Homework 03 - STAT440

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```
set.seed(42)
```

Problem 1

In this problem we will use simulation to explore the accuracy of confidence intervals. Consider a sample of n i.i.d. normal random variables with mean μ and standard deviation σ. Assume μ is unknown, but σ is known.

Part a

Generate such a sample $\mu = 1$ and $\sigma = 0.5$ for the cases $n = \{10, 100, 1000\}$. In reality this step would be done by nature, but here we can simulate it ourselves.

```
n <- c(10, 100, 1000)
samples <- vector(mode = "list", length = length(n))

mu <- 1
sigma <- 0.5

for (i in 1:length(n)) {
  samples[[i]] <- rnorm(n[i], mean = mu, sd = sigma)
}
```

Part b

Build a 95% confidence interval for each of these samples. Remember σ is assumed to be known but not μ.

Recall that when σ is known we can create our confidence interval for μ by using a transformation to the standard normal variable resulting in:

$$[\overline{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]$$

Here we use $\alpha = 0.05$, so $\frac{\alpha}{2} = 0.025$. $z_{0.025} = 1.96$.

```
z_multi <- 1.96
intervals <- vector("list", length(n))
```

```r
for (i in 1:length(n)) {
  x_bar <- mean(samples[[i]])
  diff <- z_multi * sigma / sqrt(n[i])
  intervals[[i]] <- c(x_bar - diff, x_bar + diff)
}
```

Part c

```r
num <- 1000
n <- 100
diff <- z_multi * sigma / sqrt(n)
count_in <- 0

for (i in 1:num) {
  sample <- rnorm(n, mean = mu, sd = sigma)
  x_bar <- mean(sample)
  if(x_bar - diff < mu & x_bar + diff > mu) {
    count_in <- count_in + 1
  }
}
print(count_in)
```

```
## [1] 952
```

Out of 1000 simulations of sampling 100 values from the normal distribution with $\mu = 1$ and $\sigma = 0.5$, we get μ in our 95% confidence interval 952 times, or 95.2% of the time. This makes perfect sense from the definition of a confidence interval, which states that for a given confidence level, the level represents the frequency with which a confidence interval at that level should contain the given parameter. In plain English this means that for a confidence interval at x% confidence, we should expect x% of the confidence intervals constructed at that level to contain the parameter.

Part d

If the parameter σ was not known, then we would have to use the student's t distribution to calculate the confidence interval.

Problem 2

For this problem we will use the matrix D from the previous homework.

```r
mat_v <- matrix(c(1, 2, 3, 2, 1, 0, 4, -4, 1, -1, 1, -1, 0, 3, 5), nrow = 3, ncol=5)

l2_norm <- function(vec) {
  sqrt(sum(vec^2))
}

num_cols <- dim(mat_v)[2]
mat_d <- matrix(1:25, nrow = num_cols, ncol = num_cols)
```

```
for (i in 1:num_cols) {
  for (j in 1:num_cols) {
    mat_d[i, j] <- l2_norm(mat_v[,i] - mat_v[,j])
  }
}
mat_d
```

```
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,] 0.000000 3.316625 7.000000 4.582576 2.449490
## [2,] 3.316625 0.000000 5.477226 3.162278 5.744563
## [3,] 7.000000 5.477226 0.000000 7.348469 9.000000
## [4,] 4.582576 3.162278 7.348469 0.000000 6.403124
## [5,] 2.449490 5.744563 9.000000 6.403124 0.000000
```

Part a

Create a matrix Σ whose ij^{th} element is $exp(-\tau D_{ij}^2)$ for $\tau = \frac{1}{20}$.

```
tau <- 1 / 20
mat_sig <- matrix(1:25, nrow = num_cols, ncol = num_cols)
mat_d_square <- mat_d %*% mat_d

for (i in 1:num_cols) {
  for (j in 1:num_cols) {
    mat_sig[i, j] <- exp(-1 * tau * mat_d_square[i, j])
  }
}
mat_sig
```

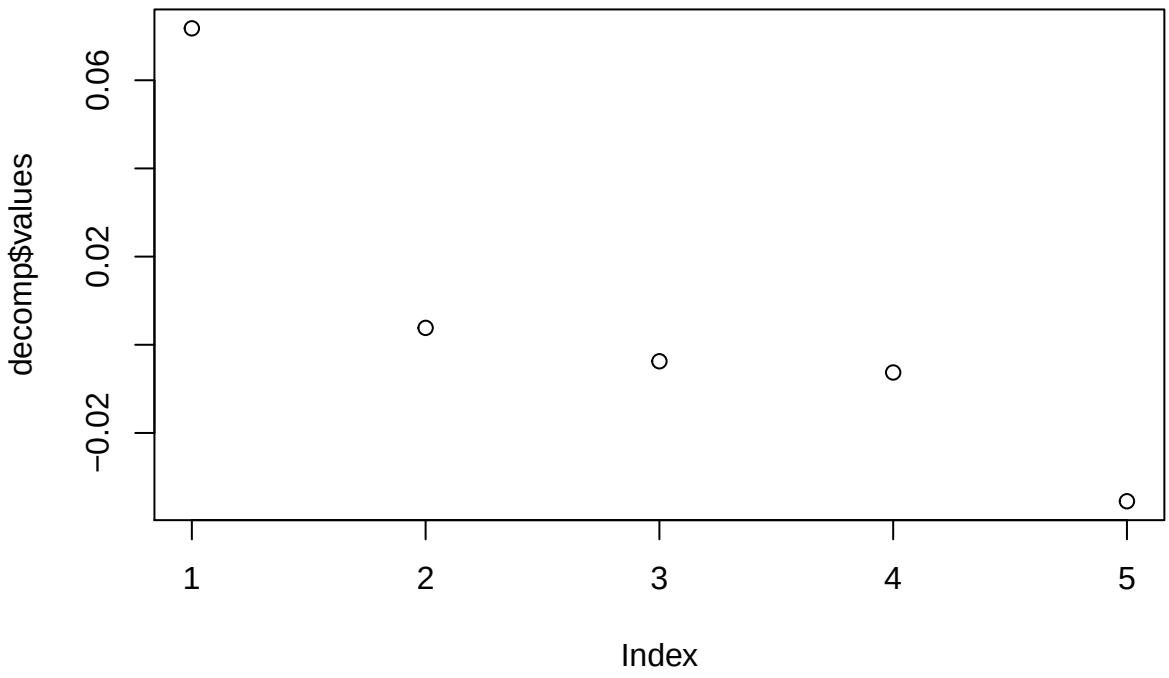
```
##             [,1]        [,2]         [,3]        [,4]         [,5]
## [1,] 0.012906813 0.035254569 2.486457e-02 0.020638686 0.0038113645
## [2,] 0.035254569 0.014995577 7.388970e-03 0.009936377 0.0205812201
## [3,] 0.024864572 0.007388970 2.254494e-05 0.004741827 0.0083692354
## [4,] 0.020638686 0.009936377 4.741827e-03 0.001836305 0.0084266593
## [5,] 0.003811364 0.020581220 8.369235e-03 0.008426659 0.0003191019
```

Part b

Compute the eigen-decomposition of Σ and plot the eigen values (sorted from largest to smallest).

```
decomp <- eigen(mat_sig)

plot(decomp$values)
```



Part c

```
frob_norm <- function(mat_input) {
  decomp <- eigen(mat_input)
  return(sqrt(sum(decomp$values^2)))
}
```

```
epsilon <- 0.5

# the eigen values are the diagonal entries of lambda
for (l in 1:length(decomp$values)) {
  diag_entries <- numeric(length(decomp$values))
  diag_entries[1:l] <- decomp$values[1:l]
  sig_l <- decomp$vectors %*% diag(diag_entries) %*% t(decomp$vectors)
  mat_diff <- mat_sig - sig_l
  err_val <- frob_norm(mat_diff) / frob_norm(mat_sig)
  print(err_val)
}
```

```
## [1] 0.452698
## [1] 0.4501939
## [1] 0.4477838
## [1] 0.440927
## [1] 3.144088e-15
```

The minimum l, so that $\frac{||\Sigma - \Sigma^l||_F}{||\Sigma||_F} < \epsilon$ is just 1. This would imply that a majority of the variance/information contained within the original matrix Σ is stored within the first eigen value/vector. Basically when you start with l = 1, you will receive a matrix that is very close to the values in Σ, and as you increase l, the matrix becomes closer to Σ itself, but the larger values contribute more information.

Part d

If $l << L$, then the using M^l would require much less memory and involve less computation time, because you would only need to store the first l eigen vectors/eigen values to calculate the matrix. This is the entire idea behind principal component analysis and dimension reduction. If our error from the Frobenius norm is small enough this replacement is justified, because we will still maintain a majority of the information/variance from the original matrix.

Problem 3

Part a

Part b

Problem 4

Problem 5

Part a

Part b