Homework 03 - STAT440

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```
set.seed(42)
```

Problem 1

In this problem we will use simulation to explore the accuracy of confidence intervals. Consider a sample of n i.i.d. normal random variables with mean μ and standard deviation σ. Assume μ is unknown, but σ is known.

Part a

Generate such a sample $\mu = 1$ and $\sigma = 0.5$ for the cases $n = \{10, 100, 1000\}$. In reality this step would be done by nature, but here we can simulate it ourselves.

```
n <- c(10, 100, 1000)
samples <- vector(mode = "list", length = length(n))

mu <- 1
sigma <- 0.5

for (i in 1:length(n)) {
  samples[[i]] <- rnorm(n[i], mean = mu, sd = sigma)
}
```

Part b

Build a 95% confidence interval for each of these samples. Remember σ is assumed to be known but not μ.

Recall that when σ is known we can create our confidence interval for μ by using a transformation to the standard normal variable resulting in:

$$[\overline{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]$$

Here we use $\alpha = 0.05$, so $\frac{\alpha}{2} = 0.025$. $z_{0.025} = 1.96$.

```
z_multi <- 1.96
intervals <- vector("list", length(n))
```

```r
for (i in 1:length(n)) {
  x_bar <- mean(samples[[i]])
  diff <- z_multi * sigma / sqrt(n[i])
  intervals[[i]] <- c(x_bar - diff, x_bar + diff)
}
```

Part c

```r
num <- 1000
n <- 100
diff <- z_multi * sigma / sqrt(n)
count_in <- 0

for (i in 1:num) {
  sample <- rnorm(n, mean = mu, sd = sigma)
  x_bar <- mean(sample)
  if(x_bar - diff < mu & x_bar + diff > mu) {
    count_in <- count_in + 1
  }
}
print(count_in)
```

```
## [1] 952
```

Out of 1000 simulations of sampling 100 values from the normal distribution with $\mu = 1$ and $\sigma = 0.5$, we get μ in our 95% confidence interval 952 times, or 95.2% of the time. This makes perfect sense from the definition of a confidence interval, which states that for a given confidence level, the level represents the frequency with which a confidence interval at that level should contain the given parameter. In plain English this means that for a confidence interval at x% confidence, we should expect x% of the confidence intervals constructed at that level to contain the parameter.

Part d

If the parameter σ was not known, then we would have to use the student's t distribution to calculate the confidence interval.

Problem 2

For this problem we will use the matrix D from the previous homework.

```r
mat_v <- matrix(c(1, 2, 3, 2, 1, 0, 4, -4, 1, -1, 1, -1, 0, 3, 5), nrow = 3, ncol=5)

l2_norm <- function(vec) {
  sqrt(sum(vec^2))
}

num_cols <- dim(mat_v)[2]
mat_d <- matrix(1:25, nrow = num_cols, ncol = num_cols)
```

```
for (i in 1:num_cols) {
  for (j in 1:num_cols) {
    mat_d[i, j] <- l2_norm(mat_v[,i] - mat_v[,j])
  }
}
mat_d
```

```
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,] 0.000000 3.316625 7.000000 4.582576 2.449490
## [2,] 3.316625 0.000000 5.477226 3.162278 5.744563
## [3,] 7.000000 5.477226 0.000000 7.348469 9.000000
## [4,] 4.582576 3.162278 7.348469 0.000000 6.403124
## [5,] 2.449490 5.744563 9.000000 6.403124 0.000000
```

Part a

Create a matrix Σ whose ij^{th} element is $exp(-\tau D_{ij}^2)$ for $\tau = \frac{1}{20}$.

```
tau <- 1 / 20
mat_sig <- matrix(1:25, nrow = num_cols, ncol = num_cols)
mat_d_square <- mat_d %*% mat_d

for (i in 1:num_cols) {
  for (j in 1:num_cols) {
    mat_sig[i, j] <- exp(-1 * tau * mat_d_square[i, j])
  }
}
mat_sig
```

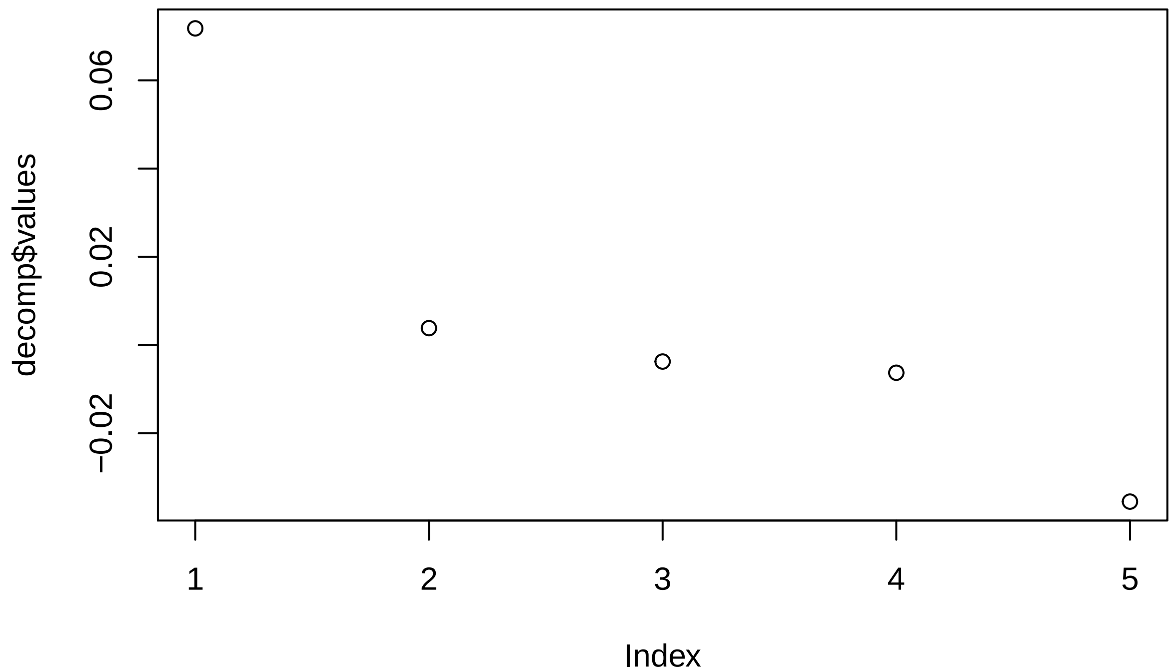
```
##             [,1]        [,2]         [,3]        [,4]         [,5]
## [1,] 0.012906813 0.035254569 2.486457e-02 0.020638686 0.0038113645
## [2,] 0.035254569 0.014995577 7.388970e-03 0.009936377 0.0205812201
## [3,] 0.024864572 0.007388970 2.254494e-05 0.004741827 0.0083692354
## [4,] 0.020638686 0.009936377 4.741827e-03 0.001836305 0.0084266593
## [5,] 0.003811364 0.020581220 8.369235e-03 0.008426659 0.0003191019
```

Part b

Compute the eigen-decomposition of Σ and plot the eigen values (sorted from largest to smallest).

```
decomp <- eigen(mat_sig)

plot(decomp$values)
```



Part c

Part d

Problem 3

Part a

Part b

Problem 4

Problem 5

Part a

Part b