Homework 03 - STAT416

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Chapter 2 Problem 72

Part a

Here we have 5 independent and identically distributed Normal random variables with a mean of 100 and variance of 100. To find the probability that at least one month of sales exceeds 115 we can find the probability that no months exceed 115 and subtract that value from one. The probability that no months exceed 115 is the value of the CDF at 115 multiplied 5 times:

$$P(X_1, X_2, X_3, X_4, X_5 \leq 115 = \Pi_{i=1}^{5} P(X_i \leq 115)$$

We can use the standard normal CDF if we transform the distribution.

$$P(X_i \leq 115) = P(\frac{X_i - \mu}{\sigma} \leq \frac{115 - 100}{10}) = P(Z \leq 1.5) = \Phi(1.5) = 0.9332$$

$$\Pi_{i=1}^{5} P(X_i \leq 115) = (0.9332)^5 = 0.7077$$

This gives us our desired value of $1 - 0.7707 = 0.2923$. The probability that at least one month exceeds 115 in sales is **29.23%**.

Part b

The probability that the total sales exceeds 530 in the next five months requires us to define a new random variable Y. $Y = \Sigma i = 1^5 X_i$, which gives a mean of $\Sigma_{i=1}^{5} \mu_{X_i} = 500$ and a variance of $\Sigma_{i=1}^{5} \sigma_{X_i}^2 = 500$. We can then search for:

$$1 - P(Y \leq 530) = 1 - P(\frac{Y - 500}{22.36} \leq \frac{530 - 500}{22.36}) = 1 - \Phi(1.34) = 1 - 0.9099 = 0.0901$$

The probability that total sales exceeds 530, $P(Y > 530)$, is **9.01%**.

Chapter 2 Problem 78

Part a

Recall the Markov inequality:

$$P(X \geq a) \leq \frac{E[x]}{a}$$

We also know that the sum of independent Poisson R.V.s is a Poisson R.V. whose mean is the sum of the individual means:

$$Y = \Sigma_{i=1}^{10} X_i = \text{Poisson}(\Sigma_{i=1}^{10} \lambda_i)$$

Here we can conclude the bound is:

$$P(Y \geq 15) \leq \frac{\Sigma_{i=1}^{10} \lambda_i}{15} = \frac{10}{15}$$

Part b

We can use the central limit theoream to approximate $P(\Sigma_{i=1}^{10} X \geq 15)$. Using the central theorem we will approximate by assuming that the sample mean is normally distributed. We make sure to transform the value to the standard normal and then use the CDF to determine the probability.

$$P(\Sigma_{i=1}^{10} X \geq 15) = 1 - P(\frac{\Sigma_{i=1}^{10} X - 10}{\sqrt{10}} \leq \frac{15 - 10}{\sqrt{10}}) = 1 - \Phi(1.58) = 1 - 0.9429 = 0.0571$$

Using CLT to approximate the probability that the sum is greater than 15, we get that the probability is **5.71%**, which is less than our bound of 66.67%.

Chapter 3 Problem 3

Compute $E[X|Y = i]$ for $i = 1, 2, 3$. Recall the formula for the conditional expectation:

$$E[X|Y = i] = \Sigma_x x P(X|Y = i)$$

For i = 1:

$$E[X|Y = 1] = 1P(1|Y = 1) + 2P(2|Y = 1) + 3P(3|Y = 1) = \frac{1}{5} + \frac{6}{5} + \frac{3}{5} = 2$$

For i = 2:

$$E[X|Y = 2] = 1P(1|Y = 2) + 2P(2|Y = 2) + 3P(3|Y = 2) = \frac{2}{3} + 0 + \frac{3}{3} = \frac{5}{3}$$

For i = 3:

$$E[X|Y = 3] = 1P(1|Y = 3) + 2P(2|Y = 3) + 3P(3|Y = 3) = 0 + \frac{6}{5} + \frac{6}{5} = \frac{12}{5}$$

Chapter 3 Problem 7

$$E[X|Y=2] = 1P(1|Y=2) + 2P(2|Y=2) = 1(\frac{1}{5}+0) + 2(0+\frac{4}{5}) = \frac{1}{5}+\frac{8}{5} = \frac{9}{5}$$

$$E[X|Y=2, Z=1] = 1P(1|Y=2, Z=1) + 2P(2|Y=2, Z=1) = 1+0 = 1$$

Chapter 3 Problem 11

Here we are finding $E[X|Y=y]$. We can use the equation for expectation of a continuous random variable:

$$E[X|Y=y] = \int x f_{X|Y}(X|Y=y)dx$$

First we need to find the conditional pdf $f_{X|Y}(X|Y=y)$. $f_{X|Y}(X|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.

$$f_Y(y) = \int_{-y}^{y} \frac{e^{-y}(y^2-x^2)}{8}dx = \frac{e^{-y}}{8}(xy^2 - \frac{1}{3}x^3)|_{-y}^{y}$$

$$f_Y(y) = \frac{e^{-y}}{8}((y^3 - \frac{1}{3}y^3) - (-y^3 + \frac{1}{3}y^3)) = \frac{e^{-y}}{6}y^3$$

Plug this back in to get the conditional pdf:

$$f_{X|Y}(X|Y=y) = \frac{\frac{e^{-y}(y^2-x^2)}{8}}{\frac{e^{-y}}{6}y^3} = \frac{3(y^2-x^2)}{4y^3} = \frac{3}{4}y^{-1} - \frac{3}{4}x^2y^{-3}$$

Now we can find the conditional expectation:

$$\int_{-y}^{y} x f_{X|Y}(X|Y=y)dx = \int_{-y}^{y}(\frac{3}{4}xy^{-1} - \frac{3}{4}x^3y^{-3})dx = \frac{3}{4}(\frac{x^2}{2y} - \frac{x^4}{4y^3})|_{-y}^{y} = \frac{3}{4}((\frac{y}{2}-\frac{y}{4}) - (\frac{y}{2}-\frac{y}{4})) = 0$$

Thus we confirm that the conditional expectation $E[X|Y=y] = 0$.

Chapter 3 Problem 15

Here we are finding $E[X^2|Y=y]$. We can use the equation for the second moment of a continuous random variable:

$$E[X|Y=y] = \int x^2 f_{X|Y}(X|Y=y)dx$$

First we need to find the conditional pdf $f_{X|Y}(X|Y=y)$. $f_{X|Y}(X|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y}dx = \frac{e^{-y}}{y}x|_0^y = e^{-y}$$

Plug this back in to get the conditional pdf:

$$f_{X|Y}(X|Y=y) == \frac{\frac{e^{-y}}{y}}{e^{-y}} = \frac{1}{y}$$

Now we can find the conditional second moment:

$$\int_0^y x^2 f_{X|Y}(X|Y=y)dx = \int_0^y \frac{x^2}{y}dx = \frac{x^3}{3y}|_0^y = \frac{y^2}{3}$$

Chapter 3 Problem 23

We are trying to find $E[N]$ where N is the number of flips until 2 out of three most recent coin flips are heads. We know that we start with 2 flips, then we can condition on the result of these two flips:

$$E[N] = 2 + E[N|H,H]P(H,H) + E[N|H,T]P(H,T) + E[N|T,H]P(T,H) + E[N|T,T]P(T,T)$$

We must define what the expectations are. First we have $E[N|H,H]$. The expected number of further flips then would be 0, because two heads have already been flipped. Next we have $E[N|T,T]$, which is $E[N]$, since it would essentially be like starting the coin flips over, since no progress towards flipping heads has been made.

Next let's look at the two more complicated values. $E[N|H,T]$ and $E[N|T,H]$ show progress toward our goal of two heads in the last three flips. For $E[N|H,T]$ we add 1 flip more. If that next flip is heads, then we met our goal, otherwise we reset. This gives us $E[N|H,T] = 1 + 0P(H) + E[N]P(T)$. For $E[N|T,H]$ we have the same 1 flip more, that either results in meeting the goal of 2 heads, or results in the previous expectation of a H,T. $E[N|T,H] = 1 + 0P(H) + E[N|H,T]P(T)$. Let's plug these values back into the main equation:

$$E[N] = 2+0p^2+(1+0P(H)+E[N]P(T))p(1-p)+(1+0P(H)+(1+0P(H)+E[N]P(T))P(T))p(1-p)+E[N]p(1-p)$$

$$E[N] = 2 + (1 + E[N](1-p))p(1-p) + (1 + (1 + E[N](1-p))(1-p))p(1-p) + E[N](1-p)$$

$$E[N] = 2 + p(1-p) + E[N]p(1-p)^2 + p(1-p) + p(1-p)^2 + E[N]p(1-p)^3 + E[N](1-p)^2$$

$$E[N](1 - p(1-p)^2 - (1-p)^2 - p(1-p)^3) = 2 + 2p(1-p) + p(1-p)^2$$

$$E[N] = \frac{2 + 2p(1-p) + p(1-p)^2}{1 - p(1-p)^2 - (1-p)^2 - p(1-p)^3}]$$

Chapter 3 Problem 24

Chapter 3 Problem 26

Problem A

Part a

Part b