Stat/Math 415 Homework 10

Due Friday Dec 13; Joseph Sepich (jps6444)

1 Problem 6.8-1

Problem constraints:

$$Y \sim Poisson(\lambda)$$
$$\theta \sim Gamma(\alpha, \beta)$$

Going off this we have:

$$P(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}$$
$$f(\theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

This second pdf would be the prior pdf for our parameter.

1.1 Part a

To get our posterior pdf we must solve for $h(\theta|y)$ where y is our data.

$$h(\theta|y) = \frac{P(y|\theta)f(\theta)}{P(y)} = \frac{P(y|\theta)f(\theta)}{\int P(y|\theta)f(\theta)d\theta}$$
$$\int P(y|\theta)f(\theta)d\theta = \int \frac{n\theta^y e^{-n\theta}}{y!} \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} d\theta$$
$$= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int \theta^{\alpha+y-1} e^{-n\theta-\theta/\beta} d\theta$$
$$= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int \theta^{\alpha+y-1} e^{-\theta(n+1/\beta)} d\theta$$

Now let's multiply by one to make the solution to the integral a trivial one, being a Gamma distribution pdf:

$$= \frac{n^y \Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}}{y!\Gamma(\alpha)\beta^\alpha} \int \frac{\theta^{\alpha+y-1} e^{-\theta(n+1/\beta)}}{\Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}} d\theta$$
$$= \frac{n^y \Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}}{y!\Gamma(\alpha)\beta^\alpha}$$
$$= \frac{n^y \Gamma(y+\alpha)}{y!\Gamma(\alpha)\beta^\alpha (n+1/\beta)^{y+\alpha}}$$
$$= \frac{n^y \Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)(n\beta+1)^{y+\alpha}}$$

Plugging back in to the posterior:

$$h(\theta|y) = \frac{P(y|\theta)f(\theta)}{\int P(y|\theta)f(\theta)d\theta}$$
$$= \frac{\frac{n\theta^y e^{-n\theta}}{y!}\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}}{\frac{n^y\Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)(n\beta+1)^{y+\alpha}}}$$
$$= \frac{\theta^{y+\alpha-1}e^{-\theta(n+1/\beta)}}{\Gamma(y+\alpha)(1/(n+1/\beta))^{y+\alpha}}$$

This means that our posterior distrbution is a Gamma distribution with parameters $\alpha + y$ and $\frac{1}{n+1/\beta}$.

1.2 Part b

Our loss fuction is $[w(y) - \theta]^2$. We want to find our point estimate w(y), which would be the expected value of our posterior distribution:

$$E[\theta|y] = \alpha + y * \frac{1}{n+1/\beta} = \frac{y+\alpha}{n+1/\beta}$$

1.3 Part c

$$w(y) = \frac{y}{n}\frac{n}{(n+1/\beta)} + \frac{\alpha\beta(1/\beta)}{(n+1/\beta)}$$
$$= \frac{y}{(n+1/\beta)} + \frac{\alpha}{(n+1/\beta)}$$
$$= \frac{y+\alpha}{n+1/\beta}$$

This shows that w(y) is the weighted average of the maximum likelihood estimate y/n and the prior mean $\alpha\beta$.

2 Problem 6.8-4

Problem constraints:

$$f(x|\theta) = 3\theta x^2 e^{-\theta x^3}$$
$$g(\theta) = \frac{\theta^{4-1}e^{-\theta/(1/4)}}{\Gamma(4)(1/4)^4} = \frac{3}{128}\theta^3 e^{-4\theta}$$

We want to find the conditional mean of θ given the data in X. This first requires that we discover the posterior distribution from the prior distribution. To do this we need the joint pdf of our random samples.

$$f(x_1, x_2, ..., x_n|\theta) = f(x_1|\theta)f(x_2|\theta)...f(x_n|\theta) = \Pi_{i=1}^n f(x_i|\theta)$$
$$\Pi_{i=1}^n f(x_i|\theta) = 3^n\theta^n e^{-\theta\Sigma_{i=1}^n x_i^3}\Pi_{i=1}^n x_i^2$$

This brigs us to our posterior pdf:

$$h(\theta|x) = \frac{f(x|\theta)g(\theta)}{f(x)} = \frac{f(x|\theta)g(\theta)}{\int f(x|\theta)g(\theta)d\theta}$$

$$\int f(x|\theta)g(\theta)d\theta = \int 3^n\theta^n e^{-\theta\Sigma_{i=1}^n x_i^3}\Pi_{i=1}^n x_i^2 \frac{3}{128}\theta^3 e^{-4\theta}d\theta$$

$$= \frac{3^{n+1}}{128}\Pi x_i^2 \int \theta^{n+3}e^{-\theta(\Sigma_{i=1}^n x_i^3+4)}d\theta$$

From the pattern in these problems, and the fact we should multiply by one to get the new Gamma posterior distribution we know that our new parameters would be:

$$n+4$$

$$\frac{1}{\Sigma_{i=1}^n X_i^3 + 4}$$

Since our expected value is the product of the parameters in the gamma distrbution our final result is:

$$\frac{n+4}{\Sigma_{i=1}^n X_i^3 + 4}$$

3 Problem 6.9-1

We know our random variable x follows the poissson distribution with probability

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

We know that our parameter θ has a gamma distribution with probability

$$f(\theta) = \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

The marginal pmf of X would be the joint pdf with its integral taken over θ:

$$k_1(\theta) = \int P(X)f(\theta)d\theta = \int \frac{\lambda^x e^{-\lambda}}{x!}\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}d\theta$$

$$= \frac{1}{x!\Gamma(\alpha)\beta^\alpha}\int \theta^{x+\alpha-1}e^{-\theta(1+1/\beta)d\theta}$$

Now let's multiply by one so our integral evaluates to a trivial one.

$$= \frac{\Gamma(x+\alpha)(\frac{1}{1+1/\beta})^{x+\alpha}}{x!\Gamma(\alpha)\beta^\alpha}\int \frac{\theta^{x+\alpha-1}e^{-\theta(1+1/\beta)d\theta}}{\Gamma(x+\alpha)(\frac{1}{1+1/\beta})^{x+\alpha}}$$

$$= \frac{\Gamma(x+\alpha)(\frac{1}{1+1/\beta})^{x+\alpha}}{x!\Gamma(\alpha)\beta^\alpha}$$

$$= \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha)\beta^\alpha(1+1/\beta)^{x+\alpha}}$$

$$= \frac{\Gamma(x+\alpha)\beta^x}{x!\Gamma(\alpha)(1+\beta)^x+\alpha}$$

4 Problem Bayesian Decision Exercise

Problem Constraints

- $p(\omega_1) = 0.9$
- $p(\omega_2) = 0.1$

4.1 Part a

Using the loss matrix we should calculate the action that minimizes risk. Risk is defined as:

$$R(a_j) = \Sigma_{i=1}^{n} \lambda(a_j|\omega_i)p(\omega_i)$$

Where $\lambda(a_j|\omega_i)$ is our loss function.

$$R(a_1) = 0 * 0.9 + 10 * 0.1 = 1$$
$$R(a_2) = 1 * 0.9 + 0 * 0.1 = 0.9$$

Since a risk of 0.9 is less than a risk of 1 the optimal decision would be to take action a_2, which is prescribe antibiotics.

4.2 Part b

We now are able to take a blood test. The posterior probabilites of failure are:

- $p(x_1|\omega_2) = 0.3$
- $p(x_2|\omega_1) = 0.2$

Since the two states cover the whole probabily space any x not in that state is in the other state:

- $p(x_1|\omega_1) = 0.7$
- $p(x_2|\omega_2) = 0.8$

We should find the optimal decision give x_1 or x_2. Our updated risk function would be:

$$R(a_j) = \Sigma_{i=1}^{n} \lambda(a_j|\omega_i)p(\omega_i|x_k)$$

First we must find $p(\omega_i|x_k)$ for both x_1 and x_2. To do this we can use Bayes rule.

$$p(\omega_i|x_k) = \frac{p(x_k|\omega_i)p(\omega_i)}{\Sigma_{i=1}^{n} p(x_k|\omega_i)p(\omega_i)}$$

$$p(\omega_1|x_1) = \frac{0.7 * 0.9}{0.7 * 0.9 + 0.3 * 0.1} = \frac{0.63}{0.66} = 0.955$$
$$p(\omega_2|x_1) = \frac{0.3 * 0.1}{0.7 * 0.9 + 0.3 * 0.1} = \frac{0.03}{0.66} = 0.045$$

$$p(\omega_1|x_1) = \frac{0.2 * 0.9}{0.2 * 0.9 + 0.8 * 0.1} = \frac{0.18}{0.26} = 0.692$$

$$p(\omega_2|x_1) = \frac{0.8 * 0.1}{0.2 * 0.9 + 0.8 * 0.1} = \frac{0.08}{0.26} = 0.308$$

For x_1:

$$R(a_1) = 0 * 0.955 + 10 * 0.045 = 0.45$$
$$R(a_2) = 1 * 0.955 + 0 * 0.045 = 0.955$$

For x_2:

$$R(a_1) = 0 * 0.692 + 10 * 0.308 = 3.08$$
$$R(a_2) = 1 * 0.692 + 0 * 0.308 = 0.692$$

According to our new risk value if the test is negative we should choose to prescribe hot tea, since 0.45 is less than 0.955. If the test is positive we should prescribe antibiotics since 3.08 is greater than 0.692.