Homework 07 - STAT440

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10/18/2020

```
set.seed(42)
```

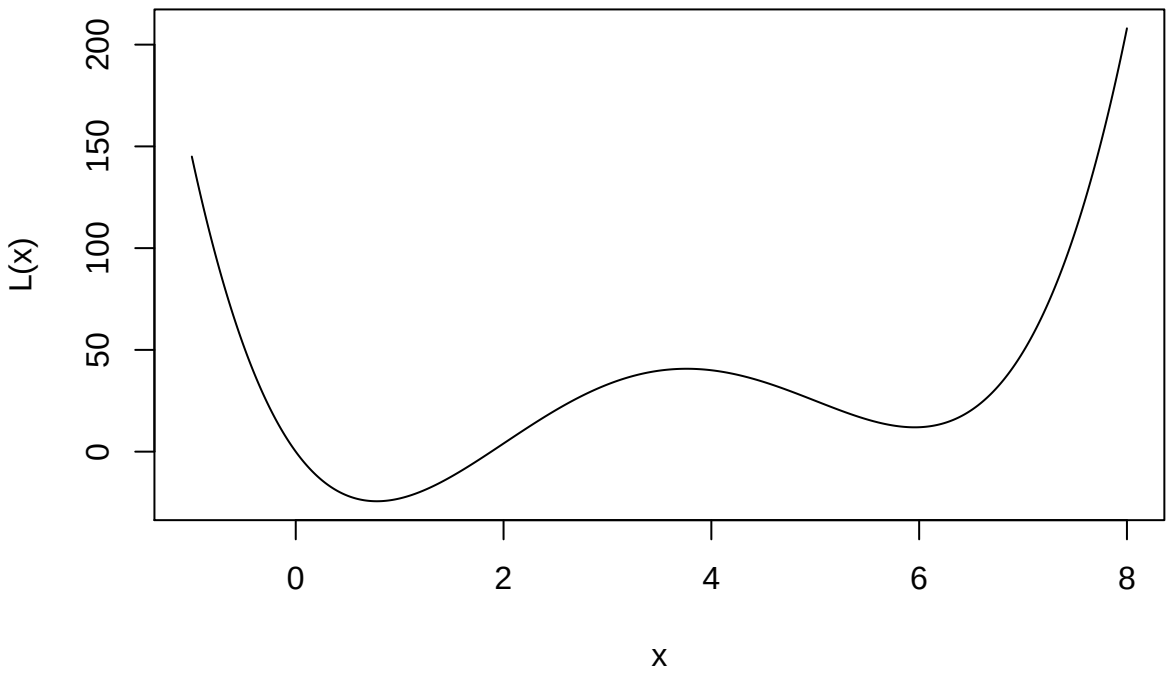
Problem 1

Define loss function in R:

```
L <- function(x) {
    x^4 - 14*x^3 + 60*x^2 - 70*x
}
```

Part a

```
# plot function
x <- seq(-1, 8, 0.001)
plot(x, L(x), type="l")
```



The function is not convex. You cannot draw a line between any two points on the line that remains above the line. The bump in the middle would prevent the point at x = 0 to a point at x = 6 from being above the function. (The second derivative is not always positive.) There are regions where it is locally convex. You could split the function approximately near where x take the value just below four and each side of the function (left and right) would be convex.

Part b

I would expect both Nelder-Mead and Newton's method to find local extrema. Since we are going to be looking for minimums I would expect both to either find the actual global minimum on the left or the local minimum on the right. The found minimum would depend on the initialization of the method. For example, if we started Newton's Method at x = 7 we would expect to find the suboptimal solution as opposed to started at x = 0, where would find the optimal solution.

Part c

```
example_nm <- function(par=c(0, 2), fn = L, return_points=FALSE) {
    # parameters
    alpha <- 1
    gamma <- 2
    rho <- 0.5
    sigma <- 0.5
```

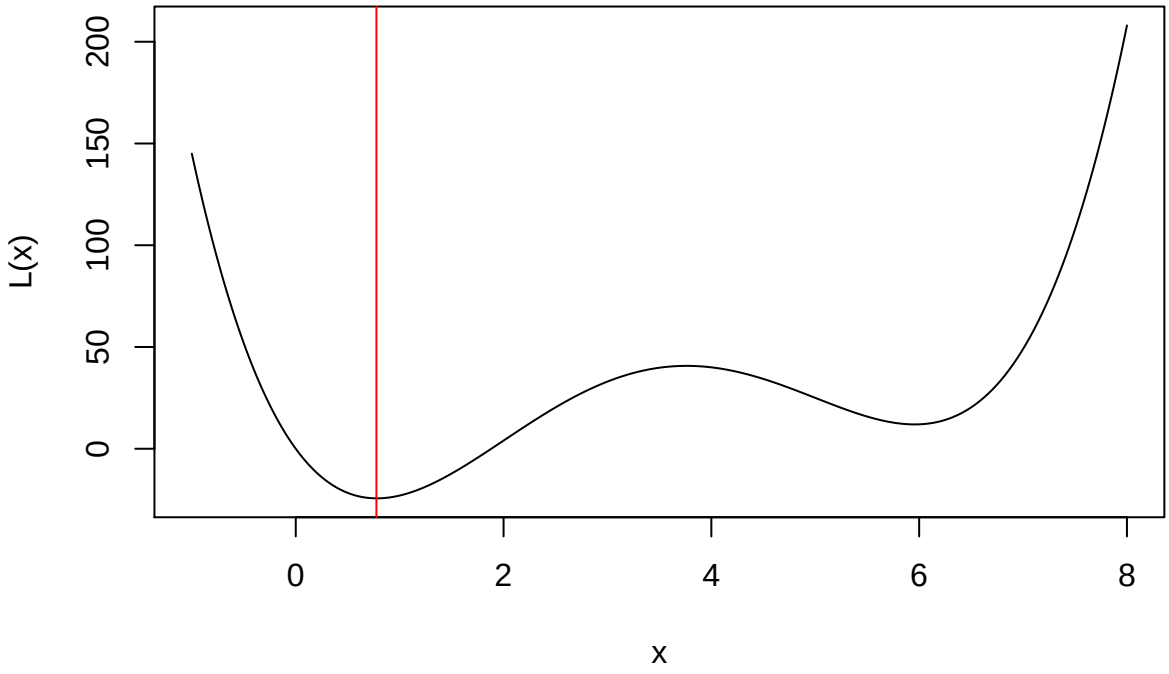
```
    # termination criteria
    max_term <- 100
    sd_threshold <- 0.01

    dim <- 2
    points <- runif(dim, min=par[1], max=par[2])

    point_mat <- NA

    iter <- 1
    while(sd(points) >= sd_threshold & iter <= max_term) {
        # Order points
        points <- points[order(L(points))]
        if (return_points) {
            if (iter == 1) {
                point_mat <- matrix(points, nrow=1, ncol=dim)
            } else {
                point_mat <- rbind(point_mat, matrix(points, nrow=1, ncol=dim))
            }
        }

        # Compute centroid - this is merely the first point (2 points only)
        centroid <- points[1:dim-1] / (dim-1)
        # Reflect about centroid
        reflected_point <- centroid +  alpha * (centroid - points[dim])
        val_r <- L(reflected_point)
        # don't need to check reflected
        # criteria; if better than x_n but not better than x_1; isn't possible
        # since x_1 = x_n here

        # Expand step
        if (val_r < L(points[1])) {
            expanded <- centroid + gamma * (reflected_point - centroid)
            if (L(expanded) < val_r) {
                points[dim] <- expanded
            } else {
                points[dim] <- reflected_point
            }
        } else {
            contracted <- centroid + rho * (reflected_point - centroid)
            if (L(contracted) < L(points[dim])) {
                # contract
                points[dim] <- contracted
            } else {
                # shrink
                points <- points[1] + sigma * (points - points[1])
            }
        }
        # next iteration
        iter <- iter + 1
    }
    if (return_points) {
        return(point_mat)
```

```
    }
    return(points[1])
}
```

```
print(example_nm())
```

```
## [1] 0.7829517
```

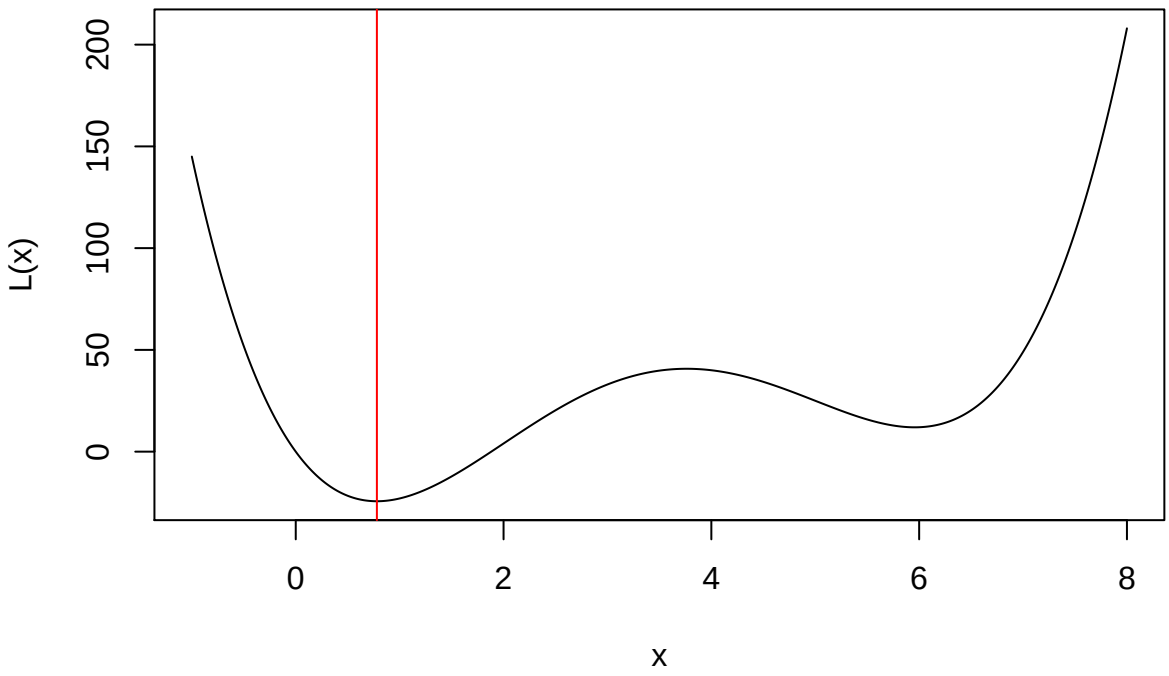
```
plot(x, L(x), type="l")
abline(v=example_nm(), col="red")
```




```
my_loss <- function(x) {
    L(x)[1]
}
val <- optim(par = c(0, 2), fn = my_loss, method="Nelder-Mead")$par[1]
print(val)
```

```
## [1] 0.7808841
```

```
plot(x, L(x), type="l")
abline(v=val, col="red")
```

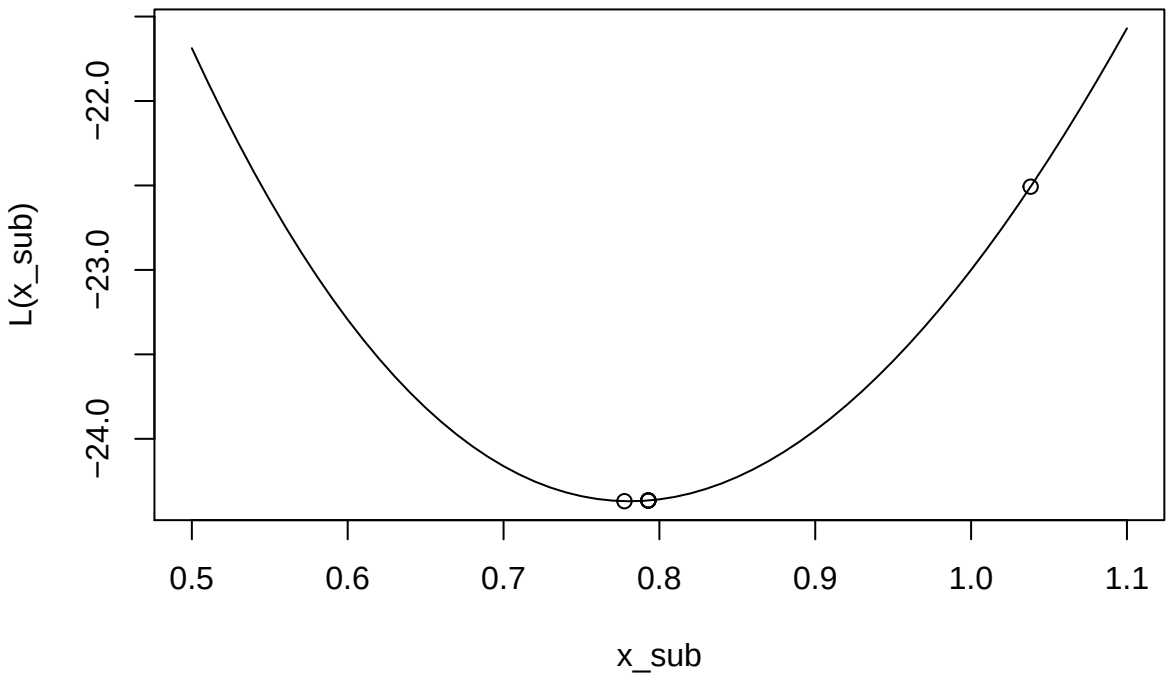


Part d

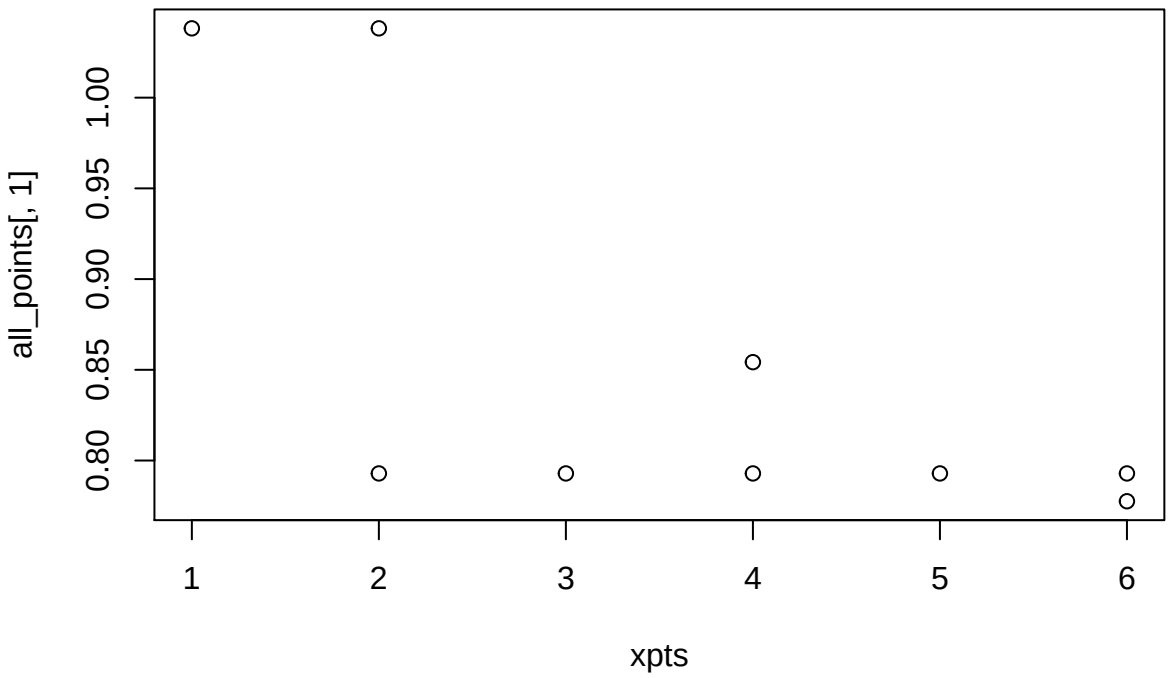
```
all_points <- example_nm(return_points = TRUE)
best_points <- all_points[,1]
print(best_points)
```

```
## [1] 1.0381919 0.7928928 0.7928928 0.7928928 0.7928928 0.7775616
```

```
x_sub <- seq(0.5, 1.1, 0.01)
plot(x_sub, L(x_sub), type="l")
points(best_points, L(best_points))
```




```
xpts <- 1:length(best_points)
plot(xpts, all_points[,1])
points(xpts, all_points[,2])
```

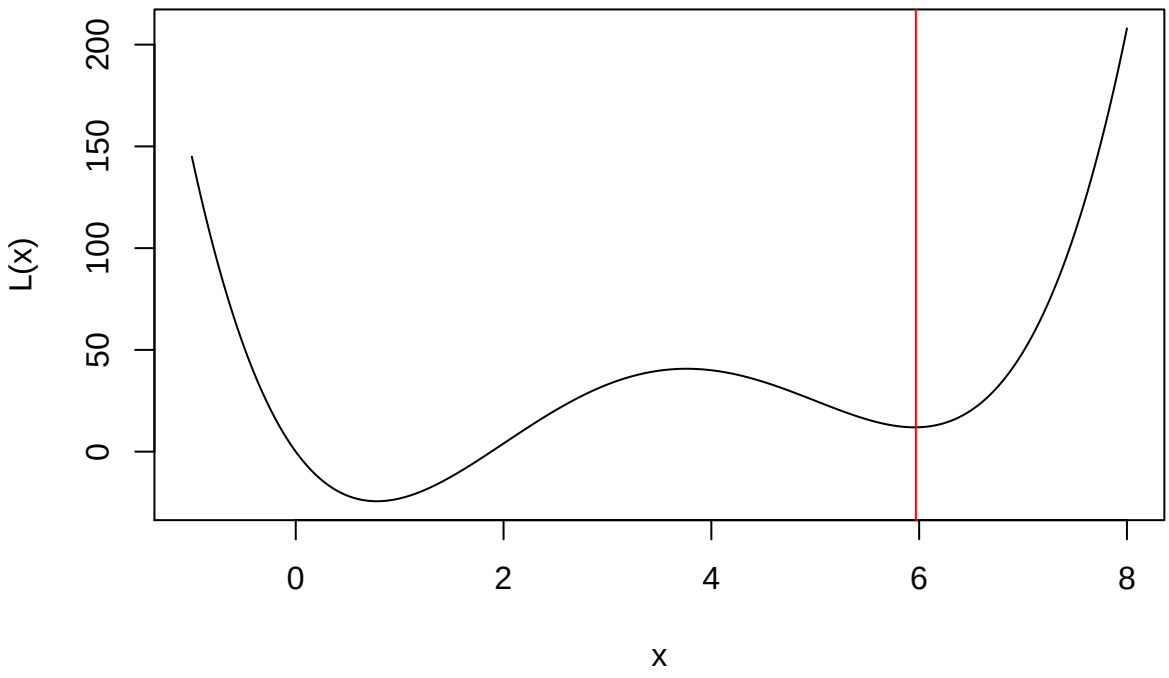


Part e

```
print(example_nm(par=c(6,8)))
```

```
## [1] 5.949562
```

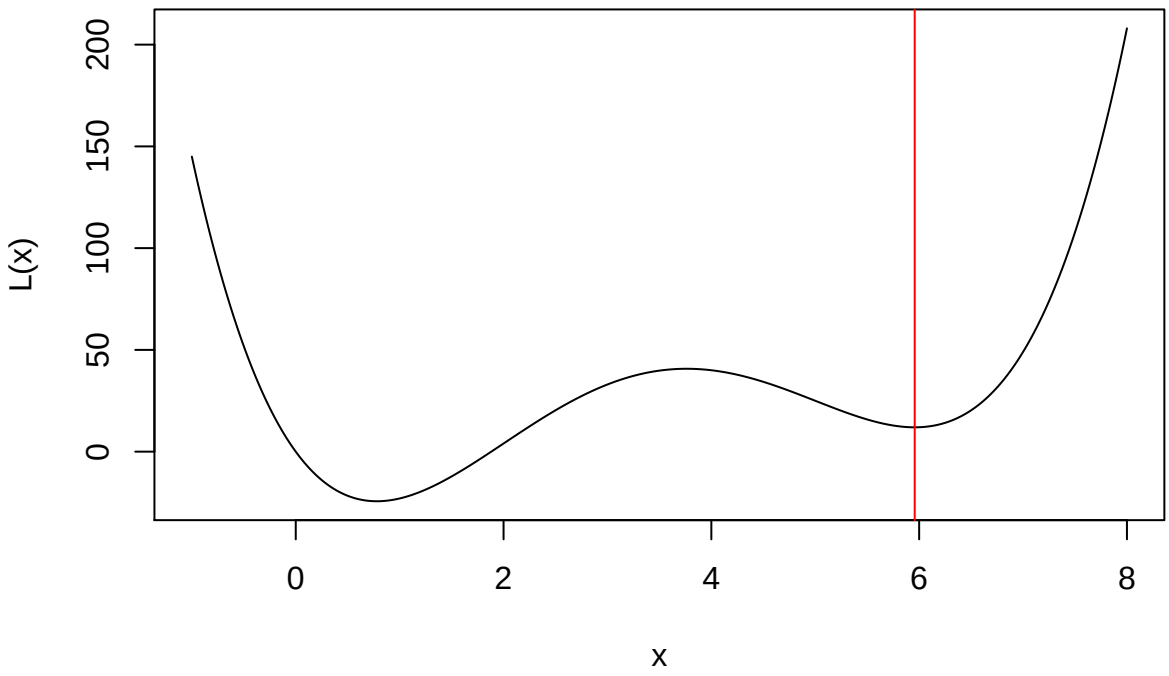
```
plot(x, L(x), type="l")
abline(v=example_nm(par=c(6,8)), col="red")
```




```
my_loss <- function(x) {
    L(x)[1]
}
val <- optim(par = c(6, 8), fn = my_loss, method="Nelder-Mead")$par[1]
print(val)
```

```
## [1] 5.957156
```

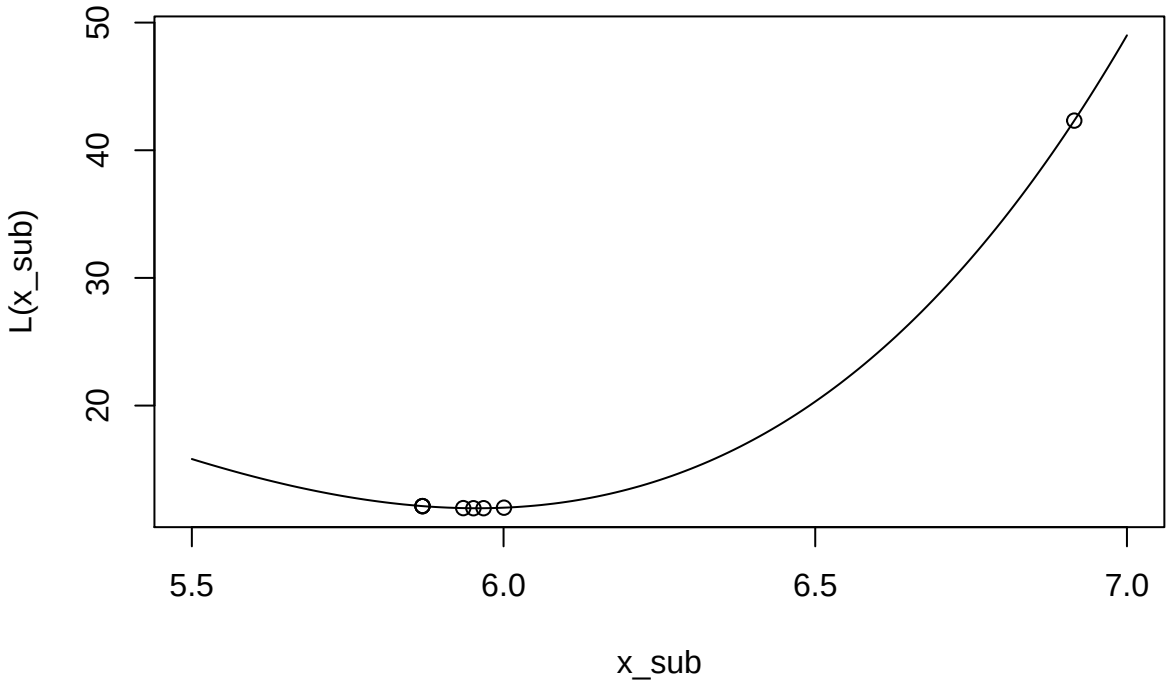
```
plot(x, L(x), type="l")
abline(v=val, col="red")
```



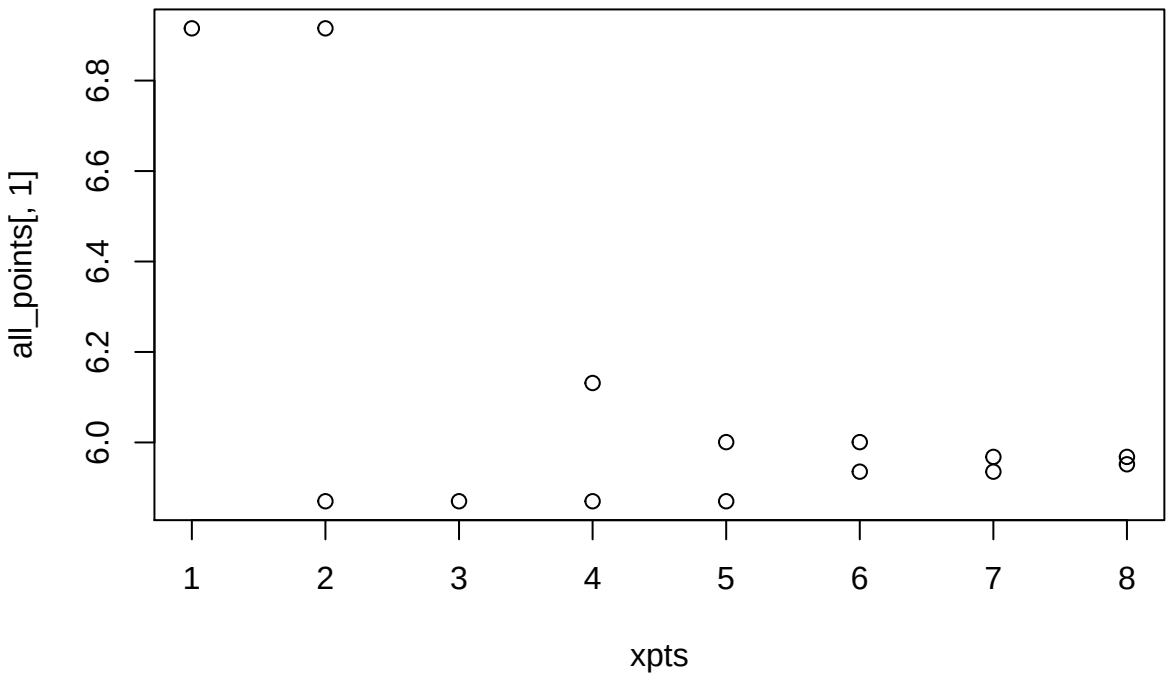

```
all_points <- example_nm(par = c(6,8), return_points = TRUE)
best_points <- all_points[,1]
print(best_points)
```

```
## [1] 6.915484 5.870002 5.870002 5.870002 6.000687 5.935344 5.968016 5.951680
```

```
x_sub <- seq(5.5, 7, 0.01)
plot(x_sub, L(x_sub), type="l")
points(best_points, L(best_points))
```




```
xpts <- 1:length(best_points)
plot(xpts, all_points[,1])
points(xpts, all_points[,2])
```



The results are what you would expect. The algorithm got closer and closer the the minimum it was near. In this part where we started on the far right of the function we go the suboptimal minimum, because the algorithm doesn't reflect/expand far enough to overcome this locality. We can see it starts to go to the other side of the local minimum here on step 3, but the loss function increases so the points come back toward the actual minimum.

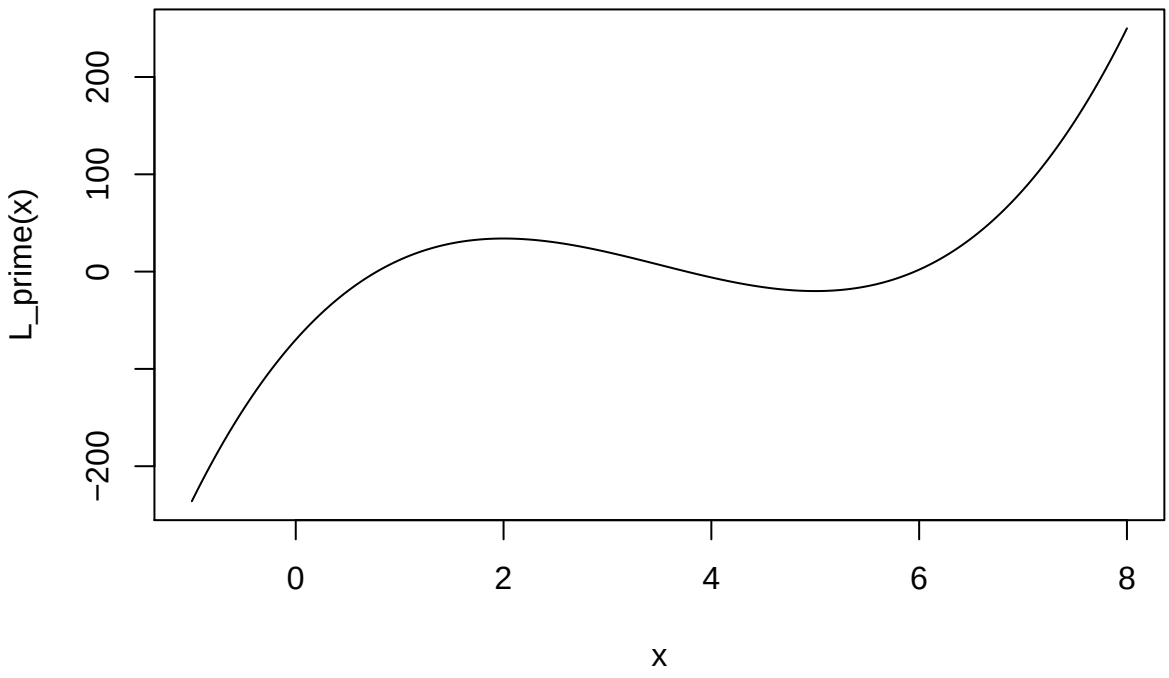
Problem 2

For Newton's Method we need the derivative of our loss function as well:

```
L_prime <- function(x) {
    4*x^3 - 14*3*x^2 + 120*x - 70
}

L_prime_2 <- function(x) {
    12*x^2 - 14*6*x + 120
}
```

If we plot this function we can see the two minima (one local, one global) where the derivative is equal to zero twice.

```
# plot function
x <- seq(-1, 8, 0.001)
plot(x, L_prime(x), type="l")
```



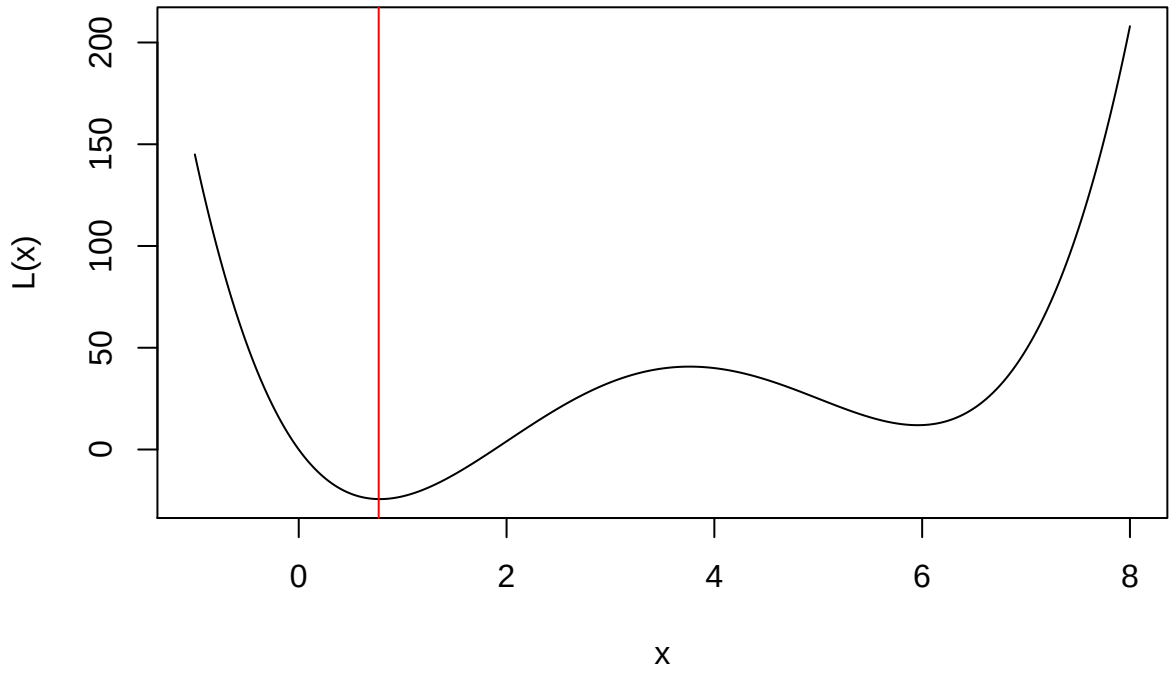

```
example_newton <- function(init_area=c(0, 2), return_points=FALSE) {
    # termination criteria
    diff_threshold <- 0.01

    point <- runif(1, init_area[1], init_area[2])
    point_mat <- NA

    diff <- 1
    iter <- 1
    while(diff > diff_threshold) {
        # store iteration
          if (return_points) {
              if (iter == 1) {
                  points <- c(point)
                  point_mat <- matrix(points, nrow=1, ncol=1)
              } else {
                  points <- c(point)
                  point_mat <- rbind(point_mat, matrix(points, nrow=1, ncol=1))
              }
          }

        point_new <- point - L_prime(point) / L_prime_2(point)
        diff <- abs(L(point) - L(point_new))
        point <- point_new
        iter <- iter + 1
    }
```

```
    if (return_points) {
        return(point_mat)
    }
    return(point)
}
```

```
print(example_newton())
```

```
## [1] 0.7808703
```

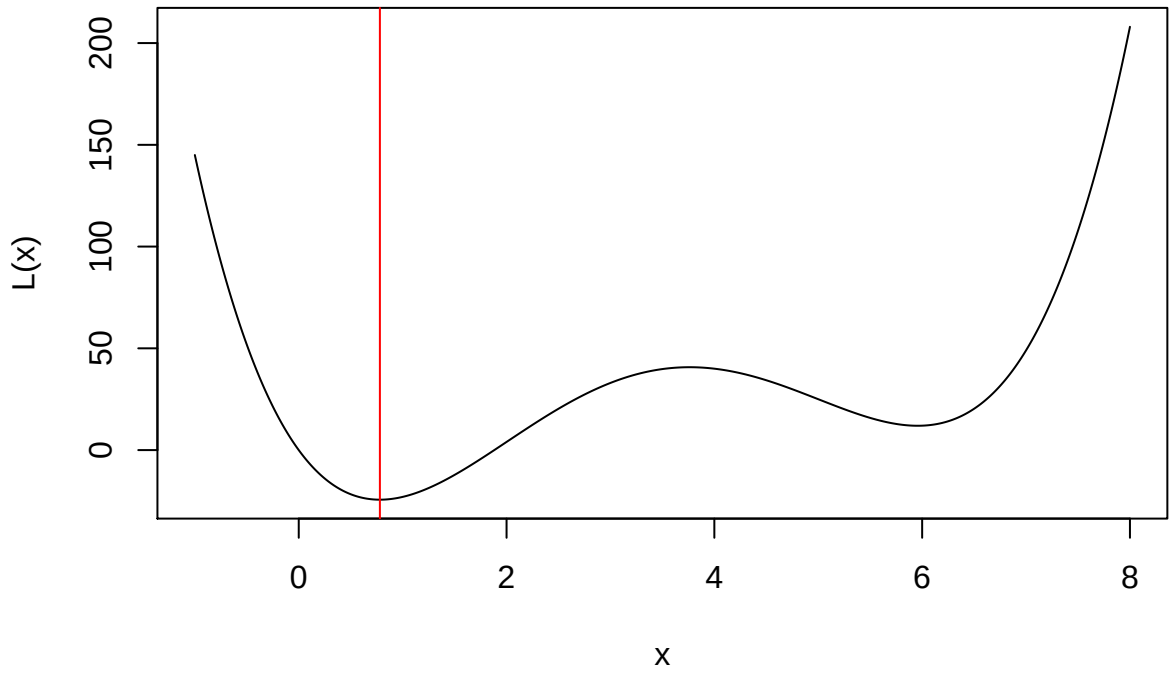
```
plot(x, L(x), type="l")
abline(v=example_nm(), col="red")
```




```
my_loss <- function(x) {
    L(x)[1]
}
val <- optim(par = c(0, 2), fn = my_loss, method="Nelder-Mead")$par[1]
print(val)
```

```
## [1] 0.7808841
```

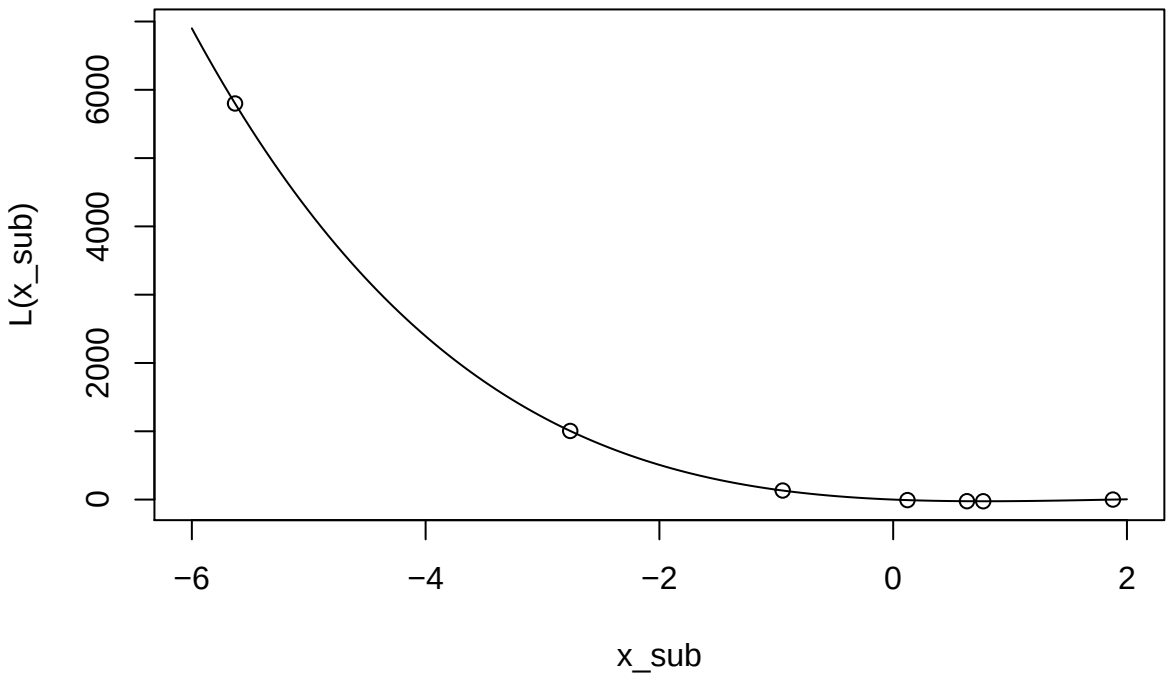
```r
plot(x, L(x), type="l")
abline(v=val, col="red")
```



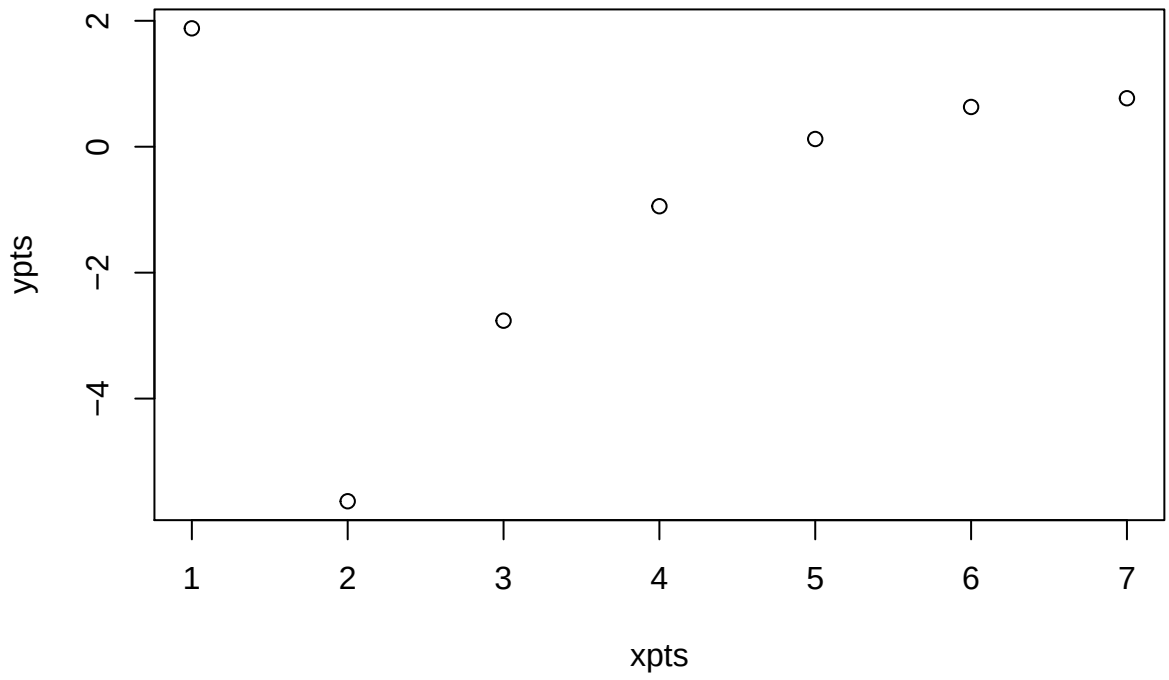

```r
all_points <- example_newton(return_points = TRUE)
best_points <- all_points[,1]
print(best_points)
```

```
## [1]  1.8800290 -5.6303208 -2.7629171 -0.9451346  0.1224503  0.6313800  0.7703465
```

```r
x_sub <- seq(-6, 2, 0.01)
plot(x_sub, L(x_sub), type="l")
points(best_points, L(best_points))
```



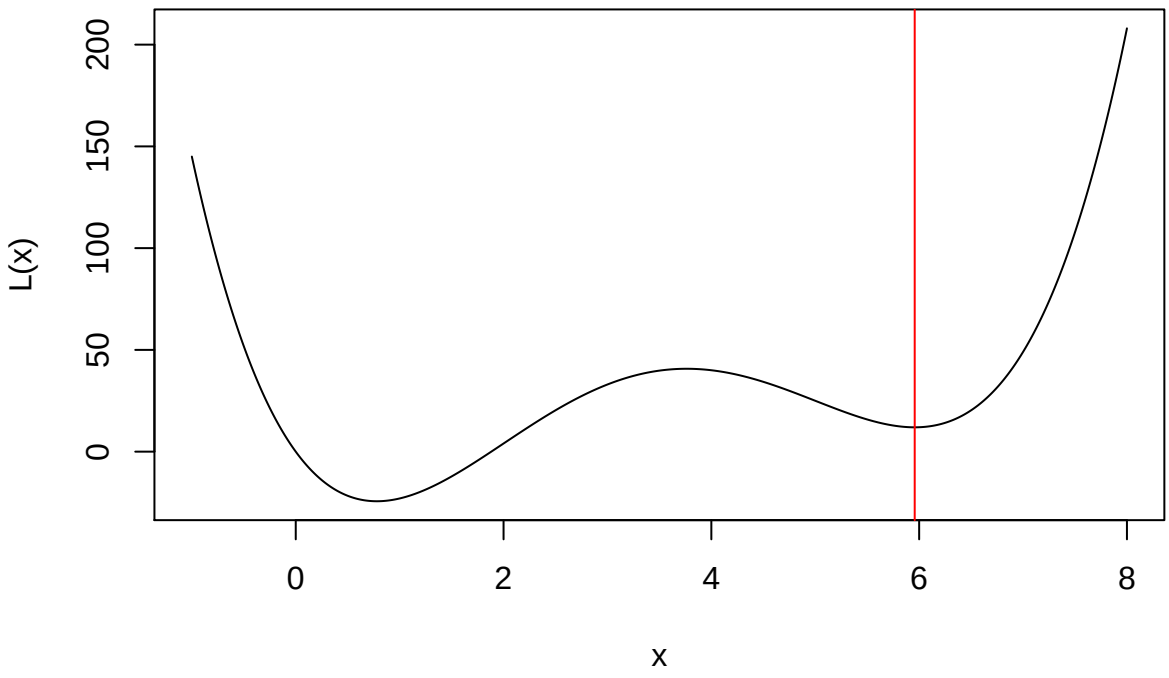

```
xpts <- 1:length(best_points)
ypts <- best_points
plot(xpts, ypts)
```




```
print(example_newton(init_area=c(6,8)))
```

```
## [1] 5.957195
```

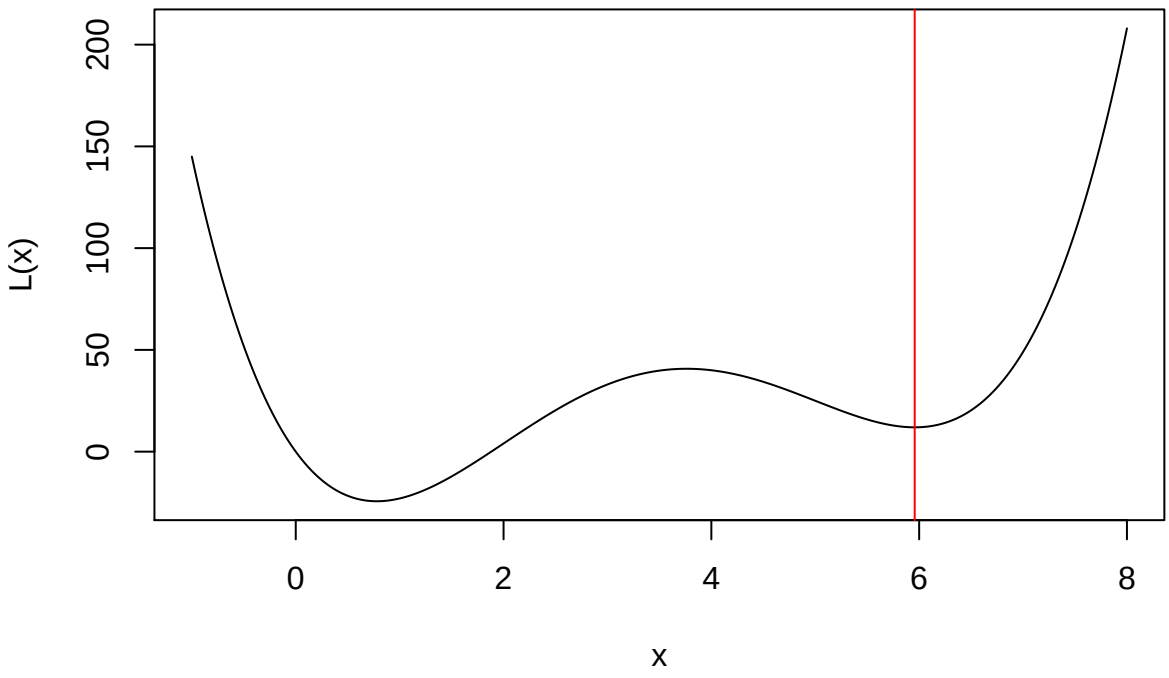
```
plot(x, L(x), type="l")
abline(v=example_newton(init_area=c(6,8)), col="red")
```




```
my_loss <- function(x) {
    L(x)[1]
}
val <- optim(par = c(6, 8), fn = my_loss, method="Nelder-Mead")$par[1]
print(val)
```

```
## [1] 5.957156
```

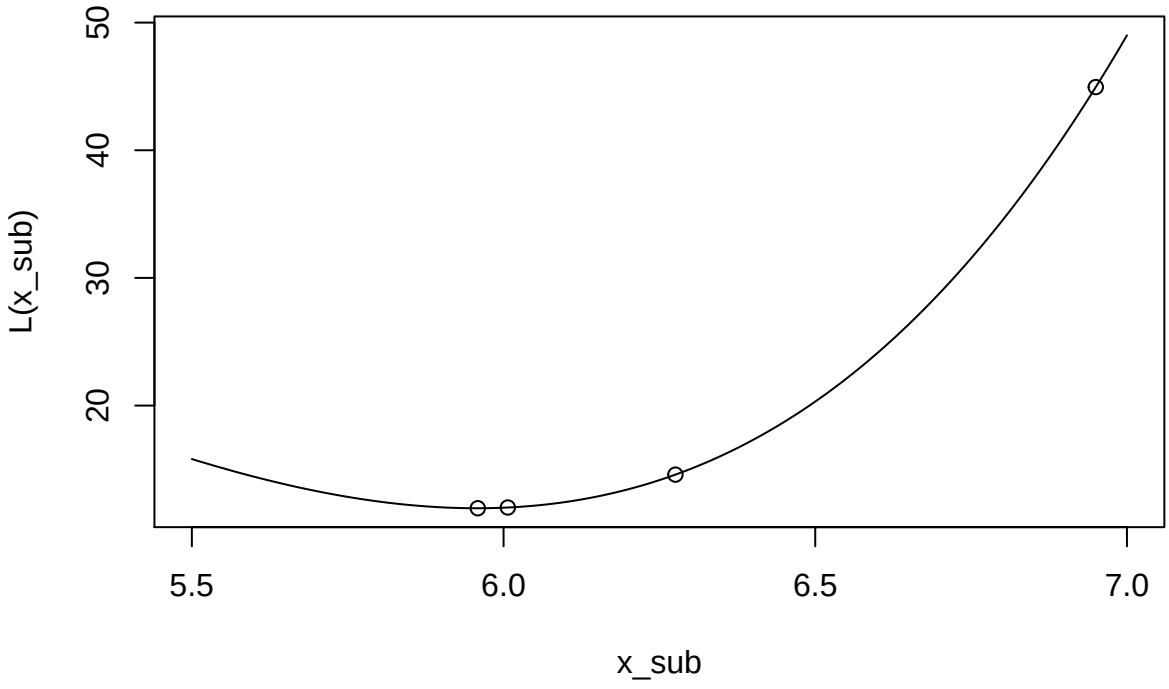
```
plot(x, L(x), type="l")
abline(v=val, col="red")
```



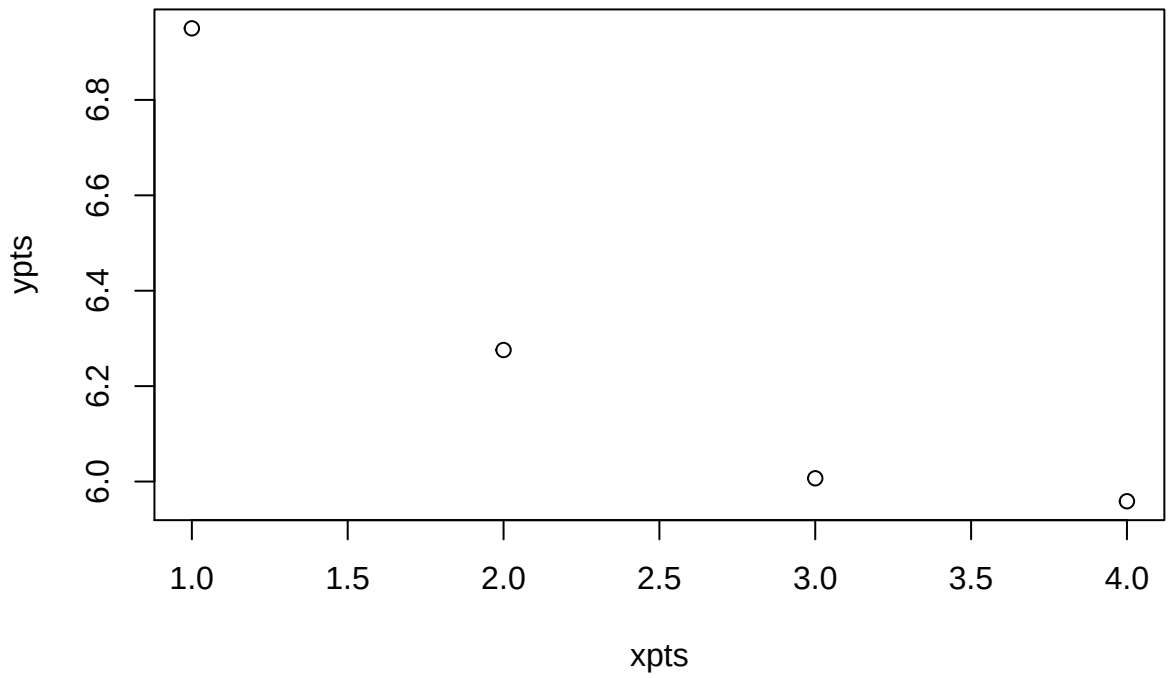

```
all_points <- example_newton(init_area = c(6,8), return_points = TRUE)
best_points <- all_points[,1]
print(best_points)
```

```
## [1] 6.949994 6.275694 6.006842 5.958716
```

```
x_sub <- seq(5.5, 7, 0.01)
plot(x_sub, L(x_sub), type="l")
points(best_points, L(best_points))
```




```
xpts <- 1:length(best_points)
ypts <- best_points
plot(xpts, ypts)
```



Again the results here with Newton's method are similar to Nelder Mead and what you expect. Unlike Nelder-Mead, Newton's method's convergence shows the effect of using the derivative of the function where it slows down in the amount it changes as it gets closer and closer to the extrema it is finding, whereas Nelder Mead is able to reflect/move right by the extrema to the other side of it. This can help in Newton's method by likely being faster than Nelder Mead, but if Nelder Mead has the right settings it could occasionally jump over to the optimum from the sub-optimum.

Problem 3