Homework 06 - STAT416

Joseph Sepich (jps6444)

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Chapter 4 Problem 17

Here we look at the infinite random walk in Example 4.19 from the book. The random variable Y_i is an indicator of whether we went to the state 1 higher than the current state or not: $Y_i = \{1, -1\}$. Suppose $p = P(Y_i = 1) > \frac{1}{2}$. We know that the state at time n can be written as $\Sigma_{i=1}^n Y_i$. The expected value of Y_i is

$$E[Y_i] = p + (p - 1) = 2p - 1$$

When $p = \frac{1}{2}$ this expected value is $2 * \frac{1}{2} - 1 = 0$, so when $p > \frac{1}{2}$ the expected value is $1 > E[Y_i] > 0$ (equal to $2p - 1$). We are looking at the value $\Sigma_{i=1}^n Y_i = n\overline{Y_n}$

Recall the strong law of large numbers which states as $n \to \infty$, then $\overline{X_n} \to \mu$.

Similarly we can say as $n \to \infty$ then $\overline{Y_n} \to 2p - 1$, so $n\overline{Y_n} = \Sigma_{i=1}^n Y_i \to \infty$, since we have infinity multiplied by a constant we get infinity. Since we just concluded that the state at time n will go to ∞ if $p > \frac{1}{2}$, then we can say 0 is only visited finitely often, and therefore must be transient. Since transient is a class property and there is only one class the whole chain must be transient.

Note that we can use similar logic when $p < \frac{1}{2}$. Here we merely get a negative constant $0 > E[Y_i] > -1$. Therefore this proof holds for $p \neq \frac{1}{2}$.

Chapter 4 Problem 18

Probability transition matrix below includes the two states coin 1 and coin 2. Note that this Markov Chain is both recurrent and irreducible. There is a single class and we know that it is recurrent, because the expected value that a tail eventually occurs follows from the Geometric distribution, which is a finite value.

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

Part a

We want to find the proportion of flips that uses coin 1, which can be denoted as π_1.

$$\pi_1 = 0.6\pi_1 + 0.5\pi_2$$
$$\pi_2 = 0.4\pi_1 + 0.5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

$$\pi_1 = \frac{0.5}{1 + 0.5 - 0.6} = \frac{0.5}{0.9} = \frac{5}{9}$$

Part b

We want to find the probability that we use coin 2 at time 5 given that we started with coin 1: $P(X_5 = 2|X_1 = 1)$. To find this we can use P^4 combined with transitions to coin 2.

```
P <- matrix(c(0.6,0.5,0.4,0.5), nrow=2, ncol=2)
P_5 <- P %*% P %*% P %*% P %*% P
P_5
```

```
##          [,1]    [,2]
## [1,] 0.55556 0.44444
## [2,] 0.55555 0.44445
```

So we get:

$$P(X_5 = 2|X_1 = 1) = P(X_5 = 2|X_4 = 1)P(X_4 = 1|X_1 = 1) + P(X_5 = 2|X_4 = 2)P(X_4 = 2|X_1 = 1)$$

$$P_{12}^5 = P_{12}P_{11}^4 + P_{22}P_{12}^4 = 0.4\frac{5}{9} + 0.5\frac{4}{9}$$

$$P_{12}^5 = \frac{4}{9}$$

Part c

We want to know the proportion of times the flips land on heads. This is simply the given weight times the proportions we found before:

$$\pi_H = 0.6\pi_1 + 0.5\pi_2 = 0.6\frac{5}{9} + 0.5\frac{4}{9}$$
$$\pi_H = \frac{5}{9}$$

Chapter 4 Problem 21

Transitional Probability P_{i_j}:

$$P_{i_j} = 1 - 1 + 3\alpha/3 = \alpha; j \neq i$$
$$P_{i,i} = 1 - 3\alpha$$

Part a

Show that $P_{1,1}^n = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$. To do this let's construct our transition probability matrix:

$$P^1 = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix}$$

Let's get $P^2_{1,1}$ first. All the diagonal entries of the output are the same, since the matrix is symmetric:

$$P^2_{1,1} = (1-3\alpha)^2 + 3\alpha^2 = 1 - 6\alpha + 12\alpha^2 = \frac{1}{4} + \frac{3}{4} - 6\alpha + 12\alpha^2 = \frac{1}{4} + \frac{3}{4}(1 - 8\alpha + 16\alpha^2) = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^2$$

Since all the non-diagonal entries are the same and the matrix is symmetric the pattern will continue so $P^2_{1,1} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$.

Chapter 4 Problem 23

Part a

Part b

Part c

Chapter 4 Problem 33

Chapter 4 Problem 36