Stat/Math 415 Homework 10

Due Friday Dec 13; Joseph Sepich (jps6444)

1 Problem 6.8-1

Problem constraints:

$$Y \sim Poisson(\lambda)$$
$$\theta \sim Gamma(\alpha, \beta)$$

Going off this we have:

$$P(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}$$
$$f(\theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

This second pdf would be the prior pdf for our parameter.

1.1 Part a

To get our posterior pdf we must solve for $h(\theta|y)$ where y is our data.

$$h(\theta|y) = \frac{P(y|\theta)f(\theta)}{P(y)} = \frac{P(y|\theta)f(\theta)}{\int P(y|\theta)f(\theta)d\theta}$$
$$\int P(y|\theta)f(\theta)d\theta = \int \frac{n\theta^y e^{-n\theta}}{y!} \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} d\theta$$
$$= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int \theta^{\alpha+y-1} e^{-n\theta-\theta/\beta} d\theta$$
$$= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int \theta^{\alpha+y-1} e^{-\theta(n+1/\beta)} d\theta$$

Now let's multiply by one to make the solution to the integral a trivial one, being a Gamma distribution pdf:

$$= \frac{n^y \Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}}{y!\Gamma(\alpha)\beta^\alpha} \int \frac{\theta^{\alpha+y-1}e^{-\theta(n+1/\beta)}}{\Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}} d\theta$$
$$= \frac{n^y \Gamma(y+\alpha)(\frac{1}{n+1/\beta})^{y+\alpha}}{y!\Gamma(\alpha)\beta^\alpha}$$
$$= \frac{n^y \Gamma(y+\alpha)}{y!\Gamma(\alpha)\beta^\alpha (n+1/\beta)^{y+\alpha}}$$
$$= \frac{n^y \Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)(n\beta+1)^{y+\alpha}}$$

Plugging back in to the posterior:

$$
\begin{aligned}
h(\theta|y) &= \frac{P(y|\theta)f(\theta)}{\int P(y|\theta)f(\theta)d\theta} \\
&= \frac{\frac{n\theta^y e^{-n\theta}}{y!}\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}}{\frac{n^y\Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)(n\beta+1)^{y+\alpha}}} \\
&= \frac{\theta^{y+\alpha-1}e^{-\theta(n+1/\beta)}}{\Gamma(y+\alpha)(1/(n+1/\beta))^{y+\alpha}}
\end{aligned}
$$

This means that our posterior distrbution is a Gamma distribution with parameters $\alpha + y$ and $\frac{1}{n+1/\beta}$.

1.2 Part b

Our loss fuction is $[w(y) - \theta]^2$. We want to find our point estimate w(y), which would be the expected value of our posterior distribution:

$$E[\theta|y] = \alpha + y * \frac{1}{n+1/\beta} = \frac{y+\alpha}{n+1/\beta}$$

1.3 Part c

$$
\begin{aligned}
w(y) &= \frac{y}{n}\frac{n}{(n+1/\beta)} + \frac{\alpha\beta(1/\beta)}{(n+1/\beta)} \\
&= \frac{y}{(n+1/\beta)} + \frac{\alpha}{(n+1/\beta)} \\
&= \frac{y+\alpha}{n+1/\beta}
\end{aligned}
$$

This shows that w(y) is the weighted average of the maximum likelihood estimate y/n and the prior mean $\alpha\beta$.

2 Problem 6.8-4

3 Problem 6.9-1

4 Problem Bayesian Decision Exercise

4.1 Part a

4.2 Part b