Homework 02 - STAT416

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Chapter 2 Problem 33

Let X be a random variable with the following probability density from $-1 < x < 1$:

$$f(x) = c(1 - x^2)$$

Part a

What is the value of c? The value of c must make the density valid. For this to be true the pdf must be positive and also integrate to 1 over its support, so let's integrate:

$$\int_{-1}^{1} c(1 - x^2)dx = c(x - \frac{1}{3}x^3)|_{-1}^{1} = c((1 - \frac{1}{3}) - (-1 + \frac{1}{3})) = c(\frac{2}{3} + \frac{2}{3}) = c\frac{4}{3}$$

This means that $c = \frac{3}{4}$ for the density to be valid. This also maintains a positive function, since $1 - x^2$ will be positive between -1 and 1 and c is also a positive number.

Part b

What is the CDF of X?

To find the CDF of X we merely need to integrate the function over its support. We already did this in the last part, so we will just paste the result here:

$$F_X(a) = c(x - \frac{1}{3}x^3)|_{-1}^{a} = \frac{3}{4}x - \frac{1}{4}x^3|_{-1}^{a} = \frac{3}{4}a - \frac{1}{4}a^3 - (-\frac{3}{4} + \frac{1}{4}) = \frac{3}{4}a - \frac{1}{4}a^3 + \frac{1}{2}$$

This gives us the CDF:

$$F_X(x) = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}$$

Chapter 2 Problem 37

Let $X_1, X_2, ..., X_n$ be independent random variables, each having a uniform distribution over (0,1). Let $M = maximum(X_1, X_2, ..., X_n)$. Show that the distribution function of M is $F_M(x) = x^n$.

We want to find $F_M(x) = P(M \leq x) = P(X_1, X_2, ..., X_N \leq x)$. This must hold true, since x in this case represents the max value. Since each random variable is independent we can multiply them to find their intersect:

$$P(X_1, X_2, ..., X_N \leq x) = \Pi_{i=1}^{n} P(X_i \leq n) = F_X(x)^n$$

Since each random variable has the same distribution we can plug in the normal distribution for parameters 0 and 1: $F_X(x; a = 0, b = 1) = \frac{x-a}{b-a} = \frac{x-0}{1-0} = x$. Plugging this into result found above we get:

$$F_M(x) = F_X(x)^n = x^n$$

Chapter 2 Problem 44

Let Y denote the number of red balls chosen after the first but before the second black ball has been chosen.

Part a

Express Y as the sum of n random variables, each of which is either 0 or 1.

Let Y_i be a random variable whose value is 1 if red ball i is taken between the first and second black ball and zero otherwise. This makes $Y = \Sigma_{i=1}^{n} Y_i$.

Part b

$$E[Y] = E[\Sigma_{i=1}^{n} Y_i] = \Sigma_{i=1}^{n} E[Y_i]$$

Let's define $p(Y_i)$. Y_i means that you are drawing a single red ball out of $m - 1$ black balls and 1 red ball. This gives us $p(Y_i) = \frac{1}{m}$. This would make $E[Y_i] = 1 * p(Y_i) + 0 * (1 - p(Y_i)) = p(Y_i)$

$$\Sigma_{i=1}^{n} \frac{1}{m} = \frac{n}{m} = E[Y]$$

Chapter 2 Problem 53

If X is uniform over (0, 1), calculate $E[X^n]$ and $Var(X^n)$. Recall that a Uniform distribution from 0 to 1, the density is $f(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1$

$$E[X^n] = \int_0^1 x^n f(x)dx = \int_0^1 x^n dx = \frac{1}{n+1} x^{n+1}|_0^1 = \frac{1}{n+1}$$

To find the variance of the n^{th} moment recall that we can define:

$$Var(X^n) = E[X^{2n}] - (E[X^n])^2$$

Let's find the two terms; the second we can pull from above.

$$E[X^{2n}] = \int_0^1 x^{2n} f(x)dx = \int_0^1 x^{2n} dx = \frac{1}{2n+1} x^{2n+1}|_0^1 = \frac{1}{2n+1}$$

$$(E[X^n])^2 = (\frac{1}{n+1})^2 = \frac{1}{n^2+2n+1}$$

Plugging this back in we get:

$$Var(X^n) = E[X^{2n}] - (E[X^n])^2 = \frac{1}{2n+1} - \frac{1}{n^2+2n+1} = \frac{n^2+2n}{2n+1(n^2+2n+1)}$$

Chapter 2 Problem 55

The following is a joint pmf for X and Y.

$$P(X=i, Y=j) = \binom{j}{i} e^{-2\lambda}\lambda^j/j!; 0 \leq i \leq j$$

Part a

Find the pmf of Y.

$$P(Y=j) = \Sigma_{i=1}^{j} P(X=i, Y=j) = \Sigma_{i=0}\binom{j}{i} e^{-2\lambda}\lambda^j/j! = e^{-2\lambda}\lambda^j/j!\Sigma_{i=0}^{j}\binom{j}{i} = 2^j e^{-2\lambda}\lambda^j/j!$$

$$P(Y=j) = 2^j e^{-2\lambda}\lambda^j/j!$$

Part b

Find the pmf of X.

$$P(X=i) = \Sigma_{j=i}^{\infty} P(X=i, Y=j) = \Sigma_{j=i}^{\infty}\binom{j}{i} e^{-2\lambda}\lambda^j/j! = e^{-2\lambda}\Sigma_{j=i}^{\infty}\binom{j}{i}\lambda^j/j!$$

$$P(X=i) = e^{-2\lambda}\Sigma_{j=i}^{\infty}\frac{j!}{i!(j-i)!}\lambda^j/j! = \frac{e^{-2\lambda}}{i!}\Sigma_{j=i}^{\infty}\frac{\lambda^j}{(j-i)!} = \frac{e^{-2\lambda}}{i!}e^{\lambda}$$

$$P(X=i) = \frac{e^{-\lambda}}{i!}$$

Chapter 2 Problem 63

Let X denote the number of white balls selected when k balls are chosen at random from an urn containing n white and m black balls.

Part a

Compute $P(X = i)$.

The total sample space is $\binom{n+m}{k}$ where we want to consider $\binom{n}{i}$ white balls chosen and $\binom{m}{k-i}$ black balls chosen. Therefore we get:

$$P(X = i) = \frac{\binom{n}{i}\binom{m}{k-i}}{\binom{n+m}{k}}$$

Part b

Compute $E[X]$.

Using X:

$$E[X] = E[\Sigma_{i=1}^{k} X_i] = \Sigma_{i=1}^{k} E[X_i]$$

We know that $E[X_i]$ is just the probability of the bernoulli variable being 1, which is the probaility that the i^{th} ball selected is white. This probability is $p(X_i) = \frac{1}{m+1}$. This gives us:

$$E[X] = \frac{k}{m+1}$$

Chapter 2 Problem 67

Calculate the moment generating function of the uniform distribution on (0, 1). Obtain the expectation and variance by differentiation.

Recall that $M_x(t) = E[e^{tx}] = \int_0^1 e^{tx} dx = \frac{e^{tx}}{t}|_0^1 = \frac{e^t - 1}{t}$

Using the MGF for (E[X]) we take the first derivative and evaluate for t=0:

$$M_x'(t = 0) = \frac{e^t}{t} - \frac{e^t - 1}{t^2}$$

Chapter 2 Problem 68

Let X and W be the working and subsequent repair times of a certain machine. Let $Y = X + W$ and suppose that the joint probability density of X and Y is

$$f_{X,Y}(x, y) = \lambda^2 e^{-\lambda y}, 0 < x < y < \infty$$

Part a

Find the density of X:

$$f_X(x) = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 e^{-\lambda y} x|_0^y = y\lambda^2 e^{-\lambda y}$$

Part b

Find the density of Y:

$$f_Y(y) = \int_x^\infty \lambda^2 e^{-\lambda y} dy = -\lambda e^{-\lambda y}|_x^\infty = \lambda e^{-\lambda x}$$

Chapter 2 Problem 70

Calculate the moment generating function of a geometric random variable.

Recall the pmf:

$$p(k) = (1-p)^{k-1} p$$

$$E[e^{tX}] = \Sigma_{i=1}^\infty (1-p)^{i-1} p e^{ti} = p \Sigma_{i=0}^\infty ((1-p)e^t)^i = \frac{p}{1-(1-p)e^t}$$

$$M_x(t) = \frac{p}{1-(1-p)e^t}$$

Problem A

Find $P(\frac{-1}{2} < X < 0$ for the R.V. in 2.33. Recall the CDF $F_X(x) = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}$.

We can find the desired value via $F(0) - F(-0.5)$.

$$F(0) - F(-\frac{1}{2}) = \frac{1}{2} - (\frac{3}{4}\frac{-1}{2} + \frac{1}{4}\frac{1}{8} + \frac{1}{2}) = \frac{12}{32} - \frac{1}{32} = \frac{11}{32}$$

The mean is the expected value:

$$\int_{-1}^1 \frac{3x}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^1 x - x^3 dx = \frac{3}{4}(\frac{x}{2}^2 - \frac{x^4}{4})|_{-1}^1 = \frac{3}{4}((\frac{1}{2} - \frac{1}{4}) - (\frac{1}{2} - \frac{1}{4})) = 0$$

The mean is 0.

The variance is the second moment in this case.

$$\int_{-1}^1 \frac{3x^2}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^1 x^2 - x^4 dx = \frac{3}{4}(\frac{1}{3}x^3 - \frac{1}{5}x^5)|_{-1}^1 = \frac{3}{4}(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}) = \frac{3}{4}(\frac{2}{3} - \frac{2}{5}) = \frac{3}{4}(\frac{4}{15}) = \frac{1}{5}$$

Problem B

Let X be an exponential random variable with parameter λ. Prove that $P(X > s+t | X > t) = P(X > s)$. for any positive values of s and t.

$$P(X > s+t | X > t) = \frac{P(X > s+t, X > t)}{P(X > t)}$$

Since t is greater than or equal to zero the intersection of the events of X > t and X > s + t both fall into s + t. This gives us:

$$P(X > s+t|X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{1-(1-e^{-\lambda s-\lambda t})}{1-(1-e^{-\lambda t})} = e^{-\lambda s-\lambda t+\lambda t} = e^{-\lambda s}$$

From this we can see that

$$P(X > s+t|X > t) = e^{-\lambda s} = 1-(1-e^{-\lambda s}) = P(X > s)$$