.,u)/P_X^D(.,v)$ is an N-martingale.

We have

$$1 + \delta F_X^{\text{FUT},j}(t,u,v) = \frac{P^D(t,u)}{P^D(t,v)}\beta_X^{\text{FUT},j}(t) = \frac{P^D(t,u)}{P^D(t,v)}\gamma(t,0,v,u)\frac{\beta_X^{\text{FUT},j}(t)}{\gamma(t,0,v,u)}$$

with the first term a N-martingale. Using the martingale properties and the deterministic hypothesis **SD**$^{\text{FUT}}$, we have

$$\frac{P^D(0,u)}{P^D(0,v)}\beta_X^{\text{FUT},j}(0) = \frac{P^D(0,u)}{P^D(0,v)}\gamma(0,0,v,u)\frac{\beta_X^{\text{FUT},j}(t)}{\gamma(t,0,v,u)}.$$

This means that $\beta_X^{\text{FUT},j}(t) = \beta_X^{\text{FUT},j}(0)\gamma(t,0,v,u)$. In the Gaussian HJM model, the hypothesis **SD**$^{\text{FUT}}$ is equivalent to the following hypothesis.

HJM1: The quantities $\beta_X^{\text{FUT},j}(t,u,v)$ are such that

$$\beta_X^{\text{FUT},j}(t,u,v) = \beta_X^{\text{FUT},j}(0,u,v)\gamma(t,0,v,u).$$

This hypothesis is equivalent to the Gaussian HJM hypothesis often done in curve construction to price futures.

Theorem 3.2 *In the futures multi-curve framework, under the hypotheses* **D**, **I**^{FUT} *and* **HJM1**, *and in the Gaussian HJM model on the discounting curve, the price of the Ibor coupon fixing in t_0 for the period $[u,v]$ is given by*

$$P^D(0,u)\frac{\beta_X^{\text{FUT},j}(0,u,v)}{\gamma(0,t_0)} - P^D(0,v)$$

$$= P^D(0,v)\delta\frac{1}{\gamma(0,t_0)}\left(F_X^{\text{FUT},j}(0) + \frac{1}{\delta}(1 - \gamma(0,t_0))\right).$$

The convexity adjustment is now done on the Ibor coupon, not on the futures. Note that the adjustment is obtained by dividing by the coefficient $\gamma(0,t_0)$ and not multiplying by it. Should the adjustment be called a *concavity adjustment*?

Zero rate collateral

There is one scenario where this framework could become the standard framework. There is a push for more standardisation of the products and their legal terms – Credit Support Annex (CSA) in particular. One particular discussion is around the CSA collateral renumeration. In the current standard terms an overnight rate (Fed Fund, Eonia, and so on) is paid. One potential solution to simplify the term of the CSAs, which has been proposed by some market participants, is to pay zero interest on the collateral. This is equivalent to a futures margining. If that proposal, which simplifies a certain number of practical problems, is put in place, the futures framework would also be the natural one for swaps with zero rate collateral.

Suppose that there is a continuous perfect collateral for swaps, the collateral is paid in cash in the currency of the swap and the rate paid on the collateral is 0. This is similar to the margining on futures: the difference of value with the previous valuation is exchanged and no interest is paid on that amount. The general *futures price process* theory, or the multi-curve framework with collateral as described in Chapter 8, applies in this case. What is the value of the new Ibor coupon with CSA at rate 0?

The coupon *pays* $I_X^j(t_0)$ in v and is a futures price process up to that date. According to the general futures price theorem its value in 0 is

$$E^N \left[I_X^j(t_0) \right] = 1 - E^N \left[1 - I_X^j(t_0) \right] = 1 - \Phi_0^j(u).$$

Note that the fact that the coupon pays in v has no impact on the valuation. The value is known at the fixing date t_0 and from that date on the required collateral is paid. The payments in advance or in arrears have the same value. Maybe there would be a legal distinction between the amount paid as collateral and the amount paid as coupon, but if we ignore that distinction, from a pure cash-flow exchange perspective, everything is exchanged as soon as the cash-flow is known.

The general collateral principle is still valid: a promise to pay tomorrow is fulfilled by paying today the discounted forward value and adapting the amount up to the final payment. The difference here is the discounting at a zero rate and the fact that no adaptation is required after the last fixing. Note nevertheless that the meaning of *expected value* is not the same with different collaterals, as the definitions of F^j are different. An adjustment, like the one described in Theorem 3.2, is required. Chapter 8 is devoted to the general analysis of the collateral impact on the multi-curve framework.

4 | Interpolation

Even if there are a lot of liquid instruments related to different indexes traded in the market, they constitute only a very small number with respect to the degrees of freedom when constructing a curve. For periods longer than two years there is often at best one yearly instrument and 250 unknowns – one for each business day of the year. The description of a curve needs to supplement those constraints in one way or another to obtain the full yield curve. This is this technique of 'filling the gaps' that we call here interpolation. The term interpolation has to be understood in a generic sense; it can be interpolation in a usual sense, like linear interpolation on given node values, or using a general parameterised function with the parameters not related at all to specific nodes.

The interpolation question has certainly been analysed in the literature over the years. Some references are described below. The criteria used to anal-yse the interpolation techniques usually fall into one of the following two categories:

1. The quality of the forward rates.
2. The quality of the implied hedging strategies.

In the first category we have a question like 'How good do the forward rates look?' This is an important question but at the same time a vague one. The quality will depend on the circumstances. For the overnight forward rate of an overnight-related curve, a good curve will probably have jumps at specific dates, like the central bank meeting dates or at the turn of the year, and be relatively flat in between. Such a curve is described in Example 5.12 in the next chapter. On the other side for forward rates in the distant future for which we have very limited infor-mation, we would like to have forward rates that are relatively stable and without large changes in the absence of clear economic reason. This is why in the previous chapter the first graph in Section 3.1 seems unattractive. The forward rates jump at the curve nodes with the node having no economic interpretation. Those two examples show that the request can be contradictory from one place to another. It is therefore almost impossible to propose one or two methods and claim that they

are the best in all circumstances. Here we describe the impact of the different types of scheme and do not suggest which one to use in all circumstances.

The second category of criterion is very important as the curves are not only 'accounting' tools to compute the present value of the financial instruments but also risk management tools to analyse the risks. We concentrate on first order risks called *delta*, *bucketed delta* or *bucketed PV01*. They are defined as the mathematical derivatives or sensitivities of the present value to the change of the inputs, in our case the first order estimate of the present value change for a change of the quotes of the instruments in the calibrating basket. These numbers are used to compute the hedges. The sensitivities to market quotes can be transformed into quantities of those instruments that will cancel the risk to the first order. Are those quantities reasonable? If a method implies that you have to use ten year swaps to hedge a seven month swap, the method probably needs to be reviewed.

The curves are used for two important functions: present value computation and hedging strategies design. The interpolation mechanism used in the curves has to be good for these two usages. It is not impossible to use two different interpolation schemes, one for each function, but this creates incoherencies and is not advisable.

The literature about yield curve interpolation is relatively large. Some of those articles review the different interpolation technique in detail with all the construction details. We will not describe the different interpolation schemes with so much detail. We refer the readers to three references for extensive reviews of the different techniques: Hagan and West (2006), Andersen (2007) and Iwashita (2013).

In Hagan and West (2006), the authors analyse numerous interpolation schemes that are used in finance for curves. They focus on linear and cubic spline methods. The interpolation is performed on the continuously compounded zero-coupon rate. As described in our first section below, we believe that selecting the quantity on which the interpolation is done is part of the modelling choice and should not be constrained a priori. Andersen (2007) analyses the use of tension splines for curve construction. The approach is described in a set-up where the present value is a linear combination of discount factors. The calibration is described in a linear algebra set-up. This approach was probably acceptable in the one-curve world but not in the multi-curve framework, where even the simplest instruments require ratios and products of the discount factors and forward rates. Nevertheless the analysis of the impact of adding tension to cubic splines is still very relevant for curve construction. Andersen (2007) also indicates that there is little econometric evidence that one transformation – zero-coupon rate, instantaneous forward, discount factor, log discount factor or other – is a better choice than any other for interpolation purposes. Nevertheless, as analysed in the paper, the choice of the quantity on which the interpolation is done can have a significant impact on the final result. That choice should be part of the analysis when selecting an interpolation scheme. Our first section is dedicated to such an analysis.

Le Floc'h (2012) analyses stability properties of the interpolation schemes. He does so for interpolations on the continuously compounded zero-coupon rate for a single curve. The analysis can be extended to the multi-curve only in the pseudo-discount factor approach. He adds to the list of interpolators used in curve construction some monotone Hermite spline. The analysis of the stability of the interpolator performed refers to the stability of the one-sided finite difference delta computation. When using a finite difference scheme to compute derivative, there is a numerical instability impact and a second order derivative impact. These two impacts can create unstable results in the sense that the sum of the derivatives with respect to each input is not equal to the derivative with respect to a parallel shift. This numerical instability does not appear in any of our implementations as we compute derivatives using algorithmic differentiation, as described in Section C.3. The derivatives are computed to machine precision and there is no numerical instability.

Hagan and West (2006) extend their analysis to the hedging strategies impact, something we also do below. They show through several examples that the impact can be significant. They also propose a different approach to sensitivities computation, which they call 'hedging under waves'. This consists in calibrating the curve with any interpolator and then computing the sensitivities by shifting the curve in triangles. This is equivalent to computing the risks in a set-up with an initial curve with the said interpolation, adding a spread curve with the same nodes, value 0 everywhere and linear interpolation and computing the sensitivities with respect to the spread curve. The sensitivities obtained are very close to the sensitivities from direct curve construction with linear interpolation.

The first analysis of the interpolation question specific to the multi-curve framework is probably Ametrano and Bianchetti (2009). They extended the results in Ametrano and Bianchetti (2013). They analyse the calibration procedure (see the next chapter) and the impact of interpolation. They insist on the smoothness of forward curve and impact of interpolation on hedging.

4.1 What to interpolate?

In the case when an interpolation is required, the main question asked is *How to interpolate*? In my opinion, one question that should come before that is *What to interpolate?* If we take the above-mentioned article Hagan and West (2006), the second line states '*a risk-free yield curve which will be continuously zero curve (because this is the standard format, for all option pricing formula*', but their second criteria in judging a curve is '*How good do the forward rates look*'. Similarly Andersen and Piterbarg (2010) restrict themselves to the interpolation of continuously compounded zero-coupon rates. As mentioned above, Andersen (2007) insists on the possibility of interpolating on several transformations. In Chapter 3 we have already described the fundamental difference between pseudo-discount factors and direct forward rate interpolation.

Using the above approach a certain description of the curve is selected but the criterion on which it is decided that a curve is good or not is based on a different description. One simple way to reduce the gap between the framework and our expectation on it is to use the same description for both. This may be viewed as the equivalent to the difference between the HJM framework and the LMM framework. Both frameworks are twin brothers and it is possible to convert one into the other. Nevertheless is it in some cases easier to express some modelling choices with one approach or the other.

In the case of curves, especially the forward curves, one has a choice to make on what quantity to interpolate. The *continuously compounded zero-coupon rate* is often selected by default as the base quantity. There are reasons for that, as some formulas are easier to express in that framework. But this easiness should not have too much of an impact on the actual choices. Another object which is also often used in interpolation is the discount factors.

These two quantities should not be the only ones available. One other possibility should be the periodically compounded rates. It should be possible to choose the annual, semi-annual or any other period rates as the base quantity.

The choices proposed above are for the discounting curve and the forward curves described through pseudo-discount factors as outlined in Chapter 3. The other approach to describing the forward curves outlined in the same chapter is to use the forward rates directly as the base quantities. That possibility did not exist in the one-curve approach as the same curve was used to represent the discount factors and the forward rates. When using a full multi-curve framework, there is no reason to restrict the choice and not offer that possibility.

In the following example we calibrated a (discounting) curve with the same data but different interpolation techniques. The three techniques have a linear flavour: linear interpolation on continuously compounded rates, linear interpolation on annually compounded rates and log-linear interpolation on discount factors. The spread between the implied continuously compounded rates are given in Figure 4.1.

As can be seen from the graph, the impact is non-negligible. The difference between linear on a continuously compounded rate and log-linear on discount factors – which is linear on rate multiplied by time – is up to almost 15 basis points in this simple example. The difference between linear on continuously compounded rates and linear on annually compounded rates is lower. It nevertheless reaches 0.1 basis points, which can not be considered as negligible for market making purposes.

4.2 Impact of interpolation

In this section we review in a way similar to Chapter 3 the impact of interpolation on forward curve appearance and hedging. We restrict ourselves to interpolation on
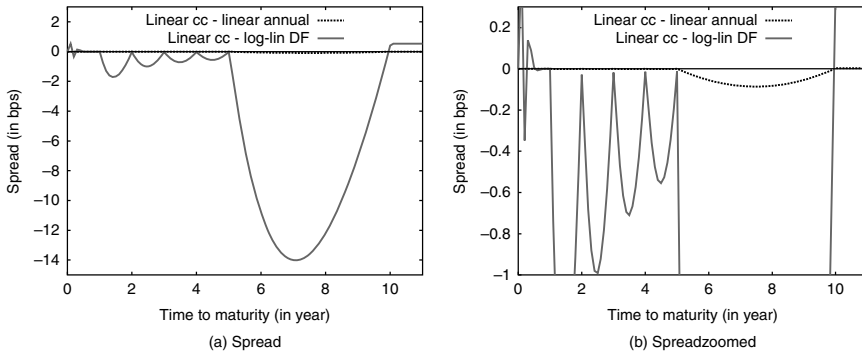
Figure 4.1 Impact of interpolation. Implied continuously compounded rates from different interpolation schemes. The black curve is the spread between the curve with linearly interpolated continuously compounded rates and the curve with linearly interpolated annually compounded rates. The grey curve is the spread between the curve with linearly interpolated continuously compounded rates and the curve with log-linearly interpolated discount factors.

the pseudo-discount factors as the comparison with direct interpolation on forward rates was done earlier.

In all the examples presented here, we use a unique interpolator for one curve. As discussed later in Section 5.9.5, in some cases it is interesting to have different interpolation schemes in different parts of the curve. The typical application of that demand is for overnight-related curves, for which we have clear economic intuition on curve jumps at the coming central bank meetings and enough information to calibrate to all of them. For longer term we only have sparser data, for example annual, and less clarity on what could happen at precise dates. The first part of the curve can be modelled with a log-linear interpolation on the discount factors, equivalent to constant short term rates, and the long part with a smoother interpolation.

It is also possible to use different interpolation schemes for different curves. As mentioned above, the overnight-related curve may be treated differently from a forward Ibor curve.

4.2.1 Forward rates quality

In this section, we use four different interpolators in the analysis. The interpolators have very different characteristics. The first interpolator is *linear on the continuously compounded zero-coupon rates*. The interpolation is local. The second interpolator is *natural cubic spline on the continuously compounded zero-coupon rates*. This interpolator is at the other end of the spectrum in terms of locality as it is non-local with dependencies on the full curve. The third interpolator is *double-quadratic on the continuously compounded zero-coupon rates*. In terms of locality, this interpolator is between the two previous ones. The dependency extends to one interval on
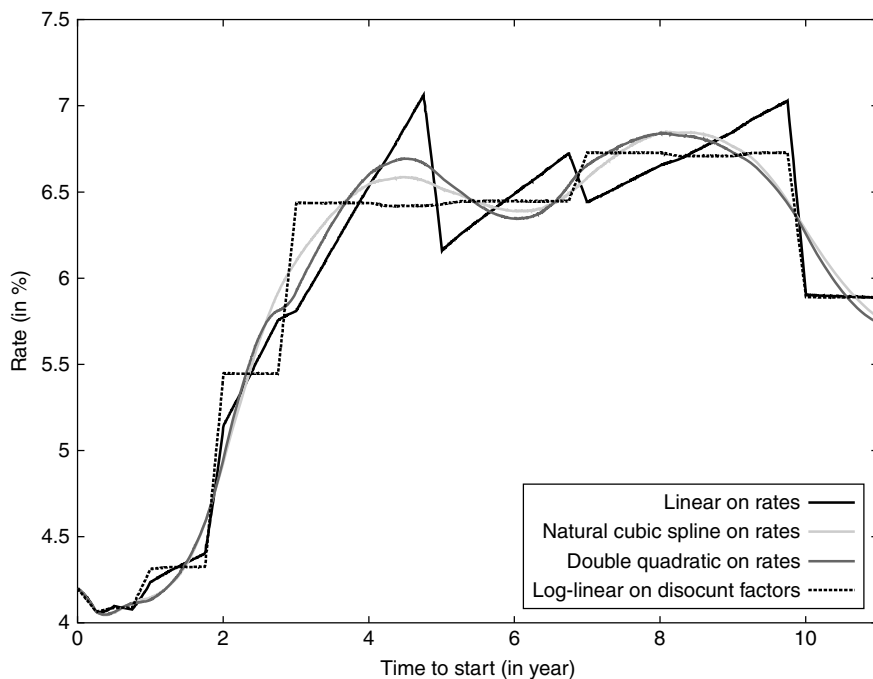
Figure 4.2 Forward rates for several interpolation schemes on pseudo-discount factors.

each side of the point but not further. The fourth interpolator is *log-linear on the discount factors*. It is linear and local as the first one, but the linear interpolation is on a different quantity, the logarithm of the discount factors.

We used the same data as in the previous section. It is data suggested by Andersen and Piterbarg (2010) which shows some of the interpolation features. The results are displayed in Figure 4.2. The graphs represent the forward three months rates computed from pseudo-discount factors.

As noted earlier, the linear interpolation implementation creates sawtooth effects with steep drops and increases. The natural cubic spline is very smooth but creates 'ripples' far away from the nodes of origin as will be obvious from the next section. The double-quadratic interpolation is somehow in between the two previous one. The log-linear interpolation on discount factors creates jumps on instantaneous forward rates. When translated into three months forward rates, it means that the rates are constant inside the intervals and the jumps from one level to the next are not instantaneous but happen over a three months period; the transitions are still fairly steep.

From a visual aspect and ignoring the economic information available for the short term, the smoother curves induced by the cubic spline and double-quadratic

interpolations look more appealing. This appearance quality has to be balanced with the next section on the quality of hedges, which often tells a very different story.

4.2.2 Hedging strategies quality

We use the same interpolation schemes as in the previous section. For each of them, we look at the hedging ratio suggested. The hedging ratios are computed as the bucketed deltas at each node divided by the sensitivity of the instrument used for the node.

If the deltas are computed for each instrument used in the curve construction, only a delta with respect to the instrument itself will appear; the other deltas will be zero. This is relatively easy to understand intuitively. If an instrument is at-the-money, that is it has a quote in line with the market quote, changing the quote of another instrument will not change the fact that this instrument is at-the-money. The quote of an instrument in the basket is not affected by the other instruments in the basket. The only unknown is the size of the sensitivity to that quote. The result will depend on the rate level and even potentially on the interpolator itself.

In Table 4.1 we have computed the market quotes deltas for a forward swap. The forward swap has a start date in 18 months and has a tenor of six years. The start date was selected to be just between two nodes. The tenor was selected in such a way that the maturity date is not on a node point but relatively close to one. The maturity is in 7.5 years, between the seven and ten year nodes but closer to the seven year node. The swap is a payer swap and the notional is 100 million.

The first part of the table reports the sensitivities to the market quotes used in the forward curve calibration. Note that not all the quotes have the same dimension. The Libor three months fixing is quoted with a day count of ACT/360 and the swaps with a day count of 30U/360. As the sensitivities are mainly on the swap nodes, the impact of this discrepancy is minor; nevertheless the total number should be taken with a pinch of salt. We have not reported the sensitivities to the market quotes used to calibrate the discounting curve. These sensitivities are real and not negligible but they are a lot smaller than those of the forward instruments quotes. We focus only on the forward curves in this analysis.

The first observation is that the bucketed delta can be largely different between the different schemes. The maximum difference on a single node is almost 15,000. If this is compared to the total sensitivity, which is around 50,000, this is a difference of around 30% of the total. Even if the bucketed deltas can be very different, it is maybe surprising to see that the total delta is almost the same for the four methods. In some sense a parallel move is a parallel move, whatever the interpolation scheme is.

What is the impact on the hedging strategies of changing the interpolation scheme? The answer can be found in the second part of the same table. We have computed the hedging strategies implied by the sensitivities. The strategies are

Table 4.1 Sensitivity to market quotes for a 100 million notional using different interpolation schemes. Hedge strategies (in millions)

Tenor	Delta (by bp)				Hedges (in millions)			
	Lin.	C. Spl.	D. Qu.	Log-lin.	Lin	C. Spl.	D. Qu.	Log-lin.
3M	−3	−1,919	−2	0	−0.1	−76.7	−0.1	0.0
6M	9	5,725	2,432	0	0.2	116.8	49.6	0.0
1Y	−7,222	−11,118	−9,946	−4,820	−74.8	−115.2	−103.1	−49.9
2Y	−7,143	−9,842	−7,716	−9,525	−37.5	−51.7	−40.5	−50.1
3Y	−27	5,366	939	0	−0.1	19.2	3.4	0.0
5Y	−58	−11,234	−6,685	0	−0.1	−25.0	−14.9	0.0
7Y	53,515	64,809	63,615	49,904	88.5	107.2	105.3	82.6
10Y	10,721	11,452	8,531	14,233	13.1	14.0	10.4	17.4
12Y	0	−3,947	−1,360	0	0.0	−4.2	−1.4	0.0
15Y	0	542	−2	0	0.0	0.5	0.0	0.0
20Y	0	−49	0	0	0.0	0.0	0.0	0.0
Total	49,793	49,784	49,805	49,792	214.6	530.6	328.7	200.0

Lin.: Linear on zero-coupon rates. C. Spl.: Natural cubic spline on zero-coupon rates. D. Qu.: Double-quadratic on zero-coupon rates. Log-lin: Log linear on the discount factors. The totals for the hedges are the totals of the absolute values. Totals for deltas are computed by adding sensitivities with different conventions and should be interpreted with caution.

obtained by dividing the sensitivity of the forward swap at each node by the sensitivity of each node instrument to its own market quote. As mentioned above, the bucketed sensitivity of a node instrument is zero for all nodes except the node of the instrument. So the sensitivity of the node instrument to the node is the same as the total sensitivity of the node instrument. In our example the sensitivities of a given instrument were only very weakly dependent on the interpolation scheme (figures not reported).

If we analyse first the two local schemes, some common sense seems to dictate the results. The start in 18 months is just in the middle of the one year and two year points. For the linear interpolation on rate, the sensitivity is split equally between the two points. In the other linear scheme, on the logarithm of the discount factors, it is the hedging quantity which is split equally between the two nodes.

When we look at the non-local schemes, the picture is very different. The natural cubic spline method implies the largest un-intuitive hedges. To hedge the start of the payer swap in 18 months, the suggested hedging is receiving 77 million at three months, paying 117 million at six months, receiving 115 million at one year and paying 52 million at two years. The wiggle patterns visible in the forward rates also appear in the hedging ratios. This wiggling pattern also means that the total notional of the hedging instruments is a lot higher than the initial instrument. The

notional of the forward swap is 100 million. It is a forward starting swap, so we can consider the notional for hedging with spot instruments purpose to be 200 million, 100 million for the start date and 100 million for the end date. The two linear schemes propose hedges with total notional close to that mark. The cubic spline approach proposes hedges for a total of more than 530 million. Even if this was a good hedge, the cost to put it in place in terms of bid/offer would be prohibitive. In the case of the cubic spline scheme, the hedging strategy proposed covers not only nodes close to the instruments cash-flows, but also nodes very far away. The hedge includes a payer 15 year swap of 0.5 million, even if the last cash-flow of the swap is in 7.5 years and there are two other nodes in between: 10 and 12 years.

5 | Curve Calibration

5.1 Introduction

Approaches to curve calibration have been described numerous times in the literature, mainly in the one-curve framework. The process is described with several names. A name often used is curve bootstrapping, which indicates that the process is recursive with one node constructed after the other. In general I prefer the terms *curve calibration* or *curve construction*. They seem more appropriate than *bootstrapping* as they better reflect the complexity of the process, especially in the multi-curve framework. I will use the term curve calibration in most of the book.

In this chapter we propose a very generic curve calibration approach taking into account all the idiosyncrasies that can appear with the multi-curve framework in different markets. To our knowledge this is the most general presentation of curve calibration available. We restrict ourselves to the *exact curve calibration* problem; this is when there is the same number of free parameters in the calibration as there are instruments used.

In the recent literature one can find a lot of particular examples of curve calibration. Some of them spend a great deal of time describing a very particular case of such a method. The methods often involve some form of bootstrapping, this is solving one part of the problem before moving to the next part of the problem. In the curve calibration case one often reduces the process to building the points on an interpolated curve successively one after the other, and the curves also successively one after the other. Apart from for an algorithmic simplification, there is no reason to impose these strong constraints. This often sounds like an 'early optimisation' process in which one decides the simple method to use before understanding the problem to solve.

The problem of curve construction is complex and should be described fully before attempting to solve it. The exogenous constraints consisting in imposing a linear relationship of present value with discount factors, restricting the curve to be interpolated on given nodes, building curves consecutively and not simultaneously or bootstrapping the points on a curve are doomed to fail at some stage.

We propose a very generic approach to the problem and, when the problems allow some simplification, we show some intermediary optimisation.

The one-curve approach to curve calibration can be found with different degrees of complexity in the literature. It was the focus of the curve calibration literature before the emergence of the multi-curve framework. A simplified approach for swap curve construction can be found in Ron (2000). The articles (Hagan and West 2006; Andersen 2007) are specifically focused on the interpolation part of the curve building. The recent book Andersen and Piterbarg (2010) presents a general and simple view on curve building.

An analysis of curve construction in the multi-curve framework is given in Ametrano and Bianchetti (2009) and Ametrano and Bianchetti (2013). The authors focus on different interpolation schemes and available data. They restrict the building procedure to building first the discounting curve and then building each forward curve separately. Our focus here is different and directed to the concurrent construction of several curves when required by the available liquid instruments. We do not impose any restriction; we allow simultaneous building of several curves (discounting and forward). Such a simultaneous construction is necessary in multiple markets; examples where this is required naturally by the structure of the most liquid instruments are given later in the chapter.

Another description of curve building in the multi-curve framework is proposed in Kenyon and Stamm (2012). Their approach is a simple bootstrapping technique with a specific local interpolation scheme. The approach can not be used if non-local interpolation schemes are used.

In a recent paper, Fries (2013) proposes a calibration approach closer to the one we describe here. The calibration is based on a root-finding process and allows for different curve descriptions, through pseudo-discount factors or what he calls *value curves*. The description restricts the curves to be interpolated curves and the node points to be maturity date or last fixing dates of the instruments. Only two legs swaps are considered as potential calibration instruments. The curves are calibrated successively (first the discounting curve and then the forward curves one by one) and the spot lags are ignored.

The general description of the curve calibration we propose involves an arbitrary number of curves and an arbitrary number of arbitrary instruments linked to those curves in an arbitrary way. The total number of parameters that describe the curves is equal to the number of calibrating instruments. Given the very generic nature of our approach and the historical approach to curve calibration it is sometime easier to describe the generality of our approach by describing the constraints we do not impose. This is why most of the introduction describes absence of constraints more than their presence.

We do not impose that only homogeneous instruments depending on only one index are used for each curve. There are liquid market instruments depending on several indexes, in particular the basis swaps, that should not be excluded from

the process. The way the instruments depend on the indexes is not restricted. In the current market, all the liquid instruments depend only on one index on each leg. But if instruments with legs depending on two or more indexes, such as a leg of averages or a leg with alternate indexes, were to become liquid tomorrow, it should be possible to incorporate them in the curve construction process.

The curve calibration process does not necessarily rely only on interpolated curves. One can imagine curves defined as parametrised functions, with their parameters calibrated or as an aggregation of functions, for example a base function plus a spread function. The necessity of those more involved structures is imposed by the multi-curve nature of the framework, where a curve cannot be built or used in isolation. Even if those requirements seem obvious when analysing the framework foundations, they are seldom fully met in implementations. We describe in Appendix C some implications of those requirements on the structure of libraries.

Given the large number of curves and the limited liquidity of some instruments in the market, we can not be selective on the instruments accepted in the curve building process. We take the instruments where the information and the liquidity is, not where it is easier for us to solve the problem. The question we face should not be simplified just because we cannot answer it in its generality. We will impose no restriction on instruments and pricing methods used. In particular we will not impose restrictions like Andersen and Piterbarg (2010, Equation 6.1) and Andersen (2007), who require a linear relation between present values and discount factors. It is certainly easier to present the problem in a linear algebra setting, but it does not represent the actual market.

In a similar way, we advise not to pre-process data before starting the curve calibration procedure. Only real market data should be used in the process. This is very important to obtain the sensitivities or greeks with respect to market quotes, not manipulated synthetic numbers.

For example, one can transform futures prices into FRA rates, using a specific convexity adjustment and an FRA pricing approach, to pre-calculate adjustments. This may speed up the construction process in some circumstances. But all the other processes that rely on the curve calibration, including the sensitivities with respect to the market quotes, will be distorted. In the futures/FRA case, the sensitivity would be to an FRA rate impossible to hedge, or worse be a misleading figure pretending to be the sensitivity to a futures but actually being something else. If the risk manager or trader is obliged to manipulate data before using them, the curve building tool has failed an important part of its specification. The curves are not only pricing tools but also risk management tools. The actual risk management can be done only with market instruments, not with fictitious or synthetic instruments. In particular a STIR futures is not an FRA with a manipulated rate, it is traded in a different market. The change of the market structure, with more collateralisation, central clearing and exchange traded instruments means also that more diversity

in instruments and pricing mechanisms is required. The pre-calibration quote manipulation is not always possible, for example with present value quoted swap futures, and this source of information should certainly not be a priori excluded from our curve calibration process. Also one should not be obliged to interpolate market rates to create fictitious instruments. Not only would such interpolation create spurious sensitivities but it would also use an interpolation mechanism different from the one used in the curve construction process itself, creating an internal incoherence. Similarly one should not be obliged to add the spread from one instrument to the rate of another to obtain a synthetic quote. If the spread based instrument is a liquid instrument it should enter directly into the curve calibration process as it will be the instrument of choice for hedging.

In general, the goal of the curve building procedure is not only the resulting curves themselves but also the related risk implications. The output of the procedure should facilitate the computation of the risk measures and provide hedging tools. The output of the curve calibration is the curve description, but also different Jacobian or transition matrices. These matrices contain the derivatives of the curve parameters to the market inputs (instruments market quotes). Appendix C describes one suggestion on how to implement a curve description and transition matrix storage.

As discussed in the chapter on interpolation, we do not impose interpolation on a specific number, for example a zero-coupon rate. In fact, we do not even impose interpolation at all. For us a curve is a generic mathematical concept which depends on parameters. One curve can be built recursively on several other curves, with a combining mechanism like additive spread; it can be an interpolated curve with the value at each node being the parameters or can be another specific parameterised curve like a Nelson-Siegel curve. By combining several of these mechanisms, one curve can have several interpolation methods in different parts and interpolation on different figures (rate or discount factor). This is not a extravagant requirement and comes naturally in some circumstances. This will be made more explicit in the examples.

As a general requirement, the selection of calibrating instrument and the selection of the node points should not be restricted for implementation reasons; only financial constraints should impact the choice. In case an interpolated curve is selected, the node points should be selected only for financial reasons. If the payoffs of instruments with the same tenor are a couple of days apart due to conventions, for example IRS and OIS, a trader or risk manager should not be forced to modify the instrument description in the curve construction to simplify the curve calibration. In some other cases, the node points will be exogenous with respect to the instruments. This is for example the case of central bank meetings that influence the short term rates. It should be possible to decide that the node points will be those meeting dates and to choose for the calibration instruments with different reference dates.

5.2 What to calibrate?

The general goal of the curve calibration is to ensure that the market instruments used in the calibration are 'at market level' when priced by the calibrated curves. The 'market level' can be seen from several points of view. The most used ways to measure how the instruments are 'at market level' are the *present value* and *par rate* or *par market quote*. By quote we mean any number that is used in the market to indicate the level of a financial instrument: rate, price, yield, clean price, and so on.

The calibration constraint used in the present value approach is that all instruments have a present value of zero when valued with the calibrated curves. This is a general approach that works in all cases. It has nevertheless a couple of practical drawbacks. The first one is that the hedgers like to see the risk figures, one of the important byproducts of the curve calibration process, in terms of sensitivity to a given change of market quote, such as one basis point. If the curves are calibrated using present value, one has to compute the market quote sensitivities as extra information. Even if one uses the present value approach internally, the par rate or equivalent approach is required from a risk perspective. The second drawback is numerical. The curve calibration process, as we describe it, is done through a root-finding algorithm. The root-finding algorithm is usually implemented with an iterative algorithm using the derivative of the target function to the inputs; in this case it would be the sensitivities of the present values to the market quotes. The ratio in maturity between the shortest and longest instrument can be from 1 to 15,000 (one day to 50 years). The sensitivity matrix, which is inverted in the root-finding process, has values covering several orders of magnitude. Its inversion is numerically less stable. For these two reasons, in practice, we prefer to implement the curve calibration using the par market quote, or a variant of it, described below.

The calibration constraint used in the par rate approach is that all quotes for the instruments computed from the curves are equal to the actual market quotes. For most instruments, this is not a problem. Nevertheless there are some instruments for which this request is not trivial. Consider a basis swap for which the first period has already been fixed. The market quote is a spread on one leg. The market quote could be computed as the opposite of the sum of the two legs without spread divided by the present value of a basis point of the spread leg. But as the first coupon has already been fixed, it is a fixed coupon, not a Ibor coupon with a spread. Saying in a generic way 'without the spread' is not obvious for all spread-like instruments.

For that reason we found it easier to work with figures we called *par market quote spreads*; this is the number that should be added to the market quote to obtain a present value of zero. The idea of present value equal to zero is similar to the first approach but achieved in a different way, through a number that has the dimension of a rate. For the basis spread mentioned above, the par market quote spread is the opposite of the sum of the two legs (including the spread) divided by the present

value of a basis point of the spread leg. Whatever the coupons on each leg are – fixed, Ibor or overnight – this quantity can be computed unambiguously. We found this measure easier to work with than the par market quote and numerically more efficient in the solver than the present value. Moreover the sensitivity to the market quote spread is the same as the sensitivity to the market quote. So the matrix used internally by the root-finding scheme is the same as the matrix we want to provide to the user to compute risk figures.

5.3 Calibration

Now that we have clarified the 'what to calibrate', we can pass to the 'how to calibrate'. In the approach we present here we deal only with exact calibration. We suppose that we have a certain number of instruments in the calibration basket and the same number of parameters in the curves. The parameters are split among the different curves. We suppose also that the multidimensional root-finding problem defined by the constraints on each instrument and the curve parameter has one non-singular solution.

It is quite difficult to describe financial conditions that lead to these mathematical hypotheses without prescribing specific curve description and instrument restriction. We would like to leave the approach as general as possible, so we refrain from going in that direction.

We index the instruments used in the calibration by k ($1 \leq k \leq n$). Let $p = (p_l)_{1 \leq l \leq n}$ denote the parameters of the curves and $S = (S_k)_{1 \leq k \leq n}$ the par market quote spread functions for the k-th instrument. The problem to solve is

$$S_k(p) = 0 \quad (1 \leq k \leq n). \tag{5.1}$$

In general, the problem is non-linear and strongly coupled. The solution has to be obtained globally for all p as one computation block.

In some special cases, the problem can be divided into sub-problems, easier to solve numerically. The most extreme simplification is when the problem can be divided into one-dimensional sub-problems. This is the case in the bootstrapping approach, where the interpolation scheme is local and the curves are not entangled. This is an extreme case, and relatively non-realistic, and we do not analyse it further.

A more frequent numerical simplification is when some curves are not fully entangled and the problem can be solved inductively, by first calibrating some curves and then using the resulting curves as an input to calibrating the next curves. In what follows we use the term *curve units* to refer to the set of curves calibrated simultaneously in one unique root-finding process. The complete set of all the related curves is called a *block*.

The standard application of this is when the curves are built one by one. More involved examples are when units of curves are first calibrated and then another

unit of several curves is calibrated based on the previous units. Let u be the number of units and n_i the total number of parameters in the first i units. The steps in the induction will look like:

$$S_k((p_l)_{0<l\leq n_1}) = 0 \quad (0 < k \leq n_1)$$

$$S_k((p_l)_{0<l\leq n_1}, (p_l)_{n_1<l\leq n_2}) = 0 \quad (n_1 < k \leq n_2)$$

$$\vdots$$

$$S_k((p_l)_{0<l\leq n_{u-1}}, (p_l)_{n_{u-1}<l\leq n}) = 0 \quad (n_{u-1} < k \leq n)$$

Each set of equations represents a unit.

Solving the above system of equations is not the only output of the calibration exercise. The output of the procedure should facilitate the computation of the risks measures and provide hedging tools. We now are more explicit in what we mean by that.

The calibrated curve is used to compute the present value of instruments (PV). We are also interested in the sensitivities of the present value to the market quotes q_k used in the curve building procedure:

$$\frac{\partial PV}{\partial q_k}.$$

This is often called the *bucketed deltas* or *partial PV01* of the instruments.

Using the calibrated curve, it is possible to compute the derivatives of the present value to the curve parameters p. Algorithmic differentiation generally helps for that. A short description of algorithmic differentiation and its application to the multi-curve framework is provided in Appendix C.3. The derivative with respect to the market quotes can be obtained through

$$\frac{\partial PV}{\partial q_k} = \sum_{l=1}^{n} \frac{\partial PV}{\partial p_l} \frac{\partial p_l}{\partial q_k}.$$

What is missing to compute the above result in practice is the matrix

$$D_q p = \left(\frac{\partial p_l}{\partial q_k} \right)_{1\leq l\leq n, 1\leq k\leq n}.$$

The figures we suggest to use for calibration are the par market quotes spread. They are the additive spreads to the market quotes, this is for q^M the market quotes and $\tilde{q}$ the computed quotes from the curves, $S(p, q^M) = \tilde{q}(p) - q^M$. We rewrite Equation (5.1) to indicate the market quote dependency explicitly. The calibrated parameters p_0 are such that

$$S(p_0, q^M) = 0.$$

Provided that some regularity conditions are satisfied around the solution (p_0, q^M), one can apply the implicit function theorem (see, for example, Mawhin (1997)). There is a function $p : \mathbb{R}^n \to \mathbb{R}^n; q \mapsto p(q)$ defined in a neighbourhood of q^M such that

$$S(p(q), q) = 0.$$

Using the implicit function theorem, the derivatives with respect to the quotes q can be obtained by

$$D_q p = -(D_p S)^{-1} D_q S = (D_p S)^{-1}$$

The last equality is obtained using the fact that the derivative of S to the quote q is -1. The use of the implicit function theorem combined with algorithmic differentiation to significantly improve the computation speed is described in Henrard (2012a).

When the root-finding problem is solved with a Newton-like algorithm, the same type of matrix is computed. The Jacobian matrix is used to find the direction of the next best guess. Computing the matrix for later use does not require developments beyond those already done for efficiently solving the root-finding algorithm.

In what follows, we will call the *Jacobian* or *transition* matrix the matrix $D_p S$ and the inverse Jacobian the matrix $D_q p = (D_p S)^{-1}$.

Once the curve calibration process is finished, we suggest storing the inverse Jacobian matrix, at least the non-zero blocks of it, for later use. A suggestion of implementation of the storing technique is proposed in Appendix C.2.

5.4 Discounting curve

The curve calibration procedure starts with the selection of the discounting curve. We could suppose that the curve is externally provided. But apart from an godly intervention, which is quite unlikely in quantitative finance, we can not really expect it. The only realistic hypothesis is to impose a relationship between some market instruments and the discounting curve.

A hypothesis often used is that the discounting is done with the curve calibrated to OIS-like instruments. This choice is not completely arbitrary and can be linked to the legal terms of standard collateral agreement that are discussed in Chapter 8. In our notation, the hypothesis reads as

$$P_X^D(s, t) = P_X^O(s, t).$$

Note that it means we have chosen to represent the overnight forward curve using pseudo-discount factors as described in Section 2.2.

Other hypotheses that are used are to impose a given spread between another curve and the discounting one, for example a fixed spread S with an Ibor curve:

$$P_X^D(s,t) = \exp(S(t-s))P_X^3(0,t).$$

The reason an extra constraint must be imposed on the discounting curve is that the pricing of single-currency derivative instruments requires at least one forward curve, and there is usually only one liquid instrument for each maturity. We are left with one constraint and two unknowns (discounting and forward curves). One could avoid the constraint if there were several liquid instruments with the same maturity, for example a coupon and zero-coupon swaps or par swaps and specific coupon swaps[1], but this is generally not the case.

The choice of discounting curves cannot be done independently in all currencies. Cross-currency products impose a link between the curves. If one risk-free discounting curve has been selected and if one considers the cross-currency products like Foreign Exchange (FX) swaps to be credit risk-free, then the FX swaps impose a link between the risk-free curves in the different currencies. This is discussed further in Section 5.8.

5.5 Index fixing

5.5.1 Ibor index fixing

The pricing formulas for the most frequently used instruments are presented in Chapter 2 and some further instruments are presented in Chapter 6. One constraint that does not appear in the list of instruments is the index fixing. In some senses this is natural as it is not an instrument sensu stricto. On the other side it is the base ingredient of the framework and should not be excluded from the curve calibration process. In the curve calibration process, it is the only data that is not an instrument. It also means that the sensitivities to this market quote cannot really be used for hedging purposes.

There are several ways to represent the fixing. One popular one is to replace the fixing by the theoretical deposit underlying the index. The synthetic deposit would be priced like a standard deposit by discounting the synthetic cash-flows but with the forward curve pseudo-discount factors instead of the risk-free discount factors. This approach creates theoretical and implementation problems. From the theoretical point of view, the pseudo-discount factors exist in only one standard approach as described in Chapter 3, but not in all approaches. In the same chapter we propose a variation for which the pseudo-discount factors are not defined at all. The

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[1] Specific coupon swaps are mostly illiquid, but there is a proposal of trading *Market Agreed Coupons* (MAC) swaps. The proposal by ISDA can be found at www2.isda.org/news/isda-publishes- market-agreed-coupon-confirmation-for-interest-rate-swaps.

problem is also at the implementation level. How do tell your implementation that you want to discount with the wrong curve? One way is to create a fake counterpart for each index, like '*USD LIBOR 3M*' for example. Create a fictitious deposit linked to that counterpart and finally indicate to your curve repository that each time a deposit with the fake counterpart is valued, it should provide the relevant forward curve instead of an actual counterpart curve. It does not represent an impossible task but appears a little bit too convoluted to price the simplest part of the curve: the current fixing.

The link between the Ibor fixing and the curve is given by Equation (2.3). That link was itself obtained by a limiting process on the coupon value when one approaches the fixing time. The same idea can be used for embedding the fixing in the curve construction. The fixing 'instrument' is priced internally as a one period swap with fixed rate equal to the index fixing rate and a floating coupon depending on the index. The fixing time of the index is set to one instant after the curve construction time; in other words it has not fixed yet, even if the real one has already fixed. In that way there is no need to create an involved pricing mechanism; one can use the swap pricing approach without changing the curve repository. The discounting curve is used to discount and the forward curve is used to estimate the forward rate. The technique can be used for present value as well as for the rate or the spread based curve calibration approaches.

5.5.2 Which fixing value?

Another question that should be asked regarding the fixing is: Which fixing value should be used? There is one instant at which the answer is clear, which is at the fixing time (like 11:00 am London time for Libor). For all other moments the answer is: 'Do your best!' There is no obligation to use the 11:00 am fixing at any other moment of the day. Should one use the morning fixing at 5:00 pm? In the instruments themselves, the Ibor coupons with fixing today are not floating coupons anymore but are now fixed coupons. They are priced by discounting and the forward curve does not affect them anymore. The forward curve has no impact on the valuation of the first payment of the floating leg. The impact of the forward curve will be for the coupons fixing tomorrow and after. To obtain the most realistic value of the coupons fixing tomorrow one should have the best estimate of what the fixing will be tomorrow. If the estimate is obtained by interpolation between today and another fixing, one should try to use a fixing for today that will imply the best estimate of tomorrow's fixing, not the best estimate of this morning's fixing, which is not used anymore. Using today's fixing is allowed, but certainly not mandatory and probably not the best choice.

The simplest approach is probably to apply an estimate of the market changes from the fixing time to the rate. This can be done for example by applying the change of rate from the OISs with same tenor as the index between the fixing time

and now. Another approach is to apply the change in the first futures rate to the rate, particularly where serial futures are traded and liquid. This somehow contradicts our suggestion not to use manipulated data. In this case the manipulation is not caused by a technical problem in our implementation that we try to avoid by changing the market description, but by a lack of market information. There is no market traded instrument that reflects exactly the current fixing, or more exactly, there is no 'current fixing' except once a day.

Obviously the same ideas apply before the fixing. We should not necessarily use the last fixing, that is, the previous business day fixing, but we should try to estimate the fixing that will be published as efficiently as possible. Again this can be done with OIS, futures or other instruments highly correlated with the rates underlying the index.

Some curve construction literature explicitly recommends using the actual fixing. For example Sadr (2009, Chapter 3) states 'Libor fixings (rather than their live quotes values) are used to construct the front end of the Libor curve'. For the reason described above, I believe that this type of statement is incorrect in two different ways: the last known fixings should not be used if they are not a good estimate of the next fixing, and there is no such thing as 'live quotes values'. The argument that this rate is 'official' is used as an argument in some places. This appears more as an accounting related argument than a quantitative financial argument. What good can come from using an 'official' rate at the wrong place? This is not a case of using the wrong figure in the wrong formula to obtain the right price but of using the wrong number in the correct formula to obtain the wrong price.

It is also possible to use one day FRAs when they are liquid. But this is more an exception than the rule. In that case, the fixing point can simply be ignored after the fixing time as it would not be used anywhere.

5.6 Root-finding and Jacobian

The curve calibration, except with the simplest bootstrapping method, requires a multidimensional root-finding algorithm. Often this is implemented with some Newton-Raphson style approach, using the Jacobian matrix as a basis for some iterative process. One initial guess is selected, the Jacobian matrix is computed and used as a tool to obtain the best first order estimate of the next guess. In this case the Jacobian matrix is the derivative matrix of the computed quotes with respect to the changes of curve parameters. Computing the matrices in a finite-difference way – by bumping each input and recomputing each implied market quote – is very time consuming, especially when many instruments are involved. For a 100 instruments case, like four curves with 25 points each, it would be a 10,000 quotes computation exercise. This would be a lot of 'lost' computations. The first requirement is thus

to have an *efficient computation of the derivatives* of the instruments quotes with respect to the curve parameters. The efficient derivative computation is often done through some form of algorithmic differentiation. A description of algorithmic differentiation is proposed in Appendix C.3. An efficient algorithmic differentiation implementation would achieve, in theory, the computation of the derivatives in less than four times the time to compute one market quote. In practice we achieve a ratio of less than four for par quotes. It means that for Jacobian calculation in the multi-curve framework, algorithmic differentiation can divide the computation time by 20.

The Jacobian matrix is not only used for the curve construction itself but also as a base of the calculation of present value derivatives with respect to the data used in curve calibration. An efficient implementation will provide the sensitivities of the instruments with respect to the parameters of the curves and multiply those by the inverse Jacobian to recover the sensitivities with respect to the market quotes. The curve building process should not only provide the curves themselves but also the full *relevant* Jacobian matrices. Several examples of Jacobian matrices and a description of what relevant means for each of them are given through the next sections. In most cases, the number of data points has been significantly reduced to be able to fit the data on the page.

5.7 Instruments and curve entanglement

One important required feature of the curve calibration process is to be able to decide on a set of instruments to build the curves without limitation on the number of curves used to price one instrument and without limitation on the entanglement of the curves. In particular we want to be able to use basis swaps that require three curves to be priced – one discounting curve and two forward curves. By entangled curves we mean curves that, given a set of market instruments, have to be built simultaneously and not successively.

Unless otherwise stated, all the examples in this chapter are calibrated with a non-local interpolation scheme – double-quadratic – and a flat extrapolation. The data have realistic levels but have been manipulated to emphasise some specific features.

The first example of Jacobian uses two unentangled curves in USD: OIS/discounting and Libor three months. The OIS curve is built using standard OIS. The case of Federal Fund swaps is discussed later in this chapter, even if the pricing of those USD specific instruments is only analysed in Chapter 6. The inverse transition matrices are given in Table 5.1. As the two curves are not entangled, it is enough for the OIS to compute only the derivatives with respect to the quotes in the curve itself. The other derivatives are 0. For the forward curve, the full matrix is required. The sensitivities of the forward curve parameters to the quotes of the OIS curve are small but not negligible.

Table 5.1 Inverse Jacobian matrix for simplified two USD curves built with two units. The matrix for the second unit contains the sensitivities with respect to the first unit quotes (dimension 5) and the second unit quotes (dimension 4)

	Unit 1: USD Dsc				
ON	1.01	0.00	0.00	0.00	0.00
OIS 1M	0.00	1.01	0.00	0.00	0.00
OIS 6M	0.01	−0.03	1.01	0.00	0.00
OIS 1Y	0.01	0.00	−0.02	0.99	0.00
OIS 2Y	0.00	0.00	0.00	−0.03	1.00

	Unit 2: USD Fwd 3M								
LIBOR 3M	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
FRA 3Mx6M	0.00	0.00	0.00	0.00	0.00	0.51	0.50	0.00	0.00
IRS 1Y	0.00	0.00	0.00	−0.01	0.00	0.00	0.00	0.99	0.00
IRS 2Y	0.00	0.00	0.00	0.00	−0.01	0.00	0.00	−0.02	1.00

A specificity of the US market is the existence of Federal Fund swaps. Those instruments are described in Section 6.4. The important point for curve calibration purposes is that the instruments swap Libor three month payments for overnight indexed payments plus a spread. The overnight payments are computed by an arithmetic average of the rate over the three month period. From a calibration perspective, it is not possible anymore to split the discounting and Libor three month curves into two distinct units. All the instruments used depend on both curves. The two curves have to be calibrated simultaneously.

A graphical representation of the inverse Jacobian matrix is proposed in Figure 5.1. In this case we have used 20 instruments for the discounting curve and 14 instruments for the forward curve. The strong dependence between the curve parameters and the market quotes of both curves is clear. In the graph we have represented by a black square all the dependencies with an absolute value above 0.10. Below that level, the magnitude of the coefficients are represented by grey levels. The points of the curves were not fully aligned to also show dependencies between nodes of different maturities. The instruments used for the discounting curve are an overnight deposit, OIS up to and including one year and Federal Fund swaps from two years to thirty years.

A typical one currency example where it is important to build two forward curves simultaneously is the AUD. The most liquid fixed for floating swaps are quarterly/quarterly up to three years and semi-annual/semi-annual from four years onward; basis swaps are available for all maturities. Building the three month forward curve involves fixed for quarterly floating swaps up to three years and basis swaps three months versus six months for longer tenors, but to build the forward six month curve up to three years the basis swaps three months versus six months
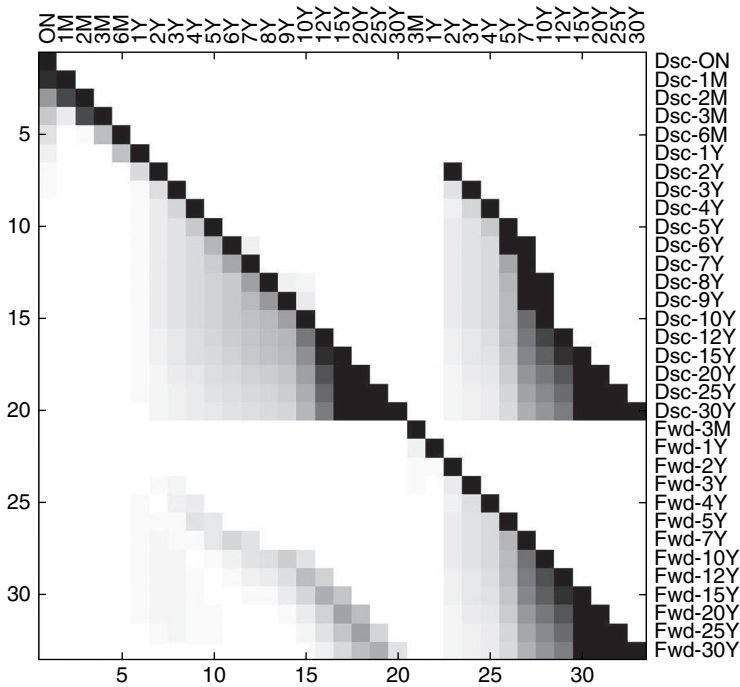
Figure 5.1 Graphical display of the inverse Jacobian for two USD curves built using Federal Funds swaps. The coefficients larger in absolute value than 0.10 are represented by black squares. Below that level, the magnitudes of the coefficients are represented by grey levels.

should be used and themselves depend on the three month index. There is no real shortcut and both forward curves need to be calibrated simultaneously.

The inverse Jacobian matrices for the AUD example are given in Table 5.2. One can see that in the second and third part of the matrices, there are significant sensitivities of some of the parameters in the three month curve to instruments in the six month curve and vice versa. Not taking into account those sensitivities does not lead to even a close approximation; it gives totally wrong numbers.

A similar entanglement appears in EUR if one wants to build the long part of the discounting/Eonia curve not with straight OIS swaps but with Euribor three month versus Eonia swaps. The curve construction involves discounting (Eonia) and forward three months simultaneously. One may add a forward six month curve if one uses standard fixed versus six months IRS and basis swaps six months versus three months plus spread.

Another scenario where simultaneous curve building is required involves foreign currency curves when the foreign discounting is linked to the discounting in the local currency. A case where three curves must be built simultaneously is the USD/JPY currency pair. The most liquid cross-currency instruments are the forex

Table 5.2 Inverse Jacobian matrices for simplified three AUD curves built with two units. Swaps (IRS) are quarterly/quarterly three months Bank Bill Swap (BBSW) or semi-annual/semi-annual six months BBSW, depending on the tenor. Basis swaps are six months BBSW plus spread versus three months BBSW

Unit 1: AUD Dsc					
ON	1.00	0.00	0.00	0.00	0.00
OIS 1M	0.06	0.94	0.00	0.00	0.00
OIS 6M	0.01	0.00	0.97	0.00	0.00
OIS 1Y	0.01	0.00	0.00	0.96	0.00
OIS 2Y	0.00	0.00	0.00	−0.04	1.00

Unit 2: AUD Fwd 3M/AUD Fwd 6M												
BBSW 3M	0.00	0.00	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.00	0.00
FRA 3Mx6M	0.00	0.00	0.00	0.00	0.00	0.50	0.49	0.00	0.00	0.00	0.00	0.00
IRS 1Y - 3M	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
BASIS 5Y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.04	−1.03	0.00	0.00	1.02
BBSW 6M	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.98	0.00	0.00
BASIS 1Y	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.99	0.00	0.00	0.99	0.00
IRS 5Y - 6M	0.00	0.00	0.00	0.00	0.01	0.00	0.00	−0.04	0.00	0.00	−0.04	1.01

swaps up to one year and the cross-currency swaps USD Libor three months versus JPY Libor three months plus spread above one year. In JPY, the standard swaps are fixed versus Libor six months. One has to build USD discounting, USD forward three months, JPY discounting, JPY forward three months and JPY forward six months as one block, with the Jacobian matrix of the last curve depending on all the other curve instruments.

One block does not mean that everything has to be done in one step. In the USD/JPY case, one could first build the USD discounting curve from OIS, then the USD forward three months curve – based on the previous curve. After that, the JPY discounting, JPY forward three months and JPY forward six months have to be calibrated simultaneously, using the already-calibrated USD curves as a basis. Even if it is possible to build all the curves simultaneously, it is numerically more efficient to split the big block into several units when possible. Having the possibility to build a set of curves together, but at the same time decide in which order to build them and which of them simultaneously, is important. Even if the actual calibration is done in several units, the curves remain one block. The sensitivity of any instrument depending on any of the curves in the block will have a sensitivity to potentially all the instruments in the block. The Jacobian matrix for each curve in the block should involve all the previous curves. In the USD/JPY example, the Jacobian associated to the JPY forward six months curve will involve five curves and not only three, even if the last unit of root-finding of that building block contains only three

curves. A three curve Jacobian may be enough to solve the root-finding problem on top of the two first curves already calibrated and build the curves, but a five curve Jacobian is required to compute the total sensitivity to all market instruments involved.

As an example of the importance in terms of performance to calibrate a block in different units, we calibrated the above USD/JPY example (five curves made of a total of 50 instruments) in one unit of five curves or as three units with one, one, and three curves. In the first case the calibration time for the five curves was 27 ms and in the second case it was 20 ms, a gain of almost 25%.

In our next example we construct curves in USD – discounting and forward three months, EUR – discounting and forward three months, and JPY – discounting, forward three months and forward six months, starting from USD and using USD swaps, cross-currency instruments and EUR and JPY swaps. This was done as one block with four units and 70 instruments. With these curves and related Jacobian we compute the sensitivity of a EUR Euribor three months versus JPY Libor three months cross-currency forward swap (6M × 5Y) to the market quotes used in the curve construction. A priori the sensitivities can be to any of the market quotes used in the construction. Moreover there will be sensitivities with values in EUR and JPY, depending on the swap leg, even for the USD curves. The sensitivities are displayed in Tables 5.3, 5.4 and 5.5. (the table was split in three to fit on the pages).

The sensitivity to the USD curves is composed of numbers of opposite sign in EUR and JPY. When the exchange rate is taken into account, the total figure is small. This is expected; the USD curves have an indirect impact on the EUR and JPY curves, so some sensitivity appears, but the total impact is relatively minor. The USD curve has an important impact on both the EUR and JPY curves, and it is a similar impact in both cases. In the EUR table, the sensitivity appears in the discounting curve for the cross-currency swaps. It is an arbitrary choice to select the cross-currency swaps for the discounting curve and the IRSs for the forward curve. If we had built the curves with the cross-currency swaps in the forward curve and the IRS in the discounting curve, the curves would have been the same but the sensitivities would have appeared in different columns. This emphasises once more that the curve calibration and the market quote sensitivities are global problems. Even if we split the sensitivity result between different curves to simplify the display, it is really only one global set of instruments that creates the curves and the sensitivities. Actually the instruments are linked more to a curve unit than to a specific curve in that unit. If curves are really entangled the disentanglement for the display will always be arbitrary.

To emphasise the arbitrariness of the choice we have calibrated the same curves with the same instruments, but now associating the cross-currency swaps to the forward curve and the IRSs to the discounting curve. We have also created a mixture of both approaches with some cross-currency swaps on one curve and some others in the other curve. The sensitivities to the EUR curves of the same EUR/JPY

Table 5.3 Sensitivities of a EUR/JPY cross-currency swap in a framework where curves have been built from cross-currency instruments. Table of the USD curve sensitivities (see next tables for EUR and JPY curves)

	USD Dsc				USD Fwd 3		
Tenor	Ins.	EUR	JPY	Tenor	Ins.	EUR	JPY
ON	Cash	0	5				
TN	Cash	0	5				
1M	OIS	0	0				
2M	OIS	0	0				
3M	OIS	11	−1,050	3M	LIBOR	0	0
6M	OIS	488	−48,651	6M	IRS	0	0
9M	OIS	−1	54				
1Y	OIS	−5	394	1Y	IRS	0	0
2Y	OIS	−15	1,129	2Y	IRS	9	−510
3Y	OIS	−22	1,667	3Y	IRS	13	−754
4Y	OIS	−28	2,196	4Y	IRS	17	−1,017
5Y	OIS	−4,996	500,047	5Y	IRS	4,922	−492,628
10Y	OIS	−544	52,348	10Y	IRS	536	−51,591

Sensitivities scaled to a one basis point market quote change. Figures are in the indicated currency, not necessarily the curve currency.

Table 5.4 Sensitivities of a EUR/JPY cross-currency swap in a framework where curves have been built from cross-currency instruments. Table of the EUR curve sensitivities (see other tables for USD and JPY curves)

	EUR Dsc			EUR Fwd 3	
Tenor	Ins.	EUR	Tenor	Ins.	EUR
ON	Cash	0			
TN	Cash	0			
1M	FX	0			
2M	FX	0			
3M	FX	−30	3M	LIBOR	−11
6M	FX	−691	6M	IRS	−485
9M	FX	1			
1Y	FX	3	1Y	IRS	−2
2Y	X Ccy	−9	2Y	IRS	−10
3Y	X Ccy	−13	3Y	IRS	−11
4Y	X Ccy	−17	4Y	IRS	−17
5Y	X Ccy	−5,069	5Y	IRS	−28
10Y	X Ccy	−557	10Y	IRS	1

Sensitivities scaled to a one basis point market quote change. Figures are in the indicated currency, not necessarily the curve currency.

Table 5.5 Sensitivities of a EUR/JPY cross-currency swap in a framework where curves have been built from cross-currency instruments. Table of the JPY curves sensitivities (see previous tables for USD and EUR curves)

JPY Dsc			JPY Fwd 3			JPY Fwd 6		
Tenor	Ins.	JPY	Tenor	Ins.	JPY	Tenor	Ins.	JPY
ON	Cash	13						
TN	Cash	13						
1M	FX	0						
2M	FX	0						
3M	FX	52	3M	LIBOR	1,048			
6M	FX	1,209	6M	Basis	−48,238	6M	LIBOR	48,901
9M	FX	−1						
1Y	X Ccy	−3	1Y	Basis	0	1Y	IRS	185
2Y	X Ccy	512	2Y	Basis	0	2Y	IRS	580
3Y	X Ccy	760	3Y	Basis	0	3Y	IRS	512
4Y	X Ccy	1,028	4Y	Basis	0	4Y	IRS	841
5Y	X Ccy	501,334	5Y	Basis	0	5Y	IRS	1,867
10Y	X Ccy	54,418	10Y	Basis	0	10Y	IRS	−261

Sensitivities scaled to a one basis point market quote change. Figures are in the indicated currency, not necessarily the curve currency.

cross-currency swap is reported in Table 5.6 with these approaches. The sensitivities are the same but appear in different columns.

Note also that not all the market quotes have the same dimension. First, even if most of them are annualised rates, they do not all have the same convention. Adding them is not really advisable. More importantly, some have completely different dimensions. This is the case for the FX swaps points, which are not annualised and are exchange rate dependent – hence the small sensitivity to the FX swap points in the JPY discounting curve. This is also the case for STIR futures prices, which have the opposite sensitivity sign to rates. If the recently created present value quoted swap futures described in Section B.13 are used for curve calibration, the sensitivity is to a present value, not to a rate.

5.8 Currency dependency

Up to now we have only mentioned the dependency of the calibrated curves to the quotes of the instruments used in the calibration. There are other indirect dependencies. In particular the multiple currency instruments, like the forex forwards

Table 5.6 Sensitivities of a EUR/JPY cross-currency swap in a framework where curves have been built from cross-currency instruments. The EUR curves sensitivities for different choices of instruments

EUR Dsc			EUR Fwd 3			EUR Dsc			EUR Fwd 3		
Ten.	Ins.	EUR	Ten.	Ins.	EUR	Ten.	Ins.	EUR	Ten.	Ins.	EUR
ON	Cash	0				ON	Cash	0			
TN	Cash	0				TN	Cash	0			
1M	FX	0				1M	FX	0			
2M	FX	0				2M	FX	0			
3M	FX	1	3M	IBOR	0	3M	FX	1	3M	IBOR	0
6M	FX	−638	6M	IRS	−447	6M	FX	−638	6M	IRS	−447
9M	FX	1				9M	FX	1			
1Y	FX	2	1Y	IRS	0	1Y	FX	2	1Y	IRS	0
2Y	IRS	−8	2Y	XCcy	−8	2Y	IRS	−8	2Y	XCcy	−8
3Y	IRS	−12	3Y	XCcy	−12	3Y	IRS	−12	3Y	XCcy	−12
4Y	IRS	−16	4Y	XCcy	−15	4Y	IRS	−16	4Y	XCcy	−15
5Y	IRS	−25	5Y	XCcy	−4,524	5Y	IRS	−25	5Y	XCcy	−4,524
10Y	IRS	1	10Y	XCcy	−506	10Y	XCcy	−506	10Y	IRS	1

Sensitivities scaled to a one basis point market quote change. Figures in the indicated currency, not necessarily the currency of the curve.

and the cross-currency swaps, also depend on exchange rates. Even if the quotations for those instruments are done with only forward points and spreads, there is an implicit forex rate. A trade on these instruments is not completed before the parties have also agreed on that secondary quote. The impact of that exchange rate is small, which is why it is not appearing directly in the main quote, but still present.

In this section we indicate a way to compute the sensitivity of the calibrated curve parameters to the embedded exchange rate.

For current interest rate market quotes q^M and spot forex rates x_0, the parameters p_0 of the curves satisfy

$$S(p_0, q^M, x_0) = 0$$

where s, p and q are of dimension n and x is of dimension[2] 1.

From the above equation, one can obtain an implicit function $p(q, x)$ valid around (q^M, x_0) which satisfies

$$S(p(q, x), q, x) = 0.$$

---

[2] It is not inconceivable, even if a little bit far fetched, that a curve construction is done with several currency pairs involved. Several forex rates would be required. In that case, each forex rate sensitivity is recorded separately and the described procedure is run several times.

The standard transition matrix computed in interest rate curve construction is $D_q p$, which is given, using the implicit function theorem, by

$$D_q p = -(D_p S)^{-1} D_q S = (D_p S)^{-1}.$$

The transition matrix $D_x p$ for the exchange rate is, using similar techniques,

$$D_x p = -(D_p S)^{-1} D_x S.$$

The value of $D_x S$ needs to be computed for each specific instrument which depends on the exchange rate.

5.9 Spread curves

One of the impacts of the multi-curve framework is obviously that multiple curves are used. One of the consequences is that there are less market instruments to construct each curve, leading regularly to sparse data problems. A second consequence is that the intuition is in some cases better on the difference between two curves than on the absolute value of each of them. One technique for curve description that helps to circumvent these two issues are *spread curves*, or curves built as a first curve plus a spread.

5.9.1 Spread-over-existing

In the sense described here a spread curve over an existing one is a curve that is constructed as another curve used in the pricing, called the underlying curve, plus a second curve representing a spread. These curves have to be constructed in the right order or simultaneously. Also one has to pay attention that when computing the sensitivities to the market quotes there is an indirect impact from the underlying curve – changing the underlying curve changes the final curve even if the spread is unchanged.

This type of construction is interesting when the data to calibrate a curve is sparse or if there is a better intuition on the spread between two curves than on the level of the curve. The next two sections provide examples of this.

5.9.2 Spread-over-existing: sparse data

In this example we suppose that the discounting and forward three months curves have been constructed; we suppose also that there is a decent quantity of data for the forward three months curve but only a limited number of forward six months related instruments. In the example we only use one fixing and three swaps with maturities 2, 5, and 10 years.

If the forward six months curve is constructed directly on the sparse data, the intuition on the spread – the spread should be positive and relatively constant – is
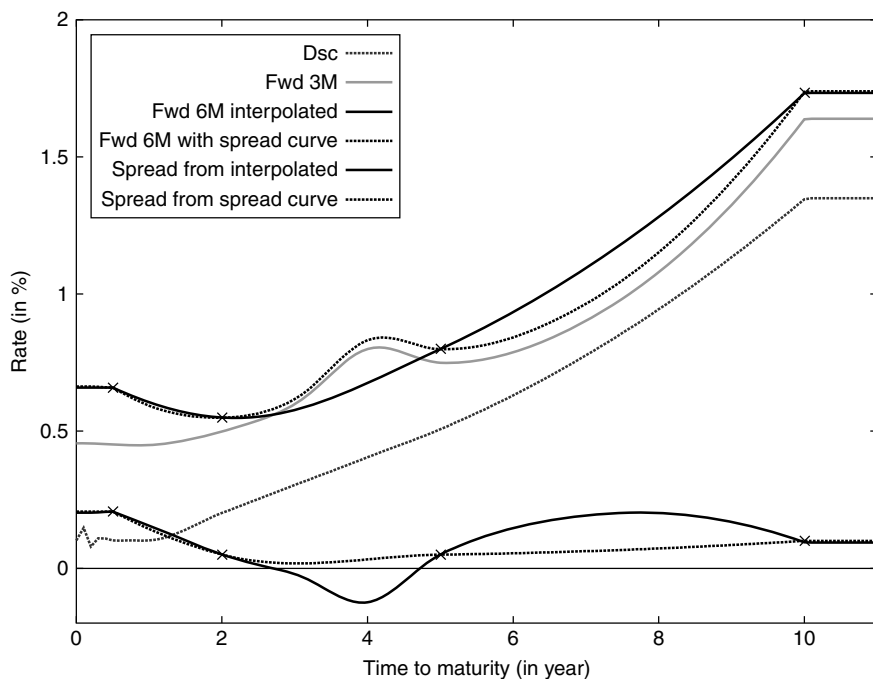
Figure 5.2 Spread-over-existing with sparse data.

not used and one can obtain strange results. The direct construction of the interpolated six months curve from the data is displayed in Figure 5.2 in a black plain line. The three months curve is in grey. The corresponding spread between the six months and the three months curves is the black line at the bottom of the figure. The crosses represent the nodes of the available six months data. The data on the three months curve has been selected to emphasise some specific features; in particular the four year data point is higher than the five year data point. The spread shows negative numbers around four years and large positive numbers around seven years, two places were there is no data. The result is not very appealing.

Another way to construct the same curves is to use the same data but to calibrate the forward six months curve as a spread above the existing three months curves with the spread interpolated. The six months curve constructed in this way is displayed in a dotted black line and the corresponding spread is the dotted line at the bottom of the figure. With this approach the result seems much more in line with intuition. The curve matches all available data and the spread is relatively stable.

5.9.3   Spread-over-existing: missing long dated data

In this example only the discounting and the forward three months curves are calibrated. The discounting curve is built with OIS but they are not available for very long swaps. We have data up to 10 years while the vanilla three months swaps are available up to 20 years. The results are presented in Figure 5.3. If the curve is simply extrapolated flat beyond ten years, we obtain the discounting curve in the black plain line and the forward curve in grey. The spread between the forward and the discounting curves is in the plain line at the bottom of the figure. Here also the spread curve seems strange, with a large bump around 15 years where we have no data for OIS.

An alternative is to build the discounting curve as a spread above the forward curve. As both the discounting curve and the forward are required to price the vanilla swaps, one can not calibrate the two curves separately; they have to be calibrated simultaneously. The discounting curve built with spread curve is displayed in the dotted line and the forward one with a different shade of grey (almost equal to the previous forward curve). The spread is in the dotted line at the bottom of the
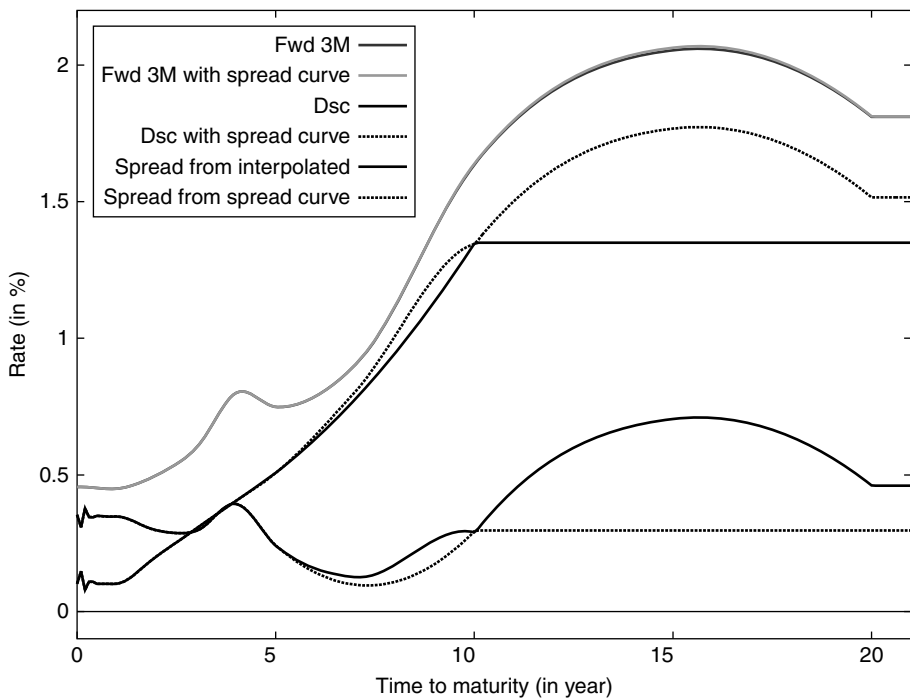
Figure 5.3 Spread-over-existing as extrapolation scheme. Discounting curve built as direct interpolation on data or as interpolated spread over the forward three months curve.

figure. As the spread is extrapolated flat, it is constant above the last data point. The result is more realistic than the previous one.

5.9.4 Static spread: turn-of-the-year effect

The spread curves can also be used to model the turn-of-year (TOY) or turn-of-quarter effects. On those dates, the short rates are often significantly higher than the days before and after. This is often due to distortion created by accounting rules. Similar phenomena, but of a lesser extent, also appear at other fixed dates, such as end of reserve periods, long holidays, and so on. An analysis of those phenomenona can be found in Hamilton (1996) and Prati et al. (2003). We would like to model this phenomenon in our curves. Ametrano and Bianchetti (2013) also analyse the TOY effect in the multi-curve framework.

In our example we suppose that the TOY (30 December 2011 to 2 January 2012) has a 25 basis point impact and the turn-of-quarter (30 March 2012 to 2 April 2012) has a 10 basis point impact. By this we mean that the short term rate for the above periods should be above the surrounding rates by the specified quantity.

We create a discount factor curve which is constant at 1 up to the last day of the year, then has a discount factor equivalent to 25 basis point overnight on the first date of the new year and from there to the last day of the quarter. From the first day of the next quarter, the discount factor is further multiplied by a number equivalent to the discount factor of 10 basis points overnight. This discount factor curve is used as an additive spread on the continuously compounded rates above an interpolated curve. This is equivalent to a multiplicative spread on discount factor. The interpolated curve is calibrated in such a way that the total curve – interpolated curve plus the TOY adjustment – prices all market instruments correctly. The overnight rates implied by the calibrated curve are displayed in Figure 5.4. The curve displayed is the curve of forward overnight rates F_X^O. Each point is an overnight rate. What appear as holes in the curve are weekends. The TOY and turn-of-quarter effects are clearly visible with points 'floating' above the rest of the curve.

The calibration is done as for any other curve, except that the fixed adjustment curve is added at construction. The TOY effect is not related to any parameter in the curve calibration process. It is considered as an exogenous effect. We insist that this is a one step calibration. The root-finding process uses the full curve, including the exogenous specific spread. There is no manipulation of data after the calibration to change some of the rates. This is very important as we want to be able to combine this feature with any of the other features described in this chapter. If a specific manipulation of data was done after the calibration, this would not be possible.

The TOY could also be incorporated through an extra instrument and its sensitivity computed if there was a market instrument related to that period.
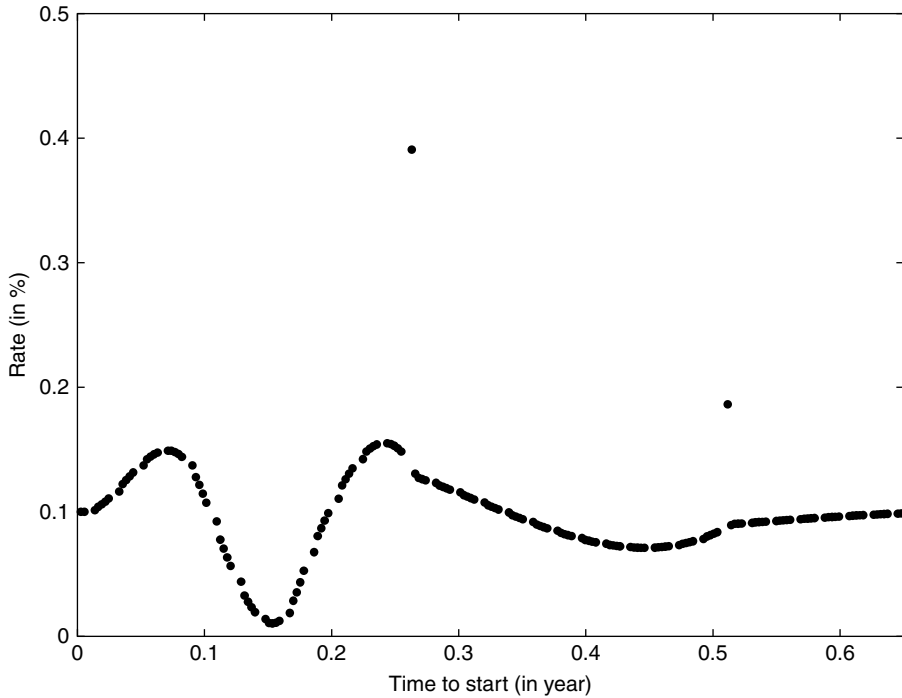
Figure 5.4 Turn-of-the-year and turn-of-the-quarter. The figure represents the curve of forward overnight rates F_X^O.

5.9.5 Multiple interpolation schemes in one curve as a spread

A priori we allow only one type of interpolation for one curve. It is nevertheless possible to create a multiple interpolation feature by describing a curve as a spread curve on two interpolated curves and defining them with different interpolation schemes. The two sub-curves will obviously be defined on disjoint sets of node points that compose the total set we would like to use for the curve.

In the example displayed in Figure 5.5 we have constructed a discounting curve using a step interpolation on the instantaneous rates or log-linear on the discount factors for the first part of the curve and a double-quadratic interpolation for the second part. Both parts are extrapolated flat. The first part has points at ON, TN, 1, 2 and 3 months. The second part has an anchor point at three months and node points at six months and above. By anchor point we mean an extra node with an imposed value of 0. Without this anchor the two curves would be invariant under translation. There would be infinitely many solutions to the root-finding problem.
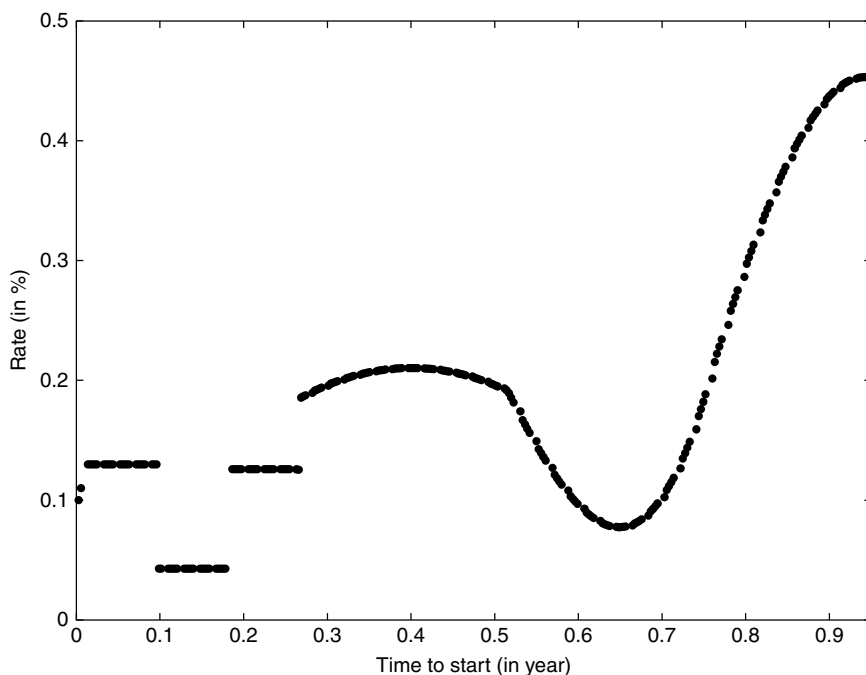
Figure 5.5 Curve using two different interpolation mechanisms: log-linear on discount factor up to three months (step function on instantaneous rates) and double-quadratic on rates above. The figure represents the curve of forward overnight rates F_X^O.

The log-linear interpolation on discount factors can be used to describe overnight-related curves with instantaneous forward rates jumping at specific dates. This can be a desired feature to represent policy committee meetings where central banks can decide on jumps of reference rates. The description of curves using that feature is presented in West (2011).

5.10 Functional curves

Another nice-to-have is the possibility to use *functional curves*. By this we mean a curve which is not interpolated but described by a parameterised function. To have an exact calibration, the number of calibrating instruments should be equal to the number of parameters. The interpolated curves can be seen as special cases of functional curves where the parameters are the node point levels.

As the number of parameters of a functional curve may not be the number of instruments one would like to use, the functional curve can be complemented by an interpolated curve with the additional number of points.
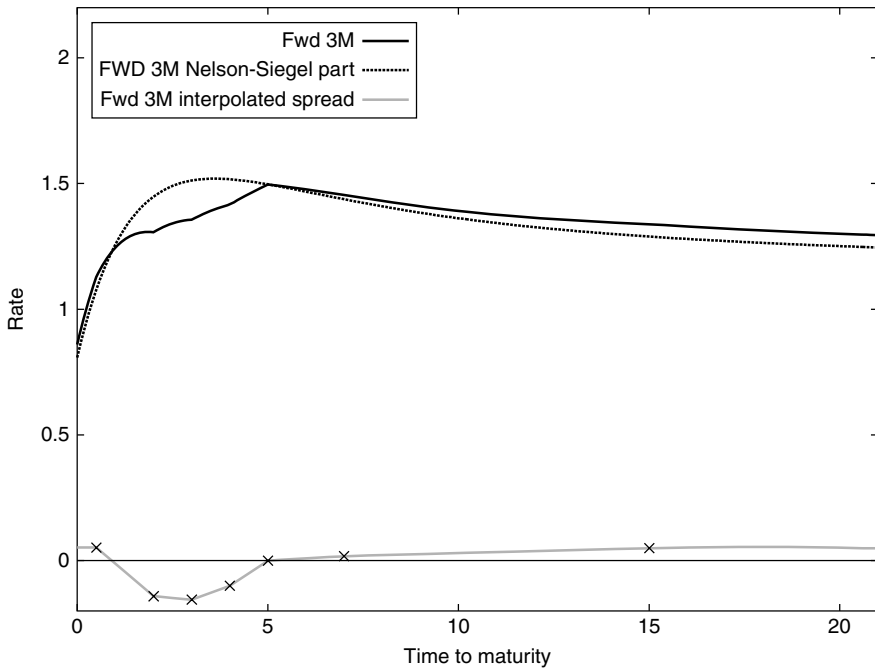
Figure 5.6 Functional curve described by a Nelson-Siegel function. Calibration with an interpolated curve as a spread.

In the example displayed in Figure 5.6, a Nelson and Siegel (1987) curve (function with four parameters) is calibrated to four instruments related to the forward three months curve: three months fixing and 1, 5 and 10 year swaps. The other points – six months, 2, 3, 4, 7 and 15 year swaps – are calibrated using an interpolated spread added to the Nelson-Siegel curve. The Nelson-Siegel part is graphed in the black dotted line and the total curve in the plain line. The spread is in grey at the bottom of the figure with the six extra nodes represented by the crosses. As both the Nelson-Siegel and the interpolated curves can be moved in a parallel way by changing the parameters, without an anchor there would be an underdetermination of the system. For that reason we anchor the interpolated spread curve to make sure it stays small. The anchoring is done by requiring that one point – in our case the 5 year point – has a value of 0 in the spread curve.

Note that the Nelson and Siegel curves, even if they have four free parameters, are not able to fit all sets of four market data points. Moreover the calibration process of those curves is not very stable. A longer analysis of the calibration with Nelson-Siegel curves and its drawbacks can be found in Annaert et al. (2012). We do not recommend using the Nelson-Siegel formula to build stable yield curves.

5.11   Interpolation on externally provided dates

On the front end of the curve, some discrete events have an important impact. Beyond the TOY effect discussed in Section 5.9.4, the central bank policy meetings are of importance. The policy rates, which directly influence the overnight market rates, are changed on those dates. It is interesting to be able to select the node of a curve to be those dates, even if the instruments used for the calibration have different maturity dates. The importance of those dates for curve construction is recognised in the literature, in particular in Pulman (2007).

> The central bank job is to take away the punch bowl just as the party gets going.
>
> *William M. Martin, 1906–1998*

In this section, we construct the OIS/discounting interpolated curve with 12 monthly OIS rates. The nodes we select are the next 12 monthly central bank meeting dates and not the maturities of the swaps. In the example, the European Central Bank (ECB) meeting dates are used. Note first that we do a global calibration of all the instruments on all the dates. The global nature of the solution allows us to provide a global exact solution even if there is more than one meeting per month. The restriction described in Pulman (2007) for those particular cases do not apply to our generalised approach.

In this example we only construct the discounting curve based on OIS; in the next example we will also incorporate a forward curve. The dates of the example have been selected to enter into the particular case described above, where two meetings are in the same month from a monthly instrument point of view. The above article (Pulman, 2007, Footnote 8) claims the case to be 'not generally soluble', but the claim is true only in the restricted bootstrapping approach he is using. In a generic root-finding framework, like the one we describe here, as long as the number of instruments is equal to the number of parameters and the system is not singular, the problem is soluble.

We insist again that all the calibration is done in one step. It is not a first calibration to the market instruments at their maturity date, an interpolation at the meeting dates and a step interpolation between the meeting dates. Such a two step approach would not reprice the initial instruments correctly. Also this exogenous date calibration is done with the same generic procedure as all other examples. This is very important as we want to be able to combine several of these effects and this would not be possible if each of them had its own specialised calibration technique.

The overnight daily rates for the first 18 months are displayed in Figure 5.7(a). The graph also represents the instrument dates (○) and the meeting dates (×) on the *X*-axis. The calibration date is 4 February 2013. In the one month to two months period, there are two meeting dates. The enlarged picture of that period is represented in Figure 5.7(b). The vertical dotted lines are the maturity dates of the
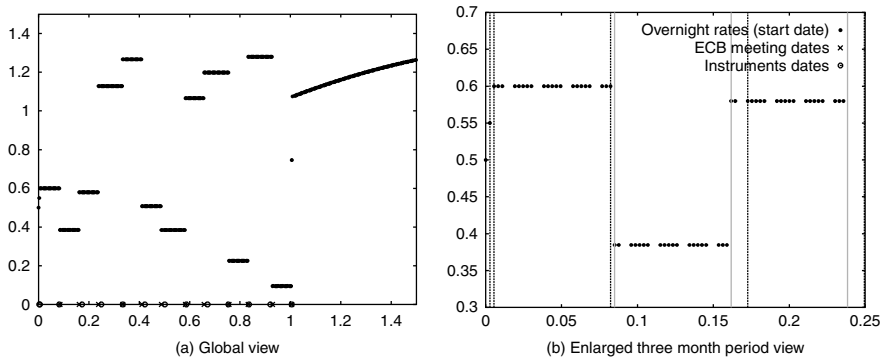
Figure 5.7 Overnight rates from a discounting curve built using central bank meeting dates. The instrument dates are ○ and the meeting dates are × on the *X*-axis.

instruments and the grey vertical lines the dates of the committee meetings. The instrument dates are 6 March 2013 and 6 April 2013 while the meeting dates are Thursday 7 March 2013 and Thursday 4 April 2013.

Note also that in this calibration, the curve is composed of three different pieces. The first part is composed of only the overnight and tom/next rates, where the curve has overnight rate steps and dates coming from the instruments. The second part is composed of the 12 monthly OIS swaps, where the curve also has overnight rate steps but the node dates are exogenous – ECB meeting dates. The data has been selected to emphasise the monthly jump but is not representing a realistic situation as the rates jump up and down at each meeting. The last part is composed of annual OISs, where the curve has smooth zero-coupon rates with dates coming from the instruments. It would be possible to combine more parts or to change the size of each of them if a special market condition required it.

5.12 Combining several effects

In the example of this section, we combine several effects described in the previous sections. We build the discounting curve from OISs with central bank meeting dates as nodes. We use a three layer curve like in the previous section. On that curve we add an exogenous TOY effect. The curve calibration date is 15 November 2012, so there is no overlap of instrument dates and meeting dates anymore, but the meeting date jumps appear in the middle of the instrument months. The overnight rates curve is reproduced in Figure 5.8. The focus on the three first months is depicted in the right part of the figure. The jump at the year-end is clearly visible.

Based on that discounting curve, we build a forward six months curve. The discounting curve is the above four piece curve – with piecewise constant overnight
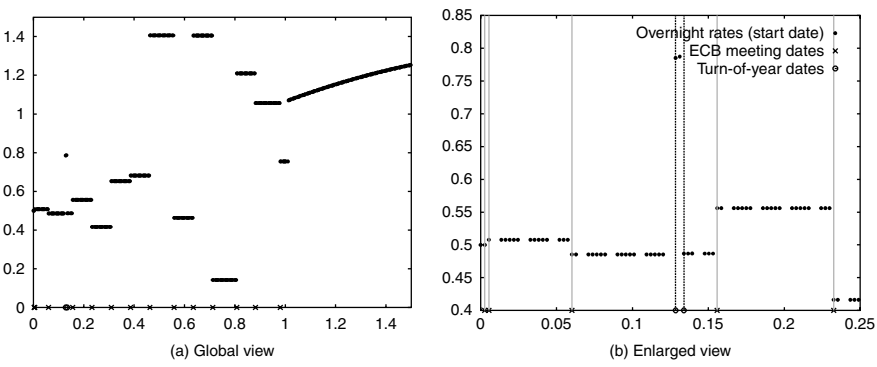
Figure 5.8 Overnight rates from a discounting curve built using central bank meeting dates and TOY effect. The end-of-year dates are represented by ∘ and the meeting dates by × on the X-axis.
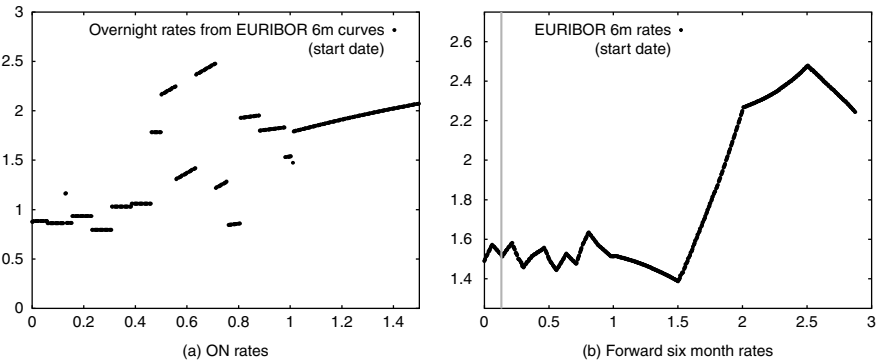
Figure 5.9 Overnight rates and forward six month rates from the forward Euribor six month curve. The curve is built as a spread over the discounting curve with central bank meeting dates.

rates – and the forward curve is built as a smooth spread – double-quadratic – above the discounting curve. The first market rates used for building the forward curve are the six months fixing, 3x9 FRA, 6x12 FRA and 2Y IRS. They are all taken with the same market rate at 1.5%. All the 'jumps' on the overnight rates computed on the forward curve come from the underlying discounting curve, not from the Euribor instruments data. We propose a graph of the forward curve overnight implied rates in Figure 5.9(a). Note that those rates have no theoretical meaning as the curve is defined only to compute the six month rates, but can be interpreted as the overnight rates that one needs to compose in order to obtain the Euribor rate. This interpretation is possible only because we have calibrated the curve using pseudo-discount factors as described in Section 3.1.

In Figure 5.9(b), the six months forward curve is displayed. Even if the market data is flat for two years, the forward rates exhibit some waves. They are the changes implied by the policy dates linked changes of the discounting curve.

5.13 Examples of standard curves

In this section, we provide examples of curves in the main currencies built using reasonably liquid instruments. The examples do not represent the only choices nor do they represent the best choices for all circumstances.

They nevertheless represent curves regularly used in practice. They provide good curves for a lot of applications. The instruments are acceptably liquid. Obviously, it is possible to combine different elements in different ways.

We first provide a couple of examples with only one currency, and later a couple of cross-currency examples.

5.13.1 One currency

5.13.1.1 USD

In USD, the OISs are relatively liquid but not on the long part of the curve. Quotes are usually provided up to ten years, even if the liquidity is lesser above five years. The standard forward curve for USD is the three month curve. In the first set of curves, described in the left part of Table 5.7, we propose to use the OIS instruments for the discounting curve and a mixture of fixing, FRAs and vanilla IRSs for the forward curve.

Even if the OISs are not liquid on the long part of the curve, it is still possible to construct an overnight curve using different instruments linked to the overnight rates, the Federal Funds swaps. These instruments are described in Section 6.4. From a curve construction point of view, the important feature is that the instrument exchanges three months Libor rates for quarterly payments related to the Federal Fund rates. It is an overnight indexed and Libor indexed instrument. If this instrument is used, the discounting/overnight forward and the Libor forward curves need to be calibrated simultaneously. On the short end of the overnight curve the Federal Fund futures are relatively liquid. They are described in Section 6.5.

Among the Libor three months related instruments, another liquid type of instrument are the STIR futures. They are futures on three months Libor[3], they are quoted up to ten years and liquid up to two or three years. In the second part of Table 5.7, we propose to build the discounting/overnight curves using Federal Funds futures, OIS, Federal Fund swaps and the forward three months curve using STIR futures and IRSs.

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[3] There also exist futures on one month Libor.

Table 5.7 First part of the table: two USD curves – discounting and Libor three months – built with OIS for the first and FRA and IRS for the second. Second part of the table: two USD curves – discounting and Libor three months – built with Fed Funds futures, OIS and Fed Fund swaps for the first and STIR futures and IRS for the second

USD: Two curves with FRAs				USD: Two curves with STIR futures			
Discounting		Forward 3M		Discounting		Forward 3M	
Label	Ins.	Label	Ins.	Label	Ins.	Label	Ins.
ON	Deposit			ON	Deposit	3M	Fixing
TN	Deposit			TN	Deposit	EDc1	STIR Futures
1M	OIS			FFc1	FF futures	EDc2	STIR futures
2M	OIS			FFc2	FF futures	EDc3	STIR Futures
3M	OIS	3M	Fixing	FFc3	FF futures	EDc4	STIR Futures
6M	OIS	3Mx6M	FRA	6M	OIS	EDc5	STIR Futures
9M	OIS	6Mx9M	FRA	9M	OIS	EDc6	STIR Futures
12M	OIS	1Y	IRS	12M	OIS	EDc7	STIR Futures
2Y	OIS	2Y	IRS	2Y	OIS	EDc8	STIR Futures
3Y	OIS	3Y	IRS	3Y	FF Swap	3Y	IRS
4Y	OIS	4Y	IRS	4Y	FF Swap	4Y	IRS
5Y	OIS	5Y	IRS	5Y	FF Swap	5Y	IRS
7Y	OIS	7Y	IRS	7Y	FF Swap	7Y	IRS
10Y	OIS	10Y	IRS	10Y	FF Swap	10Y	IRS
		15Y	IRS	15Y	FF Swap	15Y	IRS
		20Y	IRS	20Y	FF Swap	20Y	IRS

5.13.1.2 EUR

EUR is usually a little bit more involved than USD for the basic curves. The reason is that there are two dominant tenors for the forward rates: three months Euribor on the short term and six months Euribor on the long term. The exact split is not definitive but somewhere between one and three years.

An example is provided in Table 5.8. The discounting curve is based on OIS up to ten years. The forward three months curve is based on STIR futures on three month Euribor and basis swaps between Euribor six months and Euribor three months plus spread. The forward six month curve is based on FRAs and vanilla swaps. In this particular case, the three curves can be built as three different units. The order is discounting/overnight, six months forward and three months forward.

We have used different sets of tenors for the different curves. This is not an indication of fundamental differences between the curves but only to indicate that there is no calibration reason to have the same tenors on different curves. The curves are calibrated by a global root-finding exercise and there is no need to align the different

Table 5.8 Three curves in EUR: discounting, forward three months Euribor and forward six months Euribor. Curves built with basis swaps between six and three months

EUR: Three curves with futures and basis swaps					
Discounting		Forward 3M		Forward 6M	
Label	Ins.	Label	Ins.	Label	Ins.
ON	Deposit	3M	Fixing	6M	Fixing
TN	Deposit	EDc1	Futures	3Mx9M	FRA
1M	OIS	EDc2	Futures	6Mx12M	FRA
2M	OIS	EDc3	Futures	12Mx18M	FRA
3M	OIS	EDc4	Futures	2Y	IRS
6M	OIS	EDc5	Futures	3Y	IRS
9M	OIS	EDc6	Futures	4Y	IRS
12M	OIS	EDc7	Futures	5Y	IRS
2Y	OIS	EDc8	Futures	6Y	IRS
3Y	OIS	3Y	Basis	7Y	IRS
4Y	OIS	4Y	Basis	8Y	IRS
5Y	OIS	5Y	Basis	9Y	IRS
7Y	OIS	7Y	Basis	10Y	IRS
10Y	OIS	10Y	Basis	15Y	IRS
		15Y	Basis	20Y	IRS
		20Y	Basis	30Y	IRS
		30Y	Basis		

points. From a risk reading point of view, it is usually easier to have the same pillars in the risk reports. This is only for convenience and not a curve calibration requirement.

5.13.1.3 AUD

The AUD examples are always interesting due to the structure of the market. The liquid swaps are quarterly/quarterly for tenors from one to three years and semi-annual/semi-annual from four years onward. An example of a system of three curves, one discounting curve, one forward BBSW three months curve and one forward BBSW six months curve is described in Table 5.9.

The entanglement between the three and six months forward instruments means it is not possible to calibrate the three curves as three units. The two forward curves need to be calibrated simultaneously. Note also that in AUD the STIR futures are bank bill and physical delivery based. Their description is different from the one of Section B.4.

Table 5.9 Three curves in AUD: discounting, forward BBSW three months and forward BBSW six months. Note the combination of basis and vanilla swaps on both forward curves

AUD: Three curves with futures and basis swaps					
Discounting		Forward 3M		Forward 6M	
Label	Ins.	Label	Ins.	Label	Ins.
ON	Deposit				
TN	Deposit				
1M	OIS				
2M	OIS				
3M	OIS	3M	Fixing		
6M	OIS	IR-1	STIR Futures	6M	Fixing
9M	OIS	IR-2	STIR Futures	3Mx9M	FRA
12M	OIS	1Y	IRS	6Mx12M	FRA
2Y	OIS	2Y	IRS	2Y	Basis
3Y	OIS	3Y	IRS	3Y	Basis
4Y	OIS	4Y	Basis	4Y	IRS
5Y	OIS	5Y	Basis	5Y	IRS
7Y	OIS	7Y	Basis	7Y	IRS
10Y	OIS	10Y	Basis	10Y	IRS

5.13.2 Multiple currencies

5.13.2.1 USD–JPY

Suppose we want to build JPY curves based on the USD curves. We select a discounting curve and a forward three months curve from the USD section. Then we need some instruments to transfer the information from USD to JPY. The most liquid instruments to do so are the forex swaps (see Appendix B.14) up to one year and the cross-currency swaps (see Appendix B.15) from two years on. The most liquid cross-currency swaps are quarterly USD Libor versus quarterly JPY Libor plus spread. We now have a second curve in JPY, the Libor three months. The Libor three months IRS are not very liquid in JPY, the most liquid Libor three months swaps are the basis swaps versus Libor six months. We need thus to incorporate the Libor six months forward curve in our unit. The liquid instruments linked to Libor six months are the IRSs. We have obtained a unit of three curves in JPY – discounting, Libor three months and Libor six months – that can not be unentangled. A simplified version of these curves is proposed in Table 5.10.

5.13.2.2 USD–EUR

For the USD-EUR case we could repeat the USD-JPY situation, but it would not be interesting. A different construction involves a new instrument: overnight indexed

Table 5.10 Three curves in JPY: discounting, forward three months and forward six months. Curves built from multi-currency instruments

JPY: Three curves based on USD					
Discounting		**Forward 3M**		**Forward 6M**	
Label	Ins.	Label	Ins.	Label	Ins.
1M	FX swap				
3M	FX swap	3M	Fixing		
6M	FX swap	3Mx6M	FRA	6M	Fixing
12M	FX swap	9Mx12M	FRA	6Mx12M	FRA
2Y	X-ccy swap	2Y	Basis	2Y	IRS
5Y	X-ccy swap	5Y	Basis	5Y	IRS
10Y	X-ccy swap	10Y	Basis	10Y	IRS

Table 5.11 Three curves in EUR: discounting, forward three months and forward six months. Curves built from multi-currency instruments

EUR: Three curves based on USD					
Discounting		**Forward 3M**		**Forward 6M**	
Label	Ins.	Label	Ins.	Label	Ins.
1M	X-ccy ON swap				
3M	X-ccy ON swap	3M	Fixing		
6M	X-ccy ON swap	ERc1	STIR futures	6M	Fixing
		ERc2	STIR futures		
12M	X-ccy ON swap	ERc3	STIR futures	6Mx12M	FRA
		ERc4	STIR futures		
		ERc5	STIR futures		
		ERc6	STIR futures		
2Y	X-ccy ON swap	ERc7	STIR futures	2Y	IRS
5Y	X-ccy swap	5Y	Basis	5Y	IRS
10Y	X-ccy swap	10Y	Basis	10Y	IRS

cross-currency swaps. This is a cross-currency swap paying on each leg a floating coupon similar to the floating coupon of OISs. These instruments can be traded in the USD/EUR market with maturities to a couple of years. Table 5.11 describes the situation where the forex swaps and some cross-currency swaps have been replaced by cross-currency overnight indexed swaps and the forward three months curve is built with futures.

6 | More Instruments

6.1 Overnight indexed swaps

It may seem strange to start this chapter with an instrument that has already been analysed. The reason for returning to this instrument is the *payment lag*. The payment of OIS coupons does not occur on the end day of the fixing period but with a certain lag. The origin of the lag is for settlement purposes. Most of the instruments do not start immediately but with a spot lag, for the same reason the payment of OIS is not immediate. The last fixing is usually known only the business day before the end of the fixing period or even on the last day itself (see Appendix B for more details on the conventions). Without that lag, the time between the knowledge of the exact amount to be paid and the payment itself would be too short.

Because of the payment lag, the amount is not paid at its natural payment date. An contingent claim valuation should be done for that change in payment timing. This is what we analyse in this section. Without killing the suspense, we can already reveal that the impact of the lag is small. It is nevertheless important to analyse it to be able to claim so with confidence. The results of this section are mainly borrowed from Henrard (2004).

To simplify the computation, we suppose that the coupon is obtained by continuous composition of the risk-free cash account short rate with a deterministic spread; that is we suppose that the amount of the payment made in t_p is

$$\prod_{i=1}^{n} (1 + \delta_i I^O(t_{i-1}, t_i)) = N_{t_n}(N_{t_0})^{-1} \beta^O(0, t_0, t_n).$$

Henrard (2004) extends the analysis to the difference between continuous and discrete composition. Here we restrict ourselves to the payment lag analysis.

In the cash account numeraire, the present value of the coupon in 0 is

$$N_0 \, \mathrm{E}^N \left[(N_{t_p})^{-1} N_{t_n}(N_{t_0})^{-1} \beta^O(0, t_0, t_n) \right].$$

To simplify the notation, we use the convention that $\nu(s, t) = 0$ for $s > t$. This can be interpreted financially as, after the forward date is passed, the rate is known and

its volatility is 0. Using Lemma A.3 three times, we have

$$(N_{t_p})^{-1} N_{t_n} (N_{t_0})^{-1} = P^D(0, t_p) \frac{P^D(0, t_0)}{P^D(0, t_n)}$$

$$\exp \left(- \int_0^{t_p} (v(s, t_p) - v(s, t_n) + v(s, t_0)) dW_s \right.$$

$$\left. - \frac{1}{2} \int_0^{t_p} \left(|v(s, t_p)|^2 - |v(s, t_n)|^2 + |v(s, t_0)|^2 \right) \right).$$

The term inside the exponential can be written as a stochastic part minus one half of its variance and some remaining terms. The remaining terms can be reduced (after lengthly but easy computation) to

$$\zeta(t_0, t_n, t_p) = \exp \left(\int_0^{t_p} (v(s, t_n) - v(s, t_p))(v(s, t_n) - v(s, t_0)) \right) ds.$$

Theorem 6.1 *Under the hypotheses **D**, I^{CPN} and $S0^{CPN}$, the present value of the continuously compounded overnight indexed coupon described above is given by*

$$P^D(0, t_p) \frac{P^D(0, t_0)}{P^D(0, t_n)} \beta^O(0, t_0, t_n) \zeta(t_0, t_n, t_p).$$

Obviously if $t_p = t_n$, the term ζ is equal to 1, no adjustment is required and the standard market formula described in Theorem 8.7 applies. Note that the adjustment ζ is smaller than 1 for $t_p > t_n$ and a Hull–White one-factor model.

An example of the impact of the adjustment is described in Table 6.1. The example uses a flat curve at 1.50% and a Hull–White model with a mean reversion parameter $a = 0.01$ and a volatility parameter $\eta = 0.01$. The numbers are computed for coupons compounded over periods between one week and twelve months and payment lag of one, two and seven days. The impacts are for a nominal of one billion. For a one year payment period and a two day payment lag, which is the standard in USD, the impact is less than 0.003 basis points in rate.

Table 6.1 Payment delay convexity adjustment impact

		Compounding period				
		1w	1m	3m	6m	12m
Lag	1d	−0.15	−0.99	−9.06	−35.01	−137.08
	2d	−0.20	−1.97	−18.12	−70.01	−274.10
	1w	−0.35	−6.91	−63.40	−244.88	−958.65

Notional of one billion.

In the rest of the book, we will ignore the timing adjustment and use the simplified formula

$$P^D(0, t_p) \frac{P^D(0, t_0)}{P^D(0, t_n)} \beta^O(0, t_0, t_n).$$

6.2 Ibor coupons with date mismatches

In this section we analyse the impact of the date mismatches on Ibor coupon valuation. The date mismatches can come from business day conventions, as described in Appendix B.3, from short/long tenor coupons or in-arrear coupons. A short/long coupon is a coupon paying a given index for a period shorter/longer than its underlying index. A coupon paying Euribor three months on a one month period is a short coupon. An in-arrear coupon is a coupon that fixes in arrear, that is a coupon fixed at a spot lag before the payment date, not before the start date.

In all cases we have a coupon fixing in t_0 on an Ibor for the underlying period $[u, v]$ which is paid in t_p. The time t_p is potentially different from v.

If we ignore the potential notional, the coupon payoff in t_p is $\delta I_X^j(t_0)$. The forward rate can be written as

$$F_X^{\text{CPN},j}(t_0, u, v) = \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t_0, u, v)) - \frac{1}{\delta}.$$

Suppose $t_p < v$, which is always the case for date mismatches coming from business day convention adjustments in swaps. In the $P_X^D(., v)$ numeraire, the coupon present value is

$$\frac{1}{\delta} P^D(0, v) \, \mathrm{E}^v \left[\frac{P_X^D(t_0, t_p)}{P_X^D(t_0, v)} (1 + \delta F_X^{\text{CPN},j}(t_0, u, v)) \right] - \frac{1}{\delta} P_X^D(0, t_p).$$

The expected value is obtained by discounting the payoff to the fixing date t_0. We work under the constant spread hypothesis $\mathbf{SO}^{\text{CPN}}$. The investment factor can be written as

$$(1 + \delta F_X^{\text{CPN},j}(t_0, u, v)) = \beta_X^{\text{CPN},j}(u, v) \frac{P_X^D(t_0, u)}{P_X^D(t_0, v)}.$$

To value the above expected value we use Lemma A.2 twice and we have to compute

$$\frac{P_X^D(0,t_p)}{P_X^D(0,v)}\beta_X^{\text{CPN},j}(u,v)\frac{P_X^D(0,u)}{P_X^D(0,v)}$$

$$E^v\left[\exp\left(-\int_0^{t_0}(v(\tau,t_p)-v(\tau,v))\cdot dW_\tau^v-\frac{1}{2}\int_0^{t_0}|v(\tau,t_p)-v(\tau,v)|^2 ds\right)\right.$$

$$\left.\exp\left(-\int_0^{t_0}(v(\tau,u)-v(\tau,v))\cdot dW_\tau^v-\frac{1}{2}\int_0^{t_0}|v(\tau,u)-v(\tau,v)|^2 ds\right)\right].$$

The variance of the stochastic integral in the exponentials is equal to the sum of squared plus the term

$$\int_0^{t_0}(v(\tau,t_p)-v(\tau,v))(v(\tau,u)-v(\tau,v))ds.$$

The expected value is thus reduced to

$$\zeta(t_0,u,v,t_p)=\exp\left(\int_0^{t_0}(v(\tau,t_p)-v(\tau,v))(v(\tau,u)-v(\tau,v))d\tau\right).$$

Theorem 6.2 *Under the hypotheses D, I$^{\text{CPN}}$ and S0$^{\text{CPN}}$, the present value in* 0 *of the coupon with early payment date* t_p *is, in the HJM model on the discounting curve,*

$$P_X^D(0,t_p)F_X^{\text{CPN},j}(0,u,v)\zeta(t_0,u,v,t_p)+\frac{1}{\delta}P_X^D(0,t_p)(\zeta(t_0,u,v,t_p)-1).$$

When the coupon is a standard coupon with $t_p=v$, the adjustment ζ is equal to 1 and the standard formula is recovered. In all cases, $\zeta>1$ for Hull–White one-factor models.

The rate adjustment, that is the amount by which the rate should be changed in the standard valuation formula, is given by

$$\left(\frac{1}{\delta}-F_X^{\text{CPN},j}(0,u,v)\right)(\zeta-1).$$

A graphical representation of the adjustment is proposed in Figure 6.1. The model used was a constant volatility Hull–White model with a mean reversion of 0.01 and a volatility of 0.02. The volatility used is probably larger that the market volatility, which is closer to 0.01. The adjustment produced is more on the high side than a realistic estimate. The delays $v-t_p$ used are one, two and seven days. The one and two day delays are encountered once a week each. If holidays are included the differential can be larger. Some examples of larger date mismatches are provided in Section B.3. For standard coupons, the seven day example is certainly an upper limit. The payment periods used in the examples are three and six months. The adjustment is largely independent of the rate level and of the payment period.
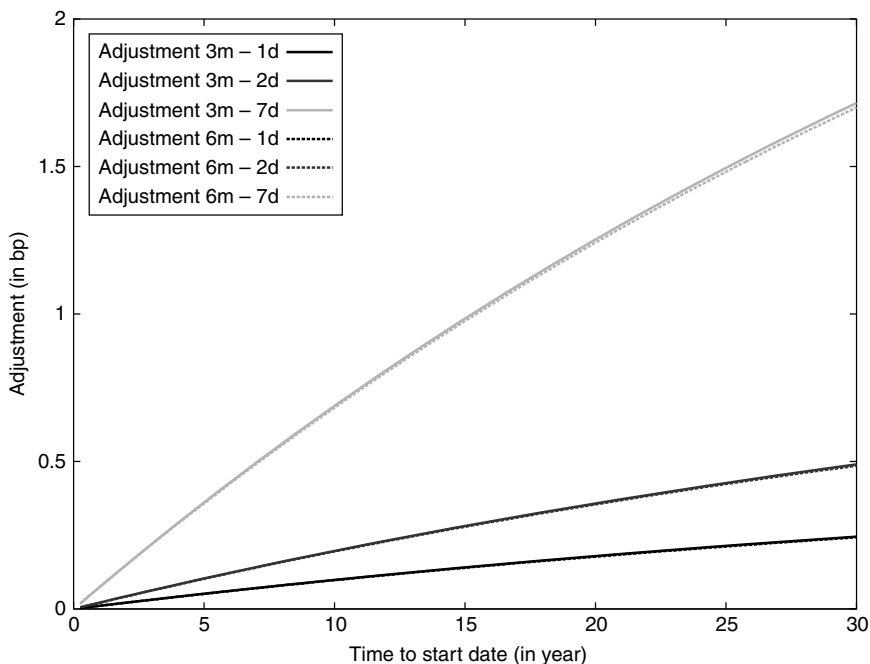
Figure 6.1 Timing adjustment for Ibor rate payments in case of dates mismatches. Impacts for payments one, two and seven days before the underlying index date. Payment periods are three and six months.

If we approximate the impact to be a one day lag in one day out of seven and a two day lag in one day out of seven and take the values from the graph, we obtain an approximate a total impact on a 30 year swap rate of 0.05 basis points. This is probably too small to significantly impact a market quote but large enough that risk managers want to check those numbers for large books. As the impacts are date dependent and a one day change in the starting date can have an important impact, it is not inconceivable for a large book with small total interest rate sensitivity to have a large timing adjustment impact.

6.3 Compounded coupons

The description of a compounded coupon is the following. The times associated are denoted t_i $(i = 0, \ldots, n)$. The fixing take place at dates $t^f_{i-1} \le t_{i-1}$. The difference between t^f_{i-1} and t_{i-1} is the spot lag. The rates for the periods $[t_{i-1}, t_i]$ are denoted $I^j_X(t^f_{i-1})$ $(i = 1, \ldots, n)$ and the accrual factors in the index convention are δ_i.

The coupon pays in t_n the amount (to be multiplied by the notional)

$$\left(\prod_{i=1}^{n}(1+\delta_i I_X^j(t_{i-1}^f))\right)-1.$$

From Equation 2.3, each fixing is linked to the curve on the fixing date by

$$I_X^j(t_{i-1}^f)=F_X^{\text{CPN},j}(t_{i-1}^f,t_{i-1},t_i).$$

In this section, we work under the standard simplifying assumption of constant spread **S0$^{\text{CPN}}$**.

Theorem 6.3 *Under the hypotheses D, I^{CPN} and S0^{CPN}, the present value of the compounded Ibor coupon described above is in t*

$$P_X^D(t,t_n)\left(\prod_{i=1}^{n}(1+\delta_i F_X^{\text{CPN},j}(t,t_{i-1},t_i))\right)-1.$$

Proof: The present value (in the $P_X^D(.,t_n)$ numeraire) is given by

$$P^D(t,t_n)\,\mathrm{E}^n\left[\left(\prod_{i=1}^{n}\left(1+\delta_i F_X^j(t_{i-1}^f,t_{i-1},t_i)\right)\right)-1\,\Big|\,\mathcal{F}_t\right] \tag{6.1}$$

Obviously the part that requires analysis is the product. Using the shortened notation β_{i-1} for $\beta_X^{\text{CPN},j}(0,t_{i-1},t_i)$, we have

$$\mathrm{E}^n\left[\prod_{i=1}^{n}\left(1+\delta_i F_X^{\text{CPN},j}(t_{i-1}^f,t_{i-1},t_i)\right)\Big|\,\mathcal{F}_t\right]$$

$$=\mathrm{E}^n\left[\prod_{i=1}^{n}\beta_{i-1}\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\Big|\,\mathcal{F}_t\right]$$

$$=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-1}\beta_{i-1}\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{n-1}\,\mathrm{E}^n\left[\frac{P^D(t_{n-1}^f,t_{n-1})}{P^D(t_{n-1}^f,t_n)}\Big|\,\mathcal{F}_{t_{n-2}^f}\right]\Big|\,\mathcal{F}_t\right]$$

$$=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-1}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-1}}^j\frac{P^D(t_{n-2}^f,t_{n-1})}{P^D(t_{n-2}^f,t_n)}\Big|\,\mathcal{F}_t\right]$$

$$=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-2}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-2}^f}^j\beta_{t_{n-1}^f}^j\frac{P^D(t_{n-2}^f,t_{n-2})}{P^D(t_{n-2}^f,t_n)}\Big|\,\mathcal{F}_t\right]$$

$$=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-2}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-2}^f}^j\beta_{t_{n-1}^f}^j\frac{P^D(t_{n-3}^f,t_{n-2})}{P^D(t_{n-3}^f,t_n)}\Big|\,\mathcal{F}_t\right].$$

In the second equality we use the fact that $P_X^D(s,t)$ is $\mathcal{F}_{t_{n-2}}$-measurable for $s \le t_{n-2}$ and that β_{n-1} is constant. In the third equality we use the martingale property of the asset $P_X^D(.,x)$ in the $P_X^D(.,t_n)$ numeraire. The fourth equality is simply the simplification of fractions. In the fifth equality we use the same technique as in the second and third but with t_{n-3}^f.

The same procedure can be repeated down for all factors in the product and we obtain

$$
E^n \left[\left(\prod_{i=1}^{n} \beta_{i-1} \right) \frac{P_X^D(t_0^f, t_0)}{P_X^D(t_0^f, t_n)} \middle| \mathcal{F}_t \right].
$$

Applying the martingale property once more we obtain for the coupon value

$$
P_X^D(t, t_n) \left(\prod_{i=1}^{n} \beta_{i-1} \frac{P_X^D(t, t_0)}{P_X^D(t, t_n)} - 1 \right)
$$

The announced result is obtained by re-decomposing the product and using the definition of β_{i-1}.   □

A compounded Ibor coupon can be priced as the discounted value of compounded forward. This is a strong result that requires the strong hypothesis **S0$^{\mathbf{CPN}}$**.

In the case of a forward curve described by pseudo-discount factors, the result can be simplified further and reads as

$$
P_X^D(t, t_n) \left(\frac{P_X^j(t, t_0)}{P_X^j(t, t_n)} - 1 \right).
$$

This means that the compounded coupon can be priced as if there were no composition but simply one payment on one rate.

6.4 Federal Funds swaps

The description of the Fed Fund swaps and their conventions can be found in Appendix B.12.

6.4.1 Approximation of the arithmetic average

When computing the price of compounded rate with pseudo-discount factors implementation, if we use the simplifying assumption of deterministic spread (which is trivially satisfied when pricing OISs with OIS discounting), we do not

need to compute every overnight rate and actually compound them. It is possible to use the direct formula of the ratio of start and end discount factors

$$P^D(0, t_n) \, \mathrm{E}^n \left[\left(\prod_{i=1}^n \frac{P^O(t_{i-1}, t_{i-1})}{P^O(t_{i-1}, t_i)} \right) - 1 \right] = P^D(0, t_n) \left(\frac{P^O(0, t_0)}{P^O(0, t_n)} - 1 \right).$$

This result is described in Section 2.7.

This type of direct computation result, which speeds-up the computation dramatically, is not available for arithmetic average. Nevertheless some approximations are available and we describe one here. It was initially introduced in Takada (2011).

Let A_c be the amount paid for the compounded rates, that is,

$$A_c = \left(\prod_{i=1}^n (1 + \delta_i I^O(t_{i-1})) \right) - 1.$$

To the first order,

$$1 + \delta_i I^O(t_{i-1}) \simeq \exp \left(\delta_i I^O(t_{i-1}) \right).$$

As δ_i is small, the approximation is relatively good. Let A_a denote the amount paid for the arithmetic average. We have

$$A_a = \sum_{i=1}^n \delta_i I^O(t_{i-1})$$

$$\simeq \ln \left(\prod_{i=1}^n (1 + \delta_i I^O(t_{i-1})) \right) = \ln (1 + A_c).$$

As A_c can be computed efficiently, if the approximation is good enough and the adjustment due to the non-linearity of the function is not too large, we have an efficient approximate way to compute the arithmetic average coupon.

In Table 6.2 we analyse the quality of the approximation for several levels of rates. The curves we use are constant (on zero-coupon continuously compounded rates) with level 1%, 5% and 10%. We have computed the difference between the exact arithmetic average and the one provided by the above approximation. The coupons are three months long with the dates computed using market conventions. We run the test with a sample of 36 different starting dates to cover different month lengths and weekend effects. Depending on the level of rates, the error is from below 0.01 basis points to 0.25 basis points. The first error is certainly small enough; it is that of a 1% curve level, which is roughly the rate level in the main currencies at the time of writing. The error for a rate level of 10% is probably beyond the level of precision one would like for market making.

For low rates ($< 5\%$), the approximation is precise enough with the error below the quotation precision of the relevant market instrument, which is at best of 0.1 basis points.

Table 6.2 Error between exact arithmetic average and proposed approximation. Results for three month coupons

Error – curve 1%		Error – curve 5%		Error – curve 10%	
Min.	Max.	Min.	Max.	Min.	Max.
0.002	0.003	0.062	0.064	0.248	0.256

Figures in basis points. Figures computed with exact market dates for a sample of 36 starting dates.

Table 6.3 Performance for exact and approximated formulas of the arithmetic average coupons

Exact formula			Approximated formula			Ratio		
PV	PVCS	C. + PV	PV	PVCS	C. + PV	PV	PVCS	C. + PV
200	215	470	5	12	250	40.0	17.9	1.9

Figures for 10,000 three month coupons. Time in milliseconds. 'PV' corresponds to the present value, 'PVCS' to the present value curve sensitivity and 'C. + PV' to the coupon construction (all daily dates) and the present value.

Now that we have established that the approximation is good enough in the current rate environment, we have to analyse if it is useful. Is the computation time of the approximation lower enough to justify the use of the approximation? The answer to that question is summarised in Table 6.3. We have analysed the performance in three ways: the calculation itself for present value and curve sensitivity (bucketed delta) and the performance of the instrument construction and the present value. The last measure requires probably a little bit of explanation. For simple instruments, the time consuming part is often to construct the dates associated to the instrument following the different conventions, not to compute the relatively simple present value. This is certainly the case for overnight-related products, where a lot of dates are calculated. If the actual value computation time was dwarfed by the instrument construction time, the approximation exercise would be relatively meaningless. Note also that the curve sensitivity is computed by algorithmic differentiation and the results are (almost) independent of the number of points used to calibrate the curve and the number of node points impacting the instrument.

The conclusion of the table is that for the computation of present value and curve sensitivity, the approximation formula is significantly faster (between 20 and 40 times). This is not surprising as we use only two dates (start and end) to compute the result instead of all the daily points (around 60 dates for three months). If we add the instrument construction time, the impact is lesser but still relatively important. The time for construction and present value is divided by two. Which of those two numbers is relevant in practice: 2 or 20? A little bit in between! If you want

only to compute the present value of a new instrument, the ratio 2 is the realistic one. But if you build curves with those instruments, the instrument description is computed once and the present value and curve sensitivities are computed numerous times. Suppose that to calibrate the curves you need for each instrument one instrument construction, ten present values, and two sensitivities, the ratio in that case would be around eight. The small approximation error is probably a price you would accept to divide the curve construction time by eight.

At this stage, the analysis covers only the *static approximation*, that is the computation of the arithmetic average when the rates are known. A second level of approximation is required as the rates in the arithmetic average are not paid at the right time; they are paid at the end of the accrual period and not at the end of their reference period.

6.4.2 Convexity adjustment

In this section we compute the Fed Fund swaps value including the convexity adjustment. The amount

$$\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1})$$

is paid in t_n.

We work in the multi-curve framework under the assumption **S0$^{\text{CPN}}$** of constant spread $\beta^O(t_{i-1}, t_i)$ between the discounting curve and the overnight forward curve. This assumption is trivially satisfied if the discounting curve is equal to the overnight forward curve. The model used is the multi-factor HJM model on the discounting curve as described in Appendix A. The forward overnight rate seen from t is denoted $F_X^O(t, t_{i-1}, t_i)$. We will abbreviate it as $F_i^O(t)$.

The value of the coupon is, using the cash account numeraire,

$$N_0 \, \mathrm{E}^N \left[(N_{t_n})^{-1} \sum_{i=1}^{n} \delta_i I_X^O(t_{i-1}) \right]$$

$$= \sum_{i=1}^{n} \mathrm{E}^N \left[(N_{t_n})^{-1} \delta_i F_i^O(t_{i-1}) \right]$$

$$= \sum_{i=1}^{n} \mathrm{E}^N \left[(N_{t_n})^{-1} \left((1 + \delta_i F_X^D(t_{i-1}, t_{i-1}, t_i)) \beta^O - 1 \right) \right].$$

We compute the value of one of these expectations. For that we use standard HJM results given by Lemma A.1 and A.3:

$$N_{t_n}^{-1} = P_X^D(0, t_n) \exp\left(- \int_0^{t_n} v(s, t_n) \cdot dW_s - \frac{1}{2} \int_0^{t_n} |v(s, t_n)|^2 \, ds \right)$$

and

$$1 + \delta_i F^D(t_{i-1}, t_{i-1}, t_i) = (1 + \delta_i F_i^D(0, t_{i-1}, t_i))$$
$$\times \exp\left(-\int_0^{t_{i-1}} (v(s, t_i) - v(s, t_{i-1})) \cdot dW_s\right.$$
$$\left. -\frac{1}{2}\int_0^{t_{i-1}} \left(|v(s, t_i)|^2 - |v(s, t_{i-1})|^2\right) ds\right).$$

In what follows, we use the usual extension of $v(s, t)$ for values $s > t$ with 0.

Combining the two HJM results above, the expected value to be computed for each fixing date is an exponential martingale plus some residual factors coming from the completion of (variance) squares. The expected value of the exponential martingales are 1. The residual terms are (after a slightly lengthy but easy computation)

$$\zeta_i = \int_0^{t_n} (v(s, t_i) - v(s, t_{i-1})) (v(s, t_n) - v(s, t_{i-1})) ds.$$

The final valuation result is given by the following theorem.

Theorem 6.4 *In the multi-curve framework under hypothesis $S0^{CPN}$ on the basis between discounting and overnight forwards, the present value of a Fed Fund swap coupon in the multi-factor HJM model on the discounting curve is*

$$P^D(0, t_n) \sum_{i=1}^{n} \left(\delta_i F^O(0, t_{i-1}, t_i)\zeta_i + (\zeta_i - 1)\right).$$

To compute the value exactly, we need to compute all the forwards and an adjustment for each of them.

6.4.3 Convexity adjustment and approximation

We can combine the approximations discussed above with a convexity adjustment. The amount paid can be approximated by

$$A_a \simeq \ln\left(\prod_{i=1}^{n}(1 + \delta_i I^O(t_i))\right) = \ln(1 + A_c).$$

When the discounting curve is the overnight forward rate curve, the discretely compounded rate itself can be approximated by a continuous composition

$$A_c \simeq \exp\left(\int_{t_0}^{t_n} r_\tau d\tau\right) - 1.$$

The present value of such a coupon is thus approximated by

$$M_0 E^M\left[(M_{t_n})^{-1}\int_{t_0}^{t_n} r_\tau d\tau\right]$$

for any numeraire M. Here we chose the t_n-forward numeraire, that is $P_X^D(.,t_n)$. The change of numeraire in the HJM model is given by

$$dW_t^{t_n} = dW_t + v(t,t_n)\,dt.$$

We use the result on the dynamic of the cash account twice (once to t_0 and once to t_n) and the above change of numeraire to obtain

$$\int_{t_0}^{t_n} r_\tau\,d\tau = \ln\left(\frac{P^D(0,t_0)}{P^D(0,t_n)}\right)\left(\int_0^{t_0} v(s,t_0)\cdot dW_s^{t_n} + \int_0^{t_n} v(s,t_n)\cdot dW_s^{t_n}\right.$$
$$\left. - \frac{1}{2}\int_0^{t_n} |(v(s,t_n) - v(s,t_0)|^2\,ds\right).$$

The expectation of the two stochastic integrals is 0 and the final result is as follows.

Theorem 6.5 *In the multi-curve framework under hypothesis $S0^{CPN}$ on the basis between discounting and overnight forwards, the approximated present value of a Fed Fund swap coupon in the multi-factor HJM model on the discounting curve is*

$$P^D(0,t_n)\left(\ln\left(\frac{P^D(0,t_0)}{P^D(0,t_n)}\right) + \zeta\right)$$

with

$$\zeta = \frac{1}{2}\int_0^{t_n} |(v(s,t_n) - v(s,t_0)|^2\,ds.$$

6.5 Federal Funds futures

The *30-Day Federal Funds futures* (simply called Fed Funds futures) are based on the monthly average of overnight Fed Funds rate for the contract month.

Let $0 < t_0 < t_1 < \cdots < t_n$ be the relevant dates for the Fed Funds futures, with t_0 the first business day of the reference month, t_{i+1} the business day following t_i and t_n the first business day of the following month. Let δ_i be the accrual factor between t_i and t_{i+1} ($0 \le i \le n-1$) and δ the accrual factor for the total period $[t_0, t_n]$.

The overnight rates fixing between t_i and t_{i+1} are denoted $I_X^O(t_i)$ with $F_X^O(t_i, t_i, t_{i+1}) = I_X^O(t_i)$. The future price on the final settlement date t_n is

$$\Phi_{t_n} = 1 - \frac{1}{\delta}\left(\sum_{i=0}^{n-1} \delta_i I_X^O(t_i)\right).$$

The margining is done on the price multiplied by the notional and divided by the one month accrual fraction ($1/12$).

The model used is the Hull–White model on the discounting curve as described in Appendix A. The result uses the deterministic spread hypothesis **S0**$^{\text{CPN}}$ between the curves.

Theorem 6.6 *In the HJM model on the discount curve in the multi-curve framework and with the deterministic hypothesis between spread and discount factor ratio* **S0**$^{\text{CPN}}$, *the price of the average overnight future for the period* $[t_0, t_n]$ *is given for* $t_j \leq t < t_{j+1}$ *($j \in \{0, 1, \ldots, n\}$) by*

$$\Phi_t = 1 - \frac{1}{\delta} \left(\sum_{i=1}^{j} \delta_i I_X^O(t_i) + \sum_{i=j+1}^{n-1} 1 - (1 + \delta_i F_X^O(t, t_i, t_{i+1})) \gamma(t, t_i, t_i, t_{i+1}) \right)$$

where γ *is given in Appendix A.*

The formula is divided into two parts to cope with the case where the averaging period has started already. In those cases the convexity adjustments γ are very small.

Proof: The generic futures pricing formula applied to the instrument is

$$\Phi_t = E^N \left[1 - \frac{1}{\delta} \left(\sum_{i=0}^{n-1} \delta_i F_X^O(t_i, t_i, t_{i+1}) \right) \middle| \mathcal{F}_t \right]$$

$$= 1 - \frac{1}{\delta} \left(\sum_{i=1}^{j} \delta_i I_X^O(t_i) + \sum_{i=j+1}^{n-1} 1 - E^N \left[1 + \delta_i F_X^O(t_i, t_i, t_{i+1}) \middle| \mathcal{F}_t \right] \right).$$

We are left with the same technical computation as in Section 2.6, once for each period. The result follows immediately. □

6.6 Deliverable swaps futures

The results presented in this section were first proposed in Henrard (2012b). The description of the instrument itself is provided in Appendix B.13.

6.6.1 Pricing

The arbitrage-free price of the future Φ_θ in θ satisfies, for PV$_\theta$ the present value of the underlying swap without up-front payment in θ and with notional 1 to the long party,

$$(1 - \Phi_\theta) N P^D(\theta, t_0) + PV_\theta N = 0$$

or

$$\Phi_\theta = 1 + \frac{PV_\theta}{P^D(\theta, t_0)}.$$

After delivery, the swap is cleared on CME clearing. The swaps are collateralised according to CME variation margin methodology. Here we suppose that the methodology is equivalent to a discounting with a given discounting curve as described in the previous chapters of the book.

The results presented here are similar to the one presented in Kennedy (2010) for SwapNote futures and in Henrard (2006b) for bond futures. They are more general as they are for multi-factor models and in the multi-curve framework.

Using the generic pricing future price process theorem (Hunt and Kennedy, 2004, Theorem 12.6),

$$\Phi_0 = \mathrm{E}^{\mathbb{N}}[\Phi_\theta].$$

Using the swap pricing formula,

$$\Phi_\theta = 1 + \sum_{i=0}^n d_\theta^i \frac{P_X^D(\theta, t_i)}{P_X^D(\theta, t_0)}.$$

The pricing formula uses the description of the swap in terms of equivalent cash-flows as provided in Section 2.4.

Using Lemma A.1, if d_θ^i is independent of the ratio of discount factors, the futures can be written as a function of a random variable X_i:

$$1 + \sum_{i=0}^n d_t^i \frac{P_X^D(0, t_i)}{P_X^D(0, t_0)} \exp\left(-\alpha(0, \theta, t_0, t_i) X_i - \frac{1}{2}\alpha^2(0, \theta, t_0, t_i)\right) \gamma(0, \theta, t_0, t_i).$$

The exponential terms have a expectation of 1. We thus obtain the following result.

Theorem 6.7 *In the multi-curve framework with independent spread hypothesis in the Gaussian HJM model, the price of a swap futures with expiry date θ is given by*

$$F_0 = 1 + \frac{1}{P_X^D(0, t_0)} \sum_{i=0}^n \mathrm{E}^{\mathbb{N}}\left[d_\theta^i\right] P_X^D(0, t_i) \gamma(0, \theta, t_0, t_i).$$

With hypothesis **S0$^{\text{CPN}}$**, the price is

$$F_0 = 1 + \frac{1}{P_X^D(0, t_0)} \sum_{i=0}^n d_0^i P_X^D(0, t_i) \gamma(0, \theta, t_0, t_i). \tag{6.2}$$

6.6.2 Numerical examples

In this section we give some numerical examples of the impact of the futures features on the price and risks in the case of constant spread hypothesis **S0$^{\text{CPN}}$**. The analysis is done in a Hull–White one-factor model. We compare the futures to the

Table 6.4 Difference in future price and swap present value for different expiries, tenor and moneyness

Expiry	Moneyness	2Y	5Y	10Y	30Y
				Tenor	
3M	−100 bps	−1.56	−3.73	−6.94	−15.85
	0 bps	−0.26	−0.62	−1.14	−2.47
	+100 bps	1.04	2.49	4.66	10.91
9M	−100 bps	−6.11	−14.51	−26.72	−59.16
	0 bps	−2.22	−5.25	−9.59	−20.73
	+100 bps	1.67	4.02	7.54	17.69

Price differences in basis points.

underlying swap. The comparison is on the price (for market making) and the curve risk (for hedging). The comparisons are done with a flat curve at 3% (continuously compounded) rate and a 2% Hull–White constant volatility.

From Equation (6.2), it is clear that the impact on the value is twofold. First is the multiplication by the common factor $1/P^D(0, t_0)$. This factor is always larger than 1 when interest is positive. This represents the fact that the profit is paid immediately (through the margin) and not at final settlement. As the futures usually have an expiry up to three or six months in the future, the impact is up to 0.5 times the short term rate. With the very low rates prevalent at the time of writing this impact is very small: below 0.10%. For higher rates, the impact can be non-negligible. When interest is at 3%, the impact on a six month future is around 1.5%.

The differences between the futures price (with the 1 subtracted) and the underlying swap present value are displayed in Table 6.4. The difference clearly depends on three items: expiry, tenor and moneyness. The moneyness is the difference between the current forward swap rate and the fixed rate embedded in the futures. If markets have moved significantly between the issuance time and the current level, the swaps can be far off-the-money.

On the curve risk side, the derivative with respect to the rate will be roughly $t_n - t_0$ for the future and t_n for the swap. The ratio swap notional/future notional will be impacted by this. Using this very crude approximation, for a three month futures, the ratio is around 1.125 for a 2 year tenor and 1.01 for a 30 year tenor. Actual ratios for different scenarios are given in Table 6.5. The ratios strongly depend on the future expiry and underlying swap tenor but very weakly on the moneyness.

The second impact is the so-called convexity adjustment $\gamma(0, \theta, t_0, t_i)$. As mentioned earlier, these adjustments are always less than 1 and this will decrease the price. The factors depend on the level of volatility (the adjustment is larger for higher volatilities) and on the time to maturity (the adjustment is larger for longer

Table 6.5 Swap/futures nominal total sensitivity hedging

Expiry	Moneyness	2Y	5Y	10Y	30Y
3M	−100 bps	1.13	1.05	1.02	1.01
	+100 bps	1.14	1.05	1.02	1.01
6M	−100 bps	1.32	1.10	1.04	1.01
	+100 bps	1.32	1.10	1.05	1.01
9M	−100 bps	1.58	1.16	1.07	1.02
	+100 bps	1.58	1.17	1.07	1.02

Hedging ratios: the notional of futures required to hedge the same amount of swap.

maturities). As the underlying tenors are between 2 and 30 years, the impact will be considerably different between the different futures. Also the adjustment factor impact will depend on the *moneyness*. As the coupon is fixed on the first trading date and the futures starts to trade several quarters before the delivery date, the difference can be important.

We also analyse the bucketed sensitivities. The difference appears mainly around the delivery date. The futures has an extra sensitivity on the discounting curve, as expected from the $1/P^D(0, t_0)$ term in Equation (6.2). In Table 6.6 we display the sensitivities of a futures on the two years swap with a coupon close to the money and an expiry in three months time (futures with delivery on 19 December 2012 valued on 20 September 2012). The sensitivities are the derivatives with respect to the rates (continuously compounded) at the node point indicated and for a notional of 1. We work in a framework where the discounting curve is calibrated to OISs and use OIS to hedge part of the difference between swaps and futures. The OIS we use is a swap with maturity on the futures delivery date. We hedge the underlying swap with a notional of 100 million using the futures and the OIS. The quantity of the futures is obtained by hedging the total sensitivity to the forward curve. The quantity of the OIS is obtained by hedging the total sensitivity of the swap and its futures hedge to the discounting curve. The residual is the result of hedging a 100 million receiver swap with the above described portfolio. The portfolio consists of selling 993 contracts[1] and receiving OIS for 102.4 million; its risk is displayed in the last columns.

Table 6.7 displays the same information for a 30 year tenor futures with expiry in six months. Note that the hedging quality on the forward curve is not uniform through the tenors. This is expected, as in Equation 6.2 the cash-flows are multiplied by convexity adjustments that are different for each cash-flow date.

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[1] The figure provided are for a not rounded hedge. In practice it is not possible to trade fractions of contracts. The figures achievable in practice will be slightly different.

Table 6.6 Swap/futures/OIS sensitivities and hedging efficiency for 2Y futures

Tenor	Futures		Swap		OIS		Residual	
	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd
3m	0.24	0.25	0.00	0.24	−0.23	0.00	0.37	−0.02
6m	0.00	−0.01	0.00	−0.01	0.00	0.00	−0.08	0.03
1y	0.01	−0.02	0.01	−0.02	0.00	0.00	−0.10	0.10
2y	0.01	−1.99	0.01	−1.98	0.00	0.00	−0.18	−0.08
5y	0.00	−0.18	0.00	−0.17	0.00	0.00	−0.01	−0.02
10y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of a 100 million receiver swap, selling 993 futures contracts and paying 102.4 million OIS up to expiry. The sensitivities of the portfolio are in USD for a one basis point move.

Table 6.7 Swap/futures/OIS sensitivities and hedging efficiency for 30Y futures

Tenor	Futures		Swap		OIS		Residual	
	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd
3m	0.01	0.01	0.00	0.01	0.00	0.00	−107.98	−0.05
6m	0.49	0.48	0.00	0.48	−0.48	0.00	219.91	−2.68
1y	0.00	−0.02	0.00	−0.02	0.00	0.00	−0.47	0.54
2y	0.00	−0.14	0.00	−0.13	0.00	0.00	−2.36	2.98
5y	0.00	−0.54	0.00	−0.53	0.00	0.00	−8.91	10.56
10y	0.02	−3.33	0.01	−3.29	0.00	0.00	−50.02	45.97
30y	−0.03	−16.96	−0.04	−16.75	0.00	0.00	−50.17	−57.31

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of a 100 million receiver swap, selling 987 futures contracts and paying 103.7 million OIS up to expiry. The sensitivities of the portfolio are in USD scaled to one basis point.

6.6.3 Curve calibration and sensitivity

Like any instrument presented in this chapter, the swap futures can be used for curve calibration. The calibration requires the instrument quotes, a curve parameterisation and the model – HJM in our case – parameters.

The curve calibration approach described in Chapter 5 is very generic and can be applied to any instrument, even newly created ones.

What is different in the swap futures case with respect to all other interest rate products we described up to now is the quote 'dimension'. Most of the interest rate products are quoted through rate or yield related quotes, the exceptions being the

forex swaps and the swap futures. The swap futures described in this section are quoted on a present-value-like basis.

When the sensitivity of a portfolio with respect to the inputs is computed, the figures obtained for swap futures will be dramatically different from the sensitivities for swaps.

6.7 Portfolio hedging

This section is not really an extra instrument but describes an approach to computing hedges with different instruments. The material could have been in another chapter but is a little bit short to be a chapter by itself. For that reason I have appended it to the end of the instruments chapter.

6.7.1 Introduction

When computing the sensitivities of a portfolio to the market quotes used in the curve construction, one implicitly computes the quantity of the instruments used in the curve construction required to hedge the portfolio. In some cases one would like to compute the (optimal) hedge with a different set of reference securities. One possibility in that case is to compute the synthetic market quotes of the reference securities, reconstruct the curves from those instruments and recompute the sensitivities.

Another possibility is to use the original sensitivities and the sensitivities of the hedging instruments and find a *good* hedge, optimal in a sense to be described. A method to do that in the case where the original sensitivities are considered independent is described in (Andersen and Piterbarg, 2010, Section 6.4). Similar techniques are used in a different context to minimise delta/normal VaR using the covariance matrix to indicate the importance of and interaction between the sensitivities. In the book by Andersen and Piterbarg (2010), the method is referred to as the *Jacobian method*. As we suggest systematically computing the Jacobian matrices of curve construction and other processes, we will refrain from using that terminology, which can be confusing, and refer to it instead as the *hedging with reference securities* method.

6.7.2 Notation

Suppose that the curves are constructed with n_m securities for which the market quotes are $m = (m_i)_{1 \leq i \leq n_m}$. The market quotes are the parameters with respect to which the sensitivities are available. In general this is a par rate, a spread or a price. Any other parameter/shock sensitivity would work from a theoretical point of view.

Suppose we have a portfolio P made of many securities. We consider it as a unique entity for hedging purposes. We want to hedge that portfolio with other reference securities. The risk of the portfolio with respect to the curve market quotes is known. It is denoted $\partial P = (\partial_i P)_{i=1,\dots n_m}$ and is the short notation for $\partial_i P = \partial_{m_i} P(m)$; ∂P is represented by a column vector.

The values of the reference instruments to be used in the hedging are denoted R_j with $j = 1, \dots, n_r$. Their sensitivity with respect to the market quotes are also supposed to be known.

The quantity of each reference security used in the hedging portfolio is denoted by $q = (q_j)_{j=1,\dots,n_r}$. The goal of this method is to find the optimal quantities q. The value of the total hedging portfolio is

$$H = H(q) = \sum_{j=1}^{n_r} q_j R_j = q^T R.$$

The sensitivity of the hedging portfolio with respect to the market quotes is, using as notation that $\partial_i R$ is the column vector with the derivatives of the different components of R with respect to m_i,

$$\partial_i H = q^T \partial_i R.$$

We denote by ∂R the matrix with the $\partial_i R$ as columns. The total sensitivity of the original portfolio P plus the hedging portfolio H is (the column vector)

$$\partial P + (q^T \partial R)^T = \partial P + \partial R^T q.$$

The quantities q are the number we are trying to find to obtain the optimal hedging portfolio. The portfolio is optimal in the sense discussed below.

6.7.3 Optimal hedging

The notion of *optimal hedging portfolio* can cover a lot of financial realities. Here we restrict ourself to the following sense.

We want, when possible, to have the individual residual sensitivities to be small (zero would be perfect). The sensitivities to some market quotes may be more important than others. We usually want to associate a weight to each of the quotes to indicate how much we want to have them small. Also there may be quotes that we consider as closely related and if each of them cannot be small then we want their total to be small. This last point is particularly important in the multi-curve framework.

To represent the related rates, we use a weight matrix. The matrix is an n_w lines and n_m columns matrix denoted W. The hedging mismatch penalty is

$$W(\partial P + \partial R^T p)$$

and we try to optimise it in the least-square sense. Some examples of possible weights and their intuition are discussed in Section 6.7.4.

The summation in the penalty computation makes sense only if all the numbers are converted to the same currency. We suppose that this has been done.

The weight matrix is related to the curves used (n_m is the number of instruments in the curve construction) but not a priori to the reference instruments used for hedging. The number of lines n_w is the number of penalties we want to minimise. In general it is larger than n_m.

With this possibility of adding weights to related securities, our analysis is more general than that of Andersen and Piterbarg (2010). Also we do not associate cost to the hedging instruments; in that respect our approach covers less applications.

The (least-square) problem we have to solve is to find the quantity vector q which minimises the quantity

$$(\partial P + \partial R^T q)^T W^T W (\partial P + \partial R^T q). \tag{6.3}$$

The solution of the above problem is given by the solution of the linear system

$$(\partial R W^T W \partial R^T) q = -\partial R W^T W \partial P. \tag{6.4}$$

6.7.4 Examples

First suppose you have a curve set made of a discounting curve and a forward curve, each of them with two points: a 1Y and a 5Y point. We order them as first the two discounting points and then the two forward points. You want, if possible to have all sensitivities small, but mainly you want the total 1Y and the total 5Y risks to be small. You can chose the following weight matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 5 \end{bmatrix}$$

The first four lines give a standard weight to each individual sensitivity. The next two lines give a larger weight to the total 1Y and 5Y sensitivities.

For the second example, suppose you have a discounting curve, a forward curve and a government curve, each of them with two points (a total of six quotes). We have a swap desk and want to hedge the swaps residual risks with bond futures, and we chose bond futures as our reference instruments. We know that a direct point by point comparison will be useless: the portfolio and the hedging instruments do not depend on the same curve. Our only interest is in the total risk for each maturity.

A potential matrix is

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}.
$$

At the opposite end of the spectrum, suppose you have a diversified portfolio of swaps and government bonds. You are concerned to have the total sensitivity of the discounting and forward curves together and the government curve on its own to be small. You can choose

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2
\end{bmatrix}.
$$

Suppose you have a unique curve with four (ordered) points and want to indicate that adjacent points compensate partially. You can use a tri-diagonal matrix like

$$
\begin{bmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 2
\end{bmatrix}.
$$

A residual sensitivity of $(1,-1,0,0)$ is in this case better than $(1,0,-1,0)$.

In the VaR-like approach, the matrix $W^T W$ would be the variance-covariance matrix. As only the matrix $W^T W$ is used in the computation, any decomposition of the covariance matrix will provide the same result. In the delta/normal approach, the α-quantile VaR is $N^{-1}(\alpha)\sqrt{p^T V p}$. The VaR is minimal when $p^T V p$ is minimal. Our optimal hedging portfolio has minimal VaR when W is such that $V = W^T W$.

7 | Options and Spread Modelling

In this chapter we describe several modelling approaches to curve and spread dynamic. The dynamic refers to the term structure of the discounting curve but also, and this is the difference with the one-curve framework, the term structure of the different forward rates or spreads. The first and simplest approach to spread modelling is the one considered previously, that is the constant spread approach described by hypothesis $\mathbf{S0^{CPN}}$.

From Section 7.4 onward, each section of this chapter proposes a different modelling approach. Each section can be read separately, even if we propose links and comparisons between the approaches. The order of presentation should not be interpreted as an order of sophistication or quality.

For historical and financial reasons, the liquid optional instruments are in separated niches. For example, Ibor and overnight-linked swaps are both very liquid even if on different parts of the curve. The swaptions on Ibor swaps are also very liquid but the swaptions on overnight-linked swaps are virtually nonexistent. Similarly, basis swaps are relatively liquid, especially since the crisis, but the swaptions on basis swaps are almost not traded at all. If one looks at the exchange traded world, the STIR futures and their options, which are the equivalent of OTC (over-the-counter) FRA and cap/floors, are liquid. But there is very little equivalent to swaps and swaptions. New instruments, like the deliverable swap futures described in Appendix B.13, have recently been added to the offering and have gained a decent liquidity. But there is virtually no equivalent to the swaptions. For that reason we propose a piece of 'financial fiction' in Section 7.8.6 by proposing a new potential exchange traded instrument that could complete the offering.

From a model calibration perspective, the absence of a certain number of natural volatility instruments means that it is often difficult to find the appropriate data. Classical term structure models apply to the discounting curve. In the collateral discounting approach, described in Chapter 8, the discounting curve is often the curve calibrated to OISs. But there are no direct OIS volatility instruments and so no way to directly calibrate those classical models. We often need additional hypotheses on the behaviour of the spread to be able to price even the simplest instruments. It will

thus be important to be able to move information from one type of instrument to another. As usual it will be important to have a coherent approach where the same model can price different instruments and be able to compare risk and hedging strategies.

7.1 Short rate models

Historically, short rate models have been heavily used to model the interest rate term structure. This approach is certainly still important to model the risk-free curve P_X^D.

The same approach cannot be extended easily to the forward curves. In the generic approach we present, the notion of short rate for the forward curves is ill-defined. To define the short rate one needs an equivalent of the instantaneous forward, which itself requires an equivalent of the discount factors to obtain the equivalent of Definition 4.1. This type of pseudo-discount factor curve is defined only in the special case described in Section 3.1, where those objects are defined to use methods similar to the one used for the discounting curve. This is useful for interpolation schemes and to define curves through spreads.

Even when the pseudo-discount factors are used, they are, by Definition 3.1, arbitrary up to the first fixing end date and even never used up to the settlement date. A short rate model on those quantities would be based on arbitrary quantities which have no impact whatsoever in the framework. Apart from the theoretical weakness of such an approach, it is difficult to imagine numerical schemes that would be stable enough to accept arbitrary number as inputs and still produce stable and meaningful outputs.

Technically, one important point on the forward curves is that the pseudo-discount factors $P_X^{\mathrm{CDF},j}(.,u)$, when they are defined, are not assets for any u in our economy. If any factor was an asset, its value today could be written using any numeraire q and its associated measure $\mathbb{Q}$ as

$$
\begin{aligned}
P_X^j(t,u) &= q_t \, \mathrm{E}^{\mathbb{Q}} \left[q_u^{-1} P_X^{\mathrm{CDF},j}(u,u) \,\middle|\, \mathcal{F}_t \right] \\
&= P_X^D(t,u) \, \mathrm{E}^u \left[P_X^D(u,u)^{-1} P_X^{\mathrm{CPN},j}(u,u) \,\middle|\, \mathcal{F}_t \right] = P_X^D(t,u)
\end{aligned}
$$

where $\mathrm{E}^u [\,]$ is the expectation associated to the $P_X^D(.,u)$ numeraire. This proves that if $P_X^{\mathrm{CPN},j}(.,u)$ was an asset, it would always be equal to $P_X^D(.,u)$. This is in contradiction to the core subject of this book, which is the differentiation between the discounting curve and the forward index estimation curve.

The same can be said in the case when the pseudo-discount factor is defined by a formula such as

$$P_X^j(t, u) = E^{\mathbb{Q}}\left[f(r_\cdot^j) \middle| \mathcal{F}_t \right]$$

for some function f of a pseudo-short rate $r_\cdot^j$.

Mal nommer les choses, c'est ajouter au malheur du monde.

Albert Camus

Personal translation: Badly naming items, it is adding to the world's tragedy.

7.2 Spreads description

Our hypothesis $\mathbf{S0}^{\mathbf{CPN}}$ is one way to link the curves together to be able to price contingent claims. Other possibilities are presented below. Most of the analysis presented below has been initially presented in Henrard (2010).

7.2.1 Swap market spread

Under this hypothesis, the spreads between the market standard tenor swap rate and other forwards are known for every fixing. Let M be the *standard market index*, the one for which we have market data, and S be the swap forward rate.

The spreads or basis

$$B^j(t_0, t_1) = S_{t_0}^j(t_0, t_1) - S_{t_0}^M(t_0, t_1) \tag{7.1}$$

are known for every fixing date t_0 and every tenor j.

Using the martingale property of the swap rate it is easy to verify that a deterministic hypothesis is equivalent to a constant forward spread.

If one has a model for S^M, the swap rates related to other Ibor indexes can be obtained by a deterministic shift. This is particularly adapted to a Bachelier type model, where the volatility would be the same for swap rates with different indexes. This can be used in a Black-like model under the understanding that if the market rate is log-normal, the others are shifted log-normal.

This hypothesis is in practice very close to $\mathbf{S0}^{\mathbf{CPN}}$, which can be viewed as an approach with known continuously compounded spreads.

7.2.2 Black spread

The simplest model used to model interest rates derivatives is the Black model on forward rates. The base equation is

$$dS_t^M = \sigma S_t^M dW_t. \tag{7.2}$$

The volatility σ is that given by the market for the standard tenor and specific expiry, tenor and strike. In this new hypothesis, the similar rate for the forward (non-market convention) curve is supposed to follow a similar equation

$$dS_t^j = \sigma S_t^j dW_t.$$

The forward swap rate follows a Black equation (between 0 and expiry) with the same volatility and Brownian motion as the rate in the market convention.

The other tenor swap rates are proportional to the market swap rate:

$$S_t^j = \frac{S_0^j}{S_0^M} S_t^M.$$

Under that approach the spread between the rates $S^M - S^j$ is not constant nor deterministic. It is a constant proportion of the rate. The spread grows (and reduces) with the rate:

$$S_t^j - S_t^M = \left(\frac{S_0^j}{S_0^M} - 1 \right)$$

7.2.3 SABR/Constant elasticity of variance (CEV) spread

In the SABR (Stochastic Alpha Beta Rho model) framework, the base equations are

$$dS_t^M = \alpha_t (S_t^M)^\beta dW_t^1. \tag{7.3}$$

$$d\alpha_t = \nu \alpha_t dW_t^2 \tag{7.4}$$

A possible spread hypothesis based on these equations is as follows.

SA: The forward swap rate follows an SABR equation (between 0 and expiry) with the same parameters and Brownian motion as the rate in the market convention.

The impact on the spreads will depend on the β parameter, like the computation of the delta in the SABR framework. A β close to 1 will give a result close to Black spread and a β close to 0 will give a result close to market spread.

In the Black and SABR approaches, the spreads would increase when the rates increase. In the recent crisis, the spreads have increased while the rates were decreasing. This approach could be counterintuitive.

The three approaches mentioned above are only convenient if the objective is only to model one (swap) rate and not the whole term structure. These approaches would not be the most convenient for exotics (Bermuda, callable, and so on). By opposition, the hypothesis $S0^{CPN}$, even if too simplistic, is a term structure hypothesis.

7.2.4 Continuously compounded constant spread

The same financial reality can be expressed in different ways, with different conventions. Traditionally each currency and instrument type has its own convention, as

described in Appendix B. A spread in one convention is not equal to the spread in another convention. The translation is often not static and depends on the rate level itself. Consequently it is not possible to use one number as a fixed spread and use it for all conventions without adjustment.

Let's look at the standard money market simple rate and see if it is possible to use a constant spread in a coherent way. Suppose one rate is denoted L and the other is F with a spread S. The basic spread equation is

$$L = F + S. \tag{7.5}$$

Is it possible to use a constant spread S in a consistent way for all tenors in a simple interest market convention?

To see that this is not the case, take two consecutive periods of length δ. The same rate F_1 is used for the two periods. The rate over the total period is F_2 and the relation between them is $1 + 2\delta F_2 = (1 + \delta F_1)^2$. Then the relation on the starting rate L is

$$1 + 2\delta F_2 + 2\delta S_2 = 1 + 2\delta L_2 = (1 + \delta L_1)^2 = (1 + \delta F_1 + \delta S_1)^2$$
$$= (1 + \delta F_1)^2 + 2\delta S_1 \left(1 + \delta F_1 - \tfrac{1}{2}\delta S_1\right)$$

This means that

$$S_2 = S_1 \left(1 + \delta F_1 - \tfrac{1}{2}\delta S_1\right)$$

To the first order in δ the spreads are the same for both tenors. Nevertheless they are not constant and the spread has a dependency on the rate level. A constant spread cannot be used in the simple interest in a coherent way. Not all current rates and forwards in the market conventions can be modelled with the same spread.

The situation is different in the continuously compounded zero-coupon framework. Let's denote by $\bar{F}$ and $\bar{L}$ the continuously compounded equivalent of F and L:

$$(1 + \delta F) = \exp(\delta \bar{F}) \quad \text{and} \quad (1 + \delta L) = \exp(\delta \bar{L})$$

The spread is expressed in that convention with $\bar{L} = \bar{F} + \bar{S}$. In this context it can be shown that a constant spread in spot and forward rates is coherent. Take three dates $t_0 < t_1 < t_2$ and the corresponding rates $F_{i,j}$ and $L_{i,j}$ between the dates t_i and t_j. The relation between the F rates is $\exp(\bar{F}_{0,2}(t_2 - t_0)) = \exp(\bar{F}_{0,1}(t_1 - t_0)) \exp(\bar{F}_{1,2}(t_2 - t_1))$, that is, $\bar{F}_{0,2}(t_2 - t_0) = \bar{F}_{0,1}(t_1 - t_0) + \bar{F}_{1,2}(t_2 - t_1)$. Then one has

$$\exp(\bar{L}_{0,2}(t_2 - t_0)) = \exp((\bar{F}_{0,2} + \bar{S})(t_2 - t_0))$$
$$= \exp((\bar{F}_{0,1} + \bar{S})(t_1 - t_0)) \exp((\bar{F}_{1,2} + \bar{S})(t_2 - t_1))$$
$$= \exp(\bar{L}_{0,1}(t_1 - t_0)) \exp(\bar{L}_{1,2}(t_2 - t_1))$$

The composition relation on $\bar{F}$ implies a similar relation on $\bar{L}$ for a constant spread.

The simplest way to represent spreads according to the above analysis is through continuously compounded rates. The investment factors can be written as

$$\exp\left((r+s)(v-u)\right) = \exp\left(r(v-u)\right)\exp\left(s(v-u)\right).$$

This is the type of description that motivated the spread definition (Definition 2.3) with $\beta(u,v) = \exp\left(s(v-u)\right)$.

7.3 Constant multiplicative spread

The simplest hypothesis to model the spread is to use a constant spread hypothesis. This simplifying assumption, denoted $\mathbf{S0^{CPN}}$, is the one we used to provide formulas for FRAs and STIR futures in Chapter 2.

The same hypothesis can be combined with different term structure models on the discounting curve to obtain models in the multi-curve framework starting from standard one-curve models. In this section we give some examples of pricing with HJM and LMM models.

7.3.1 Heath-Jarrow-Morton model

The foundations and main results for the HJM model are described in Appendix A. The model describes the evolution of the risk-free curve, that is the evolution of $P_X^D(.,u)$. Using the hypothesis $\mathbf{S0^{CPN}}$, the values of any forward rate $F_X^{CPN,j}(t,u,v)$ is provided through the formula

$$1 + \delta F_X^{CPN,j}(t,u,v) = \beta_X^{CPN,j}(0,u,v)\frac{P_X^D(t,v)}{P_X^D(t,u)}.$$

Whatever the numerical method used to obtain the values of P_X^D, being analytical formulas, numerical integration, PDE or Monte Carlo, the constant β can be multiplied and the values of $F_X^{CPN,j}$ obtained. In this way it is possible to modify all single-curve methods to also work in the multi-curve framework. The hypothesis $\mathbf{S0^{CPN}}$ is restrictive in the dynamic of the spread but has the advantage of being able to model any instrument that was modelled in the old framework. A particular case of such a modelling, for European physical delivery swaptions, is provided in Section 7.3.3.

7.3.2 Libor Market Model

The case of the Libor Market Model (LMM) is different from the HJM one, even thought the two modelling approaches are *twin brothers* (see Gątarek (2005)) in the single-curve framework. There are two ways to translate the Libor rates dynamic in the multi-curve framework: it can be translated as the dynamic of the forward

risk-free rate (F^D of Definition 2.2) or it can be translated as the dynamic of the forward coupon rate ($F^{\text{CPN},j}$ of Definition 2.1). In this section we discuss only the constant spread hypothesis and ignore the more involved models on the dynamic of both sets of rates that are presented in Section 7.9.

In this section we sketch the former, modelling the forward risk-free rates. In terms of name, this is probably not what was intended by the original model, as the starting point is not a Libor-like rate as quoted by the market, but a synthetic forward risk-free rate. Nevertheless, in terms of financial mathematics, it may be closer to the original techniques. A rate, which can be written as a ratio of discount factors, is modelled; the model is initially done on a discrete set of forward rates and is linked to an HJM model of the risk-free curve.

The starting point is any version of the single-curve LMM applied to the discounting curve. Using the modelling of those rates, one can price any risk-free dependent contingent claims. Using hypothesis **S0$^{\text{CPN}}$**, once the values of the forward risk-free rates are obtained, the actual Ibor forward rates are only one multiplication away. The Ibor forward rates are given by

$$1 + \delta F_X^{\text{CPN},j}(t, u, v) = \beta_X^{\text{CPN},j}(0, u, v)(1 + \delta F_X^D(t, u, v)).$$

In terms of implementation, the computation time of a Monte Carlo should be very similar to the one in the one-curve framework. Simply compute the relevant multiplicative spreads at the start, do the difficult Monte Carlo work on the risk-free forwards using any one-curve LMM technique and multiply by the spread at the end. The couple of extra divisions/multiplications for each cash-flow add a negligible time to the operation.

7.3.3 Swaptions in HJM

We use the cash-flow equivalent of the swap described in Section 2.4:

$$\sum_{i=0}^{\bar{n}} d_i P_X^D(t, \bar{t}_i).$$

Under hypothesis **S0$^{\text{CPN}}$**, the quantities d_i are constant.

Using this approach, the pricing of swaption in the one-factor Gaussian HJM model, originally proposed in Henrard (2003) in the one-curve framework, can be written explicitly. The Gaussian HJM framework is described in Appendix A.

Theorem 7.1. (Exact swaption price in one-factor HJM) *Suppose we work in the Gaussian HJM model with a separable volatility as described in Definition A.1 and in the multi-curve framework with hypotheses **D**, **I$^{\text{CPN}}$** and **S0$^{\text{CPN}}$**. Let $\theta \leq \bar{t}_0 < \cdots < \bar{t}_n$, $d_0 < 0$ and $d_i \geq 0$ ($1 \leq i \leq n$). The present value of a European receiver swaption (with physical delivery), with expiry θ on a swap with cash-flows equivalent $(\bar{t}_i, d_i)$ is given*

at time t by the $\mathcal{F}_t$-measurable random variable

$$\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) N(\kappa + \alpha(0,\theta,\theta,\bar{t}_i))$$

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) \exp\left(-\frac{1}{2}\alpha^2(0,\theta,\theta,\bar{t}_i) - \alpha(0,\theta,\theta,\bar{t}_i)\kappa\right) = 0. \qquad (7.6)$$

The price of the payer swaption is

$$-\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) N(-\kappa - \alpha(0,\theta,\theta,\bar{t}_i))$$

Proof: The present value of the swaption is, in the $P_X^D(.,\theta)$ numeraire,

$$P^D(0,\theta) \mathrm{E}^\theta\left[\left(\sum_{i=0}^{n} d_i P_X^D(\theta,\bar{t}_i)\right)^+\right].$$

Using Lemma A.2, the value becomes

$$\mathrm{E}^\theta\left[\left(\sum_{i=0}^{n} d_i P_X^D(0,\bar{t}_i) \exp\left(-\alpha(0,\theta,\theta,\bar{t}_i) X_{0,\theta} - \frac{1}{2}\alpha^2(0,\theta,\theta,\bar{t}_i)\right)\right)^+\right].$$

The value inside the expected value is positive when $X < \kappa$. To see this, note that $\alpha(0,\theta,\theta,.)$ is increasing. The quantity is positive when

$$\exp\left(-\alpha(0,\theta,\theta,\bar{t}_0)x\right)\left(\sum_{i=1}^{n} d_i P_X^D(0,\bar{t}_i) \exp\left(-\frac{1}{2}(\alpha^2(0,\theta,\theta,\bar{t}_i)\right)\right.$$

$$\left. \exp\left(-(\alpha(0,\theta,\theta,\bar{t}_i) - \alpha(0,\theta,\theta,\bar{t}_0))x\right) + d_0\right) > 0$$

The large sum is a sum of negative exponentials with positive coefficients. It is a strictly decreasing function with limits $+\infty$ at $-\infty$ and limit 0 at $+\infty$. It crosses the value $-d_0 > 0$ at a unique value (called κ). It is above $-d_0$ when $x < \kappa$ and below it when $x > \kappa$.

The expected value can be written in terms of integrals by

$$\sum_{i=0}^{n} d_i P_X^D(0,t_i) \int_{-\infty}^{\kappa} \exp\left(-\alpha(0,\theta,\theta,t_i)x - \frac{1}{2}\alpha^2(0,\theta,\theta,t_i)\right) \exp\left(-\frac{1}{2}x^2\right) dx.$$

After changes of variable, the integrals can be written as normal cumulative densities and the result follows. $\qquad\square$

7.4 Ibor forward rate modelling

The standard one-curve framework approach to modelling cap/floor is to notice that the forward rate is a martingale in the payment date forward numeraire. From there one can choose one's own preferred way to model the rate with a drift-less stochastic equation and obtain a coherent pricing. A very similar approach can be used in the multi-curve framework.

7.4.1 Black model

The forward rate $F_X^{\text{CPN},j}(.,u,v)$ is a martingale in the $P_X^D(.,v)$ numeraire under the hypotheses $\mathbf{D}$ and $\mathbf{I^{CPN}}$. This was described in Theorem 2.1. As F is a martingale, it can be modelled by a drift-less stochastic equation. The simplest approach is to suppose the rate follows a geometric Brownian motion:

$$dF_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(t,u,v)\eta(t,u) \cdot dW_t^v. \tag{7.7}$$

The Brownian motion W_t^v is multidimensional with the dimension of η equal to the dimension of the Brownian motion. This is often referred to as the Black (1976) model for Ibor rates.

The solution of the above equation is an exponential martingale

$$F_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(0,u,v)\exp\left(\int_0^t \eta(\tau,u) \cdot dW_\tau - \frac{1}{2}\int_0^t |\eta(\tau,u)|^2 d\tau\right).$$

Let

$$\sigma_\eta^2(t) = \int_0^t |\eta(\tau,u)|^2 d\tau$$

be the period volatility. With that notation, the solution can be written as

$$F_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(0,u,v)\exp\left(\sigma_\eta X - \frac{1}{2}\sigma_\eta^2\right)$$

for X a standard normally distributed random variable of dimension 1.

An Ibor caplet or floorlet with strike K and expiry θ on the period $[u,v]$ for an expiry equal to the fixing date of the rate $I_X^j(\theta)$ on the same period is a financial product that pays in v the amount

$$\left(\omega(I_X^j(\theta) - K)\right)^+$$

with $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet. In practice, the amount is multiplied by a notional and an accrual factor. The factors act as multipliers of the total amount. To lighten the notation, we ignore those multiplicative factors in the analysis. They should appear as a multiplicative factor also in the final result.

Theorem 7.2 *In the multi-curve framework with hypotheses* $\mathbf{D}$ *and* $\mathbf{I^{CPN}}$, *if the forward rate follows the dynamic given by Equation (7.7), the present value of a caplet/floorlet of strike K and expiry θ is given by*

$$P_X^D(0,v)\omega\left(F_X^{CPN,j}(0,u,v)N(\omega d_+) - KN(\omega d_-)\right)$$

where

$$d_\pm = \frac{\ln\left(\dfrac{F_X^{CPN,j}(0,u,v)}{K}\right) \pm \dfrac{1}{2}\sigma_\eta^2(\theta)}{\sigma_\eta(\theta)}.$$

This approach does not require any explicit hypothesis regarding the spread between the risk-free rate and the forward Ibor rate. The simple approach to modelling each rate is possible due to the definition of forward rate given by Definition 2.1. The forward rate was defined as an asset – the Ibor coupon – divided by an other asset – the discount factor – playing the role of a numeraire, facilitating the modelling of such quantities.

Note also that this approach is modelling only one specific forward rate and can be used to price instruments depending on that rate only. The approach does not describe the full forward curve but only one point on it. Also it does not describe the discounting curve dynamic, except very indirectly by saying that the forward rate dynamic is valid in a numeraire attached to the discounting curve. For this approach all the intuition and risk management techniques developed in the one-curve framework can be used with almost no change.

Even if the approach is not a full term structure and multi-curve description, it can be embedded in such models as the LMM. Some approaches to such models are described in Section 7.9.

7.4.2 SABR model

The same approach can be used for any model which is based on a stochastic equation of the forward rate in the $P_X^D(.,v)$ numeraire. In particular it can be applied to the SABR model from Hagan et al. (2002) which is very popular in interest rate modelling to describe the smile.

The term SABR is an acronym for Stochastic Alpha Beta Rho model. It is a stochastic volatility model on the forward rate:

$$dF_X^{CPN,j}(t,u,v) = \alpha_t(F_X^{CPN,j}(t,u,v))^\beta dW_t^1$$

$$d\alpha_t = v\alpha_t dW_t^2$$

with W^1 and W^2 two correlated Brownian motions with $[W^1, W^2]_t = \rho t$. The state variables are F_t, the rate, and α_t, the volatility. In some presentations the volatility is denoted σ_t, like in the Black framework (see in particular (Rebonato et al., 2009,

Section 3.2, Footnote 1)). The model parameters are β, the elasticity, v, the volatility of volatility and ρ, the correlation.

Under that approach the price of cap/floor can be approximated by a Black formula similar to the one of Theorem 7.2 but with the period volatility replaced by $\sigma(K, F_X^{CPN,j})\sqrt{\theta}$ with

$$\sigma(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left(1 + \frac{(1-\beta)^2}{24}\ln^2\frac{f}{K} + \frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}\right)}\frac{z}{x(z)}$$
$$\times\left(1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}v^2\right)\theta\right)$$

where

$$z = \frac{v}{\alpha}(fK)^{(1-\beta)/2}\ln\frac{f}{K}$$

and

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right).$$

Like for the pure Black approach the framework results in conclusions very similar to those obtained in the one-curve framework and all the techniques developed there can be reused.

7.5 Swap rate modelling

The standard approach to swap rate modelling in the one-curve framework is very similar to the one described in the previous section. One notices that the swap rate is a martingale in the present value of a basis point numeraire and proceeds from there.

The notation regarding the vanilla interest rate swaps we use here are the same as in Section 2.4. Let

$$A_t = \sum_{i=1}^{\tilde{n}} P_X^D(t, \tilde{t}_i)\tilde{\delta}_i$$

be a positive random variable called *present value of a basis point*. The quantity is also called *delivery annuity* or *level*.

The swap rate S^j described in Equation (8.19) is a martingale in the A_t numeraire under the hypotheses $\mathbf{D}$ and $\mathbf{I}^{CPN}$. To see that this is a martingale, it is enough to notice that the rate is a linear combination of assets – the floating coupons – divided by the numeraire.

As it is a martingale, it can be modelled by a drift-less stochastic equation. The simplest approach in this case is to suppose a geometric Brownian motion for the rate in the A numeraire:

$$dS_t^j = S_t^j \eta(t, u) \cdot dW_t^A. \tag{7.8}$$

with W_t^A a Brownian motion in the measure associated to the A_t numeraire. The Brownian motion is a multidimensional random variable with the dimension of η equal to the dimension of the Brownian motion. This is often referred to as the Black (1976) model for swap rates.

The solution of the above equation is an exponential martingale

$$S_t^j = S_0^j \exp\left(\int_0^t \eta(\tau, u) \cdot dW_\tau - \frac{1}{2} \int_0^t |\eta(\tau, u)|^2 d\tau \right).$$

Let σ_η, defined by

$$\sigma_\eta^2(t) = \int_0^t |\eta(\tau, u)|^2 d\tau,$$

be the period volatility. With that notation, the solution can be written as

$$S_t^j = S_0^j \exp\left(\sigma_\eta(t) X - \frac{1}{2} \sigma_\eta^2(t) \right)$$

for X a standard normally distributed random variable of dimension 1.

A physical delivery swaption with strike K and expiry θ is a financial instrument that gives its owner the right in θ to enter into a vanilla fixed for Ibor interest rate swap with fixed rate K. The details of the swap, such as its convention and tenor, are agreed between the parties when the option is written.

On the expiry date, the value of the swaption on a payer swap can be written as

$$A_t(S_\theta^j - K)^+.$$

The value is simply the positive value of a receiver par swap plus the payer swap with fixed rate K. The total value of the par swap is 0 by definition and the values of the two floating legs cancel.

From that equation, one can proceed exactly like in the cap/floor case, just replacing the discount factor numeraire by the annuity numeraire.

Theorem 7.3 *In the multi-curve framework with hypotheses D and I^{CPN}, if the swap rate follows the dynamic given by Equation (7.8), the present value of a swaption of strike K and expiry θ is given by*

$$A_0 \omega \left(S_0^j N(\omega d_+) - K N(\omega d_-) \right)$$

where

$$d_\pm = \frac{\ln\left(\frac{S_0^j}{K}\right) \pm \frac{1}{2}\sigma_\eta^2(\theta)}{\sigma_\eta(\theta)}$$

and $\omega = +1$ for a payer swaption and $\omega = -1$ for a receiver swaption.

Like in the previous section, we note that the equation was obtained without explicit hypotheses on the spread between the risk-free rates and the forward Ibor rates. The only hypothesis is the dynamic of the swap rate, involving the risk-free and Ibor curves, in a specific numeraire, itself involving the risk-free curve.

7.5.1 Modified convention

The development for the SABR model in the previous section could be repeated for the swaption. We do not do it, but simply summarise the situation by the fact that swaptions are priced with a Black-like formula using an implied volatility which is strike, forward, time and model parameters dependent. This is summarised by the present value P given by

$$P = A_0 \text{Black}(S_0, K, \sigma(\theta, T, S_0, K, p)).$$

In the above approach we have supposed that only one convention is used for the swaption underlying swaps. In practice, the conventions are often mixed in one portfolio. The standard interbank market will use only one specific convention for each currency. Those conventions for swaps are described in Section B.8. But certain swaptions may have been traded with a corporate that requested a different convention, for example to hedge a bond.

As the modelling above is a modelling of a specific swap rate, it is implicitly a modelling in a specific convention. One way to deal with the change of convention problem is to have one volatility input for each convention. This approach is not feasible in practice as the required data is not available.

To ensure absence of arbitrage when the conventions are changed, modified versions of standard figures are introduced. Let C be the standard convention.

The convention C-modified delivery annuity is

$$A_t^C = \sum_{i=1}^n \delta_i^C P_X^D(t, t_i)$$

where δ_i^C is the accrual factor in the C convention for the same dates as the original present value of a basis point. The convention-modified swap rate is

$$S_t^C = \frac{\sum_{i=1}^n P_X^D(t, t_i)\delta_i F_X^{\text{CPN},j}(t, t_{i-1}, t_i)}{A_t^C}.$$

To cover the case where the fixed rate is not the same for all coupons, a *strike equivalent* is introduced:

$$K = \frac{\sum_{i=1}^{n} \delta_i K_i P^D(0, t_i)}{A_0}.$$

Note that the strike equivalent is curve dependent.

The convention-modified strike equivalent is

$$K^C = \frac{\sum_{i=1}^{n} \delta_i K_i P^D(0, t_i)}{A_0^C}.$$

The price, taking into account the fact that the implied volatilities are provided for standard-convention instruments, is

$$P = A_0^C \text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p)).$$

Note that the convention change in the swap rate and the strike equivalent is the inverse of the one in the annuity. The modification impact appears only through the volatility; if the smile is flat, the modification has no impact. If the swaption has a standard convention, there is no impact at all.

In this way, the potential arbitrage coming from a change of convention disappears. A swaption with strike 3.60% in the ACT/360 convention has exactly the same cash-flows as a swaption with strike 3.65% in the ACT/365 convention. We have to make sure that our pricing approach provides the same prices for both and does not introduce arbitrage. The strike and forward dependency in the SABR implied volatility would create this arbitrage if the strike and forward were not adjusted for the convention.

7.6 Parsimonious HJM multi-curve framework

7.6.1 The model

Let W_t^v be a multidimensional Brownian motion in the measure associated to the numeraire $P^D(., v)$. The associated filtration is denoted $\mathcal{F}_.$.

In the parsimonious HJM approach proposed by Moreni and Pallavicini (2010), the equations for the continuously compounded instantaneous forward risk-free rates (see Section A.1 for the definition) and the market Ibor rates are

$$df(t, v) = \sigma(t, v) \cdot dW_t^v \tag{7.9}$$

$$dF_X^{\text{CPN}, j}(t, u, v) = \left(F_X^{\text{CPN}, j}(t, u, v) + a^j(v)\right) \eta^j(t, v) \cdot dW_t^v \tag{7.10}$$

The model for the risk-free rate is a Gaussian HJM model and we use the notation and results of Appendix A to analyse the approach.

In the cash account numeraire, the equations for the risk-free rates and the market Ibor forward rates are

$$df(t,v) = \sigma(t,v) \cdot v(t,v)dt + \sigma(t,v) \cdot dW_t$$

$$dF_X^{\text{CPN},j}(t,u,v) = \left(F_X^{\text{CPN},j}(t,u,v) + a^j(v) \right) \eta^j(t,v) \cdot v(t,v)dt$$

$$+ \left(F_X^{\text{CPN},j}(t,u,v) + a^j(v) \right) \eta^j(t,v) \cdot dW_t$$

The solution of the Ibor rates equations in the cash account numeraire can be written as

$$F_X^{\text{CPN},j}(t,u,v) + a^j(v) = \left(F_X^{\text{CPN},j}(s,u,v) + a^j(v) \right)$$

$$\times \exp \left(\int_s^t \eta^j(\tau,v) \cdot dW_\tau - \frac{1}{2} \int_s^t |\eta^j(\tau,v)|^2 d\tau + \int_s^t \eta^j(\tau,v) \cdot v(\tau,v)d\tau \right)$$

We give a name to several integrated volatilities. The volatility of the forward rate is

$$\sigma_\eta^2 = \sigma_\eta^2(\theta_0, \theta_1, v) = \int_{\theta_0}^{\theta_1} |\eta^j(s,v)|^2 ds.$$

Similarly the interaction between the risk-free rates and Ibor forward rates is used for different instruments:

$$\sigma_{\eta,N} = \sigma_{\eta,N}(\theta_0, \theta_1, v) = \int_{\theta_0}^{\theta_1} \eta^j(\tau,v) \cdot v(\tau,v)d\tau$$

and

$$\gamma_\eta = \gamma_\eta(\theta_0, \theta_1) = \gamma_\eta(\theta_0, \theta_1, v) = \exp \left(\sigma_{\eta,N}(\theta_0, \theta_1, v) \right).$$

Using this notation, the forward rate solution can be written in the cash account numeraire as

$$F_X^{\text{CPN},j}(t,u,v) + a^j(v) = \left(F_X^{\text{CPN},j}(s,u,v) + a^j(v) \right)$$

$$\times \exp \left(-X_\eta - \tfrac{1}{2}\sigma_\eta(s,t,v)^2 \right) \gamma_\eta(s,t) \qquad (7.11)$$

with X_η the normally distributed random variable $X_\eta = -\int_s^t \eta^j(\tau,v) \cdot dW_\tau$ independent of $\mathcal{F}_s$ with standard deviation $\sigma_\eta(s,t,v)$.

7.6.2 Parsimonious HJM analysis

With that result in mind, Equation (7.10) can be viewed as an extension of the constant spread hypothesis $\mathbf{S0}^{\text{CPN}}$. For a multiplicative spread $\beta_0^j(v)$, the equation

for the forward rate is Equation (7.10) with a displacement $a^j(v) = \beta_0^j(v)/\delta$ and a volatility $\eta^j(t, v) = v(t, v) - v(t, v)$.

Equation (7.10) can also be applied for the additive spread approach. Suppose that $F_t^j(v) = F_t^D(v) + s(v)$ with $s(v)$ a deterministic spread. Then the equation is satisfied with $a^j(v) = 1/\delta - s^j(v)$ and $\eta^j(t, v) = v(t, v) - v(t, v)$.

The choice of σ and η_i separately can lead to behaviour not in line with financial reality. Take $a^j(v) = 1/\delta$ and $\eta^j(t, v) = c(v(t, v) - v(t, v))$. The simplified intuition behind that choice is that the Ibor rates have a dynamic similar to the one coming from the short rate model but with a larger volatility (multiplied by c), one part coming from the rates and one from the credit. What is the market spread $F_X^{CPN,j}(t, u, v) - F_X^D(t, u, v)$ implied by such a model? The solution can be written explicitly in that case.

For the first graphical representation, we take $t = 1$, $\sigma_2 = 0.01$, $F_0^D = 0.02$, $F_0^j = 0.0220$ and $c = 1.05$. The results for the rates and spreads are displayed in Figure 7.1. The X-axis represents the size of the normally distributed random variable underlying the dynamic. The Y-axis represents the rate or spread in percent. The horizontal dotted line is the initial level of the spread.
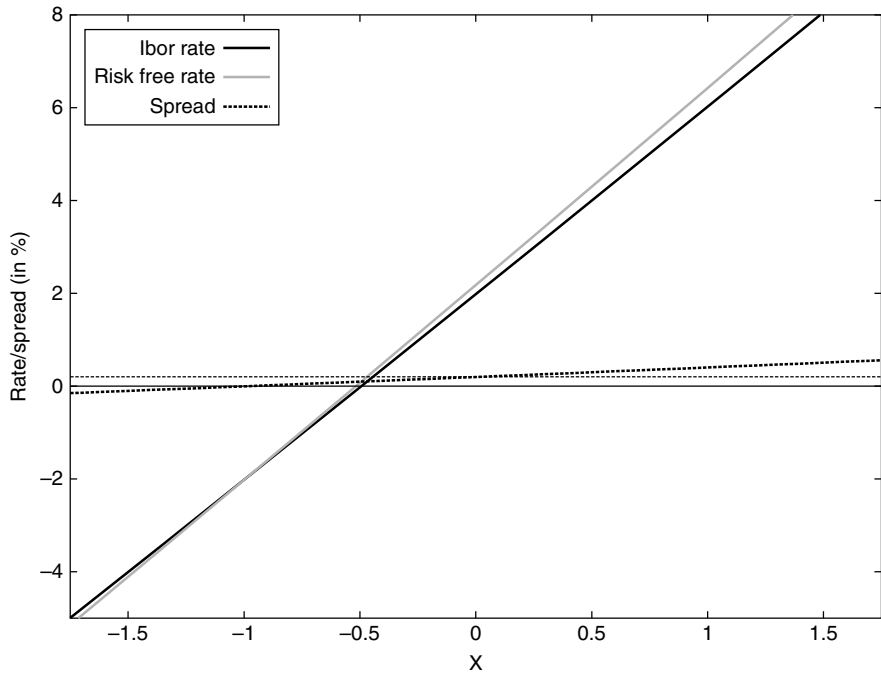

Figure 7.1 Representation of the rates relative dynamic and the implied spread dynamic in the parsimonious HJM approach. Parameter choice described in the text.
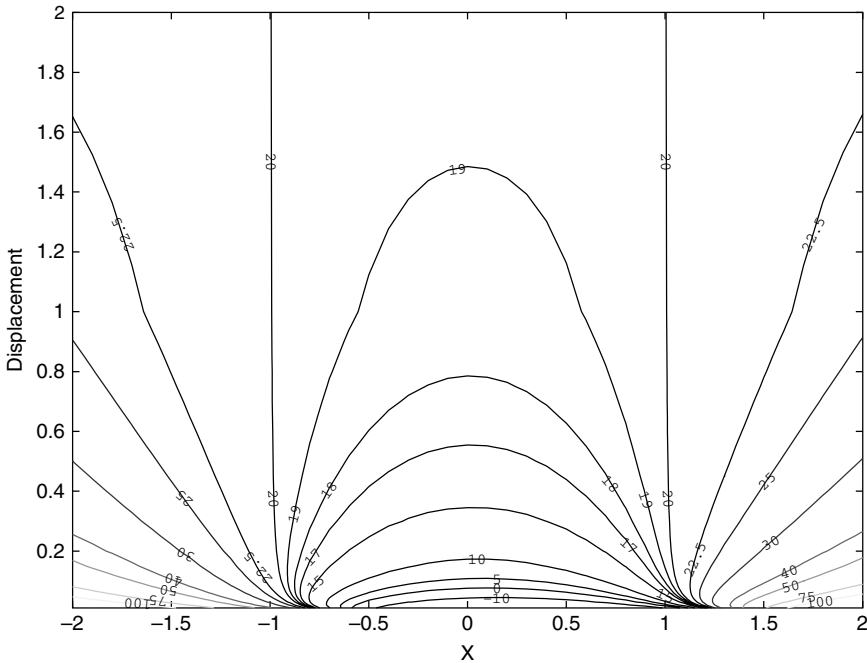
Figure 7.2 Spread levels for different values of the underlying standard normal random variable on the horizontal axis and different levels of displacement on the vertical axis. See the text for the choice of parameters.

As can be seen from the graph, the spread will depend on the level of rates. When the rate increases, the spread increases; when the rate decreases, the spread decreases. The dynamic is such that with the reasonable decrease of rate – normal distribution change of one standard deviation – the spread is becoming negative, in contradiction with financial intuition. Apart from the trivial constant spread case, it may be difficult to find actual implementation with non-trivial spread dynamic and at the same time intuitive spreads.

In the second graphical representation in Figure 7.2, we used the same starting points for the rate and the spread and the same time to expiry. This time we fix the volatility of the risk-free rate and vary the dynamic of the forward rate. The variation is obtained by changing the displacement coefficient. The coefficients used are between 0, corresponding to a log-normal model, and 2. A displacement of 4 would be equivalent to the implicit displacement of the risk-free rate of $1/\delta$ for a quarterly rate. The difference between a displacement of 2 and 4 is very minimal and would not add any information on the graph. The level of volatility η is chosen such that the approximate equivalent volatility $v(F^D(0) - 1/\delta)/(F^j(0) - a^j)$ is preserved. At the starting point, F^D and F^j have the same local volatility.

What is interesting from the graph is that even if the volatility is locally the same at the start point, the implied dynamics are very different. For small displacements, the spread can change from its initial value of 20 basis points to a range of -10 to $+100$ for not extreme movements of rate.

7.6.3 Cap/floor

The pricing of cap and floor in this approach is relatively easy. The forward rates follow a simple displaced diffusion formula in the numeraire on the discount factor to their maturity. A generalised Black formula is sufficient to price them.

Theorem 7.4 *In the parsimonious HJM framework, the present value of a cap with expiry θ and strike K on a rate over the period $[u, v]$ is given in 0 by*

$$C_0 = \left(F_X^j(0, u, v) + a^j(v) \right) N(\kappa) - \left(K + a^j(v) \right) N(\kappa - \sigma_\eta(0, \theta, v))$$

where

$$\kappa = \frac{1}{\sigma_\eta(0, \theta, v)} \ln \left(\frac{F_X^{\mathrm{CPN}, j}(0, u, v) + a^j(v)}{\tilde{K} + a^j(v)} \right) + \frac{1}{2} \sigma_\eta(0, \theta, v).$$

This is a direct application of the formula for cap/floor in the displaced diffusion framework. For cap/floor the results are similar to those proposed in Section 7.4 but with a displaced log-normal diffusion instead of a pure log-normal diffusion.

7.6.4 STIR futures

STIR futures are described in Appendix B.4. The futures is characterised by a fixing date t_0 and a reference Ibor rate on the period $[u, v]$ with an accrual factor δ. The futures price is denoted Φ_t^j.

The pricing formula for such futures is given by the following theorem.

Theorem 7.5 (STIR futures in parsimonious HJM framework). *Let $0 \le t \le t_0 \le u < v$. In the Gaussian parsimonious HJM model in the multi-curve framework with hypotheses D and I^{CPN}, the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1 - F^{\mathrm{CPN}, j}(t, u, v) \gamma(t, t_0) + a^j(v) \left(1 - \gamma_\eta(t, t_0) \right). \tag{7.12}$$

Proof: Using the generic pricing future price process theorem,

$$\Phi_t^j = \mathrm{E}^N \left[1 - I_X^j(t_0) \middle| \mathcal{F}_t \right]$$

where $\mathrm{E}^N []$ is the cash account numeraire expectation.

Using the formula for the forward rate described in the previous section, we have

$$\mathrm{E}^N \left[F_X^{\mathrm{CPN},j}(t_0, u, v) \middle| \mathcal{F}_t \right]$$

$$= \mathrm{E}^{\mathbb{N}} \left[\left(F_X^{\mathrm{CPN},j}(t, u, v) + a^j(v) \right) \exp\left(-X_\eta - \frac{1}{2}\sigma_\eta^2 \right) \gamma_\eta(t, t_0, v) - a^j(v) \middle| \mathcal{F}_t \right]$$

$$= \left(F_X^{\mathrm{CPN},j}(t, u, v) + a^j(v) \right) \gamma_\eta(t, t_0, v) - a^j(v)$$

From the above result, the futures price is easily obtained:

$$\Phi_t^j = 1 - F_X^{\mathrm{CPN},j}(t, u, v)\gamma_\eta(t, t_0, v) - a^j(v)\left(\gamma_\eta(t, t_0. v) - 1 \right).$$

□

The result is very close to the one proposed for Gaussian HJM with hypothesis **S0$^{\mathrm{CPN}}$** but with a different convexity adjustment as described by γ_η. Note also that in a Gaussian HJM model with deterministic spread, the shift a is equal to $1/\delta$. The formula proposed is a natural extension of the previous one in this more general framework.

Note that the futures price depends on the forward rate $F^{\mathrm{CPN},j}$ and also on the joined dynamic of the risk-free rate and the forward rate through γ_η.

7.6.5 STIR futures options – margin

In this section we analyse the options on STIR futures with daily margining. The options are subject to margining processes similar to the margining process on the underlying futures. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous section.

The futures options usually have an American feature. Due to the margining process, the American options have the same price as the European options. This general observation for options with continuous margining can be found in Chen and Scott (1993).

We denote by $R_t = 1 - \Phi_t^j$ the futures rate and by $\tilde{K} = 1 - K$ the strike rate.

Due to the margining process on the option, the price of the option with margining is

$$\mathrm{E}^{\mathbb{N}} \left[(\Phi_\theta - K)^+ \right] = \mathrm{E}^{\mathbb{N}} \left[(\tilde{K} - R_\theta)^+ \right].$$

Theorem 7.6 *Let $0 \leq \theta \leq t_0 \leq u < v$. The value of a STIR futures call (European or American) option of expiry θ and strike K with continuous margining in the parsimonious Gaussian HJM multi-curve framework is given in 0 by*

$$C_0 = (\tilde{K} + a^j(v))N(-\kappa_\gamma) - \left(F_X^{\mathrm{CPN},j}(0, u, v) + a^j(v) \right) \gamma_\eta(0, t_0, v)N(-\kappa_\gamma - \sigma_1(0, \theta, v))$$

where κ is defined by

$$\kappa_\gamma = \frac{1}{\sigma_\eta(0,\theta,v)} \ln \left(\gamma_\eta(0,t_0) \frac{F_X^{\text{CPN},j}(0,u,v) + a^j(v)}{\tilde{K} + a^j(v)} \right) - \frac{1}{2}\sigma_\eta(0,\theta,v).$$

The price of a STIR futures put option is given by

$$P_0 = \left(F_X^{\text{CPN},j}(0,u,v) + a^j(v) \right) \gamma_\eta(0,t_0) N(\kappa + \sigma_\eta(0,\theta,v)) - (\tilde{K} + a^j(v))N(\kappa).$$

Proof: Using the generic pricing theorem we have

$$C_0 = E^N \left[((\Phi_\theta - K)^+) \right]$$

$$= E^N \left[\left(\tilde{K} - \left(F_X^{\text{CPN},j}(\theta,u,v) + a^j(v) \right) \gamma_\eta(0,t_0,v) + a^j(v) \right)^+ \right].$$

Using Equation (7.11) once more, we have

$$C_0 = E^{\mathbb{N}} \left[\left(\left(\tilde{K} + a^j(v) \right) - \left(F_X^{\text{CPN},j}(0,u,v) + a^j(v) \right) \right. \right.$$

$$\left. \left. \times \exp\left(-X_\eta - \frac{1}{2}\sigma_\eta(0,\theta,v)^2 \right) \gamma_\eta(0,t_0,v) \right)^+ \right].$$

where we have used the fact that $\gamma_\eta(0,\theta,v)\gamma_\eta(\theta,t_0,v) = \gamma_\eta(0,t_0,v)$.

In the expectation, the parenthesis is positive when

$$\tilde{K} + a^j(v) > \gamma(0,t_0) \left(F_X^{\text{CPN},j}(0,u,v) + a^j(v) \right) \exp\left(-X_\eta - \frac{1}{2}\sigma_\eta(0,\theta,v)^2 \right)$$

or when $X_\eta > \sigma_\eta \kappa_\gamma$. With this notation, the expected value is equal to

$$\frac{1}{\sqrt{2\pi}} \int_{x > \kappa_\gamma} \left(\tilde{K} + a^j(v) \right.$$

$$\left. - \left(F_X^{\text{CPN},j}(0,u,v) + a^j(v) \right) \exp\left(-\sigma_\eta x - \frac{1}{2}\sigma_\eta^2 \right) \gamma(0,t_0) \right) \exp\left(-\frac{1}{2}x^2 \right) dx$$

$$\tag{7.13}$$

The results follow using the definition of cumulative normal density N. ☐

7.6.6 Forward rate agreement

In this section we analyse the pricing of a Forward Rate Agreement (FRA) as described in Appendix B.3.

Let t_0 be the fixing date of the FRA on a reference rate for the period $[u,v]$. The date u is the spot lag after t_0 and the end date v is a j-period after u using the

appropriate business day convention. The FRA payment is done in u, this is at the start of the reference period and the amount is, for a fixing $I_X^j(t_0)$,

$$\delta \frac{(I_X^j(t_0) - K)}{1 + \delta I_X^j(t_0)}.$$

We incorporate this analysis in this chapter as it is important for building curves from market instruments to have present values for the most liquid market instruments. The techniques used in the pricing are very similar to those used for options on futures.

With the deterministic hypothesis $\mathbf{S0^{CPN}}$, the fair rate for market FRAs is similar to the fair rate for simplified textbook FRAs. It is then possible to construct curves correctly even with the erroneous description of FRAs. The simplified version and the real version give the same market quote under hypothesis $\mathbf{S0^{CPN}}$. This luck is only valid for the curve construction process, the present values of the actual portfolios are still erroneous.

This is only by luck, not by design. If a new approach to the spread is used, like in the parsimonious HJM model, one has to analyse the present value and the par rate.

Let

$$X_1 = \int_0^{t_0} \eta^j(\tau, v) \cdot dW_\tau^v$$

and

$$X_2 = \int_0^{t_0} (v(s, u) - v(s, v)) dW_\tau^v$$

The two dimensional random variable $X = (X_1, X_2)$ has a covariance

$$\begin{pmatrix} \alpha_1^2(0, t_0, v) & \alpha_{1,2}(0, t_0, u, v) \\ \alpha_{1,2}(0, t_0, u, v) & \alpha_2^2(0, t_0, u, v) \end{pmatrix}$$

with

$$\alpha_{1,2} = \alpha_{1,2}(\theta_0, \theta_1, u, v) = \int_{\theta_0}^{\theta_1} \eta^j(\tau, v) \cdot (v(s, u) - v(s, v)) d\tau$$

From Section 7.6.1, the Ibor rate satisfies

$$F_X^{CPN,j}(t_0, u, v) + a^j = \left(F_X^{CPN,j}(0, u, v) + a^j \right) \exp \left(-X_1 - \frac{1}{2}\sigma_1^2 \right).$$

The expected value associated to the numeraire $P_X^D(., v)$ is denoted $\mathrm{E}^v[]$. The value of the FRA is given by

$$\mathrm{E}^v \left[\frac{P^D(t_0, u)}{P^D(t_0, v)} \delta \frac{(F_X^{CPN,j}(t_0, u, v) - K)}{1 + \delta F_X^{CPN,j}(t_0, u, v)} \right] = \mathrm{E}^v \left[\frac{P^D(t_0, u)}{P^D(t_0, v)} \left(1 + \frac{1 + \delta K}{1 + \delta F_X^{CPN,j}(t_0, u, v)} \right) \right]$$

All the different parts can be computed explicitly:

$$\frac{P^D(0,u)}{P^D(0,v)} \exp\left(-\frac{1}{2}\alpha_2^2\right) E^v\left[\exp\left(-X_2\right)\left(1 + \frac{1}{c_1 + c_2\exp\left(-X_1\right)}\right)\right]$$

with $c_1 = (1 - \delta a_j)/(1 + \delta K)$ and $c_2 = \delta(F_0^j + a^j)\exp\left(-\sigma_1^2/2\right)/(1 + \delta K)$.

The above expectation can be computed easily by numerical integration on a two dimensional integral. What we want to demonstrate by the above result is that even in a relatively simple model and for a relatively simple quasi-linear instrument, the valuation formulas can be complex in a multi-curve framework with spread.

7.7 Additive stochastic spread

7.7.1 The model

The additive stochastic basis approach developed by Mercurio and Xie (2012) allows for any model for the risk-free curve.

The additive spread is defined by

Definition 7.1 (Additive spread). *The additive spread between the j-Ibor forward rate and the risk-free forward rate at time t for the period $[u,v]$ is given by*

$$S_t^j(v) = S_t^j(u,v) = F_X^{\text{CPN},j}(t,u,v) - F_X^D(t,u,v).$$

The idea of the framework is to model the additive spread $S_t^j(v)$ as a function of the risk-free forward rate $F_X^D(t,u,v)$ and of an independent martingale $\mathcal{X}_t^j(v)$ through

$$S_t^j(v) = \phi_v^j(F_X^{\text{CPN},j}(t,u,v), \mathcal{X}_t^j(v)). \tag{7.14}$$

The simplest function introduced in the above article and used here is

$$S_t^j(v) = S_0^j(0) + \alpha_v^j\left(F_X^{\text{CPN},j}(t,u,v) - F_X^{\text{CPN},j}(0,u,v)\right) + \beta_v^j\left(\mathcal{X}_t^j(v) - \mathcal{X}_0^j(v)\right)$$

with $\mathcal{X}_0^j(v) = 1$. The parameter β_v^j and the volatility of $\mathcal{X}$ have similar roles in adjusting the size of the independent part of the stochastic spread.

Note that in this approach, no choice of α, β and $\mathcal{X}$ can reproduce the deterministic multiplicative spread hypothesis **S0$^{\text{CPN}}$**.

7.7.2 Model analysis

We first look at the relation between the level of rates and the implied spread. The spread is made of a function of the rate level, the one multiplied by α, and an independent part, the one multiplied by β. In Figure 7.3, we have represented some
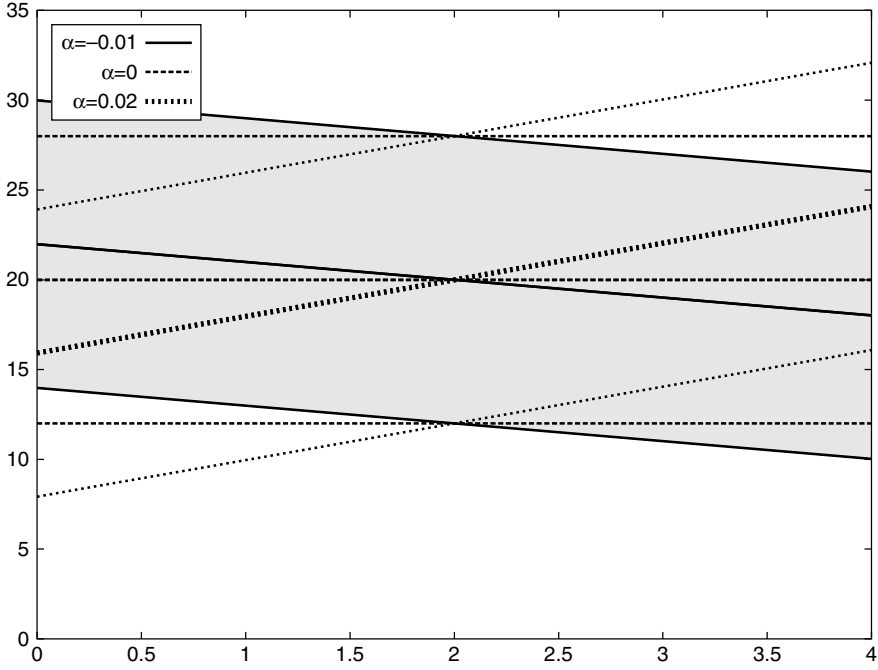
Figure 7.3 The additive spread $F_X^{\text{CPN},j}(t,u,v) - F_X^D(t,u,v)$ for different levels of risk-free rate $F_X^D(t,u,v)$. The rate is in percent and the spread in basis points.

examples of the framework with different coefficients. The starting forward risk-free rate $F^D(0)$ is 2.00%, the Ibor forward rate is $F^j(0) = 2.20\%$ and the horizon on which we look at the spread is one year. We use for $\mathcal{X}_t^j(v)$ a martingale of the form

$$\mathcal{X}_t^j(v) = \exp\left(-X_t^{\mathcal{X}}(v) - \frac{1}{2}\sigma_{\mathcal{X}}^2\right)$$

with $X^{\mathcal{X}}$ a martingale in the $P_X^D(.,v)$ numeraire normally distributed with mean 0 and variance $\sigma_{\mathcal{X}}^2$. In the example, we use $\sigma_{\mathcal{X}} = 0.0002$ and $\beta = 1$.

7.7.3 STIR futures

The pricing formula for STIR futures is given by the following theorem, which is proposed in Mercurio and Xie (2012). As for all the approaches, it is important to obtain relatively simple STIR futures formulas as the futures are one base ingredient for curve calibration.

Theorem 7.7 (STIR futures pricing in additive stochastic framework). *Let $0 \le t \le t_0 \le u < v$. In the additive stochastic basis multi-curve framework with hypotheses **D***

and I^{CPN}, the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given in t by

$$\Phi_t^j = 1 - F_X^{CPN,j}(t, u, v) \tag{7.15}$$
$$+ \frac{(1+\alpha)}{\delta} \left(E^{\mathbb{N}} \left[1 + \delta F_X^D(t_0, u, v) \middle| \mathcal{F}_t \right] - (1 + \delta F_X^D(t, u, v)) \right)$$

Proof: Using the generic pricing future price process theorem,

$$\Phi_t^j = E^N \left[1 - F_X^{CPN,j}(t_0, u, v) \middle| \mathcal{F}_t \right]$$

where $E^N[]$ is the cash account numeraire expectation. The forward rate can be written as

$$F_X^{CPN,j}(t_0, u, v) = F_X^D(t_0, u, v) + S_{t_0}^j(v)$$
$$= F_X^{CPN,j}(t, u, v) + (1 + \alpha) \left(F_X^D(t_0, u, v) - F_X^D(t, u, v) \right) + \beta \left(\mathcal{X}_{t_0}^j(v) - \mathcal{X}_t^j(v) \right).$$

As $\mathcal{X}_t^j$ is a cash account martingale, the expectation is reduced to

$$1 - F_X^{CPN,j}(t, u, v) + (1 + \alpha) \left(E^{\mathbb{N}} \left[F_X^D(t_0, u, v) \middle| \mathcal{F}_t \right] - F_X^D(t, u, v) \right).$$

$\square$

The price of a futures on the Ibor rate is reduced to the price of a forward adjusted by the convexity adjustment on the risk-free rate and by the multiplicative factor describing the volatility of the spread dependency on rates.

Theorem 7.8 *Let $0 \le t \le t_0 \le u < v$. In the additive stochastic basis for the multi-curve framework, if the risk-free rate curve follows a Gaussian HJM model as describes in Appendix A, the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1 - F_X^{CPN,j}(t, u, v) + (1 + \alpha) \left(F_X^D(t, u, v) + \frac{1}{\delta} \right) (\gamma(t, t_0, u, v) - 1). \tag{7.16}$$

The result is a direct consequence of the previous result and of the dynamic of the risk-free forward as described in Appendix A.

Note that the price depends on the forward Ibor rate $F_X^{CPN,j}$ but also on the risk-free forward rate F_X^D and the risk-free rate dynamic through γ. The fact that both $F_X^{CPN,j}$ and F_X^D appear in the pricing formula make it more difficult to price options on futures in the framework. The relatively simple results for futures options obtained in the parsimonious HJM model of Section 7.6 and in the multiplicative stochastic basis framework of Section 7.8 can not be achieved in this framework.

7.8 Multiplicative stochastic spread

In this section, we propose to analyse the multi-curve framework with a stochastic multiplicative spread approach and in the Gaussian HJM model for the risk-free curve described in Appendix A.

The idea of the stochastic spread framework is to model the multiplicative spread $\beta_t^j(v)$ as a function of the risk-free rate level and of an independent martingale $\mathcal{X}_t^j(v)$ with independent relative increments:

$$1 + \delta F_X^{\text{CPN},j}(t, u, v) = \beta^j(F_X^D(t, u, v), \mathcal{X}_t^j(v))(1 + \delta F_X^D(t, u, v)) \tag{7.17}$$

The dynamic of the risk-free curve will be based on a Brownian motion W_t which is $\mathcal{G}_t^1$-adapted. Let $\mathcal{G}_t^2$ be a filtration independent of $\mathcal{G}_t^1$. The random variable $\mathcal{X}_t^j(v)$ is a $\mathcal{G}_t^2$-adapted martingale in the $P^D(.,v)$ numeraire with independent relative increments and with $\mathcal{X}_0^j(v) = 1$. Independent relative increments for $\mathcal{X}$ means that $\mathcal{X}_t/\mathcal{X}_s$ with $t > s$ is independent of $\mathcal{G}_s^2$. The filtration generated by $\mathcal{G}_t^1$ and $\mathcal{G}_t^2$ is denoted $\mathcal{F}_t$.

Note that due to the fact that $P^D(.,v)$ and $N.$ are $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^1$ independent of $\mathcal{G}_t^2$, $\mathcal{X}_t$ is also a martingale in the $N.$ numeraire.

Obviously one cannot chose any β^j and have a coherent framework. From its definition and hypothesis $\mathbf{I}^{\text{CPN}}$, the quantity $1 + \delta F_X^{\text{CPN},j}(t, u, v)$ is a martingale in the $P^D(.,v)$-numeraire as explained in Theorem 2.1. This requirement needs to be checked for each particular function β^j.

In this section we focus on a specific form of spread. The multiplicative spread is a function of the level of the risk-free rate, an independent random variable and a deterministic function x^j:

$$\beta^j(F_X^D(t, u, v), \mathcal{X}_t^j(v)) = \beta_0^j(v)\mathcal{X}_t^j(v)x^j(t, u, v)\left(\frac{1 + \delta F_X^D(t, u, v)}{1 + \delta F_X^D(0, u, v)}\right)^{a^j(v)}. \tag{7.18}$$

The multiplicative spread is the product of the independent part and of the dependent part written as an exponent function. The function $x^j(t, u, v)$ with $x^j(0, u, v) = 1$ is deterministic and to be chosen in such a way that the martingale property is satisfied. A priori, the random variables $\mathcal{X}_t^j(v)$ and the coefficient $a^j(v)$ can be different for each index j and maturity date v. In the next sections we analyse futures and their options, which are linked to only one rate. Only one of those random variables and coefficients is used for each futures.

Note that when $a^j = 0$ and $\mathcal{X}_t^j = \mathcal{X}_0^j$, we recover the constant multiplicative spread hypothesis $\mathbf{S0}^{\text{CPN}}$.

In the applications, we will use for $\mathcal{X}_t^j(v)$ a martingale of the form

$$\mathcal{X}_t^j(v) = \exp\left(-X_t^{\mathcal{X},j}(v) - \frac{1}{2}\sigma_{\mathcal{X},j}^2(t, v)\right) \tag{7.19}$$

with $X^{\mathcal{X},j}$ normally distributed in the $P^D(.,v)$ numeraire with mean 0 and variance $\sigma^2_{\mathcal{X},j}(t,v)$. The volatility $\sigma_{\mathcal{X},j}$ is such that $\sigma_{\mathcal{X},j}(0,v)=0$. This variable could be, for example, the solution of a geometric Brownian motion equation.

The evolution of β^j_t can be written in terms of an earlier value β^j_s at s as

$$\beta^j_t(v)=\beta^j_s(v)\frac{\mathcal{X}^j_t(v)}{\mathcal{X}^j_s(v)}\frac{x^j(t,u,v)}{x^j(s,u,v)}\left(\frac{1+\delta F^D_X(t,u,v)}{1+\delta F^D_X(s,u,v)}\right)^{a^j(v)}.$$

In the Gaussian HJM framework, the risk-free forward rates are exponential martingales in the $P^D(.,v)$ numeraire as described in Lemma A.2. Using that result we have

$$E^v\left[1+\delta F^{\text{CPN},j}_X(t,u,v)\Big|\mathcal{F}_s\right]$$

$$=\frac{x^j(t,u,v)}{x^j(s,u,v)}\beta^j_s(v)(1+\delta F^D_X(s,u,v))\exp\left(-\frac{1}{2}(1+a^j)a^j\alpha^2(s,t,u,v)\right)$$

$$E^v\left[\frac{\mathcal{X}^j_t}{\mathcal{X}^j_s}\exp\left(-\alpha(s,t)(1+a^j)X^v_{s,t}-\frac{1}{2}\alpha^2(s,t,u,v)(1+a^j)^2\right)\Big|\mathcal{F}_s\right]$$

$$=\frac{x^j(t,u,v)}{x^j(s,u,v)}(1+\delta F^{\text{CPN},j}_X(s,u,v))\exp\left(-\frac{1}{2}(1+a^j)a^j\alpha^2(s,t,u,v)\right).$$

The remaining expected value in the second line is equal to 1. To obtain the equality, note that both $\mathcal{X}^j_t/\mathcal{X}^j_s$ and $X^v_{s,t}$ are independent of $\mathcal{F}_s$ and the conditional expected value becomes an expected value. Moreover due to the independence of $\mathcal{G}^1_t$ and $\mathcal{G}^2_t$, the expected value of the product is the product of the expected values. Both terms are martingales of initial value 1.

The quantity $1+\delta F^{\text{CPN},j}_X(t,u,v)$ is a martingale only if

$$x^j(t,u,v)=\exp\left(\frac{1}{2}(1+a^j)a^j\alpha^2(0,t,u,v)\right). \tag{7.20}$$

Note that x^j is linked to the dependency parameter $a^j(v)$ and the dynamic of F^D_t through the volatility $\alpha(0,t,u,v)$.

7.8.1 Spread rate dependency

The general description in Equation (7.19) of multiplicative spread is split into two parts. The first, comprising the part with an exponent a^j, is the part of the spread that depends on the level of risk-free rates. This is the systematic part, similar to the part multiplied by a coefficient α in the previous section.

The results of this section can be compared to those in the previous section by linearising Equations (7.18) and (7.19). The approximation gives

$$1+\delta F^j(t)\simeq1+\delta\left(F^j(0)-F^j(0)+a^j(F^D(t)-F^D(0))+\bar{\mathcal{X}}+F^D(t)\right)$$

for $1 + \delta\bar{\mathcal{X}} = \mathcal{X}$. This means that the hypothesis **SMS** is similar to the one in the previous section for the a^j here equal to the α there and the $\bar{\mathcal{X}}$ here equal to the $\beta\mathcal{X}$ there. As δF^D and $\bar{\mathcal{X}}$ are small with respect to 1, the linearisation is fairly precise.

We repeat the graph on dependency of spread and rate level in Figure 7.4. We use the same initial levels $F^D(0) = 2.00\%$, $F^j(0) = 2.20\%$ and the horizon of one year and values a and σ in line with that of the previous section. The figures look remarkably similar. The dynamic between the two approaches are the same to the first order. For some products, the multiplicative spread approach facilitates obtaining explicit formulas as described below.

When the coefficient a^j is 0, there is no dependency of the spread on the rate. There is no systematic change of the spread level with the rate level. When $a > 0$, the spread increases when the rate increases above its original level. When $a < 0$, the opposite behaviour appears. The framework can cope with periods where the spread is positively correlated to the level of rates or periods where the spread is negatively correlated to the level of rates. Moreover, we have two independent processes X and $\mathcal{X}$ that drive the level of risk-free rate and the non-dependent part of the spread. The total volatility of the forward rate $\alpha_{Y,j}$ described below can be
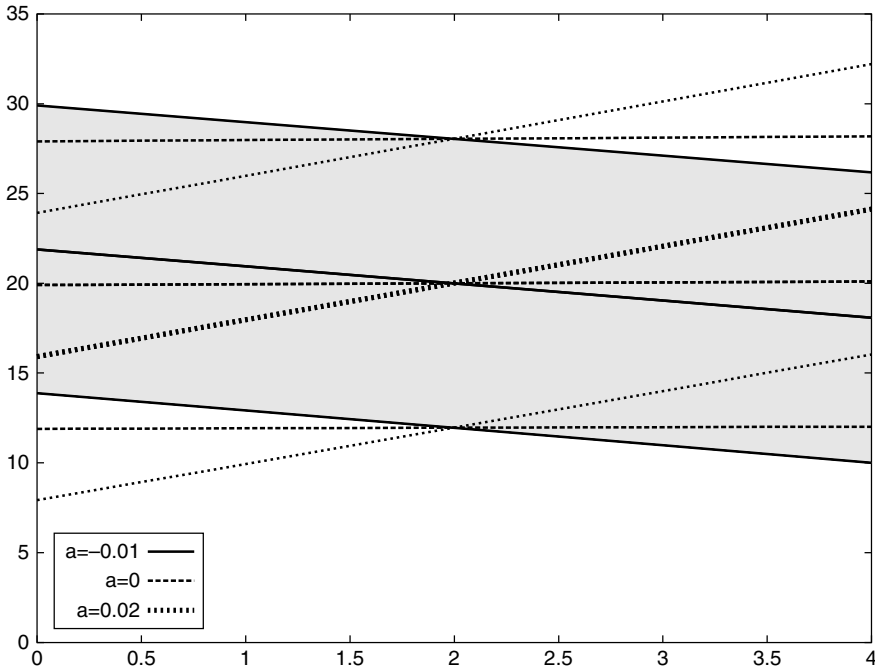

Figure 7.4 The additive spread $F_X^{\text{CPN},j}(t,u,v) - F_X^D(t,u,v)$ for different levels of risk-free rate $F_X^D(t,u,v)$ in the multiplicative stochastic spread framework. The rate is in percent and the spread in basis points.

split into risk-free rate and the credit-related spread in any way required to fit the market historical data.

Note that the exponent a is on the quantity $1 + \delta F_t^D$. This quantity is always positive as it is a ratio of assets. The quantity β^j is thus well defined for any value of a (positive or negative) and any financially possible value of F_X^D.

The independent stochastic part creates more possibility of changes in the spreads. This is why it was introduced. In the parsimonious HJM framework of Section 7.6, the spread is only a function of the level of rate.

7.8.2 Ibor rate dynamic

Using the spread dependency functions (7.18) and (7.19), the form of the independent martingale (7.20) and the dynamic of the forward risk-free rate described in Lemma A.2, the solution for the forward Ibor rate can be written in the $P_X^D(.,v)$ numeraire as

$$1 + \delta F_X^{\text{CPN},j}(\theta, u, v) = \beta_0^j \mathcal{X}_\theta^j \frac{x^j(\theta)}{(1 + \delta F_X^D(0, u, v))^a}(1 + \delta F_X^D(\theta, u, v))^{1+a}$$

$$= (1 + \delta F_X^{\text{CPN},j}(0, u, v)) \exp\left(-Y_\theta^{v,j} - \frac{1}{2}\alpha_j^2\right)$$

with $Y_\theta^{v,j} = X_\theta^{\mathcal{X},j} + (1+a)X_{0,\theta,u,v}^v$ and $\alpha_j^2(\theta) = \sigma_{\mathcal{X},j}^2(\theta) + (1+a)\alpha^2(0,\theta)$. The definition of $X_{0,\theta,u,v}^v$ is given in Appendix A. The random variable $Y_\theta^{v,j}$ is normally distributed with mean 0 and variance $\alpha_{Y,j}^2$ in the $P^D(.,v)$ numeraire.

A similar equation can be written in the cash account numeraire using Lemma A.1:

$$1 + \delta F_X^{\text{CPN},j}(\theta, u, v) = (1 + \delta F_X^{\text{CPN},j}(0, u, v)) \exp\left(-Y_\theta^j - \frac{1}{2}\alpha_j^2\right)\gamma(0, \theta, u, v)^{1+a^j}$$

with $Y_\theta^j = X_\theta^{\mathcal{X},j} + (1+a)X_{0,\theta,u,v}$.

The Ibor forward dynamic is similar to the forward risk-free dynamic but with a different volatility. The prices of cap/floor in the framework have a form very similar to the prices of cap/floor in the standard one-curve Gaussian HJM framework.

Theorem 7.9 (Cap/floor prices) *In the Gaussian HJM with multiplicative stochastic spread model in the multi-curve framework, the price of a cap of strike K and expiry θ and accrual factor δ is given by*

$$C_0 = P^D(0, v)\left((1 + \delta F_X^{\text{CPN},j}(0, u, v))N(\kappa + \alpha_j(\theta)) - (1 + \delta K)N(\kappa)\right)$$

with

$$\kappa = \frac{1}{\alpha_j(\theta)}\left(\ln\left(\frac{1 + \delta F_X^{\text{CPN},j}(0, u, v)}{1 + \delta K}\right) - \frac{1}{2}\alpha_j^2(\theta)\right).$$

7.8.3 STIR futures

The pricing of STIR futures in the multiplicative stochastic spread framework is given in the following theorem.

Theorem 7.10 (STIR futures pricing in the multiplicative spread framework). *Let $0 \leq t \leq t_0 \leq u < v$. In the Gaussian HJM with multiplicative stochastic spread multicurve framework (hypotheses **D, I** and **SMS**) the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given at t by*

$$\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t, u, v))\gamma^{1+a}(t, t_0, u, v). \tag{7.21}$$

Proof: Using the generic pricing futures price process theorem

$$\Phi_t^j = E^N \left[1 - F_X^{\text{CPN},j}(t_0, u, v) \Big| \mathcal{F}_t \right]$$

where $E^N[]$ is the cash account numeraire expectation, the future price can be written as

$$1 - F_X^{\text{CPN},j}(t_0, u, v) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\frac{\mathcal{X}_{t_0}^j}{\mathcal{X}_t^j}\frac{x(t_0)}{x(t)}\beta_t^j\frac{(1 + \delta F_X^D(t_0, u, v))^{1+a}}{(1 + \delta F_X^D(t, u, v))^a}$$

The important part in the expected value is, using Lemma A.1,

$$(1 + \delta F_X^D(t_0, u, v))^{1+a} = (1 + \delta F_X^D(t, u, v))^{1+a}$$
$$\times \exp\left(-(1+\alpha)X_{t,t_0} - \frac{1}{2}(1+a)^2\alpha^2(t, t_0) \right)\frac{x(t)}{x(t_0)}\gamma^{1+a}(t, t_0, u, v)$$

with the exponential term having an expected value of 1. This gives

$$E^{\mathbb{N}} \left[1 - F_X^{\text{CPN},j}(t_0, u, v) \Big| \mathcal{F}_t \right]$$
$$= 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^j\frac{(1 + \delta F_X^D(t, u, v)(v))^{1+a}}{(1 + \delta F_X^D(t, u, v))^a}\gamma^{1+a}$$
$$= 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t, u, v))\gamma^{1+a}$$

where we have used that $\mathcal{X}_{t_0}/\mathcal{X}_t$ and X_{t,t_0} are $\mathcal{F}_t$ independent and independent of each other. $\qquad\square$

Note that the pricing formula reduces to the one proposed in Section 2.6 in the deterministic spread hypothesis when $a = 0$. The volatility of the independent part of the spread $\mathcal{X}^j$ has no impact on the futures price.

The price of a futures on the Ibor rate is reduced to the price of a forward adjusted by the convexity adjustment on the risk-free rate and by the parameter describing the spread dependency on risk-free rates.

The futures will have different convexity adjustments for the same forward rate dynamic, this is the same volatility α_Y, but different split between the risk-free part and the independent part. At the extreme, when $a = -1$, the Ibor rate is independent of the risk-free rate and there is no convexity adjustment.

The price formula also allows us to write the forward rate as a function of the futures price

$$
F_X^{\text{CPN},j}(t, u, v) = \gamma(t, t_0, u, v)^{-1-a} \left(\frac{1}{\delta} - (1 - \Phi_t^j) \right) - \frac{1}{\delta}.
$$

7.8.4 STIR futures options – margin

In this section we analyse the options on STIR futures with continuous margining. The margined futures options are traded on LIFFE (USD, EUR, GBP, CHF) and Eurex (EUR). There is a margining process on the option itself similar to the margining process on the underlying futures. Let θ be the option expiry date and K its strike price. We also use the notation $\tilde{K} = 1 - K$, which we call strike rate. For the futures itself, we use the same notation as in the previous section.

Due to the margining process on the option, the price of the European option with margining is

$$
\mathrm{E}^N \left[(\Phi_\theta - K)^+ \right].
$$

The futures options usually have an American feature. Due to the margining process, the American options have the same price as the European options. This general observation for options with continuous margining can be found in Chen and Scott (1993). The proof of the equality is very short and is repeated below. A lower bound for the European call can be obtained using the Jensen inequality on the convex function $(.)^+$:

$$
C_0 = \mathrm{E}^N \left[(\Phi_\theta - K)^+ \right] \geq \left(\mathrm{E}^N \left[\Phi_\theta - K \right] \right)^+ = (\Phi_0 - K)^+.
$$

The value of the European call is always above the intrinsic value. It is never optimal to exercise the option early.

Theorem 7.11 (Option with continuous margining in the stochastic multiplicative spread framework) *Let $0 \leq \theta \leq t_0 \leq u < v$. The price of a STIR futures call option (European or American) of expiry θ and strike K with continuous margining in the Gaussian HJM with multiplicative stochastic spread multi-curve framework is given in 0 by*

$$
\begin{aligned}
C_0 = \frac{1}{\delta} \Big(& \left(1 + \delta \tilde{K} \right) N(-\kappa_\gamma) - \left(1 + \delta F_X^{\text{CPN},j}(0, u, v) \right) \\
& \times \gamma(0, t_0)^{1+a} N(-\kappa_\gamma - \alpha_{Y,j}(\theta)) \Big)
\end{aligned}
$$

where κ_γ is defined by

$$\kappa_\gamma = \frac{1}{\alpha_{Y,j}(\theta)} \left(\ln \left(\frac{1 + \delta F_X^{CPN,j}(0,u,v)}{1 + \delta \tilde{K}} \gamma(0,t_0)^{1+a} \right) - \frac{1}{2}\alpha_{Y,j}^2(\theta) \right).$$

The price of a STIR futures put option is given by

$$P_0 = \frac{1}{\delta} \left((1 + \delta F_X^{CPN,j}(0,u,v))\gamma(0,t_0)^{1+a}N(\kappa_\gamma + \alpha_Y) - (1 + \delta \tilde{K})N(\kappa_\gamma) \right)$$

Proof: The futures price is given in Theorem 7.10. Using the generic pricing theorem we have

$$C_0 = E^N \left[((\Phi_\theta - K)^+) \right]$$

$$= \frac{1}{\delta} E^N \left[\left((1 + \delta \tilde{K}) - \gamma(\theta,t_0)^{1+a}\beta_0 \mathcal{X}_\theta (1 + \delta F_X^D(0,u,v))\gamma(0,\theta)^{1+a} \right. \right.$$

$$\left. \left. \times \exp\left(-(1+a)X_{0,\theta} - \frac{1}{2}(1+a)^2\alpha(0,\theta)^2 \right) \right)^+ \right].$$

Using the form (7.20) for $\mathcal{X}^j$ and the definition of Y_θ^j in Section 7.8.2, the quantity in the parenthesis is positive when

$$\gamma(0,t_0)^{1+a}(1 + \delta F_X^{CPN,j}(0,u,v)) \exp\left(-Y_\theta^j - \frac{1}{2}\alpha_{Y,j}^2 \right) < 1 + \delta \tilde{K},$$

i.e. it is positive when $Y_\theta^j > \alpha_j \kappa_\gamma$.

The price is then given by

$$C_0 = \frac{1}{\delta} E^N \left[\mathbb{1}_{\{Y^j > \alpha_{Y,j}\kappa_\gamma\}} \left((1 + \delta \tilde{K}) - (1 + \delta F_X^{CPN,j}(0,u,v))\gamma(0,t_0)^{1+a} \right. \right.$$

$$\left. \left. \times \exp\left(-Y_\theta^j - \frac{1}{2}\alpha_j^2 \right) \right) \right]$$

$$= \frac{1}{\delta}\frac{1}{\sqrt{2\pi}} \int_{y > \kappa_\gamma} \left((1 + \delta \tilde{K}) \right.$$

$$\left. -(1 + \delta F_X^{CPN,j}(0,u,v))\gamma(0,t_0)^{1+a} \exp\left(-\alpha_{Y,j}y - \frac{1}{2}\alpha_j^2 \right) \right) \exp\left(-\frac{1}{2}y^2 \right) dy$$

$$= \frac{1}{\delta} \left((1 + \delta \tilde{K})N(-\kappa_\gamma) - (1 + \delta F_X^{CPN,j}(0,u,v))\gamma(0,t_0)^{1+a}N(-\kappa_\gamma - \alpha_j) \right)$$

$\square$

The pricing formula is similar to a Black formula with a price $1 + \delta \tilde{K}$, a forward $1 + \delta F^j$ and an adjustment on the forward given by the factor γ^{1+a}. The structure of the option formula is not very different from the one of the cap/floor in Theorem 7.9. The difference being the adjustment and the exercise boundary κ_γ.

Note that the adjustment factor $\gamma(0, t_0)^{1+\alpha}$ which appears in the price formula and in the formula for κ_γ is the same as the one used in the futures price. This factor can be obtained directly from the swap curve, the market price of futures and Equation (7.22) without calibrating the model itself.

Similarly the volatility parameter $\alpha_j(\theta)$ is the same as the one used in the cap/floor formula of Theorem 7.9. If the price of the cap/floor is available, the total volatility quantity α_j can be obtained directly from it. Even if a multi-factor model with time-dependent parameters is used, there is no need to calibrate each parameter individually to price the options on futures; it is enough to obtain the total volatility quantity $\alpha_j(\theta)$. Note that in practice this simplified calibration approach is valid only if the corresponding cap/floor is traded. That would not be the case for mid-curve options as 'mid-curve' cap/floors are not traded.

A pricing of options on futures coherent with swaps, futures and cap/floor can be obtained from the above instrument quotations in the proposed framework with very light calibration. The required calibration is not to fit all the model parameters but only to compute quantities deduced from the model parameters ($\gamma^{1+\alpha}$ and $\alpha_j(\theta)$) which can be read almost directly from the market quotes.

The formula in Theorem 7.11 can be written in term of futures price and strike as

$$C_0 = \left(1 - K + \frac{1}{\delta}\right) N(-\kappa_\gamma) - \left(1 - \Phi_t + \frac{1}{\delta}\right) N(-\kappa_\gamma - \alpha_j(\theta)).$$

In this way the futures options price is directly written in terms of its underlying futures.

Example

Using the above formulas we provide a sensitivity comparison between put/call on futures and cap/floor on Ibor rates. The calibration was done as described above using EUR market data from 9 August 2013. The quantity γ^{1+a} is obtained from the forward/futures difference and the quantity α_j is obtained from the cap/floor prices.

The underlying futures is the June 14 Euribor futures, this is ERM4. The futures expires in slightly more than ten months and its underlying rate has a maturity in slightly more than one year and one month.

The options are striked at 99.50%. This is close to at-the-money (ATM) with the forward rate for the same period being approximatively 0.455%. The results are displayed in Table 7.1 for ten contracts on futures options and ten million notional for the cap/floor. The table represents the market quote sensitivities, also called key

rate sensitivities, in the multi-curve framework. The discounting curve – denoted DSC in the table – is calibrated using OIS quotes and the forward curve – denoted FWD in the table – is calibrated using Euribor three months fixing and fixed versus Euribor three months swaps. The call/put on futures have a sensitivity to the discounting market quotes, even if only the forward rates appear in the formulas of Theorem 7.11. This dependency comes from the curve calibration as the forward curve depends on OIS and Euribor swaps market quotes.

Even if the two options are exercised in exactly the same conditions, that is the call is exercised if and only if the floor is exercised, the current moneyness is not exactly the same. One strike should be compared to the forward rate and the other to the futures price, which is not exactly one minus the forward rate. The difference comes from the different contractual obligations between the two instruments: one is margined, like the futures itself, and the other is not.

7.8.5 STIR futures options – premium

In this section we analyse the price of STIR futures options with up-front premium payment. Those options are traded for Eurodollar on CME and SGX and for JPY Libor on SGX. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous sections.

The price of options on futures with up-front premium payment in the Gaussian HJM model and the one-curve framework was first proposed in Henrard (2005). The extension to the multi-curve framework with deterministic spread is described in Quantitative Research (2012b). The results described here were first proposed in Henrard (2013c).

The premium is paid up-front and the value of the European call option is

$$C_0 = N_0 \, \mathrm{E}^{\mathbb{N}} \left[N_\theta^{-1} (\Phi_\theta - K)^+ \right].$$

The main stochastic processes and constant of the HJM model are described in Appendix A. The interaction between the stochastic parts of $X_{0,\theta}^N$ and Y_θ (see

Table 7.1 Call/Put and Floor/Cap comparison

Tenor	Call DSC	Call FDW	Floor DSC	Floor FDW	Put DSC	Put FDW	Cap DSC	Cap FDW
3M	0.00	0.10	0.00	0.10	0.00	−0.16	0.00	−0.16
6M	0.00	0.00	0.00	0.00	-0.01	0.00	0.00	0.00
9M	0.37	-191.89	0.38	−199.56	−0.59	312.18	−0.58	305.42
1Y	−0.57	244.34	−0.49	254.11	0.55	−397.51	0.75	−388.90
2Y	−0.05	40.47	−0.04	42.09	0.04	−65.84	0.06	−64.42

The sensitivities are to the market quotes and scaled to one basis point.

Section 7.8.2 for its definition) is given by

$$\alpha_{Nj}(t) = (1 + a^j) \int_0^t (v(\tau, u) - v(\tau, v)) \cdot v(\tau, t) d\tau.$$

There is no part dependent on $\sigma_\mathcal{X}$ as $\mathcal{X}_t^j$ is independent of W_t.

The variance/covariance matrix of the random variable $(X_{0,\theta}^N, Y_\theta^j)$ is given by

$$\Sigma = \begin{pmatrix} \alpha_N^2(\theta) & \alpha_{Nj}(\theta) \\ \alpha_{Nj}(\theta) & \alpha_j^2(\theta) \end{pmatrix}.$$

Theorem 7.12 (Options on STIR futures with premium) *Let $0 \le \theta \le t_0 \le u < v$. The value of a STIR futures call (European) option of expiry θ and strike K with up-front premium payment in the stochastic multiplicative spread framework with Gaussian HJM model on the risk-free curve is given in 0 by*

$$C_0 = \frac{1}{\delta} P^D(0, \theta) \left((1 + \delta \tilde{K}) N \left(-\kappa_\gamma + \frac{\alpha_{Nj}}{\alpha_j} \right) \right.$$

$$\left. - (1 + \delta F_X^{\mathrm{CPN},j}(0, u, v)) \gamma(0, t_0)^{1+\alpha} \exp\left(-\alpha_{Nj} \right) N \left(-\kappa_\gamma - \alpha_j + \frac{\alpha_{Nj}}{\alpha_j} \right) \right)$$

where κ_γ is defined in Theorem 7.11.

The price of a STIR futures put option is given by

$$P_0 = \frac{1}{\delta} P^D(0, \theta) \left((1 + \delta F_X^{\mathrm{CPN},j}(0, u, v)) \gamma(0, t_0)^{1+\alpha} \exp\left(-\alpha_{Nj} \right) N \left(\kappa + \alpha_j - \frac{\alpha_{Nj}}{\alpha_j} \right) \right.$$

$$\left. - (1 + \delta \tilde{K}) N \left(\kappa - \frac{\alpha_{Nj}}{\alpha_j} \right) \right)$$

Proof: Using the same formulas as in the previous section,

$$\delta C_0 = P^D(0, \theta) \, \mathrm{E}^\mathbb{N} \left[\exp \left(X_t^N - \frac{1}{2} \alpha_N^2(\theta) \right) \right.$$

$$\times \mathbb{1}_{\{Y > \sigma_Y \kappa_\gamma\}} \left((1 + \delta \tilde{K}) - (1 + \delta F_X^{\mathrm{CPN},j}(0, u, v)) \gamma(0, t_0)^{1+\alpha} \right.$$

$$\times \left. \exp \left(-Y_\theta - \frac{1}{2} \alpha_j^2(\theta) \right) \right) \Bigg]$$

$$
= P^D(0,\theta) \frac{1}{2\pi \sqrt{|\Sigma|}} \int_{x_2 > \alpha_{Y,j}\kappa_\gamma} \quad \int_\mathbb{R} \exp\left(x_1 - \frac{1}{2}\alpha_N^2(\theta)\right)
$$

$$
\times \exp\left(-\frac{1}{2}x^T \Sigma^{-1} x\right) dx_1
$$

$$
\times \left((1+\delta\tilde{K}) - (1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha} \exp\left(-x_2 - \frac{1}{2}\alpha_j^2(\theta)\right)\right) dx_2
$$

$$
= P^D(0,\theta)\left((1+\delta\tilde{K})N\left(-\kappa + \frac{\alpha_{Nj}}{\alpha_j}\right)\right.
$$

$$
\left. -(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha} \exp\left(-\alpha_{Nj}\right)N\left(-\kappa - \alpha_j + \frac{\alpha_{Nj}}{\alpha_j}\right)\right)
$$

The computation of the inner integral is done using Lemma A.4. □

As for the options with margin, most of the model constants can be deduced from futures and cap/floors; the exception in this case is α_{Nj}. The convexity adjustment γ^{1+a} is given by the futures and used in the computation of κ_γ. The cap/floor volatility α_j is used directly and in κ_γ computation.

The cap formula can be written directly as a function of the futures price with

$$
C_0 = P^D(0,\theta)\left(\left(1 - K + \frac{1}{\delta}\right)N\left(-\kappa_\gamma + \frac{\alpha_{Nj}}{\alpha_j}\right)\right.
$$

$$
\left. -\left(1 - \Phi_0^j + \frac{1}{\delta}\right)\exp\left(-\alpha_{Nj}\right)N\left(-\kappa_\gamma - \alpha_j + \frac{\alpha_{Nj}}{\alpha_j}\right)\right).
$$

7.8.6 Ceci n'est pas une option (This is not an option)

Ceci n'est pas une pipe. (This is not a pipe)

La trahison des images, René Magritte, 1928.

Even if new options on STIR futures have been added recently to the exchange traded offering, the offer is still in some way incomplete as described earlier. The equivalent of swaption is certainly a missing instrument in the exchange traded offering[1]. It is not an option for traders to trade the equivalent of swaptions in the futures world, hence the title of this section.

The filling of the missing offering could be achieved through options on deliverable swap futures or through the options on packs and bundles. To our knowledge none of these products are offered on any exchange. Given the push for more standardisation and exchange traded products, the question is which exchange will be the first to offer such a product?

---

[1] Some options on swap futures exists, but they are mostly illiquid.

> Messieurs les Anglais, tirez les premiers!
>
> *Attributed by Voltaire to the (French) Comte d'Auteroche*
> *at the battle of Fontenoy, 1745*

The XXIst century corporate warfare is probably not as courteous as the XVIIIth century real warfare and there will probably be no similar offer, but the question is open: who will shoot first?

On the other side the exchange traded offering is in some sense larger than the OTC one. There is no equivalent in the OTC world to the mid-curve options. These options are interesting from a calibration perspective as they give the view of the market on the short term volatility of forward rates further down the curve. The availability of even a more complete offer would allow a larger set of financial risks to be hedged, and enable quantitative analysts to better calibrate their models.

To obtain an explicit formulas for options on packs and bundles, we add an *separability hypothesis* on the stochastic part of the rate evolution $Y_\theta^j(v)$.

Sep: The random variables $Y_\theta^j(v)$ satisfy

$$Y_\theta^j(v) = \alpha_j(\theta, v) \tilde{Y}_\theta^j \tag{7.22}$$

with $\tilde{Y}_\theta^j$ standard normally distributed and identical for all v.

We will give below financial circumstances where this hypothesis is satisfied. The condition is inspired by the separability condition described in Appendix A. With this hypothesis all periods randomness behave similarly and the model is essentially one-factor for each forward curve. There is still a multi-curve framework with stochastic spread, but each forward curve made of the base discounting curve and its own spread acts like in a one-factor model, with all the stochasticity summarised by only one random variable: $\tilde{Y}_\theta^j$.

The options we want to price are margined options on packs or on bundles. The packs and bundles are described in Section B.4. Both results can be written in the same way. Let n be the number of futures in the pack or bundle, that is $n = 4$ for a pack and $n = 4 \times k$ for a k-year bundle. The option expiration date is denoted θ and the strike is denoted $n \times K$. We write the strike in the form $n \times K$ to emphasise that there are n futures composing the pack or bundle and they have an average strike of K. In this way, the final result is easier to compare to the one futures result. The dates related to the n underlying futures are denoted t_i for the last trading dates and $[u_i, v_i]$ for the corresponding Ibor periods with accrual factors δ_i.

For a call, the quantity we want to compute is

$$C_0 = E^N \left[\left(\sum_{i=1}^n \Phi_\theta^j(v_i) - nK \right)^+ \right].$$

Using hypothesis **Sep**, the result on the futures price and on the dynamic of $1 + \delta F_X^{\text{CPN},j}(\theta, u, v_i)$ in the cash account numeraire, the quantity in the parentheses is equal to

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i F_X^{\text{CPN},j}(\theta, u, v_i))\gamma^{1+a}(\theta, t_i, u_i, v_i)$$

$$= \sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i F_X^{\text{CPN},j}(0, u, v_i))$$

$$\times \exp\left(-\alpha_{Y,j}(\theta, v_i)\tilde{Y}_\theta^j - \frac{1}{2}\alpha_{Y,j}^2(\theta, v_i)\right)\gamma^{1+a}(0, t_i, u_i, v_i)$$

In a way very similar to what is done for swaptions in Henrard (2003) and in Section 7.3.3, we define κ as the solution of

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i F_X^{\text{CPN},j}(0, u, v_i))$$

$$\times \exp\left(-\alpha_{Y,j}(\theta, v_i)\kappa - \frac{1}{2}\alpha_{Y,j}^2(\theta, v_i)\right)\gamma^{1+a}(0, t_i, u_i, v_i) = 0$$

Like in the above-mentioned result it can be proved that the solution of the equation is unique and non singular. Any numerical method will solve it very efficiently.

The quantity in the expected value above is positive when $\tilde{Y}_\theta^j > \kappa$. Using that result and computing the expected value of the quantity dependent on a normal standard random variable using the usual integral approach, one gets the following theorem.

Theorem 7.13 *The price of the option on pack or bundle, in the multi-curve framework D–I^{CPN} with multiplicative stochastic spread SMS under the separability hypothesis Sep is given by*

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i \tilde{K})N(-\kappa)$$

$$- \sum_{i=1}^{n} \frac{1}{\delta_i}(1 + \delta_i F_X^{\text{CPN},j}(0, u, v_i))\gamma^{1+a}(0, t_i, u_i, v_i)N(-\kappa - \alpha_{Y,j}(\theta, v_i)).$$

where κ is the solution of Equation (7.24).

The ingredients of this formula are, like for the options on futures, the convexity coefficient γ^{1+a} and the volatilities $\alpha_{Y,j}$. Here we require several of them. Several futures and (mid-curve) cap/floors are required to calibrate the different parts of the formula.

To obtain a particular case of the separability condition **Sep**, we base the hypothesis on the features of the one-factor extended Vasicek (or Hull-White) model.

SepV: The HJM volatility ν satisfies

$$\nu(\tau, u) - \nu(\tau, v) = H(u, v)g(\tau) \tag{7.23}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ and the spread random variable $X^{\mathcal{X}}$ satisfies

$$X_\theta^{\mathcal{X}}(u, v) = H(u, v) \int_0^\theta g^{\mathcal{X}}(\tau) \cdot dW_\tau^{\mathcal{X}} \tag{7.24}$$

where H is the same function as above and $W_\tau^{\mathcal{X}}$ is a Brownian motion independent of W_τ.

7.9   Libor market model on multiple curves

The results described in this section were originally presented in Mercurio (2010c). The starting point of the approach is that, as described in Theorem 2.1, the forward rates $F_X^{\text{CPN},j}(., u, v)$ are martingales in the $P_X^D(., v)$-numeraire. The modelling of the forward rates using an LMM approach can be done like in the one-curve approach, except that on top of the Libor forward rates, one also needs to model the risk-free curve.

7.9.1   The model

We describe the model for a unique index j. As for any LMM like approach, one selects a set of dates $0 < t_0 < \cdots < t_M$ which are j-period apart, that is $t_{k+1} = t_k + j$.

Note that in Mercurio (2010c) the quantity he calls $L_k^i(t)$ corresponds to our $F_X^{\text{CPN},i}(t, t_{k-1}, t_k)$. The notation is different but the meaning is the same. Similarly what he denotes $F_k^D(t)$ we denote $F_X^D(t, t_{k-1}, t_k)$.

The equation for the forward (Ibor) rates is, in the $P^D(., t_k)$ numeraire,

$$dF_X^{\text{CPN},j}(t, t_{k-1}, t_k) = \sigma_k(t) F_X^{\text{CPN},j}(t, t_{k-1}, t_k) dZ_k^k(t)$$

where Z_k^k is a Brownian motion associated to the relevant numeraire.

Similarly each forward risk-free rate follows a geometric Brownian motion in the associated $P^D(., t_k)$ numeraire,

$$dF_X^D(t, t_{k-1}, t_k) = \sigma_k^D(t) F_X^D(t, t_{k-1}, t_k) dZ_k^{k,D}(t)$$

where $Z_k^{k,D}$ is a Brownian motion associated to the relevant numeraire.

The correlations between the different Brownian motions are given by

$$dZ_k^k(t)dZ_l^l(t) = \rho_{k,l}dt$$
$$dZ_k^{k,D}(t)dZ_l^k(t) = \rho_{k,l}^D dt$$
$$dZ_k^{k,D}(t)dZ_l^{l,D}(t) = \rho_{k,l}^{D,D} dt$$

Like in the standard LMM, it is useful to write all the equations under the same numeraire. The change of numeraire is similar to the one-curve one. We give the equations only for the case where the last time, t_M, is used as numeraire:

$$dF_X^{\text{CPN},j}(t, t_{k-1}, t_k) = \sigma_k(t)F_X^{\text{CPN},j}(t, t_{k-1}, t_k)$$
$$\left(-\sum_{h=k+1}^{m} \frac{\rho_{k,h}^D \delta_h \sigma_h^D(t) F_X^D(t, t_{k-1}, t_k)}{1 + \delta_h F_X^D(t, t_{k-1}, t_k)} dt + dZ_k^m(t) \right)$$
$$dF_X^D(t, t_{k-1}, t_k) = \sigma_k^D(t)F_X^D(t, t_{k-1}, t_k)$$
$$\left(-\sum_{h=k+1}^{m} \frac{\rho_{k,h}^{D,D} \delta_h \sigma_h^D(t) F_X^D(t, t_{k-1}, t_k)}{1 + \delta_h F_X^D(t, t_{k-1}, t_k)} dt + dZ_k^{m,D}(t) \right).$$

7.9.2 Pricing caplets and floorlets

As the main hypothesis is a log-normal equation for the forward rates in the forward numeraire, the pricing of caplet and floorlet is straightforward and corresponds, like in the one-curve framework, to the Black price. The present value formulas are presented in Section 7.4.

Like most of the term structure models in the multi-curve framework, this approach has many parameters and most of them cannot be calibrated using standard market instruments. There are no optional instruments on risk-free rates to calibrate σ_k^D and no real hybrid instruments to estimate the correlations ρ^D and ρ^{DD}.

7.9.3 Pricing swaptions

Like in the one-curve LMM, analytical approximation for the implied volatility of swaptions can be derived in this approach. To describe the swap underlying the swaption, we use the same notation as in Section 2.4.

The forward rate S_t^j can be written as a linear combination of the forward rates with

$$S_t^j = \sum_{i=1}^{n} \omega_i(t) F_X^{\text{CPN},j}(t, t_{i-1}, t_i)$$

and

$$\omega_i(t) = \frac{P_X^D(t, t_i)\delta_i}{\sum_{i=1}^{\tilde{n}} P_X^D(t, \tilde{t}_i)\tilde{\delta}_i}.$$

The weights ω are not constants but depend on the risk-free discount factors. The standard initial freeze approximation technique is used. The weights are replaced by their initial values $\omega_i(0)$. The approximation of the swap rate is

$$S_t^j \simeq \sum_{i=1}^{n} \omega_i(0) F_X^{\text{CPN},j}(t, t_{i-1}, t_i).$$

This approximation leads to the equation for the swap rate

$$dS_t^j \simeq \sum_{i=1}^{n} \omega_i(0)\sigma_i(t) F_X^{\text{CPN},j}(t, t_{i-1}, t_i) dZ_k^m(t) + \cdots dt.$$

We know that in the present value of a basis point A_t numeraire the swap rate is a martingale, so we don't worry about the drift term at this point.

We would like to obtain a log-normal equation for the swap rate of the type

$$dS_t^j = S_t^j \sigma_S(t) dZ_S(t). \tag{7.25}$$

We equate the instantaneous quadratic variations in the previous two equations and we obtain

$$(S_t^j \sigma_S(t))^2 = \sum_{i,k=1}^{n} \omega_i(0)\omega_k(0)\sigma_i(t)\sigma_k(t) F_X^{\text{CPN},j}(t, t_{i-1}, t_i) F_X^{\text{CPN},j}(t, t_{k-1}, t_k)\rho_{i,k}.$$

Using the initial freeze technique again, this time on $F_X^{\text{CPN},j}$ and S_t^j, we obtain the following volatility

$$\sigma_S(t) = \frac{\sqrt{\sum_{i,k=1}^{n} \omega_i(0)\omega_k(0)\sigma_i(t)\sigma_k(t) F_X^{\text{CPN},j}(0, t_{i-1}, t_i) F_X^{\text{CPN},j}(0, t_{k-1}, t_k)\rho_{i,k}}}{S_0^j}.$$

From Equation (7.26), the price of the swaption is obtained with the usual Black formula as

$$P = A_t \text{Black}(S_0^j, K, \sigma_S(t)).$$

8 | Collateral and Funding

8.1 Introduction

The post-crisis multi-curve framework, as described in this book, relies on credit risk-free discounting curves and does not incorporate collateral and funding.

The goal in this chapter is to extend the multi-curve framework to incorporate collateral and funding. We prove generic collateral results that extend those presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012) and specialise them to the multi-curve framework. Using techniques similar to the ones developed in Henrard (2013c) for the multi-curve framework with stochastic spread we propose explicit formulas to evaluate the impact of the collateral corrections or convexity adjustments. Our analysis of collateral framework has also benefited from the insights of McCloud (2013b), Ametrano and Bianchetti (2013), Pallavicini and Brigo (2013) and Fujii and Takahashi (2013).

The question of the impact of collateral on pricing of derivatives products has certainly been analysed in the literature, especially since the beginning of the 2007 crisis. Some research predates the crisis and we cite in that category Johannes and Sundaresan (2007). Obviously with the increasing importance of collateral and the widening of spreads between different rates, the literature on the subject has boomed since the start of the crisis and we cite as a short list (Fujii et al. 2010a, Fujii et al. 2010b, Fujii and Takahashi 2010, Brigo et al. 2011, Piterbarg 2010, Pallavicini et al. 2011, Brigo et al. 2012a, Brigo et al. 2012b, Castagna 2012a, McCloud 2013a, Crépey 2012 and Han et al. 2013).

Our first goal is to compute the theoretically consistent amount to be transferred for a given collateral agreement. By theoretically consistent amount we mean that the amount transferred is the exact compensation for the rest of the contract, including the impact of the collateral itself. If at any date a party breaks the contract, the contract ceases to exist and the collateral is kept by the other party; the two parties do not occur any financial loss. There is a feedback loop between the

collateral contract details and the value computed. The collateral contract creates extra cash-flows with respect to the stand alone derivative contract. The daily payments of collateral and potentially of interest have to be taken into account when computing the 'value'. The term of 'value' has to be reviewed in line with the above description. As explained at the end of the introduction, we will use the term 'quote' to describe the theoretical number computed.

Some of the above-mentioned articles and books propose frameworks significantly more general than the one we describe below, in particular regarding funding and credit risk. We concentrate mainly on the collateral impact. On the other hand our definition of collateral situation is *more general* than the ones used in the above literature as it incorporates not only collateral, repo, no-collateral and futures but also proportional under- and over-collateral, collateral by other assets, with and without interest, collateral in foreign currency and collateral with haircut. Our definition of collateral is given in the first section. The specificity of our approach is also to describe the impacts of those generic results for the multi-curve framework and to provide explicit results on the convexity adjustments on the Ibor forward rates.

The starting point of the multi-curve framework is the existence of (credit) risk-free fixed cash-flows (and the associated rates). Part of the collateral approach can be described in theory without supposing the existence of such rates; see for example Macey (2011) and Piterbarg (2012). Pallavicini and Brigo (2013) suppose the existence of a risk-free rate but obtain pricing formulas based on the market quoted instruments only. It is probably debatable from a philosophical point of view if risk-free rates exist and if we need them at all. Nevertheless from a practical point of view, in some cases it is easier to suppose they exist, even if we don't suppose they are used in actual market contracts. We will do so when it simplifies the approach.

The multi-curve framework is the foundation for pricing Ibor related instruments. Most of the liquid instruments are nowadays collateralised in one way or another. It is thus paramount to have a detailed framework that combines collateral and multi-curve frameworks. We provide such a framework here. Similar definitions related to a multi-curve framework with collateral can be found in Fries (2013).

The combination of collateral and multi-curve framework has often been reduced to an 'OIS discounting' formula. I will not use that term here as it is, in my opinion, a misleading term. A collateral at overnight is not enough to justify the use of the technique. It requires several other conditions, including OISs themselves to be collateralised with the same rule, a condition not always satisfied in practice. The exact requirements for 'OIS discounting' are detailed later in the chapter.

It is important to insist that our approach to collateralised instrument valuation supposes a feedback loop. The quote computed from the perfect collateral framework should be used to transfer the collateral amount. If the collateral transferred is computed using a different rule, like an old one-curve framework valuation or an

accounting rule, then it does not fit our definition of collateral. We are describing a world where the computations of quantitative finance are imposed on the rest of the process, and in particular on back-offices and accounting departments. In other words, our collateral world is 'perfect' if quants are ruling the world!

An approach with only one collateral rule for each currency is not general enough to cover all practical cases. Collateral with foreign currencies, often described in the literature, is only one special case. Collateral with bonds, leading to 'collateral square'-type results, is often used; some contracts impose an haircut. Modified collateral rates, like overnight plus a fixed spread, are also found in practice. This chapter provides answers for those different cases.

The collateral practice leads to a somewhat confusing terminology. In this chapter we use a term which has to be has explained in the collateral framework. Due to collateral rules, when an asset is purchased with perfect collateral, the amount obtained from the market quote is paid but returned immediately through the collateral, resulting in a net zero payment at the start of the trade. Only the change of the figure and interests on the collateral account are exchanged (daily). We will use the more neutral term *quote* or *collateral quote* to designate that quantity. Personally I prefer the term *reference number* or *reference index* but it is a little bit too long to use. The quote could be shifted by an arbitrary figure without impact on the economy if the reference amount for the collateral was similarly adapted. This consideration on prices was originally proposed in Henrard (2006b) for futures and later adapted for collateral in Piterbarg (2012).

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view its pricing can be viewed as the pricing with a very specific collateral rule paying a rate of 0. Understanding the pricing of those instruments in a multi-curve and collateral framework is the first step to a more general understanding of change of collateral. We dedicate one part of the 'Modelling with collateral' section to those instruments.

8.2 Collateral: rate, asset or both

In general, we are interested in analysing the collateral impact not only in one currency but in a multi-currency framework. The currencies will be denoted X and Y. In the first part of this section we restrict ourself to a unique currency and will extend the results to collateral with assets and more than one currency later.

8.2.1 Definition of collateral rate

We need to define *collateral rate c for a quote V*. In this section, all the cash-flows are expressed in the same currency, which we call the domestic currency. If

a cash-flow is paid in another currency, it is supposed to be converted into the domestic currency at the exchange rate prevailing at the cash-flow time.

Definition 8.1 Collateral rate *The quote V_t has a collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - c_t V_t dt \qquad (8.1)$$

without any other payments (neither up-front nor final payment) can be created.

In the naming convention of Duffie (2001), which is also used in Brigo et al. (2012b), this means that it is possible to create an asset package with price process constant at 0 and a dividend process D_t given above.

The dividend payment does not necessarily come from one asset on its own; it can be provided by a package of instruments, including collateral contracts, which, combined together, create the above dividend cash-flow. This is the case for example for uncollateralised transactions where the dividend is created by a combination of an asset and its funding.

This may look like a convoluted way to describe collateral but it covers several financial realities and allows us to have only one treatment for all of them. Several financial realities are covered, as described below; nevertheless by abuse of language, we refer to it everywhere with the name *collateral rate*. To our knowledge, this generalised definition of collateral has not been used before, except in Crépey (2012) in a different setting. We will create even more general definitions of collateral in the next sections to include collateral by assets, including foreign currencies, with or without interest payments. In some sense, through that definition we bundle together the instrument and its financing (collateral in this case).

La mathématique est l'art de donner le même nom à des choses différentes.

Henri Poincaré.

Meta translation: Mathematics is the art of giving twice the same quote in different chapters.

In case of perfect cash collateral in domestic currency, if one purchases an asset, he pays V_t but the amount is immediately paid back to him as collateral. So there is no cash-flow at the start. At the end of the interval of time dt, he pays the interest $c_t V_t dt$ on the collateral and receives at the same time as new collateral the change of value of the asset dV_t. At any stage the contract can be cancelled without further cash-flow; the collateral compensates for the instrument value due to the perfect collateral. The resulting cash-flows satisfy our definition above.

The situation of an asset for which a special repo rate is available is also covered by the definition. One can purchase the asset paying the amount V_t, immediately repoing it out, receiving the same[1] cash-flow V_t. No net cash-flow occurs for the

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[1] Here we ignore haircut. See Section 52.5 for the haircut treatment.

buying party. At the end, the buyer receives the asset back and pays the repo (principal plus interest), $-V_t(1 + c_t dt)$. He can sell the asset and receive the proceeds $V_t + dV_t$. In total the continuous cash-flow is $dV_t - c_t V_t dt$. This situation also falls within our general collateral definition.

The situation where there is no special repo for that asset and the asset can be financed only by a naked borrowing at rate c_t is also covered. The rate is the risk-free rate if we suppose that it is possible to borrow at that rate or the funding rate if not. Here we ignore the possibility of our own default. The funding rate is becoming the equivalent of the risk-free rate, the one at which one can borrow or lend. This equivalence was one of the starting arguments in Henrard (2007). One purchases the asset at V_t, borrowing the cash. No net cash-flow is paid at the start by the purchasing party. The payment at the end of the dt period is the borrowing reimbursement, $-V_t(1 + c_t dt)$, and the simultaneous sell of the asset for the price $V_t + dV_t$. Again the total net payment is $dV_t - c_t V_t dt$. This situation also falls in our general definition. Note that the funding cash account itself is an asset with collateral rate equal to the funding rate. The account satisfies $dV_t = c_t V_t dt$, which means that the continuous payment $dV_t - c_t V_t dt$ is a continuous payment of 0.

Our definition also covers the case of proportional cash collateral (under-collateral or over-collateral). Suppose that the collateral value in cash is αV_t. If $\alpha = 1$ we have the perfect collateral case, if $\alpha = 0$ we have the no-collateral case, if $0 < \alpha < 1$ it is a proportional under-collateral, and if $\alpha > 1$ it is a proportional over-collateral. The situation is covered by the above definition with an equivalent collateral rate of $\alpha c + (1 - \alpha)f$, where c is the collateral rate paid on the cash and f is the risk-free/funding rate for the extra amount borrowed or invested to fill the collateral account. The proportional over-/under-collateral is discussed in Pallavicini and Brigo (2013) in the Central Counterpart (CCP)s framework. Our approach does not cover the case where the over-/under-collateral is not proportional. This would be the case for a CCP where an initial margin is required which is risk proportional, not value proportional. Pallavicini (2013) described initial amount corresponding to initial margins in a very general setting including credit risk.

Finally we mention the margined futures and futures options. In a future, one can enter into a contract without any up-front payment. There is an initial margin, but we ignore it here; it corresponds to an imperfect collateral which is beyond the scope of our analysis. The next futures payment is done at each margining time, where the amount dV_t is paid. The futures situation enters into our collateral definition with $c_t = 0$. This equivalence between futures and zero rate collateral is mentioned in Johannes and Sundaresan (2007), Henrard (2012d) and Piterbarg (2012).

Obviously in the five cases, the collateral rate, denoted c_t in all cases, will not be the same. In the first case, it is the collateral rate as agreed in the contract terms and conditions (often through a CSA); in the second case, it is the market repo

rate for the particular asset; in the third case, it is the risk-free rate or funding rate specific to the institution; and in the futures case it is 0. The unique notation covers different situations, allowing us to deal with all of them in the same unified framework. A unified approach between repo and funding rate is implicit in (Han et al., 2013, Section 3); we prefer to have all our hypotheses and choices explicit.

One asset can have a collateral rate that changes from one category to another through time. A fully collateralised swap at overnight rate will have a collateral rate equal to the overnight rate up to its payment. After that, the payment exits the legal framework of the collateral agreement and becomes a non-collateral cash, with funding rate. With our extended definition of collateral, an instrument can still be modelled after a coupon payment. With a restricted version of collateral, at the first coupon payment, the instrument exits the framework and the theoretical world ends; the modelling is not possible anymore.

Without our generalised definition of collateral, it is difficult to deal with instruments paying an actual cash-flow. For example the generic dynamic given by Equation (8.2) is written as a diffusion and would not be valid at a coupon payment time. At the payment time, there would be a jump in the quote. For the framework to be coherent, the amount paid needs to stay in the framework and an uncollateralised amount needs to satisfy the collateral definition. Otherwise, after the end of the contract there is no collateral in place and no equation left. The time horizon for each instrument would be its payment date. This means also that the pricing would not be truly additive; the equations would be meaningful only to the first payment date in the package. But if the equations are meaningful only to the first payment date, what is the value of the instrument at that date? The value cannot be computed by discounting from the end date of the other payments, as our economy would not exist any more after the first payment date. The pricing of a swap as the sum of its coupons would then be debatable. The additive property can be brought in at the cost of extending the valuation beyond the maturity and this can be done by reintroducing the funding rate and using a generalised definition of collateral.

Note also that we use the term cash-flow to denote any change of value. It can be achieved by an actual cash-flow, a cash-flow in another currency converted in the domestic currency or a change of value which could be realised by selling part of the asset. All the values are no-arbitrage values, not accounting figures. We do not allow hiding profit, losses or future cash-flows in banking book like places or historical cost accounting.

The first assumption of the multi-curve framework concerns risk-free discounting. Here we do not work with a risk-free rate but with collateral. Our hypothesis is the existence of collateralised assets.

A: There exist packages in our economy satisfying Definition 8.1.

At this stage this hypothesis is very vague, but we will give more precision later.

The general filtration of the economy is denoted $(\mathcal{F}_t)_{t \in [0,T]}$. Later we will introduce other filtrations to describe the randomness of some specific components. When we do so we will do it through sub-filtrations $\mathcal{G}_t^i$ $(i = 1, \ldots)$.

Suppose that the continuously compounded rate paid on the collateral is the rate c_t and let

$$N_t^c = \exp\left(\int_0^t c_\tau \, d\tau\right)$$

be the collateral cash account. The rate c can be stochastic (and in most cases will be). Even if N_t^c is called the collateral cash account, we do not suppose that we can invest or borrow through that account freely. Only the cash used for collateral will be in that account, according to the quotes associated with the collateralised trades.

Results similar to those described below can be found, when applied to a restricted definition of collateral, in Macey (2011), Crépey (2012), Piterbarg (2012), Pallavicini et al. (2012) and Fujii and Takahashi (2013). The proof approaches in those papers are each quite different to the others, but the results specialised to a common framework are similar. Note that Macey (2011) restricts the collateral to be the overnight index, a restriction we do not impose. In our proof, we follow the general techniques used in Macey (2011) and later in Piterbarg (2012) even if we add some ingredients missing in the original proofs' sketches and apply them to the more general definitions of collateral introduced here. Crépey (2012) on the other hand proposes a more general equation than a diffusion equation on the underlying asset.

Let $V_u^{X,c,Y}$ be the *quote* in currency X at time u of an asset with collateral in currency Y at rate c. At some stage, the value $V_u^{X,c,Y}$ will be independent of the collateral. For example a fixed cash-flow has a value on its payment date independent of the collateral currency and rate and equal to the amount paid. In that case we don't indicate the currency and the collateral rate and denote the price $V_u^{X,c,Y} = V_u$. When the asset and collateral currencies are the same and the context is clear, we write $V_u^{X,c,X} = V_u^c$.

We now develop the generic pricing formula under collateral. The proof is based on portfolio hedging, not dissimilar to the one used in standard Black-Scholes pricing formula. In this case we hedge one package generating the dividend (8.1) with n other packages as a source of randomness. The dimension n is the dimension of the Brownian motion underlying the economy. This extra asset is important to create the hedging procedure. In the Black-Scholes proof the underlying asset of the option and a cash account are used; here in general we don't have the same flexibility to access the collateralised account as the instrument and the collateral account are tied together.

8.2.2 Single currency cash collateral

Suppose that we have $n+1$ collateralised assets with collateral rates c. By rate c, we mean a vector c with one rate for each asset. The rates are potentially different for each asset. The collateral cash account for multiple assets and rates has also to be understood as a multidimensional value with each element corresponding to one asset and given by the above defined cash account. When a multiplication of vector is done element by element, like a multidimensional rate multiplied by a multidimensional quote, we use the notation $*$, like $c_t * V_t^c$.

The quotes of the assets depend on an n-dimensional Brownian motion W_t through the equation

$$dV_t^c = \mu_t dt + \Sigma_t dW_t. \tag{8.2}$$

The quotes V and the drift μ have dimension $n+1$, the volatility matrix has dimension $n+1 \times n$. Suppose moreover that the matrix composed of the n first lines of Σ_t is invertible for all t. This means that we really have n assets and not a lower number with some of them recombined in a different way. We denote the square matrix with the first n lines by $\bar{\Sigma}_t$. Similarly, for any vector x of dimension $n+1$, we denote by $\bar{x}$ its first n elements. The i-th line of Σ_t is denoted Σ_t^i. The drift and the volatility are potentially stochastic (and potentially dependent on V itself). We suppose also that $\mathcal{F}_t$ is the natural filtration of W_t and μ and Σ are 'sufficiently regular' and in particular that they are $\mathcal{F}_t$-adapted. The measure in which those equations are valid can be the physical measure or any other measure; we don't restrict the result to the physical measure like Macey (2011) and Piterbarg (2012). The only important point at this stage is that the measure is the same for all instruments.

Like in the Black-Scholes approach, the instrument valuation is based on a replicating portfolio approach by eliminating risk. The portfolios used for hedging will be self-financing. The intuition of self-financing is that there is no cash appearing or disappearing from the portfolio. The change of value of the portfolio is triggered only by the change of value of the assets in the portfolios and the dividends.

Definition 8.2 *The portfolio with $\mathcal{F}_t$-adapted quantities w_t in each package, where the packages have only a dividend process D_t and a zero price process, is said to be self-financing when*

$$w_t^T dD_t = 0.$$

The self-financing property is not strictly speaking required for some of the computations in the proof, but without the property or something equivalent there is not actual proof. The strategy described is a valid replication strategy only if it is self-financing at the portfolio level. It is paramount to verify this requirement to obtain a replicable price.

Several different proof strategies can be used for the result below. I like the one used below because the hedging strategy to eliminate the risk is explicit. Given the volatility matrix Σ_t, the hedging strategy is explicit.

The self-financing definition above is the one used in Duffie (2001) and Brigo et al. (2012b) adapted to our situation. Our description starts from (asset) packages with zero price process and a dividend process. The equality between the gain process and the price process reduces in our case to zero dividend process as defined above.

The main result for pricing under collateral is provided below. The result is written as a generic formula for one asset, even if the proof requires several assets for hedging purposes.

We want to prove that any $\mathcal{F}_t$-measurable quote V_u^{n+1} is replicable and we want to describe the formula for the quote at earlier times.

In our framework, replicating the quote V_u^{n+1} means finding $\mathcal{F}_t$-measurable quantities $\bar{w}_t$ and a random variable Z_t such that the portfolio $(\bar{w}_t, 1)$ is self-financing and $Z_u = V_u^{n+1}$.

Theorem 8.1 (Cash collateral formula). *Under the Brownian motion hypothesis (8.2), any $\mathcal{F}_u$-measurable quote V_u with collateral rate c is replicable and the quote at time t satisfies*

$$V_t = N_t^c \, \mathrm{E}^{\mathbb{X}} \left[(N_u^c)^{-1} V_u \big| \mathcal{F}_t \right] \tag{8.3}$$

for some measure $\mathbb{X}$ (identical for all assets, but potentially currency dependent).

Proof: Suppose that V_u^{n+1} is replicable. Then there exists $\mathcal{F}_t$-measurable quantities $\bar{w}_t$ such that the strategy $(\bar{w}_t, 1)$ is self-financing. This means

$$0 = w_t^T dD_t = w_t^T (dV_t - c_t * V_t dt) = w^T (\mu_t - c_t * V_t) dt + w_t^T \Sigma_t dW_t.$$

We can deduce from there a property of w and the fact that $w^T (\mu_t - c_t * V_t) = 0$. Let $x = \mu_t - c_t * V_t$. As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have $0 = w_t^T x = -\Sigma_t^{n+1} \lambda_t + x^{n+1}$. This shows that $\mu_t - c_t * V_t = \Sigma_t \lambda_t$. The equations for V_t become

$$dV_t = (\Sigma_t \lambda_t + c_t * V_t) dt + \Sigma_t dW_t = c_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt)$$

$$= c_t * V_t dt + \Sigma_t dW_t^{\mathbb{X}} \tag{8.4}$$

where $dW_t^{\mathbb{X}}$ is a new Brownian motion defined by $dW_t + \lambda_t dt$. In the measure associated to that Brownian motion, each quote V_t^i has a drift c_t^i. In that measure, each price rebased by its collateral account $N_t^{c^i}$ is a martingale and the result on the quote formula follows.

We still need to prove that any $\mathcal{F}_u$-measurable quote V_u^{n+1} can be replicated. Let $\tilde{Z}_t = \mathrm{E}^{\mathbb{X}} \left[(N_u^{c^{n+1}})^{-1} V_u^{n+1} \big| \mathcal{F}_t \right]$. The random variable $\tilde{Z}_t$ is an $\mathbb{X}$-martingale.

By the martingale representation theorem, there exists K_t, $\mathcal{F}_t$-adapted, such that $d\tilde{Z}_t = K_t dW_t^X$. We define $\bar{w}_t = -N_t^{c^{n+1}} K_t (\bar{Z}_t)^{-1}$. The quantity is such that $\bar{w}_t d\bar{D}_t = \bar{w}_t \bar{\Sigma}_t dW_t^X = -N_t^{c^{n+1}} K_t dW_t^X$. Let $Z_t = \tilde{Z}_t N_t^{c^{n+1}}$. Then $dZ_t = c_t^{n+1} Z_t dt + N_t^{c^{n+1}} K_t dW_t^X$. The portfolio $(\bar{w}_t, 1)$ with the last component in the package of quote Z_t, is self-financing. The final quote Z_u is $N_u^{c^{n+1}} \mathrm{E}^X \left[(N_u^{c^{n+1}})^{-1} V_u^{n+1} \middle| \mathcal{F}_u \right] = V_u^{n+1}$. This proves that V_u^{n+1} is replicable. □

At this stage the question is: what is this still mysterious measure X? To analyse it, remember that we used $n+1$ assets to create a hedged portfolio but we have only a dimension n source of randomness. Remember that λ_t is defined by

$$\lambda_t = (\bar{\Sigma}_t)^{-1} (\bar{\mu}_t - \bar{c}_t * \bar{V}_t).$$

The quantity λ_t that defines the new Brownian motion and the measure is uniquely defined by any n assets with volatility matrix of full rank. It is possible to remove the last asset and replace it by any other asset (with its own collateral rule). If the asset is such that another sub-matrix of full rank n including that asset exists, we can change the order of the lines and have the same resulting λ_t from another set of assets. We can do this for any number of assets and replace all the original assets by new assets, one by one, without changing λ_t. The λ_t is intrinsic to the economy and the starting measure. If the starting measure is the real economy one, the quantity λ_t is usually called the price of risk.

We can describe the elements of the proof using an economic explanation and we do so in the following lines. The assets depend on several underlying 'economic' factors W_t. Each asset depends on the economic factors with different weights, represented by the matrix Σ_t.

In absence of arbitrage, the growth rate of the assets is necessarily linked to those risk weights. Each risk has an intrinsic growth rate, called the price of risk and denoted λ_t. The intrinsic growth rates associated to the risks are added to the risk-less growth represented by the rate c. In the collateral case the risk-less growth is the collateral rate paid for that particular asset; in absence of collateral, the growth is at risk-free rate. If the intrinsic growth rate of each risk factor was not respected, it would be possible to combine the assets to eliminate the risk and keep the growth mismatch. This would create a risk-less portfolio 'printing' money.

Note also that there is no relation imposed between the different growth rates λ associated to the different risk factors. All the risk factors are represented by standard Brownian motions with the same volatility $\sqrt{t}$. The risks, measured by their volatility, are of the same size, but the returns are (potentially) very different.

A result similar to Theorem 8.1 can be obtained starting from different hypotheses. This is for example the approach chosen in (Crépey, 2012, Section 4.1). The assumption is to suppose the existence of a measure X equivalent to the economic probability such that the gain process (denoted dD_t here and $d\mathcal{M}_t$ in the above

article) is an $\mathbb{X}$-martingale. Such a hypothesis guarantees that there is no arbitrage opportunity. If the dividend process is a martingale, the quote process, which is

$$dV_t = c_t V_t dt + dD_t$$

has the same form as described above and the result is proved like in the last paragraph of our proof.

The approach is more generic as it only supposes a martingale property, and not a special form of martingale as with our $\Sigma_t dW_t$ involving a Brownian motion, and infers the arbitrage-free property from it. On the other hand our approach proves the existence of such a martingale (under some conditions) and gives an explicit hedging strategy through the quantities w_t.

There is one type of collateral situation for which we know the formula: when there exists a risk-free rate and uncollateralised asset. In that case, the formula above is valid for $\mathbb{X}$, the measure associated to the risk-free cash account numeraire, and c, the risk-free rate. The measure $\mathbb{X}$ is the one associated to the risk-free cash account if we suppose that such a concept exists.

Here we don't explicitly need this existence, so we keep the vague name, using only the currency as indication. Later in specific cases, for the valuation of futures in particular, we may reintroduce the risk-free rate existence.

It is also important to notice that the quantity N_t^c is not the numeraire associated to the standard measure $\mathbb{X}$. If the measure were the one associated to the numeraire N_t^c, Formula (8.3) would be the standard formula without consideration of collateral. Moreover there are infinitely many N_t^c, one for each rate c, and only one measure $\mathbb{X}$.

Also the quantities V_t^c analysed and that we call *quotes* are not 'present values' in the traditional sense of quantitative finance. They are quantities used in the collateral framework but cannot be used outside their context. In particular a quote rescaled by a numeraire will never be a martingale (except in the particular case where the collateral is the risk-free rate). What we have is that the quote rebased by the collateral cash account associated to that particular quote is a martingale. The pricing situation is more complex than without collateral and always involves three elements: the quote, the collateral discounting and a numeraire, as opposed to only two, the price and the numeraire, in standard quantitative finance.

From the collateral account, we define the collateral pseudo-discount factors.

Definition 8.3 (Collateral pseudo-discount factors). *The collateral (pseudo-) discount factors for collateral rate c paid in currency X are defined by*

$$P_X^c(t, u) = N_t^c \mathrm{E}^{\mathbb{X}} \left[(N_u^c)^{-1} \big| \mathcal{F}_t \right]. \tag{8.5}$$

The quantity $P_X^c(t, u)$ is a positive random variable. The quantity is the quote at time t for paying a cash-flow 1 (in currency X) at time u collateralised in the currency of the cash-flow at rate c_t. Note that N_t^c is not currency dependent but P_X^c is

currency dependent. The currency dependency appears through the expected value $E^{\mathbb{X}}[.]$.

Let $E^{c,v}[]$ be the expectation associated to the numeraire $P_X^c(.,v)N_.^{\mathbb{X}}(N_.^c)^{-1}$. This is the numeraire such that

$$N_t^c E^{\mathbb{X}}\left[(N_u^c)^{-1} Y_u \big| \mathcal{F}_t\right] = P_X^c(t,v) E^{c,v}\left[(P_X^c(u,v))^{-1} Y_u \big| \mathcal{F}_t\right] \tag{8.6}$$

for any random variable Y. The associated measure is called a c-collateralised v-forward measure by Pallavicini and Brigo (2013). We use the same denomination here. In particular we have that any quote with collateral c scaled by $P_X^c(t,v)$ is a martingale in that measure. Moreover a random variable is a martingale in the $N^{\mathbb{X}}$ numeraire measure when rescaled by the collateral account if and only if it is a martingale in the c-collateralised v-forward measure when rescaled by the collateral pseudo-discount factor. This equivalence will be used when we model the collateral rates later.

The change of numeraire for a time u has density L_u given by

$$L_u^{c,v} = \frac{P_X^c(u,v)(N_u^c)^{-1}}{P_X^c(0,v)(N_0^c)^{-1}}, \tag{8.7}$$

that is, the change of numeraire measure can be written as

$$E^{\mathbb{X}}[Z_u] = E^{c,v}\left[L_u^{c,v} Z_u\right].$$

This density, together with Girsanov's theorem, will be used when we obtain explicit change of collateral adjustments in Section 8.4.3.

When the collateral rate is the overnight index in the currency of the payment, the valuation formula (8.6) is often called 'OIS discounting'. This terminology is correct in the sense that an overnight-related discounting is used to discount the pay-off inside the expected value. Nevertheless the terminology is hiding several facts. The first is that the expectation is done under a measure for which the discounting may *not be the numeraire*. The result is not the same as replacing the risk-free rate of the no-collateral approach by the overnight rate. Moreover, for the term to be right one needs not only the instrument priced to be collateral at overnight, but also to link the OIS quotes to overnight collateral account. This last part requires the OISs themselves to be collateral with the same rule and a formula that links collateral pseudo-discount factors to OIS instruments. We provide the latter in Theorem 8.7. The 'OIS discounting' does not depend only on the collateral rule of the instrument priced, but also on the OIS collateral rule, the OIS pricing formula and the pricing in a specific numeraire. For those reasons, the term 'OIS discounting' is misleading. We will refrain from using the term further.

To have symmetrical results later, we define forward rates for the collateral curve.

Definition 8.4 (Forward collateral rate). *The collateral forward rate over the period* $[u, v]$ *is given at time t by*

$$F_X^c(t, u, v) = \frac{1}{\delta} \left(\frac{P_X^c(t, u)}{P_X^c(t, v)} - 1 \right). \tag{8.8}$$

This means that

$$1 + \delta F_X^c(t, u, v) = \frac{P_X^c(t, u)}{P_X^c(t, v)}.$$

We call the above quantity the 'investment factor', as opposed to the discount factor; it represents the amount obtained at the end of the period if 1 is invested at the start with the rate.

Using the definition of c-collateral v-forward measure and the remark above, the investment factor $1 + \delta F_X^c(t, u, v)$ is a martingale in the c-collateral v-forward measure. Even if this result seems trivial at this stage, because of its importance for rate modelling, we write it as a theorem.

Theorem 8.2 (Martingale property of forward collateral rate). *In the collateral framework, the collateral forward rate $F_X^c(t, u, v)$ is a martingale in the c-collateral v-forward measure.*

8.2.3 Change of collateral: independent spread

We have the general formula for computing the price in a framework with collateral. A natural question is to ask what is the relation between the prices of assets with same final price V_u but different collateral rules.

Suppose that we have two collateral rates with the relation

$$c_t^2 = c_t^1 + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and V and c_1 both $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. Often the spread will be constant or deterministic. The situation can be, for example, that the rate paid on the collateral is the overnight index plus a fixed spread.

By definition of the spread, the collateral accounts are such that $N_t^{c2} = N_t^{c1} N_t^s$. Suppose that the price at time u is collateral independent (for example it is the cash-flow date). With this simple relation between the rates, the relation between the quotes is

$$\begin{aligned} V_0^{c^2} &= N_0^{c1} N_0^s \, \mathrm{E}^X \left[(N_u^{c1} N_u^s)^{-1} V_u \right] \\ &= N_0^s \, \mathrm{E}^X \left[(N_u^s)^{-1} \big| \mathcal{F}_t \right] N_0^{c1} \, \mathrm{E}^X \left[(N_u^{c1})^{-1} V_u \right] \\ &= P_X^s(0, u) V_0^{c1} \end{aligned} \tag{8.9}$$

where for the second equality we have used the independence property of the variables.

This can be interpreted as, when there is a spread paid on the collateral and the spread is independent, the quote is the original quote discounted by the spread.

Note also that we have

$$P_X^{c_2}(0, u) = P_X^s(0, u)P_X^{c_1}(0, u) \tag{8.10}$$

as a special case of the above result.

8.2.4 Simplified example of hedging

Suppose that we use two deterministic collateral rates $c_i(t)$ $(i = 1, 2)$ and that the quotes of the related assets can be written as

$$V_t^{c_i} = P_X^{c_i}(t, v)F_t(v).$$

The quantity $F_t(v)$ can be interpreted as a forward. This situation will be the one we will deal with in Section 8.3 when we analyse the multi-curve framework with collateral. To fit in our framework, the quantity F_t is supposed to follow a one-dimensional diffusion equation

$$dF_t(v) = \cdots dt + \sigma_t dW_t^{\mathbb{X}}.$$

The value of the drift is not important for what we want to show here. Due to the deterministic nature of the collateral rates, it can be proved that the drift is 0. The pseudo-discount factors satisfy the equations $dP_X^{c_i}(t, v) = c_i(t)P_X^{c_i}(t, v)dt$. The quotes satisfy

$$dV_t^{c_i} = c_i(t)P_X^{c_i}(t, v)F_t(v)dt + P_X^{c_i}(t, v)\sigma_t dW_t^{\mathbb{X}}.$$

In this setting, we can describe the hedging quantities w_t explicitly. To hedge one unit of $V_t^{c_2}$, we need

$$\bar{w}_t = -\frac{P_X^{c_2}(t, v)}{P_X^{c_1}(t, v)}$$

units of $V_t^{c_1}$. The quantity depends in theory on σ, but as the same σ is used in both equations, they simplify. If we buy one unit of $V_t^{c_2}$, we hedge it with $\bar{w}_t$ units of $V_t^{c_1}$.

We look at two scenarios around this hedging. First suppose that the value F changes instantaneously (no interest payment involved). The change of the second quote is $P^{c_2}(t, v)dF_t$ while the first is $P^{c_1}(t, v)dF_t$ to be multiplied by the quantity $\bar{w}_t$ and the two changes cancel. As expected from a perfect hedging, no profit is incurred when the underlying changes.

In the second scenario, the value F does not change over a certain period but the time is passing and collateral interest should be paid. For the second instrument,

the interest on the cash account is $P^{c_2}(t,v)F_t(v)c_2 dt$ and the change of quote is $dV_t^{c_2} = c_2(t)P^{c_2}(t,v)F_t(v)dt$. The two compensate exactly. The interest payment on the collateral account is compensated by the carry on the instrument.

In summary, we have two assets with different collateral, the change of value of one of them is exactly the change of the other and the supplementary cash required by one collateral account is provided by the cash paid by the other one. The interest payment on each account is compensated by the carry on the instrument itself. This is due to the pricing formula used, not to an external financial mechanism that provides the funding.

The hedging, even in this simplistic case of linear products and deterministic collateral rates, is dynamic. The hedging of one unit of the second product is not done with one unit of the first one, but by the quantity $\bar{w}_t$, which is changing through time.

8.2.5 Collateral with asset

In some cases the collateral posted is not cash but a different asset. The asset itself is subject to collateral (remember that no collateral is considered as a form of collateral in our generalised definition). This can be described as *collateral square*[2].

This is the case with interest rate swaps. The standard is cash collateral at overnight rate but in some cases the collateral consists of treasuries. This is often the case with asset managers who prefer to deposit assets instead of cash. This section describes how to coherently price assets collateralised with other assets.

We need to define asset collateral. As before, we do it through a definition of continuous dividends. We denote by V_t^C the quote for an asset with collateral with quote C.

Definition 8.5 (Collateral with asset). *The quote V_t has an asset with quote C_t as collateral if a continuous dividend*

$$dD_t = dV_t - \frac{V_t}{C_t}dC_t$$

without any other payments (neither up-front nor final payment) can be created.

The explanation of the definition is the following. If the collateral is an asset with associated quote C_t, the collateral cash-flow is the change of asset dV_t. There is no payment of interest per se but the change of the value of the collateral asset needs to be compensated. The quantity of asset with value C_t to cover the value V_t is V_t/C_t. The change of value of the asset is dC_t. When the collateral asset increases in value we have to release the extra value to the counterpart, hence a negative sign in front of the change of value. In the analysis, we always suppose that the collateral can

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[2] The name was suggested by [Brigo 2013].

be rehypothecated. In particular, if you buy the asset and receive the other asset as collateral in guarantee, you can sell the collateral to cancel the initial cash-flows.

With that definition, a quote has a collateral rate c_t if the quote has the collateral account as a collateral asset. The new definition is a natural extension of our general definition of collateral with cash paying a given rate.

Like before, we suppose we have $n + 1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (8.2). We suppose that the first n assets have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset; we suppose that the last asset is collateralised by one of the previous assets. Without loss of generality, we suppose the collateral is the first asset. The continuous cash-flow that can be created by the last asset with collateral the first asset is

$$dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1.$$

Theorem 8.3 (Asset collateral formula). *In presence of collateral with asset of quote C_t which has itself a collateral rate c_t, the quote at time t for an asset with quote V_u^C at time u is*

$$V_t^C = N_t^c \, E^{\mathbb{X}} \left[(N_u^c)^{-1} V_u^C \middle| \mathcal{F}_t \right]. \tag{8.11}$$

Proof: Let

$$\bar{w}_t^T = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equations of V,

$$\bar{w}_t^T dD_t = \bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1$$

$$= \left(\bar{w}_t^T (\bar{\mu}_t - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_t^1 \right) \right) dt$$

$$+ \left(\bar{w}_t \bar{\Sigma}_t + \Sigma_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 \right) dW_t$$

By definition of w, the coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend cannot have only drift and be non-zero. Consequently we get the drift $w_t^T x = 0$ with

$$x = \begin{pmatrix} \bar{\mu}_t - \bar{c} * \bar{V} \\ \mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_t^1 \end{pmatrix}.$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definitions of w and λ, we have

$$0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x^{n+1} = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) \lambda_t + x^{n+1}$$

and so $x^{n+1} = \left(\Sigma^{n+1} - V_t^{n+1}/V_t^1 \Sigma^1 \right) \lambda_t$. From the definition of λ we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_1, we obtain $\mu_t^{n+1} = \Sigma_t^{n+1} \lambda_t + c_t^1 * V_t^{n+1}$.

Define $c_t^{n+1} = c_t^1$. The equations for V_t become

$$dV_t = c_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) = c_t * V_t dt + \Sigma_t dW_t^X. \tag{8.12}$$

In the measure associated to the W_t^X Brownian motion, each asset has a drift c^i equal to its collateral rate. The drift for the $n+1$-th asset is c_t^1, that is, the collateral rate of the collateral asset. $\square$

Note that the definition of λ_t depends only on $\bar{\Sigma}_t$, $\bar{\mu}$ and $\bar{c}$, that is, only on the first n instruments equation, and collateral is used to define the change of measure. The new Brownian motion W^X and the associated measure $\mathbb{X}$ are the same as those obtained in the cash only collateral case. Both formulas use the same expectation.

The result could be described as the quote of an asset collateralised by another asset is obtained by discounting the quote at the rate of the collateral collateral, that is by discounting by the collateral rate of the collateral. Hence the name 'collateral square' mentioned earlier.

8.2.6 Collateral with asset and rate payment

In this section we extend the analysis to the combination of the two previous cases and add a gearing factor. The collateral is composed of another asset with gearing and at the same time there is the payment of some rate on the collateral value. The gearing factor is used to model haircut as described later. For a collateral asset αC and a collateral rate c, we denote the quote by $V_t^{\alpha, C, c}$.

The payment of interest on the collateral is required for foreign currency collateral. The currency itself changes value and collateral has to be added or removed with that change of value. The foreign currency is an asset when viewed from the domestic currency and not simple cash. At the same time a conventional interest has to be paid on the amount. Note that even if the rate is paid in a foreign currency it is still paid on the full collateral value, which is V_t. There is no need to know the exchange rate to know the domestic currency value of interest paid; it will be $c_t V_t dt$.

To our knowledge, this generalised definition of collateral has not been used before.

Definition 8.6 (Collateral with asset and rate). *The asset with quote V_t has a collateral asset C_t and a collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - \alpha \frac{V_t}{C_t} dC_t - c_t V_t dt$$

without any other payments (neither up-front nor final payment) can be created.

As hinted above, the definition covers collateral with haircut. Suppose that collateral asset C is accepted with a haircut h, for a value V_t; one has to post assets for a total value of $V_t/(1-h)$ in asset C. To use the notation of the definition, a gearing factor of $\alpha = 1/(1-h)$ is required. What is the dividend of such a collateral process? At t, one party pays the instrument V_t, receives the collateral for a total value αV_t, sells it and invests the difference $(\alpha - 1)V_t$ in cash at the funding rate f. After the time dt, he sells the instrument for $V_t + dV_t$, repays to collateral for $\alpha V_t/C_t(C_t + dC_t)$ and receives from the cash investment $(\alpha - 1)V_t(1 + fdt)$. The total dividend created is $dV_t - \alpha V_t/C_t dC_t + (\alpha - 1)fV_t dt$. The definition is satisfied with $\alpha = 1/(1-h)$ and $c = (\alpha - 1)f$.

The same result can be obtained for foreign currency for which haircut is required and a conventional rate $\bar{c}$ is paid. The definition is satisfied with the same gearing factor $\alpha = 1/(1-h)$ and $c = \alpha\bar{c} + (1-\alpha)f$. Obviously, if there is no haircut, $\alpha = 1$ and the implied collateral rate is simply the actual collateral rate.

Like before, we suppose we have $n+1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (8.2). We suppose that the first n assets have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset. We suppose that it is collateralised by an asset and the rate c_t^{n+1}. Without loss of generality, we suppose that the collateral asset is the first one. So the continuous payment that can be created by the last asset is

$$dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt.$$

Theorem 8.4 (Geared asset and rate collateral formula). *Suppose the asset has collateral with asset C_t, gearing α and rate c_t^{n+1} and the collateral asset itself has a collateral rate c_t^1. Then the quote at time t of an asset with quote $V_u^{\alpha,C,c^{n+1}}$ at time u is*

$$V_t^{\alpha,C,c^{n+1}} = N_t^{\alpha(c^1+c^{n+1})} \mathbb{E}^{\mathbb{X}} \left[(N_u^{\alpha(c^1+c^{n+1})})^{-1} V_u^{\alpha,C,c^{n+1}} \Big| \mathcal{F}_t \right]. \tag{8.13}$$

Proof: Let

$$\bar{w}_t^T = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equation of the assets,

$$\bar{w}_t^T dD_t = \bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt$$

$$= \left(\bar{w}_t^T (\bar{\mu}_t - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_t^1 - c_t^{n+1} V_t^{n+1} \right) \right) dt$$

$$+ \left(\bar{w}_t \bar{\Sigma}_t + \Sigma_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 \right) dW_t$$

By the choice of $\bar{w}_t$, the coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend cannot have only drift and be non-zero. Consequently we get the drift $w_t^T x = 0$ with

$$x = \begin{pmatrix} \bar{\mu}_t - \bar{c}_t * \bar{V}_t \\ \mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_t^1 - c_t^{n+1} V_t^{n+1} \end{pmatrix}.$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have

$$0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x_{n+1} = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) \lambda_t + x_{n+1}$$

and so $x^{n+1} = \left(\Sigma^{n+1} - \alpha V_t^{n+1}/V_t^1 \Sigma^1 \right) \lambda_t$. By construction we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_t^1, we obtain $\mu_t^{n+1} = \Sigma_t^{n+1} \lambda_t + (c_t^1 + c_t^{n+1}) V_t^{n+1}$.

Define $\tilde{c} = (\bar{c}_t, \alpha(c_t^1 + c_t^{n+1}))$. The equations for V_t become

$$dV_t = \tilde{c}_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) = \tilde{c}_t * V_t dt + \Sigma_t dW_t^{\mathbb{X}}. \tag{8.14}$$

In the numeraire associated to that Brownian motion $W^{\mathbb{X}}$, each asset has a drift $\tilde{c}_t^i$ corresponding to its collateral rate. The drift for the $n+1$-th asset is $\alpha(c_t^1 + c_t^{n+1})$, that is, the collateral rate of the collateral asset plus the collateral rate itself multiplied by the gearing factor. □

An approach covering collateral by other assets is described in McCloud (2013c), where he defines a generalised collateral rate similar to our $c^1 + c^{n+1}$. The result he proposes does not cover haircut.

The above result can be applied to collateral in foreign currencies. For a foreign currency, the asset collateral rate in the sense of Definition 8.1 is the forex swap annualised points. For a domestic currency X and a foreign currency Y, we

denote the points by $p_t^{X,Y}$. Note that the market quoted forex forward points are not annualised and not relative (they are absolute numbers to be added to the current exchange rate).

The collateral rate for foreign currency is obtained in the following way. Buying one unit of foreign currency can be covered by an instantaneous forex swap, selling the foreign currency in the near leg and buying the same quantity in the far leg. If the forward points are annualised and relative, the swap far leg will pay the exchange rate f_t times $1 + p_t^{X,Y} dt$. At the end of the period, one can sell the foreign currency unit of the far leg at $f_t + df_t$. The total cash-flow is then $df_t - f_t p_t^{X,Y} dt$. By Definition 8.1, $p_t^{X,Y}$ is the collateral rate of the foreign currency asset Y.

When risk-free rates exist in both currencies, one has $p_t^{X,Y} = r_t^X - r_t^Y$.

If we define the mixed currency account for the collateral of an asset in currency X with a rate c in currency Y by

$$N_t^{X,c,Y} = \exp\left(\int_0^t c_\tau + p_\tau^{X,Y} d\tau \right).$$

our result specialised to the currency case without haircut reads like (Pallavicini et al., 2012, Proposition 3.7) and (Fujii and Takahashi, 2013, Theorem 6.1).

Theorem 8.5 (Foreign currency collateral formula). *In presence of collateral in currency Y with rate c_t, the quote in currency X at time t of an instrument with quote $V_u^{X,c,Y}$ at time u is*

$$V_t^{X,c,Y} = N_t^{X,c,Y} \mathrm{E}^{\mathbb{X}}\left[(N_u^{X,c,Y})^{-1} V_u^{X,c,Y} \middle| \mathcal{F}_t \right] \tag{8.15}$$

Note that this is simply the result for single-currency collateral with the collateral being another currency. The cross-currency value $V_t^{X,c,Y}$ is simply the single-currency value $V_t^{X,c+p^{X,Y},X}$.

Like in Definition 8.3, we can define a pseudo-discount factor associated to the foreign currency cash collateral.

Definition 8.7 (Foreign currency collateral pseudo-discount factors). *The foreign currency collateral (pseudo-)discount factors for collateral rate c paid in currency Y for an instrument in currency X are defined by*

$$P^{X,c,Y}(t,u) = N_t^{X,c,Y} \mathrm{E}^{\mathbb{X}}\left[(N_u^{X,c,Y})^{-1} \middle| \mathcal{F}_t \right]. \tag{8.16}$$

Suppose that forex swaps between currency X and Y are quoted in the market with the characteristics that forex swaps are for the period $[0, t_0]$ with points $p^{Y,X,c,X}(t_0)$ paid in currency Y and the collateral on the swap is paid in currency X at the rate c. The leg in currency X has cash-flows -1 in 0 and 1 in t_0, its value is $-1 + P_X^c(0, t_0)$. The cash-flows in currency Y are 1 in 0 and $-(1 + p^{Y,X,c,X}(t_0) t_0)$ paid in t_0. Because the cash-flows are fixed, the can be valued

using pseudo-discount factors and the value of the Y leg converted into currency X is $1 - (1 + p^{Y,X,c,X}(t_0)t_0)P^{Y,c,X}(0,t_0)$. If we suppose that the pseudo-discount factor $P_X^c(0,t_0)$ is known, we obtain one (linear) equation with one unknown to solve to obtain $P^{Y,c,X}(0,t_0)$: $P_X^c(0,t_0) - (1 + p^{Y,X,c,X}(t_0)t_0)P^{Y,c,X}(0,t_0) = 0$. A similar result can be obtained if the forex points are paid in currency X: $(1 + p^{X,Y,c,X}(t_0)t_0)P_X^c(0,t_0) - P^{Y,c,X}(t_0) = 0$.

This is exactly the same equation as needs to be solved in the multi-curve framework without collateral to compute one discounting curve from the discounting curve in one currency to the discounting curve in another currency through forex swaps (see for example Henrard (2010)).

With this result, from EUR/USD forex swaps with points in USD and collateralised in USD at Fed Funds rates, and from the pseudo-discount factors for USD collateralised at Fed Funds, we can obtain the pseudo-discount factors for EUR collateralised in USD at Fed Funds.

In McCloud (2013a) and McCloud (2013c), a quantification of the convexity adjustments for forex forwards is provided. For cases when forex swaps with collateral in one of the two currencies are not quoted but other forex swaps are quoted, one may want to estimate (in our notation) $P^{X,c^Y,Y}$ from $P^{X,c^Z,Z}$ and hypothesise on the different spreads, rate and forex dynamics. The paper above provides such a formula using Hull-White like dynamics for the rates.

8.3 Multi-curve framework with collateral

8.3.1 Forward index rate and spread

We apply the general collateral results to the multi-curve framework. Like in Chapter 2, the multi-curve framework with collateral supposes the existence of a specific set of assets, the Ibor coupons.

I: A j floating coupon in currency X with collateral in currency Y at rate c is an asset for each tenor j, each fixing date t_0, each collateral rate c and each currency X and Y.

The curve description approach is based on the following definitions.

Definition 8.8 (Forward index rate with collateral). *The* forward curve $F_t^{X,c,Y,j}$ *(θ, u, v) is the continuous function of t such that,*

$$P^{X,c,Y}(t,v)\delta F^{X,c,Y,j}(t,u,v) \tag{8.17}$$

is the quote in t of the j-Ibor coupon in currency X with fixing date t_0, start date u, maturity date v ($t \le t_0 \le u = \text{Spot}(t_0) < v$) and accrual factor δ collateralised at rate c in currency Y.

Let $I_X^j(t_0)$ denote the fixing rate published on t_0 for the index in currency X for the period $[u, v]$ of length j. The fixing rate is independent of the collateral. On the fixing date we have

$$F^{X,c,Y,j}(t_0, u, v) = I_X^j(t_0)$$

for any collateral rate c and currency Y.

An IRS is described by a set of fixed coupons or cash-flows c_i at dates $\tilde{t}_i$ ($1 \leq i \leq \tilde{n}$). For those flows, the collateral pseudo-discounting curve is used. The IRS also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ ($1 \leq i \leq n$). The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS in $t < t_0$ is

$$\sum_{i=1}^{\tilde{n}} c_i P^{X,c,Y}(t, \tilde{t}_i) - \sum_{i=1}^{n} P_X^{X,c,Y}(t, t_i) \delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i). \tag{8.18}$$

In the textbook one curve pricing approach, the IRSs are usually priced through either the *discounting forward rate* approach or the *cash-flow equivalent* approach. The *discounting forward rate* approach is similar to the above formula.

As with the one-curve framework, we can define a forward swap rate. This is the fixed rate for which the vanilla IRS price is 0:

$$S_t^{X,c,Y,j} = \frac{\sum_{i=1}^{n} P^{X,c,Y}(t, t_i) \delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} P^{X,c,Y}(t, \tilde{t}_i) \tilde{\delta}_i}. \tag{8.19}$$

8.3.2 Same currency collateral

Suppose that the coupon currency X is the same as the collateral currency Y. We use the short notation for the forward: $F^{X,c,X,j}(t, u, v) = F_X^{c,j}(t, u, v)$. This last notation is closer to the multi-curve framework notation.

In the c-collateral v-forward numeraire described after Definition 8.3, the collateralised forward satisfies

$$F_X^{c,j}(t, u, v) = E^{c,v}\left[F_X^{c,j}(s, u, v) \middle| \mathcal{F}_t \right], \tag{8.20}$$

this is the forward rate and is a martingale.

Like in Henrard (2010) and Henrard (2013a), we can define the multiplicative spreads between the different curves.

Definition 8.9 (Multiplicative spread). *The multiplicative spread $\beta^{A|B}$ between the forward rate $A \in \{D; c; c, j\}$ and the forward rate B is defined by*

$$(1 + \delta F^A(t, u, v)) \beta_t^{A|B}(u, v) = 1 + \delta F^B(t, u, v). \tag{8.21}$$

Like in the standard multi-curve framework, we write the spread in a multiplicative way. The multiplicative spread is natural in interest rate modelling as it corresponds to the composition of rates.

Using that notation the quote of a collateralised coupon becomes

$$P_X^c(t,v)(1+\delta F_X^c(t,u,v))\beta_t^{c|c,j}(u,v) - P_X^c(t,v) = P_X^c(t,u)\beta_t^{c|c,j}(u,v) - P_X^c(t,v).$$

We have a formula similar to the one-curve framework where the value of a coupon is obtained by discounting the notional at start date and at end date. Here the start date amount is adjusted (by the ratio β) to take into account the spread between the discounting curve and the index projection curve.

8.3.3 Overnight indexed swaps

Up to now we have defined the collateral discount factors, but have not described how to obtain them from the market. The most commonly used collateral rates are overnight rates. There exist liquid market instruments with pay-off linked to those rates; they are called *overnight indexed swaps* (OIS). We show how to price OISs in an overnight collateral framework.

The proof below is adapted from Quantitative Research (2012a), where a similar proof is provided for compounded coupons with deterministic spread between the discounting and forward curves. In Fujii et al. (2010a) a similar result is provided for continuously compounded rates (not periodically compounded) and no spread. A similar-looking result is also proposed in (Macey, 2011, Section 3.3.1) but from undefined notation.

The overnight rate for the period $[u,v]$ is fixed in u and is denoted $I_X^O(u,v)$. The date v is u plus one business day. The associated accrual factor is δ.

Suppose the rate paid on the collateral account is the overnight rate plus a deterministic spread, which means that N_v is $\mathcal{F}_u$-adapted and

$$N_v^c(N_u^c)^{-1}\beta_u^{c|O}(u,v) = 1 + \delta I_X^O(u,v)$$

with $\beta_u^{c|O}$ deterministic. Usually β will come from a constant spread like $\beta_u = 1 + \delta s$ or $\beta_u(u,v) = \exp(s(v-u))$ with s a constant.

Using Equation (8.5) defining the collateral discount factor and the above equality, we have

$$P_X^c(u,v)(1+\delta F_X^{c,O}(u,u,v)) = P_X^c(u,v)(1+\delta I_X^O(u)) = \beta_u^{c|O}(u,v).$$

By Definition 8.8 with the overnight index O, the above quantity in u is the price of an asset (fixed amount plus overnight coupon). Its price satisfies the generic collateral pricing and

$$P_X^c(s,v)(1+\delta F^{c,O}(s,u,v)) = N_s^c \, \mathrm{E}^X\left[(N_u^c)^{-1}\big|\mathcal{F}_s\right]\beta_u^{c|O}(u,v) = P_X^c(s,u)\beta_u^{c|O}(u,v).$$

The forward index rate collateralised by the same rate satisfies the familiar looking equation as follows.

Theorem 8.6 (One period overnight forward rate value). *The rate of a one period (that is with $v = u+1$ day) overnight coupon, when the collateral rate is the overnight rate multiplied by a deterministic spread, is*

$$1 + \delta F_X^{c,O}(s,u,v) = \frac{P_X^c(s,u)}{P_X^c(s,v)} \beta_u^{c|O}(u,v).$$

The description of an overnight indexed (compounded) coupon is as follows. The times associated are denoted $(t_i)_{i=0,\ldots,n}$. The rate for the period $[t_{i-1}, t_i]$ is fixed in t_{i-1} $(i = 1,\ldots,n)$. The accrual factor in the index convention for the period $[t_{i-1}, t_i]$ is δ_i.

The overnight coupon pays the amount

$$\left(\prod_{i=1}^{n} (1 + \delta_i I_X^O(t_{i-1})) \right) - 1$$

in t_p. In this section we suppose that $t_p = t_n$.

Using the result above, we have, for the expected value of the product in the numeraire defined by Equation 8.6),

$$E^{c,t_n} \left[\left(\prod_{i=1}^{n} (1 + \delta_i F^{c,O}(t_{i-1}, t_{i-1}, t_i)) \right) \right]$$

$$= E^{c,t_n} \left[\left(\prod_{i=1}^{n-1} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} E^{c,t_n} \left[\frac{P_X^c(t_{n-1}, t_{n-1})}{P_X^c(t_{n-1}, t_n)} \middle| \mathcal{F}_{t_{n-2}} \right] \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i)$$

$$= E^{c,t_n} \left[\left(\prod_{i=1}^{n-1} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-2}, t_{n-1})}{P_X^c(t_{n-2}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i)$$

$$= E^{c,t_n} \left[\left(\prod_{i=1}^{n-2} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-2}, t_{n-2})}{P_X^c(t_{n-2}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i)$$

$$= E^{c,t_n} \left[\left(\prod_{i=1}^{n-2} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-3}, t_{n-2})}{P_X^c(t_{n-3}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i).$$

In the first equality we use Theorem 8.6, the fact that F_s^O is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and the tower property. In the second equality, we use the martingale property of the c-quotes rescaled by the c-pseudo-discount factors with maturity t_n in the $E^{c,t_n}[]$ measure. The third equality is simply the simplification of fractions. In the fourth equality we use the same technique as in the first and second but with $\mathcal{F}_{t_{n-3}}$. The same procedure can be repeated down for all factors in the product and we obtain

$$E^{c,t_n} \left[\frac{P_X^c(t_0, t_0)}{P_X^c(t_0, t_n)} \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i).$$

After applying once more the martingale property and the definition of the numeraire, we obtain the following result.

Theorem 8.7 *Under the hypotheses **A** and **I**, the price of the overnight indexed (compounded) coupon described above in the overnight collateral framework is given by*

$$P_X^c(0, t_0) \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}(t_{i-1}, t_i) - P_X^c(0, t_n).$$

The result means that for collateral equal to the overnight index plus a deterministic spread, the OIS curve can be built like in the one-curve framework (taking the spread into account). The formula is very simple and resembles the one for a deposit. Once we have this first building block, we can, like in the multi-curve framework without collateral, start to build index curves with collateral. The index (forward) curves are the curves as described in Definition 8.8 for which the model quote is equal to its market quote (or for which the model par rate is equal to the market par rate).

In this situation the tools developed for curve calibration in the multi-curve framework without collateral can still be used. We refer in particular to Ametrano and Bianchetti (2013) and Henrard (2013a) for more details on challenges of the curve calibration.

As mentioned earlier the 'OIS discounting' framework requires not only that the instrument analysed is collateralised at overnight, but also that the OISs themselves are collateralised at overnight and that the above formula is correct. Not all those conditions are satisfied in all markets. For example in SGD, there is a (quite illiquid) market on OIS with fixing on SONAR. But most of the cash collateral pays an interest indexed on overnight SIBOR, a different index (Tee (2013)). The 'OIS discounting' rule cannot be applied sensu stricto in the SGD market.

8.3.4 Change of collateral: independent spread

Suppose that, like in Section 8.2.3, the spread between two collateral rates is independent of the initial rate, that is

$$c_t^2 = c_t^1 + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and I_X^j and c_1 are $\mathcal{G}_t^1$ adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. The forward rates are defined through the value of coupons and are, by Equation (8.9) and (8.10), linked by

$$P_X^{c_2}(0, v)\delta F^{c_2, j}(0, u, v) = P_X^s(0, v)P_X^{c_1}(0, v)\delta F^{c_1, j}(0, u, v) = P_X^{c_2}(0, v)\delta F^{c_1, j}(0, u, v)$$

This means that the forward rates are the same:

$$F^{c_2, j}(0, u, v) = F^{c_1, j}(0, u, v)$$

If a forward curve has been calibrated for a given collateral rate it can be used for other collateral rates if the spread is independent. Obviously the equality does not extend to quotes, which are discounting dependent. In particular it does not extend to swap rates. The swap rates do not depend only on forward rates but also on discounting. We will show in Section 8.3.8 that the dependency is actually quite mild. We will come back to not independent spreads later and introduce the convexity adjustment required in those cases.

8.3.5 Collateral in foreign currency and independent spread

In this section we want to price interest rate products linear on Ibor rates but with collateral in a foreign currency. At a later stage we want to price cross-currency swaps between the domestic currency and a foreign currency with the collateral in the domestic currency (a foreign currency for the foreign currency leg).

Using the previous notation we want to price $V^{Y,c,X}$. Let the standard collateral used in Y be denoted c^Y. In particular the currency Y IRS quoted in the market uses that collateral rate. From the previous development one can obtain the curve $F^{Y,c^Y,Y,j}$ and $P_Y^{c^Y}$ from the market. Suppose that the standard collateral rule in currency X refers to the rate c^X. In particular we can obtain the discounting curve $P_X^{c^X}$ from the market. Remember that the discounting in currency Y for collateral in a currency X at rate c^X is done with the rate $c_t^X + p_t^{Y,X}$. Suppose that the collateral and forward points are, like in Section 8.2.3, such that

$$c_t^X + p_t^{Y,X} = c_t^Y + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and $I^{Y,j}$ and $c^X + p^{Y,X}$ $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. We don't know s yet (nor $p^{X,Y}$) but we suppose that the rates have the above relationship. For a price with common value V_u we have, using the remark after Theorem 8.5 and Equation (8.9),

$$V_0^{Y,c^X,X} = P_Y^s(0,v)N_0^{c^Y} \, \mathrm{E}^Y \left[(N_v^{c^Y})^{-1} I_Y^j(t_0) \right]$$

$$= P_Y^s(0,v) P_Y^{c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v)$$

$$= P_Y^{s+c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v).$$

The collapse of $P_Y^s P_Y^{c^Y}$ into $P_Y^{s+c^Y}$ is due to the independence hypothesis as noted in Equation (8.10).

The value of a fixed collateralised cash-flow N at time v can be written, by definition, as

$$V_0^{Y,c^X,X} = P_Y^{s+c^Y}(0,v)N.$$

In cross-currency swaps, the pricing of fixed coupons is required for the potential spreads and for the notional payments.

Suppose that cross-currency swaps are collateralised in domestic currency. By providing the spread in the cross-currency swap, the market is providing the price of the foreign leg with collateral in the domestic currency. On the theoretical side, the price can be obtained from two curves: the new discounting curve including the standard collateral and the unknown spread $P^{Y,s+c^Y}$ and the forward rate curve $F^{Y,c^Y,Y}$. The second curve is known from the foreign currency swap market. We are thus faced with solving a 'one curve and one set of constraints' problem. This new curve calibration can be done using the same techniques used for standard multi-curve framework calibration. The inputs are different but the numerical techniques are the same.

The main challenge is probably to keep track of the numerous inputs and relations between them to create a large Jacobian/transition matrix which covers all the inputs and outputs. Such a mechanism was described in Quantitative Research (2013) for the multi-curve framework. We extend it below to the collateral multi-currency framework.

8.3.6 Keeping track of transition matrices

To obtain the transition matrices between the market quotes and the curve parameters, one can directly compute a huge matrix with all the market quotes as input and all the curve parameters as the output of a very large root-solving problem.

This is possible, but with many currencies, many collaterals and many ways to combine them, one easily obtains 10 curves with 20 parameters each. Solving a 200 by 200 non-linear system of equations and computing the associated Jacobian matrix is probably not the most efficient way to achieve the result. As described above the calibration can be done in an inductive way for most of the curves. The last curves calibrated depend on all the previous curves, but at each step only a reduced system has to be solved. Usually the sub-system, which we call unit in the sequel, contains only one to three curves and not ten or more like the full system.

Suppose that the curves are obtained as a multidimensional root-finding process in an inductive way. The market quotes for instrument n are denoted m_n. The parameters to describe the calibrated curves are denoted p_n. There is the same number of market instruments as number of parameters for the curves. The parameters are grouped in units with indices $[n_{i-1}, n_i)$ where $n_{-1} = n_0 = 0$ and there are $n_i - n_{i-1}$ instruments and parameters in each unit.

The goal is to obtain the calibrated curves $(p_n)_{n\in[0,N)}$ from the market rates $(m_n)_{n\in[0,N)}$ but also the generalised transition matrix

$$\left(D_{m_l} p_n\right)_{n\in[0,N),l\in[0,N)}.$$

Due to the way the curves are built, we know that a good part of the matrix is full of zeros; the non-zero part, which we are interested in, is made of the sub-matrices

$$\left(D_{m_l} p_n\right)_{n\in[n_{i-1},n_i),l\in[0,n_i)}.$$

The equations solved to obtain the parameters at the i-th step are:

$$s[n_{i-1},n_i)(p[0,n_{i-1}),p[n_{i-1},n_i)) = 0$$

where the notation $x[i,j]$ represents the vector with values $(x_k)_{k\in[i,j)}$. The functions s_i are the spreads between the market quotes and the quotes computed from the same instrument implied by the curves with parameters $p[0,n_i)$.

Suppose that at each step, the previous step transition matrices $D_{m_l} p_n$ are available for $n \in [n_{i-2},n_{i-1})$ and $l \in [0,n_{i-1})$. We can compute $D_{p_n} m_l = D_{p_n} s_l$ for $n,l \in [n_{i-1},n_i)$ directly when solving the root-finding for the current unit. Moreover

$$\left(D_{m_l} p_n\right)_{n,l\in[n_{i-1},n_i)} = \left(\left(D_{p_n} m_l\right)_{n,l\in[n_{i-1},n_i)}\right)^{-1}.$$

So we have the derivative with respect to the market quotes of the current unit. For the previous market quotes, we use composition. Suppose that we have $D_{p_l} p_n$ for $n \in [n_{i-1},n_i)$ and $l \in [0,n_{i-1})$. Then

$$\left(D_{m_l} p_n\right)_{n\in[n_{i-1},n_i),l\in[0,n_{i-1})} = \left(D_{p_l} p_n\right)_{n\in[n_{i-1},n_i),l\in[0,n_{i-1})} \left(D_{m_k} p_l\right)_{l\in[0,n_{i-1}),k\in[0,n_{i-1})}$$

The second factor is provided by the previous steps; we still have to obtain the first factor. Using the implicit function theorem, we have

$$\left(D_{p_l} p_n\right)_{n\in[n_{i-1},n_i),l\in[0,n_{i-1})} = -\left(\left(D_{p_n} s_k\right)_{k\in[n_{i-1},n_i),n\in[n_{i-1},n_i)}\right)^{-1}$$

$$\left(D_{p_l} s_k\right)_{k\in[n_{i-1},n_i),l\in[0,n_{i-1})}.$$

We now have all the required results to keep track of the full transition matrix.

In the following example we use an implementation[3] of the above construction. We build the USD Fed Fund, USD Libor 3M, EUR Eonia and EUR Euribor 3M as in the standard multi-curve framework. The USD Libor 3M is the one for collateral at Fed Fund and the EUR Euribor 3M is the one for the collateral at Eonia. We now want to construct the EUR discounting curve for collateral in USD at Fed Fund. We use the above developments with the hypothesis of independence to calibrate the curve itself and to build the associated extended transition matrix.

The transition matrix will depend on all the previous curves. The EUR Dsc USD FedFund parameters depend on five curves' market rates. Using the notation of our implementation, we have the summary dependency in Table 8.1. It is interpreted

---

[3] The implementation we used is the OpenGamma OG-Analytics library. It is open source and available at developers.opengamma.com/downloads.

Table 8.1 Summarised representation of the dependency of the collateral curve to the other curves of the example

USD Dsc FedFund=[0, 13],

EUR Dsc Eonia=[13, 12],

USD Fwd Libor3M FedFund=[25, 8],

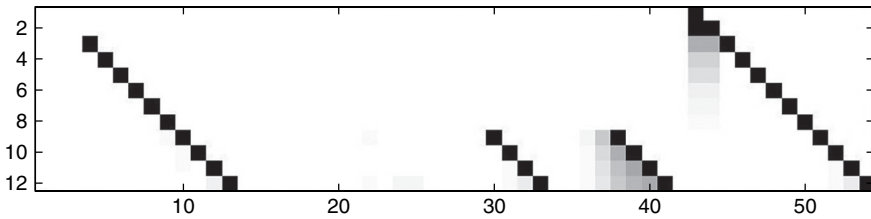EUR Fwd Euribor3M USD FedFund=[33, 8],

EUR Dsc USD FedFund=[41, 13]

Figure 8.1 Visual representation of the transition matrix of the EUR discounting for collateral in USD with Fed Fund rates.

as follows: the generalised transition matrix depends on five curves (with the listed name), for each curve the numbers are the index of the first parameter of the curve and the number of parameters in the curve. Note that even if we have reused the Euribor 3M curves built with Eonia collateral for the forwards, we give to the copy a different name (EUR Fwd Euribor3M USD FedFund), even if we do not compute convexity adjustment for that curve. In this simplified example we have 54 parameters.

The total transition matrix for the last curve has 13 rows and 54 columns and is too large to be represented directly in this document. Instead we give a graphical representation of it in Figure 8.1. The dark squares indicate a strong dependency (absolute value above 0.10) and the grey squares a low dependency down to the white squares where the dependency is 0.

The discounting curve in EUR for collateral in USD obviously depends on the cross-currency instruments (FX swaps and cross-currency swaps), but also on the USD discounting (OIS), USD single-currency swaps and EUR single-currency swaps.

Using this huge quantity of information and managing the dependencies is probably the main challenge in collateralised cross-currency trading and risk management.

8.3.7 Collateral in foreign currency: cross-currency swaps

At the end of Section 8.2.6, we described how forex swap can be used to transfer information on collateral discounting from one currency to another. When the forex swaps between currencies X and Y are collateralised in X cash at rate c, their information can be used to obtain the pseudo-discount factors $P^{Y,c,X}$. In the multi-curve framework with collateral, this is only one small part of the requirement, as we also need the forward rates $F^{Y,c,X,j}$ to be able to price Ibor coupon swaps.

There are several ways to obtain these curves. One is to suppose some independence hypothesis like in Section 8.3.5 and use single-currency curve $F_X^{c,j}$. This is a hypothesis based approach without guarantee it will work in practice. A second approach is to compute the convexity adjustment like we will do in Section 8.4.3. This is a model based approach. The most direct approach is to use market quoted instruments to calibrate the figures. This approach works for a limited list of indexes and currencies for which such information is available.

This is the case when cross-currency swaps between two currencies with cash collateral in one of them are quoted on top of the forex swaps described above. Suppose cross-currency swaps between index j^X in currency X and index j^Y in currency Y (plus spread S) with collateral at rate c in currency X are quoted in the market. This would be the case for example for USD/EUR cross-currency swaps collateralised in USD. Using forward notation, the price of such a cross-currency swap is

$$\sum_{i=1}^{n} P_X^c(0, t_i) \delta_i F_X^{c,j^X}(0, t_{i-1}, t_i) - \sum_{i=1}^{n} P^{Y,c,X}(0, t_i) \delta_i (F^{Y,c,X,j^Y}(0, t_{i-1}, t_i) + S).$$

From the c-OIS and j^X-IRS c-collateralised in X, we can obtain P_X^c and F_X^{c,j^X}. From the forex swaps we can obtain the pseudo-discount factors $P^{Y,c,X}$. We are left with one unknown, F^{Y,c,X,j^Y}, and one constraint, which is the present value using market quote spread S should be 0. We write the explanation as a consecutive curve calibration exercise, but it can obviously be solved as a simultaneous curve calibration process if required.

This approach does not require any hypothesis but requires instruments quoted in the market. In practice those instruments are available only for a small set of currencies and usually only for one index by currency. The most common cross-currency swaps are based on three month tenors.

To come back to our USD/EUR example, this approach would be suitable to obtain the forward curve for Euribor three months collateralised at Federal Funds in USD, if we consider that the cross-currency market quotes are valid for USD Federal Funds collateral. The other indexes, like Euribor six months or Euribor one month, need to be obtained in a different way.

8.3.8 Collateral in foreign currency: correct curves with wrong instruments

We work in a set-up similar to the conclusion of Section 8.3.5 and foreign currency collateral. Suppose that cross-currency swaps are collateralised in domestic currency and the foreign currency swaps are collateralised in foreign currency. Can we calibrate the discount factors and the forward rates directly and not by reusing the forward rates of a first calibration like before?

A priori we can not combine the domestic currency collateralised leg of the cross-currency swaps with foreign currency collateralised instruments to coherently calibrate the curves.

The start of the answer, on the short part of the curve, is yes. If the forward rates are by hypothesis the same, the quotes of the FRAs collateralised in foreign currency are the same as the quote of the FRAs collateralised in domestic currency. The present value is collateral dependent, but the quote, which is the K such that the present value is zero:

$$P_X^c(0,v) \frac{\delta}{\beta} (F_0^{c,j}(0,u,v) - K) = 0$$

is the same for any P_X^c. The part of the curve where FRAs are used to build the foreign currency forward curve can be combined with forex swaps and cross-currency swaps in a coherent way. The quotes are the same for any collateralisation (under the 'same forward' hypothesis).

When we move to the part of the curve calibrated to vanilla swaps, the same is not true. The swap rate is a weighted average of the forward rates (this is good) but the weights are discount factor dependent (this is bad). Using the notation of Equation (8.19),

$$S_t^{X,c,Y,j} = \frac{\sum_{i=1}^n P_X^{c+p^{X,Y}}(t,t_i)\delta_i F^{X,c,Y,j}(t,t_{i-1},t_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i}.$$

This weight is independent of the discounting in the very particular case where the forward curve is constant and floating and fixed payments have the same frequency and convention (like GBP and AUD). When the forward rates are not constant, is the difference acceptable for typical change of collateral rate and a typical curve shape? To answer that question, we have to define 'acceptable' and 'typical'. Here we loosely define 'totally acceptable' as within the tick value of the instruments and 'acceptable' as within the bid-offer of the instrument. For the graphs, we set the tick value at 0.1 basis points and the bid-offer at one basis point.

For the typical value of the spread, we have to remember what the spread is. It is not the difference between the interest rates in the two currencies but the difference between the collateral in one currency plus the forex implied forward points and the collateral in the other currency. If the risk-free rates exists and collateral is done
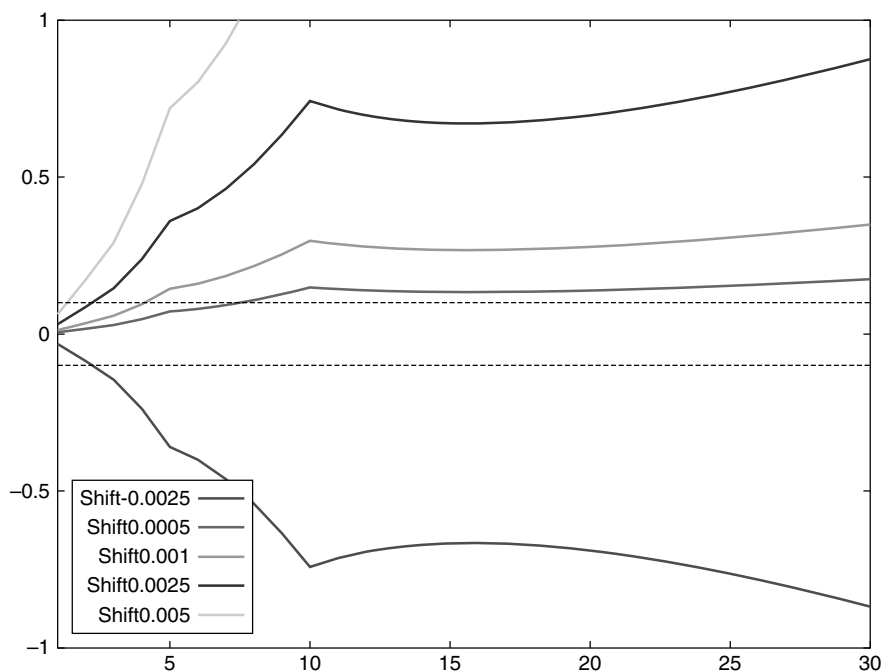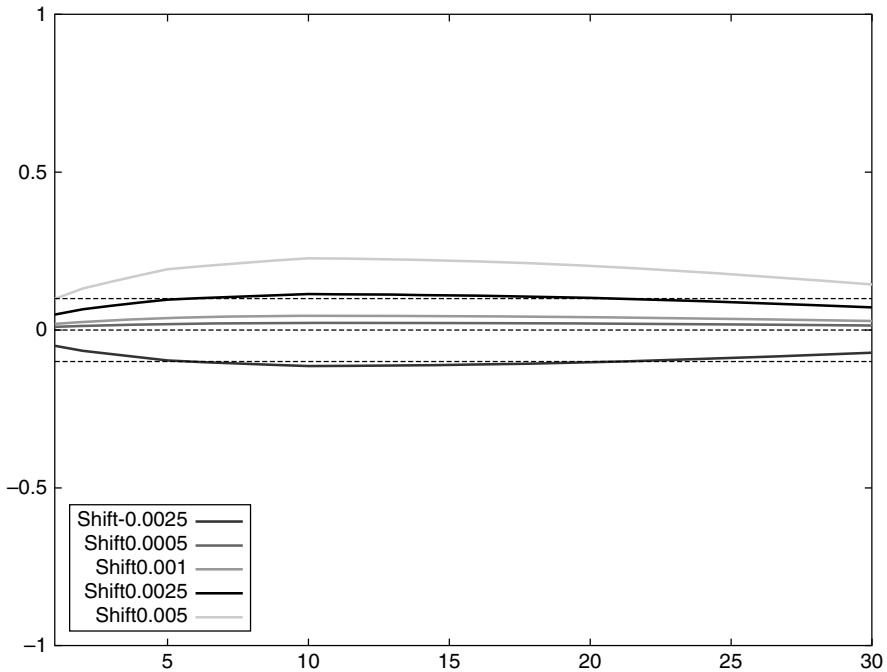
Figure 8.2 Differences in swap rates for different collateral discounting curves. Swap with tenors between 1 and 30 years. Figures for GBP swaps.

at risk-free rate, the spread is 0. The spread can be estimated by the cross-currency OIS spread. The market where the spread is the most liquid is EUR/USD and the spread is currently between -7 bps (short term) and -27.5 bps (long term). The spreads were generally lower, that is, more negative, by as much as 20 bp over last year. For the GBP/USD the spread is currently between -2 and -15 basis points. We can think of 'typical' as anything between -30 and $+30$ bps. Before the crisis, those spreads were typically -5 to $+5$ basis points.

In Figure 8.2, the difference for swap rates for different collateral discounting curves are displayed. The figures are computed for GBP swaps (with standard GBP conventions). The initial forward pseudo-discount factors curve is very steep with short rates at 1.00% and long term rates at 4.00%. The difference is very acceptable for spreads up to 5 bps. It is also acceptable for spreads below 25 bps.

The lower difference for shorter swaps is understandable as for shorter swaps the forward rates have less variability and the approximation is better. This leads to a possible strategy to reduce the difference: using shorter swaps for all the curves. This can be achieved by using forward swaps. In Figure 8.3(a) the difference is displayed for forward swaps of tenor one year (the X-axis represents the start tenor of the swap). For those short swaps, the difference is totally acceptable. Even if the
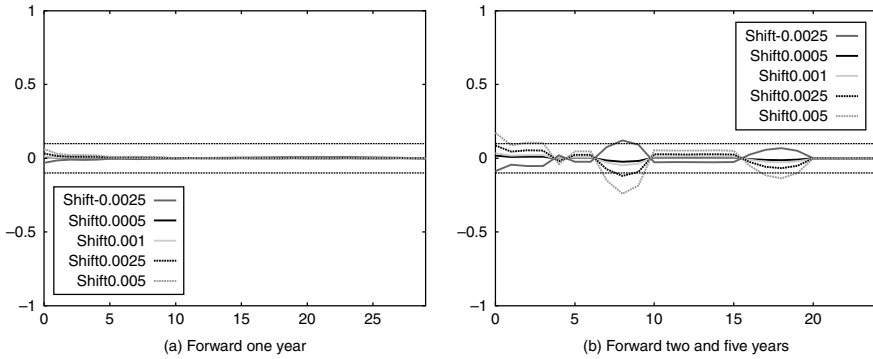
Figure 8.3 Differences in swap rates for different collateral discounting curves. Forward swaps with start year given by the X-axis and tenor given in the subfigure. Figures for GBP swaps.

instruments are not correct (they are collateralised in the domestic currency) they produce the correct information (the correct swap rates for swap collateralised in foreign currency) for any collateralisation spread.

Using one year swaps is maybe too much too ask. In Figure 8.3 we have displayed the difference for swaps with tenor 2 years for the starting years 0 to 7 and forward swaps with tenor 5 years for longer starting periods. The differences are totally acceptable for most of the starting points and tenors. The domestic currency collateral pseudo-discount factors in foreign currency can be computed from the foreign currency (forward) swap rates and forex or cross-currency swaps collateralised in domestic currency with small differences.

The same technique can be used, even more efficiently, for basis swap. The spread in a basis swap (quoted on leg j^1) is computed from the curves by

$$
S_t^{X,c,Y,j^1,j^2}
$$

$$
= \frac{\sum_{i=1}^{n} P_X^{c+p^{X,Y}}(t,t_i)\delta_i F^{X,c,Y,j^2}(t,t_{i-1},t_i) - \sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i F^{X,c,Y,j^1}(t,\tilde{t}_{i-1},\tilde{t}_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i}.
$$

This time, the results will be independent of the level of rate if the spread between the two forward curves is relatively constant. The shape of the curve is not the most important feature; the shape of the spread curve is more important. The spread curve is usually more stable than the forward curve itself.

In Figure 8.4 we display the impact of change of discounting curve on computing the basis swap spread for GBP Libor three months plus spread versus GBP Libor six months swaps. The initial spread between the curves is 15 basis points at the front end and 10 basis points at the long end of the curve. The difference between the quotes with discounting curve as far as 50 basis points apart is still acceptable, with the maximum difference being 0.2 bp for a 50 basis points change. When

Figure 8.4 Differences in basis swap spreads for different collateral discounting curves. Swap with tenors between 1 and 30 years. Figures for GBP swaps.

the discounting curves are less than 25 basis points apart, the difference is very acceptable.

In general, as a first good approximation, when curves have been built for one tenor and collateral in another currency, it is relatively easy to build the curves with other tenors using the same currency basis swaps. Any adjustment required on one leg is also required on the other leg; the adjustments tend to cancel. The at-the-money market quotes for basis swaps with one collateral are very likely to be close to the at-the-money quote with the other collateral. The remark is valid only of at-the-money quotes, not for out-of-the-money collateral quotes, as described in Equation (8.13) which are strongly discounting curve dependent.

8.3.9 Change of collateral: general case

In this section we give a general description of the pricing of swap-related products for general change of collateral (including foreign currency collateral). The result is written as a general *convexity adjustment* result. To be really useful in practice, the result should be made more explicit with a modelling of the different components

and an estimate of the adjustment value. This will be done for an HJM model on the collateral curve and stochastic basis in Section 8.4.3.

Theorem 8.8 *The forward rate with collateral $c+s$ can be written as a function of the forward rate with collateral c through*

$$F_X^{c+s,j}(t,u,v) = F_X^{c,j}(t,u,v)\left(1+\gamma^{c,v,s}(t,u,v)\right)$$

with

$$\gamma^{c,v,s}(t,u,v) = \frac{P_X^c(t,v)}{P_X^{c+s}(t,v)}\frac{\mathrm{cov}^{c,v}\left[N_t^s(N_v^s)^{-1},I_X^j(t_0)\middle|\mathcal{F}_t\right]}{F_X^{c,j}(t,u,v)}.$$

Proof: The value of a coupon with collateral c_t+s_t is given, by the definition of $F_X^{c+s,j}$, by

$$P_X^{c+s}(t,v)F_X^{c+s,j}(t,u,v).$$

Using Theorem 8.1, the price is also

$$
\begin{aligned}
V_t^{c+s} &= N_t^{c+s}\,\mathrm{E}^X\left[(N_v^{c+s})^{-1}I_X^j(t_0)\middle|\mathcal{F}_t\right]\\
&= N_t^s P_X^c(t,v)\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}I_X^j(t_0)\middle|\mathcal{F}_t\right]\\
&= N_t^s\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]P_X^c(t,v)F_X^{c,j}(t,u,v)\\
&\quad\left(1+\frac{\mathrm{cov}^{c,v}\left[(N_v^s)^{-1},I_X^j(t_0)\middle|\mathcal{F}_t\right]}{\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]F_X^{c,j}(t,u,v)}\right).
\end{aligned}
$$

From the definition of $\mathrm{E}^{c,v}[]$, we have

$$
\begin{aligned}
N_t^s\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right] &= N_t^s(P_X^c(t,v))^{-1}P_X^c(t,v)\,\mathrm{E}^{c,v}\left[(P_X^c(v,v))^{-1}(N_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= N_t^s(P_X^c(t,v))^{-1}N_t^c\,\mathrm{E}^X\left[(N_v^c N_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= P_X^{c+s}(t,v)(P_X^c(t,v))^{-1}.
\end{aligned}
$$

Combining the two results, we have

$$F_X^{c+s,j}(t,u,v) = F_X^c(t,u,v)\left(1+\gamma^{c,v,s}(t,u,v)\right)$$

as announced. □

8.4 Modelling with collateral: collateral HJM model

8.4.1 Collateral curve

The results described in this section are inspired by Fujii et al. (2011), Fujii and Takahashi (2013) and Pallavicini and Brigo (2013). From the definition of $P_X^c(t,v)$, the ratio $P_X^c(t,v)/N_t^c$ is a martingale in the $\mathbb{X}$ measure. If we model the ratio by a diffusion equation, we need an equation without drift. We write it, in a way reminiscent of Heath et al. (1992) modelling of the risk-free curve, as

$$d\left(\frac{P_X^c(t,v)}{N_t^c}\right) = -\frac{P_X^c(t,v)}{N_t^c}v^c(t,v) \cdot dW_t^{\mathbb{X}} \tag{8.22}$$

where $W_t^{\mathbb{X}}$ is an n-dimensional Brownian motion in the standard measure $\mathbb{X}$. The volatility v^c can be stochastic. In all cases, we have $v^c(t,t) = 0$. If we write the equation for $P_X^c(t,v)$, taking into account the definition of N^c, we have

$$dP_X^c(t,v) = P_X^c(t,v)c_t \, dt - P_X^c(t,v)v^c(t,v) \cdot dW_t^{\mathbb{X}}. \tag{8.23}$$

Using the above equation, the definition of the collateral forward rate, Definition 8.4, and Ito's lemma, we obtain the following equation for the collateral investment factor $1 + \delta F_t^c(u,v)$:

$$d(1 + \delta F_t^c(u,v))$$
$$= \frac{dP^c(t,u)}{P^c(t,v)} - \frac{P^c(t,u)}{(P^c(t,v))^2}dP^c(t,v)$$
$$+ \left(-\frac{P^c(t,u)}{P^c(t,v)}v^c(t,u).v^c(t,v) + \frac{P^c(t,u)}{P^c(t,v)}|v^c(t,v)|^2\right) dt$$
$$= (1 + \delta F_t^c(u,v))\left((v^c(t,v) - v^c(t,u)) \cdot dW_t^{\mathbb{X}} + v^c(t,v) \cdot (v^c(t,v) - v^c(t,u))dt\right).$$

This leads to the following solution for the collateral forward rates.

Lemma 8.1. (HJM dynamic of collateral forward rates). *In the collateral HJM model, the collateral forward rates satisfy in the standard measure $\mathbb{X}$, for $t \geq s$,*

$$1 + \delta F_t^c(u,v) = (1 + \delta F_s^c(u,v)) \exp\left(-\alpha_c(s,t,u,v)X_{s,t}^{c,\mathbb{X}} - \tfrac{1}{2}\alpha_c^2(s,t,u,v)\right)$$

$$\gamma_c(s,t,u,v) \tag{8.24}$$

with

$$\alpha_c(s,t,u,v)X_{s,t}^{c,\mathbb{X}} = \int_s^t (v^c(\tau,u) - v^c(\tau,v)) \cdot dW_\tau^{\mathbb{X}},$$

$$\alpha_c^2(s,t,u,v) = \int_s^t |v^c(\tau,u) - v^c(\tau,v)|^2 d\tau$$

and

$$\gamma_c(s,t,u,v) = \exp\left(\int_s^t v^c(\tau,v) \cdot (v^c(\tau,v) - v^c(\tau,u))d\tau\right).$$

The result above is in the standard measure $\mathbb{X}$. The same result could be written in the measure associated to $E^{c,v}[]$ in which case there would be no adjustment γ_c as the investment factor is a martingale in that measure.

By Equation (8.22), the quantity $P_{\mathbb{X}}^c(.,v)/N_{\cdot}^c$ is an exponential martingale and the change of numeraire density defined by Equation (8.7) is

$$L_u^{c,v} = \exp\left(-\int_0^u v^c(\tau,v) \cdot dW_\tau^{\mathbb{X}} - \frac{1}{2}\int_0^u |v^c(\tau,v)|^2 d\tau\right).$$

By Girsanov's theorem (see for example (Lamberton and Lapeyre, 1997, Théorème 2.2)), the Brownian motion for the measure associated to $E^{c,v}[]$ is

$$W_t^{c,v} = W_t^{\mathbb{X}} + \int_0^t v^c(\tau,v)\, d\tau.$$

The change of numeraire between $W^{c_1,v}$ and $W^{c_2,v}$ for two collateral rates c_1 and c_2 is

$$W_t^{c_1,v} = W_t^{c_2,v} + \int_0^t \left(v^{c_1}(\tau,v) - v^{c_2}(\tau,v)\right) d\tau.$$

The same lemma can be written in terms of the discount factors, instead of the investment factor $1 + \delta F^c$. By inverting the above lemma, we have the following.

Lemma 8.2. (HJM dynamic of collateral discount factors). *In the collateral HJM model, the collateral discount factors satisfy, for $t \geq s$,*

$$\frac{P_{\mathbb{X}}^c(t,v)}{P_{\mathbb{X}}^c(t,u)} = \frac{P_{\mathbb{X}}^c(s,v)}{P_{\mathbb{X}}^c(s,u)} \exp\left(\alpha_c(s,t,u,v)X_{s,t}^{c,\mathbb{X}} - \frac{1}{2}\alpha_c^2(s,t,u,v)\right)\gamma_c(s,t,v,u).$$

$$(8.25)$$

We now detail a specific model where the above equation for $P_{\mathbb{X}}^c$ is satisfied.

When the discount curve $P_{\mathbb{X}}^c(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, the equivalent of the instantaneous forward rate can be defined for the collateral pseudo-discount factors.

Definition 8.10 *Suppose that the pseudo-discount factor curve $P_{\mathbb{X}}^c(t,.)$ is absolutely continuous. The instantaneous forward collateral rate $f_{\mathbb{X}}^c(t,u)$ is defined by*

$$P_{\mathbb{X}}^c(t,u) = \exp\left(-\int_t^u f_{\mathbb{X}}^c(\tau,u)d\tau\right).$$

This is equivalent to, for a weak differentiation,

$$f_X^c(t, u) = -D_u \ln (P_X^c(t, u)).$$

Let $\sigma^c : \mathbb{R}^2 \to \mathbb{R}^n$ and

$$v^c(t, u) = \int_t^u \sigma^c(t, \tau) d\tau.$$

From Equation (8.23) and Ito's lemma, we have

$$\ln (P_X^c(t, u)) - \ln (P_X^c(0, u)) = \int_0^t \left(c_\tau - \frac{1}{2} v^2(\tau, u) \right) d\tau + v(\tau, u) \cdot dW_\tau^{\mathbb{X}}.$$

Taking the opposite of the derivative with respect to u on both sides, and exchanging integral and derivative in the right-hand side term, we obtain

$$df_X^c(t, u) = \sigma^c(t, u) \cdot v^c(t, u) dt + \sigma^c(t, u) \cdot dW_t^{\mathbb{X}}.$$

in the standard measure.

We defined the short rate associated with the instantaneous forward rate by $r_t^c = f_X^c(t, t)$. We will show below that the short rate r^c is equal to the collateral rate c.

By definition of r^c,

$$r_\tau^c = f_X^c(\tau, \tau) = f_X^c(u, \tau) + \int_u^\tau df_X^c(s, \tau) ds$$

$$= f_X^c(u, \tau) + \int_u^\tau v^c(s, \tau) \cdot D_2 v^c(s, \tau) ds + \int_u^\tau D_2 v^c(s, \tau) \cdot dW_s^{\mathbb{X}}.$$

Then using Fubini, we have

$$\int_u^v r_\tau^c d\tau = \int_u^v f_X^c(u, \tau) d\tau + \frac{1}{2} \int_u^v |v^c(s, v)|^2 ds + \int_u^v v^c(s, v) \cdot dW_s^{\mathbb{X}}$$

where we have used the fact that $v^c(s, s) = 0$.

The result is obtained by taking the exponential of the last equality and using the definition of f_X^c:

$$N_u^{r^c} (N_v^{r^c})^{-1} = \exp \left(-\int_u^v r_\tau^c d\tau \right) = P_X^c(u, v)$$

$$\times \exp \left(-\int_u^v v^c(s, v) \cdot dW_s^{\mathbb{X}} - \frac{1}{2} \int_u^v |v^c(s, v)|^2 ds \right).$$

Using the above equality and the above lemma for pseudo-discount factors, we have that

$$\frac{P_X^c(t, v)}{N_t^{r^c}} = \frac{P_X^c(0, v)}{N_0^{r^c}} \exp \left(-\int_0^t v^c(s, v) \cdot dW_s^{\mathbb{X}} - \frac{1}{2} \int_0^t |v^c(s, v)|^2 ds \right).$$

This implies that

$$dP_X^c(t,v) = P_X^c(t,v)r_t^c - P_X^c(t,v)v^c(t,v) \cdot dW_t^X.$$

Combined with Equation (8.23), it proves that $r_t^c = c_t$.

Lemma 8.3. (HJM dynamic of collateral cash account). *In the collateral HJM model, the collateral cash account N^c satisfies the equation*

$$N_u^c(N_v^c)^{-1} = P_X^c(u,v)\exp\left(-\int_u^v v^c(s,v) \cdot dW_s^X - \frac{1}{2}\int_u^v |v^c(s,v)|^2 ds\right). \quad (8.26)$$

The equivalence between the collateral rate c and the short term rate r^c can be obtained more directly (Fujii 2013). Using the definition of f_X^c, of P_X^c and exchanging differentiation and expectation, we have

$$f_X^c(t,u) = -D_u \ln\left(P_X^c(t,u)\right)$$

$$= \frac{1}{P_X^c(t,u)}\mathrm{E}^X\left[\exp\left(-\int_t^u c_\tau d\tau\right)c_u\Big|\mathcal{F}_t\right]$$

If one takes $u = t$ in the above equation, one obtains

$$r^c(t) = f_X^c(t,t) = \mathrm{E}^X[c(t)|\mathcal{F}_t] = c(t).$$

Using the above expression of $f_X^c(t,u)$ as an expectation and changing the measure, one has

$$f_X^c(t,u) = \mathrm{E}^{c,u}[c(u)|\mathcal{F}_t].$$

The instantaneous forward collateral rate is the expected value in the c-collateral u-forward measure of the collateral rate.

Spread description

Suppose that $c_2 = c_1 + s$ and c_1 and c_2 follow collateral HJM models with volatilities v^{c_1} and v^{c_2} respectively. Then the spread s satisfies

$$(N_t^s)^{-1} = (N_t^{c_2})^{-1}(N_t^{c_1}) = \frac{P_X^{c_2}(0,t)}{P_X^{c_1}(0,t)}\exp\left(-\int_0^t (v^{c_2}(\tau,t) - v^{c_1}(\tau,t)) \cdot dW_\tau^X\right.$$

$$\left.-\frac{1}{2}\int_0^t |v^{c_2}(\tau,t)|^2 - |v^{c_1}(\tau,t)|^2 ds\right).$$

The dynamic of the spread $s = c_2 - c_1$ is linked to the volatility

$$v^s(\tau,v) = v^{c_2}(\tau,t) - v^{c_1}(\tau,t).$$

Nevertheless note that in general

$$P^s(0,t) = \mathrm{E}^X\left[(N_t^s)^{-1}\right] \neq \frac{P^{c_2}(0,t)}{P^{c_1}(0,t)}$$

as the second part of the exponential is not the variance of the first. The equality is satisfied for all t only if $(v^{c_2}(\tau, t) - v^{c_1}(\tau, t)) \cdot v^{c_1}(\tau, t) = 0$ for almost all τ, that is, v^s is orthogonal to v^{c_1}. In some senses we have obtained the converse of Equation (8.10).

Theorem 8.9 [Orthogonality of volatilities]. *In the HJM collateral framework, the pseudo-discount factor of the spread satisfies $P_X^{c_2}(0, t) = P_X^{c_1}(0, t) P_X^s(0, t)$ for all t if and only if the volatility of the spread $v^s(\tau, t)$ is orthogonal to the volatility $v^{c_1}(\tau, t)$ for almost all (τ, t) with $\tau \leq t$.*

8.4.2 STIR futures

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view it involves a mechanism of margin payment equivalent to a zero rate collateral. An understanding of the collateral impact would not be complete without a detailed analysis of those instruments. Technically, this is a simplified version of the change of collateral presented later.

In this section we provide an explicit pricing formula for STIR futures in a Gaussian HJM model using as data the collateral curve and the collateralised forward curve. By Gaussian HJM, we mean we use the curve modelling described in the previous section with v^c deterministic. The results described are very close to those described in Henrard (2013c) and in large part based on Henrard (2005).

For this section, we denote by $\mathcal{G}_t^1$ the filtration generated by the collateral rate and the N^X numeraire. That is N_t^c, $P_x^c(t, c)$ and N_t^X are $\mathcal{G}_t^1$-adapted. As F^c is a martingale in the $E^{c,v}$ numeraire,

$$1 + \delta F_t^c(u, v) = (1 + \delta F_s^c(u, v)) \exp\left(-\alpha_c(s, t, u, v) X_{s,t}^{c,v} - \tfrac{1}{2} \alpha_c^2(s, t, u, v)\right) \quad (8.27)$$

with

$$\alpha_c(s, t, u, v) X_{s,t}^{c,v} = \int_s^t (v^c(\tau, u) - v^c(\tau, v)) dW_\tau^{c,v}.$$

and $W^{c,v}$ the Brownian motion associated to the $E^{c,v}$ [] measure.

The main hypothesis on the index basis is as follows.

SjS: The multiplicative spread $\beta_t^{c|c,j}$ is written as

$$\beta_t^{c|c,j}(u, v) = \beta_0^{c|c,j} \mathcal{X}_t^{c|c,j} \left(\frac{1 + \delta F_t^c}{1 + \delta F_0^c}\right)^a x^{c|c,j}(t, u, v)$$

with $\mathcal{X}_t$ $\mathcal{G}_t^2$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$ and $\mathcal{X}_t$ a $\mathbb{X}$-martingale with independent relative increments.

By independent relative increments, we mean that $\mathcal{X}_t/\mathcal{X}_s$ is independent of $\mathcal{F}_s$. In particular, this will be the case if $\mathcal{X}_t$ is a geometric Brownian motion. The deterministic function $x^{c|c,j}(t)$ is selected such that the rate $F^{c,j}$ is a martingale in $E^{c,v}[]$ as required.

In our modelling assumptions, we do not allow the spread β to depend on the risk-free rate, if such a rate exists. The spread depends only on the collateral rate c and a random variable $\mathcal{X}$ independent of c and of the standard numeraire N^X. We restrict the dependency on quantities provided by the market.

Note that as $\mathcal{X}_t$ is a martingale in $E^X[]$ and $\mathcal{X}_t$ is $\mathcal{G}_t^2$-adapted and the numeraire of $\mathbb{X}$ and $E^{c,v}[]$ are $\mathcal{G}_t^1$ adapted, then $\mathcal{X}_t$ is also a martingale in $E^{c,v}[]$.

The choice of $x^{c|c,j}$ is similar to the one in Henrard (2013c) and is obtained, for $t < s$, through

$$
\begin{aligned}
1 + \delta F^{c,j}(t) &= E^{c,v}\left[1 + \delta F^{c,j}(s)\,\Big|\,\mathcal{F}_t\right] \\
&= E^{c,v}\left[\frac{(1+\delta F^c(s))^{1+a}}{(1+\delta F^c(t))^a}\beta_t\frac{\mathcal{X}_s}{\mathcal{X}_t}\frac{x(s)}{x(t)}\,\Big|\,\mathcal{F}_t\right] \\
&= E^{c,v}\left[\exp\left(-\frac{1}{2}(1+a)\alpha_c(t,s)X_{t,s}^{c,v} - \frac{1}{2}(1+a)^2\alpha_c^2(t,s)\right)\frac{\mathcal{X}_s}{\mathcal{X}_t}\,\Big|\,\mathcal{F}_t\right] \\
&\quad \times (1+\delta F^{c,j}(t))\frac{x^{c|c,j}(s)}{x^{c|c,j}(t)}\exp\left(\frac{1}{2}(1+a)a\alpha_c^2(t,s)\right)
\end{aligned}
$$

Given that the exponential is a martingale in the $E^{c,v}[]$ numeraire and $X_{s,t}$ and $\mathcal{X}_t/\mathcal{X}_s$ are independent of $\mathcal{F}_s$, the expected value is 1. The equality is true only if

$$
x^{c|c,j}(t,u,v) = \exp\left(-\frac{1}{2}(1+a)a\alpha_c^2(0,t,u,v)\right).
$$

Theorem 8.10 (Futures price in collateral framework with stochastic spread). *Let $0 \le t \le t_0 \le u \le v$. In the multi-curve and collateral framework with the collateral HJM model and **SjS** hypothesis on the spread, the price of the futures fixing in t_0 for the period $[u,v]$ with accrual factor δ is given in t by*

$$
\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1+\delta F_X^{c,j}(t,u,v))\gamma_c^{1+a}(t,t_0,u,v). \tag{8.28}
$$

Proof: We use the generic pricing futures price process theorem which is equivalent to the collateral pricing formula with 0 collateral rate,

$$
\Phi_t^j = E^{\mathbb{X}}\left[1 - I_X^j(t_0)\,\Big|\,\mathcal{F}_t\right].
$$

The Ibor rate can be written as

$$
1 - I_X^j(t_0) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\frac{\left(1+\delta F_X^c(t_0,u,v)\right)^{1+a}}{(1+\delta F_X^c(t,u,v))^a}\frac{x^{c|c,j}(t_0)}{x^{c|c,j}(t)}.
$$

Using Lemma 8.1 and the definition of $x^{c|c,j}$ we have

$$1 - I_X^j(t_0) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\left(1 + \delta F_X^c(t,u,v)\right)$$

$$\times \exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\gamma_c^{1+a}(t,t_0,u,v).$$

The variables X_{t,t_0}^c and $\mathcal{X}_{t_0}/\mathcal{X}_t$ are independent of $\mathcal{F}_t$ and independent of each other, so

$$\mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\bigg|\mathcal{F}_t\right]$$

$$= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\right]$$

$$= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\right]\mathrm{E}^N\left[\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\right] = 1$$

From there the result is direct. □

As both collateralised Ibor coupon based instruments, like swaps, and futures are liquid in most markets, the above result can be used to calibrate part of the model. With the market providing information on both the collateralised forward rates $F_X^{c,j}$ and the futures prices Φ_t^j, one can obtain the market implied quantity γ_c^{1+a}. The quantity can be used as an input to the pricing of other instruments. In particular Henrard (2013c) shows how to use that information (together with cap/floor prices) to obtain the price of futures options coherent with both futures and cap/floor market and very light calibration.

8.4.3 Change of collateral: explicit formulas in collateral HJM

Suppose that we have two collateral rates c_1 and c_2. We suppose that both those rates satisfy the HJM model described in Section 8.4.1 with volatilities v^{c_1} and v^{c_2}. Let $\mathcal{G}_t^1$ be a filtration such that N^X and the rates c_1, c_2, F^{c^1} and F^{c^2} are $\mathcal{G}_t^1$-adapted.

Our goal is to write the forward rate with respect to one collateral $F^{c_2,j}$ as a function of the known rate with the other collateral $F^{c_1,j}$ and some explicit adjustments. In practice this would be used in particular for computing the forward in one currency for collateralisation in a foreign currency from the forwards with the natural collateralisation in the domestic currency.

To achieve that result, we use a modelling framework of stochastic spreads very similar to the one of the previous section for STIR futures. We work directly with the forward rate $F^{c_1,j}$ and write it through its dependent and independent parts.

We use the following hypothesis with $\mathcal{G}_t^2$ a filtration independent of $\mathcal{G}_t^1$

SjS2: The forward rate $F^{c_1,j}$ is written as

$$\frac{1+\delta F^{c_1,j}(t,u,v)}{1+\delta F^{c_1,j}(0,u,v)} = \mathcal{X}_t^{c_1,j,c_2} x^{c_1,j,c_2}(t,u,v) \left(\frac{1+\delta F^{c_1}(t,u,v)}{1+\delta F^{c_1}(0,u,v)}\right)^{a_1} \left(\frac{1+\delta F^{c_2}(t,u,v)}{1+\delta F^{c_2}(0,u,v)}\right)^{a_2}$$

with $\mathcal{X}_t^{c_1,j,c_2}$ a $\mathcal{G}_t^2$-adapted random variable and a martingale in the measure $\mathbb{X}$. The deterministic function x^{c_1,j,c_2} will be selected to ensure that the forward rate $F^{c_1,j}$ is a martingale in $\mathrm{E}^{c,v}[]$. Note that, like in the previous section, due to the fact that $\mathcal{X}^{c_1,j,c_2}$ is an $\mathbb{X}$-martingale and due to the independence, $\mathcal{X}^{c_1,j,c_2}$ is also a $\mathrm{E}^{c_1,v}[]$-martingale.

We could have equivalently, like in the previous section, written the hypothesis on $\beta^{c_1|c_1,j}$. We prefer the more direct hypothesis here as we are looking at results on $F^{c,j}$.

We need to indicate F^{c_2} explicitly in the dependency and not hide it partly in $\mathcal{X}$ like in the previous section. The reason is that the adjustment will depend on the co-movement of Ibor rates and the two pseudo-discount curves, as described in Section 8.3.9. Even if the rate $F^{c_1,j}$ can be written as function of F^{c_1} and an independent part, we need to extract from that independent part what is related to the new collateral c_2.

The choice of a_1 and a_2 will never be unique. If F^{c_1} and F^{c_2} are correlated, we can transfer some dependency from a_1 to a_2 and stuff the remaining independent part in $\mathcal{X}$.

The developments are based on the two following results. The first one is equivalent to Equation (8.24) and the second one describes the dynamic of the collateral account given by Lemma 8.3 for each rate in the standard expectation:

$$1 + \delta F^{c_i}(t,u,v) = (1 + \delta F^{c_i}(s,u,v)) \exp\left(-\int_s^t (v^{c_i}(\tau,u) - v^{c_i}(\tau,v))dW_\tau^{\mathbb{X}}\right.$$
$$\left. - \frac{1}{2}\int_s^t (|v^{c_i}(\tau,u) - v^{c_i}(\tau,v)|^2)d\tau\right)\gamma_{c_i}(s,t)$$

and

$$(N_v^{c_i})^{-1} = P_X^{c_i}(0,v)\exp\left(-\int_0^v v^{c_i}(\tau,v)\cdot dW_\tau^{\mathbb{X}} - \frac{1}{2}\int_0^v |v^{c_i}(\tau,v)|^2 d\tau\right).$$

We first obtain the value of x^{c_1,j,c_2} like in the previous section. We know that $1 + \delta F^{c_1,j}$ is a martingale in $\mathrm{E}^{c_1,v}[]$. Using exactly the same technique as in the previous

section, we obtain, after a lengthy but relatively straightforward computation

$$x^{c_1,j,c_2}(t) = \exp\left(-\frac{1}{2}\left(\int_0^t a_1(a_1-1)|v^{c_1}(\tau,u)-v^{c_1}(\tau,v)|^2 + a_2(a_2-1)\right.\right.$$
$$\times |v^{c_2}(\tau,u)-v^{c_2}(\tau,v)|^2$$
$$+2a_1a_2(v^{c_1}(\tau,u)-v^{c_1}(\tau,v))\cdot(v^{c_2}(\tau,u)-v^{c_2}(\tau,v))$$
$$\left.\left.-2a_2(v^{c_2}(\tau,u)-v^{c_2}(\tau,v))\cdot(v^{c_2}(\tau,v)-v^{c_1}(\tau,v))d\tau\right)\right)$$

If we take $a_2 = 0$ and $a_1 = a + 1$, we obtain $x^{c_1,j,c_2}(t) = x^{c|c,j}(t)$ for the function defined in the previous section.

To obtain the relation between $F_X^{c_1,j}$ and $F_X^{c_2,j}$ we use the definition of forward rate and the fact that $F_X^{c_2,j}(t_0) = F_X^{c_1,j}(t_0)$ as both are fixing on the same index. We have

$$P^{c_2}(0,v)(1+\delta F^{c_2,j}(0)) = N_0^{c_2} E^{\mathbb{X}}\left[(N_v^{c_2})^{-1}(1+\delta F^{c_1,j}(t_0))\right].$$

We can then replace $N_v^{c_2}$ using the result above and develop $F^{c_1,j}$ using the hypothesis **SjS2** and writing the value $F^{c_i}(t_0)$ as a function of $F^{c_i}(0)$ using the rate dynamic from above.

We obtain

$$P^{c_2}(0,v)(1+\delta F^{c_2,j}(0)) = P^{c_2}(0,v)(1+\delta F^{c_1,j}(0))\xi(0,t_0,a_1,a_2)$$

with

$$\xi(0,t_0,a_1,a_2) = \exp\left(\int_0^{t_0}(v^{c_2}(\tau,v)-v^{c_1}(\tau,v))\right.$$
$$\left.\cdot(a_1(v^{c_1}(\tau,u)-v^{c_1}(\tau,v))+a_2(v^{c_2}(\tau,u)-v^{c_2}(\tau,v)))d\tau\right).$$

Theorem 8.11 (Change of collateral in HJM and stochastic spread). *In the HJM model for both collateral, if the forward rate spread satisfies the hypothesis **SjS2**, the forward rates in the different collaterals are linked by*

$$(1+\delta F_0^{c_2,j}(t_0,u,v)) = (1+\delta F_0^{c_1,j}(t_0,u,v))\xi(0,t_0,a_1,a_2)$$

From a practical point of view, the above result is as explicit as the computation of ξ.

Special cases

If $v^{c_2} = 0$, that is, c_2 is deterministic, then $\xi = \gamma_{c_1}^{a_1}$ as described in the STIR futures section. When one collateral rate is deterministic or 0, the problem is reduced to the same as the futures margining, which is equivalent to a collateral of 0.

If $v^{c_2} = v^{c_1}$, $\xi = 1$ and there is no convexity adjustment. If the rates are moving together, then we are in a case similar to a deterministic spread.

Suppose that v^{c_2} has two components associated to independent motions: $v^{c_2} = (\tilde{v}^{c_1}, \tilde{v}^s)$ with $v^{c_1} = (\tilde{v}^{c_1}, 0)$, with s interpreted as a spread (see the end of Section 8.4.1 for more on spread in the collateral HJM framework). This means that the quantities associated to F^{c_1} evolve with two independent parts. Suppose also that, like in Section 8.3.4, the filtration underlying the spread is independent of the forward rate $F^{c_1,j}$. If there is independence, it is natural to have $a_2 = 0$, that is, no dependent part between $F^{c_1,j}$ and s through F^{c_2}. In the description of ξ, the first factor is the difference between v^{c_2} and v^{c_1}, which is v^s here. The scalar product in the integral of ξ is thus the scalar product between two orthogonal vectors. This proves that there is no adjustment in that situation, as already proved in Section 8.3.4.

Appendix A
Gaussian HJM

In this appendix, we describe a generic multi-factor Gaussian HJM model on the discounting curve. The model describes only the risk-free discounting curve. Additional hypotheses on the forward curves or on the relationship between the multiple curves will be required when the model is used in a multi-curve framework.

A.1 Model

A term structure model describes the behaviour of $P_X^D(t, u)$, the price in t of the risk-free zero-coupon bond paying 1 in u ($0 \leq t, u \leq T$). To be able to model curves with the usual tools of quantitative finance one needs to fix an upper bound on the time frame on which the modelling is done. This is for example described in (Hunt and Kennedy, 2004, Section 7.4.4). This is why we impose an upper bound T. If the bound T is taken sufficiently large, it will have no impact in practice.

We suppose that the discount curve $P_X^D(t, .)$ is absolutely continuous. This is always the case in practice as the curve is constructed by some kind of regular interpolation or through parameterised functions as described in Chapter 5. When the curve is absolutely continuous, there exists $f(t, u)$, called the *instantaneous forward rate*, such that

$$P_X^D(t, u) = \exp\left(-\int_t^u f(t, \tau) d\tau\right). \tag{A.1}$$

The short rate associated to the curve is denoted $(r_t)_{0 \leq t \leq T}$ with $r_t = f(t, t)$. The cash account numeraire is

$$N_t = \exp\left(\int_0^t r_\tau d\tau\right).$$

The idea of Heath et al. (1992) is to model f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u) \cdot dW_t$$

for some suitable μ and σ and to deduce the behaviour of P_X^D from there. The stochastic variable W_t is an n – dimensional standard Brownian motion. The volatility σ is a function $\sigma : \mathbb{R}^2 \to \mathbb{R}^n$. The function is defined on $\{(t, u) \in \mathbb{R}^2 : t \geq 0$ and $u \geq t\}$.

To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of Hunt and Kennedy (2004).

To simplify the writing, the notation

$$v(t, u) = \int_t^u \sigma(t, \tau) d\tau$$

is used throughout the book. The integrated volatility v also has values in $\mathbb{R}^n$. Note that $v(t, t) = 0$.

The arbitrage-free equations of the model in the cash account numeraire measure associated to N_t are

$$df(t, u) = \sigma(t, u) \cdot v(t, u) dt + \sigma(t, u) \cdot dW_t. \tag{A.2}$$

Let W_t^v be the Brownian motion associated to the measure for the $P_X^D(., v)$ numeraire. The relation between dW_t and dW_t^v is

$$dW_t^v = dW_t + v(t, v) dt.$$

In this book we restrict ourselves to the case where the following hypothesis is true.

G: The function $\sigma(t, u)$ (and v) is a deterministic function of the current time and the maturity u.

This case is generally called the *Gaussian HJM* model, a name used in the Appendix title. With this hypothesis the model is very tractable, as will be seen in the next sections.

A.2 Generic results

The forward volatility of a re-based zero-coupon bond is the positive number defined by

$$\alpha(\theta_0, \theta_1, u, v)^2 = \int_{\theta_0}^{\theta_1} |(v(s, v) - v(s, u))|^2 \, ds.$$

The description of how we arrived at that number is provided below.

The value of the zero-coupon bonds rebased by another bond can be evolved explicitly in the numeraire associated to the second bond. As an asset rebased by the numeraire is a martingale, it is not surprising that the zero-coupon bond solution is the easiest to write in that context.

Formulas similar to the one described below can be found in different articles, in particular in Henrard (2006a) for the Gaussian one-factor HJM and in (Brody and Hughston, 2004, (3.3),(3.4)) in the framework of coherent interest-rate models.

Note the following equality that will be used several times:

$$\int_{\theta_0}^{\theta_1} \left(|v(s,v)|^2 - |v(s,u)|^2 \right) ds = \alpha^2(\theta_1,\theta_2,u,v) + 2\int_{\theta_0}^{\theta_1} v(s,u) \cdot (v(s,v) - v(s,u)) ds.$$

The quantity

$$\gamma(\theta_0,\theta_1,u,v) = \exp\left(\int_{\theta_0}^{\theta_1} v(s,u) \cdot (v(s,v) - v(s,u)) ds \right) \tag{A.3}$$

will also be used on several occasions, in particular for convexity adjustments, and deserves a name of its own.

Lemma A.1. (Discount factors HJM dynamic). *Let $0 \le t \le s \le u, v$. In an HJM multi-factor model, the ratio of zero-coupon bonds can be written in the cash account numeraire as*

$$\frac{P_X^D(s,v)}{P_X^D(s,u)} = \frac{P_X^D(t,v)}{P_X^D(t,u)} \exp\left(-\alpha(t,s,u,v)X_{t,s,u,v} - \frac{1}{2}\alpha^2(t,s,u,v) \right) \gamma(t,s,u,v) \tag{A.4}$$

for a standard normally distributed random variable $X_{t,s,u,v}$ independent of $\mathcal{F}_t$.

Proof: By definition of the forward rate and its equation,

$$P_X^D(t,u) = \exp\left(-\int_t^u f(t,\tau)d\tau \right)$$

$$= \exp\left(-\int_t^u \left[f(0,\tau) + \int_0^t v(s,\tau) \cdot D_2 v(s,\tau)ds + \int_0^t D_2 v(s,\tau) \cdot dW_s \right] d\tau \right).$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$P_X^D(t,u) = \frac{P_X^D(0,u)}{P_X^D(0,t)} \exp\left(-\int_0^t \int_t^u v(s,\tau) \cdot D_2 v(s,\tau)d\tau \, ds \right.$$

$$\left. -\int_0^t \int_t^u D_2 v(s,\tau)d\tau \cdot dW_s \right)$$

$$= \frac{P_X^D(0,u)}{P_X^D(0,t)} \exp\left(-\frac{1}{2}\int_0^t \left(|v(s,u)|^2 - |v(s,t)|^2 \right) ds \right.$$

$$\left. -\int_0^t (v(s,u) - v(s,t)) \cdot dW_s \right).$$

The result is obtained using the equality mentioned before the theorem and the definition of α and γ. □

It is also useful to write the same result in the $P_X^D(.,u)$ numeraire.

Lemma A.2. (Discount factors HJM dynamic). *Let $0 \leq t \leq s \leq u, v$. In an HJM multi-factor model, the ratio of zero-coupon bonds can be written in the $P_X^D(., u)$ numeraire as*

$$\frac{P_X^D(s, v)}{P_X^D(s, u)} = \frac{P_X^D(t, v)}{P_X^D(t, u)} \exp\left(-\alpha(t, s, u, v)X_{t,s,u,v}^u - \frac{1}{2}\alpha^2(t, s, u, v)\right) \qquad (A.5)$$

for a standard normally distributed random variable $X_{t,s,u,v}^u$ independent of $\mathcal{F}_t$.

The result can be obtained using the change of numeraire described above. Alternatively one can notice that the ratio of discount factors should be a martingale as it is the ratio of an asset by the numeraire. The volatility part should be the same as in the previous result, only the mean is changed.

In the general HJM framework the volatility term is not deterministic and can depend on P_X^D. For deterministic volatility the lemmas above provide a solution to the present value problem. When the volatility is state dependent saying that the price *is equal* to the formula presented is a misuse of language. The result is another equation satisfied by the price; the differential equation is replaced by an integral equation.

The cash account value itself can be written explicitly as a function of the Brownian motion.

Lemma A.3. (Cash account HJM dynamic). *Let $0 \leq u \leq v$. In the HJM framework*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_\tau \, d\tau\right)$$

$$= P_X^D(u, v) \times \exp\left(-\int_u^v v(s, v) \cdot dW_s - \frac{1}{2}\int_u^v |v(s, v)|^2 ds\right).$$

Proof: The computation of this lemma is similar to that of Lemma A.1. By definition of r,

$$r_\tau = f(\tau, \tau) = f(u, \tau) + \int_u^\tau df(s, \tau) ds$$

$$= f(u, \tau) + \int_u^\tau v(s, \tau) \cdot D_2 v(s, \tau) ds + \int_u^\tau D_2 v(s, \tau) \cdot dW_s.$$

Then using Fubini, we have

$$\int_u^v r_\tau \, d\tau = \int_u^v f(u, \tau) d\tau + \frac{1}{2}\int_u^v |v(s, v)|^2 ds + \int_u^v v(s, v) \cdot dW_s$$

where we have used the fact that $v(s, s) = 0$.

The result is obtained by taking the exponential of the last equality and using the definition of f. $\qquad \square$

In some circumstances, we will use the following notation for the cash account variables:

$$X_{u,v}^N = \int_u^v v(\tau, v) \cdot dW_\tau$$

and

$$\alpha_N^2(u, v) = \int_u^v |v(\tau, v)|^2 d\tau.$$

A.3 Special cases

A.3.1 Separability

The separability hypothesis on the HJM volatility is defined as follows.

Definition A.1 Separability hypothesis. *The function σ is said to be* separable *if $\sigma(t, u) = g(t)h(u)$ for some positive functions g and h.*

When σ is separable, the zero-coupon volatility v can be written as

$$v(t, u) = g(t)(H(u) - H(t))$$

for H a primitive of h. It means also that in Lemma A.1 and A.2, the random variables $X_{t,s,u,v}$ are the same for all u and v, that is $X_{t,s,u,v} = X_{t,s}$. This will be important when analysing instruments dependent on more than one date, like swaptions and coupons with date mismatches. The random variables depend on the starting and end date of the randomness period but not on the maturity dates of the cash flow payments.

A.3.2 Hull–White

The Hull and White (1990) model, also called extended Vasicek model, is a special case of a one-factor Gaussian HJM model. It is a one-factor model with $\sigma(s, t) = \eta(s) \exp(-a(t-s))$ and $v(s, t) = (1 - \exp(-a(t-s)))\eta(s)/a$. The model satisfies the separability hypothesis.

The model is often used with a piecewise constant volatility. By this we mean that there exists $0 = \tau_0 < \tau_1 < \cdots < \tau_n = +\infty$ such that $\eta(s) = \eta_i$ for $\tau_{i-1} \le s < \tau_i$.

With the piecewise constant volatility, the value of α and γ can be written explicitly. The expiry dates θ_1 and θ_2 are between some of the dates defining the piecewise constant function. The dates are denoted $\tau_p \le \theta_0 < \theta_1 \le \tau_q$. To shorten the notation an intermediary notation is used: $r_p = \theta_0 < r_l = \tau_l < r_q = \theta_1$. With this notation,

one has

$$\alpha^2(\theta_0, \theta_1, u, v) = \left(\exp(-au) - \exp(-av)\right)^2$$

$$\frac{1}{2a^3} \sum_{l=p}^{q-1} \eta_l^2 \left(\exp(2ar_{l+1}) - \exp(2ar_l)\right).$$

The value of γ, the convexity adjustment, can also be written explicitly for that special case. With the same notation as above, we have

$$\ln \gamma(\theta_0, \theta_1, u, v) = \frac{1}{2a^3}(\exp(-au) - \exp(-av))$$

$$\sum_{l=p}^{q-1} \eta_l^2(\exp(ar_{l+1}) - \exp(ar_l))(2 - \exp(-a(u - r_{l+1})) - \exp(-a(u - r_l))).$$

The Hull–White model can also be written as a short rate model. A good reference for the description of the short rate approach is (Brigo and Mercurio, 2006, Section 3.3). The book description refers to the case of the constant volatility model. The stochastic (one-factor) equation for the short rate is, in the cash account numeraire,

$$dr_t = (\theta(t) - ar_t) \, dt + \eta(t) dW_t. \tag{A.6}$$

For a constant volatility η, the mean reversion level is given by

$$\theta(t) = D_2 f(0, t) + af(0, t) + \frac{\eta^2}{2a}(1 - \exp(-2at)).$$

A.3.3 G2++

The term G2++ stands for Gaussian two factors additive model. A description of the model can be found in Brigo and Mercurio (2006).

The model has two mean reversions a_1 and a_2 and a correlation ρ. The stochastic two factors equations for the short rate are, in the cash-account numeraire,

$$dx_t^i = -a_i x_t^i dt + \eta_i(t) d\bar{W}_t^i$$

$$r_t = x_t^1 + x_t^2 + \varphi(t), \qquad r(0) = r_0 \tag{A.7}$$

where $(\bar{W}_t^1, \bar{W}_t^2)$ is a two-dimensional Brownian motion with instantaneous correlation ρ.

The G2++ can be written as an HJM model. It can be written as a model with correlated Brownian motions $\bar{W}_t = (\bar{W}_t^1, \bar{W}_t^2)$. The correlation matrix between the two motions is denoted

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and can be decomposed as $R = A^T A$ with

$$A = \begin{pmatrix} \sqrt{1-\rho^2} & 0 \\ \rho & 1 \end{pmatrix}$$

The equations in the correlated Brownian motions are

$$df(t,u) = \bar{\sigma}^T(t,u)\rho\bar{v}(t,u)dt + \bar{\sigma}^T(t,u)d\bar{W}_t.$$

If we take $\bar{\sigma}^T = \sigma^T A^{-1}$ and $\bar{v}^T = v^T A^{-1}$, the equation above reduces to the standard HJM Equation (A.2) .

For the G2++ model the volatility functions are

$$\bar{\sigma}_i(s,u) = \eta_i(s)\exp(-a_i(u-s)).$$

The equivalence of G2++ to the special HJM case described above is analysed in (Brigo and Mercurio, 2006, Section 5.2). The bond volatilities are

$$\bar{v}_i(s,u) = \frac{\eta_i(s)}{a_i}\left(1 - \exp(-a_i(u-s))\right).$$

A.4 Monte Carlo (Hull–White)

To speed-up Monte-Carlo simulations on several periods, it is useful to decompose the volatility into a maturity dependent part and an expiry part. Let $g(s) = \eta(s)\exp(as)$ and $h(t) = \exp(at)$. We define $H(u) = \int_0^u h(t)dt = \exp(au)/a$. The expiry dependent part is

$$\bar{g}(\theta_0,\theta_1) = \int_{\theta_0}^{\theta_1} g^2(s)ds = \frac{1}{2a}\sum_{l=p}^{q-1}\eta_l^2\left(\exp(2ar_{l+1}) - \exp(2ar_l)\right).$$

The volatility is

$$\alpha(\theta_0,\theta_1,u,v) = \sqrt{\bar{g}(\theta_0,\theta_1)}(H(v) - H(u)).$$

Let $s_0 = 0 < s_1 \cdots < s_i < \cdots$ be a set of times and $Z_i = \int_{s_{i-1}}^{s_i} g(s)dW_s$. The stochastic variables can be written as

$$\alpha(0,s_i,u,v)X_{0,s_i} = (H(v) - H(u))Y_i$$

where $Y_i = \sum_{j=1}^i Z_j$. The random variables Y_i are such that $Y_i = Y_{i-1} + Z_i$ with Z_i independent normally distributed with variance $\bar{g}(\tau_{i-1},\tau_i)$.

The variables Y_i are not independent; they can be viewed as equal on some time interval with one of them continuing after. The covariance between Y_i and Y_j (with $i \le j$) is the variance of Y_i and is

$$\sum_{k=1}^i \int_{s_{k-1}}^{s_k} g^2(s)ds = \int_0^{s_i} g^2(s)ds = \gamma(0,s_i).$$

As the variables are not independent anymore, we simulate independent variables and multiply them by the Cholesky decomposition of the covariance matrix.

It is therefore very easy to generate Monte Carlo simulations at important dates. There is no need to generate intermediary points as the solutions are explicit. One can also generate the values at the different dates directly from 0 without looking at the intermediary values. The dependence between the different times is provided by the correlation structure.

A.5 Miscellaneous

The following technical lemma is not particularly relevant to the other information in this chapter, but has been included here for the lack of a better place.

Lemma A.4. (Normal integral).

$$\frac{1}{\sqrt{2\pi}\sqrt{|\Sigma|}} \int_{\mathbb{R}} \exp\left(x_2 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}x^T \Sigma^{-1} x\right) dx_2 = \frac{1}{\sigma_1} \exp\left(-\frac{1}{2\sigma_1^2}(x_1 - \sigma_{12})^2\right).$$

Appendix B
Conventions

In the multi-curve framework, more curves are constructed and more types of instruments are required to obtain those curves. The interactions between the different instruments also plays an important role. It is therefore paramount to incorporate all the instruments with their exact conventions. Some of these details are neglected in textbooks, rendering them almost useless in practice.

This chapter describes the detailed conventions for the instruments used in curve calibration. It is a description of the term and conditions of the instrument, at least those important for the pricing, but not a description of the way to price them. The pricing of the instruments is discussed in other chapters and in particular in Chapter 2 and in Chapter 6.

An important part of this chapter content is borrowed from the OpenGamma *Interest Rate Instruments and Market Conventions Guide* (Quantitative-Research (2012c)). The booklet, which is now at its second version, covers a lot more instruments and currencies than those described in this appendix. Here we restrict ourselves to the instruments used in curve calibration and to the main currencies (in alphabetical order: CHF, EUR, GBP, JPY, USD).

In this appendix we do not describe some of the building blocks on which the conventions are based: day-count conventions, business day conventions, exchanges on which some of the instruments are traded and bodies involved in setting the fundamental indexes used through this book. For those details, we refer to the above-mentioned guide.

B.1 Ibor indexes

Ibor-like indexes are often related to interbank lending with maturities between one day and one year. It is usually computed as the trimmed average between rates contributed by participating banks. The rates are bank's estimates but usually do not refer to actual transactions.

The most common usages of these indexes in interest rate derivatives are in swaps and STIR futures fixing.

The main currencies' Ibor-like indexes and their main characteristics are summarised in Table B.1.

Table B.1 Ibor-like indexes

Currency	Name	Maturities	Convention	Spot lag	Bbg
CHF	LIBOR	O/N–12M	ACT/360	2	SF00xxx
EUR	EURIBOR	1W–12M	ACT/360	2	EUR0xxx
EUR	LIBOR	O/N–12M	ACT/360	2	EU00xxx
GBP	LIBOR	O/N–12M	ACT/365	0	BP00xxx
JPY	LIBOR	O/N–12M	ACT/360	2	JY00xxx
JPY	Japan TIBOR	1W–12M	ACT/365	2	
JPY	Euroyen TIBOR	1W–12M	ACT/360	2	
USD	LIBOR	O/N–12M	ACT/360	2	US00xxx

In the Bloomberg code, the `xxx` should be replaced by the tenor (T/N, 01W, 11M, and so on) and followed by `␣Index`.

B.1.1 LIBOR

LIBOR is the acronym for London Interbank Offered Rate. Up to 2013, LIBOR was calculated on behalf of the British Bankers' Association by Reuters. Major banks submit their estimated cost of borrowing unsecured funds for several tenors and currencies.

Up to 2013, there were 15 tenors in 10 currencies (AUD, CAD, DKK, EUR, JPY, NZD, GBP, SEK, CHF, USD). Some of them were phased out in the first half of 2013. By May, BBA had cut back to 42 LIBOR rates. Rates are published for six currencies (EUR, EUR same day, JPY, GBP, CHF, USD). Only seven tenors are covered: overnight/spot-next, one-week, and one, two, three, six and 12 months. In July 2013 it was announced that the NYSE Euronext will be in charge of the administration through it's subsidiary *NYSE Euronext Rates Administration Limited.* In the mean time NYSE Euronext has been bought by ICE and the LIBOR is now administrated by ICE.

The conventions are the same for most of the currencies. *For all currencies other than EUR and GBP the period between fixing date and value date will be two London business days. However, if that day is not both a London business day and a business day in the principal financial centre of the currency concerned, the next following day that is a business day in both centres shall be the Value Date.* The business day convention is *modified following* and the *end-of-month* rule applies. The dates for the deposit underlying the index are calculated in the calendar of the principal financial centre of the currency concerned. For all currencies except GBP, the day-count convention is ACT/360.

Reference: http://www.bbalibor.com/technical-aspects/fixing-value-and-maturity.

Reference: http://www.nyx.com/libor

B.1.2 GBP-LIBOR

The fixing date and value date are the same (0 day spot lag). The day-count convention is ACT/365.

B.1.3 EUR-LIBOR

The value date is two TARGET business days after the fixing date. The dates of the deposit underlying the index are calculated according to the TARGET calendar.

B.1.4 EURIBOR

EURIBOR is an acronym for EURo InterBank Offered Rate. The day-count convention is ACT/360 and the spot lag is two days. The business day convention is *modified following* and the *end-of-month* rule applies. There are 43 contributor banks. The rates are published at 11:00 a.m. (CET). The dates of the deposit underlying the index are calculated according to the TARGET calendar.
Reference: http://www.euribor-ebf.eu/euribor-org/about-euribor.html

B.1.5 JPY-TIBOR

TIBOR is the acronym for Tokyo Interbank Offered Rate. It is published by the Japanese Bankers Association. There are two types of TIBOR: The 'Japanese Yen TIBOR' rates reflect prevailing rates on the unsecured Japanese market; the 'Euroyen TIBOR' rates reflect the Japan offshore market. The JBA TIBOR is calculated by JBA as a prevailing market rate based on quotes for 13 different maturities (1 week, 1-12 months) provided by reference banks as of 11:00 a.m. Tokyo time each Tokyo business day. The day-count convention is ACT/365 for the domestic market and ACT/360 for the Euroyen market.
Reference: http://www.zenginkyo.or.jp/en/tibor/the_jba_tibor/

B.2 Overnight indexes

Overnight indexes are indexes related to interbank lending on a one day horizon. Most indexes are for overnight loans and some for tomorrow/next loans. The rates are computed as a weighted average of actual transactions.

The most common usage of these indexes in interest rate derivatives is in overnight indexed swaps (see Section B.9) and in collateral interest payment (see Chapter 8).

The standard overnight indexes for the main currencies and their characteristics are summarised in Table B.2.

Table B.2 Overnight indexes for the main currencies

Currency	Name	Reference	Convention	Publication lag
CHF	TOIS	TN	ACT/360	0
EUR	EONIA	ON	ACT/360	0
GBP	SONIA	ON	ACT/365	0
JPY	TONAR	ON	ACT/365	1
USD	Effective Federal Funds	ON	ACT/360	1

Publication lag is the number of days between the start date of the period and the rate publication. A lag of 0 means on the start date, a lag of 1 means on the period end date.

B.2.1 USD-Effective Federal Funds Rate

The daily effective Federal Funds rate is a volume-weighted average of rates on trades arranged by major brokers. The effective rate is calculated by the Federal Reserve Bank of New York using data provided by the brokers and is subject to revision. The rate is published in the morning (between 7:00 and 8:30) of the period end date. The day-count convention is ACT/360.

Reference: http://www.newyorkfed.org/markets/omo/dmm/fedfundsdata.cfm

B.2.2 EUR-EONIA

EONIA is the acronym of *Euro OverNight Index Average*. It is computed as a weighted average of all overnight unsecured lending transactions undertaken in the interbank market, initiated within the Euro area by the contributing banks. It is calculated by the European Central Bank. The rate is published in the evening (around 19:00 CET) of the period start date. The day-count convention is ACT/360.

Reference: http://www.euribor-ebf.eu/euribor-eonia-org/about-eonia.html

B.2.3 GBP-SONIA

SONIA is the acronym of *Sterling OverNight Index Average*. It is the weighted average rate of all unsecured sterling overnight cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all counterparts in a minimum deal size of GBP 25 million. The rate is published in the evening (around 17:00 CET) of the period start date. The day count convention is ACT/365.
Reference: http://www.bba.org.uk/policy/article/
sterling-overnight-index-average-sonia-a-guide/benchmarks/

B.2.4 JPY-TONAR-Uncollateralised overnight call rate

TONAR is the acronym of *Tokyo OverNight Average Rate*. It is the weighted average rate of all unsecured overnight cash transactions between financial institutions. The rate is published by the Bank of Japan (BOJ). The day-count convention is ACT/365.

A provisional result is published on the evening (at 17:15 JST except on the last business day of the month where it is 18:15 JST) of the period start. The final result is published in the morning (10:00 JST) of the end date.

Reference: http://www.boj.or.jp/en/statistics/market/short/mutan/

B.3 Forward rate agreement

Forward rate agreements (FRA) are over-the-counter (OTC) contracts linked to an Ibor-like index. At the trade date a *reference rate* (R), a *start period*, and a *reference index* are agreed. The *end period* is equal to the start period plus the index tenor (a six month start period and a three month tenor give a nine month end period). The instrument *reference dates* are computed in the following way. Its *start date* is computed from today by adding the index spot lag and then the start period (using the business day convention and calendar of the index). Its *end date* is computed from today by adding the index spot lag and then the end period. The *fixing date* (or exercise date) is the spot lag before the start date. The accrual factor between the start date and the end date (in the index day count) is denoted δ. In some (rare) cases the dates described above are not computed but decided arbitrarily by the counterparties (usually changing the dates by one day or two for convenience reasons).

The FRA *settlement date* is the start date (not the end date). On the settlement date the payoff is, for the FRA buyer,

$$\delta \frac{L_\theta - R}{1 + \delta L_\theta}$$

where L_θ is the value of the reference index on the fixing date. The payoff for the FRA seller is obviously the same amount with an opposite sign. The term to describe such a settlement method is often referred to as *FRA discounting*.

The term *FRA buyer* can be interpreted in the following way: the buyer pays a known fixed price R and in exchange receives a good (the index rate L_θ). The *FRA seller* obviously has the opposite exposure.

Note that all the cash-flows are settled on the start date and not on the end date. In some accounting schemes, the payment is accrued between the start date and the end date. The instrument stays 'alive' from an accounting point of view even if it has already fully settled.

Note also that the FRA's end date may be (slightly) different from the end date of the theoretical deposit underlying the Ibor rate. This potential mismatch comes from a difference in adjustment of the non-good business days between the different ways to compute the periods. To use the example above, a period of six months followed by a period of three months is not always equal to a period of nine months.

Table B.3 FRA dates with differences between end of the accrual period and end of the underlying fixing deposit period

FRA	Trade date	Spot date	Fixing	Start accr.	End accr.	End fixing
1Mx4M	9-Sep-13	11-Sep-13	9-Oct-13	11-Oct-13	13-Jan-14	13-Jan-14
1Mx4M	10-Sep-13	12-Sep-13	10-Oct-13	14-Oct-13	13-Jan-14	14-Jan-14
1Mx2M	10-Sep-13	12-Sep-13	10-Oct-13	14-Oct-13	12-Nov-13	14-Nov-13
1Mx4M	14-Mar-14	18-Mar-14	16-Apr-14	22-Apr-14	18-Jul-14	22-Jul-14
1Mx3M	14-Mar-14	18-Mar-14	16-Apr-14	22-Apr-14	18-Jun-14	23-Jun-14

Dates in TARGET calendar.

Several cases of dates mismatch are proposed in Table B.3. The difference in the table is up to five days.

FRA can also be traded as International Money Market (IMM) FRA, that is, FRA with accrual dates equal to consecutive IMM dates as used in STIR futures (see Section B.4). The underlying Ibor rate has as tenor the one relevant for the IMM dates frequency (three months Ibor for quarterly dates and one month Ibor for monthly dates).

B.4 STIR futures

The futures type described in this chapter are the Ibor-based futures, also called Short Term Interest Rate (STIR) Futures. They all have the same settlement mechanism but differ on the notional, the underlying STIR fixing and exchange on which they are traded.

The dates related to these futures are based on the third Wednesday of the month[1], which is the *start date* of the Ibor rate underlying the future. The rate is fixed at a *spot lag* prior to that date (see Table B.1 for the different conventions); the fixing usually takes place on the Monday (or on the Wednesday itself for GBP). The fixing date is also the *last trading date* for the future. The index underlying the futures is a three or one month index. The three month futures are the most liquid.

The margining process works in the following way. For a given *closing price* or *settlement price* (as published by the exchange), the daily margin paid is that price minus the *reference price* multiplied by the notional and by the accrual factor of the future. Equivalently it is the price difference multiplied by one hundred and by the *point value*, the point value being the margin associated with a one (percentage) point change in the price. The reference price is the trade price on the trade date and the previous closing price on the subsequent dates. The *tick value* is the value of

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[1] When the day is a non-good business day, it is adjusted to the following day.

the smallest increment in price. The price usually changes in 1.0 or 0.5 basis point increments.

We denote the futures price in t by Φ_t^j with j the tenor of the underlying index. On the fixing date at the moment of the publication of the underlying Ibor rates $I_X^j(t_0)$, the future price is $\Phi_{t_0}^j = 1 - I_X^j(t_0)$. Before that moment, the price evolves with demand and offer.

The three month STIR futures are in some places described as '90 day deposit futures'. The origin of the 90 days can be traced back to the futures accrual factor which is 0.25 and correspond to a 90 day period in the ACT/360 convention, the most used day-count convention for money market instruments. But there is no actual reference to a 90 day period in the contract itself. It appears to be an apocryphal description of the instrument trying to invent a explanation behind a conventional number. Moreover that description would be wrong for the GBP futures for which the day-count convention is ACT/365. To avoid introducing an unnecessary confusion for no benefit, we will refrain from using that terminology.

The futures are designated by character codes. The first part depends on the data provider and is usually two to four characters. The main codes are given in Table B.4. The second part describes the month, with the codes given in Table B.5, and the year, with its last digit. For example the March 2013 three month USD LIBOR futures is called ED H4; the June 2014 Eurex Euribor futures is called FPM5. As interest rate futures are quoted for fixing dates up to 10 years in the future only, there is no ambiguity by using only one figure for the year. Note also that it means that when a future reaches its last trading date, a new one is created a couple of days later with the same name but for a maturity 10 years in the future.

Table B.4 Interest rate futures on Ibor details and codes

Currency	Tenor	Exchange	Underlying	Notional	Bloomberg
CHF	3M	Liffe	LIBOR	1,000,000	ES
EUR	3M	Eurex	EURIBOR	1,000,000	FP
EUR	3M	Liffe	EURIBOR	1,000,000	ER
GBP	3M	Liffe	LIBOR	500,000	L_
JPY	3M	SGX/CME	TIBOR	100,000,000	EY
JPY	3M	SGX	LIBOR	100,000,000	EF
USD	3M	CME	LIBOR	1,000,000	ED
USD	1M	CME	LIBOR	3,000,000	EM
USD	3M	SGX	LIBOR	1,000,000	DE
USD	3M	Liffe	LIBOR	1,000,000	LEDA

Table B.5 Rate futures month codes

Month	Code	Month	Code	Month	Code
January	F	February	G	March	H
April	J	May	K	June	M
July	N	August	Q	September	U
October	V	November	X	December	Z

The futures are not only traded individually but also by *packs*, which are groups of four consecutive futures. The packs are referred to by colours. The colours are white (first four futures), red (next four), green, blue, gold, purple, orange, pink, silver and copper.

The futures are also traded as *n-year bundles*, which are groups of the futures up to *n* years from now. The 1-year bundle contains the first four futures, the 2-year bundle the first eight futures, and so on.

If a pack or bundle is traded, the resulting position is a position in all the futures composing the pack or bundle. The futures traded through pack and bundle are fungible with the futures traded individually. Packs and bundles are not quoted directly in price but in average change of price since the last settlement price. A pack or bundle is quote as '$\pm p$' ticks. The trade price of each of the futures of the pack or bundle will be the previous settlement price of that futures plus or minus the quoted p ticks.

B.4.1   USD

USD interest rate futures are traded on CME, on SGX and on Liffe. For three months futures, the nominal is USD 1,000,000 and the accrual factor is 1/4. The fixing index is LIBOR. For one month futures, the nominal is USD 3,000,000 and the accrual factor is 1/12. In both cases, the nominal multiplied by the accrual factor, called the *unit amount*, is USD 250,000.

B.4.2   EUR

The EUR three month interest rate futures are traded on Liffe and on Eurex. The nominal is EUR 1,000,000 and the accrual factor is 1/4. The fixing index is Euribor.

B.4.3   GBP

The GBP three month interest rate futures are traded on Liffe. The nominal is GBP 500,000 and the accrual factor is 1/4. The fixing index is LIBOR.

B.4.4 JPY

The JPY three month interest rate futures are traded on CME and on SGX for Tibor-based futures and on SGX for LIBOR-based futures. The nominal is JPY 100,000,000 and the accrual factor is 1/4.

B.4.5 CHF

CHF interest rate futures are traded on Liffe. The nominal is CHF 1,000,000 and the accrual factor is 1/4. The fixing index is LIBOR.

B.5 Coupons

The coupons are the building blocks of the legs and swaps described in the next sections. We use five main types of coupon: fixed, simple Ibor-linked, compounded Ibor-linked, overnight compounded and overnight arithmetic average –a USD specific coupon used in Federal Funds swaps. The ways to compute the start and end dates of the coupons are described in the appropriate leg or swap sections.

B.5.1 Fixed coupons

A fixed coupon is simply the payment of a fixed amount at a fixed date. The amount is usually written as a multiple of an accrual factor. The accrual factor for the period in the appropriate day-count convention is denoted δ. The amount paid at the payment date for a fixed rate of R and a notional of N is $N\delta R$. Most of the time, the payment date is the end accrual date, but not in all cases. The most frequent case with a difference between these two dates that of the overnight indexed swaps described later.

B.5.2 Simple Ibor floating coupons

An Ibor coupon or Ibor floating coupon is the payment of an amount linked to an Ibor index fixed at a given date t_0 and paid at another given date t_p. The amount is written as a multiple of an accrual factor and the index fixing. Let δ denote the accrual factor for the period in the payment day-count convention . The amount paid at the payment date for a fixing rate $I_X^j(t_0)$ and a notional of N is $N\delta I_X^j(t_0)$.

   In the most standard coupons, the fixing date is the spot lag associated to the index before the start accrual date, the payment date is the end accrual date and the accrual period is equal to the index tenor. There can be up to a couple of days difference, similarly to the difference described for FRAs; this will be the case for all Ibor floating coupons used in the curves construction.

Some Ibor floating coupons also have a *spread*. This means that a fixed payment is added to the variable amount described above. The total payment is, for a spread s, $N\delta(I_X^j(t_0) + s)$.

B.5.3 Compounded Ibor floating coupons

The Ibor floating coupons described above have one payment for one fixing. Other types of Ibor coupons compound several fixings for one payment.

The accrual period is divided into several sub-periods with one fixing associated to each sub-period. The associated dates are denoted $(t_i)_{i=0,\ldots,n}$. The fixing rate for each sub-period $[t_{i-1}, t_i]$ with fixing in t_{i-1}^f is denoted $I_X^j(t_{i-1}^f)$ $(i = 1, \ldots, n)$ and the accrual factors in the index convention δ_i. The total accrual factor for the full period $[t_0, t_n]$ is denoted δ and is equal to $\sum_{i=1}^n \delta_i$.

For a notional of N, the coupon pays in t_p the amount

$$N\left(\left(\prod_{i=1}^n (1 + \delta_i I_X^j(t_{t-1}^f))\right) - 1\right).$$

Compounded floating coupons can also have spread. The International Swap and Derivative Association (ISDA) definition references three different ways to compound spread (see Mengle (2009)). They are called *Compounding*, *Flat Compounding* and *Compounding Treating Spread as Simple Interest*. We describe only two of them.

B.5.3.1 *Compounding*

In this method, both the rate and the spread are compounded. For a spread s the amount paid is

$$N\left(\left(\prod_{i=1}^n (1 + \delta_i(I_X^j(t_{t-1}^f) + s))\right) - 1\right).$$

B.5.3.2 *Compounding treating spread as simple interest*

In this method, the spread is added at the end using a simple interest method:

$$N\left(\left(\prod_{i=1}^n (1 + \delta_i I_X^j(t_{t-1}^f))\right) - 1 + \delta s\right).$$

Compound interests are the most powerful force in the universe.

Attributed to Albert Einstein – 1879–1955

B.5.4 Compounded overnight indexed coupons

Let t_0 be the coupon fixing period start date and t_n be its fixing period end date. Let $t_0 < t_1 < \cdots < t_n$ be all the successive good business days in the coupon fixing period. Let δ_i be the accrual factor between t_{i-1} and t_i ($1 \leq i \leq n$) and δ the accrual factor for the total period $[t_0, t_n]$. The overnight rates between t_{i-1} and t_i are given in t_{i-1} by $I_X^j(t_{t-1}^f)$. The paid amount is

$$\left(\prod_{i=1}^{n} (1 + \delta_i I_X^j(t_{t-1}^f)) \right) - 1$$

multiplied by the notional. This corresponds to the composition of the overnight rates over the period.

The payment is usually not done on the fixing period end date t_n, but at a certain lag after the last fixing publication date. The reason for the lag is that the actual amount is only known at the very end of the period; the payment lag allows for a smooth settlement. The standard conventions for the main currencies are given in Table B.7.

The description of compounded overnight coupons are in some places described as *interest accrued through geometric averaging of the floating index rate.* This is for example the case on Wikipedia (as of December 2012), other web sites and in a Suisse (2001) note Wikepedia refers to. This description should be understood as a *poetic licence* on the meaning of geometric average and not as a precise description. The computation of the rate involves a multiplication, like the computation of a geometric average. But the multiplication is not applied on the rates themselves and no root of the result is taken to obtain the final rate. The licence with the mathematical definition of geometric average is also pointed out in West (2011).

B.6 Legs

The dates on different instruments are spaced by a given payment period. Due to holidays, conventions and broken periods, the way to compute these dates should be detailed. The description below refers to the usual method; as the products are OTC, any variant is possible if agreed by the parties.

The dates are computed from the start (or settlement) date. The last date will be the start date plus the total length (tenor) of the leg. The intermediary dates are spaced by the given payment period, except potentially one when the total tenor is not a multiple of the payment period. The non-standard period is usually the first one. For example a 15 month leg with a 6 month payment period will pay after $3(15 - 2 \times 6)$, 9 and 15 months. The dates will be adjusted by the business day convention and the end-of-month rule. All the dates are first computed without

adjustment and then all the dates are adjusted. This means that if a swap starts on the 5th of the month and its maturity is on a Saturday and adjusted to the Monday (the 7th), the intermediary payments take place on the 5th (potentially adjusted) of the intermediary months, not on the 7th.

The non-standard period is called the *stub*. It can be *short* (shorter than one period) or *long* (between one and two periods). The reason the non-standard period is usually the first one is that once that period is finished, the instrument has the same date as a standard one. The non-standard instrument remains non-standard for the shortest period of time as possible. If the stub was the last period, it would never become a standard one.

The term *roll* (like in '*29 roll*') is also used. It means that the (unadjusted) dates will be on the given day. When it is used, it is often around the month end to clarify the payment schedule.

Ibor legs

Due to the way the payment dates of legs are computed, for Ibor legs there can be a mismatch between the payment dates and the dates for which the fixing is valid. This type of mismatch is the same as the one described for FRAs.

B.7 Swaps

The *start*, *effective* or *settlement date*, of a swap is usually a certain lag (called spot lag) after the trade date. Usually the lag for the swaps is equal to the lag for the related Ibor index. The start date can also be forward. In that case the start date is the trade date, plus the forward period plus the spot lag. The forward period is a given number of months or of years.

B.8 Interest rate swaps: fixed for Ibor

The payments on the fixed leg are regularly spaced by a given period, most of them with a six month or twelve month period.

In fixed for floating swaps, the terms *payer* and *receiver* refer to the fixed leg. A swap is a payer for one party if that party pays the fixed leg (and receives the floating leg). A payer swap for one party is a receiver swap for the other party.

Like for FRAs, the terms *buyer* and *seller* are also used. The swap buyer buys the floating leg for a given fixed price; he is the fixed leg (and swap) payer.

In a vanilla IRS, all the coupons have the same notional and all the coupons on the fixed leg have the same rate.

Table B.6 Vanilla swap conventions

Currency	Spot Lag	Fixed Leg		Floating Leg		
		Period	Convention	Reference	Period	Convention
CHF: 1Y	2	1Y	30/360	LIBOR	3M	ACT/360
CHF: >1Y	2	1Y	30/360	LIBOR	6M	ACT/360
EUR: 1Y	2	1Y	30/360	Euribor	3M	ACT/360
EUR: >1Y	2	1Y	30/360	Euribor	6M	ACT/360
GBP: 1Y	0	1Y	ACT/365	LIBOR	3M	ACT/365
GBP: >1Y	0	6M	ACT/365	LIBOR	6M	ACT/365
JPY	2	6M	ACT/365	Tibor	3M	ACT/365
JPY	2	6M	ACT/365	LIBOR	6M	ACT/360
USD	2	6M	30/360	LIBOR	3M	ACT/360
USD (London)	2	1Y	ACT/360	LIBOR	3M	ACT/360

The spot lag is the lag in days between the trade date and the swap start or effective date.

The payments on the floating leg are also regularly spaced, most of them with a three month or six month period. The period between the payments is equal to the Ibor index tenor. The fixing date for floating payment is the index spot lag before the period start date. The lag is the one given by the index and is usually the same as the swap spot lag, which means that the first fixing takes place on the trade date.

Standard conventions for vanilla swaps in different currencies are provided in Table B.6.

In USD, the swaps are also traded as spread over the treasuries. This quotation mechanism is mainly used for 2, 3, 5, 10 and 30 year swaps where the on-the-run treasuries are the most liquid. The spread is the difference between the conventional yield of the corresponding on-the-run treasury and the swap rate. No adjustment is made for the difference in convention or difference in maturity.

B.9 Overnight indexed swaps

The overnight indexed swaps (OIS) exchange a leg of fixed payments for a leg of floating payments linked to an overnight index.

The start (or settlement date) of the swap is a certain lag (called spot lag) after the trade date. The most common lag is two business days.

The payments on the fixed leg are regularly spaced by a given period. Most of the OISs have one payment if shorter than one year and a twelve month period for longer swaps. The payments on the floating leg are also regularly spaced, usually on

Table B.7 Overnight indexed swap conventions

		Fixed Leg		Floating Leg		
Currency	Spot	Period	Convention	Reference	Convention	Pay lag
CHF ≤ 1Y	2	tenor	ACT/360	TOIS	ACT/360	2
CHF > 1Y	2	1Y	ACT/360	TOIS	ACT/360	2
EUR ≤ 1Y	2	tenor	ACT/360	EONIA	ACT/360	2
EUR > 1Y	2	1Y	ACT/360	EONIA	ACT/360	2
GBP ≤ 1Y	0	tenor	ACT/365	SONIA	ACT/365	1
GBP > 1Y	0	1Y	ACT/365	SONIA	ACT/365	1
JPY ≤ 1Y	2	tenor	ACT/365	TONAR	ACT/365	1
JPY > 1Y	2	1Y	ACT/365	TONAR	ACT/365	1
USD ≤ 1Y	2	tenor	ACT/360	Fed Fund	ACT/360	2
USD > 1Y	2	1Y	ACT/360	Fed Fund	ACT/360	2

The spot lag is the lag in days between the trade date and the swap start date. The pay lag is the lag in days between the last fixing publication and the payment.

the same date as the fixed leg. The amount paid on the floating leg is computed by composing the rates.

The standard conventions for OIS are provided in Table B.7.

B.9.1 CHF

In CHF the payment is one day after the end of the fixing period. This one day is computed as the last publication date, which is at the start of the last period and one day before the end of the last period, plus two lag days. Note that in CHF, the reference index is a tomorrow/next index.

B.9.2 EUR

In EUR the payment is one day after the end of the fixing period. This one day is computed as the last publication date, which is at the start of the last period and one day before the end of the last period, plus two lag days.

B.9.3 GBP

In GBP the payment is on the end of the fixing period, that is, on the fixing period end date.

B.9.4 JPY

In JPY the payment is two days after the end of the fixing period which is also two days after the last index publication.

B.9.5 USD

In USD the payment is two days after the end of the fixing period. These two days are computed from the last publication date, which is at the end of the last period. In USD above two or three years, the OISs are not the most liquid instruments related to the overnight index. The Federal Fund swaps (see Section B.12) are usually more liquid.

B.9.6 Committee meetings

A somehow popular choice of start or end date for OIS swaps are the dates of the relevant central bank committee meetings. The links to pages with the dates of the main currencies' central bank meetings are provided below.
ECB: http://www.ecb.int/events/calendar/mgcgc/html/index.en.html
BoE: http://www.bankofengland.co.uk/monetarypolicy/Pages/decisions.aspx
Fed: http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm

B.10   Basis swap: Ibor for Ibor

In a basis swap, both legs are floating legs and depend on an Ibor index in the same currency (see Section B.15 for swaps with legs in different currencies). In most cases, the indexes have different tenors. A spread above the Ibor index is paid on one of the legs. The quoting convention is to quote the spread on the shorter tenor leg.

Suppose you trade a swap USD LIBOR 3M vs USD LIBOR 6M quoted at 12 (bps) for ten million paying the three months. You will pay on a quarterly basis the USD LIBOR three month rate plus the spread of 12 bps and receive on a semi-annual basis the USD LIBOR six month rate.

This is the convention for almost all currencies, with the notable exception of EUR. In EUR, the basis swaps are conventionally quoted as two swaps. A quote of EURIBOR 3M vs EURIBOR 6M quoted at 12 (bps) for ten million paying the three months has the following meaning. You enter with the counterpart into two swaps fixed against Ibor. In the first swap you receive a fixed rate and pay the three month EURIBOR. In the second swap, you pay the same fixed rate plus the spread of 12 bps and receive the EURIBOR six months. Note that with that convention the spread is paid on an annual basis, like the standard fixed leg of a fixed versus Ibor swap. Even if the quote refers to the spread of a three months versus six months swap, the actual spread is paid annually with the fixed leg convention.

The composition of Ibor index described in Section B.5.3 is not restricted to fixed for Ibor swaps. Some basis swaps are also traded on a compounded basis to align the payment of both legs. For example a basis swap LIBOR one month versus LIBOR three months can be quoted with the one month LIBOR compounded over three

periods and paid quarterly in line with the three months period. Note that the exact convention on the spread compounding needs to be made precise for the trade.

B.11 Basis swap: Ibor for overnight

The OISs are very liquid for the short term, and certainly up to one year or two years. For longer maturities, these swaps are less liquid. In some currencies for longer maturities the basis swaps Ibor for overnight are more liquid. These swaps have two floating legs and exchange Ibor indexed payments for an overnight index composed over the same period. As with the fixed for overnight swaps, a payment lag in the payment is applied on each leg. Both payments (or in practice the netted amount) are paid on the same day.

One exception to this mechanism is the USD where the most liquid medium term OISs are the Federal Fund swaps, which have a different payment mechanism and are described in the next section.

B.12 Federal Funds swaps

The Federal Fund swaps, or simply *Feds* or *Fed swaps*, like the basis swaps Ibor for overnight, exchange an Ibor payment for an overnight index based payment. The difference comes from the way the payment on the overnight leg is computed. In this case the payment is computed as the arithmetic sum of the overnight interest (USD-Effective Federal Funds rate) on the notional (also described as the arithmetic average of the rate). This payment mechanism is similar to the one for the Federal Fund futures. The quarterly coupon payment on the overnight leg is not equal to a three month OIS. In some cases the Federal Funds swaps are traded versus on-month LIBOR. This type of swap is less liquid.

Let $0 \leq t_0 < t_1 < \cdots < t_n < t_n$ be the relevant dates (all good business dates) in the overnight coupon period. Let δ_i be the accrual factor between t_{i-1} and t_i ($1 \leq i \leq n$) and δ the total accrual factor for the total period $[t_0, t_n]$. The overnight rates between t_{i-1} and t_i are given in t_{i-1} by $I_X^O(t_{i-1})$. The paid amount is $\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1})$ multiplied by the notional.

A special final Federal Funds effective fixing or rate cut-off is applied to the last two fixing days. In formula it means that the amount paid is

$$\sum_{i=1}^{n-1} \delta_i I_X^O(t_{i-1}) + \delta_n I_X^O(t_{n-1}).$$

It is possible to trade absent the rate cut-off, but this requires the counterpart to make the payment on the same day as the last fixing information is published.

The swaps are quoted with a spread on the overnight leg. A quote of x – often in basis points – means the swap exchanges LIBOR for overnight average plus a spread of x. The spread is usually positive. The computation of the interest on the floating overnight leg is additive with simple compounding and the spread is also additive with simple compounding. There is no alternative for the composition technique like in the Ibor compounding case. The payment, up to the final day repeated fixing option, is

$$\left(\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1}) \right) + \delta s.$$

B.13 Present value quoted deliverable swap futures

The futures nominal is USD 100,000 per contract; the notional is denoted N in this section. The margining feature is the future-type daily margin on the quoted price[2]. The underlying swap has the standard conventions for USD swaps: semi-annual bond basis versus LIBOR 3M. The futures are quoted for swaps with tenors 2, 5, 10 and 30 years. The underlying swap has a fixed rate as decided by the exchange on the first trading date of the contract. The rate is changed in increments of 25 basis points. Note, the rate is fixed at a predefined value, like the reference coupon of bond futures.

The *delivery dates* follow the quarterly cycle standard to interest rate futures. The *delivery date* is the third Wednesday of the quarterly months (March, June, September, December). The *last trading date* or *expiry date* is two business days prior to that date, usually on the Monday. The expiry date is denoted θ here.

On the expiry date, the parties agree to enter into a swap where the party long the futures receives fixed on the swap and the party short the futures pays fixed. The *effective date* of the swap, denoted t_0, is the delivery date. The fixed rate of the swap is the one attached to the swap futures. The swap also has an up-front payment on the delivery date. The up-front payment is obtained from the futures settlement price on the last trading date, denoted F_θ. The amount received by the long party is $(1 - F_\theta) * N$[3].

The 1 in the formula is a cosmetic feature. One could not add it to compute the up-front amount and keep the same economic reality. The only thing that would change is that the future *price* would be around 0 and not around 1 and the range

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[2] Note that the price is quoted in (percentage) points and 32nd of points, like the bond futures contracts.
[3] If the amount is negative, it should be interpreted as the absolute value is paid by the long party.

of price would be different from the other futures. Personally I would have preferred it without the 1, but it is a personal taste.

Regarding the long/short choice in the contract definition (long is receiving fixed), it can be related to bond futures. The interest rate sensitivity is similar between the two types of futures. Being long the future is similar to being long a bond and short the rate/yield.

B.14 Forex swaps

Among foreign exchange products, the most liquid are certainly the spot forex transactions. But they contain no direct information about interest rate and are not used directly to build curves; we do not analyse them further.

The second next most liquid forex products are the forex swaps. They exchange notional amounts in two currencies at two different dates, receiving one currency at one date and paying it at the other date. The forex swaps are more liquid than the forex forwards. Even if from a theoretical point of view the forex swaps can be viewed as the sum of a forex spot and a forex forward, from a practical point of view, the forex forward is viewed as the difference of a forex swap and a forex spot. The reason is the following. The forex spot deal bears a forex risk (and provides forex information) and no interest rate risk. The forex swaps bear interest rate risk (see below) and almost no forex risk. The forex forwards are hybrid products with forex and interest rate risks. From a business perspective, the desks dealing with interest rate risks and forex risks are often split. When dealing with forex forward, the resulting risk has to be split between two centres of expertise, hence the reduced liquidity.

The quotation of forex swaps is done through *swap points*, also called *forward points*. For a typical forex swap, a notional N is agreed in one currency and is exchanged at the spot date for a notional in the other currency at the prevailing spot exchange rate x. That exchange is called the *near leg*. In some cases the agreed exchange rate is not in line with the current exchange rate, but this is usually done for accounting purposes or to hide some worrying economic reality. A second exchange is done at the forward date. The opposite of the notional (if you are receiving in the near leg, you pay in this one) is exchange at a rate computed as the initial rate plus the forward points p. The forward points are not annualised or relative to the exchange rate. This exchange is called the *far leg*. The cash-flows are described in Table B.8.

If the interest rate is neglected, one can see that the total cash-flow in the first currency is zero. There is very little total exposure to the first currency. The total cash-flow in the second currency is given by Np and is small. There is also very little total exposure to the second currency. The present value is how much value there

Table B.8 Vanilla swap conventions

Date	Currency 1	Currency 2
Near date	N	$-N \times x$
Far date	$-N$	$N \times (x+p)$

The near date is usually the spot date and the exchange rate the one prevailing at the trade moment.

Table B.9 Conventional currency strength

Strength 1	Strength 2	Strength 3	Strength 4	Strength 5
EUR	GBP	USD	CHF	JPY

is in receiving and paying the same amount at different dates in the first currency with respect to receiving and paying the very similar amounts at different dates in the second currency. The answer lies in the difference in interest rate between the two currencies. From a pure etymological perspective, these instruments would be better called *cross-currency interest rate swaps*, but this is not the terminology used in the market.

For this reason forex swaps are an important source of information for curve calibration in a multi-currency setting. They allow us to transfer interest rate information from one currency to another.

Another important practical question is in which order the forex products should be quoted. Do you quote EUR/USD or USD/EUR? Do you have to declare it for each transaction? For a forex spot trade, an EUR/USD transaction can be written as a USD/EUR transaction easily, but for forex swaps, where one currency has the same cash-flow amounts in both legs, an EUR/USD forex swap can not be transformed into a USD/EUR forex swap. The market has adopted a standard quotation order based on the *strength* of currencies. The strength is not an economic assessment of the current situation but a conventional one. The order, from the strongest to the weakest, of the five main currencies is given in Table B.9.

B.15 Cross-currency swaps: Ibor for Ibor

In this section we restrict ourselves to cross-currency swaps for which both legs are floating legs linked to Ibor indexes.

The notional is not the same on both legs as they are in different currencies. The notional on one leg is usually the notional in the other leg translated into the other currency through the exchange rate. The rate is often the exchange rate at the

moment of the trade as agreed between the parties. The notional is paid on both legs, at the start and at the end of the swap.

In some cases the FX rates used are not in line with the market rate. Usually this is to disguise some debts from accounting rules. This type of cross-currency swap at non-market exchange rates were famously used by Greece to hide some of its debt when it entered the Euro. The swaps used for curve construction are done at the market exchange rate.

There also exists some cross-currency swaps with forex rate reset, called *market-to-market cross-currency swap*. At each payment date, the exchange rate between the legs is reset to the exchange rate at the time of reset and full notionals are exchanged. This feature is created to reduce the credit risk created by the movement of exchange rates.

Both legs of the swap are linked to an Ibor-like index. In the standard swaps, the Ibor tenor on both legs is the same. The payments are done on the same days for both legs to reduce the credit risk. It means that the payment calendar is the joint calendar of both currencies involved in the swap.

The most liquid cross-currency swaps exchange three month payments. Even if one currency has a six month index as its most used index, the cross-currency swap use three month payments. This is in particular the case of USD/JPY and USD/EUR swaps that use three month payments, even if the six month EUR Euribor and six month JPY LIBOR are the standard floating references for the IRS in those currencies.

The cross-currency swaps also pay a spread on one of the legs. In which currency the spread is paid depends on the currency pairs. For almost all swaps involving USD, the spread is paid in the other currency.

Appendix C
Implementation in a library

The third subtitle of this book is 'implementation'. Throughout this book we have focused on the practical aspects of the framework, how to deal with the market idiosyncrasies of each instrument and currency. In this section we describe some of the most important concepts that need to be put in place in an IT library that implements the multi-curve framework.

The description is done using the Java terminology and pseudo-code. The concepts extend to any programming language. Note that the code is always 'pseudo-code', this is code that gives all the important information but that may in some cases not compile or work. The reason for presenting only pseudo-code is to avoid filling the book with listings of redundant information. Also, in some cases to fit the code on the page, some semantic simplifications are introduced.

A full implementation is available in an open source version to which I'm an important contributor. The framework is the OpenGamma platform and the project containing the quantitative finance specific code is OG-Analytics. The code is available at http://developers.opengamma.com/downloads.

C.1 Curve universe object

The set of curves used in the multi-curve framework is not simply a set of independent curves, there is an important conceptual link between the curves and other financial concepts. These financial links should be represented into the code structure.

The other financial objects closely related to curves are:

Currency: For each currency there is one discounting curve. The link between the currency and its discounting curve should be explicit.

Ibor Index: For each Ibor index there is one curve. The same curve can be potentially shared by several indexes, such as between EURIBOR and EUR LIBOR. From a purist point of view this may not be desirable, as each index is different and has its own curve, but from a practical point of view the market information may be limited and some simplifications may be necessary.

Overnight Index: For each overnight index there is one curve. Note that in some cases, like for OIS discounting, the same curve can be used by a currency and an index. This flexibility should be available.

Listing C.1 Multi-curve framework provider

```java
public interface MulticurveProviderInterface {
  double getDiscountFactor(Currency ccy, double time);
  double getForwardRate(IndexIbor index, double startTime,
    double endTime, double accrualFactor);
  double getForwardRate(IndexON index, double startTime,
    double endTime, double accrualFactor);
  double getFxRate(Currency ccy1, Currency ccy2);
}
```

Forex rate: When the curves are used in a multi-currency environment it is important that the forex rates used for valuation are the same as those used in the curve construction.

The multi-curve object should provide the financially meaningful numbers. For a discounting curve associated to a currency, the object should provide the discount factor at each time. For a forward curve associated to an index, the object should provide the forward rate between two dates (maybe dependent on the accrual factor between them). The access to these numbers should be independent of the way the data is stored or the curve implemented. The different implementations described in Chapter 3 should have no impact on the user. For example the link between pseudo-discount factor and forward rate (Equation 3.1) should not be repeated in the valuation code each time a forward is required. The method needing the forward should simply request the forward. The model knows that a forward is required and asks for it, but it has no knowledge of how the curve is stored. The two should be kept separated.

This notion of a 'service' being available independently of the internal representation of the data is well represented in Java by an interface. The interface to multi-curve frameworks have been implemented under different names in different libraries. I have encountered the names *universe*, *market* and *provider*. In the code below we use the term provider. The simplified code of a multi-curve provider is in Listing C.1.

One single payment may depend on several indexes, depending on the contract description and the collateral rules. It is not the user's job to provide a restricted set of curves for a specific valuation. He should give the provider and the exact description of the instrument. It is to the pricing method to chose the correct curves for a given financial reality. Before the crisis, a lot of implementations were limited to one or two curves (usually one for discounting and one for forward). The description of an instrument would explicitly include the link to the two curves. Such an approach

Listing C.2 Present value method for an Ibor floating coupon

```
public MultipleCurrencyAmount presentValue(CouponIbor cpn,
    MulticurveProviderInterface multicurve) {
  double forward = multicurve.getForwardRate(cpn.getIndex(),
    cpn.getFixingPeriodStartTime(),
    cpn.getFixingPeriodEndTime(),
    cpn.getFixingAccrualFactor());
  double df = multicurve.getDiscountFactor(cpn.getCurrency(),
    cpn.getPaymentTime());
  double pv = cpn.getNotional() * cpn.getPaymentAccrualFactor()
    * forward * df;
  return MultipleCurrencyAmount.of(cpn.getCurrency(), pv);
}
```

considerably limits what can be modelled correctly. All multi-Ibor coupons and collateralised instruments are mis-priced because the instrument description and its modelling are not separated cleanly. In such a framework where the instrument itself is aware of the modelling through the names of the curves, a model change means from a database/code perspective that everything needs to be changed, even objects not related to the model.

With the separated approach we describe the pricing code becomes very short and clear. For example the present value of an Ibor coupon is proposed in Listing C.2

Note that the result of the present value computation is a Multiple Currency Amount and not a simple double. In general we work with multiple currencies, in particular for cross-currency swaps. It is important to explicitly state for each number to which currency it is associated.

C.2   Curve calibration

Given the requirement of multiple curves for different purposes, there is the parallel requirement of more instruments to build the curves. One cannot restrict the instrument used to build the curves to a small set, like deposit, FRA, futures and IRS. All the instruments that convey information on the market should be available for curve construction purposes.

The instruments I have used to build curves recently are: deposit, Ibor fixing, OIS (fixed tenor and central bank meetings dates), FRA, IRS (fixed versus Ibor, fixed tenor and IMM dates), basis swaps (Ibor versus Ibor, overnight versus Ibor), Fed Fund swaps, FX swaps, cross-currency swaps, STIR futures, Fed Fund futures and swap futures. The interaction between instruments and curves may be complex. One instrument often depends on three curves. The idiosyncrasies of some markets may be that different products depending on different indexes are liquid

for different tenors. Those particularities require a very flexible curve calibration mechanism.

The flexibility means that one should be allowed to chose any set of instrument with any dependency on index and use them for curve calibration. The only mechanism that appears to satisfy these requirements is a large root-finding procedure. The root-finding is not for one individual curve but for a full universe of curves. It is in some cases possible to split the large root-finding into sub-problems solved separately. This simplification is described in Section 5.3. This is more a particular case than the general approach. Splitting the problem into sub-problems usually provides computation time improvement. But it introduces some extra house keeping problems. As mentioned in the curve calibration chapter, calibrating the curves to compute instruments present values is only a small part of the problem. A more important one, and usually more computationally intensive, is to compute the derivatives or sensitivities of the present value to all the market data input used in the curve construction.

The way to do this is usually by implementing some version of algorithmic differentiation (see next section). The first building block of algorithmic differentiation is the ability to compute the derivative of any sub-process used in the computation of the final numbers. In the curve construction case, one has to compute the Jacobian matrices of sensitivities of the curve internal parameters to the market quotes. Given the complexity of the interactions of all the curves, the matrix is usually not stored as a large unique matrix. That would mean storing a square matrix of dimension several hundred and carrying it around, with the indication of the order of the market quotes, even if only one curve is used for a particular computation. Our suggested implementation is to compute the partial matrix related to each curve and store one such matrix for each curve.

The information required for one curve is the list of curves on which its parameters depend and the number of those parameters. We call it a CurveBlock. Its code is provided in Listing C.3. An example of content for such an object is proposed in Table 8.1. We describe the curves simply by their name (String). It would be better to describe them by a proper identifier; this is not analysed here.

Once we have the list of curves a curve relates to, we need the bundle of those relationships for all curves and the actual Jacobian matrices. We propose to store this information in a bundle we call CurveBlockBundle. Its code is provided in Listing C.4. An example of the type of matrices stored can be found in Table 5.1 and Table 5.2.

A theoretical description of how to produce the partial Jacobian matrices without recomputing the full Jacobian matrix for each curve is proposed in Section 8.3.6.

Listing C.3 Curve building block with reference to other curves and their market quote sizes label

```java
public class CurveBlock {
  /**
   * List of curves in the block as a map:
   *   Curve name to a pair of integers:
   *   1) Start index of the curve in the list of all
           parameters
   *   2) Number of parameters in the curve
   */
  private Map<String, Pair<Integer, Integer>> _unit;

  public CurveBlock(Map<String, Pair<Integer,
    Integer>> unit) {
    _unit = new Map<>();
    _unit.putAll(unit);
  }

  public Integer getStart(String name) {
    return _unit.get(name).getFirst();
  }

  public Integer getNbParameters(String name) {
    return _unit.get(name).getSecond();
  }
}
```

Listing C.4 Bundle of all curve building blocks and their Jacobian matrices

```java
public class CurveBlockBundle {

  private Map<String, Pair<CurveBlock, Matrix2D>> _bundle;

  public CurveBlockBundle(Map<String, Pair<CurveBlock,
    Matrix2D>> bundle) {
    _bundle = new Map<>();
    _bundle.putAll(bundle);
  }

  public void add(String name, CurveBlock block,
    Matrix2D mat)
    _bundle.put(name, Pair.of(block, mat));
  }

  public Pair<CurveBlock, Matrix2D> getBlock(String name)
    return _bundle.get(name);
  }
}
```

C.3 Algorithmic differentiation

Computing the first order derivative of results with respect to the input is an important task of any financial software. In finance, these derivatives are known as *the greeks*. In the multi-curve framework this is an even more important task as there are more inputs. As described in the curve construction chapter, the number of inputs can be large and often the different curves are entangled and cannot be split. Even the simplest instruments depend on many inputs. A standard IRS, priced using two curves, can depend on more than 40 points. When curves are built through cross-currency instruments, a simple cross-currency swap can depend on 100 inputs. In the simplified example described in Section 5.7, we used 50 points. This multiplication of input is a feature of the multi-curve framework and should certainly not be overlooked. The number of required outputs in the computations increases with the complexity of the curve construction.

A traditional approach to first order derivative computation is to use the *bump and recompute* approach, also called *finite difference*. It is based on the definition of derivative through the limit of differentiation ratio.

Definition C.1. (Differentiation ratio). *The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction e_i is defined, when it exists, by the limit*

$$D_{e_i}f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon e_i) - f(x)}{\epsilon}.$$

By definition the above limiting process is true in theory. In practice, in a finite precision arithmetic, the above equality is not true. When decreasing the bump size, the machine precision is reached and the results diverge. This comes from the fact that the limit is of the style '0/0'. For standard implementations, the level at which the divergence starts is around 1.0×10^{-10}. This is the best case, when the process to compute f does not contain heavy numerical procedure. The user is squeezed between taking a small bump to obtain the theoretical convergence and a larger one in order to avoid the numerical instability problem.

An example of instability for small increments is proposed in Figure C.1. This particular example is a swaption present value in the Hull-White model. Any pricing will present a similar picture. The error decreases when the bump size is decreased, up to a certain point where the numerical instability kicks in and the number become meaningless. The level where the numerical instability kicks in will depend on the algorithm used in the computation.

The *bump and recompute* approach is the standard approach in most implementations. As it is the dominant approach, often the numerical method is confused with its goal. There is a confusion between the cause and the consequence. For example, (Traven 2010, Section 4) presents the the bump and recompute method as the definition of delta risk. (Sadr 2009, Chapter 3) also uses the bump and
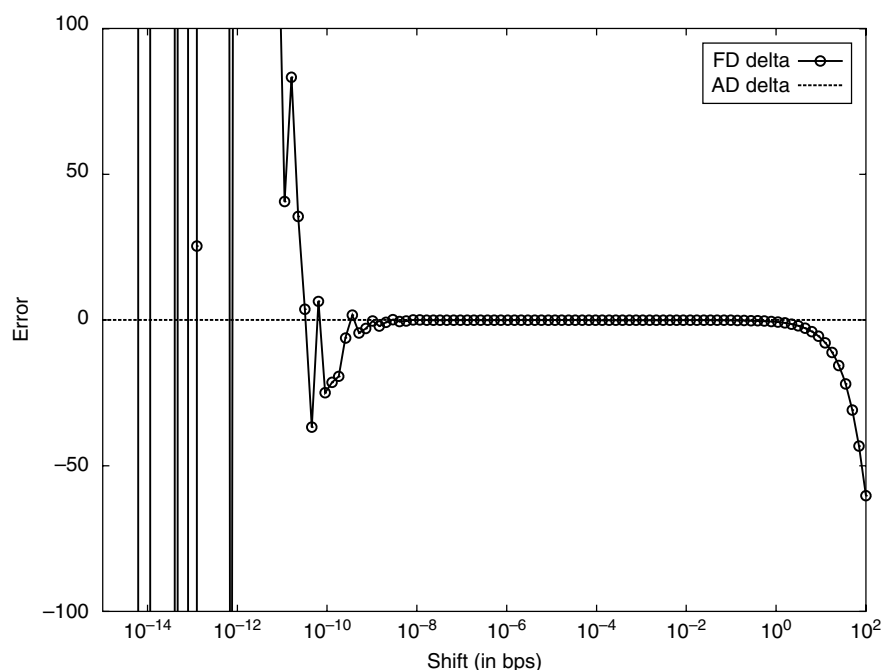
Figure C.1 Instability of finite difference computation of derivative. The horizontal axis represents the bump size ϵ in logarithmic scale. The vertical axis is the error. The true value is represented by the horizontal line.

recompute method as the definition of PV01. This last reference even includes in the definition the fact that the bump should be one sided and of particular size (one basis point). Under that definition, partial PV01 does not add to the total PV01 as the definition explicitly includes some convexity and higher order effects. Le Floc'h (2012) analyses the quality of interpolation methods for curve construction through their stability in the above numerical procedure. The stability question appears only if a poor procedure is used in the computation.

The bump and recompute approach becomes more and more expensive when one implements a full scope multi-curve framework and even more so in its collateral version. Not only because several curves are used for one instrument but also because of the dependency between them. An efficient implementation of the derivatives computation is not a nice-to-have anymore, but a fundamental requirement for a near real time system.

The algorithmic differentiation approach appears as the only viable approach. The theoretical foundation of the method can be found in the classical book Griewank and Walther (2008). The approach, including outside the multi-curve framework, has taken on more importance in finance. One of the articles that

relaunched interest in the approach in finance was probably Giles and Glasserman (2006). More recent descriptions of implementations in finance can be found in Capriotti (2011) and Capriotti and Giles (2011).

One of the main results of algorithmic differentiation is that in its *adjoint* or *reverse* mode, the relative cost between the computation of the function and its derivatives (F+D) and the function (F) alone is (see (Griewank and Walther, 2008, Section 4.6))

$$\frac{\text{Cost}(F+D)}{\text{Cost}(F)} \leq \omega_A$$

with $\omega_A \in [3, 4]$ for a function with an output of dimension 1 (like a present value).

The theoretical performance can be achieved in practice when a library is built with the algorithmic differentiation requirement in mind. When the performance is achieved in practice, this represents in the multi-curve framework an improvement of a factor between 5 and 25. For very large portfolios, it can be the difference between a one hour computation time and a one day computation time.

With an efficient implementation of algorithmic differentiation, the time to compute the present value and the curve sensitivities should not be above four times the time of computing the present value alone. The sensitivities include risk figures like the bucketed deltas, even if there are several curves and a lot of instruments used to calibrate the curves. The factor four is obviously not equal to one, which would be a free lunch, but is close enough that if the real-time pricing is achievable, the real-time sensitivities computation is also possible.

C.3.1 Principles of algorithmic differentiation

The starting point is the algorithm to compute a value:

$$z = f(a).$$

We consider only the case where f has its values in $\mathbb{R}$. This is the traditional case in quantitative finance, where the first goal is to compute the present value of a financial instrument. The function inputs are $a = a[0 : p_a]$, a vector of dimension $p_a + 1$. The function output is z, a single double, that is z is of dimension 1.

The code is summarised in Table C.1. The code starts by initialising the internal variables with the inputs. Then the actual algorithm take place. We restrict ourself to the case where the algorithm is composed of a list of consecutive assignments. Each line of code is potentially using all the previous variables, those coming from the inputs and the intermediary variables. This algorithm is suppose to be implemented in the code.

The goal is to compute the derivatives of z with respect to the different inputs a_i:

$$\frac{\partial}{\partial a_i} f(a) = \frac{\partial}{\partial a_i} z.$$

Table C.1 The generic code for a function computing a single value

Initialisation	$[j = -p_a : 0]$	$b[j]$	$= a[j + p_a]$
Algorithm	$[j = 1 : p_b]$	$b[j]$	$= g_j(b[-p_a : j-1])$
Value		z	$= b[p_b]$

The result of this computation is a vector of size $p_a + 1$, like the input vector a.

There are two modes to viewing this problem. The first mode puts emphasis on 'with respect to a_i'; this is called the *standard* or *forward* mode. One starts from the inputs a_i and at each line of code in the value computation associates $p_a + 1$ lines of code which compute the different derivatives of each intermediary variable with respect to the different inputs.

The second mode put emphasis on 'of z'; this is called the *adjoint* of *reverse* mode. One starts from the output and reads the code in the reverse order. At each line of code we associate a new line of code which computes the derivative of the output with respect to the intermediary variables of that line.

In each case, to be able to compute the derivative of each line of code, we need to have the derivatives $\partial g_j / \partial b_i$ ($-p_a \leq i < j$) available. In practice, this means that the function g_j is a known mathematical function, like exp, or a function previously built in our library for which we have already developed an algorithmic differentiation version. The algorithmic differentiation works satisfactorily only when it is built in the library from scratch. All the building blocks, like interpolators and Black formula, need to have an algorithmic differentiation version before starting with the more complex algorithms and models. There is no point in trying to implement the approach for the most time consuming algorithm if one has not already spent the time implementing the approach for all the fundamental blocks. This is a bottom-up approach.

The final result is the same in both cases: the derivatives of the output with respect to all the inputs. From the brief description above, one can guess the efficiency of the two procedures. In the forward mode the computation time is proportional to the number of inputs while in the reverse mode the computation time is proportional to the number of outputs, which is 1 in our problem. In general in finance, where one present value is computed from numerous inputs, the adjoint mode is significantly more efficient. We do not discuss the forward mode further here.

Let's look at the adjoint mode in more detail. Our goal is to compute $\partial z / \partial a_i$. We achieve that by computing for each j ($-p_a \leq j \leq p$) the value

$$\bar{b}[j] = \frac{\partial}{\partial b[j]} z.$$

Table C.2 The generic code for a function computing a single value and its algorithmic differentiation code

Init	$[j = -p_a : 0]$	$b[j]$	$= a[j + p_a]$
Algorithm	$[j = 1 : p_b]$	$b[j]$	$= g_j(b[-p_a : j-1])$
Value		z	$= b[p_b]$
Value		$\bar{z}$	$= 1.0$
Value		$\bar{b}[p_b]$	$= 1.0$
Algorithm	$[j = p_b - 1 : -1 : -p_a]$	$\bar{b}[j]$	$= \sum_{k=j+1}^{p_b} \dfrac{\partial}{\partial b_k} z \dfrac{\partial}{\partial b_j} b_k$
			$= \sum_{k=j+1}^{p_b} \bar{b}[k] \dfrac{\partial}{\partial b_j} g_k$
Init	$[i = 0 : p_a]$	$\dfrac{\partial}{\partial a_i} z$	$= \bar{b}[i - p_a + 1]$

The summary of the algorithm is presented in Table C.2. We start with the easy part. For $j = p_b$, the derivative of z with respect to b_j is simply the derivative of z with respect to itself, which is 1. This is the starting point of a recursive algorithm.

From there we read the code in the reverse order and use the derivative composition. Each intermediary variable $b[j]$ is used only in the lines that follow in the computation. The derivative $\bar{b}[j]$ is given by

$$\bar{b}[j] = \frac{\partial}{\partial b_j} z = \sum_{k=j+1}^{p_b} \frac{\partial}{\partial b_k} z \cdot \frac{\partial}{\partial b_j} b_k = \sum_{k=j+1}^{p_b} \bar{b}[k] \cdot \frac{\partial}{\partial b_j} g_k.$$

From the recursive approach, the values $\bar{b}[k]$ for $k > j$ are already known. The derivatives of the functions g_k can be obtained from the implementation of algorithmic differentiation for building blocks or as known mathematical functions. Using that approach one can recursively obtain the values $\bar{b}[j]$ down to $\bar{b}[j]$ for $j = -p_a, \ldots, 0$. These numbers are equal to the derivatives of z with respect to a_i ($i = 0, \ldots, p_a; j = i - p_a + 1$). This concludes the algorithm.

C.3.2 Example: curve calibration

The first example of computation efficiency concerns an indirect application of algorithmic differentiation. In the curve calibration, we use a multidimensional root-finding algorithm. The implementation uses the derivative of the function to iterate the point in the root-finding algorithm. The derivative can be computed through finite difference or explicitly by algorithmic differentiation. We compare the finite difference performance to the adjoint algorithmic differentiation performance.

The example refers to the building of three curves in EUR (discounting, forward three months and forward six months). We use OIS, fixed versus three months

Table C.3 Time to build curves with and without algorithmic differentiation

Algorithmic differentiation	Number of units		
	1	3	Ratio
No	109.1	41.5	2.63
Yes	33.3	20.2	1.65
Ratio	3.28	2.05	5.40

Time in milliseconds.

IRS and fixed versus six months IRS. The choice of instruments is done in such a way that the curves are not entangled and can be built one by one; no simultaneous building is required. With that choice, we can analyse two items: the interest of having a flexible implementation allowing for simultaneous or successive curve building and the algorithmic differentiation impact. Note that the algorithmic differentiation impact is indirect as the derivative of the objective function is only a small part of the computation.

The example uses three curves with 20 instruments each. The results are presented in Table C.3. When three units are used the ratio between the version without algorithmic differentiation and the version with algorithmic differentiation is 2.05; when the three curves are built simultaneously, the ratio is 3.28. In the most complex problems, the algorithmic differentiation, even if used indirectly, has a real impact.

The possibility of splitting the problem into several units is assessed by the last column. The ratio between one unit and three is 1.65 with algorithmic differentiation and 2.63 without. The computation of the derivative being very efficient and only slightly impacted by the number of inputs, the difference is not huge but far from trivial. In the absence of efficient derivative computation, the ratio is significantly larger (2.63). Finally, the last ratio is between the number with one unit and no algorithmic differentiation and with three units and algorithmic differentiation. The combination of these two numerical improvements reduces the computation time more than fivefold.

C.3.3 Bucketed delta computation

As described earlier, computing the risk of a book with respect to all the market quotes is a requirement of any implementation. The next item is to apply algorithmic differentiation to the multi-curve risk computation. We suggest doing so in three steps.

The first step is to move from the market quotes to the parameterised curves. The output of that step is obviously the parameterised curves, but also the Jacobian or transition matrices, as explained in Section C.2.

Listing C.5 Multicurve sensitivity implementation

```
public class MulticurveSensi {

  private Map<String, List<DoublesPair>> _sensiYieldDisc;

  private Map<String, List<ForwardSensi>> _sensiForward;

  MulticurveSensi(Map<String, List<DoublesPair>>
    sensiYieldDisc,
    Map<String, List<ForwardSensi>> sensiForward) {
    _sensiYieldDisc = sensiYieldDisc;
    _sensiForward = sensiForward;
  }

  public MulticurveSensi plus(MulticurveSensi other) {
    Map<String, List<DoublesPair>> resultDsc
      = SensiUtils.plus(_sensiYieldDisc, other._sensiYieldDisc);
    Map<String, List<ForwardSensi>> resultFwd
      = SensiUtils.plusFwd(_sensiForward, other._sensiForward);
    return new MulticurveSensi(resultDsc, resultFwd);
  }

  public MulticurveSensi multipliedBy(double factor) {
    final Map<String, List<DoublesPair>> resultDsc
      = SensiUtils.multipliedBy(_sensiYieldDisc, factor);
    final Map<String, List<ForwardSensitivity>> resultFwd
      = SensiUtils.multipliedByFwd(_sensiForward, factor);
    return new MulticurveSensi(resultDsc, resultFwd);
  }
}
```

The last step concerns the computation of the sensitivities of a specific instrument with respect to some curve information. The way we suggest to implement it is the following. For the discounting curve, the sensitivity is computed for each payment date (not only the dates used as node points if the curve is interpolated). This ensure that no information is lost at this stage. We compute the sensitivity with respect to each payment. Similarly, for each forward rate, the sensitivity to that forward rate is computed, again without any projection to any curve node. The information so obtained is larger than the information coming from the sensitivity to the curve parameters, so can be viewed as too large. But at this stage there is no reason to lose information on the exact dates of the sensitivity. Producing that information is important in some circumstances; one may want to check the sensitivity to a rate on a particular date.

In the implementation we create an object called MulticurveSensi that contains these sensitivities. The pseudo-code to the class is given in Listing C.5.

Listing C.6 Forward rate sensitivity object implementation

```
public class ForwardSensi {
  private double startTime;
  private double endTime;
  private double accrualFactor;
  private double value;
}
```

The sensitivity to the discounting curves is stored, for each curve represented by its name, as a List of DoublesPair. For each cash-flow date, represented by the first double, the sensitivity, represented by the second double, is stored. One needs a convention on how the sensitivity is represented. In our implementation, we decided to describe the sensitivity as the sensitivity with respect to the zero-coupon continuously compounded rate equivalent to the discount factor. To be fully compliant with the theory, we should have represented it as the sensitivity to the discount factor itself. But there, like in many other places, the *evolution* was stronger than the *foundation* and we kept the rate (or yield) as the storing base.

The sensitivity to the forward curve is stored, for each curve represented by its name, as a List of ForwardSensi. The last object is provided in Listing C.6. It simply collects the sensitivity to a particular forward rate with the rate itself described by its start time, end time and accrual factor. Again to be in line with the theory, it would be enough to store only the start time, as the end time and accrual factor are implicit from the description of the relevant index. But it is a lot more efficient to compute those numbers only once and store them than to have the curve recomputing them time and time again when required.

An example of the implementation of the curve sensitivity computation for an Ibor coupon is proposed in Listing C.7. Note that, like for the present value, the output is a multiple currency object, as the sensitivities can be with respect to several currencies. The sensitivities can also be in a currency different to the curve currency as described in Section 5.7.

In between these two steps lies the missing link. From the sensitivity to each cash-flow, and the curve description, we need to be able to compute the sensitivity to the curve parameters. This is a job for the curve itself to provide that sensitivity. The implementation will be strongly curve representation dependent. The sensitivity to a interpolated curve on continuously compounded rates will be different from a parameterised curve like Nelson-Siegel or a spread curve built from two curves interpolated on discount factors.

To achieve that result, the curve interface described in Listing C.1 is completed by the methods described in Listing C.8. For a given sensitivity (to discounting yield or forward rates), the method returns the sensitivity to the curve parameters. In this

Listing C.7 Curve sensitivity computation for a Ibor coupon

```
public
MultipleCurrencyMulticurveSensipresentValueCurveSensitivity(
    CouponIbor cpn, MulticurveProviderInterface multicurve) {
  double forward = multicurve.getForwardRate(cpn.getIndex(),
    cpn.getFixingPeriodStartTime(),
    cpn.getFixingPeriodEndTime(),
    cpn.getFixingAccrualFactor());
  double df = multicurve.getDiscountFactor(cpn.getCurrency(),
    cpn.getPaymentTime());
  double pv = cpn.getNotional() * cpn.getPaymentAccrualFactor()
    forward * df;
  // Backward sweep
  double pvBar = 1.0;
  double forwardBar = coupon.getNotional()
    * coupon.getPaymentYearFraction() * df * pvBar;
  double dfBar = coupon.getNotional()
    * coupon.getPaymentYearFraction() * forward * pvBar;
  Map<String, List<DoublesPair>> mapDsc = new Map<>();
  List<DoublesPair> listDiscounting = new List<>();
  listDiscounting.add(Pair.of(coupon.getPaymentTime(),
    -coupon.getPaymentTime() * df * dfBar));
  mapDsc.put(multicurve.getName(coupon.getCurrency()), listDiscounting);
  Map<String, List<ForwardSensitivity>> mapFwd = new Map<>();
  List<ForwardSensitivity> listFwd = new List<>();
  listFwd.add(new ForwardSensitivity(coupon.getFixingPeriodStartTime(),
    coupon.getFixingPeriodEndTime(), coupon.getFixingAccrualFactor(),
    forwardBar));
  mapFwd.put(multicurve.getName(coupon.getIndex()),
    listForward);
  return MultipleCurrencyMulticurveSensitivity.of(coupon.getCurrency(),
  MulticurveSensitivity.of(mapDsc, mapFwd));
  }
```

Listing C.8 Methods for the computation of parameters sensitivities

```
double[] parameterSensi(String name,
  List<DoublesPair> pointSensi);

double[] parameterForwardSensi(String name,
  List<ForwardSensi> pointSensi);
```

case, for each curve, the number of parameters is known and independent of the instrument. It can be returned as a double array with a known size.

Bibliography

Ametrano, F. and Bianchetti, M. (2009). *Modeling Interest Rate*, chapter Bootstrapping the illiquidity: Multiple yield curves construction for market coherent forward rates estimation. Risk Books, Incisive Media. 7, 22, 43, 51

Ametrano, F. and Bianchetti, M. (2013). Everything You Always Wanted to know About Multiple Interest Rate Curve Bootstrapping But Were Afraid to Ask. SSRN Working paper 2219548. Banca IMI/Banca Intesa Sanpaolo. 43, 51, 72, 145, 169

Andersen, L. and Piterbarg, V. (2010). *Interest Rate Modeling – Volume I: Foundations and Vanilla Models*. Atlantic Financial Press. 16, 33, 43, 46, 51, 52, 101, 103

Andersen, L. B. (2007). Discount curve construction with tension splines. *Review of Derivatives Research*, 10(3):227–67. 42, 43, 51, 52

Annaert, J., Claes, A., De Ceuster, M., and Zhang, H. (2012). Estimating the yield curve using the Nelson-Siegel model: A ridge regression approach. Working Paper Series 2054689, SSRN. 75

Baviera, R. and Cassaro, A. (2012). Mr. Crabs bootstrap. Technical report, Politecnico di Milano. 33

Bianchetti, M. (2010). Two curves, one price. *Risk*, 74–80. 7, 22

Bianchetti, M. and Carlicchi, M. (2011). Interest rates after the credit crunch: Multiple curve vanilla derivatives and sabr. Capco Journal of Financial Transformation, Applied Finance, 32, Capco. 7

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics*, 3(12):167–79. 113, 116

Boenkost, W. and Schmidt, W. (2004). Cross currency swap valuation. Working Paper 2, HfB – Business School of Finance & Management. 5, 6

Brigo, D. (2013). Personnal communication. 159

Brigo, D., Buescu, C., Pallavicini, A., and Liu, Q. (2012a). Illustrating a problem in the self-financing condition in two 2010–2011 papers on funding, collateral and discounting. Working Paper Series 2103121, SSRN. 145, 148, 153

Brigo, D., Capponi, A., and Pallavicini, A. (2012b). Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance*. To appear. 145

Brigo, D., Capponi, A., Pallavicini, A., and Papatheodorou, V. (2011). Collateral margining in arbitrage-free counterparty valuation adjustment including re-hypothecation and netting. Working paper series, SSRN. 145

Brigo, D. and Mercurio, F. (2006). *Interest Rate Models, Theory and Practice*. Springer Finance. Springer, second edition. 195, 196

Brigo, D., Morini, M., and Pallavicini, A. (2013). *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. Wiley Finance. Wiley. 11

Brody, D. C. and Hughston, L. P. (2004). Chaos and coherence: a new framework for interest-rate modelling. *Proc. R. Soc. Lond. A.*, 460:85–110. 191

Cantor, G. (1867). De aequationibus secundi gradus indeterminatis, PhD thesis.

Capriotti, L. (2011). Fast Greeks by algorithmic differentiation. *The Journal of Computational Finance*, 14(3):3–35. 225

Capriotti, L. and Giles, M. (2011). Algorithmic Differentiation: Adjoint Greeks Made Easy. *Risk*, 24(9):96–102. 225

Carver, L. (2013). Futures vs FRAs inversion baffles market participants. *Risk magazine*. 26

Castagna, A. (2012). Pricing of collateralized derivatives contracts when more than one currency are involved: Liquidity and funding value adjustments. Working Paper Series 2073300, SSRN. 145

Chen, R.-R. and Scott, L. (1993). Pring interest rate futures options with futures-style margining. *The Journal of Futures Market*, 13(1):15–22. 123, 134

Chibane, M., Selvaraj, J., and Sheldon, G. (2009). Building curves on a good basis. Technical report, Shinsei Bank. Available at SSRN: http://ssrn.com/abstract=1394267. 7, 22

Crépey, S. (2012). Bilateral counterparty risk under funding constraints – part I: Pricing. *Mathematical Finance*. doi: 10.1111/mafi.12004. 145, 148, 151, 154

Crépey, S. (2013). Lois: credit and liquidity. *Risk*, 26(6):78–82. 11

Duffie, D. (2001). *Dynamic Asset Pricing Theory*. Princeton University Press, third edition. 148, 153

Filipovic, D. and Trolle, A. (2013). The term structure of interbank risk. *Journal of Financial Economics*, 109(3):707–33. 11

Fries, C. (2013). Curves and term structure models. definition, calibration and application of rate curves and term structure market models. Working papers series 2194907, SSRN. 31, 51, 146

Fruchard, E., Zammouri, C., and Willems, E. (1995). Basis for change. *Risk Magazine*, 8(10):70–5. 5

Fujii, M. (2013). Personnal communication. 183

Fujii, M., Shimada, Y., and Takahashi, A. (2010a). A note on Construction of Multiple Swap curves with and without Collateral. Working paper, SSRN. 145, 167

Fujii, M., Shimada, Y., and Takahashi, A. (2010b). Collateral Posting and Choice of Collateral Currency. Working Paper CARF-F-216, CARF. 145

Fujii, M., Shimada, Y., and Takahashi, A. (2011). A Market Model of Interest Rates with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies. *Wilmott Magazine*, 54:61–73. 180

Fujii, M. and Takahashi, A. (2010). Choice of Collateral Currency. Working Paper CARF-F-239, CARF. 145

Fujii, M. and Takahashi, A. (2013). *Interest Rate Modelling After The Financial Crisis*, chapter Interest Rate Modelling under Full Collateralisation, 241–82. Risk. 145, 151, 164, 180

Gątarek, D. (2005). Nonparametric calibration of forward rate models. Technical report, NumeriX. 110

Giles, M. and Glasserman, P. (2006). Smoking adjoints: fast Monte Carlo Greeks. *Risk*, 19:88–92. 225

Griewank, A. and Walther, A. (2008). *Evaluating derivatives: principles and techniques of algorithmic differentiation*. SIAM, second edition. 224, 225

Hagan, P., Kumar, D., Lesniewski, A., and Woodward, D. (2002). Managing smile risk. *Wilmott Magazine*, Sep:84–108. 114

Hagan, P. S. and West, G. (2006). Interpolation methods for curve construction. *Applied Mathematical Finance*, 13(2):89–129. 36, 42, 43, 51

Hamilton, J. D. (1996). The Daily Market for Fed Funds. *Journal of Political Economy*, 114(1):26–56. 72

Han, M., He, Y., and Zhang, H. (2013). A note on discounting and funding value adjustment for derivatives. Working Paper Series 2220371, SSRN. 145, 150

Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105. 180, 190

Henrard, M. (2003). Explicit bond option and swaption formula in Heath-Jarrow-Morton one-factor model. *International Journal of Theoretical and Applied Finance*, 6(1):57–72. 111, 141

Henrard, M. (2004). Overnight Indexed Swaps and Floored Compounded Instrument in HJM one-factor model. Ewp-fin 0402008, Economics Working Paper Archive. 26, 84

Henrard, M. (2005). Eurodollar Futures and Options: Convexity Adjustment in HJM One-Factor Model. Working paper 682343, SSRN. Available at ssrn.com/abstract=682343. 24, 137, 184

Henrard, M. (2006a). A semi-explicit approach to Canary swaptions in HJM one-factor model. *Applied Mathematical Finance*, 13(1):1–18. Preprint available at IDEAS: http://ideas.repec.org/p/wpa/wuwpfi/0310008.html. 191

Henrard, M. (2006b). Bonds futures and their options: more than the cheapest-to-deliver; quality option and margining. *Journal of Fixed Income*, 16(2):62–75. 97, 147

Henrard, M. (2007). The Irony in the derivatives discounting. *Wilmott Magazine*, 30:92–8. 6, 20, 23, 24, 149

Henrard, M. (2010). The Irony in the derivatives discounting – Part II: the crisis. *Wilmott Journal*, 2(6):301–16. 8, 107, 165, 166

Henrard, M. (2012a). Adjoint Algorithmic Differentiation: Calibration and Implicit Function Theorem. *Journal of Computational Finance*, to appear. 57

Henrard, M. (2012b). Deliverable Interest Rate Swap Futures: pricing in Gaussian HJM model. Quantitative Research 5, OpenGamma. Available at docs.opengamma.com. 96

Henrard, M. (2012c). Multi-curves: Variations on a Theme. Quantitative Research 6, OpenGamma. Available at docs.opengamma.com. 9, 31

Henrard, M. (2012d). My future is not convex. Quantitative Research 3, OpenGamma. Available at ssrn.com/abstract=2053657. 149

Henrard, M. (2013a). *Interest Rate Modelling After The Financial Crisis*, chapter Irony in derivative discounting: after the crisis. Risk. 8, 166, 169

Henrard, M. (2013b). Multi-curve Framework with Collateral. Quantitative Research 13, OpenGamma. Available at docs.opengamma.com. 10

Henrard, M. (2013c). Multi-curves Framework with Stochastic Spread: A Coherent Approach to STIR Futures and Their Options. Quantitative Research 11, OpenGamma. Available at docs.opengamma.com. 137, 145, 184, 185, 186

Hull, J. and White, A. (1990). Pricing interest rate derivatives securities. *The Review of Financial Studies*, 3:573–92. 194

Hull, J. C. (2006). *Options, futures, and other derivatives*. Prentice Hall, sixth edition. 4

Hunt, P. J. and Kennedy, J. E. (2004). *Financial Derivatives in Theory and Practice*. Wiley series in probability and statistics. Wiley, second edition. 16, 25, 37, 97, 190, 191

Iwashita, Y. (2013). Piecewise Polynomial Interpolations. Technical report, OpenGamma. 42

Jäckel, P. and Kawai, A. (2005). The future is convex. *Wilmott Magazine*, 1–13. 24

Jerassy-Etzion, Y. (2010). *Stripping The Yield Curve With Maximally Smooth Forward Curves*. PhD thesis, Florida State University.

Johannes, M. and Sundaresan, S. (2007). The Impact of Collateralization on Swap Rates. *The Journal of Finance*, LXII(1):383–410. 145, 149

Kennedy, G. J. (2010). Swap Futures in HJM One-Factor Models. Working Paper Series 1648419, SSRN. Available at : http://ssrn.com/abstract=1648419. 97

Kenyon, C. and Stamm, R. (2012). *Discounting, LIBOR, CVA and Funding: Interest Rate and Credit Pricing*. Palgrave Macmillan. 51

Kijima, M., Tanaka, K., and Wong, T. (2009). A multi-quality model of interest rates. *Quantitative Finance*, 133–45. 7, 9, 10, 24, 145

Kirikos, G. and Novak, D. (1997). Convexity conundrums. *Risk*, 60–1. 24

Lamberton, D. and Lapeyre, B. (1997). *Introduction au calcul stochastique appliqué à la finance*. Ellipses. 181

Le Floc'h, F. (2012). Stable interpolation for the yield curve. Working Paper Series 2175002, SSRN. 42, 224

Macey, G. (2011). Pricing with Standard CSA Defined by Currency Buckets. Working Paper SSRN 1945769, HSBC. 10, 145, 146, 151, 152, 167

Mawhin, J. (1997). *Analyse: Fondements, Techniques, Evolution*. De Boek. 57

McCloud, P. (2013a). Collateral convexity complexity. *Risk Magazine*, 26(5):60–4. 145, 165

McCloud, P. (2013b). Collateral convexity of Libor and FX forwards. Presentation at 2nd interest rate conference – WBS, Nomura. 145

McCloud, P. (2013c). Collateral volatility. Working papers series 2260480, SSRN. 163, 165

Mengle, D. (2009). *Alternative compounding methods for over-the-counter derivative transactions*. ISDA. 207

Mercurio, F. (2009). Interest rates and the credit crunch: new formulas and market models. Technical report, QFR, Bloomberg. 7, 22, 24

Mercurio, F. (2010a). A LIBOR Market Model with Stochastic Basis. *Risk*, 23(12):84–9. 7, 24

Mercurio, F. (2010b). A LIBOR Market Model with Stochastic Basis. Working paper., Bloomberg L.P. Available at: http://ssrn.com/abstract=1583081. 25

Mercurio, F. (2010c). Moderne Libor Market Model: using different curves for projectiong rates and for discounting. *International Journal of Theoretical and Applied Finance*, 13(1):113–37. 7, 142

Mercurio, F. and Xie, Z. (2012). The Basis Goes Stochastic. *Risk*, 25(12):78–83. 25, 126, 127

Moreni, N. and Pallavicini, A. (2010). Parsimonious HJM Modelling for Multiple Yield-Curve Dynamics. Working paper, SSRN. Available at http://ssrn.com/abstract=1699300. 7, 118

Morini, M. (2009). Solving the puzzle in the interest rate market. Working paper, IMI Bank Intesa San Paolo. Available at SSRN: http://ssrn.com/abstract=1506046. 7, 11

Morini, M. (2011). *Model Risk. A practical Guide for Quants, Traders and Validors*. Wiley Finance. 5

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *The journal of Business*, 60(4):473–89. 75

Pallavicini, A. (2013). Pricing of interest-rate derivatives cleared by a CCP. In *The 9th Fixed Income Conference*. WBS. 149

Pallavicini, A. and Brigo, D. (2013). Interest-rate modelling in collateralized markets: multiple curves, credit-liquidity effects, CCPs. Working paper series, SSRN. 145, 146, 149, 156, 180

Pallavicini, A., Perini, D., and Brigo, D. (2011). Funding valuation adjustment: FVA consistent with CVA, DVA, WWA, collateral, netting and re-hypothecation. Working paper series, SSRN. 145

Pallavicini, A., Perini, D., and Brigo, D. (2012). Funding, Collateral and Hedging: uncovering the mechanics and the subtleties of funding valuation adjustments. Working Paper Series 2161528, SSRN. 10, 145, 151, 164

Piterbarg, V. (2010). Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, 23(2):97–102. 7, 145

Piterbarg, V. (2012). Cooking with collateral. *Risk*, 25(8):58–63. 10, 145, 146, 147, 149, 151, 152

Piterbarg, V. and Renedo, M. (2004). Eurodollar futures convexity adjustments in stochastic volatility models. Working Paper 610223, SSRN. Available at SSRN: http://ssrn.com/abstract=610223. 24

Pozdnyakov, V. and Steele, M. (2009). Convexity bias in eurodollar futures prices: a dimesion-free HJM criterion. *Methodology and Computing in Applied Probability*, 11(4):551–60. 26

Prati, A., Bartolini, L., and Bertola, G. (2003). The overnight interbank market: Evidence from the G-7 and the euro zone. *Journal of Banking & Finance,* 27(10):2045–83. 72

Pulman, C. (2007). The Short-End of the Curve. Fixed income research, Lehman Brothers. 76

Quantitative Research (2012a). Compounded Swaps in multi-curves framework. OpenGamma Documentation 19, OpenGamma. Version 1.1. Available at http://docs.opengamma.com/display/DOC/Quantitative+Documentation. 167

Quantitative Research (2012b). Interest Rate Futures and their options: Some Pricing Approaches. Documentation 6, OpenGamma. Version 1.5. Available at http://docs.opengamma.com/display/DOC/Quantitative+Documentation. 137

Quantitative Research (2012c). *Interest Rate Instruments and Market Conventions Guide*. OpenGamma, 1.0 edition. Available at http://docs.opengamma.com/display/DOC/Analytics. 1, 198, 214

Quantitative Research (2012d). Overnight Indexed Swaps in multi-curves framework. OG Notes, OpenGamma. Available at http://docs.opengamma.com/display/DOC/Analytics. 26

Quantitative Research (2013). Curve Calibration in Practice: Requirements and Nice-to-haves. Documentation 20, OpenGamma. Version 1.3. 171

Rebonato, R., Mckay, K., and White, R. (2009). *The SABR Libor Market Model*. Wiley. 114

Ron, U. (2000). A Practical Guide to Swap Curve Construction. Working Paper 2000-17, Bank of Canada. 51

Sadr, A. (2009). *Interest Rate Swaps and Their Derivatives: a practitioner's guide*. Wiley Finance. 4, 60, 223

Suisse, C. (2001). Overnight Indexed Swap. European strategy and trades, Credit Suisse. 208

Takada, K. (2011). Valuation of Arithmetic Average of Fed Funds Rates and Construction of the US dollar Swap Yield Curve. Working Paper Series 1981668, SSRN. 91

Taleb, N. N. (2012). *Antigragile: How to Live in a World We Don't Understand.* Allen Lane. 3

Tee, C. W. (2013). Personnal communication. 169

Traven, S. (2010). Pricing and Hedging Linear Derivatives with an Arbitrage-Free Set of Interest Rate Curves. Technical report, Barclays Capital. 7, 223

Tuckman, B. and Porfirio, P. (2003). Interest rate parity, money market basis swaps and cross-currency basis swaps. Fixed income liquid markets research, Lehman Brothers. 5, 6

West, G. (2011). Post-Crisis methods for bootstrapping the yield curve. Technical report, Financial Modelling Agency, South Africa. 74, 208

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