Homework 09 - STAT416

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Problem 21

Problem Constraints

- server $i; i \in (1, 2)$
- service times: $\text{Exp}(\mu_i)$
- servuce acts as queue (first 1 then 2)
- you enter with server 1 busy

Since an exponential distribution is memoryless we do not need to know how long the first server is busy before we arrived. The total amount of time spent in the system will be the time spent in service with server 1 and then with server 2. If server 2 is slower than server 1 when you are served then you must wait for server 2 to finish before transferring. Our total time is defined as the summation of:

1. Expected waiting time for server 1
2. Expected time until next completed server
3. Expected remaining time (wait + served by 2)

The first quantity is merely the expected value of an exponential distribution: $E[W_1] = E[X_1] = \frac{1}{\mu_1}$. The second quantity we want to find the minimum of the two servers. This we defined as the exponential random variable with the sum of parameters $\mu_1 + \mu_2$. This gives the expected wait time of $E[W_2] = E[min(X_1 + X_2)] = E[Z] = \frac{1}{\mu_1 + \mu_2}$

The third quantity has two cases, which correponds to who finished first. If server 2 finishes first, then we expect to wait the amount of time the first server has left (memoryless, so its just the expectation) plus the expected amount of time of server 2. If server 2 finishes last, then we must wait for server 2 twice, which is merely the expected time of server 2 to finish twice. This gives us the following value:

$$E[W_3] = E[W_3|X_1 < X_2]P(X_1 < X_2) + E[W_3|X_2 < X_1]P(X_2 < X_1)$$
$$E[W_3] = (\frac{2}{\mu_2})\frac{\mu_1}{\mu_1 + \mu_2} + (\frac{1}{\mu_1} + \frac{1}{\mu_2})\frac{\mu_2}{\mu_1 + \mu_2}$$

This gives us our overall expected time in the system:

$$E[W] = E[W_1] + E[W_2] + E[W_3]$$
$$E[W] = \frac{1}{\mu_1} + \frac{1}{\mu_1 + \mu_2} + (\frac{2}{\mu_2})\frac{\mu_1}{\mu_1 + \mu_2} + (\frac{1}{\mu_1} + \frac{1}{\mu_2})\frac{\mu_2}{\mu_1 + \mu_2}$$
$$E[W] = \frac{2}{\mu_1} + \frac{1}{\mu_1 + \mu_2} + \frac{2\mu_1}{\mu_2(\mu_1 + \mu_2)}$$

Problem 30

Problem Constraints

- Cat and Dog
- Lifetime are exponential with rate $\lambda_i; i = c, d$

We must find the addtional lifetime of one pet, given the other has died.

$$E[L] = E[L_c|D = d]P(X_d < X_c) + E[L_d|D = c]P(X_c < X_d)$$
$$E[L] = \frac{1}{\lambda_c}\frac{\lambda_d}{\lambda_c + \lambda_d} + \frac{1}{\lambda_d}\frac{\lambda_c}{\lambda_c + \lambda_d}$$
$$E[L] = \frac{\lambda_d\lambda_d}{\lambda_d\lambda_c(\lambda_c + \lambda_d)} + \frac{\lambda_c\lambda_c}{\lambda_c\lambda_d(\lambda_c + \lambda_d)}$$
$$E[L] = \frac{\lambda_c^2 + \lambda_d^2}{\lambda_c\lambda_d(\lambda_c + \lambda_d)}$$

Problem 34

Problem Constraints

- A and B need kidneys (ordered A then B)
- death occurs Exp(μ_i)
- kidneys arrive Poisson(λ)

Part a

A will receive a kidney if A lives longer than the amount of time before the first kidney arrives.

$$P(\text{A gets kidney}) = P(T_1 < X_A)$$

From lemma 5.2 we know that T_1 is exponentially distributed with rate λ. Therefore we can use our known formula for comparing two exponential random variables:

$$P(T_1 < X_A) = \frac{\lambda}{\lambda + \mu_A}$$

Part b

B will receive a kidney in two cases. A dies first and the first kidney comes before B dies or two kidneys arrive before either person dies. These can be described using the minimum of the three exponential random variables T_1, X_A, X_B, which is an exponential random variables with rate $\lambda + \mu_A + \mu_b$. (Note that we use Proposition 5.4 to expand the exponential distributed R.V. to T_2).

$$P(\text{B gets kidney}) = P(T_1 < X_B)P(X_A = min(T_1, X_A, X_B)) + P(T_2 < X_B)P(T_1 = min(T_1, X_A, X_B))$$
$$P(\text{B gets kidney}) = \frac{\lambda}{\lambda + \mu_B}\frac{\mu_A}{\lambda + \mu_A + \mu_B} + \frac{\lambda}{\lambda + \mu_b}\frac{\mu_A}{\lambda + \mu_A + \mu_B}$$
$$P(\text{B gets kidney}) = (\frac{\lambda}{\lambda + \mu_B})(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B})$$

Part c

Both A and B will not receive a kidney if the first kidney arrives after they both dies. This means we want to determine the probability:

$$P(X_A, X_B < T_1) = P(X_A < T_1)P(X_B < T_1) = \frac{\mu_A}{\lambda + \mu_A}\frac{\mu_B}{\lambda + \mu_B}$$

Part d

Both A and B will receive new kidneys if the first kidney arrives before A dies and the second kidney arrives before B dies:

$$P(\text{Both get kidney}) = P(T_2 < X_B)P(T_1 = min(T_1, X_A, X_B)) = \frac{\lambda}{\lambda + \mu_b}\frac{\mu_A}{\lambda + \mu_A + \mu_B}$$

Problem 35

FINISH THIS

$$P(t < T_1 < t + h | T_1 > t) = \lambda_{T_1}(t)h + o(h)$$
$$P(t < T_1 < t + h | T_1 > t) = \frac{P(t < T_1 < t + h, T_1 > t)}{P(T_1 > t)}$$

Recall that $P(T_1 > t) = P(N(t) = 0)$

$$P(t < T_1 < t + h | T_1 > t) = \frac{P(N(t+h) = 1, N(t) = 0)}{P(N(t) = 0)} = \frac{P(N(t+h) - N(t) = 1)}{P(N(t) = 0)}$$
$$\lambda_{T_1}(t)h + o(h) = \frac{\lambda + o(h)}{e^{-\lambda t}}$$
$$1 = e^{-\lambda t} + \lambda + \lambda_{T_1}(t)h$$

Problem 38

Part a

Since N_1 and N_2 are independent Poisson processes, the join probability is simply the product of the marginal probabilities:

$$P(N_1(t) = n, N_2(t) = m) = P(N_1(t) = n)P(N_2(t) = m)$$

We know that the rates for each process are the sum of their component properties rates: $\lambda_{N_1} = \lambda_1 + \lambda_2$ and $\lambda_{N_2} = \lambda_2 + \lambda_3$.

$$P(N_1(t) = n) = e^{-\lambda_{N_1} t}(\lambda_{N_1} t)^n / n! = e^{-(\lambda_1 + \lambda_2)t}((\lambda_1 + \lambda_2)t)^n / n!$$
$$P(N_1(t) = n) = e^{-\lambda_{N_2} t}(\lambda_{N_2} t)^m / m! = e^{-(\lambda_2 + \lambda_3)t}((\lambda_2 + \lambda_3)t)^m / m!$$

$$P(N_1(t) = n, N_2(t) = m) = e^{-(\lambda_1 + 2\lambda_2 + \lambda_3)t}((\lambda_1 + \lambda_2)t)^n((\lambda_2 + \lambda_3)t)^m / (n!m!)$$

Part b

$$Cov(N_1(t), N_2(t)) = E[N_1(t)N_2(t)] - E[N_1(t)]E[N_2(t)]$$

We already know $E[N_1] = \lambda_1 + \lambda_2$ and $E[N_2] = \lambda_2 + \lambda_3$.

$$E[N_1 N_2] = \Sigma P(N_1, N_2) S_1^i S_2^j = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)\lambda_2$$

$$Cov(N_1, N_2) = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)\lambda_2 - (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3) = \lambda_2$$

Problem 39

Here we have a Poisson process with $\lambda = 2.5$. After 196 events a person dies.

Part a

The mean lifetime of an individual is the mean time it takes for 196 events in this process. This can be denoted as:

$$E[S_{196}]$$

Recall that S_n follows a gamma distribution with parameters $n = 196$ and $\lambda = 2.5$.

Thus we get $E[S_{196}] = 196/2.5 = 78.4$. The mean lifetime is 78.4 years.

Part b

Again we can find the variance of a lifetime of an individual as the varaince to the Gamma distributed R.V. S_{196}. This is:

$$Var(S_{196}) = \frac{196}{(2.5^2)} = 31.36$$

Part c

$$P(S_{196} < 67.2) + F_{S_{196}}(67.2)$$

Recall the form of our gamma R.V.:

$$f(s) = 2.5e^{-2.5s}\frac{(2.5s)^{195}}{195!}$$

GAMMA CDF

Part d

Part e

Problem 41

Since the first event of each component process is exponentially distributed then we know the $P(T_{11} < T_{12}) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Problem 43

Problem Constraints

- Customers arrive at rate λ
- Customer in system departs when one arrives
- Server 1 then 2
- Exponential serve times μ_i

We want to know the proportion of entering customers that complete their service with server 2. This is equal to the probability any given customer is fully served. This occurs when both servers finish before a new arrival comes. A new arrival comes with rate λ in a Poisson process.

$$P(\text{customer is served}) = P(W_1 + W_2 < T_1)$$

We can also rephrase the sum $W_1 + W_2$ as a Gamma distribution with parameters $n = 2$ and $\lambda = \mu_1 + \mu_2$. Now consider a Poisson process with rate $\lambda + \mu_1 + \mu_2$ with subprocesses $W_1 + W_2$ and T_1. An arrival is of type $W_1 + W_2$ with probability $\frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2}$ and the arrival is of type T_1 with probability $\frac{\lambda}{\lambda+\mu_1+\mu_2}$. Now we can say what is the probability that all the arrivals before time $t = 3$ come from the first type $W_1 + W_2$. We stated with happens with probability $\frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2}$. Therefore the proportion of customers that complete their service is:

$$\frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2}$$

Problem 44

- Cars pass Poisson process rate λ
- Wait until no cars in next T units

Part a

The probability that the waiting time is zero is the probability that no cars arrive in the next T time units:

$$P(T_1 > T) = 1 - P(T_1 < T)$$

We know that T_1 is exponentially distributed, so we can use the CDF.

$$P(T_1 > T) = e^{-\lambda T}$$

Part b

$$E[W] = E[E[W|T_1]] = \int_0^\infty E[W|T_1]\lambda e^{-\lambda x} dx$$

Basically we keep waiting $E[W]$ until the next arrival time T_n is longer than T. We know that all these arrival time R.V. T are i.i.d. with exponential distributions, so we get:

$$E[W] = E[W]F_T(T) + E[T_1] = E[W]\int_0^T \lambda e^{-\lambda x} dx + \int_0^T \lambda x e^{-\lambda x} dx$$

$$\int_0^T \lambda e^{-\lambda x} dx = 1 - e^{-\lambda T}$$

$$\int_0^T \lambda x e^{-\lambda x} dx = \frac{e^{-\lambda T}(e^{\lambda T} - \lambda T - 1)}{\lambda}$$

$$E[W] - E[W](1 - e^{\lambda T}) = \frac{e^{-\lambda T}(e^{\lambda T} - \lambda T - 1)}{\lambda}$$

$$E[W](e^{\lambda T}) = \frac{e^{-\lambda T}(e^{\lambda T} - \lambda T - 1)}{\lambda}$$

$$E[W] = \frac{(e^{\lambda T} - \lambda T - 1)}{\lambda}$$