Homework 08 - STAT416

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Chapter 4 Problem 70

Part a

In this Markov chain the state is the number of m black balls in the first urn. This means that there are $m+1$ total states: $0, 1, 2..., m$.

The state i can only transition from i to either $i-1$, i, or $i+1$. Losing one black ball means pulling a black ball from urn 1 and a white ball from urn 2. This occurs with probability $P_{i,i-1} = \frac{i}{m}\frac{i}{m} = \frac{i^2}{m^2}$. Gaining one black ball means pulling a white ball from urn 1 and a black ball from urn 2. This occurs with probability $P_{i,i+1} = \frac{m-i}{m}\frac{m-i}{m} = \frac{(m-1)^2}{m^2}$. Maintaining the same number of black balls in urn 1 involves pulling the same color from both urns. This happens with probability $P_{i,i} = \frac{i}{m}\frac{m-i}{m} + \frac{m-i}{m}\frac{i}{m} = \frac{2i(m-i)}{m^2}$.

Part b

I think that the limiting probabilities of this chain would be the possible ways to have i black balls in urn 1 out of the possible ways to divide the balls.

Part c

$$\pi_i = \pi_{i-1}\frac{(m-1)^2}{m^2} + \pi_i\frac{2i(m-i)}{m^2} + \pi_{i+1}\frac{i^2}{m^2}$$

This chain is isolated to going up or down in the current state and does not involve loops, so it is reversible. If that is true then:

$$\pi_i P_{i,i+1} = \pi_{i+1}P_{i+1,i}$$
$$\frac{\pi_{i+1}}{\pi_i} = \frac{P_{i,i+1}}{P_{i+1,i}} = (\frac{m-i}{i+1})^2$$

If this is true we can then go back and prove π_i by induction.

$$\frac{\pi_1}{\pi_0} = m^2$$
$$\frac{\pi_2}{\pi_0} = (\frac{m}{1}\frac{m-1}{2})^2 = \binom{m}{2}^2$$

Continuing this by induction we get

$$\frac{\pi_i}{\pi_0} = \binom{m}{i}^2$$

From this result we then know how often π_i will happen. We just add in the denominator, which includes all possible options:

$$\pi_i = \frac{\binom{m}{i}^2}{\binom{m}{0}^2 + \binom{m}{1}^2 + ... + \binom{m}{m}^2}$$

Chapter 4 Problem 72

We know that in a time reversible Markov chain the following rate between i and j is equal to the rate between j and i and j and k:

$$\pi_i P_{ij} = \pi_j P_{ji}$$
$$\pi_j P_{jk} = \pi_k P_{kj}$$

The transition rate of i to j to k would be represented as:

$$\pi_i P_{ij} P_{jk}$$

We are trying to prove that

$$\pi_i P_{ij} P_{jk} = \pi_k P_{kj} P_{ji}$$

We can substitute using the first two equations:

$$\pi_i P_{ij} P_{jk} = \pi_i P_{ij} \pi_k P_{kj}/\pi_j = \pi_j P_{ji} \pi_k P_{kj}/\pi_j = \pi_k P_{kj} P_{ji}$$

Therefore from substitution of the adjacent rates of transition we can conclude that the rate of transition from i to j to k is the same as from k to j to i.

Chapter 4 Problem 73

Part a

There are k possible states in this Markov chain. State i as stated in the problem corresponds to player i winning the game. There are two cases when a game is played. The current winner/state could win or the challenger (randomly selected palyer) could win. The transition probability if player i wins is $P_{ii} = \frac{v_i}{(k-1)(v_i+v_j)}$, which is dependent on challenger j. The transition probability if a challenger player j wins is $P_{ij} = \frac{v_j}{(k-1)(v_i+v_j)}$.

Part b

The stationary probability involes two components. Player j is challenge as the winner and remains the winner, or player j challenges player i and becomes the winner.

$$\pi_j = \pi_j \Sigma_i \frac{v_j}{(v_i + v_j)} + (1 - \pi_j)\Sigma_i \frac{v_j}{(v_i + v_j)}$$
$$\pi_j = \Sigma_i \frac{v_j}{(v_i + v_j)}(\pi_j + (1 - \pi_j))$$
$$\pi_j = \Sigma_i \frac{v_j}{(v_i + v_j)}$$
$$\Sigma_j \pi_j = 1$$

Part c

$$Q_{ij} = P_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$
$$\frac{v_j}{(k-1)(v_i + v_j)} = \frac{\pi_j \frac{v_i}{(k-1)(v_i+v_j)}}{\pi_i}$$
$$v_j = \frac{v_i \pi_j}{\pi_i}$$
$$\frac{v_j}{v_i} = \frac{\Sigma_i \frac{v_j}{(v_i+v_j)}}{\Sigma_i \frac{v_i}{(v_i+v_j)}}$$
$$1 = 1$$

Part d

The proportion of games won by player j is the stationary probability of player j.

$$\pi_j = \Sigma_i \frac{v_j}{(v_i + v_j)}$$

Part e

The proportion of games involving player j is the stationary probability, plus when j is picked and loses.

$$\pi_j + (1 - \pi_j)P_{ij}$$
$$\Sigma_i \frac{v_j}{(v_i + v_j)} + (1 - \Sigma_i \frac{v_j}{(v_i + v_j)})\Sigma_i \frac{v_i}{(v_i + v_j)}$$

Chapter 5 Problem 1

$$T \;\; Exp(\lambda = 2)$$

Part a

$$P(T > \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - 1 + e^{-1} = e^{-1}$$

Part b

Here we can apply the memoryless property:

$$P(T > 12 + \frac{1}{2} | T > 12) = P(T > \frac{1}{2}) = e^{-1}$$

Chapter 5 Problem 4

Part a

If the service time of each clerk is exactly 10 minutes, then the probability A is still in the post office after B and C have left is zero. In this case A and B would enter, be served in ten minutes and leave, then C would be served for 10 minutes and leave.

Part b

A could would only still be in the post office after B and C are service if the value of service A is $i = 3$. The sum of B and C would have to be 2.

$$P(A > B + C) = \frac{1}{3}(\frac{1}{3}\frac{1}{3}) = \frac{1}{27}$$

Part c

Service times are Exponentially distributed with parameter μ. Recall that the sum of exponentially distributed random variables is a

$$P(A > B + C) = \int_0^\infty P(A > B + C | C = x) f_c(x) dx = \int_0^\infty P(A > B + x) \mu e^{-\mu x} dx$$

$$P(A > B + C) = \int_0^\infty \frac{1}{2} e^{-\mu x} \mu e^{-\mu x} dx = \frac{\mu}{2} \int_0^\infty e^{-2\mu x} dx = \frac{\mu}{2} (\frac{-e^{-2\mu x}}{2\mu})|_0^\infty = \frac{1}{4}$$

Chapter 5 Problem 12

Part a

$$P(X_1 < X_2 < X_3) = P(X_1 < X_2) P(X_2 < X_3) = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)}$$

Chapter 5 Problem 14

Part a

We want to find the probability that A arrives before and departs after B. This implies that the arrival time for A is smaller than B, but the sum of arrival time and time spent for A is greater than B.

$$P(A<B)P(A+S_A>B+S_B|B>A,S_A>B)=P(A<B)P(B<S_A)P(S_A<S_B)=\frac{\lambda_A\lambda_b\mu_A}{(\lambda_A+\lambda_B)(\lambda_B+\mu_A)(\mu_A+\mu_B)}$$

Chapter 5 Part 19

$$E[R_A]=E[R_A|W=A]+E[R_A|W=B]=E[R_A|W=A]+0=E[R_A|W=A]$$
$$E[R_A|W=A]=Re^{-\alpha t}P(A<B)=\frac{\lambda_A Re^{-\alpha t}}{\lambda_A+\lambda_B}$$

We expect A to earn $\frac{\lambda_A Re^{-\alpha t}}{\lambda_A+\lambda_B}$.

Problem A

Recall that a Markov chain is reversible if $Q_{ij}=P_{ij}=\pi_j P_{ji}/\pi_i$. We had calculated the stationary probabilities.

```
pi_0 <- c(-0.3,0.2,1)
pi_1 <- c(0.2,-0.4,1)
pi_2 <- c(0.1,0.4,1)
A <- cbind(pi_0, pi_1, pi_2)
b <- c(0,0,1)
solve(A,b)
```

```
##      pi_0      pi_1      pi_2
## 0.3529412 0.4117647 0.2352941
```

We can see if this equality holds:

```
stationary <- solve(A,b)
P <- matrix(c(0.7,0.2,0.1,0.2,0.6,0.4,0.1,0.2,0.5), nrow=3)
for (i in 1:3) {
  for (j in 1:3) {
    #print(P[i,j])
    equivalent <- stationary[j]*P[j,i]/stationary[i]
    print(equivalent == P[i,j])
  }
}
```

```
## pi_0
## TRUE
##  pi_1
```

```
## FALSE
##  pi_2
## FALSE
##  pi_0
## FALSE
## pi_1
## TRUE
##  pi_2
## FALSE
##  pi_0
## FALSE
##  pi_1
## FALSE
## pi_2
## TRUE
```

We can see that the Markov chain is not in fact time reversible. An example is going from state 1 to state 2.