Homework 08 - STAT416

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Chapter 4 Problem 70

Part a

In this Markov chain the state is the number of m black balls in the first urn. This means that there are $m+1$ total states: $0, 1, 2..., m$.

The state i can only transition from i to either $i-1$, i, or $i+1$. Losing one black ball means pulling a black ball from urn 1 and a white ball from urn 2. This occurs with probability $P_{i,i-1} = \frac{i}{m}\frac{i}{m} = \frac{i^2}{m^2}$. Gaining one black ball means pulling a white ball from urn 1 and a black ball from urn 2. This occurs with probability $P_{i,i+1} = \frac{m-i}{m}\frac{m-i}{m} = \frac{(m-1)^2}{m^2}$. Maintaining the same number of black balls in urn 1 involves pulling the same color from both urns. This happens with probability $P_{i,i} = \frac{i}{m}\frac{m-i}{m} + \frac{m-i}{m}\frac{i}{m} = \frac{2i(m-i)}{m^2}$.

Part b

I think that the limiting probabilities of this chain would be the possible ways to have i black balls in urn 1 out of the possible ways to divide the balls.

Part c

$$\pi_i = \pi_{i-1}\frac{(m-1)^2}{m^2} + \pi_i\frac{2i(m-i)}{m^2} + \pi_{i+1}\frac{i^2}{m^2}$$

This chain is isolated to going up or down in the current state and does not involve loops, so it is reversible. If that is true then:

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

$$\frac{\pi_{i+1}}{\pi_i} = \frac{P_{i,i+1}}{P_{i+1,i}} = (\frac{m-i}{i+1})^2$$

If this is true we can then go back and prove π_i by induction.

$$\frac{\pi_1}{\pi_0} = m^2$$

$$\frac{\pi_2}{\pi_0} = (\frac{m}{1}\frac{m-1}{2})^2 = \binom{m}{2}^2$$

Continuing this by induction we get

$$\frac{\pi_i}{\pi_0} = \binom{m}{i}^2$$

From this result we then know how often π_i will happen. We just add in the denominator, which includes all possible options:

$$\pi_i = \frac{\binom{m}{i}^2}{\binom{m}{0}^2 + \binom{m}{1}^2 + ... + \binom{m}{m}^2}$$

Chapter 4 Problem 72

We know that in a time reversible Markov chain the following rate between i and j is equal to the rate between j and i and j and k:

$$\pi_i P_{ij} = \pi_j P_{ji}$$
$$\pi_j P_{jk} = \pi_k P_{kj}$$

The transition rate of i to j to k would be represented as:

$$\pi_i P_{ij} P_{jk}$$

We are trying to prove that

$$\pi_i P_{ij} P_{jk} = \pi_k P_{kj} P_{ji}$$

We can substitute using the first two equations:

$$\pi_i P_{ij} P_{jk} = \pi_i P_{ij} \pi_k P_{kj} / \pi_j = \pi_j P_{ji} \pi_k P_{kj} / \pi_j = \pi_k P_{kj} P_{ji}$$

Therefore from substitution of the adjacent rates of transition we can conclude that the rate of transition from i to j to k is the same as from k to j to i.

Chapter 4 Problem 73

Chapter 5 Problem 1

Chapter 5 Problem 4

Chapter 5 Problem 12

Part a

Chapter 5 Problem 14

Part a

Chapter 5 Part 19

Problem A