Homework 11 - STAT416

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All questions come from Chapter 6.

Problem 5

Problem Constraints

- N individuals
- Contact occurs with rate λ (poisson)
- Contact equally likely to involve any $\binom{N}{2}$ people
- probability p of infection (with infected and non-infected)
- $X(t)$ is number of infected people

Part a

Yes, $X(t)$ is a continuous time Markov chain. The number of infected people only depends on the previous number of infected people and therefore satisfies the Markovian property. The number of infected people will only be one more than the previous number and depends on when a contact occurs.

Part b

This Markov chain is a birth process. As time increases the state can only go from state i to state i + 1. Since the process can never decrease it is a pure birth process with $\mu_n = 0$.

Part c

With $X(0) = 1$ we want to determine the amount of time it takes for all N individuals to become infected. At time t we take 2 individuals that come into contact. There are $\binom{N}{2}$ possible pairs, but if the pair have the same status, then nothing happens to the state. If one and only one is infected then the other is infected as well and the state increases by one with probability p.

We can define our birth rate as a function of the rate of contact times the probability of infection times the probability of exactly one infected and one non-infected person being chosen.

$$\lambda_n = \lambda p \frac{i(N-i)}{\binom{N}{2}}$$

Since the time between infections are exponentially distributed with this rate λ_n we can calcuate the mean time until total infection by summing the component means:

$$\Sigma_{i=1}^{N-1}\frac{1}{\lambda_n} = \frac{\binom{N}{2}}{\lambda p}\Sigma_{i=1}^{N-1}\frac{1}{i(N-i)}$$

Problem 13

Problem Constraints

- Barbershop with 2 chairs
- Customers arrive Poisson process rate of 3 per hour
- Serving times Exponential with mean 1/4

To solve the questions model the process as a Markov chain with $X(t)$ representing the number of customers in the shop at time t. The state of this Markov chain can take on the values 0, 1, or 2, since the shop can serve a max of 2 customers.

Part a

While we are below capacity $X(t) \in \{1, 2\}$ the state can increase by 1 with a rate of 3 per hour and while there are customers we can decrease the state by 1 with a rate of 4 per hour. We can use these rates to get the following matrix representing the rate of change from state i (row) to j (col):

$$P = \begin{bmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{bmatrix}$$

Using this system we can solve for the stationary probabilities:

$$\pi_1 = \frac{3}{4}\pi_0$$

$$\pi_2 = \frac{3}{4}\pi_1 1$$

This results in the probailities:

$$\pi_0 = \frac{4^2}{4^3 - 3^3}$$

$$\pi_1 = \frac{3 * 4}{4^3 - 3^3}$$

$$\pi_2 = \frac{3^2}{4^3 - 3^3}$$

We can use these probabilities to find the average number of customers in the shop:

$$\overline{n} = 0\pi_0 + 1\pi_1 + 2\pi_2 = \frac{3 * 4}{4^3 - 3^3} + 2\frac{3^2}{4^3 - 3^3} \approx 0.811$$

The average number of customers in the barbershop in the long run is 0.811.

Part b

The proportion of potential customers that enter the shop is equal to the proportion of time the the shop is not at capacity. This is:

$$\pi_0 + \pi_1 = \frac{4^2}{4^3 - 3^3} + \frac{3 * 4}{4^3 - 3^3} \approx 0.757$$

Part c

We can find the additional amount of business by adjusting our system of equations for when the barber works double speed:

$$\pi_0 = 5\frac{8^2}{8^3 - 3^3}$$
$$\pi_1 = \frac{3 * 40}{8^3 - 3^3}$$
$$\pi_2 = 5\frac{3^2}{8^3 - 3^3}$$

These new systems give the proportion of customers entering the shop to be:

$$5\frac{8^2}{8^3 - 3^3} + \frac{3 * 40}{8^3 - 3^3} \approx 0.907$$

The shop has 90.7 percent of the potential customers entering the shop when working twice as fast versus 75.7 percent of customers. This is a roughly 19.8 percent increase in business.

Problem 15

Problem 17

Problem 20

Problem 21

Problem 23