Homework 05 - STAT416

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Chapter 4 Problem 5

We are given both the initial probability of starting at state i along with the 1 step transition probability matrix. We can use both of these to determine $P(X_3) = \alpha P^3$.

$$\alpha_1 = \alpha_0 P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{6} & \frac{11}{24} \end{bmatrix}$$

$$\alpha_2 = \alpha_1 P = \begin{bmatrix} \frac{3}{8} & \frac{1}{6} & \frac{11}{24} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} & \frac{13}{72} & \frac{29}{72} \end{bmatrix}$$

$$\alpha_3 = \alpha_2 P = \begin{bmatrix} \frac{5}{12} & \frac{13}{72} & \frac{29}{72} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{59}{144} & \frac{43}{216} & \frac{169}{432} \end{bmatrix}$$

$$E[X_3] = 0 + \frac{43}{216} + \frac{169}{216} = \frac{212}{216} \approx 0.9815$$

Chapter 4 Problem 8

Part c

To find the probability that the second ball selected is red we need to know the probability of the state at time 2 (from which the ball will be selected). We use our initial probabilities combined with our transition probability matrix:

$$\alpha_2 = \alpha_1 P = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.15 & 0.6 & 0.25 \\ 0 & 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.6 & 0.25 \end{bmatrix}$$

So the probability of drawing a red ball second when there is 0 red balls is 0, 0.5 when there is 1 red ball, and 1 when there is 2 red balls. This gives us the probability $0 * 0.15 + 0.5 * 0.6 + 1 * 0.25 = 0.25 + 0.3 = 0.55$. The probability of selecting a red ball second is **55%**.

Part d

Calculating the probability of selecting a red ball fourth follows the same logic as the previous part. We can pickup with α_2.

$$\alpha_3 = \alpha_2 P = \begin{bmatrix}0.15 & 0.6 & 0.25\end{bmatrix} \begin{bmatrix}0.5 & 0.5 & 0 \\ 0.15 & 0.6 & 0.25 \\ 0 & 0.3 & 0.7\end{bmatrix} = \begin{bmatrix}0.165 & 0.51 & 0.325\end{bmatrix}$$

$$\alpha_4 = \alpha_3 P = \begin{bmatrix}0.165 & 0.51 & 0.325\end{bmatrix} \begin{bmatrix}0.5 & 0.5 & 0 \\ 0.15 & 0.6 & 0.25 \\ 0 & 0.3 & 0.7\end{bmatrix} = \begin{bmatrix}0.159 & 0.486 & 0.355\end{bmatrix}$$

This gives us the probability $0 * 0.159 + 0.5 * 0.486 + 1 * 0.355 = 0.243 + 0.355 = 0.598$. The probability of selecting a red ball fourth is **59.8%**. It makes sense that the likelihood of selecting red goes up with time as you are more likely to replace a red ball with a red ball while replace blue ball has equal probability for each.

Chapter 4 Problem 10

The probability that we desire from Gary given our Markov chain $\{X_n, n \geq 0\}$ is $P(X_{i+k} \notin \{G\}\text{for k} = 1, 2, 3 | X_i = C)$. This is also the complement that is every in a glum mood: $1 - P(X_{i+k} \in \{G\}\text{for k} = 1, 2, 3 | X_i = C)$. Here we will define our Markov chain $\{W_n, n \geq 0\}$. Its states are the states the same as X_n, but as soon as it enters the state Glum it stays there. For this chain we get the transition probability matrix:

$$Q = \begin{bmatrix}0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1\end{bmatrix}$$

All we need to do now is calculate Q^3_{CG}:

```
Q <- matrix(c(0.5, 0.3, 0, 0.4, 0.4, 0, 0.1, 0.3, 1), nrow=3, ncol=3)
Q
```

```
##      [,1] [,2] [,3]
## [1,]  0.5  0.4  0.1
## [2,]  0.3  0.4  0.3
## [3,]  0.0  0.0  1.0
```

```
Q %*% Q %*% Q
```

```
##       [,1]  [,2]  [,3]
## [1,] 0.293 0.292 0.415
## [2,] 0.219 0.220 0.561
## [3,] 0.000 0.000 1.000
```

The probability of Gary never being in a glum mood the following three days, given that he starts cheerful is 1 - 0.415 = **58.5%**.

Chapter 4 Problem 15

Chapter 4 Problem 16

Problem A

Problem B